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ON COQUASITRIANGULAR POINTED MAJID ALGEBRAS

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We study coquasitriangular pointed Majid algebras and braided pointed tensor categories via the quiver approaches. The class of Hopf quivers whose path coalgebras admit coquasitriangular Majid algebras is classified. The quiver settings for general coquasitriangular pointed Majid algebras are also provided. Through this, some examples and classification results are obtained.

Key Words: Braided tensor category; Coquasitriangular Majid algebra; Hopf quiver; Quasitriangular quasi-Hopf algebra.

2000 Mathematics Subject Classification: 16W30; 16G20

1. INTRODUCTION

Quasitriangular quasi-Hopf algebras were introduced and profoundly studied by Drinfeld in a series of papers [5–7]. These are a natural generalization of quasitriangular Hopf algebras which play an essential role in his theory of quantum groups [4]. They turn out to have deep connections with tensor categories, conformal field theory, knot invariants, Grothendieck-Teichmüller group, multiple zeta value, and so on.

In [18], Majid showed by a categorical formulation that any quasi-Hopf algebra can be associated to a quasitriangular quasi-Hopf algebra by a quantum double construction which generalizes the well-known one of Drinfeld's for Hopf algebras [4]. However, there are not many “natural” examples of quasitriangular quasi-Hopf algebras in literature other than the quasi-triangular quasi-Hopf QUE-algebras of Drinfeld [5] and the twisted quantum double of finite groups of Dijgraaf–Pasquier–Roche [3] as well as their various generalizations. In particular, the fundamental problem of classification and construction of quasitriangular quasi-Hopf algebras is still widely open. It is our expectation that there will be a nice theory for the systematic construction and classification of some interesting classes of quasitriangular quasi-Hopf algebras and the associated braided tensor categories, for instance, an extension of the classification theory of finite-dimensional triangular Hopf algebras over the field of complex numbers due to Etingof and Gelaki (see [8] and references therein) into the quasitriangular quasi-Hopf setting.

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This article is devoted to the study of quasitriangular quasi-Hopf algebras and braided tensor categories via the quiver approaches initiated in [11, 12, 14, 15]. Note that the notion of quasi-Hopf algebras is not self-dual, so there is a dual notion which is called Majid algebra [20]. In this terminology, the dual of a quasitriangular quasi-Hopf algebra is called a coquasitriangular Majid algebra. As in previous works [11, 12, 14, 15], we will work mainly on this dual setting since this allows a wider scope and the convenience of exposition (see [11, 12] for an explanation). A recent work of the authors [14] shows that the coquasitriangularity of pointed Hopf algebras can be described by combinatorial property of Hopf quivers. Moreover, the quiver setting helps to give a complete classification of finite-dimensional coquasitriangular pointed Hopf algebras over an algebraically closed field of characteristic 0. The basic aim of the present paper is to extend the study in [14] to the “quasi” situation.

We start by showing that the path coalgebra kQ of a quiver Q admits a coquasitriangular Majid algebra structure if and only if Q is a Hopf quiver of the form $Q(G, R)$ with G abelian. Next we give a classification of the set of graded coquasitriangular Majid structures on a given connected Hopf quiver of this form. Then we show that, for a general coquasitriangular pointed Majid algebra, its graded version induced by coradical filtration can be viewed as a large sub-structure of a graded coquasitriangular Majid structure on some unique Hopf quiver defined in the previous step. So far a quiver setting for the class of coquasitriangular pointed Majid algebras is built up. Finally, we use the quiver setting to provide some examples and classification results.

Throughout the article, we work over a field k . Vector spaces, algebras, coalgebras, linear mappings, and unadorned \otimes are over k . The readers are referred to [17, 20] for general knowledge of quasi-Hopf and Majid algebras, and to [1] for that of quivers and their applications to associative algebras and representation theory. We turn to [11, 12] frequently for definitions, notations, and results of the quiver setting of Majid algebras.

2. MAJID ALGEBRAS AND THEIR QUIVER SETTING

In this section we recall the definition of Majid algebras and the quiver framework of pointed Majid algebras for the convenience of the readers.

2.1. Majid Algebras

A Majid algebra is a coalgebra (H, Δ, ε) equipped with a compatible quasi-algebra structure and a quasi-antipode. Namely, there exist two coalgebra homomorphisms

$$M : H \otimes H \longrightarrow H, \quad a \otimes b \mapsto ab, \quad \mu : k \longrightarrow H, \quad \lambda \mapsto \lambda 1_H,$$

a convolution-invertible map $\Phi : H^{\otimes 3} \longrightarrow k$ called reassociator, a coalgebra antimorphism $\mathcal{S} : H \longrightarrow H$, and two functionals $\alpha, \beta : H \longrightarrow k$ such that for all

