Algebr Represent Theor (2013) 16:771–791
DOI 10.1007/s10468-011-9331-1

Basic Hopf Algebras of Tame Type

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Received: 10 April 2010 / Accepted: 9 October 2011 / Published online: 12 January 2012

Abstract In continuation of the articles (Liu J Algebra 299:841–853, 2006; Huang, J Algebra 321:2650–2669, 2009) we classify all finite-dimensional basic Hopf algebras of tame type over an algebraically closed field of characteristic 0 in this paper. As consequences, we show the following statements: (1) the representation dimension of a tame basic Hopf algebra is exactly 3, (2) for a basic Hopf algebra H, if $\mathrm{C}(H) \geq 3$ then it is wild. These conclusions verify a folklore conjecture and one of Rickard's statements for the class of finite-dimensional basic Hopf algebras.

Keywords Basic Hopf algebra · Representation type · Representation dimension · Complexity

Mathematics Subject Classifications (2010) 16G60 · 16W30

1 Introduction

Throughout this paper k denotes an algebraically closed field and all spaces are k-spaces. By an algebra we mean a finite-dimensional associative algebra with identity element. We freely use the results, notations, and conventions of [35].

According to the fundamental result of Drozd [13], every finite dimensional algebra exactly belongs to one of following three kinds of algebras: algebras of finite representation type, algebras of tame type and wild algebras. For the algebras of the former two kinds, a classification of indecomposable modules seems feasible. By contrast, the module category of a wild algebra, being "complicated" at least as

Dedicated to Professor Mitsuhiro Takeuchi in honor of his distinguished career.

Presented by Michel Van den Bergh.

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that of any other algebra, can't afford such a classification. Inspired by the Drozd's trichotomy, one is often interested in classifying a given kind of algebras according to their representation type. Such classification for finite-dimensional Hopf algebras has received considerable attention. For modular group algebras of finite groups, a block of such a modular group algebra is of finite representation type if and only if the corresponding defect groups are cyclic while it is tame if and only if char $k = 2$ and its defects groups are dihedral, semidihedral or generalized quaternion. See [7, 9, 14, 24]. In the case of small quantum groups, i.e., Frobenius–Lusztig kernels, the only tame one is $\mathbf{u}_q(\mathfrak{sl}_2)$ and the others are all wild [12, 40, 41]. The classification for finite-dimensional cocommutative Hopf algebras, i.e., finite algebraic groups, of finite representation type and tame type was given by Farnsteiner and his cooperators recently [17–21].

The class of basic Hopf algebras and their duals, pointed Hopf algebras, have been studied intensively by many authors. See, for example, [3, 4, 23]. Our intention is to classify finite dimensional basic Hopf algebras through their representation type. The class of basic Hopf algebras of finite representation type has been given in [29] and all radically graded basic Hopf algebras of tame type were classified recently [25]. So, in order to give a complete classification of tame basic Hopf algebras, one needs to "lift" radically graded basic Hopf algebras to the general case at first. Experiences tell us that it is easier to deal with lifting for the dual case, i.e., pointed Hopf algebras. In fact, the lifting method for pointed Hopf algebras has been developed by Andruskiewitsch and Schneider [2–4]. Through Hopf-Galois approach, Masuoka showed that the lifting developed by Andruskiewitsch and Schneider are special cases of 2-cocycle deformations and gave many interesting applications [32–34]. In this paper, we will explain such lifting through path coalgebras, an intuitive way. Upon this lifting way, we get all liftings of the tame graded basic Hopf algebras and all of them are shown to be tame (this is also a consequence of degeneration theory [22]). In fact, we indeed show that the algebraic structures of these lifting are the same as that of their graded versions.

One of the difficulties of this paper is to determine whether tame basic Hopf algebras can be degenerated to wild algebras. In this paper, we will show that it will not happen for finite-dimensional basic Hopf algebras, that is, we show that the tameness of a basic Hopf algebra H implies the tameness of $\mathrm{gr}\, H$. Thus the lifting we get before is a complete classification of tame basic Hopf algebras over an algebraically closed field of characteristic 0. In particular, the connected tame basic Hopf algebras are shown to be the dual of small quantum groups of dihedral type, which is not included in the list of Andruskiewitsch and Schneider [4].

As a consequence of this classification, some representation properties of basic Hopf algebras are determined. We hope the representation properties we chose can not only help us to give information about basic Hopf algebras but also to verify some open problems in the representation theory of finite-dimensional algebras. A folklore conjecture states that if an algebra A is tame, then its representation dimension, introduced by Auslander [5] to measure how far an artin algebra is from being of finite representation type, is 3. We will show that this is true for basic Hopf algebras. As another unsettled case, Rickard [38] established an important result implying that the complexity $\mathrm{C}(M)$ of a finite-dimensional module M over a self-injective tame algebra A is bounded by 2, but there is a gap in his proof. Farnsteiner showed that the Rickard's conclusion is true for cocommutative Hopf algebras [16].

As a consequence, we will show that the Rickard's conclusion is also true for basic Hopf algebras.

All preliminary notions and results that are relevant for our purpose are summarized in Section 2. The main task of Section 3 is to give all liftings of tame graded basic Hopf algebras. Section 4 tells us that the tameness of a finite-dimensional basic Hopf algebras H is an invariant in the process of deformation of H. The complete classification of tame basic Hopf algebras is formulated in Section 5 at last. In the last section, the representation dimensions and complexities of tame basic Hopf algebras are determined.

2 Preliminaries

Throughout we will be working over an algebraically closed field k of characteristic 0. All spaces are k-spaces. For short, $\otimes_k$ is just denoted by $\otimes$.

2.1 Path Coalgebras

Given a quiver $Q = (Q_0, Q_1)$ with Q_0 the set of vertices and Q_1 the set of arrows, denote by kQ, kQ^a, and kQ^c, the k-space with basis the set of all paths in Q, the path algebra of Q, and the path coalgebra of Q, respectively. Note that they are all graded with respect to length grading. For $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote respectively the starting and ending vertex of α.

Recall that the comultiplication of the path coalgebra kQ^c is defined by

$$\Delta(p) = \alpha_l \cdots \alpha_1 \otimes s(\alpha_1) + \sum_{i=1}^{l-1} \alpha_l \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 + t(\alpha_l) \otimes \alpha_l \cdots \alpha_1$$

for each path $p = \alpha_l \cdots \alpha_1$ with each $\alpha_i \in Q_1$; and $\varepsilon(p) = 0$ for $l \geq 1$ and 1 if $l = 0$ ($l = 0$ means p is a vertex). This is a pointed coalgebra.

