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THE CASIMIR NUMBER AND THE DETERMINANT OF A FUSION CATEGORY

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Abstract. Let $\mathcal{C}$ be a fusion category over an algebraically closed field $\Bbbk$ of arbitrary characteristic. Two numerical invariants of $\mathcal{C}$, that is, the Casimir number and the determinant of $\mathcal{C}$ are considered in this paper. These two numbers are both positive integers and admit the property that the Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ over any field K is semisimple if and only if any of these numbers is not zero in K. This shows that these two numbers have the same prime factors. If moreover $\mathcal{C}$ is pivotal, it gives a numerical criterion that $\mathcal{C}$ is nondegenerate if and only if any of these numbers is not zero in $\Bbbk$. For the case that $\mathcal{C}$ is a spherical fusion category over the field $\mathbb{C}$ of complex numbers, these two numbers and the Frobenius–Schur exponent of $\mathcal{C}$ share the same prime factors. This may be thought of as another version of the Cauchy theorem for spherical fusion categories.

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1. Introduction. A fusion category $\mathcal{C}$ over a field $\Bbbk$ is called *nondegenerate* if the global dimension $\dim(\mathcal{C})$ of $\mathcal{C}$ is not zero in $\Bbbk$. Since $\dim(\mathcal{C})$ is automatically not zero in a field $\Bbbk$ of characteristic zero, this notation is only considered in a field $\Bbbk$ of positive characteristic. A crucial property of nondegenerate fusion categories is that they can be lifted to the case of characteristic zero (see, e.g., [**6**, Section 9]). It is interesting to know when a fusion category over a field of positive characteristic is nondegenerate. Ostrik stated that a spherical fusion category $\mathcal{C}$ over a field $\Bbbk$ is nondegenerated, if the Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$ is semisimple (see [**15**, Proposition 2.9]). It has been proved by Shimizu that a pivotal fusion category $\mathcal{C}$ over an algebraically closed field $\Bbbk$ is nondegenerate if and only if its Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$ is semisimple (see [**17**, Theorem 6.5]).

In this paper, we first pay attention to the question when the Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$ is semisimple for any fusion category $\mathcal{C}$ over an algebraically closed field $\Bbbk$. To solve this question, in Section 2, we associate any fusion category $\mathcal{C}$ with two positive integers: the Casimir number $m_{\mathcal{C}}$ and the determinant $d_{\mathcal{C}}$. These two numbers are numerical invariants of $\mathcal{C}$ and provide a semisimplicity criterion on the Grothendieck algebra of $\mathcal{C}$. Namely, the Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ over any field K is semisimple if and only if any one of these two numbers is not zero in K. This leads to a result that the Casimir number $m_{\mathcal{C}}$ and the determinant $d_{\mathcal{C}}$ have the same prime factors. The semisimplicity criterion for the Grothendieck algebra of $\mathcal{C}$ together with Shimizu's work [17, Theorem 6.5] gives a numerical criterion for a pivotal fusion category to be nondegenerate. That is, a pivotal fusion category $\mathcal{C}$ over a field $\Bbbk$ is nondegenerate if and only if any of the numbers $m_{\mathcal{C}}$ and $d_{\mathcal{C}}$ is not zero in $\Bbbk$.

As these two numbers $m_{\mathcal{C}}$ and $d_{\mathcal{C}}$ have the same prime factors, in Section 3, we only focus on the determinant $d_{\mathcal{C}}$ of a fusion category $\mathcal{C}$. We give some results concerning prime factors of determinants of representation categories of semisimple Hopf algebras. In particular, for a semisimple and cosemisimple Hopf algebra H, we show that the determinant of the representation category of the Drinfeld double $D(H)$ of H and the dimension $\dim_{\Bbbk} H$ of H share the same prime factors. We also reveal a relationship between the determinant $d_{\mathcal{C}}$ of a fusion category $\mathcal{C}$ and the determinant $d_{\widetilde{\mathcal{C}}}$ of the pivotalization $\widetilde{\mathcal{C}}$ of $\mathcal{C}$. We show that the former is a factor of the latter. This gives a result that any nondegenerate fusion category over a field $\Bbbk$ has a nonzero determinant in $\Bbbk$. However, the converse is not known to be true.

The Frobenius–Schur exponent of a spherical fusion category $\mathcal{C}$ has been defined in [12, Definition 5.1] as a minimal positive integer satisfying certain properties. In the case that the ground field is the field $\mathbb{C}$ of complex numbers, the Cauchy theorem for spherical fusion categories asserts that the prime ideals dividing the global dimension $\dim(\mathcal{C})$ of $\mathcal{C}$ and those dividing the Frobenius–Schur exponent of $\mathcal{C}$ are the same in the ring of algebraic integers [1, Theorem 3.9]. We prove in Section 4 that the determinant $d_{\mathcal{C}}$ and the Frobenius–Schur exponent of $\mathcal{C}$ have the same prime factors. This may be considered as an integer version of the Cauchy theorem for spherical fusion categories.

2. Numerical invariants. In this section, all fusion categories and Hopf algebras are defined over an algebraically closed field $\Bbbk$ of arbitrary characteristic. We first introduce some numerical invariants of a fusion category $\mathcal{C}$, and then use them to describe when the Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ over any field K is semisimple.

Let $\mathcal{C}$ be a fusion category over $\Bbbk$ and $\{X_i\}_{i\in I}$ the set of isomorphism classes of simple objects of $\mathcal{C}$. The Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ of $\mathcal{C}$ is an associative unital ring with a multiplication induced by the tensor product on $\mathcal{C}$, namely,

$$X_iX_j = \sum_{k\in I} N_{ij}^k X_k,$$

where N_{ij}^k, called the *fusion coefficient* of $\mathrm{Gr}(\mathcal{C})$, is the multiplicity of X_k in the Jordan–Hölder series of $X_i \otimes X_j$. The duality functor $*$ of $\mathcal{C}$ induces an involution on $\mathrm{Gr}(\mathcal{C})$, namely, $(X_iX_j)^* = X_j^*X_i^*$ and $(X_i)^{**} = X_i$ for $i, j \in I$. We write $(X_i)^* = X_{i^*}$ for convenience. In view of this, the duality functor $*$ induces a permutation on the index set I.

