LOCALIZATION SEQUENCE IN K-THEORY

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Abstract: We discuss Quillen's localization sequence for schemes in this note. Many contents are copied from De Jong's book. Quillen's paper can be downloaded at http://www.math.ucla.edu/ justinshih/quillen_notes_v1.03j.pdf.

1. **Some definitions**

Recall that an abelian category is a category $\mathcal{C}$, such that: for each $A,\ B \in \mathrm{Ob}\mathcal{C}$, $\mathrm{Hom}(A,\ B)$ has a structure of an abelian group, and the composition law is linear; finite direct sums exists; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and finally, every morphism can be factored into an epimorphism followed by a monomorphism. Define a category to be well- powered when the subobjects of each object $a \in \mathcal{A}$ can be indexed by a small set.

Definition 1.1. Let $\mathcal{A}$ be an abelian category. A *Serre subcategory* of $\mathcal{A}$ is a nonempty full subcategory $\mathcal{C}$ of $\mathcal{A}$, such that given an exact sequence

$$A \longrightarrow B \longrightarrow C$$

with $A, C \in \mathrm{Ob}(\mathcal{C})$, then also $B \in \mathrm{Ob}(\mathcal{C})$.

Note that the last condition is equivalent to the condition: if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact in $\mathcal{A}$, then $B \in \mathcal{C}$ if and only if A and C are in $\mathcal{C}$ on the first page of Swan's book [1].

Lemma 1.2. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C}$ be a subcategory of $\mathcal{A}$. Then $\mathcal{C}$ is a Serre subcategory if and only if the following conditions are satisfied:*

(1) $0 \in Ob(\mathcal{C})$,
(2) *$\mathcal{C}$ is a full subcategory of $\mathcal{A}$,*
(3) *if $C \in Ob(\mathcal{C})$ and $A \in \mathcal{A}$ is isomorphic to a subobject or a quotient object of C, then also $A \in Ob(\mathcal{C})$,*
(4) *if $A \in \mathcal{A}$ is an extension of objects of $\mathcal{C}$ then also $A \in Ob(\mathcal{C})$.*

Moreover, a Serre subcategory is an abelian category such that the inclusion functor is exact.

Lemma 1.3. *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ forms a Serre subcategory of $\mathcal{A}$.*

Definition 1.4. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an exact functor. Then the full subcategory of objects C of $\mathcal{A}$ such that $F(C) = 0$ is called the *kernel of the functor* F, and is sometimes denoted $\mathrm{Ker}(F)$.

Lemma 1.5. *Let $\mathcal{A}$ be an abelian category. Let $\mathcal{C} \subset \mathcal{A}$ be a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor*

$$F : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{C}$$

which is essentially surjective and whose kernel is $\mathcal{C}$. The category $\mathcal{A}/\mathcal{C}$ and the functor F are characterized by the following universal property: For any exact functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\mathcal{C} \subset Ker(G)$ there exists a factorization $G = H \circ F$ for some functor $H : \mathcal{A}/\mathcal{C} \longrightarrow \mathcal{B}$.

Let F be an exact covariant functor from $\mathcal{A}$ to $\mathcal{B}$, where $\mathcal{A}$, $\mathcal{B}$ are abelian categories. Then $\mathcal{C} = \ker F$ is a Serre subcategory and F factors

$$\mathcal{A} \xrightarrow{T} \mathcal{A}/\mathcal{C} \xrightarrow{G} \mathcal{B}.$$

Theorem 1.6. *Suppose that in the above situation, we also have*

(1) *For any $B \in \mathcal{B}$, there is an $A \in \mathcal{A}$ such that FA is isomorphic to B, and*
(2) *if $f : FA \longrightarrow FA'$, then there exists an A'', $h : A'' \longrightarrow A$, and $g : A'' \longrightarrow A'$ such that the following diagram is commutative*

$$\begin{tikzcd} & FA'' \arrow[dl, "Fh"'] \arrow[dr, "Fg"] & \\ FA \arrow[rr, "f"'] & & FA'. \end{tikzcd}$$

and Fh is an isomorphism. Then $G : \mathcal{A}/\mathcal{C} \longrightarrow B$ is an equivalence.

Example 1.7. Let X be a noetherian scheme, $\mathcal{P}$ the category of vector bundles over X (=locally free sheaves of $\mathcal{O}_X$-modules of finite rank) equipped with the usual notion of exact sequences. Let $\mathcal{M}$ be the abelian category of coherent sheaves on X. Let $\mathcal{M}_p(X)$ denote the Serre subcategory of $\mathcal{M}(X)$ consisting of those coherent sheaves whose support is of codimension $\geq p$. (The codimension of a closed subset Z of X is the infimum of the dimension of the local rings $\mathcal{O}_{X,\ z}$ where z runs over the generic points of Z.) Let X_p be the set of points of codimension p in X.

Then $\mathcal{M}_p(X) = \varinjlim(Z)$, where Z run over the closed sub-scheme of codimension $\geq p$. For the definition of direct limit of categories, one can see Srinivas's algebraic K-theory (appendix). Hence by (2.9) of Quillen's paper, we have $K_i(\mathcal{M}_p(X)) = colim(K_i(Z))$. And we have the equivalence of categories

$$\mathcal{M}_p(X)/\mathcal{M}_{p+1}(X) \simeq \coprod_{x\in X_p} \bigcup_n Modf(\mathcal{O}_{X,x}/(\mathrm{Rad}\mathcal{O}_{X,x})^n),$$

Where $Modf$ means the category of finitely generated modules. This equivalence can be easily proved by Theorem 1.6.

2. Quillen's localization sequence

Theorem 2.1 (Localization)**.** *Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$, let $\mathcal{A}/\mathcal{B}$ be the associated quotient abelian category (see for example [Gabriel], [Swan]), and let $e: \mathcal{B} \longrightarrow \mathcal{A}$, $s: \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B}$ denote the canonical functors. Then there is a long exact sequence*

$$\cdots \xrightarrow{s_*} K_1(\mathcal{A}/\mathcal{B}) \longrightarrow K_0(\mathcal{B}) \xrightarrow{e_*} K_0(\mathcal{A}) \xrightarrow{s_*} K_0(\mathcal{A}/\mathcal{B}) \longrightarrow 0.$$

Corollary 2.2. *Let notations be as in section 1. Then*

$$\cdots \longrightarrow K_i(\mathcal{M}_{p+1}(X)) \longrightarrow K_i(\mathcal{M}_p(X)) \longrightarrow \coprod_{x\in X_p} K_i k(x) \longrightarrow K_{i-1}(\mathcal{M}_{p+1}(X)) \longrightarrow \cdots$$

is an exact sequence.

Let $X = \mathrm{Spec}(A)$, where A is a Dedekind domain, and $p = 0$. Then we have an exact sequence

$$\cdots \longrightarrow \coprod_{\mathfrak{p}} K_{i-1}(A/\mathfrak{p}) \longrightarrow K_i(A) \longrightarrow K_i(F) \longrightarrow \coprod_{\mathfrak{p}} K_{i-1}(A/\mathfrak{p}) \longrightarrow \cdots.$$

Since K_{2i} of finite fields are trivial. We have $K_{2i}(A) \longrightarrow K_{2i}(F)$ is injective. In fact, Soule prove that $K_{2i-1}(A) \longrightarrow K_{2i-1}(F)$ is also injective if A is the ring of integers of a number field.

References

[1] Swan, Algebraic K-theory, LNM 76.