UNIVERSAL OBJECT OF A FUNCTOR

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Abstract: The definition and several examples are given. One can see the reference [1] for details..

1. **The definition and examples**

Definition 1.1. Let $F:\ \mathcal{C}^{opp} \longrightarrow (Sets)$ be a functor, i.e., a contravariant functor from $\mathcal{C}$ to the category of sets. A universal object for F is a pair $(X,\ \xi)$ consisting of an object X of C, and an element $\xi \in F(X)$ with the property that for each object U of $\mathcal{C}$ and each $\sigma \in F(U)$, there is a unique arrow $f:\ U \longrightarrow X$ such that $F(f)(\xi) = \sigma \in F(U)$.

Example 1.2. Let F be a functor from $(Sets)^{opp} \longrightarrow (Sets)$ which maps each set A to the set of all subsets of A. If $A = \{a,\ b\}$, then $F(A) = \{\emptyset,\ \{a\},\ \{b\}, A\}$.

The universal object of F is $(\{0,\ 1\}, \{1\})$.

Example 1.3. Let F be the functor $Sch^{opp} \longrightarrow (Sets)$ which maps each scheme to the set of isomorphism classes of triples $(E,\ e_1,\ e_2)$ where E is an elliptic curve over S and $e_1,\ e_2$ is a pair of sections of E over S which forms a $\mathbb{Z}/N$-basis of ${}_NE = \mathrm{Ker}(N : E \longrightarrow E)$.

The universal object of F is $(Y(N),\ (E,\ e_1,\ e_2))$, where $Y(N) = (\Gamma(N)\backslash\mathcal{H}) \times (\mathbb{Z}/N\mathbb{Z})^\times$, this E is called the universal elliptic curve over $Y(N)$. Note that we don't need the Weil pairing of $(e_1,\ e_2)$ to be some fixed N-th root of unity. Hence $(\mathbb{Z}/N\mathbb{Z})^\times$ must be added. $Y(N)$ is the modular curve over $\mathbb{Q}$ of level N without cusps, which represents the functor F.

Example 1.4. Let F be the functor $Sch^{opp} \longrightarrow (Sets)$ which maps S to $\Gamma(S)^\times$. The universal object of F is $(\mathbb{G}_m,\ x)$, where $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[x,\ x^{-1}])$ and x is invertible in $\mathbb{Z}[x,\ x^{-1}]$.

References

[1] Fundamental Algebraic Geometry: Grothendieck's FGA Explained.