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Quasitriangular Structures on Abelian Extensions of $\mathbb{Z}_2$ I

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Abstract. In this paper, we study quasitriangular structures on a class of semisimple Hopf algebras $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ constructed through abelian extensions of $\Bbbk\mathbb{Z}_2$ by $\Bbbk^G$ for an abelian group G. We find that there is an analogy between these quasitriangular structures and the solutions of a linear system.

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1 Introduction

Quasitriangular Hopf algebra is undoubtedly important and has been studied extensively in past years. It is of interest to know how to construct a Hopf algebra, determine whether it is quasitriangular and, moreover, describe all possible quasitriangular structures. Some researches related to this topic can be found in [2–4, 6].

In this paper we study quasitriangular structures on a class of semisimple Hopf algebras H arising from exact factorizations of finite groups:

$$\Bbbk^G \xrightarrow{\iota} H \xrightarrow{\pi} \Bbbk\mathbb{Z}_2, \tag{1.1}$$

where G is an abelian group. The well-known 8-dimensional Kac-Paljutkin algebra K_8 is an example of this kind. We can write $H = \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$, associated with appropriate cohomology data σ and τ (see Section 2 for the definition).

In the paper [10], the authors have shown that there are only two types of quasitriangular structures on $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$: one is called trivial and the other non-trivial. The present work can be regarded as a continuation of [10]. As the trivial

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quasitriangular structures can be given quite easily, we focus on the non-trivial quasitriangular structures. We first show that there is a division-like operation on the set of non-trivial quasitriangular structures. Using the division-like operation, we divide the solutions of non-trivial quasitriangular structures in two steps by analogy with the solutions of a system of linear equations. One step is to give all the general solutions, while the other step is to find a special solution; the definitions of a general solution and a special solution are given in Section 3.

This paper is organized as follows. In Section 2, we recall the definition of Hopf algebras $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and give some examples of them. After that we review some main results of [10] about the form of the quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. In Section 3, we prove that there is a division-like operation on the set of non-trivial quasitriangular structures. Then we observe that a non-trivial quasitriangular structure of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ can be expressed as a combination of a general solution and a fixed special solution. Moreover, we show that the set of general solutions has a natural group structure.

Throughout the paper we work over an algebraically closed field $\Bbbk$ of characteristic 0. All Hopf algebras in this paper are finite-dimensional. The symbol δ in Section 2 means the classical Kronecker symbol.

2 Abelian Extensions of $\mathbb{Z}_2$

In this section, we recall the definition of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, and then we give some examples of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ for guiding our further research.

• **The definition of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.**

Definition 2.1. A *short exact sequence of Hopf algebras* is a sequence of Hopf algebras and Hopf algebra maps

$$K \xrightarrow{\iota} H \xrightarrow{\pi} A \tag{2.1}$$

such that

(i) ι is injective,
(ii) π is surjective, and
(iii) $\ker(\pi) = HK^+$, where K^+ is the kernel of the counit of K.

In this situation H is said to be an extension of A by K [5, Definiton 1.4]. An extension (2.1) above such that K is commutative and A is cocommutative is called abelian. In this paper, we only study the following special abelian extensions:

$$\Bbbk^G \xrightarrow{\iota} H \xrightarrow{\pi} \Bbbk\mathbb{Z}_2,$$

where G is a finite abelian group. Abelian extensions were classified by Masuoka (see [5, Proposition 1.5]), and the above H can be expressed as $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, which is defined as follows.

Let $\mathbb{Z}_2 = \{1, x\}$ be the cyclic group of order 2 and let G be a finite group. To give the description of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we need the following data:

(i) $\triangleleft\colon \mathbb{Z}_2 \to \mathrm{Aut}(G)$ is an injective group homomorphism;
(ii) $\sigma\colon G \to \Bbbk^\times$ is a map such that $\sigma(g \triangleleft x) = \sigma(g)$ for $g \in G$ and $\sigma(1) = 1$;

(iii) $\tau\colon G\times G\to \Bbbk^\times$ is a unital 2-cocycle and satisfies

$$\sigma(gh)\sigma(g)^{-1}\sigma(h)^{-1}=\tau(g,h)\tau(g\triangleleft x,\, h\triangleleft x)\quad \text{for } g,h\in G.$$

The aim of (i) is to avoid making a commutative algebra (in such a case all quasitriangular structures are given by bicharacters and, thus, are known).

Definition 2.2. [1, Section 2.2] As an algebra, the Hopf algebra $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is generated by $\{e_g,x\}_{g\in G}$ satisfying

$$e_ge_h=\delta_{g,h}e_g,\quad xe_g=e_{g\triangleleft x}x,\quad x^2=\sum_{g\in G}\sigma(g)e_g,\quad g,h\in G.$$

The coproduct, counit and antipode are given by

$$\Delta(e_g)=\sum_{h,k\in G,\, hk=g}e_h\otimes e_k,\quad \Delta(x)=\Big[\sum_{g,h\in G}\tau(g,h)e_g\otimes e_h\Big](x\otimes x),$$
$$\epsilon(x)=1,\quad \epsilon(e_g)=\delta_{g,1}1,$$
$$\mathcal{S}(x)=\sum_{g\in G}\sigma(g)^{-1}\tau(g,g^{-1})^{-1}e_{(g\triangleleft x)^{-1}}x,\quad \mathcal{S}(e_g)=e_{g^{-1}},\quad g\in G.$$

The following are some examples of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and we will discuss them in the next sections.

