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Classification of affine prime regular Hopf algebras of GK-dimension one ☆

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A B S T R A C T

The classification of affine prime regular Hopf algebras of GK-dimension one is completed. As consequences, 1) we give a negative answer to Question 7.1 posed in [5] and 2) we show that there do exist prime regular Hopf algebras of GK-dimension one which are not pointed.

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1. Introduction

Throughout this paper, k denotes an algebraically closed field of characteristic 0, all vector spaces are over k, and all Hopf algebras are affine and noetherian.

In recent years, Hopf algebras of finite Gelfand–Kirillov dimension (GK-dimension for short) have been studied intensively [1,2,5,6,11,13,18–20]. In particular, the classification of Hopf algebras of low GK-dimension was carried over and got substantial progress. Lu, Wu and Zhang initiated the program of classifying Hopf algebras of GK-dimension one [13]. Brown and Zhang [5] made further efforts in this direction and classified all prime regular Hopf algebras H of GK-dimension one under the hypothesis:

$$\mathrm{im}(H) = 1 \quad \text{or} \quad \mathrm{im}(H) = \mathrm{io}(H) \tag{$*$}$$

(see Section 2 for the definition of $\mathrm{im}(H)$ and $\mathrm{io}(H)$).

For Hopf algebras H of GK-dimension two, all known classification results are given under the condition of H being domains. In [6], Goodearl and Zhang classified all Hopf algebras H of GK-dimension two which are domains and satisfy the condition $\mathrm{Ext}^1_H(k,k) \neq 0$. For those with vanishing Ext-groups, some interesting examples were constructed by Wang–Zhang–Zhuang [19] and they conjectured these examples together with Hopf algebras given in [6] exhausted all Hopf algebra domains with GK-dimension two. In order to study Hopf algebras H of GK-dimensions three and four, a more restrictive but natural condition was added: H is connected, that is, the coradical of H is 1-dimensional. All connected Hopf algebras with GK-dimension three were classified by Zhuang in [22]. Recently, Wang, Zhang and Zhuang [20] completed the classification of connected Hopf algebras of GK-dimension four.

In another direction, the pointed Hopf algebra domains of finite GK-dimension with generic infinitesimal braiding have been classified by Andruskiewitsch and Schneider [2] and Andruskiewitsch and Angiono [1].

Philosophically, prime regular Hopf algebras of finite GK-dimension can be regarded as "noncommutative" counterparts of connected algebraic groups. It is well-known that there are only two connected algebraic groups of dimension one. This fact makes us believe that there should be a complete classification of prime regular Hopf algebras of GK-dimension one. In this paper, we will go on with Brown–Zhang's works [5] and finish the classification of all prime regular Hopf algebras of GK-dimension one.

As the first step, we construct a new class of prime regular Hopf algebras $D(m,d,\xi)$ of GK-dimension one, which implies that the condition $(*)$ is really necessary for Brown–Zhang's classification. In fact, Brown and Zhang [5] gave four classes of Hopf algebras of GK-dimension one: the coordinate algebras of connected algebraic groups of dimension one, the infinite dihedral group algebra, infinite dimensional Taft algebras and generalized Liu algebras, and proved that all prime regular Hopf algebras H of GK-dimension one satisfying condition $(*)$ belong to these four classes. Naturally, they raised the following open question (Question 7.1 in [5]):

(**Q**) *Does their result still hold without the hypothesis* (∗)?

Our new examples in Section 4 of this paper provide a negative answer to this question.

Secondly, we will prove our main result (see Theorem 8.3) which states that our new examples together with the four classes of Hopf algebras given in [5] form a complete list, up to isomorphisms of Hopf algebras, of all prime regular Hopf algebras of GK-dimension one. The key idea to prove the main result is not complicated: let H be a prime regular Hopf algebra of GK-dimension one which doesn't satisfy the condition (∗). From this Hopf algebra H, we can construct a Hopf subalgebra $\widetilde{H}$ which will be shown to meet the condition (∗). Thus the classification result given in [5] can be applied. At last, we show that $\widetilde{H}$ determines the structure of H entirely.

The precess to realize our idea, which motivates our discovery of new examples, turns out to be much more complicated than our expectation: According to Brown–Zhang's classification result, there is a dichotomy on $\widetilde{H}$: $\widetilde{H}$ is either primitive or group-like (see Definition 6.1). When $\widetilde{H}$ is primitive, we find that H must be an infinite dimensional Taft algebra. The difficult part is group-like case. In this case, we gradually realize that there is an essential difference between the situation $\frac{\mathrm{io}(H)}{\mathrm{im}\,(H)} > 2$ and the situation $\frac{\mathrm{io}(H)}{\mathrm{im}\,(H)} = 2$. The last situation becomes very delicate: More generators and subtle relations are allowed to appear. Ultimately, this leads us to find the final missing piece in the puzzle of prime regular Hopf algebras of GK-dimension one.

In practice, the assumption "pointed" is always added when we want to classify Hopf algebras of lower GK-dimensions. As a matter of fact, all known examples are pointed and it is widely believed that, at least for prime regular Hopf algebras of GK-dimension one, these Hopf algebras should be pointed automatically. Our new examples will change this naive understanding since all the new examples are not pointed!

The paper is organized as follows. Necessary definitions, known examples and preliminary results are collected in Section 2. Some combinatorial relations, which are crucial to the following analysis, will be given in Section 3. Section 4 is devoted to constructing the new examples $D(m, d, \xi)$ of prime regular Hopf algebras of GK-dimension one. The proof of $D(m, d, \xi)$ being a Hopf algebra is nontrivial and the combinatorial equations given in Section 3 are used extensively in the proof. In Section 5, the definition of the Hopf subalgebra $\widetilde{H}$ and some basic properties of $\widetilde{H}$ are given. In particular, we show that $\widetilde{H}$ is a prime regular Hopf algebra satisfying the condition (∗). Thus $\widetilde{H}$ is either primitive or group-like. By the results of the last three sections, we can reconstruct H from $\widetilde{H}$. In details, the general relations between $\widetilde{H}$ and H are built in Section 6. Section 7 is designed to analyse the primitive case and H is shown to be an infinite dimensional Taft algebra. We consider the group-like case in the last section, and as the desired conclusion, we show that H is isomorphic to some $D(m, d, \xi)$. Finally, the main result, that is, the classification result, and its proof are also formulated in this section.

2. Preliminaries

In this section we recall the urgent needs around affine noetherian Hopf algebras for completeness and the convenience of the reader. About general background knowledge, the reader is referred to [15] for Hopf algebras, [14] for noetherian rings and [3,13,5] for exposition about noetherian Hopf algebras.

Usually we are working on left modules. Let A^{op} denote the opposite algebra of A. Throughout, we use the symbols Δ, ϵ and S respectively, for the coproduct, counit and antipode of a Hopf algebra H, and the Sweedler's notation for coproduct $\Delta(h) = \sum h_1 \otimes h_2$ $(h \in H)$ will be used freely.

2.1. Stuffs from ring theory

In this paper, a ring R is called *regular* if it has finite global dimension, and it is *prime* if 0 is a prime ideal.

• *PI-degree.* If Z is an Ore domain, then the *rank* of a Z-module M is defined to be the $Q(Z)$-dimension of $Q(Z) \otimes_Z M$, where $Q(Z)$ is the quotient division ring of Z. Let R be an algebra satisfying a polynomial identity (PI for short). The PI-degree of R is defined to be

$$\text{PI-deg}(R) = \min\{n | R \hookrightarrow M_n(C) \text{ for some commutative ring } C\}$$

(see [14, Chapter 13]). If R is a prime PI ring with center Z, then the PI-degree of R equals the square root of the rank of R over Z.

•*Artin–Schelter condition.* Recall that an algebra A is said to be *augmented* if there is an algebra morphism $\epsilon : \ A \to k$. Let (A, ϵ) be an augmented noetherian algebra. Then R is *Artin–Schelter Gorenstein*, we usually abbreviate to *AS-Gorenstein*, if

(AS1) $\text{injdim}_A A = d < \infty$,
(AS2) $\dim_k \text{Ext}^d_A({}_Ak, {}_AA) = 1$ and $\dim_k \text{Ext}^i_A({}_Ak, {}_AA) = 0$ for all $i \neq d$,
(AS3) the right A-module versions of (AS1, AS2) hold.

The following result is the combination of [21, Theorem 0.1] and [21, Theorem 0.2 (1)], which shows that a large number of Hopf algebras are AS-Gorenstein.

Lemma 2.1. *Each affine noetherian PI Hopf algebra is AS-Gorenstein.*

2.2. Homological integrals

The concept *homological integral* can be defined for an AS-Gorenstein augmented algebra.

Definition 2.2. (See [5, Definition 1.3].) Let (A, ϵ) be a noetherian augmented algebra and suppose that A is AS-Gorenstein of injective dimension d. Any non-zero element of the one-dimensional A-bimodule $\mathrm{Ext}^d_A({}_Ak, {}_AA)$ is called a *left homological integral* of A. We write $\int^l_A = \mathrm{Ext}^d_A({}_Ak, {}_AA)$. Any non-zero element in $\mathrm{Ext}^d_{A^{op}}(k_A, A_A)$ is called a *right homological integral* of A. We write $\int^r_A = \mathrm{Ext}^d_{A^{op}}(k_A, A_A)$. By abusing the language we also call $\int^l_A$ and $\int^r_A$ the left and the right homological integrals of A respectively.

• *Winding automorphisms.* Let H be an affine noetherian PI Hopf algebra. By Lemma 2.1, it is AS-Gorenstein and thus has left homological integrals $\int^l_H$. Let $\pi : H \to H/\mathrm{r.ann}(\int^l_H)$ be the canonical algebra homomorphism, where $\mathrm{r.ann}(\int^l_H)$ denotes the set of right annihilators of $\int^l_H$ in H. We write Ξ^l_π for the *left winding automorphism* of H associated to π, namely

$$\Xi^l_\pi(a) := \sum \pi(a_1)a_2 \qquad \text{for } a \in H.$$

Similarly we use Ξ^r_π for the right winding automorphism of H associated to π, that is,

$$\Xi^r_\pi(a) := \sum a_1\pi(a_2) \qquad \text{for } a \in H.$$

Let G^l_π and G^r_π be the subgroups of $\mathrm{Aut}_{k\text{-alg}}(H)$ generated by Ξ^l_π and Ξ^r_π, respectively.

• *Integral order and integral minor.* With the same notions as above, the *integral order* $\mathrm{io}(H)$ of H is defined by the order of the group G^l_π:

$$\mathrm{io}(H) := |G^l_\pi|. \tag{2.1}$$

As noted in [5, Lemma 2.1], we always have $|G^l_\pi| = |G^r_\pi|$. So the above definition is independent of the choice of G^l_π or G^r_π. In addition, if H is prime regular of GK-dimension one, then [13, Theorem 7.1] implies

$$\text{PI-deg}(H) = \mathrm{io}(H).$$

The *integral minor* of H, denoted by $\mathrm{im}(H)$, is defined by

$$\mathrm{im}(H) := |G^l_\pi / G^l_\pi \cap G^r_\pi|. \tag{2.2}$$

Remark 2.3. Crudely speaking, $\mathrm{io}(H)$ is a measure of the commutativity of H and $\mathrm{im}(H)$ is a measure of the cocommutativity of H. In fact, for a prime regular Hopf algebra H

of GK-dimension one, we have $\mathrm{io}(H) = 1$ if and only if H is commutative (see [13, Corollary 7.8]) and $\mathrm{im}(H) = 1$ if and only if H is cocommutative (see [5, Section 4]).

• *Strongly graded and bigraded properties.* Let H be a prime regular Hopf algebra of GK-dimension one. By [5, Theorem 2.5], $|G_\pi^l|$ is finite, say n. Therefore, the character group $\widehat{G_\pi^l} := \mathrm{Hom}_{k\text{-alg}}(kG_\pi^l, k)$ of G_π^l is isomorphic to G_π^l. Similarly, the character group $\widehat{G_\pi^r}$ of G_π^r is isomorphic to G_π^r.

Fix a primitive nth root ζ of 1 in k, and define $\chi \in \widehat{G_\pi^l}$ and $\eta \in \widehat{G_\pi^r}$ by setting

$$\chi(\Xi_\pi^l) = \zeta \quad \text{and} \quad \eta(\Xi_\pi^r) = \zeta.$$

Thus $\widehat{G_\pi^l} = \{\chi^i | 0 \leqslant i \leqslant n-1\}$ and $\widehat{G_\pi^r} = \{\eta^j | 0 \leqslant j \leqslant n-1\}$. For each $0 \leqslant i, j \leqslant n-1$, let

$$H_i^l := \{a \in H | \Xi_\pi^l(a) = \chi^i(\Xi_\pi^l)a\}$$

and

$$H_j^r := \{a \in H | \Xi_\pi^r(a) = \eta^j(\Xi_\pi^r)a\}.$$

The following lemma is [5, Theorem 2.5 (b)].

Lemma 2.4. (1) $H = \bigoplus_{\chi^i \in \widehat{G_\pi^l}} H_i^l$ *is strongly* $\widehat{G_\pi^l}$*-graded.*

(2) $H = \bigoplus_{\eta^j \in \widehat{G_\pi^r}} H_j^r$ *is strongly* $\widehat{G_\pi^r}$*-graded.*

It is clear that $\Xi_\pi^l \Xi_\pi^r = \Xi_\pi^r \Xi_\pi^l$, so H_i^l is stable under the action of G_π^r. Consequently, the $\widehat{G_\pi^l}$- and $\widehat{G_\pi^r}$-gradings on H are *compatible* in the sense that

$$H_i^l = \bigoplus_{0 \leqslant j \leqslant n-1} (H_i^l \cap H_j^r) \quad \text{and} \quad H_j^r = \bigoplus_{0 \leqslant i \leqslant n-1} (H_i^l \cap H_j^r)$$

for all i, j. Then H is a bigraded algebra:

$$H = \bigoplus_{0 \leqslant i,j \leqslant n-1} H_{ij}, \tag{2.3}$$

where $H_{ij} = H_i^l \cap H_j^r$. And we write $H_0 = H_{00}$ for convenience.

For later use, we collect some properties about H which are [5, Proposition 2.1 (c)(e)] and [5, Theorem 2.5 (f)].

Lemma 2.5. *Let H be an affine prime regular Hopf algebra of GK-dimensional one. Then*

(a) $\Delta(H_i^l) \subseteq H_i^l \otimes H$ *and* $\Delta(H_j^r) \subseteq H \otimes H_j^r$*; thus H_i^l is a right coideal of H and H_j^r is a left coideal of H.*

(b) $\Xi_\pi^r S = S(\Xi_\pi^l)^{-1}$*, where* $(\Xi_\pi^l)^{-1} = \Xi_{\pi\circ S}^l$.

(c) H_0^l, H_0^r *and* H_0 *are affine commutative Dedekind domains with* $H_0^l \cong H_0^r$.

Note that only (c) needs all the hypotheses of H, (a) and (b) hold if H has finite integral order.

Remark 2.6. (1) By [16,17], prime affine algebras of GK-dimension one are noetherian and PI automatically. So "*noetherian*" and "*PI*" do not appear in the title of this paper.

(2) If H is an affine prime regular Hopf algebra of GK-dimensional one, then $\text{gl.dim}H = 1$. Indeed, assume that $\text{gl.dim}H = d$. Wu and Zhang [21] proved that every noetherian affine PI Hopf algebra is Cohen–Macaulay and this forces $d = 1$.

2.3. Known examples

The following examples appeared in [5] already and we recall them for completeness.

• *Connected algebraic groups of dimension one.* It is well-known that there are precisely two connected algebraic groups of dimension one (see, say [7, Theorem 20.5]) over an algebraically closed field k. Therefore, there are precisely two commutative k-affine domains of GK-dimension one which admit a structure of Hopf algebra, namely $H_1 = k[x]$ and $H_2 = k[x^{\pm 1}]$. For H_1, x is a primitive element, and for H_2, x is a group-like element. Commutativity and cocommutativity imply that $\text{io}(H_i) = \text{im}(H_i) = 1$ for $i = 1, 2$.

• *Infinite dihedral group algebra.* Let $\mathbb{D}$ denote the infinite dihedral group $\langle g, x | g^2 = 1, gxg = x^{-1}\rangle$. Both g and x are group-like elements in the group algebra $k\mathbb{D}$. By cocommutativity, $\text{im}(k\mathbb{D}) = 1$. Using [13, Lemma 2.6], one sees that as a right H-module,

$$\int_{k\mathbb{D}}^{l} \cong k\mathbb{D}/\langle x - 1, g + 1\rangle.$$

This implies $\text{io}(k\mathbb{D}) = 2$.

• *Infinite dimensional Taft algebras.* Let n and t be integers with $n > 1$ and $0 \leqslant t \leqslant n-1$. Fix a primitive nth root ξ of 1. Let $H(n, t, \xi)$ be the algebra generated by x and g subject to the relations

$$g^n = 1 \quad \text{and} \quad xg = \xi gx.$$

Then $H(n, t, \xi)$ is a Hopf algebra with coalgebra structure given by

$$\Delta(g) = g \otimes g,\ \epsilon(g) = 1 \quad \text{and} \quad \Delta(x) = x \otimes g^t + 1 \otimes x,\ \epsilon(x) = 0,$$

and with

$$S(g) = g^{-1} \quad \text{and} \quad S(x) = -xg^{-t}.$$

As computed in [5, Subsection 3.3], we have

$$\int_H^l \cong H/\langle x, g-\xi^{-1}\rangle,$$

and the corresponding homomorphism π yields left and right winding automorphisms

$$\Xi_\pi^l : \begin{cases} x \longmapsto x, \\ g \longmapsto \xi^{-1} g, \end{cases} \quad \text{and} \quad \Xi_\pi^r : \begin{cases} x \longmapsto \xi^{-t} x, \\ g \longmapsto \xi^{-1} g. \end{cases}$$

So that $G_\pi^l = \langle \Xi_\pi^l \rangle$ and $G_\pi^r = \langle \Xi_\pi^r \rangle$ have order n. If $\gcd(n,t) = 1$, then $G_\pi^l \cap G_\pi^r = \{1\}$ and [5, Propositon 3.3] implies that there exists a primitive nth root η of 1 such that $H(n,t,\xi) \cong H(n,1,\eta)$ as Hopf algebras. If $\gcd(n,t) \neq 1$, let $m := n/\gcd(n,t)$, then $G_\pi^l \cap G_\pi^r = \langle (\Xi_\pi^l)^m \rangle$.

Thus we have $\mathrm{io}(H(n,t,\xi)) = n$ and $\mathrm{im}(H(n,t,\xi)) = m$ for any t. In particular, $\mathrm{im}(H(n,0,\xi)) = 1$, $\mathrm{im}(H(n,1,\xi)) = n$ and $\mathrm{im}(H(n,t,\xi)) = m = n/t$ when $t|n$.

Now assume $t|n$ and $m = n/t$, and let $H := H(n,t,\xi)$, we calculate the homogeneous parts H_i^l, H_j^r and H_{ij} for our later arguments. The gradings start from fixing a primitive nth root ζ of 1. We choose $\zeta = \xi^{-1}$. By the expressions of Ξ_π^l and Ξ_π^r, it is not difficult to find that

$$H_i^l = k[x]g^i \quad \text{and} \quad H_j^r = k[xg^{-t}]g^j \tag{2.4}$$

for all $0 \leqslant i,j \leqslant n-1$. Thus we have

$$H_{00} = k[x^m] \quad \text{and} \quad H_{i,i+jt} = k[x^m]x^j g^i \tag{2.5}$$

for all $0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant m-1$. Moreover we can see that

$$H_{ij} = 0 \ \text{ if } \ i-j \not\equiv 0 \pmod t$$

for all $0 \leqslant i,j \leqslant n-1$.

