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Diagonability of idempotent matrices over noncommutative rings

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Abstract

Let R be an arbitrary ring. In this paper, the following statements are proved: (a) Each idempotent matrix over R can be diagonalized if and only if each idempotent matrix over R has a characteristic vector. (b) An idempotent matrix over R can be diagonalized under a similarity transformation if and only if it is equivalent to a diagonal matrix. (a) and (b) generalize Foster's and Steger's theorems to arbitrary rings. We give some new results about 0-similarity of idempotent matrices over R. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

In 1945, Foster examined the following questions: for a commutative ring R, when can we find an invertible matrix P over R such that $PAP^{-1} = \mathrm{diag}\{e_1, \ldots, e_n\}$ for a given idempotent matrix A over R? The problem concerns not only matrix theory but also module theory and algebraic K-theory. He proved the following theorem (cf. [1, Theorem 10]).

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Foster's theorem. *The following are equivalent for a commutative ring R with identity*:

(a) *Each idempotent matrix over R is diagonalizable under a similarity transformation.*

(b) *Each idempotent matrix over R has a characteristic vector.*

In 1966, Steger in [2] (or, see [3, IV.52 Theorem]) utilized Foster's theorem to prove the following theorem.

Steger's theorem. *Let R be a commutative ring with identity and A be an $n \times n$ idempotent matrix over R. If there exist invertible matrices P and Q such that PAQ is a diagonal matrix, then there is an invertible matrix U over R such that UAU^{-1} is a diagonal matrix.*

In this paper, we will demonstrate that Foster's theorem and Steger's theorem can be generalized to an arbitrary ring with identity.

Let R be a ring with identity, a and $b \in R$, we say that a is equivalent to b, denoted by $a \simeq b$, if there exist invertible elements u and $v \in R$ such that $uav = b$; a is called similar to b, denoted by $a \sim b$, if there exists an invertible element $u \in R$ such that $uau^{-1} = b$. Let $A \in R^{m\times n}$, $B \in R^{s\times t}$, we say that A is 0-equivalent to B, denoted by $A \overset{0}{\simeq} B$, if there exist sufficiently large integers $p \geqslant \max\{m, n\}$ and $q \geqslant \max\{n, t\}$, $P \in GL(p, R)$ and $Q \in GL(q,R)$ such that

$$P\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} Q = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}_{p\times q}.$$

We say that A is 0-similar to B, denoted by $A \overset{0}{\sim} B$, if there exist sufficiently large integers $p \geqslant \max\{m, n, s, t\}$ and $P \in GL(p, R)$ such that

$$P\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}.$$

By [4, Lemma 1.2.1], $A \overset{0}{\sim} B$ if and only if the corresponding finitely generated projective R-modules are isomorphic. One can find also the definition of 0-similarity in [3]. It is obvious that "similar $\Longrightarrow$ 0-similar" and "equivalent $\Longrightarrow$ 0-equivalent". Theorems 10 and 11 give two equivalent conditions for two matrices to be 0-similar.

2. Main results

Lemma 1. *Let R be a ring, $a, b \in R$ with $a^2 = a$ and $aba = a$, then $a \sim ab \sim ba$.*

Proof. Since $(a-ab)^2 = a - ab - aba + abab = 0$, let $t = 1 - a + ab$, then t is invertible and $t^{-1} = 1 + a - ab$. So $tabt^{-1} = (1 - a + ab)ab(1 + a - ab) = a$, hence $a \sim ab$. Similarly, it can be proved that $a \sim ba$. □

Theorem 2. *Let R be a ring, $a, b \in R$ with $a^2 = a$ and $b^2 = b$, then $a \sim b$ if and only if $a \simeq b$.*

Proof. It is only needed to prove that "$a \simeq b \Longrightarrow a \sim b$". Suppose that there exist invertible elements p and $q \in R$ such that $paq = b$. Let $s = q^{-1}p^{-1}$, then $pap^{-1} = paqq^{-1}p^{-1} = bs$, so $a \sim bs$ and $bsb = b$. By Lemma 1, $bs \sim b$, so $a \sim bs \sim b$. □

