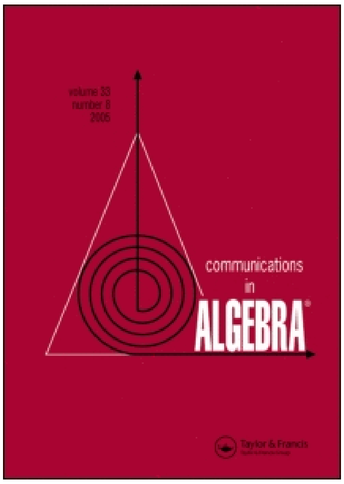

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IMAGINARY QUADRATIC FIELDS WITH ONO NUMBER 3

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In this article, we prove that an imaginary quadratic field F has the ideal class group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if the Ono number of F is 3 and F has exactly 3 ramified primes under the Extended Riemann Hypothesis (ERH). In addition, we give the list of all imaginary quadratic fields with Ono number 3.

Key Words: Ideal class group; Ono number.

2000 Mathematics Subject Classification: 11R11; 11R29.

1. INTRODUCTION

Let F be an imaginary quadratic field with discriminant D, and

$$f_D(x) = \begin{cases} x^2 + x + \dfrac{1-D}{4}, & \text{if } D \equiv 1 \mod 4 \\ x^2 - \dfrac{D}{4}, & \text{if } D \equiv 0 \mod 4. \end{cases}$$

The Ono number p_D of F is defined as $p_D = \max\{\Omega(f_D(i)),\ i \in \mathbb{Z} \cap [0, \frac{-D}{4} - 1]\}$, where $\Omega(f_D(i))$ is the number of prime factors (counting multiplicity) of $f_D(i)$. One can see [7] for details. Let h_D be the class number of $\mathbb{Q}(\sqrt{D})$. Then by the Frobenius–Rabinowitch Theorem [3, 8], $h_D = 1$ if and only if $p_D = 1$. Sasaki proved in [12] that $h_D \geq p_D$ and $h_D = 2$ if and only if $p_D = 2$. Sairaiji and Shimizu proved some very useful inequalities between class numbers and Ono numbers in [9, 10]. Cohen and Sonn conjectured in [2] that $h_D = 3$ if and only if $p_D = 3$ and $|D|$ is a prime. Their conjecture is proved to be true in [4] under Extended Riemann Hypothesis (ERH).

We will show in this article that the Ono number is not only related to the class number, but also related to the structure of the ideal class group. Let $\mathscr{C}_D$ be the ideal class group of F, t_D the number of ramified primes in F. We will prove that $\mathscr{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if and only if $p_D = 3$ and $t_D = 3$. In addition, we give the list of all imaginary quadratic fields with Ono number 3.

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2. MAIN RESULTS

We use the same notations as in Section 1. Let q_D be the smallest prime number which splits completely in $\mathbb{Q}(\sqrt{D})$. By genus theory of Gauss, if $\mathscr{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then $t_D = 3$. By Theorem 2.2 of [11] (or cf. [6]), $p_D = t_D$ if the exponent of $\mathscr{C}_D$ is less than or equal to 2. Thus we have $p_D = 3$. Next we will prove that if $t_D = 3$ and $p_D = 3$, then $\mathscr{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Theorem 2.1 (ERH). *If* $p_D = 3$, *then* $|D| \leq 9.662 \times 10^{10}$.

Proof. Assume that $p_D = 3$. By Theorem 2.1 of [10] (cf. [5] p. 111) or Lemma 17 and Theorem 18 of [2], $p_D > \log_{q_D}(|D|/4 - 1)$. Thus we have $|D/4| < q_D^3$. By Theorem 3.3 of [10],

$$p_D \geq \frac{\log\log 163}{\log 163} \frac{\log|D|}{\log\log|D|}.$$

Hence we have $|D| \leq 10^{15}$. One can find an upper bound of q_D by Theorem 5.1 of [1]. The upper bound in Theorem 5.1 of [1] is made to work for all Galois extensions. When the extension is $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$, that bound is improved in the Table 3 of [1]. By the Table 3 of [1], $q_D \leq (1.881 \cdot \log|D| + 6.18)^2$ if $25 \leq \log|D| \leq 100$. So we have $|D/4| < (1.881 \cdot \log|D| + 6.18)^6$ which implies that $|D| \leq 9.662 \times 10^{10}$. □

By GP/PARI, one can prove that if $|D| \leq 9.662 \times 10^{10}$, then $q_D \leq 167$. In fact, there are only two imaginary quadratic fields satisfying $|D| \leq 9.662 \times 10^{10}$ and $q_D > 157$. They are $q_{-30942935860} = 167$, $q_{-18936628027} = 163$.

Theorem 2.2 (ERH). *If* $p_D = 3$, *then* $|D| < 18629852$.

Proof. Since $q_D \leq 167$ and $|D/4| < q_D^3$ (Theorem 2.1 of [10]), we have $|D| < 4 \cdot 167^3 = 18629852$. □

We can determine all imaginary quadratic fields with Ono number 3 with a help of computer. We give the complete list on next page. By that table, we have the following theorem.