• *Generalized Liu algebras.* Let n and ω be positive integers. The generalized Liu algebra, denoted by $B(n,\omega,\gamma)$, is generated by $x^{\pm 1}, g$ and y, subject to the relations

$$\begin{cases} xx^{-1} = x^{-1}x = 1, \quad xg = gx, \quad xy = yx, \\ yg = \gamma gy, \\ y^n = 1 - x^\omega = 1 - g^n, \end{cases}$$

where γ is a primitive nth root of 1. The comultiplication, counit and antipode of $B(n,\omega,\gamma)$ are given by

$$\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g, \quad \Delta(y) = y \otimes g + 1 \otimes y,$$
$$\epsilon(x) = 1, \quad \epsilon(g) = 1, \quad \epsilon(y) = 0,$$

and

$$S(x) = x^{-1}, \quad S(g) = g^{-1} \quad S(x) = -yg^{-1}.$$

Let $B := B(n, \omega, \gamma)$. Using [13, Lemma 2.6], we get $\int_B^l = B/\langle y, x-1, g-\gamma^{-1}\rangle$. The corresponding homomorphism π yields left and right winding automorphisms

$$\Xi_\pi^l : \begin{cases} x \longmapsto x, \\ g \longmapsto \gamma^{-1} g, \\ y \longmapsto y, \end{cases} \quad \text{and} \quad \Xi_\pi^r : \begin{cases} x \longmapsto x, \\ g \longmapsto \gamma^{-1} g, \\ y \longmapsto \gamma^{-1} y. \end{cases}$$

Clearly these automorphisms have order n and $G_\pi^l \cap G_\pi^r = \{1\}$, whence $\mathrm{io}(B) = \mathrm{im}(B) = n$.

Choosing $\zeta = \gamma^{-1}$ for defining gradings of B, we can see

$$B_i^l = k\langle x^{\pm 1}, y\rangle g^i \quad \text{and} \quad B_j^r = k\langle x^{\pm 1}, yg^{-1}\rangle g^j \tag{2.6}$$

for all $0 \leqslant i, j \leqslant n-1$. Thus

$$B_0 = k[x^{\pm 1}] \quad \text{and} \quad B_{ij} = k[x^{\pm 1}] y^{j-i} g^i, \tag{2.7}$$

where $j - i$ is interpreted $\bmod n$.

Now, the main results of [5] can be formulated in the following form (for details, see [5, Proposition 3.1, Theorems 4.1 and 6.1]).

Theorem 2.7. *Assume H is an affine prime regular Hopf algebra of GK-dimensional one.*
(a) *If* $\mathrm{io}(H) = \mathrm{im}(H) = 1$, *then* $H \cong k[x]$ *or* $H \cong k[x^{\pm 1}]$*;*
(b) *If* $\mathrm{io}(H) = n > 1$, $\mathrm{im}(H) = 1$, *then* $H \cong H(n, 0, \xi)$ *or* $H \cong k\mathbb{D}$*;*
(c) *If* $\mathrm{io}(H) = \mathrm{im}(H) = n$, *then* $H \cong H(n, 1, \xi)$ *or* $H \cong B(n, \omega, \gamma)$*.*

In [5, Theorem 6.1], the case (c) in the above theorem is expressed in a more general and convenient form. For our purpose, we state the general form as follows.

Lemma 2.8. *Let H be an affine prime regular Hopf algebra of GK-dimension one. Assume that there exists an algebra homomorphism $\mu : H \to k$ such that $n :=$ PI-deg$(H) = |G_\mu^l|$ and $G_\mu^l \cap G_\mu^r = \{1\}$, where G_μ^l and G_μ^r are the groups of left and right winding automorphisms associated to μ. Then H is isomorphic as a Hopf algebra either to the Taft algebra $H(n, 1, \xi)$ or to the generalized Liu algebra $B(n, \omega, \gamma)$. As a consequence,* $\mathrm{io}(H) = \mathrm{im}(H)$*.*

3. Some combinatorial equations

We collect some combinatorial equations in this section. These equations turn out to be important for the following analysis.

Throughout this section, m, d are two natural numbers, γ is an mth primitive root of 1 and ξ an element in k satisfying $\xi^m = -1$. For each $i \in \mathbb{Z}$, ϕ_i is a polynomial defined by

$$\phi_i := 1 - \gamma^{-i-1}x^d.$$

Take t to be an arbitrary integer, define $\bar{t}$ to be the unique element in $\{0, 1, \dots, m-1\}$ satisfying $\bar{t} \equiv t \pmod m$. Then we have

$$\phi_t = \phi_{\bar{t}}$$

since $\gamma^m = 1$.

With this observation, we can use

$$]s, t[$$

to denote the resulted polynomial by omitting all items *from* $\phi_{\bar{s}}$ *to* $\phi_{\bar{t}}$ in $\phi_0\phi_1 \cdots \phi_{m-1}$, that is

$$]s, t[= \begin{cases} \phi_{\bar{t}+1} \cdots \phi_{m-1}\phi_0 \cdots \phi_{\bar{s}-1}, & \text{if } \bar{t} \geqslant \bar{s} \\ 1, & \text{if } \bar{s} = \bar{t} + 1 \\ \phi_{\bar{t}+1} \cdots \phi_{\bar{s}-1}, & \text{if } \bar{s} \geqslant \bar{t} + 2. \end{cases} \tag{3.1}$$

For example, $]-1, -1[=]m-1, m-1[= \phi_0\phi_1 \cdots \phi_{m-2}$.

To study equations with omitting items, the following formula is useful for us.

Lemma 3.1. *(See [8, Proposition IV.2.7].) Fix an invertible element q of the field k. For any scalar a we have*

$$(a - z)(a - qz) \cdots (a - q^{n-1}z) = \sum_{l=0}^{n} (-1)^l \binom{n}{l}_q q^{\frac{l(l-1)}{2}} a^{n-l} z^l.$$

Lemma 3.2. *With notions defined as above, we have*

$$\sum_{j=0}^{m-1} \gamma^{-j}\ ]j-1, j-1[\ = m x^{(m-1)d}.$$

Proof. By Lemma 3.1, $\phi_0 \cdots \phi_{m-1} = (1 - x^{dm})$. Note that

$$\sum_{j=0}^{m-1} \,]j-1, j-1[\, = \sum_{j=0}^{m-1} 1 + \gamma^{-j}x^d + \gamma^{-2j}x^{2d} + \cdots + \gamma^{-(m-1)j}x^{(m-1)d} = m.$$

So

$$\begin{aligned} &\sum_{j=0}^{m-1} \,]j-1, j-1[\, - \sum_{j=0}^{m-1} \gamma^{-j}x^d \,]j-1, j-1[\\ &= \sum_{j=0}^{m-1} (1 - \gamma^{-j}x^d) \,]j-1, j-1[\\ &= \sum_{j=0}^{m-1} \phi_0\phi_1 \cdots \phi_{m-1} \\ &= m(1 - x^{md}). \end{aligned}$$

Therefore, $\sum_{j=0}^{m-1} \gamma^{-j} \,]j-1, j-1[\, = mx^{(m-1)d}$. □

If we omit two items, then we have

Lemma 3.3.

$$\sum_{j=0}^{m-1} \gamma^{-j} \,]j-2, j-1[\, = 0.$$

Proof. By Lemma 3.1,

$$\begin{aligned}]j-2, j-1[&= (1 - \gamma^{-j-1}x^d)(1 - \gamma^{-(j+1)-1}x^d) \cdots (1 - \gamma^{-(m+j-3)-1}x^d) \\ &= \sum_{l=0}^{m-2} (-1)^l \binom{m-2}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l-1)}{2}} (\gamma^{-j-1}x^d)^l \\ &= \sum_{l=0}^{m-2} (-1)^l \binom{m-2}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2} - lj} x^{ld}. \end{aligned}$$

So, to get the result, it is sufficient to show that

$$\sum_{j=0}^{m-1} \gamma^{-j}\gamma^{-lj} = \sum_{j=0}^{m-1} \gamma^{-(l+1)j} = 0$$

for all $0 \leqslant l \leqslant m-2$. Since $1 \leqslant l+1 \leqslant m-1$, it is clear that $\sum_{j=0}^{m-1} \gamma^{-(l+1)j} = 0$. Thus the conclusion is established. □

As a direct consequence of this lemma, we get the following basic observation.

Corollary 3.4.

$$\sum_{j=0}^{m-1} \xi^{2j}\gamma^{-2j} \,]j-2, j-1[= 0$$

if and only if $\xi^2 = \gamma$, *where* ξ *is an element in* k *satisfying* $\xi^m = -1$.

Proof. Let $\theta = \xi^2\gamma^{-2}$, then θ is an mth root of 1. By the proof of the above lemma, it is sufficient to verify that $\theta\gamma^{-s} \neq 1$ for all $0 \leqslant s \leqslant m-2$. It follows that $\theta = \gamma^{-1}$, i.e., $\xi^2 = \gamma$. □

Lemma 3.5. *Fix* i *such that* $1 \leqslant i \leqslant m-1$ *and let* $1 \leqslant i' \leqslant i$. *Then*

$$\sum_{j=0}^{m-1} \gamma^{-i'j} \,]j-1-i, j-1[= 0.$$

Proof. Using Lemma 3.1, we have

$$\begin{aligned}
&]j-1-i, j-1[\\
&= \phi_j \cdots \phi_{m-1}\phi_0 \cdots \phi_{j-i-2}\\
&= (1-\gamma^{-j-1}x^d)(1-\gamma^{-(j+1)-1}x^d)\cdots(1-\gamma^{-(m+j-i-2)-1}x^d)\\
&= \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l-1)}{2}} (\gamma^{-j-1}x^d)^l\\
&= \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-lj} x^{ld}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{j=0}^{m-1} \gamma^{-i'j} \,]j-1-i, j-1[\\
&= \sum_{j=0}^{m-1} \gamma^{-i'j} \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-lj} x^{ld}\\
&= \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}} x^{ld} \sum_{j=0}^{m-1} \gamma^{-(i'+l)j}.
\end{aligned}$$

Since $0 \leqslant l \leqslant m-1-i$, $1 \leqslant i' \leqslant i'+l \leqslant m-1-i+i' \leqslant m-1$. Then we have $\sum_{j=0}^{m-1} \gamma^{-(i'+l)j} = 0$ for all $0 \leqslant l \leqslant m-1-i$ and $1 \leqslant i' \leqslant i$. This ends the proof. □

The next technical result is also needed.

Lemma 3.6. *Let $0 \leqslant t \leqslant i+j \leqslant m-1$, $0 \leqslant l \leqslant m-1-i-j$ and let q be a primitive mth root of 1. Then*

$$\begin{aligned} & q^{\frac{(l+t)(l+t+1)}{2}+t(i+j-t)} \cdot (-1)^{l+t} \binom{m-1-t}{l}_q \binom{m-1+t-i-j}{l+t}_q \\ & = \binom{i+j}{t}_q \binom{m-1-i-j}{l}_q . \end{aligned}$$

Proof. Since

$$\begin{aligned} & \binom{m-1-t}{l}_q \binom{m-1+t-i-j}{l+t}_q \\ & = \frac{(m-1-t)!_q}{(l)!_q(m-1-t-l)!_q} \cdot \frac{(m-1+t-i-j)!_q}{(l+t)!_q(m-1-l-i-j)!_q} \end{aligned}$$

and

$$\binom{i+j}{t}_q \binom{m-1-i-j}{l}_q = \frac{(i+j)!_q}{(t)!_q(i+j-t)!_q} \cdot \frac{(m-1-i-j)!_q}{(l)!_q(m-1-l-i-j)!_q},$$

we have

$$\begin{aligned} & \frac{\binom{m-1-t}{l}_q \binom{m-1+t-i-j}{l+t}_q}{\binom{i+j}{t}_q \binom{m-1-i-j}{l}_q} \\ & = \frac{(m-l-t)_q(m-l+1-t)_q \cdots (m-1-t)_q}{(t+1)_q(t+2)_q \cdots (t+l)_q} \\ & \quad \cdot \frac{(m-i-j)_q(m-i-j+1)_q \cdots (m-i-j+t-1)_q}{(i+j-t+1)_q(i+j-t+2)_q \cdots (i+j)_q}. \end{aligned}$$

Note that for every number c, $0 \leqslant c \leqslant m-1$,

$$\begin{aligned} (m-c)_q &= 1+q+\cdots+q^{m-1-c} \\ &= -(q^{m-c}+q^{m-c+1}\cdots+q^{m-1}) = -q^{m-c}(1+q+\cdots+q^{c-1}) \\ &= -q^{m-c}(c)_q. \end{aligned}$$

Thus

$$\frac{(m-l-t)_q(m-l+1-t)_q \cdots (m-1-t)_q}{(t+1)_q(t+2)_q \cdots (t+l)_q} = (-1)^l q^{-\frac{l(1+l+2t)}{2}}$$

and

$$\frac{(m-i-j)_q(m-i-j+1)_q\cdots(m-i-j+t-1)_q}{(i+j-t+1)_q(i+j-t+2)_q\cdots(i+j)_q}=(-1)^t q^{-\frac{t(1-t+2(i+j))}{2}}.$$

Therefore,

$$\begin{aligned}\frac{\binom{m-1-t}{l}_q\binom{m-1+t-i-j}{l+t}_q}{\binom{i+j}{t}_q\binom{m-1-i-j}{l}_q}&=(-1)^{l+t}q^{-\frac{l(1+l+2t)+t(1-t+2(i+j))}{2}}\\&=(-1)^{l+t}q^{-\frac{(l+t)(l+t+1)}{2}-t(i+j-t)}.\end{aligned}$$

This completes the proof. □

4. New examples

In this section, we will introduce a new class of algebras $D(m,d,\xi)$ and show that these algebras are prime regular Hopf algebras of GK-dimension one. Some properties about $D(m,d,\xi)$ are established, in particular, we will show that $D(m,d,\xi)$ is not a pointed Hopf algebra.

4.1. Definition of the Hopf algebra $D(m,d,\xi)$

As before, let m,d be two natural numbers satisfying that $(1+m)d$ is even and ξ a primitive $2m$th root of 1. Define

$$\omega := md, \quad \gamma := \xi^2.$$

• *The algebra structure.* As an algebra, $D(m,d,\xi)$ is generated by $x^{\pm1}, g^{\pm1}, y, u_0, u_1, \cdots, u_{m-1}$, subject to the following relations

$$\begin{gathered}xx^{-1}=x^{-1}x=1,\ gg^{-1}=g^{-1}g=1,\ xg=gx,\ xy=yx,\\ yg=\gamma gy,\ y^m=1-x^{\omega}=1-g^m,\end{gathered}\tag{4.1}$$

$$xu_i=u_ix^{-1},\ yu_i=\phi_iu_{i+1}=\xi x^du_iy,\ u_ig=\gamma^ix^{-2d}gu_i,\tag{4.2}$$

$$u_iu_j=\begin{cases}(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\phi_{i+1}\cdots\phi_{m-2-j}y^{i+j}g,\\ \quad \text{if } i+j\leqslant m-2,\\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}y^{i+j}g,\\ \quad \text{if } i+j=m-1,\\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\cdots\phi_{m-1}\phi_0\cdots\phi_{m-2-j}y^{i+j-m}g,\\ \quad \text{otherwise,}\end{cases}\tag{4.3}$$

where $\phi_i=1-\gamma^{-i-1}x^d$ and $0\leqslant i,j\leqslant m-1$.

Since $\gamma = \xi^2$, the expression $(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}$ in Equation (4.3) equals $(-1)^{-j}\xi^{j^2}$. We still use this expression $(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}$ because it is convenient for the further computations involving coproduct and antipode.

To give a unified expression for the last terrible relation (4.3), we have the following observations. On one hand, as observed at the beginning of Section 3, if we still define $\phi_t = 1 - \gamma^{-t-1}x^d$ for any $t \in \mathbb{Z}$, then

$$\phi_t = \phi_{\bar{t}},$$

where $\bar{t} \equiv t \pmod m$. For any $i, j \in \mathbb{Z}$, we have

$$]{-1-j}, i-1[\, = \begin{cases} \phi_{\bar{i}} \cdots \phi_{m-1}\phi_0 \cdots \phi_{m-2-\bar{j}}, & \text{if } \bar{i} + \bar{j} \geqslant m \\ 1, & \text{if } \bar{i} + \bar{j} = m-1 \\ \phi_{\bar{i}} \cdots \phi_{m-2-\bar{j}}, & \text{if } \bar{i} + \bar{j} \leqslant m-2, \end{cases}$$

by (3.1).

For the convenience of our later computations, the next notion is also useful for us,

$$[s, t] := \begin{cases} \phi_{\bar{s}}\phi_{\bar{s}+1} \cdots \phi_{\bar{t}}, & \text{if } \bar{t} \geqslant \bar{s} \\ 1, & \text{if } \bar{s} = \bar{t} + 1 \\ \phi_{\bar{s}} \cdots \phi_{m-1}\phi_0 \cdots \phi_{\bar{t}}, & \text{if } \bar{s} \geqslant \bar{t} + 2. \end{cases}$$

In fact, $[s, t]$ can be considered as the resulted polynomial (except the case $\bar{s} = \bar{t} + 1$) by preserving all items *from* $\phi_{\bar{s}}$ *to* $\phi_{\bar{t}}$ in $\phi_0\phi_1 \cdots \phi_{m-1}$. So, by definition, we have

$$[i, m-2-j] = \,]{-1-j}, i-1[. \tag{4.4}$$

On the other hand, we find that

$$(-1)^{-km-j}\xi^{-km-j}\gamma^{\frac{(km+j)(km+j+1)}{2}} = (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \tag{4.5}$$

for any $k \in \mathbb{Z}$. Therefore, if we define

$$u_s := u_{\bar{s}},$$

then the relation (4.3) can be replaced by

$$\begin{aligned} u_i u_j &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \,]{-1-j}, i-1[\, y^{\overline{i+j}} g \\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \, [i, m-2-j] \, y^{\overline{i+j}} g \end{aligned} \tag{4.6}$$

for all $i, j \in \mathbb{Z}$.

We give a bigrading on this algebra for use later. Define the following two algebra automorphisms of $D(m,d,\xi)$:

$$\Xi_\pi^l : \begin{cases} x \longmapsto x, \\ y \longmapsto y, \\ g \longmapsto \gamma^{-1}g, \\ u_i \longmapsto \xi^{-1}u_i, \end{cases} \quad \text{and} \quad \Xi_\pi^r : \begin{cases} x \longmapsto x, \\ y \longmapsto \gamma^{-1}y, \\ g \longmapsto \gamma^{-1}g, \\ u_i \longmapsto \xi^{-(2i+1)}u_i. \end{cases}$$

It is straightforward to show that Ξ_π^l and Ξ_π^r are indeed algebra automorphisms of $D(m,d,\xi)$ and these automorphisms have order $2m$ by noting that ξ is a primitive $2m$th root of 1 and $u_i \neq 0$ in $D(m,d,\xi)$ for all i (if one $u_i = 0$ in $D(m,d,\xi)$, then $y^{\overline{i+j}}g = 0$ by (4.3), which is absurd). Choosing $\zeta = \xi^{-1}$, define

$$D_i^l := \{h \in D(m,d,\xi) | \Xi_\pi^l(h) = \zeta^i h\}, \quad D_j^r := \{h \in D(m,d,\xi) | \Xi_\pi^r(h) = \zeta^j h\}$$

for $0 \leqslant i,j \leqslant 2m-1$. Direct computations show that

$$D_i^l = \begin{cases} k\langle x^{\pm 1}, y\rangle g^{\frac{i}{2}}, & i = \text{even}, \\ \sum_{s=0}^{m-1} k[x^{\pm 1}] g^{\frac{i-1}{2}} u_s, & i = \text{odd}, \end{cases}$$

and

$$D_j^r = \begin{cases} k\langle x^{\pm 1}, yg^{-1}\rangle g^{\frac{j}{2}}, & j = \text{even}, \\ \sum_{s=0}^{m-1} k[x^{\pm 1}] g^s u_{\frac{j-1}{2}-s}, & j = \text{odd}. \end{cases}$$

Therefore

$$D_{ij} := D_i^l \cap D_j^r = \begin{cases} k[x^{\pm 1}] y^{\frac{j-i}{2}} g^{\frac{i}{2}}, & i,j = \text{even}, \\ k[x^{\pm 1}] g^{\frac{i-1}{2}} u_{\frac{j-i}{2}}, & i,j = \text{odd}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.7}$$

Since $\sum_{i,j} D_{ij} = D(m,d,\xi)$, we have

$$D(m,d,\xi) = \bigoplus_{i,j=0}^{2m-1} D_{ij} \tag{4.8}$$

which is a bigrading on $D(m,d,\xi)$ automatically.