Let C be a coalgebra. The set of group-like elements is defined to be

$$G(C) := \{c \in C | \Delta(c) = c \otimes c,\ c \neq 0\}.$$

It is clear $\varepsilon(c) = 1$ for $c \in G(C)$. For $x, y \in G(C)$, denote by

$$P_{x,y}(C) := \{c \in C | \Delta(c) = c \otimes x + y \otimes c\}$$

the set of x, y-primitive elements in C. It is clear that $\varepsilon(c) = 0$ for $c \in P_{x,y}(C)$. Note that $k(x - y) \subseteq P_{x,y}(C)$. An element $c \in P_{x,y}(C)$ is *non-trivial* if $c \notin k(x - y)$. Note that $G(kQ^c) = Q_0$; and

Lemma 2.1 (Lemma 1.1 in [10]) *For $x, y \in Q_0$, we have $P_{x,y}(C) = y(kQ_1)x \oplus k(x - y)$ where $y(kQ_1)x$ denotes the k-space spanned by all arrows from x to y. In particular, there is a non-trivial x, y-primitive element in kQ^c if and only if there is an arrow from x to y in Q.*

For a coalgebra C, one can construct its coradically graded coalgebra $\mathrm{gr}\, C$ (see Chapter 5 in [35]). Chin and Montgomery showed the following result [11]:

Lemma 2.2 *Let C be a pointed coalgebra, then there exists a unique quiver $Q(C)$ such that C can be embedded into the path coalgebra $kQ(C)^c$ as a large subcoalgebra.*

This unique quiver $Q(C)$ is called its *dual Gabriel quiver*. Here "large" means that C contains all group-like elements and x, y-primitive elements of $kQ(C)^c$ for $x, y \in Q(C)_0$. For a pointed coalgebra, it is easy too see that

Lemma 2.3 $Q(C) = Q(\mathrm{gr}\, C)$.

Let H be a pointed Hopf algebra, then H can be embedded into a path coalgebra $kQ(H)^c$.

2.2 Bosonization

Let H, H_0 be Hopf algebras and $\pi : H \to H_0$ and $\iota : H_0 \to H$ Hopf homomorphisms. Assume that $\pi\iota = id_{H_0}$, so that π is surjective and ι is injective. Define

$$R_H := H^{co\pi} = \{h \in H|(id \otimes \pi)\Delta(h) = h \otimes 1\}.$$

By a result of Radford's (see Theorem 3 of [37]),

$$H \cong R_H \times H_0 \quad \text{as Hopf algebras}$$

where "$\times$" is called *biproduct* in [37] and *bosonization* in [31]. Now let H be a basic Hopf algebra, so the Jacobson radical J_H is a Hopf ideal automatically [23]. Thus $\mathrm{gr}H := H/J_H \oplus J_H/J_H^2 \oplus \cdots$ is a graded Hopf algebra. Clearly, $H/J_H = \mathrm{gr}H(0)$ is a Hopf subalgebra of $\mathrm{gr}H$ and there is a natural Hopf algebra epimorphism $\pi : \mathrm{gr}H \to H/J_H$ with a retraction of the inclusion. We can then apply the above discussion. Let $R_H = \{h \in H|(id \otimes \pi)\Delta(h) = h \otimes 1\}$, then $\mathrm{gr}H$ can be reconstructed from R_H and H/J_H as a bosonization

$$\mathrm{gr}H \cong R_H \times H/J_H.$$

The main result of [25] can be stated in the following way.

Lemma 2.4 *Let H be a radically graded basic Hopf algebra of tame type. Then*

$$H \cong k\langle x, y\rangle/I \times (kG)^*$$

for some finite group G and $I = (x^2, y^2, (xy)^m - c(yx)^m)$ for some $m \geq 1$ and $c \in k$.

2.3 Representation Type

A finite-dimensional algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are a finite number of A-$k[T]$-bimodules M_i which are free of finite rank as right $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{k[T]} k[T]/(T-\lambda)$ for $\lambda \in k$. We say that A is of *wild type* or A is a *wild* algebra if there is a finitely generated A-$k\langle X, Y\rangle$-bimodule B which is free as a right $k\langle X, Y\rangle$-module such that the functor $B \otimes_{k\langle X,Y\rangle}$—from mod-$k\langle X, Y\rangle$, the category of finitely generated $k\langle X, Y\rangle$-modules, to mod-A, the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms. See [14] for more details. For other unexplained notations about representation theory of finite-dimensional algebras in this paper, see [6, 14].

3 Lifting of Connected Tame Graded Basic Hopf Algebras

All connected tame radically graded basic Hopf algebras were classified in [25] (Theorems 4.9, 4.16). The main result of this section is to give all possible liftings of these tame graded basic Hopf algebras. Our basic idea is quite simple: We dualize the radically graded tame basic Hopf algebras at first, then we get some coradically graded pointed Hopf algebra; Lift such pointed Hopf algebras and then dualize them back. In this procedure, there are two difficulties we need to overcome. One is to find suitable generators of the dual Hopf algebras and another is to make sure that all lifting we get are still tame.

We recall a Hopf algebra is called *connected* if it is connected as an algebra. In [25], all connected radically graded tame basic Hopf algebras are given by using covering quivers and allowable bimodules on them. For our purpose, we describe such Hopf algebras through generators and relations. Consider the following three kinds of basic Hopf algebras:

Type 1 $(\mathrm{gr}\, I_1)^*$ Let $W = \mathbb{Z}_n = \langle g | g^n = 1\rangle$ be a cyclic group of order n with n even and q an n-th primitive root of unity. The Hopf algebra $(\mathrm{gr}\, I_1)^*$ is defined to be an associative algebra generated by elements x, y and g, with relations

$$g^n = 1,\ x^2 = y^2 = xy + yx = 0,\ gxg^{-1} = qx,\ gyg^{-1} = qy.$$

The comultiplication Δ, counit ε, and antipode S are given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g^{\frac{n}{2}} \otimes x, \quad \Delta(y) = y \otimes 1 + g^{\frac{n}{2}} \otimes y,$$

$$\varepsilon(g) = 1, \qquad \varepsilon(x) = \varepsilon(y) = 0$$

$$S(g) = g^{-1}, \qquad S(x) = -g^{\frac{n}{2}} x, \qquad S(y) = -g^{\frac{n}{2}} y.$$

Type 2 $(\mathrm{gr}\, I^1_m)^*$ Let $W = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} = \langle g, h | g^{n_1} = h^{n_2} = 1, gh = hg\rangle$ be the direct product of two cyclic groups of orders n_1, n_2 respectively with n_1, n_2 even. Let q, p be n_1-th and n_2-th primitive roots of unity respectively. Take two integers m_1, m_2 with $m_1 | n_1, m_2 | n_2$ and set

$q_2 := q^{m_1},\ p_1 := p^{m_2}$. Assume that p_1q_2 is an m-th primitive root of unity. Then the basic Hopf algebra $(\mathrm{gr}\, I^1_m)^*$ is defined to be an associative algebra generated by elements x, y and g, h, with relations

$$g^{n_1} = h^{n_2} = 1,\ x^2 = y^2 = (xy)^m + (-q_2)^m (yx)^m = 0,$$

$$gxg^{-1} = qx,\ gyg^{-1} = y,\ hxh^{-1} = x,\ hyh^{-1} = py.$$

The comultiplication Δ, counit ε, and antipode S are given by

$$\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h$$

$$\Delta(x) = x \otimes 1 + g^{\frac{n_1}{2}} h^{m_2} \otimes x, \quad \Delta(y) = y \otimes 1 + g^{m_1} h^{\frac{n_2}{2}} \otimes y,$$

$$\varepsilon(g) = \varepsilon(h) = 1, \quad \varepsilon(x) = \varepsilon(y) = 0$$

$$S(g) = g^{-1}, \quad S(h) = h^{-1}, \quad S(x) = -g^{\frac{n_1}{2}} h^{-m_2} x, \quad S(y) = -g^{-m_1} h^{\frac{n_2}{2}} y.$$