Proposition 3. *Let R be a ring and a, b be idempotents of R. If $(a-b)^2 = 0$, then $a \sim ab \sim ba \sim b$.*

Proof. Since $(a-b)^2 = a^2 - ab - ba + b^2 = 0$, so we have $a + b = ab + ba$ and $a(a+b) = a^2b + aba$, i.e., $a + ab = ab + aba$ which implies $a = aba$. Similarly, we have $b = bab$. So by Lemma 1, $a \sim ab \sim ba \sim b$. □

Theorem 4. *Let A be an idempotent matrix over a ring R. If A is equivalent to a block diagonal matrix $B = \mathrm{diag}\{B_1, B_2, \ldots, B_m\}$, then for any $1 \leqslant i \leqslant m$, there exist matrices S_{ii} such that $A \sim D = \mathrm{diag}\{B_1S_{11}, B_2S_{22}, \ldots, B_mS_{mm}\}$. Moreover,*
1. *$B_i \neq 0 \Longleftrightarrow B_iS_{ii} \neq 0, \quad i = 1, 2, \ldots, m$.*
2. *If $B_i^2 = B_i$, S_{ii} can be chosen to be the identity matrix.*

Proof. Assume that there exist $P, Q \in GL(n, R)$ such that $PAQ = B$. Let $S = Q^{-1}P^{-1} \in GL(n, R)$, then $P^{-1}AP = PAQQ^{-1}P^{-1} = BS$. Let $S = (s_{ij})_{m\times m}$ be the block matrix with the same block type of B, then $A \sim BS$ and $(BS)^2 = BS$, $BSB = B$. So we have $B_iS_{ii}B_i = B_i$, $B_iS_{ij}B_j = 0$, $1 \leqslant i \neq j \leqslant n$. Let $D = \mathrm{diag}\{B_1S_{11}, \ldots, B_mS_{mm}\}$, then $D^2 = D$ and $DB = B$, $BSD = D$. Since $(D - BS)^2 = D^2 - DBS - BSD + (BS)^2 = 0$, $(I - (D - BS))^{-1} = I + (D - BS)$. So let $T = I - (D - BS)$, then $TBST^{-1} = (I - (D - BS))BS(I + (D - BS)) = D$ which implies $A \sim BS \sim D$.

Observe that $B_iS_{ii}B_i = B_i$, so $B_i \neq 0 \Longleftrightarrow B_iS_i \neq 0$. To show (2), since $B_i^2 = B_i$, by Lemma 1, $B_i \sim B_iS_{ii}$. So S_{ii} can be changed as an identity matrix. □

The following corollary is a generalization of Steger's theorem.

Corollary 5. *Let A be an $n \times n$ idempotent matrix over a ring R. If A is equivalent to a diagonal matrix, then A is similar to a diagonal matrix.*

Using Theorem 4, we obtain the following corollaries about the diagonability of idempotent matrices.

Corollary 6. *Let A be an $n \times n$ idempotent matrix over a ring R. If A has an invertible $k \times k$ submatrix, $1 \leqslant k \leqslant n$, then $A \sim \operatorname{diag}\{I_k, B\}$.*

Proof. By elementary transformations, the invertible $k \times k$ submatrix can be put at the left-up corner of A, so A is equivalent to $\operatorname{diag}\{I_k, B\}$. By Theorem 4, A is similar to $\operatorname{diag}\{I_k, B_1\}$. □

Corollary 7. *Let R be a ring, and*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a block idempotent matrix over R with $A_{11}^2 = A_{11}$, then $A \simeq \operatorname{diag}\{A_{11}, A_{22}^2\}$ moreover $A \sim \{A_{11}, B_{22}\}$, where B_{22} is an idempotent matrix.