Let $D := D(m,d,\xi)$, then $D \otimes D$ is graded naturally by inheriting the grading defined above. In particular, for any $h \in D \otimes D$, we use

$$h_{(s_1,t_1)\otimes(s_2,t_2)}$$

to denote the homogeneous part of h in $D_{s_1,t_1} \otimes D_{s_2,t_2}$. This notion will be used freely in the proof of Proposition 4.2.

• *The coalgebra structure and the antipode.* The coproduct Δ, the counit ϵ and the antipode S of $D(m,d,\xi)$ are given by

$$\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g, \quad \Delta(y) = y \otimes g + 1 \otimes y,$$
$$\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j};$$
$$\epsilon(x) = \epsilon(g) = \epsilon(u_0) = 1, \quad \epsilon(y) = \epsilon(u_s) = 0;$$
$$S(x) = x^{-1}, \quad S(g) = g^{-1}, \quad S(y) = -yg^{-1},$$
$$S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id+\frac{3}{2}(1-m)d} g^{m-i-1} u_i, \tag{4.9}$$

for $0 \leq i \leq m-1$ and $1 \leqslant s \leqslant m-1$.

Since $g^m = x^{md}$ and (4.5), the definition about $S(u_i)$ still holds for any integer i, that is, (4.9) can be replaced in a more convenient way:

$$S(u_s) = (-1)^s \xi^{-s} \gamma^{-\frac{s(s+1)}{2}} x^{sd+\frac{3}{2}(1-m)d} g^{m-s-1} u_s \tag{4.10}$$

for all $s \in \mathbb{Z}$.

Remark 4.1. Recall that $\xi^2 = \gamma$. It is not hard to see that the subalgebra of $D(m,d,\xi)$ generated by $x^{\pm 1}, g^{\pm 1}, y$ is exact the generalized Liu algebra $B(m,\omega,\gamma)$. Indeed, by Equations (4.7) and (4.8), $B_{ij} = D_{2i,2j}$ for all $0 \leqslant i,j \leqslant m-1$. By definition, $D(m,d,\xi)$ is affine. Moreover, $D(m,d,\xi)$ is a finitely generated $k[x^{\pm 1}]$-module by Equation (4.7). Thus $D(m,d,\xi)$ has GK-dimension one. At the same time, note that $Z = k[z_s | z_s := x^s + x^{-s}, s \in \mathbb{N}_0]$ lies in the center of $D(m,d,\xi)$, which implies that $D(m,d,\xi)$ is PI. In one word, $D(m,d,\xi)$ is affine, PI and has GK-dimension one, and contains $B(m,\omega,\gamma)$ as a Hopf subalgebra.

4.2. $D(m,d,\xi)$ is a Hopf algebra

The main aim of this subsection is to show that $D(m,d,\xi)$ is indeed a Hopf algebra.

Proposition 4.2. *The algebra $D(m,d,\xi)$ defined above is a Hopf algebra.*

Proof. The proof is standard but not easy. For completeness and the convenience of the reader, we give the proof here. As usual, we decompose the proof into several steps. Since the subalgebra generated by $x^{\pm 1}, y, g$ is just the generalized Liu algebra $B(m,\omega,\gamma)$, which is a Hopf algebra already, we only need to verify the related relations in $D(m,d,\xi)$ where u_i are involved.

• *Step* 1 (Δ and ϵ are algebra homomorphisms).

First of all, it is clear that ϵ is an algebra homomorphism. Since x and g are group-like elements, the verifications of $\Delta(x)\Delta(u_i) = \Delta(u_i)\Delta(x^{-1})$ and $\Delta(u_i)\Delta(g) = \gamma^i\Delta(x^{-2d})\Delta(g)\Delta(u_i)$ are simple and so they are omitted.

(1) *The proof of* $\Delta(\phi_i)\Delta(u_{i+1}) = \Delta(y)\Delta(u_i) = \xi\Delta(x^d)\Delta(u_i)\Delta(y)$.

By definition $\Delta(u_i) = \sum_{j=0}^{m-1}\gamma^{j(i-j)}u_j \otimes x^{-jd}g^j u_{i-j}$ for all $0 \leqslant i \leqslant m-1$, we have

$$\begin{aligned}\Delta(\phi_i)\Delta(u_{i+1}) = (1\otimes 1 - \gamma^{-i-1}x^d \otimes x^d)\sum_{j=0}^{m-1}\gamma^{j(i+1-j)}u_j \otimes x^{-jd}g^j u_{i+1-j}\\ = \sum_{r=0}^{m-1}\gamma^{r(i+1-r)}u_r \otimes x^{-rd}g^r u_{i+1-r}\\ - \sum_{l=0}^{m-1}\gamma^{l(i+1-l)-i-1}x^d u_l \otimes x^{(1-l)d}g^j u_{i+1-l}.\end{aligned}$$

And

$$\begin{aligned}\Delta(y)\Delta(u_i) &= (y\otimes g + 1\otimes y)\sum_{j=0}^{m-1}\gamma^{j(i-j)}u_j \otimes x^{-jd}g^j u_{i-j}\\ &= \sum_{l=0}^{m-1}\gamma^{l(i-l)}yu_l \otimes x^{-ld}g^{l+1}u_{i-l}\\ &\quad + \sum_{r=0}^{m-1}\gamma^{r(i-r)}u_r \otimes yx^{-rd}g^r u_{i-r}\\ &= \sum_{l=0}^{m-1}\gamma^{l(i-l)}u_{l+1} \otimes x^{-ld}g^{l+1}u_{i-l}\\ &\quad - \sum_{l=0}^{m-1}\gamma^{l(i-l)-l-1}x^d u_{l+1} \otimes x^{-ld}g^{l+1}u_{i-l}\\ &\quad + \sum_{r=0}^{m-1}\gamma^{r(i-r+1)}u_r \otimes x^{-rd}g^r u_{i+1-r}\\ &\quad - \sum_{r=0}^{m-1}\gamma^{(r-1)(i+1-r)}u_r \otimes x^{(1-r)d}g^r u_{i+1-r}\\ &= \sum_{r=0}^{m-1}\gamma^{r(i-r+1)}u_r \otimes x^{-rd}g^r u_{i+1-r}\\ &\quad - \sum_{l=0}^{m-1}\gamma^{l(i-l)-l-1}x^d u_{l+1} \otimes x^{-ld}g^{l+1}u_{i-l}.\end{aligned}$$

Hence $\Delta(\phi_i)\Delta(u_{i+1}) = \Delta(y)\Delta(u_i)$. Similarly,

$$\begin{aligned}
\xi\Delta(x^d)\Delta(u_i)\Delta(y) &= \xi \cdot (x^d \otimes x^d)(\sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j})(y \otimes g + 1 \otimes y) \\
&= \sum_{s=0}^{m-1} \gamma^{s(i-s)} y u_s \otimes x^{(1-s)d} g^s u_{i-s} g \\
&\quad + \sum_{t=0}^{m-1} \gamma^{t(i-t)} x^d u_t \otimes x^{-td} g^t y u_{i-t} \\
&= \sum_{s=0}^{m-1} \gamma^{(s+1)(i-s)} u_{s+1} \otimes x^{-(s+1)d} g^{s+1} u_{i-s} \\
&\quad - \sum_{s=0}^{m-1} \gamma^{(s+1)(i-s-1)} x^d u_{s+1} \otimes x^{-(s+1)d} g^{s+1} u_{i-s} \\
&\quad + \sum_{t=0}^{m-1} \gamma^{t(i-t)} x^d u_t \otimes x^{-td} g^t u_{i+1-t} \\
&\quad - \sum_{t=0}^{m-1} \gamma^{(t-1)(i-t)-1} x^d u_t \otimes x^{(1-t)d} g^t u_{i+1-t} \\
&= \sum_{s=0}^{m-1} \gamma^{(s+1)(i-s)} u_{s+1} \otimes x^{-(s+1)d} g^{s+1} u_{i-s} \\
&\quad - \sum_{t=0}^{m-1} \gamma^{(t-1)(i-t)-1} x^d u_t \otimes x^{(1-t)d} g^t u_{i+1-t},
\end{aligned}$$

which equals $\Delta(\phi_i)\Delta(u_{i+1})$ clearly.

(2) *The proof of* $\Delta(u_i u_j) = \Delta(u_i)\Delta(u_j)$.

We have that

$$\begin{aligned}
\Delta(u_i)\Delta(u_j) &= \sum_{s=0}^{m-1} \gamma^{s(i-s)} u_s \otimes x^{-sd} g^s u_{i-s} \cdot \sum_{t=0}^{m-1} \gamma^{t(j-t)} u_t \otimes x^{-td} g^t u_{j-t} \\
&= \sum_{t=0}^{m-1} \sum_{s=0}^{m-1} \gamma^{s(i-s)} u_s \gamma^{(t-s)(j-t+s)} u_{t-s} \otimes x^{-sd} g^s u_{i-s} x^{-(t-s)d} g^{t-s} u_{j-t+s} \\
&= \sum_{t=0}^{m-1} \sum_{s=0}^{m-1} \gamma^{(t-s)(j-t+s)+(i-s)t} u_s u_{t-s} \otimes x^{-td} g^t u_{i-s} u_{j-t+s}.
\end{aligned}$$

By the bigrading given in (4.8), we can find that for each $0 \leqslant t \leqslant m-1$,

$$\sum_{s=0}^{m-1} \gamma^{(t-s)(j-t+s)+(i-s)t} u_s u_{t-s} \otimes x^{-td} g^t u_{i-s} u_{j-t+s} \in D_{2,2+2t} \otimes D_{2+2t,2+2(i+j)},$$

where the suffixes in $D_{2,2+2t} \otimes D_{2+2t,2+2(i+j)}$ are interpreted mod $2m$.

Note that

$$u_s u_{t-s} = (-1)^{-(t-s)} \xi^{-(t-s)} \gamma^{\frac{(t-s)(t-s+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \, [s, m-2-t+s] \, y^t g$$

and

$$\begin{aligned} u_{i-s} u_{j-t+s} = (-1)^{-(j-t+s)} \xi^{-(j-t+s)} \gamma^{\frac{(j-t+s)(j-t+s+1)}{2}} \\ \times \frac{1}{m} x^{-\frac{1+m}{2}d} \, [i-s, m-2+t-j-s] \, y^{\overline{i+j-t}} g. \end{aligned}$$

Using Lemma 3.1, we get

$$\begin{aligned} [s, m-2-t+s] &= (1-\gamma^{-s-1}x^d)(1-\gamma^{-s-2}x^d)\cdots(1-\gamma^{-(m-2-t+s)-1}x^d) \\ &= \sum_{l=0}^{m-1-t} (-1)^l \binom{m-1-t}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l-1)}{2}} (\gamma^{-s-1}x^d)^l \\ &= \sum_{l=0}^{m-1-t} (-1)^l \binom{m-1-t}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-sl} x^{ld}, \end{aligned}$$

and

$$\begin{aligned} &[i-s, m-2+t-j-s] \\ &= (1-\gamma^{-(i-s)-1}x^d)(1-\gamma^{-(i-s+1)-1}x^d)\cdots(1-\gamma^{-(i-s+m-2-\overline{i+j-t})-1}x^d) \\ &= \sum_{r=0}^{m-1-\overline{i+j-t}} (-1)^r \binom{m-1-\overline{i+j-t}}{r}_{\gamma^{-1}} \gamma^{-\frac{r(r-1)}{2}} (\gamma^{s-i-1}x^d)^r \\ &= \sum_{r=0}^{m-1-\overline{i+j-t}} (-1)^r \binom{m-1-\overline{i+j-t}}{r}_{\gamma^{-1}} \gamma^{-\frac{r(r+1)}{2}+(s-i)r} x^{rd}. \end{aligned}$$

Then for each $0 \leqslant t \leqslant m-1$,

$$\begin{aligned} &\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} \\ &= \sum_{s=0}^{m-1} \gamma^{(t-s)(j-t+s)+(i-s)t} u_s u_{t-s} \otimes x^{-td} g^t u_{i-s} u_{j-t+s} \\ &= \sum_{s=0}^{m-1} \gamma^{(t-s)(j-t+s)+(i-s)t} (-1)^{-(t-s)} \xi^{-(t-s)} \gamma^{\frac{(t-s)(t-s+1)}{2}} \end{aligned}$$

$$\times \frac{1}{m} x^{-\frac{1+m}{2}d} \, [s, m-2-t+s] \, y^t g$$
$$\otimes x^{-td} g^t (-1)^{-(j-t+s)} \xi^{-(j-t+s)} \gamma^{\frac{(j-t+s)(j-t+s+1)}{2}}$$
$$\times \frac{1}{m} x^{-\frac{1+m}{2}d} \, [i-s, m-2+t-j-s] \, y^{\overline{i+j-t}} g$$
$$= (-1)^{-j} \xi^{-j} \frac{1}{m^2} (\sum_{s=0}^{m-1} \gamma^{\frac{j^2+j}{2}+(i-s)t-t(i+j-t)} \, [s, m-2-t+s]$$
$$\otimes x^{-td} \, [i-s, m-2+t-j-s] \,)$$
$$(x^{-\frac{1+m}{2}d} y^t g \otimes x^{-\frac{1+m}{2}d} y^{\overline{i+j-t}} g^{t+1})$$
$$= (-1)^{-j} \xi^{-j} \frac{1}{m^2} \sum_{l=0}^{m-1-t} \sum_{r=0}^{m-1-\overline{i+j-t}} \gamma^{\frac{j(j+1)-l(l+1)-r(r+1)}{2} - t(j-t) - ir} (-1)^{l+r}$$
$$\times \binom{m-1-t}{l}_{\gamma^{-1}} \binom{m-1-\overline{i+j-t}}{r}_{\gamma^{-1}} \sum_{s=0}^{m-1} \gamma^{(r-l-t)s} \cdot (x^{ld} \otimes x^{(r-t)d})$$
$$\cdot (x^{-\frac{1+m}{2}d} y^t g \otimes x^{-\frac{1+m}{2}d} y^{\overline{i+j-t}} g^{t+1}).$$

Meanwhile, $u_i u_j = (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \, [i, m-2-j] \, y^{\overline{i+j}} g$. Since

$$\begin{aligned}
\Delta(y^{\overline{i+j}}) &= (1 \otimes y + y \otimes g)^{\overline{i+j}} \\
&= \sum_{t=0}^{\overline{i+j}} \binom{\overline{i+j}}{t}_{\gamma^{-1}} (1 \otimes y)^{\overline{i+j}-t} \cdot (y \otimes g)^t \\
&= \sum_{t=0}^{\overline{i+j}} \binom{\overline{i+j}}{t}_{\gamma^{-1}} y^t \otimes y^{\overline{i+j}-t} g^t
\end{aligned}$$

and

$$\begin{aligned}
\Delta([i, m-2-j]) &= (1 \otimes 1 - \gamma^{-i-1} x^d \otimes x^d)(1 \otimes 1 - \gamma^{-i-2} x^d \otimes x^d) \cdots \\
&\quad \times (1 \otimes 1 - \gamma^{-(i+m-2-\overline{i+j})-1} x^d \otimes x^d) \\
&= \sum_{l=0}^{m-1-\overline{i+j}} (-1)^l \binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l-1)}{2}} (\gamma^{-i-1} x^d \otimes x^d)^l \\
&= \sum_{l=0}^{m-1-\overline{i+j}} (-1)^l \binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2} - il} \cdot x^{ld} \otimes x^{ld},
\end{aligned}$$

we get

$$\begin{aligned}\Delta(u_iu_j) &= \Delta((-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\,[i,m-2-j]\,y^{\overline{i+j}}g)\\
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}\Delta(x^{-\frac{1+m}{2}d})\Delta(\,[i,m-2-j]\,)\Delta(y^{\overline{i+j}})\Delta(g)\\
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}\sum_{l=0}^{m-1-\overline{i+j}}(-1)^l\binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}}\gamma^{-\frac{l(l+1)}{2}-il}\\
&\quad\times\sum_{t=0}^{\overline{i+j}}\binom{\overline{i+j}}{t}_{\gamma^{-1}}(x^{-\frac{1+m}{2}d}\otimes x^{-\frac{1+m}{2}d})\cdot(x^{ld}\otimes x^{ld})\\
&\quad\cdot(y^t\otimes y^{\overline{i+j-t}}g^t)\cdot(g\otimes g)\\
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}\sum_{t=0}^{\overline{i+j}}\cdot\sum_{l=0}^{m-1-\overline{i+j}}(-1)^l\binom{\overline{i+j}}{t}_{\gamma^{-1}}\binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}}\\
&\quad\times\gamma^{-\frac{l(l+1)}{2}-il}(x^{ld}\otimes x^{ld})\cdot(x^{-\frac{1+m}{2}d}y^tg\otimes x^{-\frac{1+m}{2}d}y^{\overline{i+j-t}}g^{t+1}).\end{aligned}$$

For each $0\leqslant t\leqslant\overline{i+j}$, $(x^{ld}\otimes x^{ld})\cdot(x^{-\frac{1+m}{2}d}y^tg\otimes x^{-\frac{1+m}{2}d}y^{\overline{i+j-t}}g^{t+1})\in D_{2,2+2t}\otimes D_{2+2t,2+2(i+j)}$ for any l. So,

$$\begin{aligned}&\Delta(u_iu_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))}\\
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}\sum_{l=0}^{m-1-\overline{i+j}}(-1)^l\binom{\overline{i+j}}{t}_{\gamma^{-1}}\binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}}\gamma^{-\frac{l(l+1)}{2}-il}\\
&\quad\times(x^{ld}\otimes x^{ld})\cdot(x^{-\frac{1+m}{2}d}y^tg\otimes x^{-\frac{1+m}{2}d}y^{\overline{i+j-t}}g^{t+1}).\end{aligned}$$

By the graded structure of $D\otimes D$, $\Delta(u_i)\Delta(u_j)=\Delta(u_iu_j)$ if and only if

$$\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))}=0 \tag{4.11}$$

for all $\overline{i+j}+1\leqslant t\leqslant m-1$ and

$$\Delta(u_iu_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))}=\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} \tag{4.12}$$

for all $0\leqslant t\leqslant\overline{i+j}$.

