K_1 GROUP OF FINITE DIMENSIONAL PATH ALGEBRA

*Xuejun Guo and **Libin Li

*(Department of Mathematics
University of Science and Technology of China
Hefei, Anhui 230026, P.R.China
E-mail: guoxj@mail.ustc.edu.cn)
**(Department of Mathematics, Yangzhou University
Yangzhou, Jiangsu, 205009, P.R.China
E-mail: libinli@mail.ustc.edu.cn)

Abstract. In this paper, by calculating the commutator subgroup of the unit group of finite path algebra $k\Delta$ and the unit group abelianized, we explicitly characterize the K_1 group of finite dimensional path algebra over an arbitrary field.

Keywords: path algebra, K_1 group

1991 MR Subject Classification: 19B99, 16L30

1. Introduction

Given a connected quiver $\Delta = (\Delta_0, \ \Delta_1)$, where Δ_0 is the set of all vertices and Δ_1 is the set of all arrows. Let k be a field, we will denote by $k\Delta$ the path algebra of Δ. In $k\Delta$, we will denote by $k\Delta^+$ the ideal generated by all arrows and so $(k\Delta^+)^n$ is the ideal generated by all paths of length $\geq n$ (cf [1] for more details). For any path $\alpha \in k\Delta$, we will denote by $s(\alpha)$ the start point of α and $e(\alpha)$ the end point of α.

We know that $k\Delta$ can be seen as a graded algebra, i.e., $k\Delta = \oplus_{i=0}^{\infty} A_i$, where A_i is the k-submodule generated by paths whose length are i. If $k\Delta$ is finite dimensional, then there exists a minimum l such that $k\Delta = \oplus_{i=0}^{l} A_i$. Throughout this paper, $k\Delta$ will be assumed to be finite dimensional, i.e., Δ is a finite quiver (This means that Δ_0 and Δ_1 are both finite sets and Δ contains no oriented cycles).

Project supported by the National Natural Science Foundation of China and Doctoral Foundation of Education of China

It is well known that path algebra of quiver plays an important role in representation theory. How to describe the K-group of path algebra is an interesting problem. The K_0 group of $k\Delta$ is $\mathbb{Z}^n$, where n is the number of isomorphism classes of simple $k\Delta$-modules. The image of a module M in $K_0(k\Delta)$ is its dimension vector. One can see the definition of dimension vector and more details in Chapter 2.4 of [1].

Let R be an arbitrary ring. We will denote by $GL_n(R)$ the set of all $n \times n$ invertible matrices and $E_n(R)$ the set of all elementary $n \times n$ matrices. There is a natural way embedding $GL_n(R)$ into $GL_{n+1}(R)$ and embedding $E_n(R)$ in to $E_{n+1}(R)$. Recall that the stable general linear group $GL(R)$ and the stable elementary linear group $E(R)$ are the direct limits of $GL_n(R)$ and $E_n(R)$ respectively, i.e., $GL(R) = \underrightarrow{lim} GL_n(R)$ and $E(R) = \underrightarrow{lim} E_n(R)$. The K_1 group of R is defined to be $GL(R)/E(R)$ (cf [2], [3], [4] for more details).

Calculating the K_1 group of a given ring is generally difficult. In this note, we will characterize the K_1 group of finite dimensional path algebra $k\Delta$ explicitly. First, in part 2, we give the structure of the commutator subgroup of unit group of finite dimensional path algebra (Proposition 2.1 and 2.2). By these two propositions, we can get the unit group abelianized (Proposition 3.1 and 3.3). In part 4, we give the K_1 group for any finite dimensional path algebra (Theorem 4.3).

2. The Commutator Subgroup of the unit group of $k\Delta$

Let R be a ring, we denote by $U(R)$ its group of units. If G is a group, we denote by $G^{'}$ its commutator subgroup. For any finite dimensional path algebra $k\Delta$, the Jacobson radical $k\Delta^+$ is nilpotent. So the unit group of $k\Delta$ is consisting of the elements which have the form $u+\alpha$, where $u \in A_0$ is invertible and $\alpha \in k\Delta^+$. Let $\Delta_0 = \{e_1, \ ..., \ e_n\}$, notice that $u \in A_0$ is invertible if and only if u has the form $u = \sum_{i=1}^n c_i e_i$, where each $c_i \in k^\times$, $k^\times$ denote the multiplicative group of field k.