Type 3 $(\mathrm{gr}\, I^2_m)^*$ Let $W = \mathbb{Z}_n = \langle g | g^n = 1\rangle$ be a cyclic group of order n with n is even. Assume g^i, g^j generate W and thus $(i, j) = 1$ which implies there exist $s, t \in \mathbb{Z}$ such that

$$si + tj = 1.$$

Assume that the orders of g^i, g^j are n_1, n_2 respectively with n_1, n_2 are all even. Let q, p be n_1-th and n_2-th primitive root of unity respectively. Take two integers m_1, m_2 with $m_1 | n_1, m_2 | n_2$ and set $q_2 := q^{m_1},\ p_1 := p^{m_2}$. Assume that p_1q_2 is an m-th primitive root of unity. Then the basic Hopf algebra $(\mathrm{gr}\, I^2_m)^*$ is defined to be an associative algebra generated by elements x, y and g, with relations

$$g^n = 1, x^2 = y^2 = (xy)^m + (-q_2)^m (yx)^m = 0, gxg^{-1} = q^{is} x, gyg^{-1} = p^{jt} y,$$

The comultiplication Δ, counit ε, and antipode S are given by

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1 + g^{\frac{n}{2} + jm_2} \otimes x, \quad \Delta(y) = y \otimes 1 + g^{im_1 + \frac{n}{2}} \otimes y,$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = \varepsilon(y) = 0$$

$$S(g) = g^{-1}, \quad S(x) = -g^{-(\frac{n}{2} + jm_2)} x, \quad S(y) = -g^{-(im_1 + \frac{n}{2})} y.$$

Lemma 3.1 (Theorems 4.9, 4.16 in [25]) *Let H be a connected radically graded tame basic Hopf algebra over an algebraically closed field k with characteristic* 0*. Then as a Hopf algebra it is isomorphic to* $(\mathrm{gr}\, I_1)^*$ *or* $(\mathrm{gr}\, I^1_m)^*$ *or* $(\mathrm{gr}\, I^2_m)^*$.

Let H be a finite-dimensional Hopf algebra. Assume its Jacobson radical is a Hopf ideal, then the radically graded version $\mathrm{gr}\, H := H/J_H \oplus J_H/J_H^2 \oplus \cdots$ is a Hopf algebra too. Consider its dual H^* and we know its coradical H_0^* is a sub Hopf algebra now. Thus its coradically graded version $\operatorname{gr} H^* := H_0^* \oplus H_1^*/H_0^* \oplus \cdots$ is also a Hopf algebra. Here we use the notation "gr" to denote both radically graded versions and coradically graded versions and we can discriminate the exact meaning of this notation by using the context.

Lemma 3.2 *Let H be as above. Then H is a lifting of* $\operatorname{gr} H$ *if and only if H^* is a lifting of* $\operatorname{gr} H^*$.

Proof It is enough to show that $(\operatorname{gr} H)^* \cong \operatorname{gr} H^*$ as Hopf algebras. This is indeed a direct consequence of Proposition 5.2.9 in [35]. □

Type 1 Let n be an even number. Define the Hopf algebra $\operatorname{gr} I_1$ now. As an associative algebra, it is generated by X, Y and G with relations

$$G^n = 1,\ X^2 = Y^2 = XY + YX = 0,\ GXG^{-1} = -X,\ GYG^{-1} = -Y.$$

The comultiplication Δ, counit ε and antipode S are defined by

$$\Delta(G) = G \otimes G,\quad \Delta(X) = X \otimes 1 + G \otimes X,\quad \Delta(Y) = Y \otimes 1 + G \otimes Y,$$

$$\varepsilon(G) = 1, \qquad \varepsilon(X) = \varepsilon(Y) = 0,$$

$$S(G) = G^{-1}, \qquad S(X) = -G^{-1}X, \qquad S(Y) = -G^{-1}Y.$$

It is tedious to show that $\operatorname{gr} I_1$ is indeed a Hopf algebra.

Proposition 3.3 *As a Hopf algebra, the dual Hopf algebra of* $(\operatorname{gr} I_1)^*$ *is isomorphic to* $\operatorname{gr} I_1$.

We will give a relatively detailed proof of this proposition, as an example to illustrate how to get the dual Hopf algebras. By the diamond lemma [8], $(\operatorname{gr} I_1)^*$ has a basis $\{g^l x^i y^j | 0 \le l \le n-1, 0 \le i, j \le 1\}$ and denote by $(g^l x^i y^j)^*$ the element of the dual of $(\operatorname{gr} I_1)^*$ which sends $g^l x^i y^j$ to 1 and the other elements in this basis to 0. Recall q is an n-th primitive root of unity and define

$$G := \sum_{i=0}^{n-1} q^i (g^i)^*,\quad X := \sum_{i=0}^{n-1} q^i (g^i x)^*,\quad Y := \sum_{i=0}^{n-1} q^i (g^i y)^*.$$

Lemma 3.4 *The elements G, X, Y generate the dual Hopf algebra of* $(\operatorname{gr} I_1)^*$.

Proof Denote the sub algebra generated by G, X, Y by H. It is enough to show $(g^l x^i y^j)^* \in H$ for $0 \le l \le n-1, 0 \le i, j \le 1$. Indeed, it is easy to see that G is a character of $\mathbb{Z}_n$ and also generates a cyclic group of order n. This implies that $(g^i)^* \in H$ for $0 \le i \le n-1$. Direct computations show that $(g^i)^*(g^j x)^* \neq 0$ if and only if $i = j + \frac{n}{2}$ and in this case we have

$$(g^{i+\frac{n}{2}})^*(g^i x)^* = (g^i x)^*.$$

Thus

$$(g^0)^* X = -(g^{\frac{n}{2}} x)^*, \ (g^1)^* X = -q(g^{1+\frac{n}{2}} x)^*, \cdots$$

which implies that $(g^i x)^* \in H$ for $0 \le i \le n-1$. Similarly, $(g^i y)^* \in H$ for $0 \le i \le n-1$.

Let us compute $(g^i x)^*(g^j y)^*$. In order to make $(g^i x)^*(g^j y)^*(g^l x^s y^t) \neq 0$, we must have $s = t = 1$. And,

$$\Delta(g^l xy) = g^l xy \otimes g^l - g^{l+\frac{n}{2}} x \otimes g^l y + g^{l+\frac{n}{2}} y \otimes g^l x + g^l \otimes g^l xy.$$

Thus $(g^i x)^*(g^j y)^* \neq 0$ if and only if $i = j + \frac{n}{2}$ and in this case

$$(g^{i+\frac{n}{2}} x)^*(g^i y)^* = -(g^i xy)^*.$$

So $(g^i xy)^* \in H$ for $0 \le i \le n-1$. □

Lemma 3.5 *With G, X, Y defined as above, we have*

$$X^2 = Y^2 = 0, \quad XY = -YX, \quad GXG^{-1} = -X, \quad GYG^{-1} = -Y.$$

Proof It is easy to see that $X^2 = Y^2 = 0$. Using the proof of the last lemma,

$$\begin{aligned} XY &= \sum_{i,j} q^{i+j}(g^i x)^*(g^j y)^* \\ &= \sum_j -q^{j+\frac{n}{2}+j}(g^j xy)^* \\ &= \sum_j q^{2j}(g^j xy)^*. \end{aligned}$$

Similarly, $YX = \sum_j -q^{2j}(g^j xy)^*$. Thus $XY = -YX$.