$a, b, c, d \in H$ the following equalities hold:

$$a_1(b_1c_1)\Phi(a_2, b_2, c_2) = \Phi(a_1, b_1, c_1)(a_2b_2)c_2, \tag{2.1}$$

$$1_H a = a = a 1_H, \tag{2.2}$$

$$\Phi(a_1, b_1, c_1d_1)\Phi(a_2b_2, c_2, d_2) \tag{2.3}$$

$$= \Phi(b_1, c_1, d_1)\Phi(a_1, b_2c_2, d_2)\Phi(a_2, b_3, c_3),$$

$$\Phi(a, 1_H, b) = \varepsilon(a)\varepsilon(b), \tag{2.4}$$

$$\mathcal{S}(a_1)\alpha(a_2)a_3 = \alpha(a)1_H, \quad a_1\beta(a_2)\mathcal{S}(a_3) = \beta(a)1_H, \tag{2.5}$$

$$\Phi(a_1, \mathcal{S}(a_3), a_5)\beta(a_2)\alpha(a_4) \tag{2.6}$$

$$= \Phi^{-1}(\mathcal{S}(a_1), a_3, \mathcal{S}(a_5))\alpha(a_2)\beta(a_4) = \varepsilon(a).$$

Here and below we use the Sweedler sigma notation $\Delta(a) = a_1 \otimes a_2$ for the coproduct and $a_1 \otimes a_2 \otimes \dots \otimes a_{n+1}$ for the result of the n -iterated application of Δ on a .

A Majid algebra H with reassociator Φ is said to be coquasitriangular, if there is a convolution-invertible map $\mathcal{R} : H \otimes H \rightarrow k$ such that

$$\mathcal{R}(ab, c) = \Phi(c_1, b_1, a_1)\mathcal{R}(a_2, c_2)\Phi^{-1}(a_3, c_3, b_2) \tag{2.7}$$

$$\times \mathcal{R}(b_3, c_4)\Phi(a_4, b_4, c_5),$$

$$\mathcal{R}(a, bc) = \Phi^{-1}(b_1, c_1, a_1)\mathcal{R}(a_2, c_2)\Phi(b_2, a_3, c_3) \tag{2.8}$$

$$\times \mathcal{R}(a_4, b_3)\Phi^{-1}(a_5, b_4, c_4),$$

$$b_1a_1\mathcal{R}(a_2, b_2) = \mathcal{R}(a_1, b_1)a_2b_2 \tag{2.9}$$

for all $a, b, c \in H$. The map \mathcal{R} is called a coquasitriangular structure of H . A coquasitriangular Majid algebra (H, Φ, \mathcal{R}) is called cotriangular if

$$\mathcal{R}(a, b)\mathcal{R}(b, a) = \varepsilon(a)\varepsilon(b) \tag{2.10}$$

for all $a, b \in H$.

2.2. Hopf Quivers

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t : Q_1 \rightarrow Q_0$ are two maps assigning respectively the source and the target for each arrow. A path of length $l \geq 1$ in the quiver Q is a finitely ordered sequence of l arrows $a_l \dots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $1 \leq i \leq l - 1$. By convention a vertex is said to be a trivial path of length 0. Let Q_n denote the set of paths of length n in Q . There is a natural path coalgebra structure on the path space kQ with coproduct defined by splitting of paths.

According to [2], a quiver Q is said to be a Hopf quiver if the corresponding path coalgebra kQ admits a graded Hopf algebra structure. Hopf quivers can be determined by ramification data of groups as follows. Let G be a group and denote

its set of conjugacy classes by \mathcal{C} . A ramification datum R of the group G is a formal sum $\sum_{C \in \mathcal{C}} R_C C$ of conjugacy classes with coefficients in $\mathbb{N} = \{0, 1, 2, \dots\}$. The corresponding Hopf quiver $Q = Q(G, R)$ is defined as follows: the set of vertices Q_0 is G , and for each $x \in G$ and $c \in \mathcal{C}$, there are R_c arrows going from x to cx .

2.3. Quiver Setting for Majid Algebras

It is shown in [11] that a path coalgebra kQ admits a graded Majid algebra structure if and only if the quiver Q is a Hopf quiver. Moreover, given a Hopf quiver $Q = Q(G, R)$, the set of graded Majid algebra structures on kQ with $kQ_0 = (kG, \Phi)$ as Majid algebras is in one-to-one correspondence with the set of (kG, Φ) -Majid bimodule structures on kQ_1 . *In this article, we always ignore the difference of various quasi-antipodes of a Majid algebra, since they are essentially equivalent according to Drinfeld [5] (Proposition 1.1).*

Recall that, if M is a (kG, Φ) -Majid bimodule, then the underlying bicomodule structure makes it a G -bigraded space $M = \bigoplus_{g, h \in G} {}^g M^h$ with (g, h) -isotypic component ${}^g M^h = \{m \in M \mid \delta_{\ell}(m) = g \otimes m, \delta_r(m) = m \otimes h\}$; meanwhile, the quasi-bimodule structure maps satisfy the following equalities:

$$e(fm) = \frac{\Phi(e, f, g)}{\Phi(e, f, h)}(ef)m, \quad (2.11)$$

$$(me)f = \frac{\Phi(h, e, f)}{\Phi(g, e, f)}m(ef), \quad (2.12)$$

$$(em)f = \frac{\Phi(e, h, f)}{\Phi(e, g, f)}e(mf), \quad (2.13)$$

for all $e, f, g, h \in G$ and $m \in {}^g M^h$.

