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The Grothendieck algebras of certain smash product semisimple Hopf algebras

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ABSTRACT

Let H be a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of positive characteristic p. Under the conditions that $p > \dim_{\Bbbk}(H)^{1/2}$ and $p \nmid 2\dim_{\Bbbk}(H)$, we determine all non-isomorphic irreducible representations of the smash product semisimple Hopf algebra $H\#\Bbbk G$, where G is a cyclic group of order $n := 2\dim_{\Bbbk}(H)$. We endow the Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H), *)$ of H with a new multiplication $\star$ and show that the Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *)$ of $H\#\Bbbk G$ is isomorphic to $(\mathrm{G}_{\Bbbk}(H), *)^{\oplus \frac{n}{2}} \bigoplus (\mathrm{G}_{\Bbbk}(H), \star)^{\oplus \frac{n}{2}}$ as algebras. This reveals a relationship between the Grothendieck algebra of $H\#\Bbbk G$ and that of H.

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1. Introduction

The Grothendieck rings of finite dimensional semisimple or cosemisimple Hopf algebras have been studied by Nichols and Richmond [11], Nikshych [12], Kashina [5], Chen, Yang and Wang [2, 17], etc. For a finite dimensional semisimple Hopf algebra H, the category $\mathrm{Rep}(H)$ of finite dimensional representations of H is a fusion category and the Grothendieck ring $\mathrm{G}_0(H)$ of H can be used to study the fusion category $\mathrm{Rep}(H)$. For instance, the knowledge of the structure of the Grothendieck ring $\mathrm{G}_0(H)$ allows to determine all fusion subcategories of $\mathrm{Rep}(H)$, which correspond to the so-called based subrings of $\mathrm{G}_0(H)$. Also, the Grothendieck ring $\mathrm{G}_0(H)$ reveals the decompositions of the tensor products of irreducible representations into a direct sum of irreducible representations.

For a semisimple Hopf algebra H with antipode S over a field $\Bbbk$, it is known that S^2 is an inner automorphism of H (see [7]). Here an inner automorphism is understood to be the conjugation by an invertible element of H. If the ground field $\Bbbk$ has positive characteristic p, whether or not S^2 can be given by conjugation with a group-like element is not completely solved (this problem is closely related to the Kaplansky's fifth conjecture). However, such a Hopf algebra H can be embedded into another finite dimensional Hopf algebra $H\#\Bbbk G$, namely, the smash product of H and a group algebra $\Bbbk G$, in which the square of the antipode is the conjugation with a group-like element. We refer to [6, 8, 15] for such Hopf algebras and related researches.

If H is a semisimple involutory Hopf algebra, namely, a semisimple Hopf algebra with $S^2 = id$, the smash product Hopf algebra $H\#\Bbbk G$ considered here is nothing but the usual tensor product Hopf algebra $H \otimes \Bbbk G$. In this case, the representations of $H \otimes \Bbbk G$ can be stemmed directly from the representations of H and those of $\Bbbk G$. Also, the Grothendieck algebra of $H \otimes \Bbbk G$ is the usual tensor product of the Grothendieck algebra of H and that of $\Bbbk G$. However, if H is not necessarily involutory (although the Kaplansky's fifth conjecture states that a semisimple Hopf algebra is necessarily involutory), the relationship between the Grothendieck algebra of $H\#\Bbbk G$ and that of H is not clear.

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The purpose of this paper is to study representations of the smash product semisimple Hopf algebra $H\#\Bbbk G$ and to establish a relationship between the Grothendieck algebra of $H\#\Bbbk G$ and that of H, where H is a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of positive characteristic $p > \dim_{\Bbbk}(H)^{1/2}$ with $p \nmid 2\dim_{\Bbbk}(H)$ and G is a cyclic group of order $2\dim_{\Bbbk}(H)$. It is worthy mentioning that such a Hopf algebra H is not known to be involutory unless the characteristic p is larger than a certain number (see [3, 15]).

The paper is organized as follows: In Section 2, we give some basic results on semisimple Hopf algebras. In Section 3, we describe some properties of certain elements $\mathbf{u}$ and $\mathbf{v}$, where the two are the same up to a central element. In Section 4, using the element $\mathbf{v}$ we determine all non-isomorphic irreducible representations of $H\#\Bbbk G$ by virtue of those irreducible representations of H and those of $\Bbbk G$. We also describe the dual of these irreducible representations of $H\#\Bbbk G$. In Section 5, to investigate the Grothendieck algebra of $H\#\Bbbk G$, we endow the Grothendieck algebra $\mathrm{G}_{\Bbbk}(H)$ of H with a new multiplication $\star$ so as to obtain a new algebra $(\mathrm{G}_{\Bbbk}(H), \star)$. This algebra $(\mathrm{G}_{\Bbbk}(H), \star)$ is nothing but the usual Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H), *)$ if H is involutory. We show that the Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *)$ of $H\#\Bbbk G$ has the direct sum decomposition

$$(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *) \cong (\mathrm{G}_{\Bbbk}(H), *)^{\oplus \frac{n}{2}} \bigoplus (\mathrm{G}_{\Bbbk}(H), \star)^{\oplus \frac{n}{2}},$$

where $n = 2\dim_{\Bbbk}(H)$. This reveals a relationship between the Grothendieck algebra of $H\#\Bbbk G$ and that of H. Moreover, we find a fusion subcategory $\mathcal{C}$ of $\mathrm{Rep}(H\#\Bbbk G)$ with its Grothendieck algebra $(\mathrm{G}_{\Bbbk}(\mathcal{C}), *)$ being

$$(\mathrm{G}_{\Bbbk}(\mathcal{C}), *) \cong (\mathrm{G}_{\Bbbk}(H), *) \bigoplus (\mathrm{G}_{\Bbbk}(H), \star).$$

In view of this, the Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *)$ is isomorphic to the direct sum $(\mathrm{G}_{\Bbbk}(\mathcal{C}), *)^{\oplus \frac{n}{2}}$.

2. Preliminaries

Throughout this paper, H is a finite dimensional semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of positive characteristic p, with counit ε, comultiplication Δ and antipode S. We will use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for $a \in H$. We denote by Λ and λ the left and right integrals of H and H^* respectively so that $\lambda(\Lambda) = 1$. Since the semisimple Hopf algebra H is unimodular, the left and right integrals of H are the same. We refer to [9] for basic theory of Hopf algebras.

We denote $\{V_i \mid 0 \le i \le m-1\}$ the set of all simple left H-modules up to isomorphism and $\{e_i \mid 0 \le i \le m-1\}$ the set of all central primitive idempotents of H. Note that V_0 is the trivial H-module $\Bbbk$ and e_0 is the idempotent $\Lambda/\varepsilon(\Lambda)$. The character of any simple H-module V_i is denoted by χ_i for $0 \le i \le m-1$ and the character of the left regular module H is denoted by χ_H. Obviously, $\chi_H = \sum_{i=0}^{m-1} \dim_{\Bbbk}(V_i)\chi_i$.

Recall that $S^2(a) = uau^{-1}$ for $a \in H$ and a certain unit $u \in H$. For any simple H-module V_i and any $\varphi \in \mathrm{End}_{\Bbbk}(V_i)$, we define the map $\mathcal{I}(\varphi) \in \mathrm{End}_{\Bbbk}(V_i)$ by

$$\mathcal{I}(\varphi)(v) = \Lambda_{(1)}\varphi(u^{-1}S(\Lambda_{(2)})v) \text{ for } v \in V_i.$$

Since $\mathcal{I}(\varphi)$ lies in $\mathrm{End}_H(V_i) \cong \Bbbk$, there exists a unique element $c_i \in \Bbbk$ such that

$$\mathcal{I}(\varphi) = c_i \mathrm{tr}(\varphi) id_{V_i} \text{ for all } \varphi \in \mathrm{End}_{\Bbbk}(V_i).$$

Such an element c_i, depending only on the isomorphism class of V_i, is called the Schur element associated to V_i (see [4, Theorem 7.2.1]). Since H is semisimple, it follows from [4, Theorem 7.2.6] that the Schur element $c_i \neq 0$ in $\Bbbk$.

