Computing the tame kernel of $\mathbb{Q}(\zeta_8)$

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Abstract: In this paper, it is proved that the tame kernel of $\mathbb{Q}(\zeta_8)$ is trivial.
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1. Introduction

Let F be a number field and $K_2\mathcal{O}_F$ the tame kernel of F. Although we know many properties about $K_2\mathcal{O}_F$, it is still a difficult problem to determine its structure, even for a quadratic field F. Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field, we know that

$K_2\mathcal{O}_F$ is trivial for $d = -1,\ -2,\ -3,\ -5,\ -6,\ -11,\ -19$ (see [5], [7], [8], [9]) and $K_2\mathcal{O}_F \simeq \mathbb{Z}/2\mathbb{Z}$ for $d = -7, -15, -35$ (see [5], [8]). The latest computation of $K_2\mathcal{O}_F$ on imaginary quadratic fields can be found in [2], [3] and [4]. In this paper, we prove that $K_2\mathcal{O}_F$ is trivial for $F = \mathbb{Q}(\zeta_8)$, where ζ_8 is a primitive $8th$ root of unity. This F is totally imaginary with degree 4.

2. Preliminaries

Let F be a number field with $\mathcal{O}_F$ the ring of integers of F and let S_∞ denote the set of archimedean places of F. If S is a set of places containing S_∞, we put $\mathcal{O}_S = \{a \in F|\ v(a) \geq 0,\ for\ all\ v \notin S\}$, which is the ring of S-integers. Assume that $\mathcal{P}$ is the maximal ideal corresponding to $v \notin S$ and let $k(v) = \mathcal{O}_S/\mathcal{P}$. Put $N(v) = |k(v)|$.

We shall write $K_2^S F$ for the subgroup of K_2F generated by $\{x,\ y\}$, where $x,\ y \in \mathcal{O}_S^* = U$. We can list the finite places of F, $v_1,\ v_2,\ ...,\ v_n,\ ...$, so that $N(v_i) \leq N(v_{i+1})$ for all i. Put $S_m = S_\infty \cup \{v_1,\ ...,\ v_m\}$. Let $S = S_m$, $v = v_{m+1} \notin S$, $S' = S_{m+1} = S \cup \{v\}$ and $U = \mathcal{O}_S^*$, let $\mathcal{P}$ be the maximal ideal corresponding to v, let $k = k(v) = \mathcal{O}_S/\mathcal{P}$ and let k^* denote the multiplicative group of k. Let

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∂_v be the tame map corresponding to v. In [1], Bass and Tate show that for a sufficiently large m, the induced homomorphism

$$\partial_v : \quad K_2^{S'} F / K_2^S F \longrightarrow k^*$$

is an isomorphism, which implies that

$$K_2 \mathcal{O}_F = \ker \Big(\partial : \quad K_2^{S_m} F \longrightarrow \coprod_{v \in S_m \setminus S_\infty} k^*(v) \Big).$$

So if we can make m relatively small and get sufficiently many relations satisfied by elements of $K_2^{S_m} F$, then we can determine the tame kernel of F.

Suppose $\mathcal{P}$ is generated by π. Let U_1 be the subgroup of U generated by $(1+\pi U) \cap U$ and β be the natural quotient map from U to k^*. In [5], Tate proved the following lemma which is very useful to prove the surjectivity of ∂_v.

Lemma 2.1. *Suppose that W, C, G are subsets of U such that*
(1) $W \subset CU_1$ *and* W *generates* U,
(2) $CG \subset CU_1$ *and* $\beta(G)$ *generates* k^*,
(3) $1 \in C \cap \ker \beta \subset U_1$.
Then ∂_v is bijective.

3. **Computation of $K_2 \mathbb{Z}[\zeta_8]$**

In this section, we will abbreviate ζ_8 to ζ. Let $F = \mathbb{Q}(\zeta)$. Then $\{1, \ \zeta, \ \zeta^2, \ \zeta^3\}$ is an integral basis of $\mathcal{O}_F$. We have that $h(F) = 1$ and $D(F) = 256$. There are four embeddings of F into $\mathbb{C}$: identity and its conjugation $\zeta \mapsto \overline{\zeta}$ $\sigma : \ \zeta \mapsto \zeta^3$ and its conjugation $\overline{\sigma} : \ \zeta \mapsto \zeta^5$. It is obvious that for every $x = a + b\zeta + c\zeta^2 + d\zeta^3$,

$$N(x) = (a^2 + c^2)^2 + (b^2 + d^2)^2 + 4bd(a^2 - c^2) + 4ac(d^2 - b^2).$$

Let $\varepsilon = 1 + \zeta + \zeta^2$. Then the unit group U of $\mathcal{O}_F$ is generated by ζ and ε.