A Majid algebra is said to be pointed, if its underlying coalgebra is pointed. Given a pointed Majid algebra (H, Φ) , let $\{H_n\}_{n \geq 0}$ be its coradical filtration. Then the corresponding coradically graded coalgebra $\text{gr}(H) = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \dots$ has an induced graded Majid algebra structure with graded reassociator $\text{gr}(\Phi)$ satisfying $\text{gr}(\Phi)(\bar{a}, \bar{b}, \bar{c}) = 0$ for all homogeneous $\bar{a}, \bar{b}, \bar{c} \in \text{gr}(H)$ unless they all lie in H_0 . In particular, H_0 is a sub-Majid algebra and turns out to be the group algebra kG of the group $G = G(H)$, the set of group-like elements of H . In addition, the restriction of Φ to the coradical of H is a 3-cocycle on G .

Thanks to the Gabriel type theorem in [11] (Theorem 3.4), for an arbitrary pointed Majid algebra H , its graded version $\text{gr}(H)$ can be realized as a large sub-Majid algebra of some graded Majid algebra structure on a unique Hopf quiver. By “large” it is meant the sub-Majid algebra contains the set of vertices and arrows of the Hopf quiver.

For a coquasitriangular pointed Majid algebra (H, Φ, \mathcal{R}) , let $(\text{gr}(H), \text{gr}(\Phi))$ be as above. Define the function $\text{gr}(\mathcal{R}) : \text{gr}(H) \otimes \text{gr}(H) \rightarrow k$, for all homogeneous elements $g, h \in \text{gr}(H)$, by

$$\text{gr}(\mathcal{R})(g, h) = \begin{cases} \mathcal{R}(g, h), & \text{if } g, h \in H_0; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following straightforward fact which is useful later on.

Lemma 2.1. *The coradically graded version $(\text{gr}(H), \text{gr}(\Phi), \text{gr}(\mathcal{R}))$ is still a coquasitriangular Majid algebra.*

2.4. Multiplication Formula for Quiver Majid Algebras

It is shown in [11] that the path multiplication formula of graded Majid algebras on Hopf quivers can be given via quantum shuffle product as in [2].

Suppose that Q is a Hopf quiver with a necessary kQ_0 -Majid bimodule structure on kQ_1 . Let $p \in Q_i$ be a path. An n -thin split of it is a sequence (p_1, \dots, p_n) of vertices and arrows such that the concatenation $p_n \cdots p_1$ is exactly p . These n -thin splits are in one-to-one correspondence with the n -sequences of $(n-l)$ 0's and l 1's. Denote the set of such sequences by D_i^n . Clearly, $|D_i^n| = \binom{n}{l}$. For $d = (d_1, \dots, d_n) \in D_i^n$, the corresponding n -thin split is written as $dp = ((dp)_1, \dots, (dp)_n)$, in which $(dp)_i$ is a vertex if $d_i = 0$ and an arrow if $d_i = 1$. Let $\alpha = a_m \cdots a_1$ and $\beta = b_n \cdots b_1$ be paths of length m and n , respectively. Let $d \in D_m^{m+n}$ and $\bar{d} \in D_n^{m+n}$ the complement sequence which is obtained from d by replacing each 0 by 1 and each 1 by 0. Define an element

$$(\alpha\beta)_d = [(d\alpha)_{m+n}(\bar{d}\beta)_{m+n}] \cdots [(d\alpha)_1(\bar{d}\beta)_1]$$

in kQ_{m+n} , where $[(d\alpha)_i(\bar{d}\beta)_i]$ is understood as the action of kQ_0 -Majid bimodule on kQ_1 and these terms in different brackets are put together by cotensor product, or equivalently concatenation. In terms of these notations, the formula of the product of α and β is given as follows:

$$\alpha\beta = \sum_{d \in D_m^{m+n}} (\alpha\beta)_d. \quad (2.14)$$

3. COQUASITRIANGULAR MAJID ALGEBRAS ON QUIVERS

In this section, we determine those quivers whose path coalgebras admit coquasitriangular Majid algebra structures. A classification of the set of graded coquasitriangular structures on such quivers is also obtained.

3.1.

Our first step is to determine the condition on a quiver Q such that its path coalgebra kQ admits a coquasitriangular Majid algebra structure.