We denote $\mathbf{u} := S(\Lambda_{(2)})\Lambda_{(1)}$. The relationship between $\mathbf{u}$ and the unit u can be found in [16, Proposition 3.1]:

$$\mathbf{u} = u \sum_{i=0}^{m-1} \dim_{\Bbbk}(V_i) c_i e_i. \tag{2.1}$$

For any map $\varphi \in \mathrm{End}_{\Bbbk}(H)$, the trace of φ is $\mathrm{tr}(\varphi) = \lambda(\varphi(S(\Lambda_{(2)}))\Lambda_{(1)})$ (see [13, Theorem 2]). Taking into account that $\varphi = L_a$, where L_a is the left multiplication operator of H by a, we have

$$\chi_H(a) = \mathrm{tr}(L_a) = \lambda(aS(\Lambda_{(2)})\Lambda_{(1)}) = \lambda(a\mathbf{u}) = (\mathbf{u} \rightharpoonup \lambda)(a).$$

Note that the right integral λ of H^* satisfies $\lambda(ab) = \lambda(S^2(b)a)$ for all $a, b \in H$ (see [13, Theorem 3(a)]) and $S^2(\mathbf{u}) = \mathbf{u}$. Then

$$\chi_H(a) = \lambda(a\mathbf{u}) = \lambda(S^2(\mathbf{u})a) = \lambda(\mathbf{u}a) = (\lambda \leftharpoonup \mathbf{u})(a).$$

Thus, we have

$$\chi_H = \mathbf{u} \rightharpoonup \lambda = \lambda \leftharpoonup \mathbf{u}. \tag{2.2}$$

3. Some properties of the elements u and v

In this section we will describe some properties of the element $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$ and a certain element $\mathbf{v}$, where the element $\mathbf{v}$ will be used in the forthcoming section to study representations of a certain smash product Hopf algebra. The following proposition gives some equivalent conditions for $\mathbf{u}$ being invertible:

Proposition 3.1. *Let H be a finite dimensional semisimple Hopf algebra over the field $\Bbbk$ of positive characteristic p. The following statements are equivalent:*

(1) The element $\mathbf{u}$ is invertible.
(2) For any simple H-module V_i, $p \nmid \dim_{\Bbbk}(V_i)$.
(3) The regular character χ_H of H is non-degenerate in the sense that if $\chi_H(ab) = 0$ for all $a \in H$, then $b = 0$.

Proof. (1) $\Leftrightarrow$ (2). It follows from (2.1) that Part (1) and Part (2) are equivalent.

(1) $\Rightarrow$ (3). Since λ is non-degenerate and $\mathbf{u}$ is invertible, it follows from (2.2) that χ_H is non-degenerate.

(3) $\Rightarrow$ (2). It follows from $\chi_H = \sum_{i=0}^{m-1} \dim_{\Bbbk}(V_i)\chi_i$ that the non-degeneracy of χ_H implies that $p \nmid \dim_{\Bbbk}(V_i)$ for any simple H-module V_i. □

Remark 3.2. (1) Recall that the (left) annihilator of $\mathbf{u}$ in H is the set $\mathrm{ann}(\mathbf{u}) := \{b \in H \mid b\mathbf{u} = 0\}$. Using the non-degeneracy of λ we may see from (2.2) that the set $\mathrm{ann}(\mathbf{u})$ coincides with the radical of χ_H defined by $\chi_H^{\perp} := \{b \in H \mid \chi_H(ab) = 0 \text{ for all } a \in H\}$.

(2) If $S^2 = id$, then

$$\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)} = S(\Lambda_{(2)})S^2(\Lambda_{(1)}) = S(S(\Lambda_{(1)})\Lambda_{(2)}) = \varepsilon(\Lambda) \neq 0.$$

Namely, $\mathbf{u}$ is a nonzero scalar. Conversely, if $\mathbf{u}$ is a nonzero scalar, then u is central by (2.1). It follows from $S^2(a) = uau^{-1}$ that $S^2 = id$.

(3) If $\mathbf{u}$ is invertible, then $\mathbf{u}$ and u are the same up to the central invertible element $\sum_{i=0}^{m-1} \dim_{\Bbbk}(V_i)c_ie_i$ by (2.1). Hence, $S^2(a) = uau^{-1}$ implies that $S^2(a) = \mathbf{u}a\mathbf{u}^{-1}$ for all $a \in H$.

(4) If the characteristic $p > \dim_{\Bbbk}(H)^{1/2}$, it follows that

$$p^2 > \dim_{\Bbbk}(H) = \sum_{i=0}^{m-1} \dim_{\Bbbk}(V_i)^2 \geq \dim_{\Bbbk}(V_i)^2.$$

Hence $p \nmid \dim_{\Bbbk}(V_i)$ for $0 \leq i \leq m-1$. In this case, $\mathbf{u}$ is invertible by Proposition 3.1.

Next, we assume that the field $\Bbbk$ has positive characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. By Remark 3.2, we have

- $S^2(a) = \mathbf{u}a\mathbf{u}^{-1}$ for all $a \in H$;
- $\dim_{\Bbbk}(V_i) \neq 0$ in $\Bbbk$ for $0 \leq i \leq m-1$.

The following result can be found in [16, Proposition 3.3], so we omit the proof.

Proposition 3.3. *The element* $\mathbf{u}$ *satisfies the following properties:*

(1) $\mathbf{u} = \chi_H(\Lambda_{(1)})S(\Lambda_{(2)})$.
(2) $\Lambda_{(1)}\mathbf{u}^{-1}S(\Lambda_{(2)}) = 1$.
(3) $\lambda(e_i) = \dim_{\Bbbk}(V_i)\chi_i(\mathbf{u}^{-1}) \neq 0$.
(4) $\mathbf{u}S(\mathbf{u}) = S(\mathbf{u})\mathbf{u} = \varepsilon(\Lambda)\sum_{i=0}^{m-1}\frac{\dim_{\Bbbk}(V_i)^2}{\lambda(e_i)}e_i$.
(5) $S(\mathbf{u}^{-1})\mathbf{u} = \mathbf{u}S(\mathbf{u}^{-1})$, *which is the distinguished group-like element* g_0 *of* H.

Recall that the dual module V_i^* is also a simple H-module for $0 \leq i \leq m-1$. This induces a permutation $*$ on the index set $\{0, 1, \cdots, m-1\}$ defined by $i^* = j$ if $V_i^* \cong V_j$. The permutation $*$ satisfies that $i^{**} = i$, $S(e_i) = e_{i^*}$, $\dim_{\Bbbk}(V_{i^*}) = \dim_{\Bbbk}(V_i)$ and by [16, Corollary 3.4] that $\lambda(e_{i^*}) = \lambda(e_i)$ for $0 \leq i \leq m-1$.

We denote η_i to be a square root of $\lambda(e_i)/\varepsilon(\Lambda)$ for $0 \leq i \leq m-1$. Note that $1/\varepsilon(\Lambda)^2 = \lambda(e_0)/\varepsilon(\Lambda)$ and $\eta_{i^*}^2 = \lambda(e_{i^*})/\varepsilon(\Lambda) = \lambda(e_i)/\varepsilon(\Lambda) = \eta_i^2$. In view of this, we further assume that

- $\eta_0 = 1/\varepsilon(\Lambda)$ and
- $\eta_i = \eta_{i^*}$ for $0 \leq i \leq m-1$.