By the expression of $\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))}$, we can find that it is zero if $\sum_{s=0}^{m-1}\gamma^{(r-l-t)s}=0$. Note that in the case of $\overline{i+j}+1\leqslant t\leqslant m-1$, $m-1-\overline{i+j-t}=t-1-\overline{i+j}$. So $0\leqslant r\leqslant t-1-\overline{i+j}$ and thus $1-m\leqslant r-l-t\leqslant -1-\overline{i+j}$. This means that in this case we always have

$$\sum_{s=0}^{m-1}\gamma^{(r-l-t)s}=0,$$

which implies (4.11).

Now let $0 \leqslant t \leqslant \overline{i+j}$. Then $1-m \leqslant r-l-t \leqslant m-1-\overline{\overline{i+j}-t}-t < m$. As discussed above, $\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} = 0$ if $r-l-t \neq 0$. So we only need to verify the case when $r = l+t$. At this time, $0 \leqslant l \leqslant m-1-\overline{i+j}$. Then (4.12) holds if and only if

$$\gamma^{-\frac{(l+t)(l+t+1)}{2}-t(i+j-t)} \cdot (-1)^{l+t} \binom{m-1-t}{l}_{\gamma^{-1}} \binom{m-1-\overline{i+j-t}}{l+t}_{\gamma^{-1}}$$
$$= \binom{\overline{i+j}}{t}_{\gamma^{-1}} \binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}},$$

which is just Lemma 3.6 by setting $q = \gamma^{-1}$.

- *Step* 2 (Coassociative and counit.)
  Indeed, for each $0 \leqslant i \leqslant m-1$

$$\begin{aligned}(\Delta \otimes \mathrm{Id})\Delta(u_i) &= (\Delta \otimes \mathrm{Id})(\sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j}) \\ &= \sum_{j=0}^{m-1} \gamma^{j(i-j)} (\sum_{s=0}^{m-1} \gamma^{s(j-s)} u_s \otimes x^{-sd} g^s u_{j-s}) \otimes x^{-jd} g^j u_{i-j} \\ &= \sum_{j,s=0}^{m-1} \gamma^{j(i-j)+s(j-s)} u_s \otimes x^{-sd} g^s u_{j-s} \otimes x^{-jd} g^j u_{i-j},\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)\Delta(u_i) &= (\mathrm{Id} \otimes \Delta)(\sum_{s=0}^{m-1} \gamma^{s(i-s)} u_s \otimes x^{-sd} g^s u_{i-s}) \\ &= \sum_{s=0}^{m-1} \gamma^{s(i-s)} u_s \otimes (\sum_{t=0}^{m-1} \gamma^{t(i-s-t)} x^{-sd} g^s u_t \otimes x^{-sd} g^s x^{-td} g^t u_{i-s-t}) \\ &= \sum_{s,t=0}^{m-1} \gamma^{s(i-s)+t(i-s-t)} u_s \otimes x^{-sd} g^s u_t \otimes x^{-(s+t)d} g^{(s+t)} u_{i-s-t}.\end{aligned}$$

It is not hard to see that $(\Delta \otimes \mathrm{Id})\Delta(u_i) = (\mathrm{Id} \otimes \Delta)\Delta(u_i)$ for all $0 \leqslant i \leqslant m-1$. The verification of $(\epsilon \otimes \mathrm{Id})\Delta(u_i) = (\mathrm{Id} \otimes \epsilon)\Delta(u_i) = u_i$ is easy and it is omitted.

- *Step* 3 (Antipode is an algebra anti-homomorphism.)
  Because x and g are group-like elements, we only check

$$S(u_{i+1})S(\phi_i) = S(u_i)S(y) = \xi S(y)S(u_i)S(x^d)$$

and

$$S(u_i u_j) = S(u_j)S(u_i)$$

here.

(1) *The proof of* $S(u_{i+1})S(\phi_i) = S(u_i)S(y) = \xi S(y)S(u_i)S(x^d)$.

Since $u_i S(\phi_j) = \phi_j u_i$ for all i, j,

$$\begin{aligned} S(u_{i+1})S(\phi_i) &= (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d+\frac{3}{2}(1-m)d}g^{m-i-2}u_{i+1}S(\phi_i) \\ &= \phi_i S(u_{i+1}). \end{aligned}$$

Through direct calculation, we have

$$\begin{aligned} S(u_i)S(y) &= (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-i-1}u_i \cdot (-yg^{-1}) \\ &= (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{i(i+1)}{2}}x^{(i-1)d+\frac{3}{2}(1-m)d}g^{m-i-1}yu_i g^{-1} \\ &= \phi_i \cdot (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d+\frac{3}{2}(1-m)d}g^{m-i-2}u_{i+1} \\ &= \phi_i S(u_{i+1}), \end{aligned}$$

and

$$\begin{aligned} \xi S(y)S(u_i)S(x^d) &= -\xi yg^{-1}\cdot(-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-i-1}u_i x^{-d} \\ &= (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d+\frac{3}{2}(1-m)d}g^{m-i-2}yu_i \\ &= \phi_i S(u_{i+1}). \end{aligned}$$

(2) *The proof of* $S(u_i u_j) = S(u_j)S(u_i)$.

Define $\overline{\phi_s} := 1 - \gamma^{-s-1}x^{-d}$ for all $s \in \mathbb{Z}$. Using this notion,

$$x^d\overline{\phi_s} = x^d(1-\gamma^{-s-1}x^{-d}) = -\gamma^{-s-1}(1-\gamma^{-(m-s-2)-1}x^d) = -\gamma^{-s-1}\phi_{m-s-2}.$$

And so

$$\begin{aligned} S(u_i u_j) &= S((-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\,[i, m-2-j]\,y^{\overline{i+j}}g) \\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}S(g)S(y^{i+j})S([i, m-2-j])S(x^{-\frac{1+m}{2}d}) \\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}g^{-1}(-yg^{-1})^{\overline{i+j}}S([i, m-2-j])x^{\frac{1+m}{2}d} \\ &= (-1)^{\overline{i+j}-j}\xi^{-j}\gamma^{\frac{j(j+1)+(\overline{i+j})(\overline{i+j}+1)}{2}}\frac{1}{m}x^{\frac{1+m}{2}d}S([i, m-2-j])y^{\overline{i+j}}g^{-(\overline{i+j}+1)} \\ &= (-1)^{m-1-j}\xi^{-j}\gamma^{\frac{j(j+1)+(\overline{i+j})(\overline{i+j}+1)+(m-1-\overline{i+j})(-m-2i+\overline{i+j})}{2}} \\ &\quad \times \frac{1}{m}x^{\frac{1+m}{2}d-(m-1-\overline{i+j})d}[j, m-2-i]y^{\overline{i+j}}g^{-(\overline{i+j}+1)} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j^2+j}{2}+i(\overline{i+j}+1)} \\
&\quad\times \frac{1}{m}x^{\frac{1+m}{2}d-(m-1-\overline{i+j})d}\ [j,m-2-i]\ y^{\overline{i+j}}g^{-(\overline{i+j}+1)} \\
&= (-1)^{j}\xi^{-j}\gamma^{\frac{j^2+j}{2}+i(i+j+1)}\frac{1}{m}x^{\frac{3-m}{2}d+(i+j)d}\ [j,m-2-i]\ y^{\overline{i+j}}g^{-(i+j+1)},
\end{aligned}
$$

$$
\begin{aligned}
S(u_j)S(u_i) &= (-1)^{j}\xi^{-j}\gamma^{-\frac{j(j+1)}{2}}x^{jd+\frac{3}{2}(1-m)d}g^{m-j-1}u_j \\
&\quad\cdot(-1)^{i}\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-i-1}u_i \\
&= (-1)^{i+j}\xi^{-i-j}\gamma^{-\frac{i(i+1)+j(j+1)}{2}}x^{(j-i)d}g^{m-j-1}u_jg^{m-i-1}u_i \\
&= (-1)^{i+j}\xi^{-i-j}\gamma^{-\frac{i(i+1)+j(j+1)}{2}-j(i+1)}x^{(i+j+2)d}g^{-(i+j+2)}u_ju_i \\
&= (-1)^{j}\xi^{-2i-j}\gamma^{-\frac{j(j+1)}{2}-j(i+1)+(i+j)(i+j+2)} \\
&\quad\times \frac{1}{m}x^{-\frac{1+m}{2}d+(i+j+2)d}\ [i,m-2-j]\ y^{\overline{i+j}}g^{-(i+j+1)} \\
&= (-1)^{j}\xi^{-j}\gamma^{\frac{j^2+j}{2}+i(i+j+1)}\frac{1}{m}x^{\frac{3-m}{2}d+(i+j)d}\ [i,m-2-j]\ y^{\overline{i+j}}g^{-(i+j+1)}.
\end{aligned}
$$

The proof is done.

- *Step* 4 ($(S*\mathrm{Id})(u_i) = (\mathrm{Id}*S)(u_i) = \epsilon(u_i)$).
  In fact,

$$
\begin{aligned}
(S*\mathrm{Id})(u_0) &= \sum_{j=0}^{m-1} S(\gamma^{-j^2}u_j)x^{-jd}g^ju_{-j} \\
&= \sum_{j=0}^{m-1}\gamma^{-j^2}(-1)^j\xi^{-j}\gamma^{-\frac{j(j+1)}{2}}x^{jd+\frac{3}{2}(1-m)d}g^{m-j-1}u_jx^{-jd}g^ju_{-j} \\
&= x^{\frac{3}{2}(1-m)d}g^{m-1}(\sum_{j=0}^{m-1}(-1)^j\xi^{-j}\gamma^{-\frac{j(j+1)}{2}}u_ju_{-j}) \\
&= x^{\frac{3}{2}(1-m)d}g^{m-1}(\sum_{j=0}^{m-1}\gamma^{-j}\frac{1}{m}x^{-\frac{1+m}{2}d}\ [j,m-2+j]\ g) \\
&= \frac{1}{m}x^{(1-m)d}(\sum_{j=0}^{m-1}\gamma^{-j}\ ]j-1,j-1[\ ) \\
&= 1 \quad (\text{by Lemma 3.2}) \\
&= \epsilon(u_0).
\end{aligned}
$$

And,

$$\begin{aligned}
(\mathrm{Id} * S)(u_0) &= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j S(x^{-jd} g^j u_{-j}) \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j S(u_{-j}) S(g^j) x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j (-1)^{-j} \xi^j \gamma^{\frac{j(-j+1)}{2}} x^{-jd+\frac{3}{2}(1-m)d} g^{m+j-1} u_{-j} g^{-j} x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j (-1)^{-j} \xi^j \gamma^{\frac{j-j^2}{2}+j^2} x^{\frac{3}{2}(1-m)d} g^{m-1} u_{-j} \\
&= x^{\frac{1-m}{2}d} g^{m-1} \sum_{j=0}^{m-1} (-1)^{-j} \xi^j \gamma^{-\frac{j^2+j}{2}} u_j u_{-j} \\
&= \frac{1}{m} \sum_{j=0}^{m-1} \xi^{2j} \gamma^{-j}\ [j, m-2+j] \\
&= \frac{1}{m} \cdot \sum_{j=0}^{m-1}\ ]j-1, j-1[\\
&= 1 \quad \text{(by the proof of Lemma 3.2)} \\
&= \epsilon(u_0).
\end{aligned}$$

For $1 \leqslant i \leqslant m-1$,

$$\begin{aligned}
(S * \mathrm{Id})(u_i) &= \sum_{j=0}^{m-1} \gamma^{j(i-j)} S(u_j) x^{-jd} g^j u_{i-j} \\
&= \sum_{j=0}^{m-1} \gamma^{j(i-j)} (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} x^{jd+\frac{3}{2}(1-m)d} g^{m-1-j} u_j x^{-jd} g^j u_{i-j} \\
&= x^{\frac{3}{2}(1-m)d} g^{m-1} (\sum_{j=0}^{m-1} (-1)^j \xi^{-j} \gamma^{ij-\frac{j(j+1)}{2}} u_j u_{i-j}) \\
&= (-1)^{-i} \xi^{-i} \gamma^{\frac{i(i+3)}{2}} \frac{1}{m} x^{(1-m)d} y^i (\sum_{j=0}^{m-1} \gamma^{-j}\ [j, m-2-i+j]) \\
&= (-1)^{-i} \xi^{-i} \gamma^{\frac{i(i+3)}{2}} \frac{1}{m} x^{(1-m)d} y^i (\sum_{j=0}^{m-1} \gamma^{-j}\ ]j-1-i, j-1[) \\
&= 0 \quad \text{(by Lemma 3.5)} \\
&= \epsilon(u_i),
\end{aligned}$$

$$
\begin{aligned}
(\mathrm{Id} * S)(u_i) &= \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j S(u_{i-j}) g^{-j} x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j (-1)^{i-j} \xi^{j-i} \gamma^{-\frac{(i-j)(i-j+1)}{2}} x^{(i-j)d+\frac{3}{2}(1-m)d} \\
&\quad \times g^{m+j-i-1} u_{i-j} g^{-j} x^{jd} \\
&= x^{id+\frac{1-m}{2}d} g^{m-1-i} (\sum_{j=0}^{m-1} (-1)^{i-j} \xi^{j-i} \gamma^{-\frac{i(i+1)+j(j+1)}{2}} u_j u_{i-j}) \\
&= \xi^{-2i} \gamma^{i(i+1)} \frac{1}{m} x^{(i-m)d} y^i g^{m-i} (\sum_{j=0}^{m-1} \xi^{2j} \gamma^{-(i+1)j} [j, m-2-i+j]) \\
&= \xi^{-2i} \gamma^{i(i+1)} \frac{1}{m} x^{(i-m)d} y^i g^{m-i} (\sum_{j=0}^{m-1} \gamma^{-ij} \ [j, m-2-i+j]) \\
&= \xi^{-2i} \gamma^{i(i+1)} \frac{1}{m} x^{(i-m)d} y^i g^{m-i} (\sum_{j=0}^{m-1} \gamma^{-ij} \ ]j-1-i, j-1[) \\
&= 0 \quad (\text{by Lemma 3.5}) \\
&= \epsilon(u_i).
\end{aligned}
$$

By steps 1, 2, 3, 4, $D(m, d, \xi)$ is a Hopf algebra. □

4.3. Properties of $D(m, d, \xi)$

For short, let $D := D(m, d, \xi)$. For this Hopf algebra, the following observation is not hard.

Lemma 4.3. *$D(m, d, \xi)$ is PI, affine and has GK-dimension one.*

Proof. See Remark 4.1. □

Lemma 4.4. gl.dim$D(m, d, \xi) = 1$.

Proof. Now we find that $D = \bigoplus_{0 \leqslant i \leqslant 2m-1} D_i^l$ satisfies all conditions stated in [5, Proposition 5.1] and thus by [5, Proposition 5.1(a)] every nonzero homogeneous element is regular (and thus a nonzero-divisor). In particular, $y \in D_0^l$ is a regular element of D. It is not hard to see that (y) is a Hopf ideal. Let $D' := D/(y)$. Then D' is a finite dimensional Hopf algebra. We will show that D' is semisimple at first. For notational convenience, the images of x, g, u_i in D' are still written as x, g and u_i. One can check that

$$\int_{D'}^{l} := \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j + \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0$$

is a non-zero left integral of D'. Indeed, it is not hard to see that $x\int_{D'}^{l} = g\int_{D'}^{l} = \int_{D'}^{l}$ and the following relations

$$x^{md} = g^m = 1,\ u_j u_0 = 0,\ u_i = \gamma^{-i}x^d u_i,\ xu_i = u_i x^{-1},\ u_i g = \gamma^i x^{-2d} g u_i$$
$$u_0^2 = \frac{1}{m} x^{-\frac{1+m}{2}d}\phi_0 \cdots \phi_{m-2} g$$

hold in D' for all $0 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant m-1$. Here we only explain the equation $u_i = \gamma^{-i}x^d u_i$ since the others are clear. Since $(1-\gamma^{-i}x^d)u_i = \phi_{i-1}u_i = yu_{i-1} = 0$ in D', we have $u_i = \gamma^{-i}x^d u_i$. Therefore,

$$\begin{aligned} u_0 \cdot \int_{D'}^{l} &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0 + \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0^2 \\ &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0 + \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j \\ &= \epsilon(u_0)\cdot \int_{D'}^{l}, \end{aligned}$$

and

$$\begin{aligned} u_s \cdot \int_{D'}^{l} &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} \gamma^{sj} x^i g^j u_s \\ &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} \gamma^{sj} x^i g^j \gamma^{-s} x^d u_s \\ &= \gamma^{-s}\sum_{i=0}^{md-1}\sum_{j=0}^{m-1} \gamma^{sj} x^i g^j u_s, \end{aligned}$$

which implies that $\sum_{i=0}^{md-1}\sum_{j=0}^{m-1} \gamma^{sj} x^i g^j u_s = 0$, and so $u_s \cdot \int_{D'}^{l} = 0 = \epsilon(u_s)\int_{D'}^{l}$ for all $1 \leqslant s \leqslant m-1$. Clearly, $\epsilon(\int_{D'}^{l}) = 2m^2 d \neq 0$. So D' is semisimple.

Secondly, let M be the trivial D-module k. By [12] or [4, Corollary 1.4], it is enough to show that $\mathrm{p.dim}_D M = 1$. Since y is a regular element (i.e., not a zero divisor) of D and $yM = 0$, M cannot be a submodule of a free D-module, which implies $\mathrm{p.dim}_D M \neq 0$. Since D' is semisimple, we get $\mathrm{p.dim}_{D'} M = 0$. By standard method of “change of rings” (see e.g., [9, Lemma 5.26]), $\mathrm{p.dim}_D M = 1$. □

Lemma 4.5. $\mathrm{io}(D) = 2m, \mathrm{im}(D) = m$.

Proof. By the proof of Lemma 4.4 D' is semisimple, then D' is unimodular. Note that the left and right homological integrals agree with the classical left and right integrals respectively when the Hopf algebra is of finite dimensional. So D' is also unimodular for homological integrals, that is, the left homological integral of D', also denoted by $\int_{D'}^l$, is isomorphic to k as D'-bimodule. Thus we have $\int_{D'}^l = D'/(x-1, g-1, u_0-1, u_1, \cdots, u_{m-1})$. Using [13, Lemma 2.6],

$$\int_D^l = (\int_{D'}^l)^{\tau^{-1}} = D/(y, x-1, g-\gamma^{-1}, u_0-\xi^{-1}, u_1, u_2, \cdots, u_{m-1}),$$

where τ is the algebra automorphism of D such that $yh = \tau(h)y$ for all $h \in D$. The corresponding homomorphism π yields left and right winding automorphisms

$$\Xi_\pi^l : \begin{cases} x \longmapsto x, \\ y \longmapsto y, \\ g \longmapsto \gamma^{-1} g, \\ u_i \longmapsto \xi^{-1} u_i, \end{cases} \quad \text{and} \quad \Xi_\pi^r : \begin{cases} x \longmapsto x, \\ y \longmapsto \gamma^{-1} y. \\ g \longmapsto \gamma^{-1} g, \\ u_i \longmapsto \xi^{-(2i+1)} u_i. \end{cases}$$

Clearly these automorphisms have order $2m$ and $G_\pi^l \cap G_\pi^r = \langle (\Xi_\pi^l)^m \rangle$. Then we have $\mathrm{io}(D) = 2m$ and $\mathrm{im}(D) = m$ by $|G_\pi^l \cap G_\pi^r| = 2$. □

Remark 4.6. These winding automorphisms are just the automorphisms constructed in Subsection 4.1 and the corresponding gradings on D have been exhibited there.