Similar to the proof of the last lemma, one can show that

$$(g^i)^*(g^j x)^*(g^k)^* \neq 0 \Leftrightarrow i = j + \frac{n}{2} = k + \frac{n}{2}$$

and in this case it is equal to $(g^j x)^*$. Therefore,

$$\begin{aligned} GXG^{-1} &= \sum_{i,j,k} q^i q^j q^{-k}(g^i)^*(g^j x)^*(g^k)^* \\ &= \sum_j q^{j+\frac{n}{2}}(g^j x)^* \\ &= -X. \end{aligned}$$

Similarly, $GYG^{-1} = -Y$. □

Lemma 3.6 *With G, X, Y defined as above, the comultiplications of G, X, Y are given by*

$$\Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes 1 + G \otimes X, \quad \Delta(Y) = Y \otimes 1 + G \otimes Y.$$

Proof It is not hard to show that $\Delta((g^i)^*) = \sum_{j+k\equiv i}(g^j)^* \otimes (g^k)^*$ and

$$\Delta((g^i x)^*) = \sum_{j+k\equiv i} q^{-k}(g^j x)^* \otimes (g^k)^* + \sum_{j+k\equiv i}(g^j)^* \otimes (g^k x)^*,$$

$$\Delta((g^i y)^*) = \sum_{j+k\equiv i} q^{-k}(g^j y)^* \otimes (g^k)^* + \sum_{j+k\equiv i}(g^j)^* \otimes (g^k y)^*.$$

Thus

$$\Delta(G) = \sum_{j,k} q^{j+k}(g^j)^* \otimes (g^k)^* = G \otimes G$$

and

$$\begin{aligned}\Delta(X) &= \sum_i \Delta(q^i(g^i x)^*)\\ &= \sum_i q^i\Big(\sum_{j+k\equiv i} q^{-k}(g^j x)^* \otimes (g^k)^* + \sum_{j+k\equiv i}(g^j)^* \otimes (g^k x)^*\Big)\\ &= \sum_{j,k} q^{j+k}q^{-k}(g^j x)^* \otimes (g^k)^* + \sum_{j,k} q^{j+k}(g^j x)^* \otimes (g^k)^*\\ &= X \otimes 1 + G \otimes X.\end{aligned}$$

Similarly, $\Delta(Y) = Y \otimes 1 + G \otimes Y$. □

Combining Lemmas 3.4–3.6, we give the proof of Proposition 3.3. Now we consider lifting of the pointed Hopf algebra gr I_1. Let $\lambda_1, \lambda_2, \lambda_3 \in k$ and define a pointed Hopf algebra $I_1(\lambda_1, \lambda_2, \lambda_3)$ as follows. As an associative algebra, it is generated by G, X, Y with relations

$$G^n = 1, \quad GXG^{-1} = -X, \quad GYG^{-1} = -Y,$$

$$X^2 = \lambda_1(G^2 - 1), \quad Y^2 = \lambda_2(G^2 - 1), \quad XY + YX = \lambda_3(G^2 - 1).$$

The comultiplication Δ, counit ε and antipode S are defined by

$$\Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes 1 + G \otimes X, \quad \Delta(Y) = Y \otimes 1 + G \otimes Y,$$

$$\varepsilon(G) = 1, \qquad \varepsilon(X) = \varepsilon(Y) = 0,$$

$$S(G) = G^{-1}, \qquad S(X) = -G^{-1}X, \qquad S(Y) = -G^{-1}Y.$$

Proposition 3.7 *Suppose that A is a lifting of* gr I_1. *Then $A \cong I_1(\lambda_1, \lambda_2, \lambda_3)$ for some $\lambda_1, \lambda_2, \lambda_3 \in k$.*

Proof Clearly, the coradical A_0 is the sub Hopf algebra generated by G. By Lemma 6.1 in [4], there are elements $X, Y \in A$ corresponding to $X, Y \in \mathrm{gr}\, I_1$ and satisfy

$$\Delta(X) = G \otimes X + X \otimes 1, \quad \Delta(Y) = G \otimes Y + Y \otimes 1,$$

$$GXG^{-1} = -X, \quad GYG^{-1} = -Y.$$

By $\mathrm{gr}\, I_1$ is generated by group-like elements and skew primitive elements, we know A is indeed generated by G, X and Y. Direct computations show that the elements X^2, Y^2 and $XY + YX$ are all $1, G^2$-primitive elements. By Lemma 2.3, there are no arrows from 1 to G^2 and thus Lemma 2.1 implies that there are no non-trivial $1, G^2$-primitive elements. Therefore, there are $\lambda_1, \lambda_2, \lambda_3 \in k$ such that

$$X^2 = \lambda_1(G^2 - 1), \quad Y^2 = \lambda_2(G^2 - 1), \quad XY + YX = \lambda_3(G^2 - 1).$$

So we have a surjective Hopf morphism from $I_1(\lambda_1, \lambda_2, \lambda_3)$ to A. Since A is a lifting of $\mathrm{gr}\, I_1$,

$$\dim_k A = \dim_k \mathrm{gr}\, I_1 = \dim_k I_1(\lambda_1, \lambda_2, \lambda_3).$$

Therefore, the surjective Hopf morphism must be an isomorphism. □

Theorem 3.8 *Let H be a lifting of radically graded Hopf algebra* $(\mathrm{gr}\, I_1)^*$. *Then as a Hopf algebra*

$$H \cong (I_1(\lambda_1, \lambda_2, \lambda_3))^*$$

for some $\lambda_1, \lambda_2, \lambda_3 \in k$.

Proof It is a direct consequence of Proposition 3.7 and Lemma 3.2. □

Remark 3.9 If we define

$$g := \sum_{i=0}^{n-1} q^i (G^i)^*, \quad x := \sum_{i=0}^{n-1} (-1)^i (G^i X)^*, \quad y := \sum_{i=0}^{n-1} (-1)^i (G^i Y)^*$$

where q is the n-th primitive root of unity given in the definition of $(\mathrm{gr}\, I_1)^*$. Just like the proofs of Lemmas 3.4–3.6, one can show that $(I_1(\lambda_1, \lambda_2, \lambda_3))^*$ is generated by g, x, y and satisfy

$$g^n = 1, \; x^2 = y^2 = xy + yx = 0, \; gxg^{-1} = qx, \; gyg^{-1} = qy.$$

The comultiplications Δ for x, y are

$$\Delta(x) = x \otimes 1 + g^{\frac{n}{2}} \otimes x, \quad \Delta(y) = y \otimes 1 + g^{\frac{n}{2}} \otimes y.$$

But the comultiplication for g is complicated. By this, we know that the algebraic structure is **not** changed through a lifting. Thus, all $(I_1(\lambda_1, \lambda_2, \lambda_3))^*$ are still tame.