There is an associative symmetric and nondegenerate $\mathbb{Z}$-bilinear form $(-,-)$ on $\mathrm{Gr}(\mathcal{C})$ defined by

$$(X_i, X_j) = \dim_{\Bbbk} \mathrm{Hom}(X_i, X_j^*) = \delta_{i,j^*},$$

where δ_{i,j^*} is the Kronecker symbol. This form is also $*$-invariant, namely, $(X_i, X_j) = (X_i^*, X_j^*)$ for all $i, j \in I$. Thus, $\mathrm{Gr}(\mathcal{C})$ is a symmetric $*$-algebra over $\mathbb{Z}$. The pair of dual bases with respect to the form $(-,-)$ is the set $\{X_i, X_{i^*}\}_{i\in I}$ satisfying the following equality:

$$\sum_{i\in I} X_i \otimes X_{i^*} = \sum_{i\in I} X_{i^*} \otimes X_i.$$

Note that $N_{ij}^k = (X_iX_j, X_{k^*})$ hold for all $i, j, k \in I$. It follows from

$$(X_iX_j, X_{k^*}) = (X_{i^*}X_k, X_{j^*}) = (X_kX_{j^*}, X_{i^*})$$

that $N_{ij}^k = N_{i^*k}^j = N_{kj^*}^i$. Using this equality one is able to check the following two equalities:

$$\sum_{i\in I} X_jX_i \otimes X_{i^*} = \sum_{i\in I} X_i \otimes X_{i^*}X_j, \tag{2.1}$$

$$\sum_{i\in I} X_iX_j \otimes X_{i^*} = \sum_{i\in I} X_i \otimes X_jX_{i^*}. \tag{2.2}$$

The *Casimir operator* (see, e.g., [9, Section 3.1]) of the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ is the map c from $\mathrm{Gr}(\mathcal{C})$ to its center $Z(\mathrm{Gr}(\mathcal{C}))$ given by

$$c(a) = \sum_{i\in I} X_iaX_{i^*} \text{ for } a \in \mathrm{Gr}(\mathcal{C}).$$

The element $c(1) = \sum_{i\in I} X_iX_{i^*}$, depending on $(-,-)$ only up to a central unit of $\mathrm{Gr}(\mathcal{C})$ (see [9, Section 1.2.5]), is called the *Casimir element* of $\mathrm{Gr}(\mathcal{C})$. It is well known that the image $\mathrm{Im}c$ of c is an ideal of $Z(\mathrm{Gr}(\mathcal{C}))$ and is called the *Higman ideal* of $\mathrm{Gr}(\mathcal{C})$.

The element $c(1) = \sum_{i\in I} X_iX_{i^*}$, as an element in $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$, is central invertible (see the proof of [4, Lemma 9.3.10]); hence there exists a unique central invertible element b in $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $c(1)b = 1$. Suppose $b = \sum_{i\in I} \frac{m_i}{n_i} X_i$, where m_i and n_i form a pair of coprime integers for each $i \in I$. Denote by $n_{\mathcal{C}} > 0$ the least common multiple of n_i for all $i \in I$. Then $bn_{\mathcal{C}} \in \mathrm{Gr}(\mathcal{C})$ and $n_{\mathcal{C}} = c(1)bn_{\mathcal{C}} = c(bn_{\mathcal{C}})$. This means that $n_{\mathcal{C}} \in \mathbb{Z} \cap \mathrm{Im}c$, and hence $\mathbb{Z} \cap \mathrm{Im}c \neq (0)$.

Since the intersection $\mathbb{Z} \cap \mathrm{Im}c$ is a nonzero principle ideal of $\mathbb{Z}$, the positive generator of $\mathbb{Z} \cap \mathrm{Im}c$ (denoted by $m_{\mathcal{C}}$) is called the *Casimir number* of $\mathcal{C}$. Namely, $\mathbb{Z} \cap \mathrm{Im}c = (m_{\mathcal{C}})$ for $m_{\mathcal{C}} > 0$. The element a satisfying $c(a) = m_{\mathcal{C}}$ is not unique in general. It is easy to see that the element a satisfying $c(a) = m_{\mathcal{C}}$ is unique if and only if the map c is injective, if and only if $\mathrm{Gr}(\mathcal{C})$ is commutative. The Casimir number $m_{\mathcal{C}}$ always divides the number $n_{\mathcal{C}}$ since we have seen that $n_{\mathcal{C}} \in \mathbb{Z} \cap \mathrm{Im}c$. If $\mathrm{Gr}(\mathcal{C})$ is commutative, we have $m_{\mathcal{C}} = n_{\mathcal{C}}$.

Observe that the matrix $[c(1)]$ of left multiplication by $c(1)$ with respect to the basis $\{X_i\}_{i\in I}$ of $\mathrm{Gr}(\mathcal{C})$ is a positive definite integer matrix (see [9, Proposition 8]). It follows that the determinant $d_{\mathcal{C}} := \det[c(1)]$, called the *determinant* of $\mathcal{C}$, is always a positive integer.

REMARK 2.1.

(1) If two fusion categories are monoidally equivalent under a monoidal functor, then this functor induces an isomorphism preserving fusion coefficients between the Grothendieck rings of fusion categories. Thus, equivalent fusion categories lead to the same Casimir numbers and the same determinants.

(2) Let H_1 and H_2 be two finite dimensional semisimple Hopf algebras over $\Bbbk$. If H_1 and H_2 are twisted of each other in the sense that $H_1 = H_2$ as algebras and $H_2 = (H_1)_\Omega$ for some 2-pseudo-cocycle Ω, then the Grothendieck rings $\mathrm{Gr}(H_1)$ and $\mathrm{Gr}(H_2)$ share the same fusion coefficients (see [13, Theorem 4.1]). It turns out that the Casimir number or the determinant of the representation category of H_1 is the

same as that of H_2. In other words, the Casimir number or the determinant of the representation category of a semisimple Hopf algebra is stable under twisting.

(3) The notation of the Casimir number of a fusion category defined here is indeed a special case of the notation of Casimir number defined over a finite tensor category [21].