Lemma 4.7. *$D(m, d, \xi)$ is prime.*

Proof. We know that G_π^l is a finite abelian group acting faithfully on $D(m, d, \xi)$ (see the proof of Lemma 4.5). Moreover, the $\widehat{G_\pi^l}$-grading is strong and $D_0^l = k[x^{\pm 1}, y]$ is a commutative domain, which shows that D meets the initial conditions of [5, Proposition 5.1]. It follows that PI-deg$(D) \leqslant \mathrm{io}(D) = 2m$. Since D is regular, we have $\mathrm{io}(D) \leqslant$ PI-deg(D/P_0) by [13, Lemma 5.3], where P_0 is the minimal prime ideal of D contained in Ker(ϵ). It is clear that PI-deg $(D/P_0) \leqslant$ PI-deg(D). So PI-deg$(D) = \mathrm{io}(D) = 2m$ and thus [5, Proposition 5.1 (d)] applied. Therefore D is prime. □

The next proposition is a direct consequence of Lemmas 4.3, 4.5, 4.4 and 4.7.

Proposition 4.8. *$D(m, d, \xi)$ is an affine prime regular Hopf algebra of GK-dimension one with* $\mathrm{io}(D) = 2m, \mathrm{im}(D) = m$.

At the end of this section, we show that

Proposition 4.9. *D is not pointed.*

Proof. Let $f : D \to D' = D/(y)$ be the canonical Hopf epimorphism. We need to show that $u_i \neq 0$ in D' for all i firstly. Since y is a normal element of D, $u_i = 0$ in D' forces $u_i = yz_i$ for some z_i in D. By the bigrading structure of D and the fact the y is nonzero-divisor, one can see that $z_i = \alpha_i u_{i-1}$ for some $\alpha_i \in D_{00} = k[x^{\pm 1}]$. Therefore,

$$u_i = yz_i = y\alpha_i u_{i-1} = \alpha_i y u_{i-1} = \alpha_i \phi_{i-1} u_i,$$

which contradicts the fact that $\phi_{i-1} = 1 - \gamma^{-i}x^d$ is not invertible in $[x^{\pm 1}]$. Secondly, by [15, Corollary 5.3.5], D' is pointed if D is pointed. We will show that D' is not pointed. Otherwise, it is semisimple and pointed and so it is cosemisimple and pointed by [10]. This implies that D' is cocommutative which is absurd. Therefore, D is not pointed. □

A direct consequence of this proposition is that $D(m, d, \xi)$ is not isomorphic to any one of Hopf algebras listed in Subsection 2.3.

Corollary 4.10. *As a Hopf algebra, $D(m, d, \xi)$ is not isomorphic to any one of algebras listed in Subsection 2.3.*

Proof. All of Hopf algebras given in Subsection 2.3 are pointed while $D(m, d, \xi)$ is not. □

Remark 4.11. In practice, the assumption "pointed" is always added when we want to classify Hopf algebras of lower GK-dimensions. As a matter of fact, all known examples are pointed and it is widely believed that, at least for prime regular Hopf algebras of GK-dimension one, these Hopf algebras should be pointed automatically. Our new examples will change this naive understanding since all the new examples are not pointed!

5. The Hopf subalgebra $\widetilde{H}$

From now on, H always denotes a prime regular Hopf algebra of GK-dimension one with $\mathrm{io}(H) = n > \mathrm{im}(H) = m > 1$ and let $t := n/m$. The aim of this section is to construct a Hopf subalgebra $\widetilde{H}$ of H satisfying the conditions of Lemma 2.8.

Recall that the letter ζ denotes the primitive nth root of 1 in the definition of the bigrading of H given in Subsection 2.2:

$$H = \bigoplus_{0 \leqslant i,j \leqslant n-1} H_{ij},$$

where $H_{ij} = H_i^l \cap H_j^r$.

It is not hard to find that [5, Lemma 6.3] still holds in our case and so we state it without proof.

Lemma 5.1. *Let* $H = \bigoplus_{0\leqslant i,j\leqslant n-1} H_{ij}$. *Then*
(a) $S(H_i^l) = H_{-i}^r$ *and* $S(H_{ij}) = H_{-j,-i}$ *(where the suffixes are interpreted* $\bmod n$*).*
(b) *If* $i \neq j$, *then* $\epsilon(H_{ij}) = 0$.
(c) $\epsilon(H_{ii}) \neq 0$.

Lemma 5.2. *Let* Ξ_π^l *and* Ξ_π^r *be the left and right winding automorphisms defined as in Subsection 2.2 respectively. Then* $(\Xi_\pi^l)^m = (\Xi_\pi^r)^m$.

Proof. Since G_π^l and G_π^r are cyclic groups, $G_\pi^l \cap G_\pi^r$ is a cyclic subgroup of both G_π^l and G_π^r. By $\mathrm{im}(H) = |G_\pi^l/G_\pi^l \cap G_\pi^r| = m$, the order of $G_\pi^l \cap G_\pi^r$ is t. But the order t cyclic subgroup of G_π^l and G_π^r is just $\langle(\Xi_\pi^l)^m\rangle$ and $\langle(\Xi_\pi^r)^m\rangle$. So there exists an integer s, $1 \leqslant s \leqslant t-1$, such that $(\Xi_\pi^l)^m = (\Xi_\pi^r)^{sm}$. Now we show that $s = 1$. Since $\epsilon(H_{11}) \neq 0$ by Lemma 5.1, we have $H_{11} \neq 0$. Then there exists $0 \neq x \in H_{11} = \{x \in H | \Xi_\pi^l(x) = \zeta x, \Xi_\pi^r(x) = \zeta x\}$, so

$$\zeta^m x = (\Xi_\pi^l)^m(x) = (\Xi_\pi^r)^{sm}(x) = \zeta^{sm} x.$$

Therefore $\zeta^m = \zeta^{sm}$, and hence $s = 1$. □

Lemma 5.3. *For every* j *with* $1 \leqslant j \leqslant n-1$, $H_{0j} \neq 0$ *if and only if* $j \equiv 0 \pmod t$ *for all* $0 \leqslant j \leqslant n-1$.

Proof. Let $0 \neq x \in H_{0j}$. Note that $H_{0j} = \{x \in H | \Xi_\pi^l(x) = x, \Xi_\pi^r(x) = \zeta^j x\}$. By the above lemma, we have

$$x = (\Xi_\pi^l)^m(x) = (\Xi_\pi^r)^m(x) = \zeta^{jm} x.$$

This implies $\zeta^{jm} = 1$. So we get $j \equiv 0 \pmod t$. That is, if $j \not\equiv 0 \pmod t$ then $H_{0j} = 0$. Therefore we can write

$$H_0^l = \bigoplus_{0\leqslant j\leqslant m-1} H_{0,jt}.$$

Now it remains to show that each $H_{0,jt} \neq 0$ for all $0 \leqslant j \leqslant m-1$.

Let s be the minimum positive integer such that $H_{0,st} \neq 0$. Such s must exist. Indeed, if there is no such s, that is, $H_{0,jt} = 0$ for all $1 \leqslant j \leqslant m-1$. Then $H_0^l = H_0$. Dually, $H_0^r = H_0 = H_0^l$. Therefore H_0^l is a Hopf subalgebra of H by Lemma 2.5 again. By the proofs of [5, Propositions 4.2 and 4.3], we see that this can happen only in the following cases: H is isomorphic as a Hopf algebra either to the Taft algebra $H(n, 0, \xi)$ or to the infinite dihedral group algebra $k\mathbb{D}$. In either case, $\mathrm{im}(H) = 1$. This contradicts the hypothesis $\mathrm{im}(H) > 1$. Next, we show that s must be a factor of m, that is, $s|m$. If not, there exists a positive number a such that $sa < m$ and $s(a+1) > m$. Since H_0^l is a domain, $0 \neq (H_{0,st})^{a+1} \subset H_{0,st(a+1)-mt}$ and thus $s(a+1) - m < s$ is a smaller number such that $H_{0,(s(a+1)-m)t} \neq 0$ which contradicts to the minimality of s. Therefore, $H_0^l = \bigoplus_{0 \leqslant j \leqslant \frac{m}{s}-1} H_{0,jst}$ (by using the minimality of s again) and thus to show the result it is enough to show that $s = 1$.

We claim that

$$H = \bigoplus_{0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant \frac{m}{s}-1} H_{i,i+jst}. \qquad (*)$$

At first, since $H_{ii} \neq 0$ for all i by Lemma 5.1 and every nonzero homogeneous element is nonzero-divisor by [5, Proposition 5.1(a)], we have $H_{i,i+jst} \supseteq H_{ii}H_{0,jst} \neq 0$ for all $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant \frac{m}{s} - 1$. Secondly, we show that $H_{ik} \neq 0$ if and only if $st|(i-k)$. We already know the "if" part and we only need to show the "only if" part. In fact, suppose $H_{ik} \neq 0$ and let $0 \neq x \in H_{ik} = \{h \in H | \Xi_\pi^l(h) = \zeta^i h, \Xi_\pi^r(h) = \zeta^k h\}$. Applying Lemma 5.2, we have

$$\zeta^{im} x = (\Xi_\pi^l)^m(x) = (\Xi_\pi^r)^m(x) = \zeta^{km} x.$$

That is, $\zeta^{im} = \zeta^{km}$ and so $i - k \equiv 0 \pmod t$. So it is harmless to assume that $k = i + at$ for a a positive number and we need to show that $s|a$. Applying [5, Proposition 5.1(a)] again, $H_{0,at} \supseteq H_{i,i+at}H_{n-i,n-i} \neq 0$ and thus $s|a$ since we already know that $H_0^l = \bigoplus_{0 \leqslant j \leqslant \frac{m}{s}-1} H_{0,jst}$. This proves the equation $(*)$. By the equation $(*)$, it is not hard to see that $\mathrm{im}(H) = m/s$. Therefore, $s = 1$. $\square$

The proof of this lemma indeed implies the following result.

Proposition 5.4. *For every i, j with $1 \leqslant i, j \leqslant n-1$, $H_{ij} \neq 0$ if and only if $i - j \equiv 0$ $(operatornamemodt)$ for all $0 \leqslant i, j \leqslant n-1$.*

This proposition also implies that

$$H_{it}^l = \bigoplus_{0 \leqslant j \leqslant m-1} H_{it,jt} \quad \text{and} \quad H_{jt}^r = \bigoplus_{0 \leqslant i \leqslant m-1} H_{it,jt}$$

for all $0 \leqslant i, j \leqslant m-1$. Thus we can define $\widetilde{H}$ as follows:

$$\widetilde{H} := \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt} = \bigoplus_{0 \leqslant i \leqslant m-1} H_{it}^l = \bigoplus_{0 \leqslant j \leqslant m-1} H_{jt}^r. \qquad (5.1)$$

Lemma 5.5. *$\widetilde{H}$ is a Hopf subalgebra of H.*

Proof. We need to verify the algebra structure, coalgebra structure and the antipode condition of $\widetilde{H}$. Clearly, by the bigraded structure of H, it is easy to see that $\widetilde{H}$ is an algebra. Since each H_i^l is a right coideal of H and H_j^r is a left coideal of H for all $0 \leqslant i, j \leqslant n-1$ (by Lemma 2.5), $\Delta(H_{it,jt}) \subseteq \sum_{k,l} H_{kt}^l \otimes H_{lt}^r \subseteq \widetilde{H} \otimes \widetilde{H}$. So $\widetilde{H}$ is a coalgebra. Note that $S(H_{it,jt}) = H_{-jt,-it} \subset \widetilde{H}$ by Lemma 5.1, thus $\widetilde{H}$ is a Hopf subalgebra of H. □

Lemma 5.6. *$\widetilde{H}$ is regular and* gl.dim$(\widetilde{H}) = 1$.

Proof. Deducing from the fact that $H = \bigoplus_{0\leqslant i\leqslant n-1} H_i^l = \bigoplus_{0\leqslant j\leqslant n-1} H_j^r$ is a strongly graded algebra, we get a new grading for H:

$$H = \bigoplus_{0\leqslant i\leqslant t-1} \widetilde{H}H_i^l = \bigoplus_{0\leqslant i\leqslant t-1} \widetilde{H}(H_1^l)^i.$$

Since H_1^l is an affine invertible H_0^l-bimodule, $\widetilde{H}H_i^l$ is an affine invertible $\widetilde{H}$-bimodule. Therefore, H is a projective $\widetilde{H}$-module and $\widetilde{H}$ is a direct summand of H as an $\widetilde{H}$-module. Thus, if V is any $\widetilde{H}$-module, then

$$\text{p.dim}_{\widetilde{H}}(V) \leqslant \text{p.dim}_{\widetilde{H}}(H \otimes_{\widetilde{H}} V) \leqslant \text{p.dim}_H(H \otimes_{\widetilde{H}} V),$$

where the second inequality holds because H is a projective $\widetilde{H}$-module. Therefore, gl.dim$(\widetilde{H}) \leqslant 1$. Since $\widetilde{H}$ is not semisimple (otherwise, it is finite dimensional), gl.dim$\widetilde{H} = 1$. □

Lemma 5.7. *$\widetilde{H}$ is prime and* PI-deg$(\widetilde{H}) = m$.

Proof. Note that $\widetilde{H} = \bigoplus_{0\leqslant i\leqslant m-1} H_{it}^l$. Therefore, the lemma follows directly from [5, Corollary 5.1 (b)]. □

Proposition 5.8. *Let $\widetilde{H}$ be the algebra constructed as (5.1). Then it is isomorphic as a Hopf algebra either to the Taft algebra $H(m, 1, \xi)$ or to the generalized Liu algebra $B(m, \omega, \gamma)$. As a consequence,* io$(\widetilde{H}) =$ im$(\widetilde{H}) = m$.

Proof. By the above three lemmas, $\widetilde{H}$ is an affine prime regular Hopf algebra, and it is clear that $\widetilde{H}$ is of GK-dimension one. Denote the restriction of the actions of Ξ_π^l and Ξ_π^r to $\widetilde{H}$ by Γ^l and Γ^r, respectively. Since $\widetilde{H} = \bigoplus_{0\leqslant i\leqslant m-1} H_{it}^l$, we can see that for each $0 \leqslant i \leqslant m-1$ and any $0 \neq x \in H_{it}^l$,

$$(\Gamma^l)^m(x) = \zeta^{itm} x = x.$$

This implies that the group $\langle \Gamma^l \rangle$ has order m. Similarly, $|\langle \Gamma^r \rangle| = m$. We claim that

$$\langle \Gamma^l \rangle \cap \langle \Gamma^r \rangle = 1.$$

In fact, if $(\Gamma^l)^i = (\Gamma^r)^j$ for some $0 \leqslant i, j \leqslant m-1$. Choose $0 \neq x \in H_{tt}$, we find

$$\zeta^{ti}x = (\Gamma^l)^i(x) = (\Gamma^r)^j(x) = \zeta^{tj}x$$

which implies $i = j$. Let $0 \neq y \in H_{0,t}$, then

$$y = (\Gamma^l)^i(y) = (\Gamma^r)^j(y) = \zeta^{tj}y$$

forces $j = 0$. Thus we get $i = j = 0$, i.e., $\langle \Gamma^l \rangle \cap \langle \Gamma^r \rangle = 1$. Therefore, Lemma 2.8 is applied. So, $\mathrm{io}(\widetilde{H}) = \mathrm{im}(\widetilde{H}) = m$ and $\widetilde{H}$ is isomorphic as a Hopf algebra either to the Taft algebra $H(m, 1, \xi)$ or to the generalized Liu algebra $B(m, \omega, \gamma)$. $\square$

6. From $\widetilde{H}$ to H

This section is to build some general relations between $\widetilde{H}$ and H. Our final aim is to show that $\widetilde{H}$ can determine the structures of H entirely. We start with the following definition.

Definition 6.1. We call H is *primitive* (respectively, *group-like*) if $\widetilde{H}$ is primitive, i.e., $\widetilde{H} \cong H(m, 1, \xi)$ (respectively, if $\widetilde{H}$ is group-like, i.e., $\widetilde{H} \cong B(m, \omega, \gamma)$).

By Proposition 5.8, H is either primitive or group-like. By definition, $\widetilde{H} = \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt}$ is bigraded automatically. Meanwhile, $\widetilde{H}$ has left integrals $\int^l_{\widetilde{H}}$ and from which we also can construct another bigrading on $\widetilde{H}$ just as in Subsection 2.2.

Lemma 6.2. *Under a suitable choice of ζ, these two bigradings on $\widetilde{H}$ are the same.*

Proof. The proof is indeed implicit in the arguments given in [5, Section 6]. For completeness, we prove it here. We start with a general form. Let A be a prime regular Hopf algebra of GK-dimension one with an algebra homomorphism $\mu : A \to k$ (note that μ need not be the algebra homomorphism $\pi : A \to A/\mathrm{r.ann}(\int^l_A))$. Denote the groups of the left and right winding automorphisms Ξ^l_μ and Ξ^r_μ associated to $\mu : A \to k$ by G^l_μ and G^r_μ respectively. Fix a primitive nth root ζ of 1, and define $\chi \in \widehat{G^l_\mu}$ and $\eta \in \widehat{G^r_\mu}$ by setting

$$\chi(\Xi^l_\mu) = \zeta \quad \text{and} \quad \eta(\Xi^r_\mu) = \zeta.$$

Assume that G^l_μ and G^r_μ satisfy

$$|G^l_\mu| = \text{PI-deg}(A) = n \quad \text{and} \quad G^l_\mu \cap G^r_\mu = \{1\}. \tag{$\star$}$$

Then there is a bigrading on

$$A = \bigoplus_{0 \leqslant i,j \leqslant n-1} A_{ij},$$

where $A_{ij} = \{a \in A | \Xi^l_\mu(a) = \zeta^i a,\ \Xi^r_\mu(a) = \zeta^j a\}$. It is proved in [5, Section 6] that $A \cong H(n, 1, \xi)$ or $A \cong B(n, \omega, \gamma)$ and under a suitable choice of ζ, A_{ij} is exactly $H(n, 1, \xi)_{ij}$ or $B(n, \omega, \gamma)_{ij}$, where $H(n, 1, \xi)_{ij}$ and $B(n, \omega, \gamma)_{ij}$ are homogeneous components of the bigrading constructed in Subsection 2.2. Now the proof is completed by noting that the algebra homomorphism $\mu = \pi|_{\widetilde{H}} : \widetilde{H} \to k$ (where $\pi : H \to H/\mathrm{r.ann}(\int^l_H)$) induces the left and right winding automorphisms satisfying the condition $(\star)$ by Proposition 5.8. □

Remark 6.3. By Lemma 6.2, we can freely use the calculation results of H_{ij} to

$$\widetilde{H} := \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt}$$

as in Subsection 2.3. That is, if H is primitive, then we can assume that $H_{it,jt} = k[x^m]x^{j-i}g^i$; and if H is group-like, then we can take that $H_{it,jt} = k[x^{\pm 1}]y^{j-i}g^i$.

Now we denote the fraction field of H^l_0 by Q^l_0. Brown and Zhang [5, Section 5] showed that there is a delicate $\widehat{G^l_\pi}$-action on Q^l_0 defined as follows: For each $\chi^i \in \widehat{G^l_\pi}$ and $a \in Q^l_0$,

$$\kappa_i(a) := u_i a u_i^{-1}.$$

Here, to avoid confusion, we denote the automorphism of Q^l_0 corresponding to $\chi^i \in \widehat{G^l_\pi}$ by κ_i. And since H^l_i is an invertible H^l_0-module, $Q^l_0 H^l_i = Q^l_0 u_i$ for any non-zero element $u_i \in H^l_i$. Brown and Zhang proved that this action is independent of the choice of the non-zero element u_i and clearly satisfies

$$xa = \kappa_i(a)x \tag{6.1}$$

for $x \in H^l_i$ and $a \in H^l_0$. Furthermore, [5, Proposition 5.1 (b)] implies that the restriction of κ_i to H^l_0 is still an automorphism and we denote this restriction by κ^l_i for convenience. In a special situation, the following lemma is implicit in [5, Subsection 6.2].