Proposition 3.1. *Let Q be a quiver. Then kQ admits a coquasitriangular Majid algebra structure if and only if Q is a Hopf quiver of form $Q(G, R)$ where G is an abelian group and R a ramification datum.*

Proof. Assume that Q is a quiver such that kQ admits a coquasitriangular Majid structure. By Lemma 2.1, we can assume that the coquasitriangular Majid algebra is graded, namely both the reassociator Φ and the coquasitriangular structure \mathcal{R} concentrate at degree 0. Then by [11] (Theorem 3.1), in the first place Q must be a Hopf quiver, say $Q(G, R)$. Note that $kQ_0 = kG$ is a group algebra and that

(kG, Φ, \mathcal{R}) is a coquasitriangular Majid algebra. Here Φ and \mathcal{R} are understood as their restriction to the degree 0 part. Now by (2.9) we have

$$hg\mathcal{R}(g, h) = \mathcal{R}(g, h)gh$$

for all $g, h \in G$. Since \mathcal{R} is convolution-invertible, one always has $\mathcal{R}(g, h) \neq 0$ and then $gh = hg$. This proves that G is an abelian group.

Conversely, assume that Q is the Hopf quiver $Q(G, R)$ of some abelian group G with respect to a ramification datum R . Then we can take the trivial 3-cocycle Φ on G , that is, $\Phi(f, g, h) = 1$ for all $f, g, h \in G$, and then the (kG, Φ) -Majid bimodule structure on kQ_1 which corresponds to the product of a set of trivial kG -modules. For more detail, see [12] (Theorem 3.3). That implies, for all $g \in G$ and $\alpha \in Q_1$, we have $g\alpha = \alpha g$. By the product formula given in Subsection 2.4, this gives rise to a commutative graded Majid structure on kQ . In fact this is even a commutative Hopf algebra as the 3-cocycle Φ is trivial. Apparently, $(kQ, \Phi, \varepsilon \otimes \varepsilon)$ is a coquasitriangular Majid algebra. \square

3.2.

Next we turn to classify the set of graded coquasitriangular Majid algebra structures on a Hopf quiver of the form $Q(G, R)$ with G abelian and $R = \sum_{g \in G} R_g g$. By the Cartier-Gabriel decomposition theorem for pointed Majid algebras [11] (Theorem 4.1), every graded Majid algebra on a general Hopf quiver can be written as the crossed product of the sub-structure on its connected component containing the identity and a group algebra possibly twisted by a 3-cocycle. Therefore, in the following we can assume without loss of generality that the quiver $Q(G, R)$ is connected. By definition, it is clear that the Hopf quiver $Q(G, R)$ is connected if and only if the set $\{g \in G | R_g \neq 0\}$ generates the group G .

Theorem 3.2. *Let $Q = Q(G, R)$ be a connected Hopf quiver with G abelian and $R = \sum_{g \in G} R_g g$. Then the set of graded coquasitriangular Majid algebra structures on kQ with reassociator and coquasitriangular structure concentrating at degree 0 and $Q_0 \cong G$ as groups is in one-to-one correspondence with the set of pairs (Φ, \mathcal{R}) in which $\Phi : G \times G \times G \rightarrow k$ is a 3-cocycle such that*

$$\frac{\Phi(eg, f, t)\Phi(g, e, t)\Phi(e, t, f)}{\Phi(eg, t, f)\Phi(g, t, e)\Phi(e, f, t)} = \frac{\Phi(gt, e, f)\Phi(g, ef, t)\Phi(t, e, f)}{\Phi(g, e, f)\Phi(g, t, ef)}, \quad (3.1)$$

$$\frac{\Phi(e, g, t)\Phi(eg, f, t)}{\Phi(f, g, t)\Phi(eg, t, f)} = \frac{\Phi(e, gt, f)\Phi(e, fg, t)\Phi(g, f, t)}{\Phi(e, g, f)\Phi(g, t, f)\Phi(f, g, t)} \quad (3.2)$$

for all $e, f, g \in G$ and $t \in G$ with $R_t \neq 0$, and $\mathcal{R} : G \times G \rightarrow k$ is a map such that

$$\mathcal{R}(f, gh) = \mathcal{R}(f, g)\mathcal{R}(f, h) \frac{\Phi(g, f, h)}{\Phi(g, h, f)\Phi(f, g, h)}, \quad (3.3)$$

$$\mathcal{R}(fg, h) = \mathcal{R}(f, h)\mathcal{R}(g, h) \frac{\Phi(h, f, g)\Phi(f, g, h)}{\Phi(f, h, g)}, \quad (3.4)$$

$$\mathcal{R}(g, h)\mathcal{R}(h, g) = 1 \quad (3.5)$$

for all $f, g, h \in G$.