PROPOSITION 2.2. *Let $\mathcal{C}$ be a fusion category over $\Bbbk$. For any field K, the following statements are equivalent:*

(1) *The determinant $d_{\mathcal{C}} \neq 0$ in K.*
(2) *The Casimir number $m_{\mathcal{C}} \neq 0$ in K.*
(3) *The Grothendieck algebra $Gr(\mathcal{C}) \otimes_{\mathbb{Z}} K$ is semisimple.*

Proof. (1) $\Rightarrow$ (2): Let $c(1) = \sum_{i\in I} X_i X_{i^*}$ denote the Casimir element of $\mathrm{Gr}(\mathcal{C})$. Suppose the characteristic polynomial of the integer matrix $[c(1)]$ is

$$f(x) = x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1} x + \alpha_n,$$

where n is the cardinality of I, $f(x) \in \mathbb{Z}[x]$ and $\alpha_n = \pm d_{\mathcal{C}}$. By the Cayley–Hamilton's theorem, the operator of left multiplication by $c(1)$ satisfies that

$$0 = f(c(1)) = c(1)(c(1)^{n-1} + \alpha_1 c(1)^{n-2} + \cdots + \alpha_{n-1}) + \alpha_n = c(1)a + \alpha_n,$$

where $a = c(1)^{n-1} + \alpha_1 c(1)^{n-2} + \cdots + \alpha_{n-1} \in Z(\mathrm{Gr}(\mathcal{C}))$. Thus, $c(a) = c(1)a = -\alpha_n = \mp d_{\mathcal{C}} \in \mathbb{Z}$. By the definition of $m_{\mathcal{C}}$, we have $m_{\mathcal{C}} \mid d_{\mathcal{C}}$. Now $d_{\mathcal{C}} \neq 0$ in K implies that $m_{\mathcal{C}} \neq 0$ in K.

(2) $\Rightarrow$ (3): Note that there exists some $a \in \mathrm{Gr}(\mathcal{C})$ such that $\sum_{i\in I} X_i a X_{i^*} = m_{\mathcal{C}}$. Denote by $A := \mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ and consider $\sum_{i\in I} X_i \frac{a}{m_{\mathcal{C}}} \otimes X_{i^*} \in A \otimes A$. Obviously,

$$\sum_{i\in I} X_i \frac{a}{m_{\mathcal{C}}} X_{i^*} = 1$$

and

$$\sum_{i\in I} b X_i \frac{a}{m_{\mathcal{C}}} \otimes X_{i^*} = \sum_{i\in I} X_i \frac{a}{m_{\mathcal{C}}} \otimes X_{i^*} b$$

hold for any $b \in A$ (see 2.1). Thus, $\sum_{i\in I} X_i \frac{a}{m_{\mathcal{C}}} \otimes X_{i^*}$ is a separable idempotent of A, and hence A is a separable K-algebra. It is well known that any separable K-algebra is a semisimple K-algebra (see, e.g., [2]).

(3) $\Rightarrow$ (1): Let $\mathrm{Tr}(a)$ be the trace of the operator of left multiplication by $a \in A = \mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$. Since A is semisimple, the bilinear form $\langle a, b\rangle = \mathrm{Tr}(ab)$ on A is nondegenerate. This implies that the matrix $[a_{ij}]$ for $a_{ij} = \langle X_i, X_j\rangle$ is an invertible matrix in K. Let c_{ij} be the (i, j)-entry of $[c(1)]$. Then

$$\begin{aligned} c_{ij} &= \left(\sum_{k\in I} X_k X_{k^*} X_i, X_{j^*} \right) = \sum_{k\in I} (X_k X_{k^*}, X_i X_{j^*}) = \sum_{k\in I} (X_i X_{j^*}, X_k X_{k^*}) \\ &= \sum_{k\in I} (X_i X_{j^*} X_k, X_{k^*}) = \mathrm{Tr}(X_i X_{j^*}) = a_{ij^*}. \end{aligned}$$

That is, the matrix $[c(1)]$ differs from the matrix $[a_{ij}]$ only by permutations of columns. It follows that $[c(1)]$ is an invertible matrix in K and $\det[c(1)] = d_{\mathcal{C}} \neq 0$ in K. □

REMARK 2.3.

(1) The proof of (3) $\Rightarrow$ (1) in Proposition 2.2 comes from the proof of [**15**, Proposition 2.9]. From this proof one is able to see that $d_{\mathcal{C}} = \pm \det[a_{ij}]$, where $a_{ij} = \mathrm{Tr}(X_iX_j)$ for $i, j \in I$.
(2) The result that $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ is semisimple if and only if $m_{\mathcal{C}} \neq 0$ in K is essentially the Higman's theorem applied to the Frobenius algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ (see [**8**, Theorem 1] or [**9**, Proposition 6]). This result can also be deduced directly from [**21**, Theorem 3.7].
(3) Any one of the statements of Proposition 2.2 is equivalent to the result that $c(1) = \sum_{i\in I} X_iX_{i^*}$ is invertible in $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ (see [**19**, Theorem 3.8]).

If the field K is of characteristic $p > 0$, it follows from Proposition 2.2 that $p \nmid d_{\mathcal{C}}$ if and only if $p \nmid m_{\mathcal{C}}$. This gives the following relationship between the numbers $m_{\mathcal{C}}$ and $d_{\mathcal{C}}$:

COROLLARY 2.4. *The Casimir number $m_{\mathcal{C}}$ and the determinant $d_{\mathcal{C}}$ of a fusion category $\mathcal{C}$ have the same prime factors.*

Recall from [**17**, Theorem 6.5] that a pivotal fusion category $\mathcal{C}$ over a field $\Bbbk$ is nondegenerate (i.e., the global dimension $\dim(\mathcal{C})$ of $\mathcal{C}$ is not zero in $\Bbbk$) if and only if its Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$ is semisimple. This result together with Proposition 2.2 gives the following numerical characterization of a nondegenerate pivotal fusion category:

PROPOSITION 2.5. *A pivotal fusion category $\mathcal{C}$ over a field $\Bbbk$ is nondegenerate if and only if the Casimir number $m_{\mathcal{C}}$ (or equivalently, the determinant $d_{\mathcal{C}}$) is not zero in $\Bbbk$.*

The rest of this section provides some fusion categories whose determinants or Casimir numbers can be explicitly described.

EXAMPLE 2.6. Let $\mathcal{C}$ be a pointed fusion category over a field $\Bbbk$. The Grothendieck ring of $\mathcal{C}$ is the group ring $\mathbb{Z}G$ for a finite group G. The Casimir number of $\mathcal{C}$ is the order $|G|$ of G, and the determinant of $\mathcal{C}$ is $|G|^{|G|}$. It follows from Proposition 2.2 that for any field K, the K-algebra $KG = \mathbb{Z}G \otimes_{\mathbb{Z}} K$ is semisimple if and only if $|G| \neq 0$ in K. This is the Maschke's theorem for group algebras.