Lemma 6.4. *$K^l := \{\kappa^l_i\}_{0 \leqslant i \leqslant n-1}$ is a cyclic group.*

Proof. We only need to show $\kappa^l_i = (\kappa^l_1)^i$ for all i. Let x_1, x'_1 be any non-zero elements of H^l_1, for any $a \in H^l_0$,

$$x_1 x'_1 a = x_1 \kappa^l_1(a) x'_1 = \kappa^l_1(\kappa^l_1(a)) x_1 x'_1.$$

Applying the strongly graded property of H, it follows that $\kappa^l_2 = (\kappa^l_1)^2$. Similarly, $\kappa^l_i = (\kappa^l_1)^i$ for all $2 \leqslant i \leqslant n-1$. So K^l is a cyclic group. □

Meanwhile, G^r_π acts on H^l_0 as explained in Subsection 2.2. We denote by ρ^l the resulting map from G^r_π to $\mathrm{Aut}(H^l_0)$. Let $P^l_\pi := \langle \rho^l(G^r_\pi), K^l \rangle \subseteq \mathrm{Aut}(H^l_0)$. By [5, Proposition 5.2],

P_π^l is abelian and $P_\pi^l = K^l$. So the cyclic group $\rho^l(G_\pi^r)$ is a subgroup of K^l. By [5, Proposition 5.1], $Z(H) \subseteq H_0^l$. Similarly, $Z(H) \subseteq H_0^r$. Therefore, $Z(H) \subseteq H_0$ and thus H_0 is a $Z(H)$-module naturally. One of our key observations is the following lemma.

Lemma 6.5. *H_0 is a torsion-free $Z(H)$-module of rank at least t.*

Proof. Since PI-deg(H) = io$(H) = n$, $K := \{\kappa_i\}_{0\leqslant i\leqslant n-1}$ acts faithfully on Q_0^l by [5, Proposition 5.1]. So it is easy to see that K^l acts faithfully on H_0^l. Then we have a $\widehat{K^l}$-grading on H_0^l:

$$H_0^l = \bigoplus_{\chi^i\in\widehat{K^l}} (H_0^l)_i^\kappa,$$

where $(H_0^l)_i^\kappa = \{x \in H_0^l \mid g(x) = \chi^i(g)x, \forall g \in K^l\}$.

Note that the grading of H_0^l induced by the action of $\rho^l(G_\pi^r)$ is just $H_0^l = \bigoplus_{0\leqslant i\leqslant m-1} H_{0,it}$. Since $\rho^l(G_\pi^r)$ is a subgroup of the cyclic group K^l, $H_0^l = \bigoplus_{\chi^i\in\widehat{K^l}} (H_0^l)_i^\kappa$ is a refinement of $H_0^l = \bigoplus_{0\leqslant i\leqslant m-1} H_{0,it}$. Therefore

$$H_0 = (H_0^l)_0^\kappa \bigoplus (H_0^l)_m^\kappa \bigoplus \cdots \bigoplus (H_0^l)_{(t-1)m}^\kappa.$$

Because of the faithful action of K^l on H_0^l, each $(H_0^l)_i^\kappa$ is torsion-free on $(H_0^l)_0^\kappa$ of rank at least one. By [5, Proposition 5.1], $(H_0^l)_0^\kappa = Z(H)$. Thus H_0 is torsion-free over $Z(H)$ of rank at least t. □

With this observation, we have the following conclusion.

Proposition 6.6. *Each homogeneous component $H_{i,i+jt}$ of $H = \bigoplus_{0\leqslant i\leqslant n-1, 0\leqslant j\leqslant m-1} H_{i,i+jt}$ is a free H_0-module of rank one on both sides.*

Proof. We work with left modules. By [5, Proposition 5.1], each H_i^l is a torsion-free H_0^l-module. So the rank of H_i^l on H_0^l is at least one. By Proposition 5.4, each homogeneous $H_{i,i+jt}$ is a non-zero H_0-module, then it is torsion-free of rank at least one. Since $\widetilde{H}$ is isomorphic as a Hopf algebra either to the Taft algebra $H(m, 1, \xi)$ or to the generalized Liu algebra $B(m, \omega, \gamma)$, H_0 is isomorphic either to $k[x^m]$ or to $k[x^{\pm 1}]$. Both of them are principal ideal domains, so each $H_{i,i+jt}$ is free on H_0 of rank at least one which implies the rank of H over H_0 is not smaller than nm. By Lemma 6.5, the rank of H_0 over $Z(H)$ is at least t. Therefore,

$$\mathrm{rank}_{Z(H)} H \geqslant nmt = n^2.$$

Recall that the PI-degree of H is n and equals the square root of the rank of H over $Z(H)$. So the rank of H_0 over $Z(H)$ is t and each $H_{i,i+jt}$ is a free H_0-module of rank one. □

Since $H_0 = k[x^m]$ or $H_0 = k[x^{\pm 1}]$, there is a generating set $\{u_{i,i+jt} | 0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant m-1\}$ of these $H_{i,i+jt}$ satisfying

$$u_{00} = 1 \quad \text{and} \quad H_{i,i+jt} = u_{i,i+jt} H_0 = H_0 u_{i,i+jt}.$$

So, H can be written as

$$H = \bigoplus_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant m-1}} H_0 u_{i,i+jt} = \bigoplus_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant m-1}} u_{i,i+jt} H_0. \tag{6.2}$$

With these preparations, we can analyse the structure of H further by dichotomy now: either $\widetilde{H}$ is $H(m, 1, \xi)$, the primitive case; or $\widetilde{H}$ is $B(m, \omega, \gamma)$, the group-like case.

7. Primitive case

If H is primitive, $\widetilde{H} = H(m, 1, \xi)$. We will prove H is isomorphic as a Hopf algebra to $H(n, t, \theta)$, for θ some primitive nth root of 1. Recall that

$$\widetilde{H} = H(m, 1, \xi) = k\langle g, x | g^m = 1, xg = \xi gx \rangle.$$

Note that $H = \bigoplus_{0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant m-1} H_{i,i+jt}$, $\widetilde{H} = \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt}$ and $H_{it,jt} = k[x^m] x^{j-i} g^i$ (the index $j - i$ is interpreted $\bmod m$). In particular, $H_0 = k[x^m]$, $H_{0,jt} = k[x^m] x^j$ and $H_{tt} = k[x^m] g$.

By Lemma 5.1, $\epsilon(u_{11}) \neq 0$. Multiplied with a suitable scalar, we can assume that $\epsilon(u_{11}) = 1$ throughout this section.

Lemma 7.1. *Let $u := u_{11}$. Then $H_1^l = H_0^l u$, $H = \bigoplus_{0 \leqslant i \leqslant t-1} \widetilde{H} u^i$ and u is invertible.*

Proof. By the bigraded structure of H, we have

$$H_{0t} H_{11} \subseteq H_{1,1+t}, \quad H_{0,(m-1)t} H_{1,1+t} \subseteq H_{11},$$

which imply

$$H_{0t} H_{0,(m-1)t} H_{1,1+t} \subseteq H_{0t} H_{11} \subseteq H_{1,1+t}.$$

Since $H_{0t} H_{0,(m-1)t} = x^m H_0$ is a maximal ideal of $H_0 = k[x^m]$ and $H_{1,1+t}$ is a free H_0-module of rank one (by Proposition 6.6), $H_{0t} H_{0,(m-1)t} H_{1,1+t}$ is a maximal H_0-submodule of $H_{1,1+t}$. Thus $H_{0t} H_{11} = H_{0t} H_{0,(m-1)t} H_{1,1+t} = x^m H_{1,1+t}$ or $H_{0t} H_{11} = H_{1,1+t}$.

If $H_{0t} H_{11} = x^m H_{1,1+t}$, then $x u_{11} = x^m \alpha(x^m) u_{1,1+t}$ for some polynomial $\alpha(x^m) \in k[x^m]$. So

$$x(u_{11} - x^{m-1} \alpha(x^m) u_{1,1+t}) = 0.$$

Therefore, $x^m(u_{11} - x^{m-1}\alpha(x^m)u_{1,1+t}) = 0$. Note that each homogeneous $H_{i,i+jt}$ is a torsion-free H_0-module, so

$$u_{11} = x^{m-1}\alpha(x^m)u_{1,1+t}.$$

By assumption, $\epsilon(u_{11}) = 1$. But, by definition, $\epsilon(x) = 0$. This is impossible. So $H_{0t}H_{11} = H_{1,1+t}$ which implies that $H_{0,t}u_{11} = H_{1,1+t}$.

Similarly, we can show that $H_{0,it}u_{11} = H_{1,1+it}$ for $0 \leqslant i \leqslant m-1$. Thus $H_1^l = H_0^l u_{11}$. Since $H = \bigoplus_{0\leqslant i\leqslant n-1} H_i^l$ is strongly graded, u_{11} is invertible and $H_i^l = H_0^l u_{11}^i$ for all $0 \leqslant i \leqslant n-1$. Let $u := u_{11}$, then we have

$$H = \bigoplus_{0\leqslant i\leqslant t-1} \widetilde{H}u^i. \qquad \square$$

We are in a position to determine the structure of H now.

Proposition 7.2. *With above notations, we have*

$$u^t = g, \quad xu = \theta ux,$$

where θ is a primitive nth root of 1.

Proof. By $H_{0t}u = uH_{0t}$, there exists a polynomial $\beta(x^m) \in k[x^m]$ such that

$$xu = ux\beta(x^m).$$

Then

$$xu^t = u^t x\beta'(x^m)$$

for some polynomial $\beta'(x^m) \in k[x^m]$ induced by $\beta(x^m)$. Since u^t is invertible and $u^t \in H_{t,t} = k[x^m]g$, $u^t = ag$ for $0 \neq a \in k$. By assumption, $\epsilon(u) = 1$ and thus $a = 1$. Therefore, $u^t = g$. Since $xg = \xi gx$, we have $\beta'(x^m) = \xi$. Then it is easy to see that $\beta(x^m) = \theta \in k$ with $\theta^t = \xi$. Of course, $\theta^n = 1$.

The last job is to show that θ is a primitive nth root of 1. Indeed, assume θ is a primitive n'th root of 1. Then $Z(H) = k[x^{n'}]$, the center of H. Recall that the PI-degree of H equals the square root of the rank of H over $Z(H)$. So the equalities

$$n^2 = nm\,\mathrm{rank}_{Z(H)}H_0 = nmn'/m$$

hold. That is, $n' = n$ and θ is a primitive nth root of 1. $\square$

Proposition 7.3. *The element u is a group-like element of H.*

Proof. First of all $H_0^r \cong k[x] \cong H_0^l$. Then $H_0^r \otimes H_0^l \cong k[x,y]$ and the only invertible elements in $H_0^r \otimes H_0^l$ are nonzero scalars in k. Since $\Delta(u)$ and $u \otimes u$ are invertible, $\Delta(u)(u \otimes u)^{-1}$ is invertible (and hence a scalar). Thus u must be grouplike by noting that $\epsilon(u) = 1$. □

The next theorem follows from Lemma 7.1 and Propositions 7.2, 7.3 directly.

Theorem 7.4. *Let H be an affine prime regular Hopf algebra of GK-dimension one satisfying* $\mathrm{io}(H) = n > \mathrm{im}(H) = m > 1$. *If H is primitive, then H is isomorphic as a Hopf algebra to an infinite dimensional Taft algebra.*

8. Group-like case

If H is group-like, then $\widetilde{H} = B(m, \omega, \gamma)$ for γ a primitive mth root of 1 and ω a positive integer. As usual, the generators of $B(m, \omega, \gamma)$ are denoted by $x^{\pm 1}, y$ and g. By Remark 6.3, we can assume that $\widetilde{H} = \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt}$ with

$$H_{it,jt} = k[x^{\pm 1}]y^{j-i}g^i$$

(the index $j - i$ is interpreted mod m). In particular, $H_0 = k[x^{\pm 1}]$, $H_{0,jt} = k[x^{\pm 1}]y^j$ and $H_{t,t} = k[x^{\pm 1}]g$.

Set $u_i := u_{1,1+it} (0 \leqslant i \leqslant m - 1)$ for convenience. By the structure of the bigrading of H, we have

$$yu_0 = \phi_0 u_1,\ yu_1 = \phi_1 u_2, \cdots, yu_{m-2} = \phi_{m-2} u_{m-1},\ yu_{m-1} = \phi_{m-1} u_0 \tag{8.1}$$

and

$$u_0 y = \varphi_0 u_1,\ u_1 y = \varphi_1 u_2, \cdots, u_{m-2} y = \varphi_{m-2} u_{m-1},\ u_{m-1} y = \varphi_{m-1} u_0 \tag{8.2}$$

for some polynomials $\phi_i, \varphi_i \in k[x^{\pm 1}], 0 \leqslant i \leqslant m-1$. With these notions and the equality $y^m = 1 - x^\omega$, we find that

$$(1 - x^\omega)u_0 = y^m u_0 = \phi_0 \phi_1 \cdots \phi_{m-1} u_0 \tag{8.3}$$

and

$$u_0(1 - x^\omega) = u_0 y^m = \varphi_0 \varphi_1 \cdots \varphi_{m-1} u_0. \tag{8.4}$$

Proposition 8.1. *There is no group-like affine prime regular Hopf algebra of GK-dimension one H satisfying* $\mathrm{io}(H) = n > \mathrm{im}(H) = m > 1$ *and $n/m > 2$.*

Proof. Since $u_iH_0 = H_0u_i$, we have

$$u_ix = \alpha_i(x^{\pm 1})u_i \quad \text{and} \quad u_ix^{-1} = \beta_i(x^{\pm 1})u_i$$

for some $\alpha_i(x^{\pm 1}), \beta_i(x^{\pm 1}) \in k[x^{\pm 1}]$. From

$$u_i = u_ixx^{-1} = \alpha_i(x^{\pm 1})u_ix^{-1} = \alpha_i(x^{\pm 1})\beta_i(x^{\pm 1})u_i,$$

we get $\alpha_i(x^{\pm 1})\beta_i(x^{\pm 1}) = 1$ and thus $\alpha_i(x^{\pm 1}) = \lambda_i x^{a_i}$ for some $0 \neq \lambda_i \in k, 0 \neq a_i \in \mathbb{Z}$.

Note that $u_i^t \in H_{t,(1+it)t} = k[x^{\pm 1}]y^{\bar{it}}g$, where $\bar{it} \equiv it \pmod m$. So we have $u_i^t = \gamma_i(x^{\pm 1})y^{\bar{it}}g$ for some $\gamma_i(x^{\pm 1}) \in k[x^{\pm 1}]$. Hence u_i^t commutes with x. Applying $u_ix = \lambda_i x^{a_i}u_i$ to $u_i^tx = xu_i^t$, we get $\lambda_i^{\sum_{s=0}^{t-1} a_i^s} = 1$ and $x^{a_i^t} = x$. If t is odd, $a_i = 1$ and if t is even, then a_i is either 1 or -1.

Now we consider the special case $i = 0$. By $\epsilon(xu_0) = \epsilon(u_0x) \neq 0$, we find that $\lambda_0 = 1$.

If $a_0 = 1$, that is $u_0x = xu_0$, then we will see $u_ix = xu_i$ for all $0 \leqslant i \leqslant m-1$. In fact, by

$$\phi_0xu_1 = x\phi_0u_1 = xyu_0 = yxu_0 = yu_0x = \phi_0u_1x,$$

we have $u_1x = xu_1$ since $H_{1,1+t}$ is a torsion-free H_0-module. Similarly, $u_ix = xu_i$ for all $0 \leqslant i \leqslant m-1$. Then by the strongly graded structure $u_{i,i+jt} \in H_i^l = (H_1^l)^i$ and x is commutative with H_1^l, it is not hard to see that $u_{i,i+jt}x = xu_{i,i+jt}$ for all $0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant m-1$. Therefore the center $Z(H) \supseteq H_0 = k[x^{\pm 1}]$. By [5, Lemma 5.2], $Z(H) \subseteq H_0$ and thus $Z(H) = H_0 = k[x^{\pm 1}]$. This implies that

$$\text{rank}_{Z(H)}H = \text{rank}_{H_0}H = nm < n^2,$$

which contradicts the fact: the PI-degree of H is n and equals the square root of the rank of H over $Z(H)$.

If $a_0 = -1$, that is $u_0x = x^{-1}u_0$, we can deduce that $u_{i,i+jt}x = x^{-1}u_{i,i+jt}$ for all $0 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant m-1$ by using the parallel proof of the case $a_0 = 1$. For $s \in \mathbb{N}$, let $z_s := x^s + x^{-s}$. Define $k[z_s|s \geq 0]$ to be the subalgebra of $k[x^{\pm 1}]$ generated by all z_s. Note that $k[x^{\pm 1}]$ has rank 2 over $k[z_s|s \geqslant 1]$. Thus $Z(H) \supseteq k[z_s|s \geq 0]$. Using [5, Lemma 5.2] again, we have $Z(H) = k[z_s|s \geq 0]$. Hence

$$\text{rank}_{Z(H)}H = 2nm \neq n^2$$

since $n/m > 2$ by assumption. This contradicts the fact that the PI-deg$H = n$ again.

Combining these two cases, we get the desired result. □

We turn now to consider the case: $\text{io}(H) = 2\,\text{im}(H) = 2m$. In this case, $t = 2$. As discussed in the proof of Proposition 8.1, if such H exists then the following relations

$$u_ix = x^{-1}u_i \quad (0 \leqslant i \leqslant m-1) \tag{8.5}$$

hold in H. Using these relations and (8.3), we have

$$\varphi_0 \cdots \varphi_{m-1} = 1 - x^{-\omega}. \tag{8.6}$$

To determine the structure of H, we need to give some harmless assumptions on the choice of u_i $(0 \leqslant i \leqslant m-1)$: (1) We assume $\epsilon(u_0) = 1$; (2) Let $\xi_s := e^{\frac{2s\pi i}{\omega}}$ and thus $1 - x^\omega = \Pi_{s\in S}(1 - \xi_s x)$, where $S := \{0, 1, \cdots, \omega - 1\}$. Since

$$\phi_0 \cdots \phi_{m-1} = y^m = 1 - x^\omega \tag{8.7}$$

by (8.3), we have

$$\phi_i = k_i x^{c_i} \Pi_{s\in S_i}(1 - \xi_s x),$$

where $k_i \in k$, c_i is an integer and S_i is a subset of S. The second assumption is: Take the u_i's such that $\phi_i = \prod_{s\in S_i}(1 - \xi_s x)$. Due to Equation (8.7), this assumption can be realized; (3) By the strongly graded structure of H, the equality $H_2^l = H_0^l g$ and the fact that g is invertible in H, we can take $u_{j,j+2i}$ such that

$$u_{j,j+2i} = \begin{cases} g^{\frac{j-1}{2}} u_i & \text{if } j \text{ is odd,} \\ y^i g^{\frac{j}{2}} & \text{if } j \text{ is even,} \end{cases}$$

for all $2 \leqslant j \leqslant 2m - 1$. In the rest of this section, we always make these assumptions.

We still need two notations, which appeared in the proof of Proposition 4.2. For a polynomial $f = \sum a_i x^{b_i} \in k[x^{\pm 1}]$, we denote by $\bar{f}$ the polynomial $\sum a_i x^{-b_i}$. Then by (8.5), we have $f u_i = u_i \bar{f}$ and $u_i f = \bar{f} u_i$ for all $0 \leqslant i \leqslant m-1$. For any $h \in H \otimes H$, we use

$$h_{(s_1,t_1)\otimes(s_2,t_2)}$$

to denote the homogeneous part of h in $H_{s_1,t_1} \otimes H_{s_2,t_2}$. Both these notations will be used frequently in the proof of the next proposition.

