Classification of Finite-dimensional Basic Hopf Algebras According to Their Representation Type

Gongxiang Liu

Abstract. The main aim of this paper is to give the classification of finite-dimensional basic Hopf algebras according to their representation type. We attach every finite-dimensional basic Hopf algebra H a natural number n_H, which will help us to determine the representation type of H. The class of finite-dimensional basic Hopf algebras of finite representation type are determined completely. All possible structure of tame basic Hopf algebras are given.

Contents

1. Introduction 1
2. Representation type of basic Hopf algebras 5
3. Classification of basic Hopf algebras of finite representation type 9
4. Classification of basic Hopf algebras of tame type 10
Acknowledgement 19
References 19

1. Introduction

1.1. Throughout this paper k denotes an algebraically closed field. All spaces are k-spaces. By an algebra we mean an associative algebra with identity element. For an algebra A, J_A denotes its Jacobson radical. We freely use the results, notation, and conventions of [**49**].

1.2. Classification of Hopf algebras is one of central problems of Hopf algebra theory. The first celebrated result on this problem is now known as the following Cartier-Kostant-Milnor-Moore theorem.

Theorem 1.1. *A cocommutative Hopf algebra over an algebraically closed field k of characteristic 0 is a semidirect product of a group algebra and the enveloping algebra of a Lie algebra. In particular, a finite-dimensional cocommutative Hopf algebra over k is a group algebra.*

1991 *Mathematics Subject Classification.* Primary 16W30, 16G30; Secondary 16G20.
Key words and phrases. Basic Hopf algebra, representation type, Nakayama algebra, Frobenius algebra.

1.3. In recent years, some substantial classification results in infinite-dimensional case are given. All possible cotriangular Hopf algebras were determined [**24**], and a class of pointed Hopf algebras with finite Gelfand-Kirillov dimension are classified [**2**][**5**]. Lu, Wu and Zhang introduced the concept of *homological integral*, which generalizes the usual integral defined for finite-dimensional Hopf algebras to a large class of infinite-dimensional Hopf algebras, and use it to research particularly noetherian affine Hopf algebras of Gelfand-Kirillov dimension 1 [**43**]. Although these results shed some light on the structure of infinite-dimensional Hopf algebras, it is still very hard to handle infinite-dimensional Hopf algebras in general, and the classification of finite-dimensional Hopf algebras is more interest for us. We can use the following diagrams to explain the general procedure to classify finite-dimensional Hopf algebras.

$$\begin{tikzcd}
 & \text{semisimple } (\Leftrightarrow H = H/J_H) & \\
\begin{array}{c}\text{Classification of fin.-dim.}\\ \text{Hopf algebra } H\end{array} \arrow[ur] \arrow[dr] & & H/J_H \text{ is a Hopf algebra} \\
 & \text{non-semisimple} \arrow[ur] \arrow[dr] & \\
 & & \text{others}
\end{tikzcd}$$

Over the last decade, under various assumption, considerable progress has been made in classifying finite-dimensional Hopf algebras. To the author's knowledge, the classification of finite-dimensional Hopf algebras mainly consists of the following four aspects:

(1) Classification of semisimple cosemisimple Hopf algebras;
(2) Classification of non-semisimple Hopf algebras;
(3) Classification of all Hopf algebras of a prescribed dimension;
(4) Classification of triangular Hopf algebras.

1.4. For (1), if the characteristic of k is 0, then we know that a Hopf algebra is semisimple is equivalent to that it is cosemisimple by [**39**]. By a beautiful result of Etingof and Gelaki [**20**], problems in positive characteristic can be reduced to similar problems in characteristic 0. Therefore, we often consider the classification of semisimple Hopf algebras over an algebraically closed field of characteristic 0. Some specially dimensional semisimple Hopf algebras, particularly for dimensions p, p^2, p^3, pq, pq^2 [**22**][**30**][**45**][**46**][**52**][**53**][**65**] and dimensions ≤ 60 [**54**], were intensively studied. For example, semisimple Hopf algebras of dimension pq are shown to be trivial. That is, they are isomorphic to group algebras or dual group algebras. The class of semisimple Hopf algebras that are simple as Hopf algebras are researched recently [**27**][**28**]. Under a systematic study of of fusion categories, Etingof, Dikshych and Ostrik asked an interesting question (Question 8.45 in [**25**]) about the semisimple Hopf algebras: dose there exist a finite-dimensional semisimple Hopf algebra whose representation category is not group-theoretical? This question was answered affirmatively by Nikshych [**59**]. But, the classification of semisimple Hopf algebras is still a widely open question. We refer to the survey papers [**1**][**50**] for a more detailed exposition.