Example 2.3. Let n be an odd number and let i be a primitive 4th root of 1. A Hopf algebra H belonging to $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is called A_{32n^2} by us if the data $(G,\triangleleft,\sigma,\tau)$ of H satisfies the following conditions:

(i) $G=\mathbb{Z}_{4n}\times\mathbb{Z}_{4n}=\langle a,b\,|\,a^{4n}=b^{4n}=1,\, ab=ba\rangle$ and $a\triangleleft x=a^{2n+1}$, $b\triangleleft x=b$;
(ii) $\sigma(g)=1$ for $g\in G$;
(iii) $\tau(a^ib^j,a^kb^l)=\mathrm{i}^{jk}$ for $1\le i,k\le 4n$ and $1\le j,l\le 4n$.

• **Some results about quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.** Now we review some results in [10] about quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and give a necessary condition for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ preserving a quasitriangular structure.

Recall that a quasitriangular Hopf algebra is a pair (H,R), where H is a Hopf algebra and $R=\sum R^{(1)}\otimes R^{(2)}$ is an invertible element in $H\otimes H$ such that

$$(\Delta\otimes\mathrm{Id})(R)=R_{13}R_{23},\quad (\mathrm{Id}\otimes\Delta)(R)=R_{13}R_{12},\quad \Delta^{\mathrm{op}}(h)R=R\Delta(h)$$

for $h\in H$. Here, by definition, $R_{12}=\sum R^{(1)}\otimes R^{(2)}\otimes 1$, $R_{13}=\sum R^{(1)}\otimes 1\otimes R^{(2)}$ and $R_{23}=\sum 1\otimes R^{(1)}\otimes R^{(2)}$. The element R is called a universal $\mathcal{R}$-matrix of H or a quasitriangular structure on H.

The first lemma below is well known.

Lemma 2.4. [8, Proposition 12.2.11] *Let H be a Hopf algebra and $R\in H\otimes H$. For $f\in H^*$, if we define $l(f):=(f\otimes\mathrm{Id})(R)$ and $r(f):=(\mathrm{Id}\otimes f)(R)$, then the following statements are equivalent:*

(i) $(\Delta\otimes\mathrm{Id})(R)=R_{13}R_{23}$ *and* $(\mathrm{Id}\otimes\Delta)(R)=R_{13}R_{12}$.
(ii) $l(f_1)l(f_2)=l(f_1f_2)$ *and* $r(f_1)r(f_2)=r(f_2f_1)$ *for* $f_1,f_2\in H^*$.

The following lemma is shown in [10, Lemma 3.2].

Lemma 2.5. *Denote the dual basis of* $\{e_g, e_gx\}_{g\in G}$ *by* $\{E_g, X_g\}_{g\in G}$*, that is,* $E_g(e_h) = \delta_{g,h}$*,* $E_g(e_hx) = 0$*,* $X_g(e_h) = 0$*,* $X_g(e_hx) = \delta_{g,h}$ *for* $g, h \in G$*. Then the following equations hold in the dual Hopf algebra* $(\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2)^*$*:*

$$E_gE_h = E_{gh}, \quad E_gX_h = X_hE_g = 0, \quad X_gX_h = \tau(g,h)X_{gh}, \quad g, h \in G.$$

Let $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ be as before. We need the following two notions, which will be used freely throughout this paper:

$$S := \{g \mid g \in G,\ g \triangleleft x = g\},$$
$$T := \{g \mid g \in G,\ g \triangleleft x \neq g\}.$$

Let $w^1, w^2, w^3, w^4 \colon G \times G \to \Bbbk$ be four maps, and define R as follows:

$$\begin{aligned} R := &\sum_{g,h\in G} w^1(g,h)e_g \otimes e_h + \sum_{g,h\in G} w^2(g,h)e_gx \otimes e_h \\ &+ \sum_{g,h\in G} w^3(g,h)e_g \otimes e_hx + \sum_{g,h\in G} w^4(g,h)e_gx \otimes e_hx. \end{aligned}$$

The following proposition shows that universal $\mathcal{R}$-matrices of $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ have only two possible forms.

Proposition 2.6. [10, Proposition 3.6] *If R is a universal $\mathcal{R}$-matrix of* $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$*, then R must belong to one of the following two cases:*

(i) $R = \sum_{g,h\in G} w^1(g,h)e_g \otimes e_h$;

(ii) $R = \sum_{s_1,s_2\in S} w^1(s_1,s_2)e_{s_1} \otimes e_{s_2} + \sum_{s\in S,\, t\in T} w^2(s,t)e_sx \otimes e_t$ $+ \sum_{t\in T,\, s\in S} w^3(t,s)e_t \otimes e_sx + \sum_{t_1,t_2\in T} w^4(t_1,t_2)e_{t_1}x \otimes e_{t_2}x.$

Remark 2.7. For simplicity, if R has the form in the case (i) (resp., (ii)) of Proposition 2.6, we will say that R has *trivial* form (resp., *non-trivial* form). Further, if R is a universal $\mathcal{R}$-matrix and has form (i) (resp., (ii)), we call it a *trivial* (resp., *non-trivial*) quasitriangular structure. We will call the w^1 (resp., w^i $(1 \leq i \leq 4)$) in the case (i) (resp., (ii)) the associated map(s) of R.