Proposition 8.2. *Let H be a prime regular Hopf algebra with $\widetilde{H} = B(m, \omega, \gamma)$. Assume that $\operatorname{io}(H) = 2\operatorname{im}(H) > 2$, then as a Hopf algebra,*

$$H \cong D(m, d, \xi)$$

constructed as in Subsection 4.1, where m divides ω and 2 divides $d(m+1)$.

Proof. We divide the proof into several steps.

Claim 1. We have $m|\omega$ and $yu_i = \phi_i u_{i+1} = \xi x^d u_i y$ for $d = \frac{\omega}{m}$ and some $\xi \in k$ satisfying $\xi^m = -1$.

Proof of Claim 1: By associativity of the multiplication, we have many equalities:

$$\begin{aligned} yu_0y^{m-1} &= \phi_0\varphi_1\varphi_2\cdots\varphi_{m-1}u_0 \\ &= \varphi_0\phi_1\varphi_2\cdots\varphi_{m-1}u_0 \\ &\cdots \\ &= \varphi_0\varphi_1\varphi_2\cdots\phi_{m-1}u_0, \end{aligned}$$

which imply that $\phi_i\varphi_j = \varphi_i\phi_j$ for all $0 \leqslant i, j \leqslant m-1$. Using associativity again, we have

$$\begin{aligned} y^m u_0 y^{m(m-1)} &= (1-x^\omega)u_0(1-x^\omega)^{m-1} = -x^\omega(1-x^{-\omega})^m u_0 \\ &= -x^\omega(\varphi_0\varphi_1\varphi_2\cdots\varphi_{m-1})^m u_0 \\ &= (\phi_0\varphi_1\varphi_2\cdots\varphi_{m-1})^m u_0 \\ &= (\varphi_0\phi_1\varphi_2\cdots\varphi_{m-1})^m u_0 \\ &\cdots \\ &= (\varphi_0\varphi_1\varphi_2\cdots\phi_{m-1})^m u_0, \end{aligned}$$

where the fourth "=", for example, is gotten in the following way: We multiply u_0 by one y from left side at first, then multiply it with y^{m-1} from right side, then continue the procedures above. From these equalities, we have

$$\phi_i^m = -x^\omega\varphi_i^m$$

for all $0 \leqslant i \leqslant m-1$. So $\phi_i = \xi_i x^d\varphi_i$ where $d = \frac{\omega}{m}$ and $\xi_i \in k$ satisfying $\xi_i^m = -1$. Applying $\phi_i\varphi_j = \varphi_i\phi_j$, we can see $\xi_i = \xi_j$ for all $0 \leqslant i, j \leqslant m-1$, and we write $\xi := \xi_i$. Therefore we have $yu_i = \xi x^d u_i y$, where $d = \omega/m$ is an integer. □

In the following of the proof, d is fixed to be the number ω/m.

Claim 2. We have $u_ig = \lambda_i x^{-2d} g u_i$ for $\lambda_i = \pm\gamma^i$ and $0 \leqslant i \leqslant m-1$.

Proof of Claim 2: Since g is invertible in H, $u_ig = \psi_i g u_i$ for some invertible $\psi_i \in k[x^{\pm 1}]$. Then $u_ig^m = \psi_i^m g^m u_i$ yields $\psi_i^m = x^{-2\omega}$. So $\psi_i = \lambda_i x^{-2d}$ for $\lambda_i \in k$ with $\lambda_i^m = 1$. Our last task is to show that $\lambda_i = \pm\gamma^i$. To show this, we need a preparation, that is, we need to show that $u_iu_j \neq 0$ for all i, j. Otherwise, assume that there exist $i_0, j_0 \in \{0, \ldots, m-1\}$ such that $u_{i_0}u_{j_0} = 0$. Using Claim 1, we can find that $u_{i_0}u_j \equiv 0$ and $u_iu_{j_0} \equiv 0$ for all i, j. Let (u_{i_0}) and (u_{j_0}) be the ideals generated by u_{i_0} and u_{j_0} respectively. Then it is not hard to find that $(u_{i_0})(u_{j_0}) = 0$ which contradicts H being prime. So we always have

$$u_iu_j \neq 0 \tag{8.8}$$

for all $0 \leqslant i, j \leqslant m-1$.

Applying this observation, we have $0 \neq u_i^2 \in H_{2,2+4i} = k[x^{\pm 1}]y^{2i}g$, $u_i^2 g = \psi_i \overline{\psi_i} g u_i^2 = \gamma^{2i} g u_i^2$. Thus $\psi_i = \pm\gamma^i x^{-2d}$ which implies that $\lambda_i = \pm\gamma^i$. The proof is ended. $\square$

We can say more about λ_i at this stage. By $0 \neq u_i u_j g = \gamma^{i+j} g u_i u_j$, we know that $\psi_i = \gamma^i x^{-2d}$ for all i or $\psi_i = -\gamma^i x^{-2d}$ for all i. So

$$\lambda_i = \gamma^i \quad \text{or} \quad \lambda_i = -\gamma^i \tag{8.9}$$

for all $0 \leqslant i \leqslant m-1$. In fact, we will show that $\psi_i = \gamma^i x^{-2d}$ for all i later.

Claim 3. For each $0 \leqslant i \leqslant m-1$, there are $f_{ij}, h_{ij} \in k[x^{\pm 1}]$ with h_{ij} monic such that

$$\Delta(u_i) = \sum_{j=0}^{m-1} f_{ij} u_j \otimes h_{ij} g^j u_{i-j}, \tag{8.10}$$

where the following $i-j$ is interpreted $\bmod m$.

Proof of Claim 3: Since $u_i \in H_{1,1+2i}$, $\Delta(u_i) \in H_1^l \otimes H_{1+2i}^r$. Noting that $H_1^l = \bigoplus_{j=0}^{m-1} H_0 u_j$ and $H_{1+2i}^r = \bigoplus_{s=0}^{m-1} H_0 g^s u_{i-s}$, we can write

$$\Delta(u_i) = \sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (u_j \otimes g^s u_{i-s}),$$

where $F_{js}^i \in H_0 \otimes H_0$. Then we divide the proof into two steps.

- *Step* 1 ($\Delta(u_i) = \sum_{0 \leqslant j \leqslant m-1} F_{jj}^i (u_j \otimes g^j u_{i-j})$).

Recall that $u_i g = \lambda_i x^{-2d} g u_i$, where λ_i is either γ^i for all i or $-\gamma^i$ for all i. The equations

$$\begin{aligned}
\Delta(u_i g) = \Delta(u_i)\Delta(g) &= \sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (u_j \otimes g^s u_{i-s})(g \otimes g) \\
&= \sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (\lambda_j x^{-2d} g u_j \otimes \lambda_{i-s} x^{-2d} g^{s+1} u_{i-s}) \\
&= \sum_{0 \leqslant j,s \leqslant m-1} \lambda_j \lambda_{i-s} (x^{-2d} g \otimes x^{-2d} g) F_{js}^i (u_j \otimes g^s u_{i-s})
\end{aligned}$$

and

$$\begin{aligned}
\Delta(\lambda_i x^{-2d} g u_i) = \lambda_i (x^{-2d} g \otimes x^{-2d} g)\Delta(u_i)\Delta(g) &\sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (u_j \otimes g^s u_{i-s}) \\
&= \sum_{0 \leqslant j,s \leqslant m-1} \lambda_i (x^{-2d} g \otimes x^{-2d} g) F_{js}^i (u_j \otimes g^s u_{i-s})
\end{aligned}$$

imply that $\lambda_i = \lambda_j \lambda_{i-s}$ for all j, s. If $\lambda_i = -\gamma^i$ for all i, then we have $-\gamma^i = \lambda_i = \lambda_j \lambda_{i-s} = \gamma^{j+i-s}$. This implies $j = s \pm m/2$. Applying $(\epsilon \otimes \mathrm{Id})$ to $\Delta(u_i)$,

$$(\epsilon \otimes \mathrm{Id})\Delta(u_i) = (\epsilon \otimes \mathrm{Id})(F^i_{0,\ m/2})g^{m/2}u_{i-m/2} \neq u_i,$$

which is absurd. If $\lambda_i = \gamma^i$ for all i, then $\gamma^i = \lambda_i = \lambda_j \lambda_{i-s} = \gamma^{j+i-s}$. This implies $j = s$ (which is compatible with the equality $(\epsilon \otimes \mathrm{Id})\Delta(u_i) = u_i$). So we get $F^i_{js} \neq 0$ only if $j = s$ and $\lambda_i = \gamma^i$ for all i. Thus we have $\Delta(u_i) = \sum_{0 \leqslant j \leqslant m-1} F^i_{jj}(u_j \otimes g^j u_{i-j})$ for all i.

• *Step* 2 (There exist $f_{ij}, h_{ij} \in H_0$ with h_{ij} monic such that $F^i_{jj} = f_{ij} \otimes h_{ij}$ for $0 \leqslant i, j \leqslant m-1$.)

We replace F^i_{jj} by F^i_j for convenience. Since

$$\begin{aligned}(\Delta \otimes \mathrm{Id})\Delta(u_i) &= (\Delta \otimes \mathrm{Id})(\sum_{0 \leqslant j \leqslant m-1} F^i_j(u_j \otimes g^j u_{i-j}))\\ &= \sum_{0 \leqslant j \leqslant m-1} (\Delta \otimes \mathrm{Id})(F^i_j)(\sum_{0 \leqslant s \leqslant m-1} F^j_s(u_s \otimes g^s u_{j-s}) \otimes g^j u_{i-j})\\ &= \sum_{0 \leqslant j,s \leqslant m-1} (\Delta \otimes \mathrm{Id})(F^i_j)(F^j_s \otimes 1)(u_s \otimes g^s u_{j-s} \otimes g^j u_{i-j})\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)\Delta(u_i) &= (\mathrm{Id} \otimes \Delta)(\sum_{0 \leqslant j \leqslant m-1} F^i_j(u_j \otimes g^j u_{i-j}))\\ &= \sum_{0 \leqslant j \leqslant m-1} (\mathrm{Id} \otimes \Delta)(F^i_j)(u_j \otimes (\sum_{0 \leqslant s \leqslant m-1} F^{i-j}_s(g^j u_s \otimes g^{j+s} u_{i-j-s}))\\ &= \sum_{0 \leqslant j,s \leqslant m-1} (\mathrm{Id} \otimes \Delta)(F^i_s)(1 \otimes F^{i-s}_{j-s})(u_s \otimes g^s u_{j-s} \otimes g^j u_{i-j}),\end{aligned}$$

we have

$$(\Delta \otimes \mathrm{Id})(F^i_j)(F^j_s \otimes 1) = (\mathrm{Id} \otimes \Delta)(F^i_s)(1 \otimes F^{i-s}_{j-s}) \tag{8.11}$$

for all $0 \leqslant i, j, s \leqslant m-1$.

Begin with the case $i = j = s = 0$. Let $F^0_0 = \sum_{p,q} k_{pq} x^p \otimes x^q$. Comparing equation

$$\begin{aligned}(\Delta \otimes \mathrm{Id})(F^0_0)(F^0_0 \otimes 1) &= (\sum_{p,q} k_{pq} x^p \otimes x^p \otimes x^q)(\sum_{p',q'} k_{p'q'} x^{p'} \otimes x^{q'} \otimes 1)\\ &= (\sum_{p,q,p',q'} k_{pq} k_{p'q'} x^{p+p'} \otimes x^{p+q'} \otimes x^q)\end{aligned}$$

and equation

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)(F^0_0)(1 \otimes F^0_0) &= (\sum_{p,q} k_{pq} x^p \otimes x^q \otimes x^q)(\sum_{p',q'} k_{p'q'} 1 \otimes x^{p'} \otimes x^{q'})\\ &= (\sum_{p,q,p',q'} k_{pq} k_{p'q'} x^p \otimes x^{q+p'} \otimes x^{q+q'}),\end{aligned}$$

one can see that $p = q = 0$ by comparing the degrees of x in these two expressions. Then $F_0^0 = 1 \otimes 1$ by applying $(\epsilon \otimes \mathrm{Id})\Delta$ to u_0. Next, consider the case $j = s = 0$. Write $F_0^i = \sum_{p,q} k_{pq} x^p \otimes x^q$. Similarly, we have $F_0^i = x^{a_i} \otimes 1$ for some $a_i \in \mathbb{Z}$ by the equation

$$(\Delta \otimes \mathrm{Id})(F_0^i)(F_0^0 \otimes 1) = (\mathrm{Id} \otimes \Delta)(F_0^i)(1 \otimes F_0^i).$$

Finally, write $F_j^i = \sum_{p,q} k_{pq} x^p \otimes x^q$ and consider the case $s = 0$. Let $F_0^i = x^{a_i} \otimes 1$ and $F_0^j = x^{a_j} \otimes 1$. The equation

$$\begin{aligned}(\sum_{p,q} k_{pq} x^{p+a_j} \otimes x^p \otimes x^q) &= (\Delta \otimes \mathrm{Id})(F_j^i)(F_0^j \otimes 1) = (\mathrm{Id} \otimes \Delta)(F_0^i)(1 \otimes F_j^i) \\ &= (\sum_{p,q} k_{pq} x^{a_i} \otimes x^p \otimes x^q)\end{aligned}$$

shows that $p = a_i - a_j$, that is, $F_j^i = x^{c_{ij}} \otimes \beta_{ij}$ some $c_{ij} \in \mathbb{Z}, \beta_{ij} \in H_0$.

By steps 1 and 2, F_j^i can be written as $f_{ij} \otimes h_{ij}$ with h_{ij} monic after multiplying suitable scalar, where $f_{ij}, h_{ij} \in k[x^{\pm 1}]$. That is,

$$\Delta(u_i) = \sum_{j=0}^{m-1} f_{ij} u_j \otimes h_{ij} g^j u_{i-j},$$

where $f_{ij}, h_{ij} \in k[x^{\pm 1}]$ with h_{ij} monic. $\square$

Since $\lambda_i = \gamma^i$ for all i has been shown above, we can improve Claim 2 as

Claim 2'. We have $u_i g = \gamma^i x^{-2d} g u_i$ for $0 \leqslant i \leqslant m-1$.

By Claim 2', we have a unified formula in H: For all $s \in \mathbb{Z}$,

$$u_i g^s = \gamma^{is} x^{-2sd} g^s u_i. \tag{8.12}$$

Claim 4. We have $\phi_i = 1 - \gamma^{-i-1} x^d$ for $0 \leqslant i \leqslant m-1$.

Proof of Claim 4: By Claim 3, there are polynomials $f_{0j}, h_{0j}, f_{1j}, h_{1j}$, such that

$$\begin{aligned}\Delta(u_0) &= u_0 \otimes u_0 + f_{01} u_1 \otimes h_{01} g u_{m-1} + \cdots + f_{0,m-1} u_{m-1} \otimes h_{0,m-1} g^{m-1} u_1, \\ \Delta(u_1) &= f_{10} u_0 \otimes u_1 + u_1 \otimes h_{11} g u_0 + \cdots + f_{1,m-1} u_{m-1} \otimes h_{1,m-1} g^{m-1} u_2.\end{aligned}$$

Firstly, we will show $\phi_0 = 1 - \gamma^{-1} x^d$ by considering the equations

$$\Delta(y u_0)_{11 \otimes 13} = \Delta(\xi x^d u_0 y)_{11 \otimes 13} = \Delta(\phi_0 u_1)_{11 \otimes 13}.$$

Direct computations show that

$$\begin{aligned}\Delta(yu_0)_{11\otimes 13} &= u_0 \otimes yu_0 + yf_{0,m-1}u_{m-1} \otimes gh_{0,m-1}g^{m-1}u_1\\ &= u_0 \otimes \phi_0 u_1 + f_{0,m-1}\phi_{m-1}u_0 \otimes x^{md}h_{0,m-1}u_1,\\ \Delta(\xi x^d u_0 y)_{11\otimes 13} &= \xi x^d u_0 \otimes x^d u_0 y + \xi x^d f_{0,m-1}u_{m-1}y \otimes x^d h_{0,m-1}g^{m-1}u_1 g\\ &= x^d u_0 \otimes \phi_0 u_1 + f_{0,m-1}\phi_{m-1}u_0 \otimes \gamma x^{(m-1)d}h_{0,m-1}u_1.\end{aligned}$$

Owing to $\Delta(yu_0)_{11\otimes 13} = \Delta(\xi x^d u_0 y)_{11\otimes 13}$,

$$(1-x^d)u_0 \otimes \phi_0 u_1 + f_{0,m-1}\phi_{m-1}u_0 \otimes (x^d-\gamma)x^{(m-1)d}h_{0,m-1}u_1 = 0.$$

Thus we can assume $\phi_0 = c_0(x^d-\gamma)x^{(m-1)d}h_{0,m-1}$ for some $0 \neq c_0 \in k$. Then $1-x^d = -c_0^{-1}f_{0,m-1}\phi_{m-1}$. Therefore,

$$\begin{aligned}\Delta(yu_0)_{11\otimes 13} &= u_0 \otimes \phi_0 u_1 - c_0(1-x^d)u_0 \otimes \frac{1}{c_0}\frac{x^d}{x^d-\gamma}\phi_0 u_1\\ &= u_0 \otimes (1-\frac{x^d}{x^d-\gamma})\phi_0 u_1 + x^d u_0 \otimes \frac{x^d}{x^d-\gamma}\phi_0 u_1\\ &= u_0 \otimes (\frac{-\gamma}{x^d-\gamma})\phi_0 u_1 + x^d u_0 \otimes \frac{x^d}{x^d-\gamma}\phi_0 u_1,\end{aligned}$$

where $\frac{1}{x^d-\gamma}\phi_0$ is understood as $c_0 x^{(m-1)d}h_{0,m-1}$. Note that $\Delta(\phi_0 u_1)_{11\otimes 13} = \Delta(\phi_0)(f_{10}u_0 \otimes u_1)$. Comparing the first components of $\Delta(yu_0)_{11\otimes 13}$ and $\Delta(\phi_0 u_1)_{11\otimes 13}$, we get $\phi_0 = 1+\theta x^d$ for some $\theta \in k$. Then it is not hard to see that $f_{10} = 1$, $f_{0,m-1} = \gamma^{-1}$, $h_{0,m-1} = x^{-(m-1)d}$ and $\theta = -\gamma^{-1}$. So $\phi_0 = 1-\gamma^{-1}x^d$.