1.5. For (2), substantial results in this case are known for the class of pointed Hopf algebras over an algebraically closed field of characteristic 0. There are two different methods which were used to classify pointed Hopf algebras. One method was formulated by N.Andruskiewitsch and H.-J.Schneider. They reduced the study of pointed Hopf algebras to the study of Nichols algebras via bosonization given by Radford [**61**] and Majid [**44**]. This method gets great success. One of most remarkable properties of this method is that it allows Lie theory to enter into the picture through quantum groups [**7**]. Many new examples about pointed Hopf algebras were found through this way. It can also help us to give counterexamples for Kaplansky's Conjecture 10. For details see [**6**][**7**][**8**][**9**][**31**][**33**][**34**][**35**] [**36**]. Recently, Andruskiewitsch and Schneider have classified all finite-dimensional pointed Hopf algebras whose group of group-like elements $G(H)$ is abelian such that all prime divisors of the order of $G(H)$ are > 7. See [**10**].

Another method, mainly due to Pu Zhang and his co-workers, is to use quivers and their representation theory. This method depends heavily on one of Cibils-Rosso's conclusions [**14**]. One of merits of this method is that it introduces the combinatorial methods to enter into the field of the classification of pointed Hopf algebras. By using this method, the classification of so called Monomial Hopf algebras was gotten. Locally finite simple-pointed Hopf algebras can also be classified. For details see [**13**][**60**].

1.6. For (3), the start point of this direction should be the following Y.Zhu's result [**65**].

Theorem 1.2. *Let p be a prime number number and k an algebraically closed field of characteristic 0. Then a Hopf algebra of dimension p over k is necessarily semisimple and isomorphic to the group algebra of Z_p.*

By using [**8**] and [**48**], S-H. Ng [**55**] classified all Hopf algebras of dimension p^2 and showed that they are the group algebras and the Taft algebras. For dimension pq with $p \neq q$, a forklore conjecture says that such Hopf algebras are semisimple. If it is true, the results given in Subsection 1.4 imply that Hopf algebras of dimension pq, where p and q are distinct, are trivial. This conjecture was verified for some particular values of p and q [**4**][**12**][**23**][**56**][**57**][**58**]. There are other classification results in low dimension. All Hopf algebras of dimension ≤ 15 were classified and the most recent in dimension 16 is [**29**]. See [**29**] and references therein.

1.7. For (4), the works of N.Andruskiewitsch, P.Etingof and S.Gelaki should be considered the most important. P.Etingof and S.Gelaki indeed show that semisimple triangular Hopf algebras are very closed to group algebras [**21**]. The structure of minimal triangular Hopf algebras is also given [**3**].

There are some nice surveys about classification of finite-dimensional Hopf algebras, see for instance [**1**].

1.8. According to the fundamental theorem of Drozd [**17**], the category of finite-dimensional algebras over k can be divided into disjoint classes of finite representation, tame and wild algebras. This fact stimulates us to classify finite-dimensional Hopf algebras through their representation type. In order to realize this idea, we need add some conditions on the Hopf algebra H:

(1): We assume that H/J_H is a quotient Hopf algebra. By the general procedure to classify finite-dimensional Hopf algebras, this requirement is not strange. Note that H satisfies this condition if and only if the coradical of the dual Hopf algebra H^* is a Hopf subalgebra of H^*.