EXAMPLE 2.7. Let $\mathcal{C}$ be a modular category over a field $\Bbbk$ with isomorphism classes of simple objects $\{X_i\}_{i\in I}$. That is, $\mathcal{C}$ is a spherical fusion category with a braiding c such that the S-matrix $S = [s_{ij}]$ is invertible in $\Bbbk$, where $s_{ij} = Tr(c_{X_jX_i} \circ c_{X_iX_j})$ (see, e.g., [**4**, Section 8.14]). Note that $\dim(X_i) \neq 0$ in $\Bbbk$ for any $i \in I$ (see [**4**, Proposition 4.8.4]). For any $i \in I$, the map

$$h_i : X_j \mapsto \frac{s_{ij}}{\dim(X_i)} \quad \text{for } j \in I$$

defines a homomorphism from $Gr(\mathcal{C})$ to $\Bbbk$. In other words, $\{h_i(X_j)\}_{i\in I}$ consists of all eigenvalues of the matrix $[X_j]$ of left multiplication by X_j with respect to the basis $\{X_i\}_{i\in I}$ of $\mathrm{Gr}(\mathcal{C})$. Note that all eigenvalues of the matrix $[c(1)]$ are $h_i(c(1))$ for $i \in I$. Moreover,

$$h_i(c(1)) = h_i\left(\sum_{j\in I} X_jX_{j^*}\right) = \sum_{j\in I} h_i(X_j)h_i(X_{j^*}) = \sum_{j\in I} \frac{s_{ij}s_{ij^*}}{\dim(X_i)^2} = \frac{\dim(\mathcal{C})}{\dim(X_i)^2},$$

where the last equality follows from [**4**, Proposition 8.14.2]. It follows that

$$d_{\mathcal{C}} = \prod_{i\in I} h_i(c(1)) = \frac{(\dim \mathcal{C})^n}{\prod_{i\in I} \dim(X_i)^2},$$

where n is the cardinality of I.

EXAMPLE 2.8. Recall from [18] that the near-group category $\mathcal{C}$ is a rigid fusion category whose simple objects except for one are invertible. Let G be the group of isomorphism classes of invertible objects in $\mathcal{C}$ and X the isomorphism class of the remaining non-invertible simple object. The Grothendieck ring $Gr(\mathcal{C})$ of $\mathcal{C}$ obeys the following multiplication rule:

$$g \cdot h = gh,\ \ g \cdot X = X \cdot g = X,\ \ X^2 = \sum_{g \in G} g + \rho X,$$

where $g, h \in G$ and ρ is a positive integer. The matrix $[c(1)]$ of left multiplication by $c(1)$ with respect to the basis $G \cup \{X\}$ of $Gr(\mathcal{C})$ can be written explicitly as follows (see [20, Example 3.3]):

$$[c(1)] = \begin{bmatrix} M & \mathbf{u} \\ \mathbf{u}^t & \rho^2 + 2|G| \end{bmatrix},$$

where M is a square matrix of size $|G|$ whose diagonal elements are all $|G| + 1$ and off-diagonal elements are all 1, and $\mathbf{u}$ is a column vector of size $|G|$ whose elements are all ρ. It is easy to compute that

$$d_{\mathcal{C}} = \det[c(1)] = (4|G| + \rho^2)|G|^{|G|}.$$

Note that the Casimir number of $\mathcal{C}$ is a minimal positive integer $m_{\mathcal{C}}$ such that

$$\sum_{g \in G} gag^{-1} + XaX = m_{\mathcal{C}} \ \text{ for some } a \in Gr(\mathcal{C}).$$

Accordingly, the Casimir number $m_{\mathcal{C}}$ and the associated $a \in \mathrm{Gr}(\mathcal{C})$ can be determined separately as follows:

Case 1: If ρ is odd, then $a = (4|G| + \rho^2) - 2\sum_{g \in G} g - \rho X$ and $m_{\mathcal{C}} = (4|G| + \rho^2)|G|$.

Case 2: If ρ is even, then $a = \frac{1}{2}(4|G| + \rho^2) - \sum_{g \in G} g - \frac{\rho}{2} X$ and $m_{\mathcal{C}} = \frac{1}{2}(4|G| + \rho^2)|G|$.

We may see that $m_{\mathcal{C}}$ and $d_{\mathcal{C}}$ have the same prime factors no matter what ρ is odd or even.

3. Prime factors of the Casimir numbers or determinants. Denote by d_H the determinant of the representation category $\mathrm{Rep}(H)$ of a semisimple Hopf algebra H. In this section, we will give some results concerning prime factors of d_H. Obviously, these results holding for the determinant of $\mathrm{Rep}(H)$ also hold for the Casimir number of $\mathrm{Rep}(H)$ as the two numbers have the same prime factors. Then we show that the determinant $d_{\mathcal{C}}$ of a fusion category $\mathcal{C}$ divides the determinant $d_{\widetilde{\mathcal{C}}}$ of the pivotalization $\widetilde{\mathcal{C}}$. This is used to prove that any nondegenerate fusion category has a nonzero determinant.

A finite dimensional Hopf algebra H is call *pivotal* if H contains a group-like element g such that $S^2(h) = ghg^{-1}$ for all $h \in H$. The representation category $\mathrm{Rep}(H)$ of a finite dimensional semisimple pivotal Hopf algebra H is a pivotal fusion category.

PROPOSITION 3.1. *Let H be a finite dimensional semisimple pivotal Hopf algebra over $\Bbbk$. The determinant $d_H \neq 0$ in $\Bbbk$ if and only if $S^2 = id_H$ and $\dim_{\Bbbk} H \neq 0$ in $\Bbbk$.*

Proof. The determinant $d_H \neq 0$ in $\Bbbk$ if and only if $\mathrm{Rep}(H)$ is nondegenerate by Proposition 2.5, if and only if H is cosemisimple by [6, Section 9.1], if and only if $S^2 = id_H$ and $\dim_{\Bbbk} H \neq 0$ in $\Bbbk$ by [3, Corollary 3.2]. □

Since a finite dimensional semisimple and cosemisimple Hopf algebra H over $\Bbbk$ always satisfies that $S^2 = id_H$ and $\dim_{\Bbbk} H \neq 0$ (see [3, Corollary 3.2]), Proposition 3.1 has the following corollary:

COROLLARY 3.2. *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra over $\Bbbk$. The determinant d_H is always not zero in $\Bbbk$.*

The following result gives more information about the determinant d_H of a semisimple and cosemisimple Hopf algebra H under a certain hypothesis.