Secondly, we want to determine ϕ_s for $s \geqslant 1$. To attack this, we will prove the fact

$$f_{j0} = h_{j0} = 1 \tag{8.13}$$

for all $0 \leqslant j \leqslant m-1$ at the same time. We proceed by induction. We already know that $f_{00} = h_{00} = f_{10} = h_{10} = 1$. Assume that $f_{i,0} = h_{i,0} = 1$ now. Similarly, direct computations show that

$$\begin{aligned}\Delta(yu_i)_{11\otimes(1,3+2i)} &= u_0 \otimes yu_i + yf_{i,m-1}u_{m-1} \otimes gh_{i,m-1}g^{m-1}u_{i+1}\\ &= u_0 \otimes \phi_i u_{i+1} + f_{i,m-1}\phi_{m-1}u_0 \otimes x^{md}h_{i,m-1}u_{i+1},\\ \Delta(\xi x^d u_i y)_{11\otimes(1,3+2i)} &= \xi x^d u_0 \otimes x^d u_i y + \xi x^d f_{i,m-1}u_{m-1}y \otimes x^d h_{i,m-1}g^{m-1}u_{i+1}g\\ &= x^d u_0 \otimes \phi_i u_{i+1} + f_{i,m-1}\phi_{m-1}u_0 \otimes \gamma^{i+1}x^{(m-1)d}h_{i,m-1}u_{i+1}.\end{aligned}$$

By $\Delta(yu_i)_{11\otimes(1,3+2i)} = \Delta(\xi x^d u_i y)_{11\otimes(1,3+2i)}$,

$$(1-x^d)u_0 \otimes \phi_i u_{i+1} + f_{i,m-1}\phi_{m-1}u_0 \otimes (x^d-\gamma^{i+1})x^{(m-1)d}h_{i,m-1}u_{i+1} = 0.$$

Thus we can assume $\phi_i = c_i(x^d - \gamma^{i+1})x^{(m-1)d}h_{i,m-1}$ for some $0 \neq c_i \in k$. Then $1 - x^d = -c_i^{-1} f_{i,m-1}\phi_{m-1}$. Therefore

$$\begin{aligned}\Delta(yu_i)_{11\otimes(1,3+2i)} &= u_0 \otimes \phi_i u_{i+1} - c_i(1-x^d)u_0 \otimes \frac{1}{c_i}\frac{x^d}{x^d - \gamma^{i+1}}\phi_i u_{i+1} \\ &= u_0 \otimes (\frac{-\gamma^{i+1}}{x^d - \gamma^{i+1}})\phi_i u_{i+1} + x^d u_0 \otimes \frac{x^d}{x^d - \gamma^{i+1}}\phi_i u_{i+1}.\end{aligned}$$

Note that $\Delta(\phi_i u_{i+1})_{11\otimes(1,3+2i)} = \Delta(\phi_i)(f_{i+1,0}u_0 \otimes h_{i+1,0}u_{i+1})$. Comparing the first components of $\Delta(yu_i)_{11\otimes(1,3+2i)}$ and $\Delta(\phi_i u_{i+1})_{11\otimes(1,3+2i)}$, we get $\phi_i = 1 - \gamma^{-i-1}x^d$ similarly. And it is not hard to see that $f_{i+1,0} = h_{i+1,0} = 1$ and $f_{i,m-1} = \gamma^{-i-1}, h_{i,m-1} = x^{-(m-1)d}$. So we prove that $f_{i+1,0} = h_{i+1,0} = 1$ at the same time. □

Claim 5. The coproduct of H is given by

$$\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j}$$

for $0 \leqslant i \leqslant m-1$.

Proof of Claim 5: By Claim 3, $\Delta(u_i) = \sum_{j=0}^{m-1} f_{ij}u_j \otimes h_{ij}g^j u_{i-j}$. So, to show this claim, it is enough to determine the explicit form of every f_{ij} and h_{ij}. By (8.13), $f_{i,0} = h_{i,0} = 1$. We will prove that $f_{ij} = \gamma^{j(i-j)}$ and $h_{ij} = x^{-jd}$ for all $0 \leqslant i, j \leqslant m-1$ by induction. So it is enough to show that $f_{i,j+1} = \gamma^{(j+1)(i-j-1)}$ and $h_{i,j+1} = x^{-(j+1)d}$ under the hypothesis of $f_{ij} = \gamma^{j(i-j)}$ and $h_{ij} = x^{-jd}$. In fact,

$$\begin{aligned}\Delta(yu_i)_{(1,3+2j)\otimes(3+2j,3+2i)} &= yf_{ij}u_j \otimes gh_{ij}g^j u_{i-j} \\ &\quad + f_{i,j+1}u_{j+1} \otimes yh_{i,j+1}g^{j+1}u_{i-j-1} \\ &= f_{ij}yu_j \otimes h_{ij}g^{j+1}u_{i-j} \\ &\quad + f_{i,j+1}u_{j+1} \otimes \gamma^{j+1}h_{i,j+1}g^{j+1}yu_{i-j-1}, \\ \Delta(\xi x^d u_i y)_{(1,3+2j)\otimes(3+2j,3+2i)} &= \xi x^d f_{ij}u_j y \otimes x^d h_{ij}g^j u_{i-j}g \\ &\quad + \xi x^d f_{i,j+1}u_{j+1} \otimes x^d h_{i,j+1}g^{j+1}u_{i-j-1}y \\ &= f_{ij}yu_j \otimes \gamma^{i-j}x^{-d}h_{ij}g^{j+1}u_{i-j} \\ &\quad + x^d f_{i,j+1}u_{j+1} \otimes h_{i,j+1}g^{j+1}yu_{i-j-1}.\end{aligned}$$

By $\Delta(yu_i)_{(1,3+2j)\otimes(3+2j,3+2i)} = \Delta(\xi x^d u_i y)_{(1,3+2j)\otimes(3+2j,3+2i)}$,

$$f_{ij}yu_j \otimes (1 - \gamma^{i-j}x^{-d})h_{ij}g^{j+1}u_{i-j} = (x^d - \gamma^{j+1})f_{i,j+1}u_{j+1} \otimes h_{i,j+1}g^{j+1}yu_{i-j-1}.$$

By induction, we have

$$\begin{aligned}&\gamma^{j(i-j)}(1-\gamma^{-j-1}x^d)u_{j+1}\otimes(x^d-\gamma^{i-j})x^{-(j+1)d}g^{j+1}u_{i-j}\\ &=(x^d-\gamma^{j+1})f_{i,j+1}u_{j+1}\otimes(1-\gamma^{-i+j}x^d)h_{i,j+1}g^{j+1}u_{i-j}.\end{aligned}$$

This implies that $h_{i,j+1}=x^{-(j+1)d}$ and $f_{i,j+1}=\gamma^{i-j}\gamma^{-j-1}\gamma^{j(i-j)}=\gamma^{(j+1)(i-j-1)}$. □

Claim 6. For $0\leqslant i,j\leqslant m-1$, the multiplication between u_i and u_j satisfies that

$$u_iu_j=(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\ [i,m-2-j]\ y^{\overline{i+j}}g$$

for some $a\in\mathbb{Z}$ and $i+j$ is interpreted $\bmod m$.

Proof of Claim 6: We need to consider the relation between u_0^2 and u_ju_{m-j} for all $1\leqslant j\leqslant m-1$ at first.

By definition, $x^d\overline{\phi_s}=-\gamma^{-s-1}\phi_{m-s-2}$ for all s. Then

$$\begin{aligned}y^mu_0^2&=\xi^{m-j}x^{(m-j)d}y^ju_0y^{m-j}u_0=\xi^{m-j}x^{(m-j)d}\phi_0\cdots\phi_{j-1}u_j\phi_0\cdots\phi_{m-j-1}u_{m-j}\\ &=\xi^{m-j}x^{(m-j)d}\phi_0\cdots\phi_{j-1}\cdot\overline{\phi_0}\cdots\overline{\phi_{m-j-1}}u_ju_{m-j}\\ &=(-1)^{m-j}\xi^{m-j}\gamma^{-\frac{(m-j)(m-j+1)}{2}}\phi_0\cdots\phi_{m-2}\cdot\phi_{j-1}u_ju_{m-j}.\end{aligned}$$

So

$$\phi_{m-1}u_0^2=(-1)^{m-j}\xi^{m-j}\gamma^{-\frac{(m-j)(m-j+1)}{2}}\phi_{j-1}u_ju_{m-j}. \tag{8.14}$$

Since $u_0^2,u_ju_{m-j}\in H_{22}=k[x^{\pm1}]g$, we may assume $u_0^2=\alpha_0g,u_ju_{m-j}=\alpha_jg$ for some $\alpha_0,\alpha_j\in k[x^{\pm1}]$ for all $1\leqslant j\leqslant m-1$.

Then Equation (8.14) implies $\alpha_0=\alpha\phi_0\cdots\phi_{m-2}$ for some $\alpha\in k[x^{\pm1}]$. We claim α is invertible. Indeed, by $\phi_{m-1}\alpha_0=(-1)^{m-j}\xi^{m-j}\gamma^{-\frac{(m-j)(m-j+1)}{2}}\phi_{j-1}\alpha_j$, we have

$$\alpha_j=(-1)^{j-m}\xi^{j-m}\gamma^{\frac{(m-j)(m-j+1)}{2}}\alpha\ ]j-1,j-1[.$$

Then

$$H_{11}\cdot H_{11}+\sum_{j=1}^{m-1}H_{1,1+2j}\cdot H_{1,1+2(m-j)}\subseteq\alpha H_{22}.$$

By the strong grading of H,

$$H_{22}=H_{11}\cdot H_{11}+\sum_{j=1}^{m-1}H_{1,1+2j}\cdot H_{1,1+2(m-j)},$$

which shows that α must be invertible. Since $\epsilon(\alpha_0) = 1$ and $\epsilon(\phi_0 \cdots \phi_{m-2}) = m$, we may assume $\alpha_0 = \frac{1}{m} x^a \phi_0 \cdots \phi_{m-2}$ for some integer a. Thus

$$\begin{aligned} u_j u_{m-j} &= (-1)^{j-m} \xi^{j-m} \gamma^{\frac{(m-j)(m-j+1)}{2}} \frac{1}{m} x^a \;]j-1, j-1[\; g \\ &= (-1)^j \xi^j \gamma^{\frac{-j(-j+1)}{2}} \frac{1}{m} x^a \;]j-1, j-1[\; g. \end{aligned}$$

Case 1. If $0 \leqslant i+j \leqslant m-2$, then

$$\begin{aligned} y^{i+j} u_0^2 &= \xi^j x^{jd} y^i u_0 y^j u_0 \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} u_i \phi_0 \cdots \phi_{j-1} u_j \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} \cdot \overline{\phi_0} \cdots \overline{\phi_{j-1}} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{i-1} \cdot \phi_{m-1-j} \cdots \phi_{m-2} u_i u_j. \end{aligned}$$

So

$$u_i u_j = (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^a \phi_i \cdots \phi_{m-2-j} y^{i+j} g.$$

Case 2. If $i+j = m-1$, then

$$\begin{aligned} y^{i+j} u_0^2 &= \xi^j x^{jd} y^i u_0 y^j u_0 \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} u_i \phi_0 \cdots \phi_{j-1} u_j \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} \cdot \overline{\phi_0} \cdots \overline{\phi_{j-1}} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{i-1} \cdot \phi_{m-1-j} \cdots \phi_{m-2} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{m-2} u_i u_j. \end{aligned}$$

So

$$u_i u_j = (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^a y^{i+j} g.$$

Case 3. If $m \leqslant i+j \leqslant 2m-2$, then

$$\begin{aligned} y^{i+j} u_0^2 &= \xi^j x^{jd} y^i u_0 y^j u_0 \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} u_i \phi_0 \cdots \phi_{i-1} u_j \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} \cdot \overline{\phi_0} \cdots \overline{\phi_{j-1}} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{i-1} \cdot \phi_{m-1-j} \cdots \phi_{m-2} u_i u_j. \end{aligned}$$

So

$$u_i u_j = (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\phi_i\cdots\phi_{m-1}\phi_0\cdots\phi_{m-2-j}y^{i+j-m}g.$$

Using the notations introduced in Subsection 4.1, we have a unified expression:

$$\begin{aligned} u_i u_j &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\ [i, m-2-j]\ y^{\overline{i+j}}g \\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\ ]-1-j, i-1[\ y^{\overline{i+j}}g \end{aligned}$$

for all i, j. □

Claim 7. We have $\xi^2 = \gamma,\ a = -\frac{1+m}{2}d$ *and*

$$S(u_i) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-1-i}u_i$$

for $0 \leqslant i \leqslant m-1$.

Proof of Claim 7: Since $S(H_{ij}) = H_{-j,-i}$, $S(u_0) = hg^{m-1}u_0$ for some $h \in k[x^{\pm 1}]$. Combining

$$\begin{aligned} S(yu_0) &= S(u_0)S(y) = hg^{m-1}u_0 \cdot (-yg^{-1}) = -\xi x^{-d}hg^{m-1}yu_0g^{-1} \\ &= -\xi x^{-d}\phi_0 hg^{m-1}u_1g^{-1} = -\xi\gamma^{-1}x^d\phi_0 hg^{m-2}u_1 \end{aligned}$$

with

$$S(yu_0) = S(\phi_0 u_1) = S(u_1)S(\phi_0) = \phi_0 S(u_1),$$

we get $S(u_1) = -\xi\gamma^{-1}x^d hg^{m-2}u_1$. The computation above tells us that we can prove that

$$S(u_i) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id}hg^{m-1-i}u_i$$

by induction. In fact, assume that $S(u_i) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id}hg^{m-1-i}u_i$, by combining

$$\begin{aligned} S(yu_i) &= S(u_i)S(y) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id}hg^{m-1-i}u_i(-yg^{-1}) \\ &= \phi_i \cdot (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d}hg^{m-2-i}u_{i+1} \end{aligned}$$

with

$$S(yu_i) = S(\phi_i u_{i+1}) = S(u_{i+1})S(\phi_i) = \phi_i S(u_{i+1}),$$

we find that $S(u_{i+1}) = (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d}hg^{m-2-i}u_{i+1}$.

In order to determine the relationship between ξ and γ, we consider the equality $(\mathrm{Id} * S)(u_1) = 0$. By computation,

$$\begin{aligned}(\mathrm{Id} * S)(u_1) &= \sum_{j=0}^{m-1} \gamma^{j(1-j)} u_j S(x^{-jd} g^j u_{1-j}) \\ &= \sum_{j=0}^{m-1} \gamma^{j(1-j)} u_j \cdot (-1)^{1-j} \xi^{j-1} \gamma^{-\frac{(1-j)(2-j)}{2}} x^{(1-j)d} h g^{m-2+j} u_{1-j} g^{-j} x^{jd} \\ &= \sum_{j=0}^{m-1} (-1)^{1-j} \xi^{j-1} \gamma^{j(1-j)-\frac{(1-j)(2-j)}{2}} x^{(2j-1)d} \overline{h} u_j g^{m-2+j} u_{1-j} g^{-j} \\ &= \sum_{j=0}^{m-1} (-1)^{1-j} \xi^{j-1} \gamma^{-\frac{(1-j)(2-j)}{2}-2j} x^{(3-2m)d} \overline{h} g^{m-2} u_j u_{1-j} \\ &= \frac{1}{m} \xi^{-2} x^{(3-2m)d+a} \overline{h} g^{m-1} (\sum_{j=0}^{m-1} \xi^{2j} \gamma^{-2j} \,]j-2, j-1[),\end{aligned}$$

where Equation (8.12) is used. Thus

$$(\mathrm{Id} * S)(u_1) = 0 \Leftrightarrow \sum_{j=0}^{m-1} \xi^{2j} \gamma^{-2j} \,]j-2, j-1[= 0.$$

This forces $\xi^2 = \gamma$ by Corollary 3.4.

Next, we will show $h = x^{\frac{3}{2}(1-m)d}$ by the equations

$$(S * \mathrm{Id})(u_0) = (\mathrm{Id} * S)(u_0) = 1.$$

Indeed,

$$\begin{aligned}(S * \mathrm{Id})(u_0) &= \sum_{j=0}^{m-1} S(\gamma^{-j^2} u_j) x^{-jd} g^j u_{-j} \\ &= \sum_{j=0}^{m-1} \gamma^{-j^2} (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} x^{jd} h g^{m-j-1} u_j x^{-jd} g^j u_{-j} \\ &= h g^{m-1} (\sum_{j=0}^{m-1} (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} u_j u_{-j}) \\ &= h g^{m-1} (\sum_{j=0}^{m-1} (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} (-1)^{j-m} \xi^{j-m} \gamma^{\frac{1}{2}(m-j)(m-j+1)} \\ &\qquad \times \frac{1}{m} x^a \,]j-1, j-1[\, g)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m} x^a h g^m \Big(\sum_{j=0}^{m-1} (-1)^{-m} \xi^{-m} \gamma^{\frac{m(m+1)}{2}-j} \;]j-1, j-1[\Big) \\
&= \frac{1}{m} x^a h g^m \Big(\sum_{j=0}^{m-1} \gamma^{-j} \;]j-1, j-1[\Big) \\
&= x^{(2m-1)d+a} h \quad \text{(by Lemma 3.2)}, \\
(\mathrm{Id} * S)(u_0) &= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j \cdot S(x^{-jd} g^j u_{-j}) \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j \cdot S(u_{-j}) S(g^j) x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j \cdot (-1)^{-j} \xi^j \gamma^{\frac{j(1-j)}{2}} x^{-jd} h g^{m+j-1} u_{-j} g^{-j} x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j \cdot (-1)^{-j} \xi^j \gamma^{\frac{j(1-j)}{2}+j^2} h g^{m-1} u_{-j} \\
&= x^{(2-2m)d} \overline{h} g^{m-1} \Big(\sum_{j=0}^{m-1} (-1)^{-j} \xi^j \gamma^{\frac{j(1-j)}{2}-j} u_j u_{-j}\Big) \\
&= \frac{1}{m} x^{(2-m)d+a} \overline{h} \Big(\sum_{j=0}^{m-1} \xi^{2j} \gamma^{-j} \;]j-1, j-1[\Big) \\
&= x^{(2-m)d+a} \overline{h} \quad \text{(by the proof of Lemma 3.2)}.
\end{aligned}
$$

So, $(S * \mathrm{Id})(u_0) = (\mathrm{Id} * S)(u_0) = 1$ implies $h = x^{(1-2m)d-a} = x^{(2-m)d+a}$. Thus $h = x^{\frac{3}{2}(1-m)d}$ and $a = -\frac{1+m}{2}d$. Therefore, for $0 \leqslant i \leqslant m-1$,

$$S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id+\frac{3}{2}(1-m)d} g^{m-1-i} u_i. \qquad \square$$

From Claim 7, we find that $2|(1+m)d$ and we can improve Claim 6 as the following form:

Claim 6'. For $0 \leqslant i, j \leqslant m-1$, *the multiplication between* u_i *and* u_j *satisfies that*

$$u_i u_j = (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \, [i, m-2-j] \, y^{\overline{i+j}} g$$

where $i+j$ *is interpreted* $\bmod m$.

We can prove Proposition 8.2 now. By Claims 1, 2', 3, 4, 5, 6' and 7, we have a natural surjective Hopf homomorphism

$$f: \; D(m, d, \xi) \to H, \; x \mapsto x, \; y \mapsto y, \; g \mapsto g, \; u_i \mapsto u_i$$

for $0 \leqslant i \leqslant m-1$. It is not hard to see that $f|_{D_{ij}} : D_{ij} \to H_{ij}$ is an isomorphism of $k[x^{\pm 1}]$-modules for $0 \leqslant i, j \leqslant 2m-1$. So f is an isomorphism. □

We conclude this paper by giving the classification of prime regular Hopf algebras of GK-dimension one.

Theorem 8.3. *Let H be a prime regular Hopf algebra of GK-dimension one. Then it is isomorphic to one of the following:*

(1) *the Hopf algebras listed in Subsection 2.3;*

(2) *the Hopf algebras constructed in Subsection 4.1.*

Proof. By Theorem 2.7, we only need to consider the case $\mathrm{io}(H) > \mathrm{im}(H) > 1$. In this case, $\widetilde{H}$ can be constructed. By Proposition 5.8, $\widetilde{H}$ is either primitive or group-like. If $\widetilde{H}$ is primitive, then H is isomorphic to an infinite dimensional Taft algebra by Theorem 7.4. If $\widetilde{H}$ is group-like, owing to Proposition 8.1 there is no such H satisfying $\frac{\mathrm{io}(H)}{\mathrm{im}(H)} > 2$. Moreover, if $\frac{\mathrm{io}(H)}{\mathrm{im}(H)} = 2$ then H is one of the Hopf algebras constructed in Subsection 4.1 by Proposition 8.2. □

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