Type 2 We use the same notations in the definition for $(\mathrm{gr}\, I_m^1)^*$. Define the Hopf algebra $\mathrm{gr}\, I_m^1$ now. As an associative algebra, it is generated by X, Y and G, H with relations

$$G^{n_1} = H^{n_2} = 1, \quad X^2 = Y^2 = (XY)^m + (-q_2)^m (YX)^m = 0,$$

$$GXG^{-1} = -X, \quad GYG^{-1} = -q_2 Y, \quad HXH^{-1} = p_1 X, \quad HYH^{-1} = -Y.$$

The comultiplication Δ, counit ε and antipode S are defined by

$$\Delta(G) = G \otimes G, \quad \Delta(H) = H \otimes H,$$

$$\Delta(X) = X \otimes 1 + G \otimes X, \quad \Delta(Y) = Y \otimes 1 + H \otimes Y,$$

$$\varepsilon(G) = \varepsilon(H) = 1, \quad \varepsilon(X) = \varepsilon(Y) = 0,$$

$$S(G) = G^{-1}, \quad S(H) = H^{-1}, \quad S(X) = -G^{-1}X, \quad S(Y) = -H^{-1}Y.$$

For $\lambda_1, \lambda_2, \lambda_3 \in k$, define the Hopf algebra $I_m^1(\lambda_1, \lambda_2, \lambda_3)$ in the following way. It is also generated by X, Y and G, H with relations

$$G^{n_1} = H^{n_2} = 1, \quad X^2 = \lambda_1(G^2 - 1), \quad Y^2 = \lambda_2(H^2 - 1),$$

$$(XY)^m + (-q_2)^m (YX)^m = \lambda_3((GH)^m - 1),$$

$$GXG^{-1} = -X, \quad GYG^{-1} = -q_2 Y, \quad HXH^{-1} = p_1 X, \quad HYH^{-1} = -Y.$$

The comultiplication Δ, counit ε and antipode S are given in the same way as $\mathrm{gr}\, I_m^1$.

Theorem 3.10

(1) $\mathrm{gr}\, I_m^1$ *and* $\mathrm{gr}\, I_m^1(\lambda_1, \lambda_2, \lambda_3)$ *are Hopf algebras.*
(2) *The dual Hopf algebra of* $(\mathrm{gr}\, I_m^1)^*$ *is isomorphic to* $\mathrm{gr}\, I_m^1$.
(3) *Any lifting of* $\mathrm{gr}\, I_m^1$ *is isomorphic to* $I_m^1(\lambda_1, \lambda_2, \lambda_3)$ *for some* $\lambda_1, \lambda_2, \lambda_3 \in k$.
(4) *Any lifting of* $(\mathrm{gr}\, I_m^1)^*$ *is isomorphic to the dual Hopf algebra of* $I_m^1(\lambda_1, \lambda_2, \lambda_3)$ *for some* $\lambda_1, \lambda_2, \lambda_3 \in k$.

Proof (1) is straightforward. For (2), just define

$$G := \sum_{i=0,\, j=0}^{n_1, n_2} q^i (g^i h^j)^*, \quad H := \sum_{i=0,\, j=0}^{n_1, n_2} p^j (g^i h^j)^*,$$

$$X := \sum_{i=0,\, j=0}^{n_1, n_2} q^i (g^i h^j x)^*, \quad Y := \sum_{i=0,\, j=0}^{n_1, n_2} p^j (g^i h^j y)^*.$$

One can check (2) just as the proof of Proposition 3.3. (3)(4) can be obtained similarly as in Proposition 3.7 and Theorem 3.8. □

Remark 3.11 Just as the Remark 3.9, we define

$$g := \sum_{i=0,\, j=0}^{n_1,n_2} q^i (G^i H^j)^*, \quad h := \sum_{i=0,\, j=0}^{n_1,n_2} p^j (G^i H^j)^*,$$

$$x := \sum_{i=0,\, j=0}^{n_1,n_2} (-1)^i p_1^j (G^i H^j X)^*, \quad Y := \sum_{i=0,\, j=0}^{n_1,n_2} (-1)^j q_2^i (G^i H^j Y)^*.$$

One can show that g, h, x, y generate the dual Hopf algebra of $I_m^1(\lambda_1, \lambda_2, \lambda_3)$ and the algebraic structure of $(I_m^1(\lambda_1, \lambda_2, \lambda_3))^*$ is the same as that of $(\mathrm{gr}\, I_m^1)^*$. Thus $(I_m^1(\lambda_1, \lambda_2, \lambda_3))^*$ is still tame.

Type 3 We use the same notations as in the definition for $(\mathrm{gr}\, I_m^2)^*$. Define the Hopf algebra $\mathrm{gr}\, I_m^2$ now. As an associative algebra, it is generated by X, Y and G with relations

$$G^n = 1, \quad X^2 = Y^2 = (XY)^m + (-q_2)^m (YX)^m = 0,$$

$$GXG^{-1} = (-1)^s p_1^t X, \quad GYG^{-1} = -q_2^s (-1)^t Y.$$

The comultiplication Δ, counit ε and antipode S are defined by

$$\Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes 1 + G^i \otimes X, \quad \Delta(Y) = Y \otimes 1 + G^j \otimes Y,$$

$$\varepsilon(G) = 1, \quad \varepsilon(X) = \varepsilon(Y) = 0,$$

$$S(G) = G^{-1}, \quad S(X) = -G^{-i} X, \quad S(Y) = -G^{-j} Y.$$

For $\lambda_1, \lambda_2, \lambda_3 \in k$, define the Hopf algebra $I_m^1(\lambda_1, \lambda_2, \lambda_3)$ in the following way. It is also generated by X, Y and G with relations

$$G^n = 1, \quad X^2 = \lambda_1 (G^{2i} - 1), \quad Y^2 = \lambda_2 (G^{2j} - 1),$$

$$(XY)^m + (-q_2)^m (YX)^m = \lambda_3 ((G)^{(i+j)m} - 1),$$

$$GXG^{-1} = (-1)^s p_1^t X, \quad GYG^{-1} = -q_2^s (-1)^t Y.$$

The comultiplication Δ, counit ε and antipode S are given in the same way as $\mathrm{gr}\, I_m^2$.

We give the following result without proofs since its proof is totally the same as that of Theorem 3.10.

Theorem 3.12

(1) gr I_m^2 *and* gr $I_m^2(\lambda_1, \lambda_2, \lambda_3)$ *are Hopf algebras.*
(2) *The dual Hopf algebra of* (gr $I_m^2)^*$ *is isomorphic to* gr I_m^2.
(3) *Any lifting of* gr I_m^2 *is isomorphic to* $I_m^2(\lambda_1, \lambda_2, \lambda_3)$ *for some* $\lambda_1, \lambda_2, \lambda_3 \in k$.
(4) *Any lifting of* (gr $I_m^2)^*$ *is isomorphic to the dual Hopf algebra of* $I_m^2(\lambda_1, \lambda_2, \lambda_3)$ *for some* $\lambda_1, \lambda_2, \lambda_3 \in k$.