We know that 2 is totally ramified and for an arbitrary odd prime $p \in \mathbb{Z}$, p splits completely if $8 | (p-1)$, otherwise, $(p) = \mathcal{P}_1 \mathcal{P}_2$ with $N(\mathcal{P}_1) = N(\mathcal{P}_2) = p^2$.

Lemma 3.1. *For any $\alpha \neq 0$, $x \in \mathcal{O}_F$, there exists $y \in \mathcal{O}_F$ such that $N(x - y\alpha) \leq \frac{9N(\alpha)}{16}$, $|x - y\alpha| \leq (1 + \frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot \alpha|$, and $|\sigma(x) - \sigma(y)\sigma(\alpha)| \leq (1 + \frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\sigma(\alpha)|$.*

Proof. Let $x/\alpha = k_1 + k_2\zeta + k_3\zeta^2 + k_4\zeta^3$, where $k_i \in \mathbb{Q}$. For a real number t, denote by $[t]$ the nearest integer to t, and let $\{t\} = t - [t]$. Clearly $-\frac{1}{2} < \{t\} \leq \frac{1}{2}$.

We shall prove that $y := \sum_{i=1}^4 [k_i] \zeta^{i-1}$ satisfies the lemma.

Let $z = x - y\alpha = \sum_{i=1}^{4}\{k_i\}\alpha\zeta^{i-1} = \sum_{i=1}^{4} z_i\alpha\zeta^{i-1}$, where $z_i = \{k_i\}$. Then

$$\begin{aligned}N(z) =& N(\alpha)N(\sum_{i=1}^{4} z_i\zeta^{i-1})\\ =& N(\alpha)((z_1^2+z_3^2)^2+(z_2^2+z_4^2)^2+4z_2z_4(z_1^2-z_3^2)+4z_1z_3(z_4^2-z_2^2))\\ \le& N(\alpha)((z_1^2+z_3^2)^2+(z_2^2+z_4^2)^2+4|z_2||z_4||z_1^2-z_3^2|+4|z_1||z_3||z_4^2-z_2^2|).\end{aligned}$$

Suppose that $z_3^2 \ge z_1^2$ and $z_4^2 \ge z_2^2$, then

$$N(z) \le N(\alpha)((z_1^2+\frac{1}{4})^2+(z_2^2+\frac{1}{4})^2+2|z_2|(\frac{1}{4}-z_1^2)+2|z_1|(\frac{1}{4}-z_2^2)).$$

Now we look for the maximal value of

$$f := (z_1^2+\frac{1}{4})^2+(z_2^2+\frac{1}{4})^2+2|z_2|(\frac{1}{4}-z_1^2)+2|z_1|(\frac{1}{4}-z_2^2)$$

in the region $\{(z_1,\ z_2)|0 \le z_i \le \frac{1}{2},\ i=1,\ 2\}$.

Since

$$\begin{aligned}\frac{\partial f}{\partial z_1} &= 4z_1^3+z_1-4z_1z_2-2z_2^2+\frac{1}{2},\\ \frac{\partial f}{\partial z_2} &= 4z_2^3+z_2-4z_1z_2-2z_1^2+\frac{1}{2},\end{aligned}$$

we have $\frac{\partial f}{\partial z_1} = \frac{\partial f}{\partial z_2} = 0$ if and only if $z_1 = z_2 = \frac{1}{2}$. So f meets its maximal value on the boundary. By computation, we can show $f \le \frac{9}{16}$. So $N(x-y\alpha) = N(z) \le \frac{9N(\alpha)}{16}$.