Proposition 2.1. *If $k \neq \{0, \ 1\}$, then for finite dimensional path algebra $k\Delta$, $p \in U(k\Delta)^{'}$ if and only if $p = 1 + \gamma$ for some $\gamma \in k\Delta^+$.*

Proof. If $p \in U(k\Delta)^{'}$, there exist invertible elements $u, \ v \in A_0, \ \alpha, \ \beta \in k\Delta^+$ such that $p = (u+\alpha)(v+\beta)(u+\alpha)^{-1}(v+\beta)^{-1} = p_1 p_2 p_1^{-1} p_2^{-1}$. Considering the image of p, p_1, p_2 in $k\Delta/k\Delta^+$, we denote them by $\overline{p}$, $\overline{p}_1$, $\overline{p}_2$, respectively. Since $k\Delta/k\Delta^+ \simeq k^n$ is commutative, $\overline{p}_1\overline{p}_2 = \overline{p}_2\overline{p}_1$. So $\overline{p} = \overline{p}_1\overline{p}_2\overline{p}_1^{-1}\overline{p}_2^{-1} = \overline{1}$ which implies that $p = 1 + \gamma$ for some $\gamma \in k\Delta^+$.

Conversely, if $p = 1 + \gamma$, $\gamma \in k\Delta^+$. We will show by induction that p can be expressed as a product of some commutators.

Step 1 Let $p = 1 + c\alpha$, where $0 \neq c \in k$ and α is an arrow from e_i to e_j. We will show that we can choose suitable $u = \sum_{s=1}^n d_s e_s \in U(k\Delta)$ and $c_1 \in k$ such that $p=(1+c_1\alpha)(u+c_1\alpha)(1+c_1\alpha)^{-1}(u+c_1\alpha)^{-1}$. We know that

$$\begin{aligned}
&(1+c_1\alpha)(u+c_1\alpha)(1+c_1\alpha)^{-1}(u+c_1\alpha)^{-1}\\
=&(1+c_1\alpha)(u+c_1\alpha)(1-c_1\alpha)(u^{-1}-u^{-1}c_1\alpha u^{-1})\\
=&1+c_1\alpha-uc_1\alpha u^{-1} \quad (\textit{ note that there is no cycle in } k\Delta)\\
=&1+c_1\alpha-c_1(\sum_{s=1}^{n}d_se_s)\alpha(\sum_{s=1}^{n}d_se_s)\\
=&1+c_1\alpha-c_1(d_ie_i+d_je_j)\alpha(d_i^{-1}e_i+d_j^{-1}e_j)\\
=&1+c_1(1-d_id_j^{-1})\alpha.
\end{aligned}$$

Since $k\neq\{0,\ 1\}$, we can choose suitable d_i, d_j such that $1-d_id_j^{-1}\neq 0$. Let $c_1=c^{-1}(1-d_id_j^{-1})^{-1}$, we have $p=(1+c_1\alpha)(u+c_1\alpha)(1+c_1\alpha)^{-1}(u+c_1\alpha)^{-1}$ to be a simple commutator.

Step 2 Let $p=1+c\alpha\beta$, where α, β are paths and $\alpha\beta\neq 0$, $c\in k$. Since there is no cycle in Δ, we have $\beta\alpha=0$. So $p=(1+c\alpha)(1+\beta)(1-c\alpha)(1-\beta)=(1+c\alpha)(1+\beta)(1+c\alpha)^{-1}(1+\beta)^{-1}$ which is a simple commutator.

Now we can assume that any $p=1+r_1\beta_1+\cdots+r_{t-1}\beta_{t-1}\in U(k\Delta)^{'}$, where each β_i is a path and $r_i\in k$, $1\leq i\leq t-1$.

Step 3 Suppose $p=1+c_1\alpha_1+\cdots+c_t\alpha_t$, where α_i is path and $c_i\in k$, $1\leq i\leq t$. Since there is no oriented cycle in $k\Delta$, we can always choose some suitable i such that $s(\alpha_i)\notin\{e(\alpha_j):\ 1\leq j\leq t,\ j\neq i\}$. So $\alpha_j\alpha_i=0$ for any $1\leq j\leq t$, $j\neq i$. We have $p=(1+c_1\alpha_1+\cdots+c_{i-1}\alpha_{i-1}+c_{i+1}\alpha_{i+1}+\cdots+c_t\alpha_t)(1+c_i\alpha_i)$.

By induction hypothesis, we have $1+c_1\alpha_1+\cdots+c_{i-1}\alpha_{i-1}+c_{i+1}\alpha_{i+1}+\cdots+c_k\alpha_k$ and $(1+c_i\alpha_i)$ belong to $U(k\Delta)^{'}$. So one see $p=1+\gamma\in U(k\Delta)^{'}$. $\square$

Now let us consider the case $k=\{0,\ 1\}$. In this case, an element $p\in k\Delta$ is invertible if and only if $p=1+\alpha$, where $\alpha\in J$.