(2): By 1.4, the classification of semisimple Hopf algebras is still a widely open question. This suggests us that we should consider semisimple Hopf algebra H/J_H which can be described easily.

A good candidate satisfying these conditions is the class of basic Hopf algebras. That is, as an algebra, it is basic. This implies that H/J_H is a Hopf algebra automatically (see Lemma 1.1 in [**32**]). Since the field k is algebraically closed and H/J_H is a Hopf algebra, the condition "basic" implies that $H/J_H \cong (kG)^*$ for some finite group G.

Before giving the classification of finite-dimensional basic Hopf algebras, we should give an effective way to determine their representation type at first. This is indeed what we will do in the next section. Explicitly, we can attach to every finite-dimensional basic Hopf algebra H a natural number n_H and prove that (i) H is of finite representation type if and only if $n_H = 0$ or $n_H = 1$; (ii) if H is tame, then $n_H = 2$ and (iii) if $n_H \geq 3$, then H is wild.

The dual of a basic Hopf algebra is a pointed Hopf algebra, and vice versa. So, all classification results on pointed Hopf algebras (some of them mentions in subsection 1.5) can be applied by duality to basic Hopf algebras. But, I think, there is no possibility to give structures of all basic Hopf algebras. Inspired by the case of path algebras, it is quite natural to give the classification of finite-dimensional basic Hopf algebras of finite representation type and tame type, and Section 3 and Section 4 are devoted to classifying finite-dimensional basic Hopf algebras of finite representation type and tame type respectively.

1.9. A finite-dimensional algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are finite number of A-$k[T]$-bimodules M_i which are free as right $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{k[T]} k[T]/(T-\lambda)$ for $\lambda \in k$. We say that A is of *wild type* or A is a *wild* algebra if there is a finitely generated A-$k<X,Y>$-bimodule B which is free as a right $k<X,Y>$-module such that the functor $B \otimes_{k<X,Y>} -$ from mod-$k<X,Y>$, the category of finitely generated $k<X,Y>$-modules, to mod-A, the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms. See [**18**] for more details. For other unexplained notations about representation theory of finite-dimensional algebras in this paper, see [**11**][**18**].

2. Representation type of basic Hopf algebras

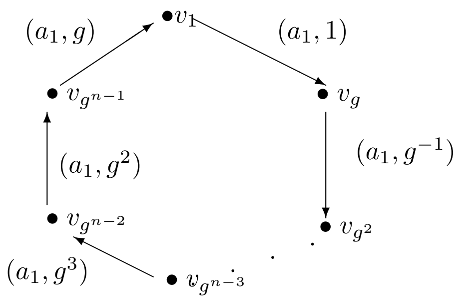
In the rest of this paper, all algebras are assumed to be finite-dimensional. In this section, the definition of the covering quiver $\Gamma_G(W)$, introduced by Green and Solberg [**32**], is given at first. Then we observe that we can associate to this covering quiver $\Gamma_G(W)$ a natural number $n_{\Gamma_G(W)}$, which can help us to determine the representation type of the finite-dimensional algebra A whose Ext-quiver is $\Gamma_G(W)$. For a finite-dimensional basic Hopf algebra H, it is known that its Ext-quiver is a covering quiver. So the above results can be applied to the case of finite-dimensional basic Hopf algebras directly.

Definition 2.1. *Let G be a finite group and let $W = (w_1, w_2, \dots, w_n)$ be a sequence of elements of G. We say W is a* weight sequence *if, for each $g \in G$, the sequences W and $(gw_1g^{-1}, gw_2g^{-1}, \dots, gw_ng^{-1})$ are the same up to a permutation. Define a quiver, denoted by $\Gamma_G(W)$, as follows. The vertices of $\Gamma_G(W)$ is the set $\{v_g\}_{g\in G}$ and the arrows are given by*

$$\{(a_i, g) : \ v_{g^{-1}} \to v_{w_i g^{-1}} | i = 1, 2, \dots, n, g \in G\}.$$

We call this quiver the covering quiver (with respect to G and W).