If R has non-trivial form, then one can see that R is invertible if and only if $w^1(s_1,s_2) \neq 0$, $w^2(s,t) \neq 0$, $w^3(t,s) \neq 0$, $w^4(t_1,t_2) \neq 0$ for $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$. Therefore, we always assume that $w^1(s_1,s_2) \neq 0$, $w^2(s,t) \neq 0$, $w^3(t,s) \neq 0$, $w^4(t_1,t_2) \neq 0$ for $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$ in the following content.

To determine all quasitriangular structures on $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$, we give the following necessary condition.

Proposition 2.8. *If* $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ *admits a quasitriangular structure, then we have* $\tau(s_1,s_2) = \tau(s_2,s_1)$ *for* $s_1, s_2 \in S$*.*

Proof. Note that the S is a subgroup of G. Consider the data $(G, \triangleleft, \sigma, \tau)$ restricted to S, and we write it as $(S, \triangleleft, \sigma|_S, \tau|_{S\times S})$. It can be seen that $(S, \triangleleft, \sigma|_S, \tau|_{S\times S})$ satisfies the compatible conditions. By Definition 2.2, a Hopf algebra is given and we denote it as $\Bbbk^S\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Let $\varphi\colon \Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2 \to \Bbbk^S\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ be a linear map which is defined as follows:

$$\varphi(e_s) = e_s, \quad \varphi(e_t) = 0, \quad \varphi(e_s x) = e_s x, \quad \varphi(e_t x) = 0,$$

where $s \in S$ and $t \in T$. Hence, it can be checked that φ is a surjective Hopf map. Assume that $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ admits a quasitriangular structure. Since $\Bbbk^S\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is a quotient of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we know that $\Bbbk^S\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ admits a quasitriangular structure. Combining the quasitriangularity of $\Bbbk^S\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ with its definition, we know that $\Bbbk^S\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is cocommutative. In particular, we have $\Delta^{\mathrm{cop}}(x) = \Delta(x)$ for $\Bbbk^S\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. This implies $\tau(s_1, s_2) = \tau(s_2, s_1)$ for $s_1, s_2 \in S$. □

If we let $\eta(g,h) = \tau(g,h)\tau(h,g)^{-1}$ for $g,h \in G$, then η is a bicharacter on G since τ is a 2-cocycle on the abelian group G, and so the necessary condition of Proposition 2.8 is equivalent to $\eta(s_1, s_2) = 1$ for $s_1, s_2 \in S$. We will often use η without explanation in the following.

Corollary 2.9. *The Hopf algebra A_{32n^2} in Example* 2.3 *admits no quasitriangular structure.*

Proof. Recall that $\eta(g,h) = \tau(g,h)\tau(h,g)^{-1}$ for $g,h \in G$. It can be seen that $a^{2n}, b \in S$ and $\eta(a^{2n}, b) = -1$. Thus, there is no quasitriangular structure by Proposition 2.8. □

The following proposition is shown in [10, Proposition 3.8].

Proposition 2.10. *If there is a non-trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, then*

(i) $|S| = |T|$;
(ii) *There is $b \in S$ such that $b^2 = 1$ and $t \triangleleft x = tb$ for $t \in T$;*
(iii) $|G| = 4m$ *for some $m \in \mathbb{N}^+$.*

Remark 2.11. Since our aim is to find all non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we agree that $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ satisfies both the condition of Proposition 2.8 and the conditions of Proposition 2.10 in all that follows.

3 Division-Like Operation

In this section, we introduce a division-like operation on the set of non-trivial quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Using the division-like operation, we prove that a non-trivial quasitriangular structure of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ can be expressed as a combination of a general solution and a fixed special solution.

Using the data $(G, \triangleleft, \sigma, \tau)$ of the Hopf algebra $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we can induce another data $(G', \triangleleft', \sigma', \tau')$ by making $G' := G$, $\triangleleft' := \triangleleft$ and $\sigma'(g) := 1$, $\tau'(g,h) := 1$ for $g,h \in G$. Then the data $(G', \triangleleft', \sigma', \tau')$ determines a Hopf algebra by Definition 2.2, and we simply denote it as $\Bbbk^G\#\Bbbk\mathbb{Z}_2$.

Assume that R and R' are non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, and suppose that the four maps associated with R (resp., R') are w^i (resp., w'^i), $1 \le i \le 4$. Then we can use R, R' to define $R'' \in (\Bbbk^G\#\Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G\#\Bbbk\mathbb{Z}_2)$ as follows.