Proposition 2.2. *if $k=\{0,\ 1\}$, then for finite dimensional path algebra $k\Delta$, $p\in U(k\Delta)^{'}$ if and only if $p=1+\gamma$ for some $\gamma\in k\Delta^{+2}$.*

Proof. If $p\in U(k\Delta)^{'}$, there exist α, $\beta\in k\Delta^+$ such that $p=(1+\alpha)(1+\beta)(1+\alpha)^{-1}(1+\beta)^{-1}$. Since α, β are nilpotent, there exist l such that $\alpha^l=\beta^l=0$. So $p=(1+\alpha)(1+\beta)(1+\sum_{i=1}^{l-1}\alpha^{l-1})(1+\sum_{i=1}^{l-1}\beta^{l-1})=1+\gamma$, where $\gamma\in(k\Delta^+)^2$.

Conversely, if $p=1+\gamma$, $\gamma\in(k\Delta^+)^2$. We will show by induction that p can be expressed as a product of some commutators. First, let $p=1+\alpha\beta$, where α, β are paths and $\alpha\beta\neq 0$. Since there is no cycle in Δ, we have $\beta\alpha=0$. So $p=(1+\alpha)(1+\beta)(1+\alpha)^{-1}(1+\beta)^{-1}$. Next, we assume that if $p=1+\alpha_1^{'}\beta_1^{'}+\cdots+\alpha_{t-1}^{'}\beta_{t-1}^{'}$, where $\alpha_i^{'}$, $\beta_i^{'}$ are paths, $1\leq i\leq t1$, then $p\in U(k\Delta)^{'}$.

Now suppose $p = 1 + \alpha_1\beta_1 + \cdots + \alpha_t\beta_t$, where α_i, β_i are paths, $1 \le i \le t$. Since $k\Delta$ is finite dimensional, we can choose some suitable i such that $s(\alpha_i\beta_i) \notin \{e(\alpha_j\beta_j) : \ 1 \le j \le t, \ j \ne i\}$. So $\alpha_j\beta_j\alpha_i\beta_i = 0$ for any $1 \le j \le t$, $j \ne i$. We have $p = (1 + \alpha_1\beta_1 + \cdots + \alpha_{i-1}\beta_{i-1} + \alpha_{i+1}\beta_{i+1} + \cdots + \alpha_t\beta_t)(1 + \alpha_i\beta_i)$. Note that p is a product of elements which are sum of 1 and no more than $t-1$ paths of length more than one. So by induction, we prove that $p = 1 + \gamma \in U(k\Delta)^{'}$. $\square$

3. Computation of $U(k\Delta)^{ab}$

We denote by $U(k\Delta)^{ab}$ the unit group $U(k\Delta)$ abelianized. In this part, we will compute $U(k\Delta)^{ab}$ for finite dimensional path algebra $k\Delta$.

If $k \ne \{0, \ 1\}$, then for any $p = u + c_1\alpha_1 + \cdots + c_k\alpha_k \in U(k\Delta)$, where $u \in A_0$ is invertible, $\alpha \in A_i$, we have $u^{-1}p = 1 + u^{-1}c_1\alpha_1 + \cdots + u^{-1}c_k\alpha_k$. So by Proposition 2.1, $u^{-1}p \in U(k\Delta)^{'}$, i.e., $p \equiv u (\mathrm{mod}\ U(k\Delta)^{'})$. We write this fact as the following proposition.

Proposition 3.1. *$U(k\Delta)^{ab} \simeq (k^{\times})^n$, where $n = |\Delta_0|$ and $k^{\times}$ denote the multiplicative group of k.* $\square$

In the case $k = \{0, \ 1\}$, $U(k\Delta)^{ab}$ is somewhat different to the corresponding part of Proposition 3.1.

Lemma 3.2. *If $k = \{0, \ 1\}$, then for any $p = 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_l \in U(k\Delta)$, $p \equiv 1 + \alpha_1 (\mathrm{mod}\ U(k\Delta)^{'})$, where $\alpha_i \in A_i$.*

Proof. Let $\gamma_1 = \alpha_2 + \cdots + \alpha_l \in U(k\Delta) \subset (k\Delta^+)^2$, then $p = 1 + \alpha_1 + \gamma_1$. Since α_1 is nilpotent, we have $(1 + \alpha_1)^{-1} = 1 - \alpha_1 + \gamma_2$, where $\gamma_2 \in (k\Delta^+)^2$. So $p(1 + \alpha_1)^{-1} = 1 + \beta$, where $\beta \in (k\Delta^+)^2$. By Proposition 2.2, we know $1 + \beta \in U(k\Delta)^{'}$. So $p \equiv 1 + \alpha_1 (\mathrm{mod}\ U(k\Delta)^{'})$. $\square$

Proposition 3.3. *If $k = \{0, \ 1\}$, then $U(k\Delta)^{ab} \simeq G_2^m$, where $G_2 = \{1, \ -1\}$ is the finite group of order 2, $m = |\Delta_1|$.*

Proof. Let α_{11}, ..., α_{1m} be the set of all arrows in $k\Delta$. For any $p \in k\Delta$, we will denote by $\overline{p}$ the image of p in $U(k\Delta)^{ab}$. Assume $p_i = 1 + \alpha_{1i}$, then $p_i^2 = 1$.