Next, let $g(z_1,z_2,z_3,z_4) = |z|^2 = (z_1+\frac{z_2-z_4}{\sqrt{2}})^2+(z_3+\frac{z_2+z_4}{\sqrt{2}})^2$, $t_1 = \frac{z_2-z_4}{\sqrt{2}}$ and $t_2 = \frac{z_2+z_4}{\sqrt{2}}$. Then

$$\begin{aligned}g(z_1,z_2,z_3,z_4) &= (z_1+t_1)^2+(z_3+t_2)^2\\ &\le (|z_1|+|t_1|)^2+(|z_3|+|t_2|)^2.\end{aligned}$$

Since $|t_1|+|t_2| = \frac{|z_2-z_4|+|z_2+z_4|}{\sqrt{2}} \le frac1\sqrt{2}$, we have

$$\begin{aligned}g &\le (|z_1|+|t_1|)^2+(|z_3|+\frac{1}{\sqrt{2}}-|t_1|)^2\\ &\le (\frac{1}{2}+|t_1|)^2+(\frac{1}{2}+\frac{1}{\sqrt{2}}-|t_1|)^2\\ &= 2|t_1|^2-\sqrt{2}|t_1|+1+\frac{\sqrt{2}}{2}\\ &\le 1+\frac{1}{\sqrt{2}}.\end{aligned}$$

So $|z| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\alpha|$. Similarly $|\sigma(z)| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot sigma(|\alpha|)$, i.e.,

$$|x - y\alpha| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\alpha|$$

$$\text{and} \qquad |\sigma(x) - \sigma(y)\sigma(\alpha)| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\sigma(\alpha)|.$$

□

Lemma 3.2. *For any prime ideal $\mathcal{P}$, there is an element α such that $\mathcal{P} = (\alpha)$, and $(\sqrt{2}-1)|\alpha| \leq |\sigma(\alpha)| \leq |\alpha|(1+\sqrt{2})$.*

Proof. Suppose that $\mathcal{P}$ is a prime ideal with generator y. Recall that $\varepsilon = 1+\zeta+\zeta^2$ is invertible in $\mathcal{O}_F$ and $|\varepsilon| = \sqrt{2}+1$, $|\sigma(\varepsilon)| = \sqrt{2}-1$. If $|\sigma(y)| > (1+\sqrt{2})|y|$, then there exists some $k \geq 0$ such that $|\sigma(y)||\sigma(\varepsilon)|^{k-1} > (1+\sqrt{2})|y||\varepsilon|^{k-1}$, while $|\sigma(y)||\sigma(\varepsilon)|^{k} \leq (1+\sqrt{2})|y||\varepsilon|^{k}$. Let $\alpha = y\varepsilon^k$. Then $(\sqrt{2}-1)|\alpha| \leq |\sigma(\alpha)| \leq |\alpha|(1+\sqrt{2})$. □

By virtue of Lemma 2.1, we will construct $W,\ C,\ G$ concretely for each S_m. For each $\mathcal{P}_i \in S_m$, choose α_i to be a generator of $\mathcal{P}_i$ such that $(\sqrt{2}-1)|\alpha_i| \leq |\sigma(\alpha_i)| \leq |\alpha_i|(1+\sqrt{2})$. Let W be the set consisting exactly of all these α_i's and all the units of $\mathcal{O}_F$. Assume $\mathcal{P} = \mathcal{P}_{m+1} = (\alpha)$, where α satisfies $(\sqrt{2}-1)|\alpha| \leq |\sigma(\alpha)| \leq |\alpha|(1+\sqrt{2})$. Then $(\sqrt{2}-1)^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}} \leq |\sigma(\alpha)| leq (\sqrt{2}+1)^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}}$. Let

$$C' = \{c \in \mathcal{O}_F |\ N(c) \leq \frac{9N(\mathcal{P})}{16},\quad |c| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\alpha|,\ |\sigma(c)| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\sigma(\alpha)|\ \},$$

where $D = D(F) = 256$. By Lemma 3.1, any integer must be congruent to some element in C' modulo $\mathcal{P}$. Let C be a subset of C' such that $1 \in C$, $0 \notin C$ and $c_1 - c_2 \notin P$ for any two different elements $c_1,\ c_2 \in C$. So (3) of Lemma 2.1 is always satisfied. Let

$$G = \{a \in \mathcal{O}_F |\ |a| \leq (\frac{2}{\pi})^{\frac{1}{2}} \cdot |D|^{\frac{1}{8}} \cdot N(\mathcal{P})^{\frac{1}{8}},\ |\sigma(a)| \leq (\frac{2}{\pi})^{\frac{1}{2}} \cdot |D|^{\frac{1}{8}} \cdot (\mathcal{P})^{\frac{1}{8}}\}.$$

Then by the GTT Theorem of [9], $\beta(G)$ generates $k^{\cdot}$ when $N(\mathcal{P}) \geq (\frac{4}{\pi^2})^2 \cdot |D| \doteq 42.05$. Let $\delta = (\frac{2}{\pi})^{\frac{1}{2}} \cdot |D|^{\frac{1}{8}}$. Then for any $g \in G$, we have $N(g) \leq \delta^4 \cdot N(\mathcal{P})^{\frac{1}{2}}$.