PROPOSITION 3.3. *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra over $\Bbbk$. If the Grothendieck ring $Gr(H)$ of H is commutative, then the determinant d_H and the dimension $\dim_{\Bbbk} H$ have the same prime factors.*

Proof. We first consider the case $\mathrm{char}(\Bbbk) = 0$. The set of isomorphism classes of simple objects of $\mathrm{Rep}(H)$ is denoted by $\{X_i\}_{i\in I}$. Since the Grothendieck ring $\mathrm{Gr}(H)$ is commutative, it follows from [9, Proposition 20] that all eigenvalues of the matrix $[c(1)]$ of left multiplication by $c(1) = \sum_{i\in I} X_i X_{i^*}$ with respect to the basis $\{X_i\}_{i\in I}$ of $\mathrm{Gr}(H)$ are positive integers, and moreover, all these eigenvalues divide $\dim_{\Bbbk} H$. In particular, $\dim_{\Bbbk} H$ itself is the largest eigenvalue of $[c(1)]$ (see [9, Proposition 8]). On the other hand, the determinant d_H is obtained by multiplying all these eigenvalues. Thus, d_H and $\dim_{\Bbbk} H$ have the same prime factors.

For the case $\mathrm{char}(\Bbbk) = p > 0$, we denote $\mathcal{O}$ the ring of Witt vectors of $\Bbbk$ and K the field of fractions of $\mathcal{O}$. For the semisimple and cosemisimple Hopf algebra H, using the lifting Theorem [3, Theorem 2.1] we may construct a Hopf algebra A over $\mathcal{O}$ which is free of rank $\dim_{\Bbbk} H$ as an $\mathcal{O}-$module such that A/pA is isomorphic to H as a Hopf algebra. The Hopf algebra $A_0 := A \otimes_{\mathcal{O}} K$ is a semisimple and cosemisimple Hopf algebra over the field K of characteristic 0 with the same Grothendieck ring as for H. It follows that the Grothendieck ring $\mathrm{Gr}(A_0)$ is commutative and the determinant d_{A_0} of $\mathrm{Rep}(A_0)$ is equal to the determinant d_H of $\mathrm{Rep}(H)$. By the same argument as for the case of $\mathrm{char}(\Bbbk) = 0$, we may see that the determinant d_{A_0} and $\dim_K A_0$ have the same prime factors. Note that $\dim_K A_0 = \dim_K(A \otimes_{\mathcal{O}} K)$ which is equal to $\dim_{\Bbbk} H$, since the Hopf algebra A over $\mathcal{O}$ is free of rank $\dim_{\Bbbk} H$ and $\mathcal{O}$ as a discrete valuation ring is a unique factorization domain. We conclude that d_H and $\dim_{\Bbbk} H$ have the same prime factors. □

Applying Proposition 3.3 to the Drinfeld double of a semisimple and cosemisimple Hopf algebra, we have the following result:

THEOREM 3.4. *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra over $\Bbbk$ and $D(H)$ the Drinfeld double of H. The determinant $d_{D(H)}$ and the dimension $\dim_{\Bbbk} H$ have the same prime factors.*

Proof. The representation category of the Drinfeld double $D(H)$ is a modular fusion category over $\Bbbk$, since $D(H)$ is a quasitriangular semisimple and cosemisimple Hopf algebra (see [10, Corollary 10.3.13]). It follows that the Grothendieck ring $\mathrm{Gr}(D(H))$ of $D(H)$ is a commutative ring. By Proposition 3.3, the determinant $d_{D(H)}$ and the dimension $\dim_{\Bbbk} D(H) = (\dim_{\Bbbk} H)^2$ have the same prime factors. This gives the desired result. □

In the sequel, we describe a relationship between the determinant $d_{\mathcal{C}}$ of a fusion category $\mathcal{C}$ and the determinant $d_{\widetilde{\mathcal{C}}}$ of $\widetilde{\mathcal{C}}$, where $\widetilde{\mathcal{C}}$ is the pivotalization of $\mathcal{C}$ stated below.

Let $\mathcal{C}$ be a fusion category over $\Bbbk$. Recall from [6, Theorem 2.6] that there exists an isomorphism $\gamma : id \to ****$ between the identity and the fourth duality tensor autoequivalences of $\mathcal{C}$. Denote by $\widetilde{\mathcal{C}} := \mathcal{C}^{\mathbb{Z}/2\mathbb{Z}}$ the corresponding equivariantization. More explicitly, simple objects of $\widetilde{\mathcal{C}}$ are pairs (X, α), where X is a simple object of $\mathcal{C}$, and $\alpha : X \to X^{**}$ satisfies $\alpha^{**}\alpha = \gamma_X$. The fusion category $\widetilde{\mathcal{C}}$ has a canonical pivotal structure which is called the *pivotalization* of $\mathcal{C}$ (see [4, Definition 7.21.9] for details). Moreover, the pivotal fusion category $\widetilde{\mathcal{C}}$ is also spherical (see [5, Corollary 7.6]).

To describe any simple object (X, α) of $\widetilde{\mathcal{C}}$, we first fix an isomorphism $\theta : X \to X^{**}$. Since $\mathrm{Hom}(X, X^{**})$ is one dimensional, we may write $\alpha = u\theta$ and $\gamma_X = v\theta^{**}\theta$ for some $u, v \in \Bbbk^\times$. Then $\alpha^{**}\alpha = \gamma_X$ implies that $u^2 = v$. Therefore, for each simple object X of $\mathcal{C}$, we only have two choices of α, and if one of them is α, then another one is $-\alpha$. In view of this, we may write $(X, \alpha) = X^+$ and $(X, -\alpha) = X^-$. It follows that $\mathbf{1}^+ = \mathbf{1}$, $\mathbf{1}^- \otimes \mathbf{1}^- = \mathbf{1}$, $\dim(\mathbf{1}^-) = -1$, and $X^\pm \otimes \mathbf{1}^- = \mathbf{1}^- \otimes X^\pm = X^\mp$ (see [16, Section 5.1]). Note that the forgetful function $F : \widetilde{\mathcal{C}} \to \mathcal{C}$, $X^\pm \mapsto X$ preserves squared norms of simple objects [4, Remark 7.21.11]. It follows that $\dim(\widetilde{\mathcal{C}}) = 2\dim(\mathcal{C})$.

If $\mathrm{char}(\Bbbk) \neq 2$, the Grothendieck algebra $\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk$ has the following decomposition:

$$\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk = e(\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk) \oplus (1 - e)(\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk), \tag{3.1}$$

where $e = \frac{\mathbf{1} - \mathbf{1}^-}{2}$ is a central idempotent element of $\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk$. It follows from [16, Section 5.1] that

$$\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk \cong (\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk)/e(\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk) \cong (1 - e)(\mathrm{Gr}(\widetilde{\mathcal{C}}) \otimes_{\mathbb{Z}} \Bbbk). \tag{3.2}$$

The determinants $d_{\mathcal{C}}$ (resp. $m_{\mathcal{C}}$) and $d_{\widetilde{\mathcal{C}}}$ (resp. $m_{\widetilde{\mathcal{C}}}$) have the following divisibility relation:

PROPOSITION 3.5. *Let $\mathcal{C}$ be a fusion category over $\Bbbk$ and $\widetilde{\mathcal{C}}$ the pivotalization of $\mathcal{C}$.*

(1) $m_{\mathcal{C}} \mid m_{\widetilde{\mathcal{C}}}$.
(2) $d_{\mathcal{C}} \mid d_{\widetilde{\mathcal{C}}}$.