Remark 3.13

(1) Just like Remarks 3.9, 3.11, the algebraic structure of $(I_m^2(\lambda_1, \lambda_2, \lambda_3))^*$ is the same as that of (gr $I_m^2)^*$ and thus all liftings of (gr $I_m^2)^*$ are still tame.
(2) In the process of the classification of finite-dimensional pointed Hopf algebras given by Andruskiewitsch and Schneider [4], they need the condition that the order of the group of group-like elements must be odd. In contrast to this requirement, we always need the order to be even. Thus the pointed Hopf algebras $I_1(\lambda_1, \lambda_2, \lambda_3)$, $I_m^1(\lambda_1, \lambda_2, \lambda_3)$ and $I_m^2(\lambda_1, \lambda_2, \lambda_3)$ constructed in this paper are not included in the list of pointed Hopf algebras given in [4]. Therefore they are new examples of non-commutative, non-cocommutative Hopf algebras, that is, quantum groups. Since the algebraic structures of them are clearly close to dihedral groups, we call them *small quantum groups of dihedral type*. Sometimes, some experts denote dihedral group of order $2m$ by I_m and this is the reason why we use the notation I_m to represent such pointed Hopf algebras.

4 The Invariance of Tameness

The main result of this section is the following theorem.

Theorem 4.1 *Let H be a finite-dimensional basic Hopf algebra and* gr H *its radically graded version. Then H is tame if and only if* gr H *is so.*

By the general degeneration theory (see [22]), the tameness of gr H indeed implies that H is tame too. Note that for a finite-dimensional tame algebra A, it is possible that gr A is a wild algebra. So the main difficulty of the proof of this theorem is to show that tame basic Hopf algebras can not be degenerated to wild algebras. To give the "only if" part of the theorem, we need some preliminaries and the main ingredient of the proof is the following result.

Lemma 4.2 (Theorem 4.15 in [28]) *Let H be a basic Hopf algebra. Assume that $dim_k H/J_H$ is invertible in k and the characteristic of k is not 2. Then H is tame if and only if* as an algebra, $H \cong k\langle x, y\rangle / I\#_\sigma (kG)^*$ *for some finite group G and some ideal I which is one of the following forms:*

(1) $I = (x^m - y^n,\ yx - ax^m,\ xy)$ *for* $a \in k$ *and* $m, n \geq 2$;
(2) $I = (x^2,\ y^2,\ (xy)^m - a(yx)^m)$ *for* $0 \neq a \in k$ *and* $m \geq 1$;
(3) $I = (x^2 - (yx)^m,\ y^2,\ (xy)^m + (yx)^m,)$ *for* $m \geq 1$;
(4) $I = (x^2 - (yx)^m,\ y^2 - (xy)^m,\ (xy)^m + (yx)^m,\ (xy)^m x)$ *for* $m \geq 1$;
(5) $I = (x^2,\ y^2,\ (xy)^m x - a(yx)^m y)$ *for* $0 \neq a \in k$ *and* $m \geq 1$;

(6) $I = (x^2 - (yx)^{m-1}y - b(xy)^m,\ y^2,\ (xy)^m - a(yx)^m)$ *for* $a,\ b \in k$ *with* $a \neq 0$ *and* $m \geq 2$*;*
(7) $I = (x^2 - (yx)^{m-1}y - b(xy)^m,\ y^2 - (xy)^m,\ (xy)^m + (yx)^m,\ (xy)^m x)$ *for* a, $b \in k$ *with* $a \neq 0$ *and* $m \geq 2$*;*
(8) $I = (x^2 - (yx)^{m-1}y - f(xy)^m,\ y^2 - (xy)^{m-1}x - e(xy)^m,\ (xy)^m - a(yx)^m,\ (xy)^m x)$ *for* $a,\ e\ f \in k$ *with* $a \neq 0$ *and* $m \geq 2$*;*
(9) $I = (x^2 - (yx)^m,\ y^2,\ (xy)^m x - a(yx)^m y)$ *for* $0 \neq a \in k$ *and* $m \geq 1$*;*
(10) $I = (x^2 - (yx)^m,\ y^2 - (xy)^m,\ (xy)^m x - a(yx)^m y,\ (xy)^{m+1})$ *for* $0 \neq a \in k$ *and* $m \geq 1$.

Remark 4.3 As pointed out by Akira Masuoka, the above crossed products are indeed smash products.

So, our idea is very simple. That is, we just have to verify the radically graded versions of algebras listed in this lemma are tame Hopf algebras or not. We divide this task into four lemmas.

Clearly, the following lemma is true.

Lemma 4.4 *The algebras* (2)*,* (5) *given in Lemma* 4.2 *are radically graded and they are tame.*

Lemma 4.5 *The algebras* (1)*,* (9)*,* (10) *given in Lemma* 4.2 *are not Hopf algebras.*

Proof If they are, then their radically graded versions are Hopf algebras too. Note that their graded versions must contain the relations $xy = 0$ or $yx = 0$ or $(xy)^m x = a(yx)^m y$ for some $0 \neq a \in k$ and $m \geq 1$. By the proofs of Lemmas 4.6, 4.7 and 4.12 in [25], such relations can not occur in Hopf ideals. □

Let H be a semisimple cosemisimple Hopf algebra and A an H-module algebra, then A and $A\#H$ have the same representation type (see Theorem 4.5 in [27]). Thus, sometimes to consider the representation type's problems of algebras $A\#(kG)^*$ appearing in Lemma 4.2, we only consider the corresponding problems for local algebras A.

Lemma 4.6 *The radically graded versions of algebras* (3)*,* (4) *given in Lemma* 4.2 *are still tame.*

Proof If $m > 1$, then their graded versions contains relations $x^2 = 0 = y^2$ and thus such graded versions are special biserial algebras (see Definition 6.1). Therefore, they are tame or of finite representation type (see [14]). Since the algebra $k[x, y]/(x^2, y^2, xy)$ is a quotient of them, they are tame.

If $m = 1$, the radically graded version of (3) is the algebra $k\langle x, y\rangle/(x^2 - yx, y^2, xy + yx)$ and the radically graded version of (4) is $k\langle x, y\rangle/(x^2 - yx, y^2 - xy, xy + yx, xyx)$. Denote them by A and B respectively. Both of them are local Frobenius algebra. It is known that for Frobenius algebras Λ, Λ and $\Lambda/soc\Lambda$

have the same representation type. Now one can find $A/socA \cong B/socB \cong k[x, y]/(x^2, y^2, xy)$ and thus both A and B are tame. □

Lemma 4.7 *The radically graded versions of algebras* (6)*,* (7) *and* (8) *given in Lemma* 4.2 *are still tame.*

Proof It is not hard to see that the radically graded versions of (6), (7) and (8) contain the relations $x^2 = y^2 = 0$ at the same time. So they are special biserial. Similar to the proof of the first part of the above lemma, they are tame. □

Proof of Theorem 4.1. "If part". A direct consequence of Geiss's degeneration theorem (see [22]).