Lemma 3.3. *Assume that $\mathcal{P} = (\alpha)$ with α satisfying $(\sqrt{2}-1)|\alpha| \leq |\sigma(\alpha)| \leq |\alpha|(1+\sqrt{2})$. Choosing $W,\ C,\ G$ as above, we have $W \subset CU_1$ if $N(\mathcal{P}) > 9$.*

Proof. By Lemma 1 of [5], $W \subset CU_1$ if $N(w-c) < N(\mathcal{P})^2$, i.e., $|w-c||\sigma(w)-\sigma(c)| < N(\mathcal{P})$ holds for any $w \in W,\ c \in C$. This can be deduced from the inequality $(|w|+|c|)(|\sigma(w)|+|\sigma(c)|) < N(\mathcal{P})$, i.e., $N(w)^{\frac{1}{2}} + |c| \cdot |\sigma(w)| + |w| \cdot |\sigma(c)| + N(c)^{\frac{1}{2}} < N(\mathcal{P})$.

Given any $c \in C$, by the construction of C, we have

$$|c| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\alpha| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot (1+\sqrt{2})^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}}.$$

on the other hand, $|c| = \frac{N(c)^{\frac{1}{2}}}{|\sigma(c)|} \geq \frac{N(c)^{\frac{1}{2}}}{(1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot |\sigma(\alpha)|} \geq \frac{N(c)^{\frac{1}{2}}}{(1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot (1+\sqrt{2})^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}}}$ $,and (\sqrt{2}-1)^{\frac{1}{2}} \cdot N(w)^{\frac{1}{4}} \leq |\sigma(w)| < N(w)^{\frac{1}{4}} \cdot (\sqrt{2}+1)^{\frac{1}{2}}. So$

$$\begin{aligned}\frac{(\sqrt{2}-1)^{\frac{1}{2}} \cdot N(c)^{\frac{1}{2}} \cdot N(w)^{\frac{1}{4}}}{(1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot (1+\sqrt{2})^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}}} \leq & |c| \cdot |\sigma(w)| \\ \leq & (\sqrt{2}+1) \cdot (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}} \cdot N(w)^{\frac{1}{4}}.\end{aligned}$$

$Since (|c| \cdot |\sigma(w)| \cdot |\sigma(c)|) = N(c)^{\frac{1}{2}} N(w)^{\frac{1}{2}} \leq N(\mathcal{P})$, and the function $x + \frac{a}{x}$ ($x \in [c_1, c_2],\ c_1 > 0$) meets its maximal value on the boundary,

$$\begin{aligned}&|c| \cdot |\sigma(w)| + |w| \cdot |\sigma(c)| \\ =&|c| \cdot |\sigma(w)| + \frac{N(c)^{\frac{1}{2}} \cdot N(w)^{\frac{1}{2}}}{|c| \cdot |\sigma(w)|} \\ \leq&(\sqrt{2}+1) \cdot (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}} \cdot N(w)^{\frac{1}{4}} + \frac{N(c)^{\frac{1}{2}} \cdot N(w)^{\frac{1}{4}}}{(\sqrt{2}+1) \cdot (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{4}}} \\ \leq&(\sqrt{2}+1) \cdot (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot N(\mathcal{P})^{\frac{1}{2}} + \frac{\frac{3}{4} N(\mathcal{P})^{\frac{1}{2}}}{(\sqrt{2}+1) \cdot (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}}} \\ =&N(\mathcal{P})^{\frac{1}{2}} \cdot ((\sqrt{2}+1) \cdot (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} + \frac{\frac{3}{4}}{(\sqrt{2}+1) \cdot (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}}}) \\ \doteq&N(\mathcal{P})^{\frac{1}{2}} \cdot (3.392)\ .\end{aligned}$$

Thus $W \subset CU_1$ if $N(\mathcal{P})^{\frac{1}{2}} + N(\mathcal{P})^{\frac{1}{2}}(3.392) + \frac{3N(\mathcal{P})^{\frac{1}{2}}}{4} < N(\mathcal{P})$. Since $N(\mathcal{P}) \in \mathbb{Z}$, we get the inequality if $N(\mathcal{P}) > 26$.