Proof. (1) Denote by $\{X_i\}_{i\in I}$ the set of isomorphism classes of simple objects of $\mathcal{C}$. Then $\{X_i^\pm\}_{i\in I}$ is the set of isomorphism classes of simple objects of $\widetilde{\mathcal{C}}$. For the Casimir number $m_{\widetilde{\mathcal{C}}}$, there exists some $a \in \mathrm{Gr}(\widetilde{\mathcal{C}})$ such that $\sum_{i\in I} X_i^\pm a(X_i^\pm)^* = m_{\widetilde{\mathcal{C}}}$. Applying the ring homomorphism $f : \mathrm{Gr}(\widetilde{\mathcal{C}}) \to \mathrm{Gr}(\mathcal{C})$ induced by the forgetful function $F : \widetilde{\mathcal{C}} \to \mathcal{C}$ to this equation, we have $\sum_{i\in I} X_i 2f(a)(X_i)^* = m_{\widetilde{\mathcal{C}}}$. It follows that $m_{\widetilde{\mathcal{C}}} \in \mathbb{Z} \cap \mathrm{Im}c = (m_{\mathcal{C}})$. This gives a proof of Part (1).

(2) In the Grothendieck ring $\mathrm{Gr}(\widetilde{\mathcal{C}})$, we suppose for any $j \in I$ that

$$\sum_{i\in I} X_i^+(X_i^+)^* X_j^+ = \sum_{i\in I} \mu_{ij} X_i^+ + \sum_{i\in I} \nu_{ij} X_i^-, \tag{3.3}$$

where $\mu_{ij}, \nu_{ij} \in \mathbb{Z}$. Then for any $j \in I$, we have

$$\sum_{i\in I} X_i^+(X_i^+)^* X_j^- = \sum_{i\in I} X_i^+(X_i^+)^* X_j^+ \mathbf{1}^- = \sum_{i\in I} \mu_{ij} X_i^- + \sum_{i\in I} \nu_{ij} X_i^+.$$

This means that, in the Grothendieck ring $\mathrm{Gr}(\widetilde{\mathcal{C}})$, the matrix of left multiplication by the Casimir element $\sum_{i\in I} X_i^\pm (X_i^\pm)^* = 2\sum_{i\in I} X_i^+(X_i^+)^*$ with respect to the basis $\{X_i^\pm\}_{i\in I}$ of $\mathrm{Gr}(\widetilde{\mathcal{C}})$ is

$$2\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A=(\mu_{ij})_{n\times n}$, $B=(\nu_{ij})_{n\times n}$, and n is the cardinality of I. Thus, the determinant of $\widetilde{\mathcal{C}}$ is

$$d_{\widetilde{\mathcal{C}}}=2^{2n}\det\begin{pmatrix} A & B \\ B & A \end{pmatrix}=2^{2n}\det(A+B)\det(A-B).$$

Applying the ring homomorphism $f:\mathrm{Gr}(\widetilde{\mathcal{C}})\to \mathrm{Gr}(\mathcal{C})$ induced from the forgetful function $F:\widetilde{\mathcal{C}}\to\mathcal{C}$ to the equation (3.3), we have that

$$\sum_{i\in I} X_i(X_i)^*X_j=\sum_{i\in I}(\mu_{ij}+\nu_{ij})X_i.$$

This shows that, in the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$, the matrix of left multiplication by the Casimir element $\sum_{i\in I}X_iX_i^*$ with respect to the basis $\{X_i\}_{i\in I}$ of $\mathrm{Gr}(\mathcal{C})$ is $A+B$. Thus, the determinant of $\mathcal{C}$ is $d_{\mathcal{C}}=\det(A+B)$, which is a factor of $d_{\widetilde{\mathcal{C}}}$. □

As a consequence, we have the following result:

PROPOSITION 3.6. *Let $\mathcal{C}$ be a fusion category over $\Bbbk$ with char$(\Bbbk)\neq 2$. If $\mathcal{C}$ is nondegenerate, then $d_{\mathcal{C}}\neq 0$ in $\Bbbk$.*

Proof. Since $\mathcal{C}$ is nondegenerate, that is, the global dimension $\dim(\mathcal{C})\neq 0$ in $\Bbbk$, it follows that $\dim(\widetilde{\mathcal{C}})=2\dim(\mathcal{C})\neq 0$. Thus, the pivotal fusion category $\widetilde{\mathcal{C}}$ is nondegenerate. It follows from Proposition 2.5 that $d_{\widetilde{\mathcal{C}}}\neq 0$ in $\Bbbk$. As a result, $d_{\mathcal{C}}\neq 0$ since $d_{\mathcal{C}}$ is a factor of $d_{\widetilde{\mathcal{C}}}$. □

We expect that the converse of Proposition 3.6 is also true. However, the proof seems too hard to be finished. What we can do is the proof of the following statement:

PROPOSITION 3.7. *Let $\mathcal{C}$ be a fusion category over $\Bbbk$ with char$(\Bbbk)\neq 2$. Then $\mathcal{C}$ is nondegenerate if and only if the subalgebra $e(Gr(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk)$ is semisimple, where $e=\frac{\mathbf{1}-\mathbf{1}^-}{2}$ is a central idempotent element of $Gr(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk$.*

Proof. The global dimension $\dim(\mathcal{C})\neq 0$ shows that $\dim(\widetilde{\mathcal{C}})=2\dim(\mathcal{C})\neq 0$. Thus, the Grothendieck algebra $\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk$ of the pivotal fusion category $\widetilde{\mathcal{C}}$ is semisimple by [**17**, Theorem 6.5]. It follows that the quotient algebra (see (3.1))

$$(\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk)/(1-e)(\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk)\cong e(\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk)$$

is semisimple. Conversely, consider $t=\sum_{i\in I}\dim(X_i^{\pm})(X_i^{\pm})^*\in\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk$. Obviously, $t\neq 0$. For any $a\in\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk$, it follows from (2.1) that

$$ta=\sum_{i\in I}\dim(X_i^{\pm})(X_i^{\pm})^*a=\sum_{i\in I}\dim(aX_i^{\pm})(X_i^{\pm})^*=\dim(a)t.$$