We denote

$$\mathbf{v} := \mathbf{u}\sum_{i=0}^{m-1}\frac{\eta_i}{\dim_{\Bbbk}(V_i)}e_i. \tag{3.1}$$

As we shall see, the element $\mathbf{v}$ plays a key role in the representation theory of a certain smash product Hopf algebra. For the element $\mathbf{v}$, we have the following result:

Proposition 3.4. *The element* $\mathbf{v}$ *satisfies the following properties:*

(1) $\varepsilon(\mathbf{v}) = 1$.
(2) $S^2(a) = \mathbf{v}a\mathbf{v}^{-1}$ *for* $a \in H$.
(3) $\mathbf{v}^2 = \mathbf{u}S(\mathbf{u}^{-1})$, *which is the distinguished group-like element* g_0 *of* H.
(4) $\mathbf{v}^n = 1$, *where* $n = 2\dim_{\Bbbk}(H)$.
(5) $\mathbf{v}^{-1} = S(\mathbf{v})$.

Proof. (1) Note that $\eta_0 = 1/\varepsilon(\Lambda)$. Applying ε to both sides of the equality (3.1), we obtain that $\varepsilon(\mathbf{v}) = 1$.

(2) Since $S^2(a) = \mathbf{u}a\mathbf{u}^{-1}$ and the elements $\mathbf{u}$ and $\mathbf{v}$ are the same up to the central unit $\sum_{i=0}^{m-1}\frac{\eta_i}{\dim_{\Bbbk}(V_i)}e_i$, it follows that $S^2(a) = \mathbf{v}a\mathbf{v}^{-1}$ for $a \in H$.

(3) Note that $\mathbf{u}^{-1}S(\mathbf{u}^{-1}) = \frac{1}{\varepsilon(\Lambda)}\sum_{i=0}^{m-1}\frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2}e_i$ by Proposition 3.3 (4). It follows that

$$\mathbf{u}S(\mathbf{u}^{-1}) = \frac{\mathbf{u}^2}{\varepsilon(\Lambda)}\sum_{i=0}^{m-1}\frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2}e_i = \mathbf{v}^2,$$

which is the distinguished group-like element g_0 of H by Proposition 3.3 (5).

(4) It can be seen from Part (3) that $\mathbf{v}^2$ is the distinguished group-like element g_0 of H, while the order of g_0 divides $\dim_{\Bbbk}(H)$. This implies that $\mathbf{v}^n = (\mathbf{v}^2)^{\dim_{\Bbbk}(H)} = 1$.

(5) Note that $S(e_i) = e_{i^*}$, $\dim_{\Bbbk}(V_{i^*}) = \dim_{\Bbbk}(V_i)$ and $\eta_i = \eta_{i^*}$ for $0 \leq i \leq m-1$. We have

$$\begin{aligned}\mathbf{v}S(\mathbf{v}) &= \mathbf{u}\left(\sum_{i=0}^{m-1}\frac{\eta_i}{\dim_{\Bbbk}(V_i)}e_i\right)S(\mathbf{u})\left(\sum_{i=0}^{m-1}\frac{\eta_i}{\dim_{\Bbbk}(V_i)}e_{i^*}\right)\\ &= \mathbf{u}S(\mathbf{u})\left(\sum_{i=0}^{m-1}\frac{\eta_i}{\dim_{\Bbbk}(V_i)}e_i\right)\left(\sum_{i=0}^{m-1}\frac{\eta_{i^*}}{\dim_{\Bbbk}(V_{i^*})}e_{i^*}\right)\end{aligned}$$

$$= \mathbf{u}S(\mathbf{u})\left(\sum_{i=0}^{m-1} \frac{\eta_i}{\dim_{\Bbbk}(V_i)} e_i\right)^2$$
$$= \mathbf{u}S(\mathbf{u})\left(\frac{1}{\varepsilon(\Lambda)} \sum_{i=0}^{m-1} \frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2} e_i\right)$$
$$= 1,$$

where the last equality follows from Proposition 3.3 (4). We obtain that $\mathbf{v}^{-1} = S(\mathbf{v})$. The proof is completed. □

4. Representations of smash product Hopf algebras

In this section, we will describe all non-isomorphic irreducible representations of a certain smash product Hopf algebra. We denote $n := 2\dim_{\Bbbk}(H)$ and assume that the field $\Bbbk$ has positive characteristic $p > \dim_{\Bbbk}(H)^{1/2}$ and $p \nmid n$.

Let G be a cyclic group of order n generated by g. The character group $\widehat{G}$ of G is also a cyclic group of order n. Let ψ be a generator of $\widehat{G}$. Then $\widehat{G} = \{\psi^j \mid 0 \le j \le n-1\}$, which is the complete set of distinct irreducible characters of simple $\Bbbk G$-modules. The simple $\Bbbk G$-module with respect to the character ψ^j is denoted by W_j for $0 \le j \le n-1$.

Since the antipode S of H satisfies $S^{2n} = id$ by Radford's formula of S^4 [14], the Hopf algebra H is a left $\Bbbk G$-module algebra whose action is given by

$$g^i \rightarrow h = S^{2i}(h) \text{ for } g^i \in G \text{ and } h \in H.$$

This reduces to a Hopf algebra $H\#\Bbbk G$ mentioned in [15]. More precisely, the Hopf algebra $H\#\Bbbk G$ is the smash product of H and $\Bbbk G$. The multiplication of $H\#\Bbbk G$ is given by

$$(a\#g^i)(b\#g^j) = a(g^i \rightarrow b)\#g^{i+j} = aS^{2i}(b)\#g^{i+j} \text{ for } a, b \in H,$$

the identity of $H\#\Bbbk G$ is $1_H\#1_{\Bbbk G}$. The comultiplication of $H\#\Bbbk G$ is given by

$$\Delta_{H\#\Bbbk G}(h\#g^i) = (h_{(1)}\#g^i) \otimes (h_{(2)}\#g^i).$$

The counit of $H\#\Bbbk G$ is $\varepsilon_{H\#\Bbbk G} = \varepsilon_H\#\varepsilon_{\Bbbk G}$ and the antipode of $H\#\Bbbk G$ is

$$S_{H\#\Bbbk G}(h\#g^i) = (1_H\#g^{-i})(S(h)\#1_{\Bbbk G}) = S^{1-2i}(h)\#g^{-i}.$$

Moreover, $1_H\#g$ is a group-like element of $H\#\Bbbk G$ that satisfies

$$S^2_{H\#\Bbbk G}(h\#g^i) = (1_H\#g)(h\#g^i)(1_H\#g)^{-1}. \tag{4.1}$$

The Hopf algebra H can be considered as a sub-Hopf algebra of $H\#\Bbbk G$ under the injective map $H \rightarrow H\#\Bbbk G$, $h \mapsto h\#1_{\Bbbk G}$.

Since Λ is an integral of H with $\varepsilon(\Lambda) \neq 0$ and $p \nmid n$, $\Lambda\#\frac{1}{n}\sum_{i=0}^{n-1} g^i$ is an integral of $H\#\Bbbk G$ with $\varepsilon_{H\#\Bbbk G}(\Lambda\#\frac{1}{n}\sum_{i=0}^{n-1} g^i) = \varepsilon(\Lambda) \neq 0$. Thus, $H\#\Bbbk G$ is a semisimple Hopf algebra over $\Bbbk$.