Proof. Since $A^2 = A$, so $A_{12}A_{21} = 0$, $A_{22}^2 - A_{12} - A_{21}A_{12} = 0$. We have

$$\begin{pmatrix} I & 0 \\ -A_{21} & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ -A_{21} & I \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{12} \end{pmatrix}$$

and

$$\begin{pmatrix} I & -A_{12} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{12} \end{pmatrix}\begin{pmatrix} I & -A_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{12} \end{pmatrix},$$

so $A \simeq \operatorname{diag}\{A_{11}, A_{22}^2\}$ then, by Theorem 4, the second part follows. □

Corollary 8. *Let A be an idempotent matrix over a ring R, and let*

$$A \sim \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

Then

$$A \sim \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}$$

Proof. Since $A^2 = A$, $B_{11}^2 = B_{11}$ and $B_{22}^2 = B_{22}$. By Corollary 7, $B \simeq \operatorname{diag}\{B_{11}, B_{22}^2\} = \operatorname{diag}\{B_{11}, B_{22}\}$, then, by Theorem 4, $A \sim \operatorname{diag}\{B_{11}, B_{22}\}$. □

Let R be an arbitrary ring. Recall that $\alpha = (a_1, \ldots, a_n) \in R^n$ is called a right unimodular vector if there exists $(b_1, \ldots, b_n) \in R^n$ such that $a_1b_1 + \cdots + a_nb_n = 1$. A right unimodular vector $(a_1, a_2, \ldots, a_n)$ in R^n is completable if it can be seen as the first row of some invertible matrix over R. Let A be an $n \times n$ matrix over R, recall that α is a characteristic vector of A if $\alpha \in R^n$ is a completable right unimodular vector and $\alpha A = \lambda\alpha$ for some λ in R (we call λ the characteristic value of α). The following theorem is a generalization of Foster's theorem.

Theorem 9. *The following are equivalent for an arbitrary ring R with identity*:
1. *Each idempotent matrix over R is diagonalizable under a similarity transformation* (*i.e. R is a projectively trivial ring*).
2. *For each nonzero projective left R-module P*, *there exist nonzero idempotent* $e_1, e_2, \dots, e_t$ *in R such that* $P \simeq Re_1 \oplus Re_2 \oplus \cdots \oplus Re_t$.
3. *Each idempotent matrix over R has a characteristic vector.*

Proof. By Lemma 1.2.1 of [4], "(1) $\Longrightarrow$ (2)" is easily got.

(1) $\Longrightarrow$ (3). Since there exists an invertible matrix P over R such that $PA = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}P$, the first row of P is a characteristic vector of A.

(3) $\Longrightarrow$ (1). Let A be an idempotent matrix over R with a characteristic vector α: $\alpha A = \lambda\alpha$, then α can be completed to $P \in GL(n, R)$, so

$$PA = \begin{pmatrix} \lambda & 0 \\ * & * \end{pmatrix} P.$$

Since A is idempotent, by Corollary 8, $A \sim \mathrm{diag}\{\lambda, B_2\}$, then by induction, the theorem is proved. □

Finally, let us discuss the 0-similarity of idempotent matrices.

Theorem 10. *Let* $A \in M_m(R)$, $B \in M_n(R)$ *be idempotent matrices. Then* $A \overset{0}{\sim} B$ *if and only if there exist* $m \times n$ *matrix P and* $n \times m$ *matrix Q over R such that* $PQ = A$, $QP = B$.

Proof. If there exists $T \in GL(k, R)$, $k \geqslant \max\{m, n\}$, such that

$$T\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}T^{-1} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}.$$

Decompose T and T^{-1} into blocks corresponding to $\mathrm{diag}\{A, 0\}$ as

$$T = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

Then we have $Q_{11}AP_{11} = B$, $P_{11}BQ_{11} = A$. Let $P = AP_{11}B$ and $Q = BQ_{11}A$, then P is an $m \times n$ matrix, Q is an $n \times m$ matrix and $PQ = A$, $QP = B$.