Proof. Assume that (kQ, Φ, \mathcal{R}) is a graded coquasitriangular Majid algebra with Φ and \mathcal{R} concentrating at degree 0 and $Q_0 \cong G$. Then the restriction to degree 0 part, namely, (kG, Φ, \mathcal{R}) , is again coquasitriangular. By definition, it is clear that Φ is a 3-cocycle on G and by (2.7)-(2.8) \mathcal{R} satisfies

$$\begin{aligned} \mathcal{R}(f, gh) &= \mathcal{R}(f, g)\mathcal{R}(f, h) \frac{\Phi(g, f, h)}{\Phi(g, h, f)\Phi(f, g, h)}, \\ \mathcal{R}(fg, h) &= \mathcal{R}(f, h)\mathcal{R}(g, h) \frac{\Phi(h, f, g)\Phi(f, g, h)}{\Phi(f, h, g)} \end{aligned}$$

for all $f, g, h \in G$. Next we verify (3.5). Choose any $g, h \in G$ with $R_g R_h \neq 0$. Then in Q there are arrows starting from the unit ϵ of G , say $\alpha : \epsilon \rightarrow g$ and $\beta : \epsilon \rightarrow h$. Then by (2.9) we have

$$\beta g \mathcal{R}(g, \epsilon) = \mathcal{R}(g, h) g \beta, \quad \alpha h \mathcal{R}(h, \epsilon) = h \alpha \mathcal{R}(h, g).$$

Here we have used the fact that Φ and \mathcal{R} concentrate at degree 0. By (2.7)-(2.8) it is easy to deduce that $\mathcal{R}(g, \epsilon) = 1 = \mathcal{R}(\epsilon, h)$ for any $g, h \in G$. Hence we have

$$\beta g = \mathcal{R}(g, h) g \beta, \quad \alpha h = h \alpha \mathcal{R}(h, g).$$

Now together with (2.9) and (2.14), we have

$$\begin{aligned} \beta \alpha &= [\beta g][\alpha] + [h \alpha][\beta] = \mathcal{R}(g, h)[g \beta][\alpha] + [h \alpha][\beta] \\ &= \mathcal{R}(g, h) \alpha \beta = \mathcal{R}(g, h)[\alpha h][\beta] + \mathcal{R}(g, h)[g \beta][\alpha] \\ &= \mathcal{R}(g, h) \mathcal{R}(h, g)[h \alpha][\beta] + \mathcal{R}(g, h)[g \beta][\alpha]. \end{aligned}$$

It follows that $\mathcal{R}(g, h)\mathcal{R}(h, g) = 1$. For any $f, g, h \in G$ with $R_f R_g R_h \neq 0$, we have

$$\begin{aligned} &\mathcal{R}(f, gh)\mathcal{R}(gh, f) \\ &= \mathcal{R}(f, g)\mathcal{R}(f, h) \frac{\Phi(g, f, h)}{\Phi(g, h, f)\Phi(f, g, h)} \mathcal{R}(g, f)\mathcal{R}(h, f) \frac{\Phi(f, g, h)\Phi(g, h, f)}{\Phi(g, f, h)} \\ &= 1. \end{aligned}$$

As the Hopf quiver Q is connected, all such f, g, h run through a generating set of G , so (3.5) follows. Finally, we prove (3.1)–(3.2). If $R_t \neq 0$, then in Q there is an arrow $\alpha : \epsilon \rightarrow t$. For any $e, f, g \in G$, by the definition of Majid algebras, we have

$$\begin{aligned} e(f(g\alpha)) &= \frac{\Phi(e, f, g\alpha)}{\Phi(e, f, g)}(ef)(g\alpha), \\ ((g\alpha)e)f &= \frac{\Phi(g, e, f)}{\Phi(gt, e, f)}(g\alpha)(ef), \\ (e(g\alpha))f &= \frac{\Phi(e, g, f)}{\Phi(e, gt, f)}e((g\alpha)f). \end{aligned}$$

Since \mathcal{R} is a coquasitriangular structure, by the first equation and (2.9) all the terms of the last two equations can be written as some scalars times $(efg)\alpha$. By comparison of the scalars, one has (3.1) and (3.2).

Conversely, we assume that (Φ, \mathcal{R}) is a pair satisfying (3.1)–(3.5). Let M be the k -space spanned by the set $\{g\alpha | g \in G, \alpha \in Q_1 \text{ with } s(\alpha) = \epsilon\}$. Set $\delta_L(g\alpha) = gt(\alpha) \otimes g\alpha$ and $\delta_R(g\alpha) = g\alpha \otimes g$. Then it is direct to verify that (M, δ_L, δ_R) is a kG -bicomodule and is isomorphic to kQ_1 . For each $f \in G$, define

$$f(g\alpha) = \Phi(f, g, t(\alpha))(fg)\alpha, \quad (g\alpha)f = \frac{\mathcal{R}(f, gt(\alpha))}{\mathcal{R}(f, g)}\Phi(f, g, t(\alpha))(fg)\alpha. \quad (3.6)$$