Suppose that $\mathcal{P}_1,\ \ldots,\ \mathcal{P}_8 = (2-i),\ \mathcal{P}_9 = (2+i)$ are the first nine prime ideals. If $\mathcal{P} = \mathcal{P}_8$, we see that for any $c \in C$, $N(c) \leq \frac{9 \cdot 25}{16}$, which implies $N(c) \leq 9$ for every $t \in \{14,\ 13,\ 12,\ 11,\ 10\}$ is not a norm from $F/\mathbb{Q}$. So the inequality also holds. If $\mathcal{P} = \mathcal{P}_9$, since $2 - i = 4(1 - \frac{1}{4} \cdot (2+i))$, we have $2 - i \in CU_1$. For $N(c) \leq 9$, the inequality holds. So we have $W \subset CU_1$ if $N(\mathcal{P}) = 25$.

We know that 17 is splitted completely. Let $\mathcal{P} = \mathcal{P}_4 = (1+2\zeta)$, $W = \{\zeta,\ \varepsilon, 1+\zeta,\ 1-\zeta-\zeta^2, \overline{1-\zeta-\zeta^2}\}$, $C = \{\zeta^i(1-\zeta-\zeta^2)^j,\ i = 0,\ 1,\ \ldots,\ 7,\ j = 0,\ 1\}$. Then we have

$$\begin{aligned}&N(w)^{\frac{1}{2}} + |c| \cdot |\sigma(w)| + |w| \cdot |\sigma(c)| + N(c)^{\frac{1}{2}} \\ <&3 + 2(1+\sqrt{2})\sqrt{3} + 3 < 17\ .\end{aligned}$$

If $\mathcal{P}=\mathcal{P}_5=(1+2\zeta^3)$, then let $W=\{\zeta,\ \varepsilon, 1+\zeta,\ 1-\zeta-\zeta^2,\ \overline{1-\zeta-\zeta^2},\ 1+2\zeta\}$, $C=\{\zeta^i(1-\zeta-\zeta^2)^j,\ i=0,\ 1,\ ...,\ 7,\ j=0,\ 1\}$. We have

$$\begin{aligned}&N(w)^{\frac{1}{2}}+|c|\cdot|\sigma(w)|+|w|\cdot|\sigma(c)|+N(c)^{\frac{1}{2}}\\ <&\sqrt{17}+\sqrt{3}(\sqrt{5+2\sqrt{2}}+\sqrt{5-2\sqrt{2}})+3<17\ .\end{aligned}$$

If $\mathcal{P}=\mathcal{P}_6$ or $\mathcal{P}_7$, the proof is the same as above.

So $W\subset CU_1$ if $N(\mathcal{P})>9$.

□

Lemma 3.4. *If $N(\mathcal{P})>9$, then $CG\subset CU_1$.*

Proof. By Lemma 1 in [5], $CG\subset CU_1$ if $N(c_1g-c_2)<N(\mathcal{P})^2$ holds for any $c_1,\ c_2\in C,\ g\in G$. This can be deduced from

$$(|c_1|\cdot|g|+|c_2|)\cdot(|\sigma(c_1)|\cdot|\sigma(g)|+|\sigma(c_2)|)<N(\mathcal{P}),$$

i.e.,

$$N(c_1)^{\frac{1}{2}}\cdot N(g)^{\frac{1}{2}}+|c_2|\cdot|\sigma(c_1)|\cdot|\sigma(g)|+|\sigma(c_2)|\cdot|c_1|\cdot|g|+N(c_2)^{\frac{1}{2}}<N(\mathcal{P}).\quad (1)$$