The R'' has a non-trivial form with associated maps v^i ($1 \leq i \leq 4$), where v^i are determined by R, R' as

$$v^1(s_1, s_2) := \frac{w^1(s_1, s_2)}{w'^1(s_1, s_2)}, \quad v^2(s, t) := \frac{w^2(s, t)}{w'^2(s, t)},$$
$$v^3(t, s) := \frac{w^3(t, s)}{w'^3(t, s)}, \quad v^4(t_1, t_2) := \frac{w^4(t_1, t_2)}{w'^4(t_1, t_2)}$$

for $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$. For simplicity, we denote R'' as $\frac{R}{R'}$.

Theorem 3.1. *The above $\frac{R}{R'}$ is a non-trivial quasitriangular structure on $\Bbbk^G \# \Bbbk\mathbb{Z}_2$.*

To show Theorem 3.1, we need the following lemmas.

Lemma 3.2. *Let $R \in (\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2)$ and assume that R has non-trivial form with associated maps w^i ($1 \leq i \leq 4$). Then $\Delta^{\mathrm{op}}(h)R = R\Delta(h)$ holds for $h \in \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ if and only if the following equations hold:*

$$w^2(s, t \triangleleft x) = w^2(s, t)\, \eta(s, t), \quad s \in S,\ t \in T, \tag{3.1}$$
$$w^3(t \triangleleft x, s) = w^3(t, s)\, \eta(t, s), \quad s \in S,\ t \in T, \tag{3.2}$$
$$\tau(t_2, t_1)\, w^4(t_1 \triangleleft x, t_2 \triangleleft x) = \tau(t_1 \triangleleft x, t_2 \triangleleft x)\, w^4(t_1, t_2), \quad t_1, t_2 \in T. \tag{3.3}$$

Proof. Since $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ is generated by $\{e_g, x \mid g \in G\}$ as an algebra, we know that $\Delta^{\mathrm{op}}(h)R = R\Delta(h)$ for $h \in \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ is equivalent to $\Delta^{\mathrm{op}}(h)R = R\Delta(h)$ for $h \in \{e_g, x \mid g \in G\}$. We first show $\Delta^{\mathrm{op}}(e_g)R = R\Delta(e_g)$ for $g \in G$. Taking $s \in S$ and $t \in T$, we directly have

$$\Delta^{\mathrm{op}}(e_s)R = \Big[\sum_{\substack{s_1, s_2 \in S,\\ s_1 s_2 = s}} w^1(s_1, s_2) e_{s_1} \otimes e_{s_2}\Big] + \Big[\sum_{\substack{t_1, t_2 \in T,\\ t_1 t_2 = s}} w^4(t_1, t_2) e_{t_1} x \otimes e_{t_2} x\Big],$$
$$R\Delta(e_s) = \Big[\sum_{\substack{s_1, s_2 \in S,\\ s_1 s_2 = s}} w^1(s_1, s_2) e_{s_1} \otimes e_{s_2}\Big] + \Big[\sum_{\substack{t_1, t_2 \in T,\\ t_1 t_2 = s}} w^4(t_1 \triangleleft x, t_2 \triangleleft x) e_{t_1 \triangleleft x} x \otimes e_{t_2 \triangleleft x} x\Big].$$

By assumption, $t_1 t_2 \in S$. Thus, we find that $t_1 t_2 = (t_1 \triangleleft x)(t_2 \triangleleft x)$. This implies $\Delta^{\mathrm{op}}(s)R = R\Delta(s)$. Similarly, we have

$$\begin{aligned}\Delta^{\mathrm{op}}(e_t)R &= R\Delta(e_t)\\ &= \Big[\sum_{\substack{s \in S,\, t' \in T,\\ st' = s}} w^2(s, t') e_s x \otimes e_{t'}\Big] + \Big[\sum_{\substack{s \in S,\, t' \in T,\\ st' = s}} w^3(t', s) e_{t'} \otimes e_s x\Big],\end{aligned}$$

but $G = S \cup T$, and so we have shown that $\Delta^{\mathrm{op}}(e_g)R = R\Delta(e_g)$ for $g \in G$. Next we prove that $\Delta^{\mathrm{op}}(x)R = R\Delta(x)$ is equivalent to the equations (3.1)–(3.3). On the one hand, we have

$$\Delta^{\mathrm{op}}(x)R = \Big[\sum_{g, h \in G} \tau(h, g) e_g \otimes e_h\Big](x \otimes x)R$$

$$\begin{aligned}
&= \Big[\sum_{s_1,s_2\in S} \tau(s_2,s_1)w^1(s_1,s_2)e_{s_1}\otimes e_{s_2} \\
&\quad + \sum_{s\in S,\, t\in T} \tau(t,s)w^2(s,t\triangleleft x)e_s x\otimes e_t \\
&\quad + \sum_{t\in T,\, s\in S} \tau(s,t)w^3(t\triangleleft x,s)e_t\otimes e_s x \\
&\quad + \sum_{t_1,t_2\in T} \tau(t_2,t_1)w^4(t_1\triangleleft x,\, t_2\triangleleft x)e_{t_1}x\otimes e_{t_2}x\Big](x\otimes x).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
R\Delta(x) &= R\Big[\sum_{g,h\in G}\tau(g,h)e_g\otimes e_h\Big](x\otimes x) \\
&= \Big[\sum_{s_1,s_2\in S} \tau(s_1,s_2)w^1(s_1,s_2)e_{s_1}\otimes e_{s_2} \\
&\quad + \sum_{s\in S,\, t\in T} \tau(s,t)w^2(s,t)e_s x\otimes e_t \\
&\quad + \sum_{t\in T,\, s\in S} \tau(t,s)w^3(t,s)e_t\otimes e_s x \\
&\quad + \sum_{t_1,t_2\in T} \tau(t_1\triangleleft x,\, t_2\triangleleft x)w^4(t_1,t_2)e_{t_1}x\otimes e_{t_2}x\Big](x\otimes x).
\end{aligned}$$