We claim that this defines (kG, Φ) -Majid bimodule on M , that is, (2.11)–(2.13) hold and the quasi-bimodule structure is compatible with the bicomodule structure. By definition (3.6), we have

$$\begin{aligned} e(f(g\alpha)) &= \Phi(f, g, t(\alpha))e((fg)\alpha) = \Phi(f, g, t(\alpha))\Phi(e, fg, t(\alpha))(efg)\alpha, \\ \frac{\Phi(e, f, gt(\alpha))}{\Phi(e, f, g)}(ef)(g\alpha) &= \frac{\Phi(e, f, gt(\alpha))}{\Phi(e, f, g)}\Phi(ef, g, t(\alpha))(efg)\alpha. \end{aligned}$$

Since Φ is a 3-cocycle, it follows that

$$e(f(g\alpha)) = \frac{\Phi(e, f, gt(\alpha))}{\Phi(e, f, g)}(ef)(g\alpha).$$

This is (2.11). Similarly, by direct calculation one can show that (3.1) and (3.2) imply respectively (2.12) and (2.13). It is clear that the quasi-bimodule structure maps are bicomodule morphisms. Now by [11] (Proposition 3.3), the (kG, Φ) -Majid bimodule structure on M can provide a graded Majid algebra structure on kQ where the reassociator is the trivial extension of Φ . That is, set $\Phi(x, y, z) = 0$ whenever one of x, y, z lies out of kQ_0 . The map \mathcal{R} is extended trivially in a similar manner. We claim that (kQ, Φ, \mathcal{R}) is coquasitriangular. Since Φ and \mathcal{R} concentrate at degree 0, the axioms (2.7)–(2.8) are direct consequence of the conditions (3.3)–(3.4). It remains to verify (2.9). We need to show that the equation

$$\beta\alpha\mathcal{R}(s(a_1), s(b_1)) = \mathcal{R}(t(a_m), t(b_n))\alpha\beta$$

holds for all paths $\alpha = a_m \cdots a_1$, $\beta = b_n \cdots b_1$. Here we use the following convention: if $m = 0$, then $\alpha \in Q_0$ and $t(\alpha) = \alpha = s(\alpha)$. When $l(\alpha) + l(\beta) \leq 1$, the equation is obvious. Now let $\alpha = a_m \cdots a_1$, $\beta = b_n \cdots b_1$ with $m + n > 1$. Then we have by the preceding cases and the product formula (2.14) that

$$\begin{aligned} \beta\alpha &= \sum_{d \in D_n^{m+n}} [(d\beta)_{m+n}(\bar{d}\alpha)_{m+n}] \cdots [(d\beta)_1(\bar{d}\alpha)_1] \\ &= \sum_{d \in D_n^{m+n}} \left[\frac{\mathcal{R}(t((\bar{d}\alpha)_{m+n}), t((d\beta)_{m+n}))}{\mathcal{R}(s((\bar{d}\alpha)_{m+n}), s((d\beta)_{m+n}))} (\bar{d}\alpha)_{m+n} (d\beta)_{m+n} \right] \cdots \\ &\quad \times \left[\frac{\mathcal{R}(t((\bar{d}\alpha)_1), t((d\beta)_1))}{\mathcal{R}(s((\bar{d}\alpha)_1), s((d\beta)_1))} (\bar{d}\alpha)_1 (d\beta)_1 \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\mathcal{R}(t(a_m), t(b_n))}{\mathcal{R}(s(a_1), s(b_1))} \sum_{d \in D_{m+n}^{m+n}} [(\bar{d}\alpha)_{m+n}(d\beta)_{m+n}] \cdots [(\bar{d}\alpha)_1(d\beta)_1] \\
&= \frac{\mathcal{R}(t(a_m), t(b_n))}{\mathcal{R}(s(a_1), s(b_1))} \sum_{d \in D_{m+n}^{m+n}} [(d\alpha)_{m+n}(\bar{d}\beta)_{m+n}] \cdots [(d\alpha)_1(\bar{d}\beta)_1] \\
&= \frac{\mathcal{R}(t(a_m), t(b_n))}{\mathcal{R}(s(a_1), s(b_1))} \alpha\beta.
\end{aligned}$$

This is exactly the desired equation. Note that in the third equality we have used the fact $t((d\beta)_i) = s((d\beta)_{i+1})$ for $i = 1, \dots, m+n-1$. Now we are done. \square

3.3.

We conclude this section by some remarks.

Remark 3.3. Keep the assumptions and notations of Subsection 3.2.

- (1) The coquasitriangular structure \mathcal{R} is sort of a “quasi” skew-symmetric bicharacter of the group G . Clearly, if Φ is trivial, then \mathcal{R} is a usual skew-symmetric bicharacter. This is the usual Hopf case as given by Theorem 3.3 in [14]. More generally, if Φ is a coboundary, then by a suitable twisting, we can also go back to the Hopf case.
- (2) The coquasitriangular structures constructed in the previous theorem are actually cotriangular by (3.5). By Lemma 2.1, all possible (not necessarily graded and concentrating at degree 0) coquasitriangular Majid algebra structures on Hopf quivers degenerate to cotriangular ones. This reduces the classification problem of general coquasitriangular Majid structures on Hopf quivers to a lifting procedure of the cotriangular ones.