For arbitrary $p = 1 + \alpha_1 + \alpha_2 + \cdots + \alpha_l \in U(k\Delta) \in k\Delta$, where $\alpha_i \in A_i$, by Lemma 3.2, $p \equiv 1 + \alpha_1 (\mathrm{mod}\ U(k\Delta)^{'})$. Suppose that $\alpha_1 = d_1 a_{11} + \cdots + d_m a_{1m}$, where d_i is 0 or 1. Since there is no oriented cycle in R, by the method used in the proof of Proposition 2.2, we can choose some suitable $\delta(1)$ such that $1 + \alpha_1 = (1 + \sum_{t=1, \ t \ne \delta(1)}^{j_1} d_t\alpha_{1t})(1 + \alpha_{1\delta(1)})$. Continuing these steps on, we get $\delta(2)$, ..., $\delta(j_1)$ such that $1 + \alpha_1 = (1 + \alpha_{1\delta(j_1)})(1 + \alpha_{1\delta(j-1)}) \cdots (1 + \alpha_{1\delta(1)})$. By this equation, $\overline{p}$ can be expressed as a product of some $\overline{p}_i$, where $\overline{p}_i$ is an element of order 2 stated above. So $U(k\Delta)^{ab} \simeq G_2^m$. $\square$

4. Computation of $K_1(k\Delta)$

Let R be a ring, we denote by $V(R)$ the subgroup of unit group $U(R)$ generated by $\{(1+ab)(1+ba)^{-1} : 1+ab \in U(R)\}$ (cf. [3], [5] for more details).

Lemma 4.1 ([3], Proposition 53). *Let R be a semilocal ring, then $K_1(R) \simeq U(R)/V(R)$.* □

Since $k\Delta/k\Delta^+ \simeq k^n$, we know that $k\Delta$ is a semilocal ring. So in order to compute $K_1(k\Delta)$, we need only to compute $U(R)/V(R)$.

Lemma 4.2. *For any finite dimensional path algebra $k\Delta$, $V(k\Delta) = U(k\Delta)^{'}$*

Proof. By Lemma 1.1 (i) of [5], $(U(k\Delta))^{'} \subset V(k\Delta)$. So we need only to prove that $V(k\Delta)/(U(k\Delta))^{'} = 0$. Let $p \in V(k\Delta)$ and suppose that $p = (1+ab)(1+ba)^{-1}$, where $a = u + \alpha_1 + \cdots + \alpha_l$, $b = v + \beta_1 + \cdots + \beta_l$, $\alpha_i, \beta_i \in A_i$, $1 \le i \le l$. For any element $x \in U(k\Delta)$, we denote by $\overline{x}$ the image of x in $U(k\Delta)/U(k\Delta)^{'}$. So $\overline{p} = \overline{(1+ab)(1+ba)^{-1}} = (\overline{1+ab})(\overline{1+ba})^{-1}$. By Part 3, we have $\overline{1+ab} = \overline{1+u(\beta_1+\cdots+\beta_l)+v(\alpha_1+\cdots+\alpha_l)} = \overline{1+ba}$. So $\overline{p} = 1$ and $V(k\Delta) = U(k\Delta)^{'}$. □

Theorem 4.3. *For any finite dimensional path algebra $k\Delta$,*

$$K_1(k\Delta) = U(k\Delta)^{ab} = \begin{cases} (k^{\times})^n, & if\ k \neq \{0,\ 1\} \\ G_2^m, & if\ k = \{0,\ 1\} \end{cases},$$

where $n = |\Delta_0|$, $m = |\Delta_1|$, $k^{\times}$ is the multiplicative group of k and G_2 is the multiplicative group $\{1, -1\}$.

Proof. By Proposition 3.1, Proposition 3.2, Lemma 4.1 and Lemma 4.2, this Theorem is got easily. □

Acknowledgments The authors were grateful to thank Professor Song Guangtian and Professor Pu Zhang for their stimulating guidance and their valuable suggestions. They would like to thank the participants for several discussion in the section on representation theory and quantum group.

References

[1] Ringel C.M. Tame algebras and integral quadratic forms, LNM 1099, Springer-Verlag, 1984

[2] Bass H. Algebraic K-theory, Benjamin, New York, 1968

[3] Silvester J.R. Introduction to algebraic K-Theory, Chapman and Hall, 1981

[4] Rosenberg J. Algebraic K-Theory and Its Applications, GTM 147, Springer-Verlag, 1995

[5] Menal P. and Moncasi J. K_1 of Von Neumann regular rings, J. Pure Appl. Algebra 1984, 33: 295-312