The representation theory of crossed product of an algebra with a group algebra has been studied in [10]. However, we do not take advantage of those notations and methods in [10] to describe $H\#\Bbbk G$-modules. Instead, since the Hopf algebra $H\#\Bbbk G$ is semisimple, we will determine all simple $H\#\Bbbk G$-modules by the study of the character of the regular representation of $H\#\Bbbk G$.

Lemma 4.1. *If V is a finite dimensional H-module and W is a finite dimensional $\Bbbk G$-module, then the vector space $V \otimes W$ is a finite dimensional $H\#\Bbbk G$-module, where the $H\#\Bbbk G$-module structure on $V \otimes W$ is given by*

$$(h\#g^k) \cdot (v \otimes w) = (h\mathbf{v}^k \cdot v) \otimes (g^k \cdot w) \text{ for } v \in V, w \in W. \tag{4.2}$$

Proof. By Proposition 3.4 (4), we have $\mathbf{v}^n = 1$. It follows that

$$(h\#g^n) \cdot (v \otimes w) = (h\mathbf{v}^n \cdot v) \otimes (g^n \cdot w) = (h \cdot v) \otimes w = (h\#1_{\Bbbk G}) \cdot (v \otimes w).$$

Hence the $H\#\Bbbk G$-module structure defined on $V \otimes W$ is compatible with the equality $h\#g^n = h\#1_{\Bbbk G}$. For $a, b \in H$, by $S^2(h) = \mathbf{v}h\mathbf{v}^{-1}$ for $h \in H$, we may check that

$$((a\#g^i)(b\#g^j)) \cdot (v \otimes w) = (a\#g^i) \cdot ((b\#g^j) \cdot (v \otimes w)).$$

Indeed,

$$\begin{aligned} ((a\#g^i)(b\#g^j)) \cdot (v \otimes w) &= (aS^{2i}(b)\#g^{i+j}) \cdot (v \otimes w) \\ &= (aS^{2i}(b)\mathbf{v}^{i+j} \cdot v) \otimes (g^{i+j} \cdot w), \end{aligned}$$

while

$$\begin{aligned} (a\#g^i) \cdot ((b\#g^j) \cdot (v \otimes w)) &= (a\#g^i) \cdot ((b\mathbf{v}^j \cdot v) \otimes (g^j \cdot w)) \\ &= (a\mathbf{v}^i \cdot (b\mathbf{v}^j \cdot v)) \otimes (g^i \cdot (g^j \cdot w)) \\ &= (a\mathbf{v}^i b\mathbf{v}^j \cdot v) \otimes (g^{i+j} \cdot w) \\ &= (aS^{2i}(b)\mathbf{v}^{i+j} \cdot v) \otimes (g^{i+j} \cdot w). \end{aligned}$$

The proof is completed. □

Lemma 4.2. *If V is a simple H-module and W is a simple $\Bbbk G$-module, then $V \otimes W$ is a simple $H\#\Bbbk G$-module.*

Proof. Note that $H\#\Bbbk G$ is a semisimple Hopf algebra over an algebraically closed field $\Bbbk$. It is sufficient to show that $\mathrm{End}_{H\#\Bbbk G}(V \otimes W) \cong \Bbbk$. Suppose that the map $\phi : V \otimes W \to V \otimes W$ is an $H\#\Bbbk G$-module morphism. Since W is one dimensional, we fix a basis w of W. The $H\#\Bbbk G$-module morphism ϕ induces an H-module morphism $\phi_0 : V \to V$ as follows: $\phi(v \otimes w) = \phi_0(v) \otimes w$ for any $v \in V$. This shows that ϕ is the identity map of $V \otimes W$ up to a scalar, since V is simple and ϕ_0 is the identity map of V up to a scalar. □

Remark 4.3. For simple H-module V_i and simple $\Bbbk G$-module W_j, it can be seen from Lemma 4.2 that $V_i \otimes W_j$ is a simple $H\#\Bbbk G$-module. Let χ_{ij} be the character associated to the simple $H\#\Bbbk G$-module $V_i \otimes W_j$. It follows from (4.2) that

$$\chi_{ij}(h \otimes g^k) = \chi_i(h\mathbf{v}^k)\psi^j(g^k) \text{ for } 0 \le i \le m-1, 0 \le j \le n-1.$$

Theorem 4.4. *The set $\{V_i \otimes W_j \mid 0 \le i \le m-1, 0 \le j \le n-1\}$ forms a complete set of non-isomorphic simple $H\#\Bbbk G$-modules.*

Proof. Note that $\Lambda\#\frac{1}{n}\sum_{i=0}^{n-1} g^i$ is a left integral of $H\#\Bbbk G$ and $\lambda\#\sum_{j=0}^{n-1}\psi^j$ is a right integral of $(H\#\Bbbk G)^*$ satisfying $(\lambda\#\sum_{j=0}^{n-1}\psi^j)(\Lambda\#\frac{1}{n}\sum_{i=0}^{n-1} g^i) = 1$. By (2.2), the characters of left regular representations of H and $H\#\Bbbk G$ are respectively given by $\chi_H = \lambda \leftharpoonup \mathbf{u}$ and $\chi_{H\#\Bbbk G} = (\lambda\#\sum_{j=0}^{n-1}\psi^j) \leftharpoonup \mathbf{u}_{H\#\Bbbk G}$, where $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$ and

$$\begin{aligned} \mathbf{u}_{H\#\Bbbk G} &= \frac{1}{n}\sum_{i=0}^{n-1} S_{H\#\Bbbk G}(\Lambda_{(2)}\#g^i)(\Lambda_{(1)}\#g^i) \\ &= \frac{1}{n}\sum_{i=0}^{n-1}(S^{1-2i}(\Lambda_{(2)})\#g^{-i})(\Lambda_{(1)}\#g^i) \end{aligned}$$

$$= \frac{1}{n}\sum_{i=0}^{n-1} S^{1-2i}(\Lambda_{(2)})S^{-2i}(\Lambda_{(1)})\#1_{\Bbbk G}$$
$$= \frac{1}{n}\sum_{i=0}^{n-1} S^{-2i}(\mathbf{u})\#1_{\Bbbk G}$$
$$= \mathbf{u}\#1_{\Bbbk G}.$$

It follows that

$$\chi_{H\#\Bbbk G} = (\lambda\#\sum_{j=0}^{n-1}\psi^j) \leftharpoonup (\mathbf{u}\#1_{\Bbbk G}) = (\lambda \leftharpoonup \mathbf{u})\#\sum_{j=0}^{n-1}\psi^j = \chi_H\#\sum_{j=0}^{n-1}\psi^j.$$

Hence,

$$(\chi_{H\#\Bbbk G})(h\#g^k) = \chi_H(h)\sum_{j=0}^{n-1}\psi^j(g^k) = \begin{cases} n\chi_H(h), & k = 0; \\ 0, & 1 \le k \le n-1. \end{cases}$$

While

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\dim_{\Bbbk}(V_i\otimes W_j)\chi_{ij}(h\#g^k) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\dim_{\Bbbk}(V_i)\chi_i(h\mathbf{v}^k)\psi^j(g^k)$$
$$= \chi_H(h\mathbf{v}^k)\sum_{j=0}^{n-1}\psi^j(g^k)$$
$$= \begin{cases} n\chi_H(h), & k = 0; \\ 0, & 1 \le k \le n-1. \end{cases}$$

We obtain that

$$\chi_{H\#\Bbbk G} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\dim_{\Bbbk}(V_i\otimes W_j)\chi_{ij}.$$

Hence, all non-isomorphic simple $H\#\Bbbk G$-modules are $V_i\otimes W_j$ for $0 \le i \le m-1$ and $0 \le j \le n-1$. □

Remark 4.5. Note that $\chi_{00} = \varepsilon_{H\#\Bbbk G}$. Hence $V_0\otimes W_0$ is the trivial $H\#\Bbbk G$-module, where V_0 is the trivial H-module and W_0 is the trivial $\Bbbk G$-module.