Example 2.1. (1): Let $G =< g >$, $g^n = 1$ and $W = (g)$. The corresponding covering quiver is



We call such quiver a *basic cycle of length n*.

(2): Let $G = K_4 = \{1, a, b, ab\}$, the Klein four group, and $W = (1)$. Then the corresponding covering quiver is

$$\bullet^1 \circlearrowleft, \quad \bullet^a \circlearrowleft, \quad \bullet^b \circlearrowleft, \quad \bullet^{ab} \circlearrowleft$$

For a covering quiver $\Gamma_G(W)$, define $n_{\Gamma_G(W)}$ to be the length of W. For an algebra A, it is Morita equivalent to a unique basic algebra $B(A)$ and for this basic algebra $B(A)$, the Gabriel's theorem says that there exists a unique quiver Q and an admissible ideal I (i.e. $J^N \subseteq I \subseteq J^2$ where J is the ideal generated by all arrows of Q) such that $B(A) \cong kQ/I$. See [**11**]. This quiver is called the *Ext-quiver* of A.

It is known that for a finite quiver Q, the path algebra kQ is of finite representation type if and only if the underlying graph $\overline{Q}$ of Q is one of Dynkin diagrams: A_n, D_n, E_6, E_7, E_8, and is of tame type if and only if the underlying graph $\overline{Q}$ is one of Euclidean diagrams: $\widetilde{A_n}$, $\widetilde{D_n}$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$. For details, see [**11**][**63**]. These facts will be used freely in the proof of the following conclusion.

Theorem 2.1. *Let $\Gamma_G(W)$ be a covering quiver, $n_{\Gamma_G(W)}$ defined as the above and assume A is an algebra with Ext-quiver $\Gamma_G(W)$. Then*

(i) *A is of finite representation type if and only if $n_{\Gamma_G(W)} = 0$ or $n_{\Gamma_G(W)} = 1$;*

(ii) *A is tame only if* $n_{\Gamma_G(W)} = 2$*;*
(iii) *If* $n_{\Gamma_G(W)} \geq 3$*, then A is wild.*

Proof. (i): "If part: " When $n_{\Gamma_G(W)} = 0$, there is no any arrow in $n_{\Gamma_G(W)}$. This implies A is a semisimple algebra and so is of finite representation type. When $n_{\Gamma_G(W)} = 1$, $\Gamma_G(W)$ is a finite union of basic cycles. It is well known that a basic algebra is Nakayama if and only if its Ext-quiver is A_n or a basic cycle. Thus the basic algebra of A is a Nakayama algebra. Since every Nakayama algebra must be of finite representation type ([**11**], p. 197) and A is Morita equivalent to its basic algebra, A is of finite representation type.

"Only if part: " It is sufficient to prove that A is not of finite representation type if $n_{\Gamma_G(W)} \geq 2$. In order to prove this, it is enough to consider the case $n_{\Gamma_G(W)} = 2$ (in fact, we will show later that if $n_{\Gamma_G(W)} \geq 3$, then A is wild). We denote the basic algebra of A by $B(A)$. We need only to prove that $B(A)$ is of infinite representation type. By the Gabriel's theorem, $k\Gamma_G(W)/I \cong B(A)$ for an admissible ideal I. Denote the ideal generating all arrows in $k\Gamma_G(W)$ by J. By the definition of admissible ideal, we have an algebra epimorphism

$$B(A) \twoheadrightarrow k\Gamma_G(W)/J^2.$$

Thus it is enough to prove that $k\Gamma_G(W)/J^2$ is not of finite representation type. Since the Jacobson radical of $k\Gamma_G(W)/J^2$ is clearly 2-nilpotent, $k\Gamma_G(W)/J^2$ is stably equivalent to the following hereditary algebra (see Theorem 2.4 in Chapter X in [**11**]):

$$\Lambda = \begin{pmatrix} k\Gamma_G(W)/J & 0 \\ J/J^2 & k\Gamma_G(W)/J \end{pmatrix}$$

The Ext-quiver of Λ is indeed the separated quiver of $\Gamma_G(W)$ (see the proof of Theorem 2.6 in Chapter X of [**11**]).