By assumption, we already have $\tau(s_1,s_2)=\tau(s_2,s_1)$ for $s_1,s_2\in S$. Therefore, $\Delta^{\mathrm{op}}(x)R = R\Delta(x)$ holds if and only if the equations (3.1)–(3.3) hold. □

In order to know whether $(\Delta\otimes\mathrm{Id})(R) = R_{13}R_{23}$ and $(\mathrm{Id}\otimes\Delta)(R) = R_{13}R_{12}$ hold, we need the following lemmas.

Lemma 3.3. *Let $R\in(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)\otimes(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)$ and assume that R has non-trivial form with associated maps w^i $(1\le i\le 4)$. Then $l(f_1)l(f_2)=l(f_1f_2)$ for $f_1,f_2\in(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$ if and only if the following equations hold for $s,s_1,s_2\in S$ and $t,t_1,t_2\in T$:*

(i) $l(E_{s_1})l(E_{s_2}) = l(E_{s_1s_2})$, $l(E_s)l(E_t)=l(E_{st})$, $l(E_{t_1})l(E_{t_2}) = l(E_{t_1t_2})$;
(ii) $l(X_{s_1})l(X_{s_2}) = l(X_{s_1s_2})$, $l(X_s)l(X_t) = l(X_sX_s)$;
(iii) $l(X_t)l(X_s) = l(X_tX_s)$, $l(X_{t_1})l(X_{t_2}) = l(X_{t_1}X_{t_2})$.

Proof. By definition, we only need to show the sufficiency. To do this, we will check the following equations:

$$l(E_g)l(X_h) = l(E_gX_h),\quad l(X_h)l(E_g) = l(X_hE_g),\quad l(E_s)l(E_t) = l(E_{st}),$$

where $g,h\in G$, $s\in S$, and $t\in T$. Since R has non-trivial form, we have

$$l(E_s) = \sum_{s'\in S} w^1(s,s')e_{s'},\quad l(E_t) = \sum_{s'\in S} w^3(t,s')e_{s'}x, \tag{3.4}$$

$$l(X_s) = \sum_{t\in S} w^2(s,t')e_{t'},\quad l(X_t) = \sum_{t'\in T} w^4(t,t')e_{t'}x. \tag{3.5}$$

Thus, we get $l(E_g)l(X_h) = l(E_gX_h) = 0$ and $l(X_h)l(E_g) = l(X_hE_g) = 0$ for g, h in G. Moreover, we have $l(E_s)l(E_t) = l(E_t)l(E_s)$ by (3.4). So $l(E_s)l(E_t) = l(E_{st})$. □

Lemma 3.4. *Let $R \in (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)$ and assume that R has nontrivial form with associated maps w^i $(1 \le i \le 4)$. Then $r(f_1)r(f_2) = r(f_2f_1)$ for $f_1, f_2 \in (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$ if and only if the following equations hold for $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$:*
(i) $r(E_{s_1})r(E_{s_2}) = r(E_{s_1s_2})$, $r(E_s)r(E_t) = r(E_{st})$, $r(E_{t_1})r(E_{t_2}) = r(E_{t_1t_2})$;
(ii) $r(X_{s_1})r(X_{s_2}) = r(X_{s_1s_2})$, $r(X_s)r(X_t) = r(X_tX_s)$;
(iii) $r(X_t)r(X_s) = r(X_sX_t)$, $r(X_{t_1})r(X_{t_2}) = r(X_{t_2}X_{t_1})$.

Proof. Similar to the proof of Lemma 3.3. □

In order to use Lemmas 3.3–3.4 more conveniently, we give some more lemmas.

Lemma 3.5. *Let $R \in (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)$ and assume that R has nontrivial form with associated maps w^i $(1 \le i \le 4)$. Then we have*
(i) $l(E_{s_1})l(E_{s_2}) = l(E_{s_1s_2}) \Leftrightarrow w^1(s_1, s)w^1(s_2, s) = w^1(s_1s_2, s)$,
(ii) $l(E_s)l(E_t) = l(E_{st}) \Leftrightarrow w^1(s, s')w^3(t, s') = w^3(st, s')$,
(iii) $l(E_{t_1})l(E_{t_2}) = l(E_{t_1t_2}) \Leftrightarrow w^3(t_1, s)w^3(t_2, s)\sigma(s) = w^1(t_1t_2, s)$,
where $s, s', s_1, s_2 \in S$ and $t, t_1, t_2 \in T$.