On the other hand, if there exist $m \times n$ matrix P and $n \times m$ matrix Q over R such that $PQ = A$, $QP = B$. Let

$$T = \begin{pmatrix} 1-A & AP \\ BQ & 1-B \end{pmatrix}.$$

It is easy to verify that

$$T^2 = \begin{pmatrix} (1-A)^2 + APBQ & (1-A)AP + AP(1-B) \\ BQ(1-A) + (1-B)BQ & BQAP + (1-B)^2 \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}.$$

Since $PQP = AP$ and $PQP = PB$, we have $AP = PB$. Similarly, we have $BQ = QA = QPQ$. So

$$\begin{aligned} T\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}T^{-1} &= \begin{pmatrix} 1-A & AP \\ BQ & 1-B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-A & AP \\ BQ & 1-B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ BQA & 0 \end{pmatrix}\begin{pmatrix} 1-A & AP \\ BQ & 1-B \end{pmatrix} \\ &= \mathrm{diag}\{0, BQAAP\} = \mathrm{diag}\{0, B\}. \end{aligned}$$

Hence $\mathrm{diag}\{A, 0\} \sim \mathrm{diag}\{0, B\} \sim \mathrm{diag}\{B, 0\}$. □

Let R be a Dedekind infinite ring (i.e., there exist a and $b \in R$ such that $ab = 1$, $ba = e \neq 1$), then by Theorem 10, 1 is 0-similar to e, but it is obvious that 1 is not similar to e.

Theorem 11. *Let $A \in M_m(R)$, $B_1 \in M_{n_1}(R)$ and $B_2 \in M_{n_2}(R)$ be idempotent matrices over a ring R. Then A is 0-similar to $B = \mathrm{diag}\{B_1, B_2\}$ if and only if A can be decomposed into the sum of two order m orthogonal idempotent matrices A_1, A_2, i.e., $A = A_1 + A_2$, moreover $A_1 \overset{0}{\sim} B_1$, $A_2 \overset{0}{\sim} B_2$.*

Proof. If $A \overset{0}{\sim} B$, by Theorem 10, there exist matrices P and Q such that $PQ = A$ and $QP = B$. Decompose P, Q into blocks as

$$P = (P_1 \quad P_2), \quad Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

then

$$PQ = (P_1 \quad P_2)\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = P_1Q_1 + P_2Q_2 = A,$$

and

$$QP = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}(P_1 \quad P_2) = \begin{pmatrix} Q_1P_1 & Q_1P_2 \\ Q_2P_1 & Q_2P_2 \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}.$$

So $Q_1P_1 = B_1$, $Q_1P_2 = 0$, $Q_2P_1 = 0$ and $Q_2P_2 = B_2$. From $A^2 = A$, we have $P_1Q_1 + P_2Q_2 = (P_1Q_1)^2 + (P_2Q_2)^2$. Times P_1Q_1 on the two sides of the equation, we have $(P_1Q_1)^2 = (P_1Q_1)^3$, so $(P_1Q_1)^2 = (P_1Q_1)^4$. Similarly we have $(P_2Q_2)^2 = (P_2Q_2)^4$. Let $A_1 = (P_1Q_1)^2$, $A_2 = (P_2Q_2)^2$. Then A_1 and A_2 are orthogonal idempotent matrices, $A = A_1 + A_2$. Since $A_1 = (P_1Q_1P_1)Q_1$ and $B_1 = Q_1(P_1Q_1P_1)$, by Theorem 10, $A_1 \overset{0}{\sim} B_1$, $A_2 \overset{0}{\sim} B_2$.

Inversely, assume that $A = A_1 + A_2$, where A_1 and A_2 are orthogonal idempotent matrices, moreover $A_1 \overset{0}{\sim} B_1$, $A_2 \overset{0}{\sim} B_2$. Let

$$S = (A_1 \quad A_2), \quad T = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

Then $ST = A_1 + A_2 = A$ and $TS = \operatorname{diag}\{A_1, A_2\}$. By Theorem 10,

$$A \overset{0}{\sim} \operatorname{diag}\{A_1, A_2\} \overset{0}{\sim} B. \qquad \square$$

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