"Only if part". It is a corollary of Lemma 4.2 and 4.4–4.7. □

5 Classification of Tame Basic Hopf Algebras

The classification of tame basic Hopf algebras can be given now. We consider the connected case at first.

Proposition 5.1 *Let H be a connected tame basic Hopf algebra. Then*

$$H \cong A^*$$

where A is a small quantum group of dihedral type given in Section 3*.*

Proof By Theorem 4.1, H is a lifting of a tame graded basic Hopf algebra. Thus the statement is just a direct consequence of Theorems 3.8, 3.10, 3.12 and Remark 3.13. □

The following theorem is the general case.

Theorem 5.2 *Suppose that H is a tame basic Hopf algebra over an algebraically closed field of characteristic* 0*. Then as a Hopf algebra*

$$H \cong (A\#_\sigma kG)^*$$

for some finite group G and a small quantum group of dihedral type A defined in Section 3*.*

Proof It is known that k is an H-module through the counit map $\varepsilon: H \to k$. We say a block of H is the *principle block* if k, as a simple H-module, belongs to this block. We denote this block by H°. By dualizing the discussion for pointed Hopf algebras (see, for example Corollary 5.6.4 in [35]), we know as an algebra H is a direct sum of copies of H°. Therefore, if H is tame, so is H°. Consider H^* and it is a pointed Hopf algebra. By Theorem 3.2 in [36],

$$H^* \cong A\#_\sigma kG$$

where A is the connected (as a coalgebra) component containing the identity element of H^*. Clearly, $A \cong (H^\circ)^*$ and thus A is a small quantum group of dihedral type by Proposition 5.1. □

Remark 5.3 Our classification is based on the main results gotten in [25] where the assumption k is an algebraically closed field of characteristic 0 is essentially needed (see Remark 5.3 (2) in [25]). Just like the fact pointed out in Remark 5.3 (2) in [25], our classification can also be established if the characteristic of k is big enough. In general, the classification of tame basic Hopf algebras (even radically graded) over an algebraically closed field of *positive* characteristic is still an open and interesting question.

6 Representation Dimensions and Complexities of Tame Basic Hopf Algebras

6.1 Representation Dimensions

At first, we show that the representation dimension of tame basic Hopf algebras is 3.

Let A be a finite-dimensional algebra, and denote by $\mathrm{mod}A$ the category of finitely generated A-modules. The *representation dimension* of A, denoted by $\mathrm{repdim}A$, is defined as

$$\mathrm{repdim}A := \inf\{\mathrm{gldim}\ \mathrm{End}_A(M) | M \text{ generates and cogenerates } \mathrm{mod}A\},$$

where gldim denotes the global dimension of an algebra. In [5], Auslander introduced the representation dimension of an Artin algebra at first to study algebras of infinite representation type. In particular, he showed that an algebra is of finite representation type if and only if its representation dimension is 2. Iyama showed that representation dimensions of Artin algebras are always finite [26] and Rouquier [39] proved that the representation dimension of the exterior algebra on a d-dimensional vector space is $d+1$.

Our basic observation is that tame basic Hopf algebras are special biserial algebras. Recall for a basic algebra A, by the Gabriel's Theorem, there is a unique quiver Q_A, and an admissible ideal I of kQ_A^a, such that $A \cong kQ_A^a/I$ (see [6]).

Definition 6.1 The algebra A is *special biserial* provided the its basic algebra kQ^a/I satisfies the following conditions:

(1) Any vertex of Q is starting point of at most two arrows.
(1') Any vertex of Q is end point of at most two arrows.
(2) Given an arrow β, there is at most one arrow γ with $s(\beta) = t(\gamma)$ and $\beta\gamma \notin I$.
(2') Given an arrow γ, there is at most one arrow β with $s(\beta) = t(\gamma)$ and $\beta\gamma \notin I$.

See [14] for this definition and related facts about special biserial algebras.

Lemma 6.2 *Let H be a tame basic Hopf algebra. Then H is special biserial.*

Proof By the proof of Theorem 5.2, as an algebra H is a direct sum of copies of H°, which is a connected tame basic Hopf algebras. By Remarks 3.9, 3.11 and 3.13, the algebraic structures of connected tame basic Hopf algebras are not changed when we apply lifting methods to them. Thus as algebras, they are isomorphic to their radically graded versions. And clearly all algebras given in Lemma 3.1 are special biserial. □

The following result was proved in [15].

Lemma 6.3 (Corollary 1.3 in [15]) *Let A be a special biserial algebra. Then* repdim $A \leq 3$.

Note that Auslander has proved that an algebra is of finite representation type if and only if its representation dimension is 2. Thus above two lemmas indeed imply our main result of this subsection.

Theorem 6.4 *Assume that H is a tame basic Hopf algebra. Then*

$$\operatorname{repdim} H = 3.$$

For the class of basic Hopf algebras, this gives a positive answer for the following conjecture.

Conjecture If A is tame, then $\operatorname{repdim} A = 3$.

6.2 Complexities

The concept of complexity of a module was first introduced by Alperin [1] in the setting of group representations and group cohomology. Let A be an associative algebra, M an A-module with minimal projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Then the *complexity* of M is defined to be the integer

$$\mathrm{C}_A(M) := \min\{c \in \mathbb{N}_0 \cup \infty \mid \exists \lambda > 0 : \dim_k P_n \leq \lambda n^{c-1},\ \forall\, n \geq 1\}.$$

And the complexity of A is the maximum of the complexities of all A-modules, that is,

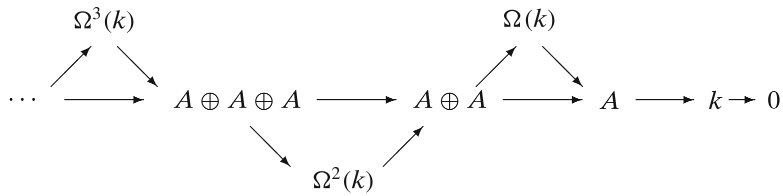
$$\mathrm{C}(A) := \sup\{\mathrm{C}_A(M) \mid M \in A\text{-mod}\}.$$

For our purpose, we need consider the following examples.

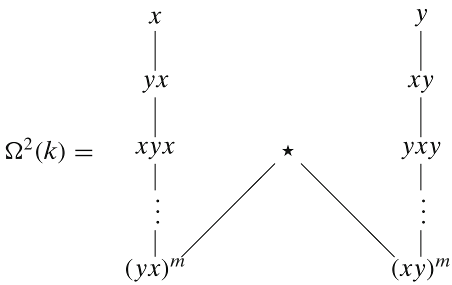
Example 6.5

(1) Let A be a selfinjective algebra of finite representation type. It is well-known that $\mathrm{C}(A) \leq 1$.