Proof. We only show (iii), and the other statements can be proved in a similar way. By (3.4), we have

$$l(E_{t_1})l(E_{t_2}) = \Big[\sum_{s\in S} w^3(t_1, s)e_sx\Big]\Big[\sum_{s\in S} w^3(t_2, s)e_sx\Big] = \sum_{s\in S} w^3(t_1, s)w^3(t_2, s)\sigma(s)e_s.$$

Since we have assumed that $|S| = |T|$, we obtain $TT = S$. Hence, $t_1t_2 \in S$ and we get

$$l(E_{t_1t_2}) = \sum_{s\in S} w^1(t_1t_2, s)e_s.$$

Thus, (iii) holds. □

Lemma 3.6. *Let $R \in (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)$ and assume that R has nontrivial form with associated maps w^i $(1 \le i \le 4)$. Then we have*
(i) $l(X_{s_1})l(X_{s_2}) = l(X_{s_1}X_{s_2}) \Leftrightarrow w^2(s_1, t)w^2(s_2, t) = \tau(s_1, s_2)w^2(s_1s_2, t)$,
(ii) $l(X_s)l(X_t) = l(X_sX_t) \Leftrightarrow w^2(s, t')w^4(t, t') = \tau(s, t)w^4(st, t')$,
(iii) $l(X_t)l(X_s) = l(X_tX_s) \Leftrightarrow w^2(s, t' \triangleleft x)w^4(t, t') = \tau(t, s)w^4(st, t')$,
(iv) $l(X_{t_1})l(X_{t_2}) = l(X_{t_1}X_{t_2}) \Leftrightarrow w^4(t_1, t')w^4(t_2, t' \triangleleft x)\sigma(t) = \tau(t_1, t_2)w^2(t_1t_2, t')$,
where $s, s', s_1, s_2 \in S$ and $t, t', t_1, t_2 \in T$.

Proof. The statements (iii) and (iv) are not obvious, and hence we only show them. By (3.5), we have

$$\begin{aligned} l(X_t)l(X_s) &= \Big[\sum_{t'\in T} w^4(t, t')e_{t'}x\Big]\Big[\sum_{t'\in T} w^2(s, t')e_{t'}\Big] \\ &= \sum_{t'\in T} w^4(t, t')w^2(s, t' \triangleleft x)e_{t'}x \end{aligned}$$

and

$$l(X_{st}) = \sum_{t' \in T} w^4(st, t') e_{t'} x.$$

Thus, (iii) holds. Directly, we have

$$\begin{aligned} l(X_{t_1}) l(X_{t_2}) &= \Big[\sum_{t' \in T} w^4(t_1, t') e_{t'} x\Big]\Big[\sum_{t' \in T} w^4(t_2, t') e_{t'} x\Big] \\ &= \sum_{t \in T} w^4(t_1, t') w^4(t_2, t') \sigma(t') e_{t'}. \end{aligned}$$

Since we have assumed that $|S| = |T|$, we know that $TT = S$. Hence, $t_1 t_2 \in S$ and we get

$$l(X_{t_1 t_2}) = \sum_{t' \in T} w^2(t_1 t_2, t') e_{t'}.$$

Thus, (iv) holds. □

The following two lemmas hold, which can be proved similarly to Lemmas 3.5 and 3.6, respectively.

Lemma 3.7. *Let $R \in (\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2)$ and assume that R has non-trivial form with associated maps w^i $(1 \le i \le 4)$. Then we have*

(i) $r(E_{s_1}) r(E_{s_2}) = r(E_{s_1 s_2}) \Leftrightarrow w^1(s, s_1) w^1(s, s_2) = w^1(s, s_1 s_2)$,
(ii) $r(E_s) r(E_t) = r(E_{st}) \Leftrightarrow w^1(s', s) w^2(s', t) = w^2(s', st)$,
(iii) $r(E_{t_1}) r(E_{t_2}) = r(E_{t_1 t_2}) \Leftrightarrow w^2(s, t_1) w^2(s, t_2) \sigma(s) = w^1(s, t_1 t_2)$,

where $s, s', s_1, s_2 \in S$ and $t, t_1, t_2 \in T$.

Lemma 3.8. *Let $R \in (\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2)$ and assume that R has non-trivial form with associated maps w^i $(1 \le i \le 4)$. Then we have*

(i) $r(X_{s_1}) r(X_{s_2}) = r(X_{s_2} X_{s_1}) \Leftrightarrow w^3(t, s_1) w^3(t, s_2) = \tau(s_2, s_1) w^3(t, s_1 s_2)$,
(ii) $r(X_s) r(X_t) = r(X_t X_s) \Leftrightarrow w^3(t', s) w^4(t', t) = \tau(t, s) w^4(t', st)$,
(iii) $r(X_t) r(X_s) = r(X_s X_t) \Leftrightarrow w^3(t' \triangleleft x, s) w^4(t', t) = \tau(s, t) w^4(t', st)$,
(iv) $r(X_{t_1}) r(X_{t_2}) = r(X_{t_2} X_{t_1}) \Leftrightarrow w^4(t', t_1) w^4(t' \triangleleft x, t_2) \sigma(t') = \tau(t_1, t_2) w^3(t', t_1 t_2)$,

where $s, s_1, s_2 \in S$ and $t, t', t_1, t_2 \in T$.