4. GENERAL COQUASITRIANGULAR POINTED MAJID ALGEBRAS

The aim of this section is to provide a quiver setting for general coquasitriangular pointed Majid algebras. Some examples and classification results are also provided via the quiver setting.

4.1.

The following is a main result which enables us to construct coradically graded coquasitriangular pointed Majid algebras exhaustively on Hopf quivers. This is a quasi analogue of Theorem 4.2 in [14] and the proof is given by adjusting the argument there into our situation.

Theorem 4.1. *Let (H, Φ, \mathcal{R}) be a coquasitriangular pointed Majid algebra, and as in Subsection 2.3 let $(\text{gr}(H), \text{gr}(\Phi), \text{gr}(\mathcal{R}))$ denote its graded version. Then there exist a unique Hopf quiver $Q = Q(G, R)$ with G abelian and a graded coquasitriangular Majid algebra structure (kQ, Ψ, \mathfrak{R}) with Ψ and \mathfrak{R} concentrating at degree 0 such that $(\text{gr}(H), \text{gr}(\Phi), \text{gr}(\mathcal{R}))$ is isomorphic to a large sub-structure of (kQ, Ψ, \mathfrak{R}) .*

Proof. Let G denote the set of group-like elements of H . Then the coradical H_0 of H is the group algebra kG . By restricting the reassociator Φ and the coquasitriangular structure \mathcal{R} , one has a sub-coquasitriangular Majid algebra (kG, Φ, \mathcal{R}) . As Proposition 3.1, we have immediately that G is an abelian group and Φ is a 3-cocycle on G . By the Gabriel type theorem for pointed Majid algebras [11], there exists a unique Hopf quiver $Q = Q(G, R)$ such that $(\text{gr}(H), \text{gr}(\Phi))$ can be viewed as a large sub-Majid algebra of the graded Majid structure (kQ, Ψ) determined by the (kG, Φ) -Majid bimodule H_1/H_0 . Note that Ψ is actually the trivial extension of the 3-cocycle Φ on G . Let \mathfrak{R} be the trivial extension of $\mathcal{R} : G \times G \rightarrow k$. By the same argument as in the proof of Theorem 3.2, one can show that (kQ, Ψ, \mathfrak{R}) is a graded coquasitriangular Majid algebra and the embedding $(\text{gr}(H), \text{gr}(\Phi)) \hookrightarrow (kQ, \Psi)$ respects the coquasitriangular structures. This completes the proof. \square

4.2.

By the quasi analogue of the Cartier–Gabriel decomposition theorem for pointed Majid algebras [11], we can focus on the connected ones (that is, those Majid algebras whose quivers are connected) without loss of generality. In that case, we can say more about their graded version.

Corollary 4.2. *Suppose that (H, Φ, \mathcal{R}) is a connected coquasitriangular pointed Majid algebra. Then its graded version $(\text{gr}(H), \text{gr}(\Phi), \text{gr}(\mathcal{R}))$ is cotriangular.*

The proof is clear by Remark 3.3 (2) and Theorem 4.1. More generally, the graded version of a non-connected coquasitriangular pointed Majid algebra can be written as the crossed product of a cotriangular one (namely, its connected component containing the identity) and a group algebra twisted by a 3-cocycle.

Recall that a tensor category is called pointed if its simple objects are invertible. See [9] for more definitions and results on finite tensor categories used below. We remark that the preceding result also implies an interesting consequence for braided pointed finite tensor categories with integral Frobenius–Perron dimensions of objects. It is well known that such tensor categories indeed correspond to the corepresentation categories of finite-dimensional coquasitriangular pointed Majid algebras. Thus Corollary 4.2 implies for any braided pointed finite tensor category \mathcal{C} with integral Frobenius–Perron dimensions of objects, its connected component (\mathcal{C} is essentially governed by its connected component, see [15] for details) containing the unit object is tensor equivalent to a deformation of a connected *symmetric* pointed finite tensor category.

4.3.

For simplicity, we assume that the ground field k is algebraically closed of characteristic 0 in the rest of the paper. As an example, let us consider the case of connected coquasitriangular pointed Majid algebras over the cyclic group $\mathbb{Z}_n = \langle g \rangle$ of order $n > 1$. The main result of this subsection is the following.

Proposition 4.3. *Assume that (H, Φ, \mathfrak{R}) is a connected coquasitriangular pointed Majid algebra with the set of group-likes equal to \mathbb{Z}_n . Then its graded version $(\text{gr}(H), \text{gr}(\Phi), \text{gr}(\mathfrak{R}))$ is twisting equivalent to a cotriangular Hopf algebra.*

Proof. First we recall a list of 3-cocycles on \mathbb{Z}_n as given in [10]. Let q be a primitive root of unity of order n . For any integer $i \in \mathbb{N}$, we denote by i' the remainder of division of i by n . A list of 3-cocycles on \mathbb{Z}_n are

$$\Phi_s(g^i, g^j, g^k) = q^{si(j+k-(j+k)/n)} \quad (4.1)$$

for all $0 \leq s \leq n-1$ and $0 \leq i, j, k \leq n-1$. Obviously, Φ_s is trivial (i.e., cohomologous to a 3-coboundary) if and only if $s = 0$.