Theorem 2.3 (ERH). *The ideal class group* $\mathscr{C}_D \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *if and only if* $t_D = 3$ *and* $p_D = 3$.

If the ideal class group $\mathscr{C}_D \simeq \mathbb{Z}/4\mathbb{Z}$, then $t_D = 2$ by genus theory. By Sasaki's inequality [12] $h_D \geq p_D$, we have $p_D = 3$ or 4. However, the inverse direction is not correct, i.e., $t_D = 2$ and $p_D = 3$ or 4 do not imply that $\mathscr{C}_D \simeq \mathbb{Z}/4\mathbb{Z}$. The last case of the table is a counterexample. In that case, $t_{-3763} = 2$ and $p_{-3763} = 3$, while $\mathscr{C}_{-3763} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$.

In the following table, (2)(2) means $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, (4) means $\mathbb{Z}/4\mathbb{Z}$, (3) means $\mathbb{Z}/3\mathbb{Z}$, and (2)(3) means $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Note that the following list also proves, the result of [4], namely, $h_D = 3$ if and only if $p_D = 3$ and $t_D = 1$.

Table 1 Table of imaginary quadratic fields with Ono number 3 (ordered by the square free part of D)

No.	$-D$	t_D	p_D	$\mathscr{C}_D$	No.	$-D$	t_D	p_D	$\mathscr{C}_D$
1	$84 = 2^2 \cdot 3 \cdot 7$	3	3	(2)(2)	27	$1012 = 2^2 \cdot 11 \cdot 23$	3	3	(2)(2)
2	23	1	3	(3)	28	283	1	3	(3)
3	$120 = 2^3 \cdot 3 \cdot 5$	3	3	(2)(2)	29	307	1	3	(3)
4	31	1	3	(3)	30	331	1	3	(3)
5	$132 = 2^2 \cdot 3 \cdot 11$	3	3	(2)(2)	31	$355 = 5 \cdot 71$	2	3	(4)
6	$168 = 2^3 \cdot 3 \cdot 7$	3	3	(2)(2)	32	379	1	3	(3)
7	$184 = 2^3 \cdot 23$	2	3	(4)	33	$435 = 3 \cdot 5 \cdot 29$	3	3	(2)(2)
8	$228 = 2^2 \cdot 3 \cdot 19$	3	3	(2)(2)	34	$483 = 3 \cdot 7 \cdot 23$	3	3	(2)(2)
9	59	1	3	(3)	35	499	1	3	(3)
10	$280 = 2^3 \cdot 5 \cdot 7$	3	3	(2)(2)	36	547	1	3	(3)
11	$292 = 2^2 \cdot 73$	2	3	(4)	37	$555 = 3 \cdot 5 \cdot 37$	3	3	(2)(2)
12	$312 = 2^3 \cdot 3 \cdot 13$	3	3	(2)(2)	38	$595 = 5 \cdot 7 \cdot 17$	3	3	(2)(2)
13	$328 = 2^3 \cdot 41$	2	3	(4)	39	$627 = 3 \cdot 11 \cdot 19$	3	3	(2)(2)
14	83	1	3	(3)	40	643	1	3	(3)
15	$340 = 2^2 \cdot 5 \cdot 17$	3	3	(2)(2)	41	$715 = 5 \cdot 11 \cdot 13$	3	3	(2)(2)
16	$372 = 2^2 \cdot 3 \cdot 31$	3	3	(2)(2)	42	$723 = 3 \cdot 241$	2	3	(4)
17	$408 = 2^3 \cdot 3 \cdot 17$	3	3	(2)(2)	43	$763 = 7 \cdot 109$	2	3	(4)
18	107	1	3	(3)	44	$795 = 3 \cdot 5 \cdot 53$	3	3	(2)(2)
19	$520 = 2^3 \cdot 5 \cdot 13$	3	3	(2)(2)	45	883	1	3	(3)
20	$532 = 2^2 \cdot 7 \cdot 19$	3	3	(2)(2)	46	907	1	3	(3)
21	139	1	3	(3)	47	$1003 = 17 \cdot 59$	2	3	(4)
22	$708 = 2^2 \cdot 3 \cdot 59$	3	3	(2)(2)	48	$1243 = 11 \cdot 113$	2	3	(4)
23	$760 = 2^3 \cdot 5 \cdot 19$	3	3	(2)(2)	49	$1387 = 19 \cdot 73$	2	3	(4)
24	$772 = 2^2 \cdot 193$	2	3	(4)	50	$1435 = 5 \cdot 7 \cdot 41$	3	3	(2)(2)
25	$195 = 3 \cdot 5 \cdot 13$	3	3	(2)(2)	51	$1555 = 5 \cdot 311$	2	3	(4)
26	211	1	3	(3)	52	$3763 = 53 \cdot 71$	2	3	(2)(3)

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