Remark 3.9. Assume that R has non-trivial form. Owing to the definition of quasitriangular structure, we know that R is a quasitriangular structure on $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ if and only if it satisfies the conditions of Lemma 3.2 and Lemmas 3.5–3.8.

Proof of Theorem 3.1. In view of the above remark, we only need to show that the maps v^i $(1 \le i \le 4)$ satisfy the conditions of Lemmas 3.2 and 3.5–3.8. By assumption, R and R' are non-trivial quasitriangular structures on $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$. Thus, w^i and w'^i $(1 \le i \le 4)$ satisfy the conditions of Lemmas 3.2 and 3.5–3.8. This implies that v^i $(1 \le i \le 4)$ satisfy the conditions of Lemmas 3.2 and 3.5–3.8. □

By virtue of Theorem 3.1, we introduce the following definition.

Definition 3.10. Let $NQ = \{$non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2\}$ and let $NQ' = \{$non-trivial quasitriangular structures on $\Bbbk^G\#\Bbbk\mathbb{Z}_2\}$. If NQ is not empty, then the map $\phi\colon NQ \times NQ \to NQ'$ defined by $\phi(R, R') = \frac{R}{R'}$ is called a *division-like operation* on NQ.

Remark 3.11. We will call a non-trivial quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$ a general solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Naturally, we will call a non-trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ a special solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. By analogy with the solutions of a linear system, one can use two steps to determine quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. One step is to give all the general solutions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, while the other step is to find a special solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Let NQ and NQ' be the sets as in Definition 3.10. Then we have

Proposition 3.12. *If $NQ \neq \emptyset$ and $R_0 \in NQ$, then the map $\varphi\colon NQ \to NQ'$ defined by $\varphi(R) = \frac{R}{R_0}$ is bijective.*

Proof. Assume that the associated maps of R_0 are w_0^i $(1 \le i \le 4)$. By Theorem 3.1, φ is well-defined. By the definition of φ, we know that φ is injective. Assume that $R' \in NQ'$ with associated maps v^i $(1 \le i \le 4)$. Therefore, we can define $R \in (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)\otimes(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)$ such that R has non-trivial form and its associated maps are $(w_0^i v^i)$ $(1 \le i \le 4)$. Similarly to the proof of Theorem 3.1, we know that $R \in NQ$. By definition, we get $\varphi(R) = R'$. This implies that φ is surjective. □

We know that the homogeneous solutions of a system of linear equations form a vector space. Similarly, all general solutions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ form a finite group. Assume that $R_0, R_0' \in NQ'$ and suppose that the four maps associated with R_0 (resp., R_0') are w_0^i (resp., $w_0'^i$) for $1 \le i \le 4$. Then we can use these maps to define four other maps v_0^i $(1 \le i \le 4)$ as follows:

$$v_0^1(s_1, s_2) := w_0^1(s_1, s_2)w_0'^1(s_1, s_2), \quad v_0^2(s, t) := w_0^2(s, t)w_0'^2(s, t),$$
$$v_0^3(t, s) := w_0^3(t, s)w_0'^3(t, s), \quad v_0^4(t_1, t_2) := w_0^4(t_1, t_2)w_0'^4(t_1, t_2),$$

where $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$. Using the maps v_0^i $(1 \le i \le 4)$, we can define $R_0'' \in (\Bbbk^G\#\Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G\#\Bbbk\mathbb{Z}_2)$ as follows. The R_0'' has non-trivial form and the associated maps of it are given by v_0^i $(1 \le i \le 4)$. For simplicity, we denote R_0'' as $R_0 \cdot R_0'$.

Proposition 3.13. *We have $R_0 \cdot R_0' \in NQ'$. Moreover, $(NQ', \cdot)$ is a finite group.*

Proof. By assumption, $R_0, R_0' \in NQ'$. Thus, R_0 and R_0' satisfy the conditions of Lemmas 3.2 and 3.5–3.8. This implies that $R_0 \cdot R_0'$ also satisfies the conditions of Lemmas 3.2 and 3.5–3.8. Hence, $R_0 \cdot R_0'$ is a quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$. To complete the proof, we only need to show that NQ' has a unit and it is finite. Define a non-trivial form R_1 by letting $w_1^i = 1$ for $1 \le i \le 4$, where w_1^i are associated maps of R_1. Obviously, R_1 is the unit of $(NQ', \cdot)$. Since $\Bbbk^G\#\Bbbk\mathbb{Z}_2$ is semisimple, we know that NQ' is finite by [7, Theorem 1]. □

Now, let us give an example to illustrate $(NQ', \cdot)$. Recall that the well-known

8-dimensional Kac algebra K_8 is isomorphic to $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ (see for example [1, Section 2.3]), and the data $(G, \triangleleft, \sigma, \tau)$ of K_8 is given as follows:

(i) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1,\ ab = ba\rangle$ and $a \triangleleft x = b$, $b \triangleleft x = a$;
(ii) $\sigma(a^i b^j) = (-1)^{ij}$, $1 \le i, j \le 2$;
(iii) $\tau(a^i b^j, a^k b^l) = (-1)^{jk}$, $1 \le i, j, k, l \le 2$.