For any simple $H\#\Bbbk G$-module $V_i\otimes W_j$, its dual module $(V_i\otimes W_j)^*$ can be described as follows:

Proposition 4.6. *We have $(V_i\otimes W_j)^* \cong V_{i^*}\otimes W_{j^*}$ for $0 \le i \le m-1, 0 \le j \le n-1$, where V_{i^*} is the dual of V_i as an H-module and W_{j^*} is the dual of W_j as a $\Bbbk G$-module.*

Proof. We need to check that $\chi_{i^*j^*} = \chi_{ij}\circ S_{H\#\Bbbk G}$ for $0 \le i \le m-1, 0 \le j \le n-1$. Note that $S(\mathbf{v}) = \mathbf{v}^{-1}$ and $S^{-2}(h) = \mathbf{v}^{-1}h\mathbf{v}$ for $h \in H$. On the one hand,

$$\chi_{i^*j^*}(h\#g^k) = \chi_{i^*}(h\mathbf{v}^k)\psi^{j^*}(g^k) = \chi_i(S(\mathbf{v})^kS(h))\psi^j(g^{-k}) = \chi_i(\mathbf{v}^{-k}S(h))\psi^j(g^{-k}).$$

On the other hand,

$$(\chi_{ij}\circ S_{H\#\Bbbk G})(h\#g^k) = \chi_{ij}(S_{H\#\Bbbk G}(h\#g^k))$$
$$= \chi_{ij}(S^{1-2k}(h)\#g^{-k})$$
$$= \chi_{ij}(\mathbf{v}^{-k}S(h)\mathbf{v}^k\#g^{-k})$$
$$= \chi_i(\mathbf{v}^{-k}S(h))\psi^j(g^{-k}).$$

We conclude that $\chi_{i^*j^*} = \chi_{ij}\circ S_{H\#\Bbbk G}$ for $0 \le i \le m-1, 0 \le j \le n-1$. □

5. The Grothendieck algebras of smash product Hopf algebras

In this section, we will investigate a relationship between the Grothendieck algebra of the smash product Hopf algebra $H\#\Bbbk G$ and that of H. We still assume that the field $\Bbbk$ has positive characteristic $p > \dim_{\Bbbk}(H)^{1/2}$ and $p \nmid n$, where $n = 2\dim_{\Bbbk}(H)$.

Recall that the Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H), *)$ of H is an associative algebra over the field $\Bbbk$ with unity ε_H under the convolution $*$, where the convolution $*$ on the basis $\{\chi_i \mid 0 \le i \le m-1\}$ of $\mathrm{G}_{\Bbbk}(H)$ is defined by

$$(\chi_i * \chi_j)(h) = (\chi_i \otimes \chi_j)(\Delta(h)) \text{ for } h \in H.$$

We define a new multiplication operator $\star$ on $\mathrm{G}_{\Bbbk}(H)$ by

$$(\chi_i \star \chi_j)(h) = (\chi_i \otimes \chi_j)\Big(\Delta(h)\Delta(\mathbf{v}^{-1})(\mathbf{v}\otimes\mathbf{v})\Big) \text{ for } h \in H.$$

Proposition 5.1. *The pair $(\mathrm{G}_{\Bbbk}(H), \star)$ is an associative algebra over the field $\Bbbk$ with unity ε_H.*

Proof. We first need to prove that $\star$ is a multiplication operator on $\mathrm{G}_{\Bbbk}(H)$. That is, $\chi_i \star \chi_j \in \mathrm{G}_{\Bbbk}(H)$ for $0 \le i, j \le m-1$. Indeed, for $a, b \in H$, using $S^2(h) = \mathbf{v}h\mathbf{v}^{-1}$ for $h \in H$, we have

$$\begin{aligned}
(\chi_i \star \chi_j)(ab) &= \chi_i(a_{(1)}b_{(1)}\mathbf{v}^{-1}{}_{(1)}\mathbf{v})\chi_j(a_{(2)}b_{(2)}\mathbf{v}^{-1}{}_{(2)}\mathbf{v})\\
&= \chi_i(a_{(1)}(b\mathbf{v}^{-1})_{(1)}\mathbf{v})\chi_j(a_{(2)}(b\mathbf{v}^{-1})_{(2)}\mathbf{v})\\
&= \chi_i(a_{(1)}(\mathbf{v}^{-1}S^2(b))_{(1)}\mathbf{v})\chi_j(a_{(2)}(\mathbf{v}^{-1}S^2(b))_{(2)}\mathbf{v})\\
&= \chi_i(a_{(1)}\mathbf{v}^{-1}{}_{(1)}S^2(b_{(1)})\mathbf{v})\chi_j(a_{(2)}\mathbf{v}^{-1}{}_{(2)}S^2(b_{(2)})\mathbf{v})\\
&= \chi_i(a_{(1)}\mathbf{v}^{-1}{}_{(1)}\mathbf{v}b_{(1)})\chi_j(a_{(2)}\mathbf{v}^{-1}{}_{(2)}\mathbf{v}b_{(2)})\\
&= \chi_i(b_{(1)}a_{(1)}\mathbf{v}^{-1}{}_{(1)}\mathbf{v})\chi_j(b_{(2)}a_{(2)}\mathbf{v}^{-1}{}_{(2)}\mathbf{v})\\
&= (\chi_i \star \chi_j)(ba).
\end{aligned}$$

It follows from [7] that $\chi_i \star \chi_j \in \mathrm{G}_{\Bbbk}(H)$ for $0 \le i, j \le m-1$. Since the map $H \to H \otimes H$, $h \mapsto \Delta(h)\Delta(\mathbf{v}^{-1})(\mathbf{v}\otimes\mathbf{v})$ is a coassociative comultiplication in H for which ε_H is still a counit (see [1, Eq.(12)]), the operator $\star$ dual to the coassociative comultiplication is an associative multiplication on $\mathrm{G}_{\Bbbk}(H)$ with unity ε_H. The associativity and unity ε_H of $\star$ on $\mathrm{G}_{\Bbbk}(H)$ can also be checked directly. Indeed, for $a \in H$, we have

$$\begin{aligned}
((\chi_i \star \chi_j) \star \chi_k)(a) &= ((\chi_i \star \chi_j) \otimes \chi_k)\Big(\Delta(a)\Delta(\mathbf{v}^{-1})(\mathbf{v}\otimes\mathbf{v})\Big)\\
&= (\chi_i \star \chi_j)(a_{(1)}\mathbf{v}^{-1}{}_{(1)}\mathbf{v})\chi_k(a_{(2)}\mathbf{v}^{-1}{}_{(2)}\mathbf{v})\\
&= \chi_i(a_{(1)}\mathbf{v}^{-1}{}_{(1)}\mathbf{v})\chi_j(a_{(2)}\mathbf{v}^{-1}{}_{(2)}\mathbf{v})\chi_k(a_{(3)}\mathbf{v}^{-1}{}_{(3)}\mathbf{v})\\
&= \chi_i(a_{(1)}\mathbf{v}^{-1}{}_{(1)}\mathbf{v})(\chi_j \star \chi_k)(a_{(2)}\mathbf{v}^{-1}{}_{(2)}\mathbf{v})\\
&= (\chi_i \star (\chi_j \star \chi_k))(a).
\end{aligned}$$