Assume $M_1,\ M_2$ are positive numbers such that $N(c_1)\le M_1,\ N(c_2)\le M_1$, $|g|\le M_2$ and $|\sigma(g)|\le M_2$. The above inequality holds if

$$M_1^{\frac{1}{2}}\cdot(N(g)^{\frac{1}{2}}+1)+M_2(|c_2|\cdot|\sigma(c_1)|+|c_1|\cdot|\sigma(c_2)|)<N(\mathcal{P}),$$

i.e.,

$$M_1^{\frac{1}{2}}(N(g)^{\frac{1}{2}}+1)+M_2(|c_2|\cdot|\sigma(c_1)|+\frac{N(c_1)^{\frac{1}{2}}\cdot N(c_2)^{\frac{1}{2}}}{|c_2|\cdot|\sigma(c_1)|})<N(\mathcal{P}).$$

In the proof of Lemma 3.3, we see that

$$\frac{N(c_2)^{1/2}}{(1+\frac{1}{\sqrt{2}})^{\frac{1}{2}}\cdot|\sigma(\alpha)|}\le|c_2|\le(1+\frac{1}{\sqrt{2}})^{\frac{1}{2}}\cdot|\alpha|,$$

and

$$\frac{N(c_1)^{1/2}}{(1+\frac{1}{\sqrt{2}})^{\frac{1}{2}}\cdot|\alpha|}\le|\sigma(c_1)|\le(1+\frac{1}{\sqrt{2}})^{\frac{1}{2}}\cdot|\sigma(\alpha)|.$$

So

$$\begin{aligned}&|c_2|\cdot|\sigma(c_1)|+\frac{N(c_1)^{\frac{1}{2}}\cdot N(c_2)^{\frac{1}{2}}}{|c_2|\cdot|\sigma(c_1)|}\\ \le&(1+\frac{1}{\sqrt{2}})\cdot N(\mathcal{P})^{\frac{1}{2}}+\frac{\frac{9}{16}N(\mathcal{P})^{\frac{1}{2}}}{1+\frac{1}{\sqrt{2}}}\\ =&2.0366N(\mathcal{P})^{\frac{1}{2}}.\end{aligned}$$

Hence (1) holds if

$$M_1^{\frac{1}{2}}(M_2^2+1)+2.0366M_2\cdot N(\mathcal{P})^{\frac{1}{2}}<N(\mathcal{P}).\quad (2)$$

Let $M_1 = \frac{9}{16} \cdot N(\mathcal{P})$ and $M_2 = \delta \cdot N(p)^{\frac{1}{8}}$. We get

$$\frac{3}{4} \cdot N(\mathcal{P})^{\frac{1}{2}} \cdot (\delta^2 N(\mathcal{P})^{\frac{1}{4}} + 1) + 2.0366\delta \cdot N(\mathcal{P})^{\frac{5}{8}} < N(\mathcal{P}).$$

Let $y = N(\mathcal{P})^{\frac{1}{8}}$. Then

$$y^4 - \frac{3}{4}\delta^2 y^2 - 2.0366\delta y - \frac{3}{4} > 0.$$

It can be easily verified that if $y > 1.945$, i.e., $N(\mathcal{P}) > 204$, the inequality holds.

Now suppose $N(\mathcal{P}) = 193$. Since $M_1 \le \frac{9}{16}N(\mathcal{P})$, the maximal possible value of M_1 is 100. Let $M_1 = 100$ and $M_2 = \delta N(\mathcal{P})^{\frac{1}{8}}$, it can be easily verified that the inequality (2) holds.

For the prime ideals $\mathcal{P}$ whose norm satisfy $9 < N(\mathcal{P}) < 193$, let $(p) = \mathcal{P} \cap \mathbb{Z}$, then $p \in \{137,\ 113,\ 97,\ 89,\ 73,\ 41,\ 13,\ 11,\ 7,\ 5,\ 17,\ 3,\ 2\}$.

If $p \in \{137,\ 113,\ 89\}$, then p splits completely and 3 generates $(\mathbb{Z}/p\mathbb{Z})^*$ which implies 3 generates $(\mathcal{O}/\mathcal{P})^*$. We let $3 = (\sqrt{2}+i)(\sqrt{2}-i)$, let $G = \{\sqrt{2}+i,\ \sqrt{2}-i\}$ and $M_2 = \sqrt{3}$, $M_1 = \frac{9}{16}N(\mathcal{P})$. We see that (2) is satisfied if $N(\mathcal{P}) > 33$. Hence for $p \in \{137,\ 113,\ 89\}$, $CG \subset CU_1$.