Example 3.14. Let $\gamma \in \Bbbk$ such that $\gamma^4 = -1$. Define R_γ as below:

$$\begin{aligned} R_\gamma := {} & [e_1 \otimes e_1 + e_1 \otimes e_{ab} + e_{ab} \otimes e_1 - e_{ab} \otimes e_{ab}] \\ & + [e_1 x \otimes e_a + e_1 x \otimes e_b - \gamma^2 e_{ab} x \otimes e_a + \gamma^2 e_{ab} x \otimes e_b] \\ & + [e_a \otimes e_1 x + e_b \otimes e_1 x + \gamma^2 e_a \otimes e_{ab} x - \gamma^2 e_b \otimes e_{ab} x] \\ & + [\gamma^{-1} e_a x \otimes e_a x + \gamma e_a x \otimes e_b x + \gamma e_b x \otimes e_a x + \gamma^{-1} e_b x \otimes e_b x]. \end{aligned}$$

The quasitriangular structures on K_8 were determined in [9]. From this result, we know that $\{R_\gamma \mid \gamma^4 = -1\}$ gives all non-trivial quasitriangular structures on K_8. By the definition of NQ', one can get $NQ' \cong \mathbb{Z}_4$ for K_8.

Lastly, we will introduce a division-like operation on the set of trivial quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ for the sake of uniformity. To do this, we first give the following proposition.

Proposition 3.15. [10, Proposition 3.10] *R is a trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ if and only if the following conditions hold:*

(i) $R = \sum_{g,h\in G} w(g,h) e_g \otimes e_h$ *for some bicharacter w on G;*
(ii) $w(g \triangleleft x, h \triangleleft x) = w(g,h)\eta(g,h)$, *where* $\eta(g,h) = \tau(g,h)\tau(h,g)^{-1}$ *for* $g, h \in G$.

Suppose $R = \sum_{g,h\in G} w(g,h) e_g \otimes e_h$, where w is a bicharacter on G satisfying the condition (ii) of Proposition 3.15. If R' is another trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ with associated map w', then we can mimic the above process to give an element $R'' \in (\Bbbk^G\#\Bbbk\mathbb{Z}_2)\otimes(\Bbbk^G\#\Bbbk\mathbb{Z}_2)$ such that $R'' = \sum_{g,h\in G} w''(g,h) e_g \otimes e_h$, where $w''(g,h) = \frac{w(g,h)}{w'(g,h)}$. By Proposition 3.15, we know that R'' is a trivial quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$. For consistency, we also write R'' as $\frac{R}{R'}$.

Definition 3.16. Let $TQ = \{$trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2\}$ and $TQ' = \{$trivial quasitriangular structures on $\Bbbk^G\#\Bbbk\mathbb{Z}_2\}$. If $TQ \neq \emptyset$, then the map $\phi\colon TQ \times TQ \to TQ'$ defined by $\phi(R, R') = \frac{R}{R'}$ is called a *division-like operation* on TQ.

Just like before, we still call a trivial quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$ a general solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and call a trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ a special solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, without distinction. Let TQ and TQ' be the sets as in Definition 3.16. Then we have the following result, for which the proof is similar to that of Proposition 3.12.

Proposition 3.17. *If $TQ \neq \emptyset$ and $R_0 \in TQ$, then the map $\varphi\colon TQ \to TQ'$ defined by $\varphi(R) = \frac{R}{R_0}$ is bijective.*

Assume that $R_0, R_0' \in TQ'$ and suppose that the map associated with R_0 (resp., R_0') is w_0 (resp., w_0'). Then we can define $R_0'' \in (\Bbbk^G \# \Bbbk\mathbb{Z}_2) \otimes (\Bbbk^G \# \Bbbk\mathbb{Z}_2)$ such that $R_0'' = \sum_{g,h \in G} w_0''(g,h) e_g \otimes e_h$, where $w_0''(g,h) = w_0(g,h) w_0'(g,h)$ for $g, h \in G$. Thus, we have the next proposition, which we can prove similarly to Proposition 3.13.

Proposition 3.18. *We have $R_0 \cdot R_0' \in NQ'$. Moreover, $(TQ', \cdot)$ is a finite group.*

Finally, we provide an example to illustrate TQ'.

Example 3.19. Let $\alpha, \beta \in \Bbbk$ such that $\alpha^2 = \beta^2 = 1$. Define

$$\begin{aligned} R_{\alpha,\beta} := {} & e_1 \otimes [e_1 + e_a + e_b + e_{ab}] + e_a \otimes [e_1 + \alpha e_a + \beta e_b + \alpha\beta e_{ab}] \\ & + e_b \otimes [e_1 - \beta e_a + \alpha e_b - \alpha\beta e_{ab}] + e_{ab} \otimes [e_1 - \alpha\beta e_a + \alpha\beta e_b - e_{ab}]. \end{aligned}$$

From the results in [9] we know that $\{R_{\alpha,\beta} \mid \alpha^2 = \beta^2 = 1\}$ gives all trivial quasitriangular structures on K_8. Using the definition of TQ', one can see that $TQ' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ for K_8.

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