Similarly, it follows from (2.2) that $at=\dim(a)t$. Thus, t is a central element of $\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk$ satisfying $t^2=\dim(t)t=\dim(\widetilde{\mathcal{C}})t=2\dim(\mathcal{C})t$. Moreover,

$$t=et+(1-e)t=et+\dim(1-e)t=et\in e(\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk).$$

If $\dim(\mathcal{C})=0$, then $t^2=0$, and hence the ideal of $e(\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk)$ generated by t is nilpotent, a contradiction to the semisimplicity of $e(\mathrm{Gr}(\widetilde{\mathcal{C}})\otimes_{\mathbb{Z}}\Bbbk)$. The proof is completed. □

4. Determinants vs. Frobenius–Schur exponents. In this section, we shall show that the determinant (or equivalently, the Casimir number) and the Frobenius–Schur exponent of a spherical fusion category over the field $\mathbb{C}$ of complex numbers have the same prime factors.

Let $\mathcal{C}$ be a fusion category over $\Bbbk$ and V a finite dimensional left $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$-module over an algebraic closure field K. For any $\varphi \in \mathrm{End}_K(V)$, we define $\mathcal{I}(\varphi) \in \mathrm{End}_K(V)$ by

$$\mathcal{I}(\varphi)(v) = \sum_{i \in I} X_i \varphi(X_{i^*} v) \text{ for } v \in V.$$

Then $\mathcal{I}(\varphi)$ lies in $\mathrm{End}_{\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K}(V)$ and does not depend on the choice of a pair of dual bases of $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ (see [7, Lemma 7.1.10]). If V is a simple $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$-module, then $\mathrm{End}_{\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K}(V) \cong K$. In this case, there exists a unique element $c_V \in K$ such that

$$\mathcal{I}(\varphi) = c_V \mathrm{Tr}(\varphi) id_V \text{ for all } \varphi \in \mathrm{End}_K(V).$$

Such an element c_V only depends on the isomorphism class of V and is called the *Schur element* associated with V (see [7, Theorem 7.2.1]). Note that the semisimplicity criterion stated in [7, Theorem 7.2.6] works for Grothendieck algebras. Namely, the Grothendieck algebra $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ is semisimple if and only if any Schur element associated with a simple module over $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ is not zero in K.

Let V be a simple $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$-module with the Schur element c_V. The character of V is denoted by χ_V. Then $\sum_{i \in I} \chi_V(X_i) X_{i^*}$ is a central element of $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$. This element acts by a scalar f_V on V and by zero on any simple module not isomorphic to V. The scalar f_V is called the *formal codegree* of V (see [14, Lemma 2.3]).

LEMMA 4.1. *Let V be a simple $Gr(\mathcal{C}) \otimes_{\mathbb{Z}} K$-module with the Schur element c_V and the formal codegree f_V. The action of $\sum_{i,j \in I} X_i X_j X_{i^*} X_{j^*}$ on V is a scalar multiple by $c_V f_V$.*

Proof. For a simple module V over $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$, there is a corresponding algebra morphism

$$\rho_V : \mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K \to \mathrm{End}_K(V), \ \ \rho_V(a)(v) = av \text{ for } a \in \mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K, \ \ v \in V.$$

Note that $\mathcal{I}(\varphi)(v) = c_V \mathrm{Tr}(\varphi) v$ holds for any $\varphi \in \mathrm{End}_K(V)$ and $v \in V$. Replacing φ and v in this equality by $\rho_V(X_j)$ and $X_{j^*} v$ respectively, we have $\mathcal{I}(\rho_V(X_j))(X_{j^*} v) = c_V \mathrm{Tr}(\rho_V(X_j)) X_{j^*} v$. Summing over all $j \in I$, we have

$$\sum_{j \in I} \mathcal{I}(\rho_V(X_j))(X_{j^*} v) = c_V \sum_{j \in I} \mathrm{Tr}(\rho_V(X_j)) X_{j^*} v.$$

Taking into account the definition of $\mathcal{I}$, we have

$$\sum_{i,j \in I} X_i \rho_V(X_j)(X_{i^*} X_{j^*} v) = c_V \sum_{j \in I} \chi_V(X_j) X_{j^*} v.$$

This gives rise to the desired result $\sum_{i,j \in I} X_i X_j X_{i^*} X_{j^*} v = c_V f_V v$ for any $v \in V$. □

Note that $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is always semisimple and $\dim(\mathcal{C})$ is always not zero in the field $\mathbb{C}$ of complex numbers. Thus, [17, Theorem 6.5] is trivial if the field $\Bbbk$ is taken to be $\mathbb{C}$. We shall give a modified version of [17, Theorem 6.5], so that we can use it to present another statement of the Cauchy theorem for spherical fusion categories.

Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$ with isomorphism classes of simple objects $\{X_i\}_{i \in I}$. The Frobenius–Schur exponent of $\mathcal{C}$ has been defined in [12, Definition 5.1] in terms of the higher Frobenius–Schur indicators of objects of $\mathcal{C}$. This exponent, denoted by N, can be regarded as the order of the twist θ of the Drinfeld center $Z(\mathcal{C})$ associated with a pivotal structure of $\mathcal{C}$ (see [12, Theorem 5.5]). Let $\xi_N \in \mathbb{C}$ be a primitive N-th root of unity. Then $\mathbb{Z}[\xi_N]$ is a Dedekind domain and every nonzero proper ideal factors into a product of prime ideal factors. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[\xi_N]$. Then $\mathfrak{p}$ is maximal since $\mathbb{Z}[\xi_N]$ is Dedekind. Thus, the quotient ring $\mathbb{Z}[\xi_N]/\mathfrak{p}$ is a field. In this case, $\dim(X) \in \mathbb{Z}[\xi_N]$ (see [12]) can be considered as an element in $\mathbb{Z}[\xi_N]/\mathfrak{p}$ in a natural way.