Therefore, $(\chi_i \star \chi_j) \star \chi_k = \chi_i \star (\chi_j \star \chi_k)$ for $0 \le i, j, k \le m-1$.

$$\begin{aligned}
(\varepsilon_H \star \chi_k)(a) &= (\varepsilon_H \otimes \chi_k)\Big(\Delta(a)\Delta(\mathbf{v}^{-1})(\mathbf{v}\otimes\mathbf{v})\Big)\\
&= \varepsilon_H(a_{(1)}\mathbf{v}^{-1}{}_{(1)}\mathbf{v})\chi_k(a_{(2)}\mathbf{v}^{-1}{}_{(2)}\mathbf{v})\\
&= \chi_k(a).
\end{aligned}$$

Hence $\varepsilon_H \star \chi_k = \chi_k$ for $0 \le k \le m-1$. It is similar that $\chi_k \star \varepsilon_H = \chi_k$ for $0 \le k \le m-1$. □

Next, we will use the algebras $(\mathrm{G}_{\Bbbk}(H), *)$ and $(\mathrm{G}_{\Bbbk}(H), \star)$ to describe the structure of the Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *)$ of $H\#\Bbbk G$. Note that $\{\chi_0, \chi_1, \cdots, \chi_{m-1}\}$ is a $\Bbbk$-basis of $\mathrm{G}_{\Bbbk}(H)$.

Suppose in $(\mathrm{G}_{\Bbbk}(H), *)$ that

$$\chi_i * \chi_j = \sum_{k=0}^{n-1} N_{ij}^k \chi_k$$

and in $(\mathrm{G}_{\Bbbk}(H), \star)$ that

$$\chi_i \star \chi_j = \sum_{k=0}^{n-1} L_{ij}^k \chi_k,$$

where N_{ij}^k and L_{ij}^k are respectively the structure coefficients of the two algebras with respect to the basis $\{\chi_0, \chi_1, \cdots, \chi_{m-1}\}$. We stress that the coefficient N_{ij}^k is the multiplicity of V_k appeared in the decomposition of tensor product $V_i \otimes V_j$ as H-modules, so each N_{ij}^k is indeed a nonnegative integer. For the coefficient L_{ij}^k, we shall see in Remark 5.3 that each L_{ij}^k is an integer.

Proposition 5.2. *We have the following equations in the Grothendieck algebra $(G_{\Bbbk}(H\#\Bbbk G), *)$:*

(1) $\chi_{ij} = \chi_{i0} * \chi_{0j} = \chi_{0j} * \chi_{i0}$ *for* $0 \le i \le m-1, 0 \le j \le n-1$.
(2) $\chi_{i0} * \chi_{j0} = \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k\frac{n}{2}}$ *for* $0 \le i,j \le m-1$.
(3) $\chi_{is} * \chi_{jt} = \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k,s+t} + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k,\frac{n}{2}+s+t}$ *for* $0 \le i,j \le m-1$ *and* $0 \le s,t \le n-1$, *where* $s+t$ *and* $\frac{n}{2}+s+t$ *are reduced modulo n.*

Proof. (1) It is direct to calculate that

$$\begin{aligned}(\chi_{i0} * \chi_{0j})(h\#g^k) &= \chi_{i0}(h_{(1)}\#g^k)\chi_{0j}(h_{(2)}\#g^k)\\ &= \chi_i(h_{(1)}\mathbf{v}^k)\psi^0(g^k)\chi_0(h_{(2)}\mathbf{v}^k)\psi^j(g^k)\\ &= \chi_i(h\mathbf{v}^k)\psi^j(g^k)\\ &= \chi_{ij}(h\#g^k).\end{aligned}$$

So we have $\chi_{i0} * \chi_{0j} = \chi_{ij}$. It is similar that $\chi_{0j} * \chi_{i0} = \chi_{ij}$.

(2) We show that the values that both sides of the desired equation taking on $h\#g^l$ are the same. Note that $\mathbf{v}^2$ is the distinguished group-like element of H and $\psi^{\frac{n}{2}}(g) = -1$. For the case $l = 2s$, we have

$$\begin{aligned}&\sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0}(h\#g^{2s}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k\frac{n}{2}}(h\#g^{2s})\\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_k(h\mathbf{v}^{2s}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_k(h\mathbf{v}^{2s})\psi^{\frac{n}{2}}(g^{2s})\\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_k(h\mathbf{v}^{2s}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_k(h\mathbf{v}^{2s})\\ &= \sum_{k=0}^{m-1} N_{ij}^k\chi_k(h\mathbf{v}^{2s}) = (\chi_i * \chi_j)(h\mathbf{v}^{2s})\\ &= \chi_i(h_{(1)}\mathbf{v}^{2s})\chi_j(h_{(2)}\mathbf{v}^{2s}) \quad (\text{since } \mathbf{v}^{2s} \text{ is a group-like element})\\ &= \chi_{i0}(h_{(1)}\#g^{2s})\chi_{j0}(h_{(2)}\#g^{2s})\\ &= (\chi_{i0} * \chi_{j0})(h\#g^{2s}).\end{aligned}$$

For the case $l = 2s + 1$, we have

$$\sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0}(h\#g^{2s+1}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k\frac{n}{2}}(h\#g^{2s+1})$$
$$= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_k(h\mathbf{v}^{2s+1}) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_k(h\mathbf{v}^{2s+1})\psi^{\frac{n}{2}}(g^{2s+1})$$
$$= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_k(h\mathbf{v}^{2s+1}) - \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_k(h\mathbf{v}^{2s+1})$$
$$= \sum_{k=0}^{m-1} L_{ij}^k\chi_k(h\mathbf{v}^{2s+1}) = (\chi_i \star \chi_j)(h\mathbf{v}^{2s+1})$$
$$= \chi_i(h_{(1)}\mathbf{v}^{2s+1})\chi_j(h_{(2)}\mathbf{v}^{2s+1}) \quad \text{(since } \mathbf{v}^{2s} \text{ is a group-like element)}$$
$$= \chi_{i0}(h_{(1)}\#g^{2s+1})\chi_{j0}(h_{(2)}\#g^{2s+1})$$
$$= (\chi_{i0} * \chi_{j0})(h\#g^{2s+1}).$$

We obtain the desired equation.

(3) Using Part (1) and Part (2) we may see that Part (3) holds. □

Remark 5.3. It follows from Proposition 5.2 (2) that the tensor product $(V_i \otimes W_0) \otimes (V_j \otimes W_0)$ has the following decomposition as $H\#\Bbbk G$-modules:

$$(V_i \otimes W_0) \otimes (V_j \otimes W_0) \cong \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)(V_k \otimes W_0) \bigoplus \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)(V_k \otimes W_{\frac{n}{2}}).$$

Thus, these coefficients $\frac{1}{2}(N_{ij}^k + L_{ij}^k)$ and $\frac{1}{2}(N_{ij}^k - L_{ij}^k)$ are both nonnegative integers. Since all N_{ij}^k are nonnegative integers, it follows that all L_{ij}^k are integers and satisfy $-N_{ij}^k \le L_{ij}^k \le N_{ij}^k$. In view of this, the multiplication operator $\star$ defined on the Grothendieck algebra $\mathrm{G}_{\Bbbk}(H)$ can be defined as well on the Grothendieck ring $G_0(H)$.

The Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *)$ is an associative unity algebra with a $\Bbbk$-basis $\{\chi_{ij} \mid 0 \le i \le m-1, 0 \le j \le n-1\}$. Denote by

$$\theta_l = \frac{1}{n}\sum_{t=0}^{n-1} \psi(g)^{-lt}\chi_{0t} \text{ for } 0 \le l \le n-1.$$

Note that $\chi_{0t} = \psi^t$ for $0 \le t \le n-1$. Thus, $\{\theta_l \mid 0 \le l \le n-1\}$ is the set of all central primitive idempotents of the algebra $\Bbbk\widehat{G}$. Moreover, we have

$$\chi_{0j} * \theta_l = \psi(g)^{jl}\theta_l \text{ and } \chi_{ij} * \theta_l = \chi_{i0} * \chi_{0j} * \theta_l = \psi(g)^{jl}\chi_{i0} * \theta_l. \tag{5.1}$$

In particular, each θ_l is a central idempotent of $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *)$. The structure of the Grothendieck algebra $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *)$ now can be described as follows:

Theorem 5.4. *We have the following algebra isomorphisms:*

(1) If l is even, then $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *) * \theta_l \cong (\mathrm{G}_{\Bbbk}(H), *)$.
(2) If l is odd, then $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *) * \theta_l \cong (\mathrm{G}_{\Bbbk}(H), \star)$.
(3) We have $(\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *) \cong (\mathrm{G}_{\Bbbk}(H), *)^{\oplus \frac{n}{2}} \bigoplus (\mathrm{G}_{\Bbbk}(H), \star)^{\oplus \frac{n}{2}}$.

Proof. (1) For the case l being even, we consider the $\Bbbk$-linear map

$$\phi_l : (\mathrm{G}_{\Bbbk}(H), *) \to (\mathrm{G}_{\Bbbk}(H\#\Bbbk G), *) * \theta_l, \;\; \chi_i \mapsto \chi_{i0} * \theta_l \;\text{ for }\; 0 \le i \le m-1.$$

It can be seen from (5.1) that ϕ_l is bijective, and moreover, $\chi_{i\frac{n}{2}} * \theta_l = \chi_{i0} * \theta_l$. Now

$$\begin{aligned}\phi_l(\chi_i * \chi_j) &= \phi_l(\sum_{k=0}^{m-1} N_{ij}^k \chi_k) = \sum_{k=0}^{m-1} N_{ij}^k \chi_{k0} * \theta_l \\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * \theta_l + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k0} * \theta_l \\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * \theta_l + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k\frac{n}{2}} * \theta_l \\ &= (\chi_{i0} * \chi_{j0}) * \theta_l \\ &= (\chi_{i0} * \theta_l) * (\chi_{j0} * \theta_l) \\ &= \phi_l(\chi_i) * \phi_l(\chi_j).\end{aligned}$$

This shows that ϕ_l is an algebra isomorphism.

(2) For the case l being odd, we consider the $\Bbbk$-linear map

$$\phi_l : (\mathrm{G}_\Bbbk(H), \star) \to (\mathrm{G}_\Bbbk(H\#\Bbbk G), *) * \theta_l, \quad \chi_i \mapsto \chi_{i0} * \theta_l \ \text{ for } \ 0 \le i \le m-1.$$

It can be seen from (5.1) that ϕ_l is bijective, and moreover, $\chi_{i\frac{n}{2}} * \theta_l = -\chi_{i0} * \theta_l$. Now

$$\begin{aligned}\phi_l(\chi_i \star \chi_j) &= \phi_l(\sum_{k=0}^{m-1} L_{ij}^k \chi_k) = \sum_{k=0}^{m-1} L_{ij}^k \chi_{k0} * \theta_l \\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * \theta_l - \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k0} * \theta_l \\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * \theta_l + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k\frac{n}{2}} * \theta_l \\ &= (\chi_{i0} * \chi_{j0}) * \theta_l \\ &= (\chi_{i0} * \theta_l) * (\chi_{j0} * \theta_l) \\ &= \phi_l(\chi_i) * \phi_l(\chi_j).\end{aligned}$$

Thus, ϕ_l is an algebra isomorphism.

(3) Let $(\mathrm{G}_\Bbbk(H), *)^{\oplus\frac{n}{2}}$ be the direct sum of $\frac{n}{2}$-folds of $(\mathrm{G}_\Bbbk(H), *)$ and $(\mathrm{G}_\Bbbk(H), \star)^{\oplus\frac{n}{2}}$ the direct sum of $\frac{n}{2}$-folds of $(\mathrm{G}_\Bbbk(H), \star)$. Since $\theta_0 + \theta_1 + \cdots + \theta_{n-1} = 1$, where 1 is the unity χ_{00} of $(\mathrm{G}_\Bbbk(H\#\Bbbk G), *)$, using Part (1) and Part (2) we obtain the following algebra isomorphism:

$$(\mathrm{G}_\Bbbk(H\#\Bbbk G), *) \cong (\mathrm{G}_\Bbbk(H), *)^{\oplus\frac{n}{2}} \bigoplus (\mathrm{G}_\Bbbk(H), \star)^{\oplus\frac{n}{2}}.$$

The proof is completed. □

Remark 5.5. If $S^2 = id$, then $\mathbf{u} = \varepsilon(\Lambda)$ and $\lambda(e_i)/\varepsilon(\Lambda) = (\dim_\Bbbk(V_i)/\varepsilon(\Lambda))^2$ by Proposition 3.3 (3). Now η_i, as a square root of $\lambda(e_i)/\varepsilon(\Lambda)$, may be chosen to be $\dim_\Bbbk(V_i)/\varepsilon(\Lambda)$. It follows that

$$\mathbf{v} = \mathbf{u} \sum_{i=0}^{m-1} \frac{\eta_i}{\dim_\Bbbk(V_i)} e_i = 1.$$

In this case, The multiplication operator $\star$ considered above is nothing but the convolution $*$ and the algebra $(\mathrm{G}_\Bbbk(H), \star)$ is nothing but the Grothendieck algebra $(\mathrm{G}_\Bbbk(H), *)$. Moreover,

$$(\mathrm{G}_\Bbbk(H\#\Bbbk G), *) \cong (\mathrm{G}_\Bbbk(H), *)^{\oplus\frac{n}{2}} \bigoplus (\mathrm{G}_\Bbbk(H), \star)^{\oplus\frac{n}{2}} \cong (\mathrm{G}_\Bbbk(H), *)^{\oplus n}.$$

Let $\mathcal{C}$ be the $\Bbbk$-linear subcategory of $\mathrm{Rep}(H\#\Bbbk G)$ spanned by objects

$$\{V_i \otimes W_0, V_i \otimes W_{\frac{n}{2}} \mid 0 \le i \le m-1\}.$$

Then $\mathcal{C}$ is closed under taking dual by Proposition 4.6. It follows from Proposition 5.2 that $\mathcal{C}$ is also closed under the tensor product of objects. More explicitly,

$$(V_i \otimes W_{\frac{n}{2}}) \otimes (V_j \otimes W_{\frac{n}{2}}) \cong (V_i \otimes W_0) \otimes (V_j \otimes W_0)$$

$$\cong \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)(V_k \otimes W_0) \bigoplus \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)(V_k \otimes W_{\frac{n}{2}}),$$

and

$$(V_i \otimes W_0) \otimes (V_j \otimes W_{\frac{n}{2}}) \cong (V_i \otimes W_{\frac{n}{2}}) \otimes (V_j \otimes W_0)$$

$$\cong \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)(V_k \otimes W_{\frac{n}{2}}) \bigoplus \bigoplus_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)(V_k \otimes W_0).$$

Hence $\mathcal{C}$ is a fusion subcategory of $\mathrm{Rep}(H\#\Bbbk G)$. Let $(\mathrm{G}_\Bbbk(\mathcal{C}), *)$ be the Grothendieck algebra of $\mathcal{C}$. Then $\{\chi_{i0}, \chi_{i\frac{n}{2}} \mid 0 \le i \le m-1\}$ forms a $\Bbbk$-basis of $(\mathrm{G}_\Bbbk(\mathcal{C}), *)$.