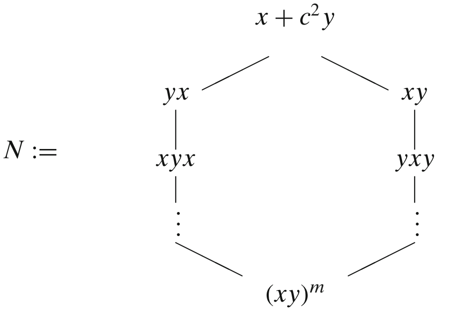
(2) Consider the algebra $A = k\langle x, y\rangle/(x^2, y^2, (xy)^m - c(yx)^m)$ for some $m \geq 1$ and $c \in k$. It is a local algebra and we denote the unique simple module by k. We can construct the minimal projective resolution of k as follows.



Here $\Omega(M)$ is the kernel of a minimal projective cover of the A-module M. It is not hard to show that



where $\star = c(yx)^{m-1}y - (xy)^{m-1}x$. Define



Then through direct computations, we have for any $i \in \mathbb{N}$

$$\Omega^i(k) = \begin{cases} \Omega^{i-1}(k) \oplus N & \textit{if } i \textit{ is odd,} \\ \Omega^{i-1}(k) \oplus \Omega(N) & \textit{if } i \textit{ is even.} \end{cases}$$

By this, we indeed have get

$$P_n \cong A^{(n+1)}$$

for a minimal projective resolution $P_\bullet \to k$. This implies that $\mathrm{C}_A(k) = 2$. Since A is local, $\mathrm{C}(A) = 2$.

The complexities of basic Hopf algebras are summarized as follows.

Theorem 6.6 *For a basic Hopf algebra H, we have*

(1) *If H is of finite representation type, then* $\mathrm{C}(H) \leq 1$.
(2) *If H is tame, then* $\mathrm{C}(H) = 2$.
(3) *If* $\mathrm{C}(H) \geq 3$, *then H is wild.*

Clearly, (3) verifies Rickard's statement (see Section 1) for basic Hopf algebras and it is a direct consequence of (1)(2). So, we need only to show (1)(2). (1) is well-known (see Example 6.5 (1)).

Attack to (2), the following result is needed. It implies that complexity is an invariant under semisimple Hopf actions.

Proposition 6.7 *Let H be a semisimple Hopf algebra and A an H-module algebra. Then*

$$\mathrm{C}(A\#H) = \mathrm{C}(A).$$

Let us fix some notation first. Let R be a ring and M, N two R-modules. If M is a direct summand of N as a R-module, then we denote it by $M|N$. To prove this proposition, we need recall the following lemma (see Lemma 4.1 in [27]).

Lemma 6.8 *Let H be a semisimple Hopf algebra and A an H-module algebra. For any finitely generated $A\#H$-module X, $X|(A\#H) \otimes_A X$.*

Proof of Proposition 6.7. Let's show that $\mathrm{C}(A\#H) \leq \mathrm{C}(A)$ at first. If $\mathrm{C}(A) = \infty$, done. So we can assume that $\mathrm{C}(A) = m$ for some $m \in \mathbb{N}$. Now let M be an $A\#H$-module. Since A is a sub algebra of $A\#H$, M can be regarded as an A-module by the restriction and we denote it by ${}_AM$. By assumption, $\mathrm{C}_A({}_AM) \leq m$. Assume

$$\cdots P_n \to P_{n-1} \to \cdots \to P_0 \twoheadrightarrow {}_A M$$

is a minimal projective resolution of ${}_AM$. Then there is a natural number β such that $\dim_k P_n \leq \beta n^{m-1}$. Since $A\#H$ is a free right A-module of rank $\dim_k H$,

$$\cdots A\#H \otimes_A P_n \to A\#H \otimes_A P_{n-1} \to \cdots \to A\#H \otimes_A P_0 \twoheadrightarrow A\#H \otimes_A M$$

is a projective resolution of $A\#H \otimes_A M$. Assume that $P'_\bullet \to M$ is a minimal resolution of M as an $A\#H$-module, then Lemma 6.8 implies that

$$\dim_k P'_n \leq \dim_k A\#H \otimes_A P_n \leq \dim_k H\beta n^{m-1}.$$

Therefore, $\mathrm{C}_{A\#H}(M) \leq m$ and so $\mathrm{C}(A\#H) \leq m$. This proves that we always have $\mathrm{C}(A\#H) \leq \mathrm{C}(A)$.

For the other direction, we note that H being semisimple implies that H is cosemisimple [30] and thus H^* is semisimple. By the proof of the above paragraph, we know that $\mathrm{C}((A\#H)\#H^*) \leq \mathrm{C}(A\#H)$. By the Blattner-Montgomery Duality Theorem (see Section 9.4 in [35]), $(A\#H)\#H^* \cong M_n(A)$ which is Morita equivalent to A. Thus $\mathrm{C}((A\#H)\#H^*) = \mathrm{C}(A)$ and so $\mathrm{C}(A) \leq \mathrm{C}(A\#H)$. □

Example 6.9 (Book algebras) Let q be an n-th primitive root of unity and m a positive integer satisfying $(m, n) = 1$. Let $H = \mathbf{h}(q, m) = k < y, x, g > /(x^n, y^n, g^n - 1, gx - qxg, gy - q^m yg, xy - yx)$ and with comultiplication, antipode and counit given by

$$\Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + g^m \otimes y, \quad \Delta(g) = g \otimes g$$

$$S(x) = -xg^{-1}, \;\; S(y) = -g^{-m} y, \;\; S(g) = g^{-1}, \;\; \varepsilon(x) = \varepsilon(y) = 0. \;\; \varepsilon(g) = 1$$

It is a Hopf algebra and is called a book algebra. It is a basic algebra since $\mathbf{h}(q, m)/J_{\mathbf{h}(q,m)}$ is a commutative semisimple algebra. For more information about book algebras, see [3]. It is shown that only when $n = 2, q = -1, m = 1$, the book algebra is tame (see Example 5.2 in [27]). In this case, $\mathbf{h}(-1, 1) \cong k < X, Y > /(X^2, Y^2, XY + YX)\#k\mathbb{Z}_n$ where $k\mathbb{Z}_n$ is the group algebra of the cyclic group $\mathbb{Z}_n$ of order n. By Example 6.5(2) and Proposition 6.7, $\mathrm{C}(\mathbf{h}(-1, 1)) = 2$.

Proof of Theorem 6.6 (2) Similar to the proof of Lemma 6.2, we only need to show that connected radically graded tame basic Hopf algebras have complexity 2. Let H be of this type, then by Lemma 2.4,

$$H \cong k\langle x, y\rangle/I \times (kG)^*,$$

where $I = (x^2, y^2, (xy)^m - c(yx)^m)$ for some $c \in k$. Note that $(kG)^*$ is semisimple and the bosonization is a special kind of smash product. Therefore, Proposition 6.7 implies that

$$\mathrm{C}(H) = \mathrm{C}(k\langle x, y\rangle/I),$$

which equals to 2 by Example 6.5 (2). The proof is done. □

Acknowledgements The author is supported by Japan Society for the Promotion of Science under the item "JSPS Postdoctoral Fellowship for Foreign Researchers" and Grant-in-Aid for Foreign JSPS Fellow. I would gratefully acknowledge JSPS. The work is also supported by Natural Science Foundation (No. 10801069). I would like thank Professor A. Masuoka for stimulating discussions and his encouragements.

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