If $p \in \{97,\ 73\}$, then $5 = (2+i)(2-i)$ generates $(\mathcal{O}/\mathcal{P})^*$. Let $G = \{2+i,\ 2-i\}$, $M_2 = \sqrt{5}$ and $M_1 = \frac{9}{16}N(\mathcal{P})$. Then (2) is satisfied if $N(\mathcal{P}) > 62$. So for $p \in \{97,\ 73\}$, $CG \subset CU_1$.

If $p = 41$, then $6 = 2\cdot 3 = (1+\zeta)(1+\zeta^3)(1+\zeta^5)(1+\zeta^7)(1-\zeta-\zeta^2)\overline{(1-\zeta-\zeta^2)}$ generates $(\mathcal{O}/\mathcal{P})^*$. Let $G = \{1+\zeta,\ 1+\zeta^3,\ 1+\zeta^5,\ 1+\zeta^7,\ 1-\zeta-\zeta^2,\ \overline{1-\zeta-\zeta^2}\}$ and $M_2 = \sqrt{2+\sqrt{2}}$, $M_1 = \frac{9}{16}N(\mathcal{P})$. Then (2) is satisfied if $N(\mathcal{P}) > 32$. So for $p = 41$, $CG \subset CU_1$.

If $p = 13$, then $N(\mathcal{P}) = 169$ and $1+\zeta$ generates $(\mathcal{O}/\mathcal{P})^*$. Let $G = 1+\zeta$, $M_1 = \frac{9}{16}N(\mathcal{P})$, $M_2 = \sqrt{2+\sqrt{2}}$. Then (2) is satisfied if $N(\mathcal{P}) > 81$. So for $p = 13$, $CG \subset CU_1$.

If $p = 11$, then $N(\mathcal{P}) = 121$. Let $G = \{1 \pm \zeta,\ \zeta,\ 1 \pm \zeta^3\}$. Then G generates $(\mathcal{O}/\mathcal{P})^*$. Let $M_1 = \frac{9}{16}N(\mathcal{P})$, $M_2 = \sqrt{2+\sqrt{2}}$. Then (2) is satisfied if $N(\mathcal{P}) > 81$. So for $p = 11$, $CG \subset CU_1$.

If $p = 7$, then $N(\mathcal{P}) = 49$. Without lost of generality, we assume $\mathcal{P} = (2+\zeta+2\zeta^2)$. Let $G = \{1+\zeta\}$. Then G generates $(\mathcal{O}/\mathcal{P})^*$. Let $M_2 = \sqrt{2+\sqrt{2}}$. Since $N(c) \le \frac{9}{16}N(\mathcal{P})$, we have $N(c) \le 27$. But neither 27 nor 26 is a norm from $F/\mathbb{Q}$. If $N(c) = 25$, we can replace c by an element of norm 2. For example, if $c = 2+i$, we can replace c by $-(\zeta+\zeta^2)$ which also satisfies the conditions in Lemma 3.1. For any $t \in \{24,\ 23,\ 22,\ 21,\ 20,\ 19\}$, t is not a norm from $F/\mathbb{Q}$. Let $M_1 = 18$. We see that (2) is satisfied. Hence for $p = 7$, $CG \subset CU_1$.

If $p = 5$, then $N(\mathcal{P}) = 25$. Without lost of generality, assume $\mathcal{P} = (2+i)$. Let $G = \{1+\zeta\}$. Then G generates $(\mathcal{O}/\mathcal{P})^*$. Since $N(c) \le \frac{9}{16}\cdot 25$, we have $N(c) \le 14$. Again, for any $t \in \{14,\ 13,\ 12,\ 11,\ 10\}$, t can not be a norm from

$F/\mathbb{Q}$. So $N(c) \leq 9$. Therefore the inequality (1) holds if

$$3 \cdot \sqrt{2} + |c_2| \cdot |\sigma(c_1)| \cdot (\sqrt{2+\sqrt{2}}) + |\sigma(c_2)| \cdot |c_1| \cdot (\sqrt{2-\sqrt{2}}) + 3 < 25 \qquad (3).$$