THEOREM 4.2. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$ with the Frobenius–Schur exponent N. Let ξ_N be a primitive N-th root of unity with $\dim(X) \in \mathbb{Z}[\xi_N]$ for any object X of $\mathcal{C}$. For any prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\xi_N]$, the determinant $d_{\mathcal{C}} \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$ if and only if the global dimension $\dim(\mathcal{C}) \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$.*

Proof. Note that the Casimir number $m_{\mathcal{C}} = \sum_{i\in I} X_i a X_{i^*}$ for some $a \in \mathrm{Gr}(\mathcal{C})$. Applying dim to this equality, we have $m_{\mathcal{C}} = \dim(\mathcal{C})\dim(a)$. Thus, if $d_{\mathcal{C}} \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$, then $m_{\mathcal{C}} \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$, and hence $\dim(\mathcal{C}) \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$. Conversely, if $\dim(\mathcal{C}) \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$, so is $\dim(\mathcal{C}) \neq 0$ in K, where K is an algebraic closure of the field $\mathbb{Z}[\xi_N]/\mathfrak{p}$. Let $Z(\mathcal{C})$ be the Drinfeld center of $\mathcal{C}$. Since $\dim(\mathcal{C}) \neq 0$ in K, it follows from [11, Section 5] that $Z(\mathcal{C})$ is a modular category and $\dim(Z(\mathcal{C})) = \dim(\mathcal{C})^2 \neq 0$ in K. If we denote $\mathrm{Irr}(Z(\mathcal{C}))$ the set of isomorphism classes of simple objects of $Z(\mathcal{C})$ and n the cardinality of $\mathrm{Irr}(Z(\mathcal{C}))$, then by Example 2.7 the determinant of $Z(\mathcal{C})$ is

$$d_{Z(\mathcal{C})} = \frac{\dim(Z(\mathcal{C}))^n}{\prod_{Y\in\mathrm{Irr}(Z(\mathcal{C}))}\dim(Y)^2} \neq 0.$$

Note that $d_{Z(\mathcal{C})}$ is the determinant of the matrix of left multiplication by the Casimir element $\sum_{Y\in\mathrm{Irr}(Z(\mathcal{C}))} YY^*$. It follows that $\sum_{Y\in\mathrm{Irr}(Z(\mathcal{C}))} YY^*$ is an invertible element in $\mathrm{Gr}(Z(\mathcal{C})) \otimes_{\mathbb{Z}} K$. Note that the forgetful tensor functor $F : Z(\mathcal{C}) \to \mathcal{C}$ induces an algebra homomorphism $f : \mathrm{Gr}(Z(\mathcal{C})) \otimes_{\mathbb{Z}} K \to \mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ whose image is contained in the center of $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$. In particular, from the proof of [14, Lemma 3.1] we may see that

$$f\left(\sum_{Y\in\mathrm{Irr}(Z(\mathcal{C}))} YY^*\right) = \sum_{i,j\in I} X_i X_j X_{i^*} X_{j^*}.$$

Thus, $\sum_{i,j\in I} X_i X_j X_{i^*} X_{j^*}$ is a central invertible element in $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$. This together with Lemma 4.1 shows that $c_V \neq 0$ for any simple $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$-module V. We conclude that $\mathrm{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} K$ is semisimple by [7, Theorem 7.2.6], and hence $d_{\mathcal{C}} \neq 0$ in K by Proposition 2.2. This gives the desired result that $d_{\mathcal{C}} \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$. □

We are now ready to state the relationship between the determinant $d_{\mathcal{C}}$ and the Frobenius–Schur exponent N of $\mathcal{C}$.

THEOREM 4.3. *Let $\mathcal{C}$ be a spherical fusion category over $\mathbb{C}$. The determinant $d_{\mathcal{C}}$ and the Frobenius–Schur exponent N of $\mathcal{C}$ have the same prime factors.*

Proof. For the Casimir number $m_{\mathcal{C}}$, there is some $a \in \mathrm{Gr}(\mathcal{C})$ such that $\sum_{i\in I} X_i a X_{i^*} = m_{\mathcal{C}}$. Applying dim to this equality, we have $\dim(\mathcal{C})\dim(a) = m_{\mathcal{C}}$ in $\mathbb{Z}[\xi_N]$. Note that (N) and $(\dim(\mathcal{C}))$ are two principal ideals of $\mathbb{Z}[\xi_N]$ having the same prime ideal factors (see [1, Theorem 3.9]). If $p \mid N$ for a prime number p, then there exists a prime ideal factor $\mathfrak{p}$ of (N) such that $\mathfrak{p} \cap \mathbb{Z} = (p)$. In this case, $\mathfrak{p}$ is also a prime ideal factor of $(\dim(\mathcal{C}))$. Moreover, $(\dim(\mathcal{C})) \cap \mathbb{Z} \subseteq \mathfrak{p} \cap \mathbb{Z} = (p)$. It follows from $m_{\mathcal{C}} = \dim(\mathcal{C})\dim(a)$

that $m_{\mathcal{C}} \in (\dim(\mathcal{C})) \cap \mathbb{Z} \subseteq (p)$, and hence $p \mid m_{\mathcal{C}}$, or equivalently $p \mid d_{\mathcal{C}}$. Conversely, if $p \nmid N$ for a prime p, we need to show that $p \nmid d_{\mathcal{C}}$. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[\xi_N]$ such that $\mathfrak{p} \cap \mathbb{Z} = (p)$. Then $(N) \nsubseteq \mathfrak{p}$ since $p \nmid N$. This implies that $(\dim(\mathcal{C})) \nsubseteq \mathfrak{p}$. Especially, $0 \neq \dim(\mathcal{C}) \in \mathbb{Z}[\xi_N]/\mathfrak{p}$. It follows from Theorem 4.2 that $d_{\mathcal{C}} \neq 0$ in $\mathbb{Z}[\xi_N]/\mathfrak{p}$. In other words, $d_{\mathcal{C}} \notin \mathfrak{p} \cap \mathbb{Z} = (p)$ and hence $p \nmid d_{\mathcal{C}}$. □

REMARK 4.4. Let (N) and $(\dim(\mathcal{C}))$ be principal ideals of $\mathbb{Z}[\xi_N]$ generated by N and $\dim(\mathcal{C})$, respectively. The statement of [**1**, Theorem 3.9] that (N) and $(\dim(\mathcal{C}))$ have the same prime ideal factors is called the Cauchy theorem for a spherical fusion category. Indeed, applying this to the case $\mathcal{C} = \text{Rep}(G)$ for a finite group G, we obtain the classical Cauchy theorem for finite groups: $\dim(\mathcal{C}) = |G|$ and $N = \exp(G)$ have the same prime factors. Now, Theorem 4.3 shows that the determinant $d_{\mathcal{C}}$ (or equivalently, the Casimir number $m_{\mathcal{C}}$) and the Frobenius–Schur exponent N of $\mathcal{C}$ have the same prime factors. This may be thought of as an integer version of the Cauchy theorem for spherical fusion categories.

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