Assume that Q is the corresponding quiver of H in the sense of Theorem 4.1. By assumption, Q is connected and is of the form $Q(\mathbb{Z}_n, R)$, where R be a ramification datum of \mathbb{Z}_n . By Theorem 3.2, the set of graded coquasitriangular Majid algebras on kQ with reassociator and coquasitriangular structure concentrating at degree 0 is equivalent to the set of pairs (Ψ, \mathfrak{R}) satisfying (3.1)–(3.5). Take $\Psi = \Phi_s$ for some $0 \leq s \leq n-1$ as given in (4.1). By (3.5), we have $\mathfrak{R}(g, g)^2 = 1$. Using induction and (3.3), one has

$$1 = \mathfrak{R}(g, g^n) = \mathfrak{R}(g, g)^n q^{-s}. \quad (4.2)$$

We claim that this indeed implies that $s = 0$. In fact, by $\mathfrak{R}(g, g)^2 = 1$ we know that $\mathfrak{R}(g, g)^n = 1$ or $\mathfrak{R}(g, g)^n = -1$. By (4.2), the first case implies that $q^{-s} = 1$ and $s = 0$. If $\mathfrak{R}(g, g)^n = -1$, then n must be odd. Also, (4.2) shows that $q^{-s} = -1$. Note that q is an n -th primitive root of unity and so $1 = (q^{-s})^n = (-1)^n = -1$. This is absurd. Thus we always have $\mathfrak{R}(g, g)^n = 1$. This claim means, Ψ can be chosen only as a 3-coboundary and thus such graded coquasitriangular Majid algebras must be twisting equivalent to cotriangular Hopf algebras by Remarks 3.3.

Now, together with Corollary 4.2, the assertion is clear. \square

As direct consequence, the pointed Majid algebras $M_+(8)$, $M_-(8)$ and $M(32)$ over \mathbb{Z}_2 given in [12], and $M(n, s, q)$ with $s \neq 0$ over \mathbb{Z}_n given in [15] are not coquasitriangular since they are nontrivial graded pointed Majid algebras, that is, pointed Majid algebras which are not twisting equivalent to Hopf algebras. Note that finite-dimensional connected graded cotriangular pointed Hopf algebras over \mathbb{Z}_n is completely classified by Corollary 6.3 of [14]. Therefore, finite-dimensional connected graded cotriangular Majid algebras over \mathbb{Z}_n are essentially known by the previous proposition.

4.4.

Of course, Proposition 4.3 also implies the corresponding consequence on connected braided pointed finite tensor categories whose invertible objects consisting of the cyclic group \mathbb{Z}_n . In particular, together with [14, 15] in a fairly straightforward way, we get a classification result for braided pointed tensor categories of finite type, i.e., in which there are only finitely many indecomposable objects.

Corollary 4.4. *Any connected braided pointed tensor category of finite type whose simple objects all have Frobenius-Perron dimension 1 is tensor equivalent to a deformation of $\text{Corep}H$ where H is a generalized Taft algebra which can be presented by generators g and x with relations*

$$g^n = 1, \quad x^2 = 0, \quad gx = -xg.$$

Here n is an even integer and $\text{Corep}H$ denotes the comodule category of H .

Proof. Let \mathcal{C} be a connected braided pointed tensor category of finite type whose simple objects all have Frobenius-Perron dimension 1. Thus we know that there is a connected coquasitriangular pointed Majid algebra H of finite corepresentation type such that $\text{Corep}H = \mathcal{C}$ (see, for example, Subsection 4.2 in [15]). All connected pointed Majid algebras of finite corepresentation type have been classified in [15] and they are shown to be pointed Majid algebras over \mathbb{Z}_n for some $n \in \mathbb{N}$. Therefore, by Proposition 4.3 one can assume that $\text{gr}H$ is a connected cotriangular pointed Hopf algebra of finite corepresentation type. It is known that a connected graded pointed algebra of finite corepresentation type is indeed a generalized Taft algebra (see [16]). Corollary 6.3 of [14] shows that this algebra must be of the form as given in this corollary. \square

By quiver representation theory, such braided tensor categories are well understood. In particular, their Auslander-Reiten quivers are truncated tubes of height 2, see, for instance, [1] and [13].

Finally, we remark that the knowledge of connected coquasitriangular pointed Majid algebras over \mathbb{Z}_n also sheds some light on the general ones over finite abelian groups. It is clear that a general Hopf quiver $Q(G, R)$ with G abelian is consisting of various sub-quivers of form $Q(\mathbb{Z}_n, r)$. Therefore, at least the local structure of a general coquasitriangular Majid algebra is known. The remaining task is the gluing of these local structures.

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