Proposition 5.6. *We have the following algebra isomorphism:*

$$(\mathrm{G}_\Bbbk(\mathcal{C}), *) \cong (\mathrm{G}_\Bbbk(H), *) \bigoplus (\mathrm{G}_\Bbbk(H), \star).$$

Proof. We denote $\theta = \frac{1}{2}(\chi_{00} + \chi_{0\frac{n}{2}})$. Then $1 - \theta = \frac{1}{2}(\chi_{00} - \chi_{0\frac{n}{2}})$, where 1 is the unity χ_{00} of $(\mathrm{G}_\Bbbk(\mathcal{C}), *)$. Note that θ and $1-\theta$ are both central idempotents of $(\mathrm{G}_\Bbbk(\mathcal{C}), *)$. In particular,

$$\chi_{i\frac{n}{2}} * \theta = \chi_{i0} * \chi_{0\frac{n}{2}} * \theta = \chi_{i0} * \theta \text{ for } 0 \le i \le m-1.$$

Consider the $\Bbbk$-linear map

$$\phi : (\mathrm{G}_\Bbbk(H), *) \to (\mathrm{G}_\Bbbk(\mathcal{C}), *) * \theta, \ \ \chi_i \mapsto \chi_{i0} * \theta \ \text{ for } \ 0 \le i \le m-1.$$

It is easy to see that ϕ is bijective and

$$\begin{aligned}
\phi(\chi_i * \chi_j) = \phi\Big(\sum_{k=0}^{m-1} N_{ij}^k \chi_k\Big) &= \sum_{k=0}^{m-1} N_{ij}^k \chi_{k0} * \theta \\
&= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * \theta + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k0} * \theta \\
&= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * \theta + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k\frac{n}{2}} * \theta \\
&= (\chi_{i0} * \chi_{j0}) * \theta \\
&= (\chi_{i0} * \theta) * (\chi_{j0} * \theta) \\
&= \phi(\chi_i) * \phi(\chi_j).
\end{aligned}$$

This shows that ϕ is an algebra isomorphism. Consider the $\Bbbk$-linear map

$$\varphi : (\mathrm{G}_\Bbbk(H), \star) \to (\mathrm{G}_\Bbbk(\mathcal{C}), *) * (1-\theta), \ \ \chi_i \mapsto \chi_{i0} * (1-\theta) \ \text{ for } \ 0 \le i \le m-1.$$

Then φ is bijective. Using $\chi_{i\frac{n}{2}} * (1-\theta) = -\chi_{i0} * (1-\theta)$ we may see that

$$\begin{aligned}\varphi(\chi_i \star \chi_j) &= \varphi(\sum_{k=0}^{m-1} L_{ij}^k \chi_k) = \sum_{k=0}^{m-1} L_{ij}^k \chi_{k0} * (1-\theta) \\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * (1-\theta) - \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k0} * (1-\theta) \\ &= \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k + L_{ij}^k)\chi_{k0} * (1-\theta) + \sum_{k=0}^{m-1} \frac{1}{2}(N_{ij}^k - L_{ij}^k)\chi_{k\frac{n}{2}} * (1-\theta) \\ &= (\chi_{i0} * \chi_{j0}) * (1-\theta) \\ &= (\chi_{i0} * (1-\theta)) * (\chi_{j0} * (1-\theta)) \\ &= \varphi(\chi_i) * \varphi(\chi_j)\end{aligned}$$

Hence, φ is an algebra isomorphism. □

Note that $\theta = \theta_0 + \theta_2 + \theta_4 + \cdots + \theta_{n-2}$ and $1 - \theta = \theta_1 + \theta_3 + \theta_5 + \cdots + \theta_{n-1}$. By Theorem 5.4 and Proposition 5.6, we have the following corollary:

Corollary 5.7. *We have algebra isomorphism:*

$$(G_{\Bbbk}(H\#\Bbbk G), *) \cong (G_{\Bbbk}(\mathcal{C}), *)^{\oplus \frac{n}{2}}.$$

Finally, we give some remarks on the pivotal (spherical) structures of the fusion categories $\mathrm{Rep}(H\#\Bbbk G)$ and $\mathcal{C}$. Since $S^2_{H\#\Bbbk G}$ is an inner automorphism of $H\#\Bbbk G$ and

$$S^2_{H\#\Bbbk G}(h\#g^i) = (1_H\#g)(h\#g^i)(1_H\#g)^{-1},$$

where $1_H\#g$ is a group-like element of $H\#\Bbbk G$, the category $\mathrm{Rep}(H\#\Bbbk G)$ is a pivotal fusion category, where the pivotal structure τ on $\mathrm{Rep}(H\#\Bbbk G)$ is the isomorphism of monoidal functors $\tau_{V\otimes W} : V \otimes W \to (V \otimes W)^{**}$ natural in $V \otimes W$. Here $\tau_{V\otimes W}(v \otimes w)$ is defined by

$$\tau_{V\otimes W}(v \otimes w)(f) = f(1_H\#g \cdot v \otimes w) = f(\mathbf{v} \cdot v \otimes g \cdot w)$$

for $v \in V, w \in W$ and $f \in (V \otimes W)^*$.

The quantum dimension of $V \otimes W \in \mathrm{Rep}(H\#\Bbbk G)$ with respect to the pivotal structure τ is denoted by $\mathbf{dim}(V \otimes W)$, which is the following composition

$$\mathbf{1} \xrightarrow{\mathrm{coev}_{(V\otimes W)}} (V \otimes W) \otimes (V \otimes W)^* \xrightarrow{\tau_{V\otimes W}\otimes id} (V \otimes W)^{**} \otimes (V \otimes W)^* \xrightarrow{\mathrm{ev}_{(V\otimes W)^*}} \mathbf{1},$$

where $\mathbf{1}$ is the trivial $H\#\Bbbk G$-module $V_0 \otimes W_0$. From this composition, we have

$$\mathbf{dim}(V \otimes W) = \chi_V(\mathbf{v})\chi_W(g).$$

Especially,

$$\mathbf{dim}(V_i \otimes W_j) = \chi_i(\mathbf{v})\psi^j(g) = \varepsilon(\Lambda)\eta_i\psi^j(g).$$

For the dual module $(V_i \otimes W_j)^* \cong V_{i^*} \otimes W_{j^*}$, we have

$$\mathbf{dim}(V_{i^*} \otimes W_{j^*}) = \varepsilon(\Lambda)\eta_{i^*}\psi^j(g^{-1}) = \varepsilon(\Lambda)\eta_i\psi^j(g^{-1}).$$

Therefore, $\mathbf{dim}(V_{i^*} \otimes W_{j^*}) = \mathbf{dim}(V_i \otimes W_j)$ if and only if $\psi^j(g) = \psi^j(g^{-1})$, if and only if $j = 0$ or $j = \frac{n}{2}$. This means that with respect to the pivotal structure τ, the fusion category $\mathrm{Rep}(H\#\Bbbk G)$ is pivotal but not spherical, while the fusion subcategory $\mathcal{C}$ of $\mathrm{Rep}(H\#\Bbbk G)$ spanned by objects $\{V_i \otimes W_0, V_i \otimes W_{\frac{n}{2}} \mid 0 \le i \le m-1\}$ is both pivotal and spherical.

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