Note that $(|c_2| \cdot |\sigma(c_1)| \cdot (\sqrt{2+\sqrt{2}}))(|\sigma(c_2)| \cdot |c_1| \cdot (\sqrt{2-\sqrt{2}})) \leq 9\sqrt{2}$ and $|c_i| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot \sqrt{5}$, $|\sigma(c_i)| \leq (1+\frac{1}{\sqrt{2}})^{\frac{1}{2}} \cdot \sqrt{5}$. Then the left side of (3) is less than

$$3(\sqrt{2}+1) + ((\sqrt{2+\sqrt{2}})(1+\frac{1}{\sqrt{2}}) \cdot 5 + \frac{9\sqrt{2}}{5(\sqrt{2+\sqrt{2}})(1+\frac{1}{\sqrt{2}})} \leq 25 \ .$$

So for $p = 5$, $CG \subset CU_1$.

If $p = 17$, then $N(\mathcal{P}) = 17$. Without lost of generality, assume $\mathcal{P} = (1+2\zeta)$. Let $C = \{\zeta^i(1-\zeta-\zeta^2)^j,\ i = 0,\ 1,\ ...,\ 7,\ j = 0,\ 1\}$, $G = \{1-\zeta-\zeta^2,\ \overline{1-\zeta-\zeta^2}\}$. If $c_1 g \equiv c_2\ (mod\ \mathcal{P})$, then $N(c_1) \neq N(c_2)$, for otherwise $g \equiv 1\ (mod\ \mathcal{P})$. So $N(c_1)N(c_2) = 9$. In the equality (1),

$$\begin{aligned} &N(c_1)^{\frac{1}{2}} \cdot N(g)^{\frac{1}{2}} + |c_2| \cdot |\sigma(c_1)| \cdot |\sigma(g)| + |\sigma(c_2)| \cdot |c_1| \cdot |g| + N(c_2)^{\frac{1}{2}} \\ <&3 \cdot 3 + 2\sqrt{3} \cdot \sqrt{3} + 1 = 16 < 17 \ . \end{aligned}$$

So for $p = 17$, $CG \subset CU_1$.

The lemma is proved. □

Combining Lemma 3.3 and Lemma 3.4, we know that (1), (2) of Lemma 2.1 are satisfied if $N(v) > 9$. So ∂v is a bijective if $N(v) > 9$.

If $p = 3$, then $(p) = \mathcal{P}\overline{\mathcal{P}}$, where $\mathcal{P} = (1-\zeta-\zeta^2)$. Let $C = \{1,\ \zeta,\ ...,\ \zeta^7\}$, $G = \{\zeta\}$ and $W = \{\zeta,\ \varepsilon,\ 1+\zeta\}$. For $\overline{\mathcal{P}}$ which is lying behind $\mathcal{P}$, we may add $1-\zeta-\zeta^2$ to W. If $p = 2$, let $C = G = \{1\}$ and $W = \{1,\ \zeta,\ \varepsilon\}$. One can verify that (1), (2), (3) in Lemma 2.1 are satisfied. Hence ∂v is a bijective.

Hence $K_2\mathcal{O}_F$ can be generated by $\{x,\ y\}$, $x,\ y \in \mathcal{O}_F^*$. However, $\mathcal{O}_F^*$ is generated by ζ and $\varepsilon = 1+\zeta+\zeta^2$. So $K_2\mathcal{O}_F$ is generated by $\{\zeta,\ \zeta\}$, $\{\zeta,\ \varepsilon\}$, $\{\varepsilon,\ \varepsilon\}$.

Theorem 3.5. *The tame kernel of* $\mathbb{Q}(\zeta_8)$ *is a trivial group.*

Proof. Since $\zeta^4 = -1$, we get

$$\{\zeta,\zeta\} = \{\zeta,-1\} = \{\zeta,\zeta^4\} = \{\zeta,\zeta\}^4 = \{\zeta,-1\}^4 = 1.$$

By $\varepsilon = 1+\zeta+\zeta^2$ and $\zeta^9 = \zeta$, we get

$$\{\zeta,\varepsilon\} = \{\zeta,(1-\zeta)\varepsilon\} = \{\zeta,1-\zeta^3\} = \{\zeta^3,1-\zeta^3\}^3 = 1.$$

On the other hand, $\{\varepsilon,\varepsilon\} = \{\varepsilon,-1\} = \{\varepsilon,\zeta\}^4 = 1$. So $\mathbb{Q}(\zeta_8)$ is a trivial group. □

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