Assume $W = (w_1, w_2)$. If $w_1 = w_2$, we can find that the separated quiver of $\Gamma_G(W)$ is a disjoint union of quivers of following form:

$$\begin{tikzcd} i \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet j' \end{tikzcd}$$

This means Λ is not of finite representation type since clearly above quiver is a Kronecher quiver which is not a Dynkin diagram (see also Theorem 2.6 in Chapter X of [**11**]).

If $w_1 \neq w_2$, $\Gamma_G(W)_s$ must contain the following sub-quiver:

$$\begin{tikzcd} \overset{i_2}{\bullet} \arrow[d] & & \overset{i_1}{\bullet} \arrow[d] \\ j_2' \bullet & \overset{1}{\bullet} \arrow[l] \arrow[r] & \bullet j_1' \end{tikzcd}$$

Here 1 is the identity element of G. If $i_1 = i_2$, $\Gamma_G(W)_s$ is not a Dynkin diagram and thus Λ is of infinite representation type. If it is not, $\Gamma_G(W)_s$ contains the following sub-quiver:

$$\begin{tikzcd}
j_3' & i_2 \arrow[l] \arrow[d] & & i_1 \arrow[r] \arrow[d] & j_4' \\
& j_2' & 1 \arrow[l] \arrow[r] & j_1' &
\end{tikzcd}$$

If $j_4' = j_l'$ for $l = 1, 2, 3$, $\Gamma_G(W)_s$ is not a Dynkin diagram and thus Λ is of infinite representation type. If it is not, repeats above process and by the definition of covering quiver, there exit i_t, i_s or j_t', j_s' satisfying $i_t = i_s$ or $j_t' = j_s'$. In a word, $\Gamma_G(W)_s$ is not a Dynkin diagram and thus Λ is of infinite representation type. A celebrated result of H.Krause [**38**] states that two stably equivalent algebras have the same representation type. Thus $k\Gamma_G(W)/J^2$ is not of finite representation type since Λ is so. Therefore $B(A)$ is not of finite representation type.

Clearly, (ii) $\Leftrightarrow$ (iii). So it is enough to prove (iii). Since $n_{\Gamma_G(W)} \geq 3$, we assume $W = (w_1, w_2, w_3, \ldots)$. Just like analysis of the "Only if part" of (i), we consider the separated quiver of $k\Gamma_G(W)/J^2$. If $w_1 = w_2$, we have the following form sub-quiver of $\Gamma_G(W)_s$:

$$\begin{tikzcd}
j_2' & \\
1 \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & j_1'
\end{tikzcd}$$

It is clearly not a Euclidean diagram and thus $k\Gamma_G(W)/J^2$ is a wild algebra. If $w_1 \neq w_2$, not loss generality, we can assume $w_i \neq w_j$ for $1 \leq i \neq j \leq 3$. This implies $\Gamma_G(W)_s$ contains the following sub-quiver

$$\begin{tikzcd}
& i_{21} \arrow[d] & j_3' & i_{11} \arrow[d] & \\
i_{22} \arrow[r] & j_2' & 1 \arrow[l] \arrow[r] \arrow[u] & j_1' & i_{12} \arrow[l]
\end{tikzcd}$$

which is clearly not Euclidean diagram and thus $k\Gamma_G(W)/J^2$ is a wild algebra. Therefore $B(A)$ and thus A is a wild algebra. $\square$

The following conclusion (see Theorem 2.3 in [**32**]) states the importance of covering quivers.

LEMMA 2.2. *Let H be a finite-dimensional basic Hopf algebra over k. Then there exists a finite group G and a weight sequence $W = (w_1, w_2, \ldots, w_n)$ of G, such that $H \cong k\Gamma_G(W)/I$ for an admissible ideal I.*

This result indeed tells us that the Ext-quiver of a basic Hopf algebra H must be a covering quiver $\Gamma_G(W)$. By this, define $n_H := n_{\Gamma_G(W)}$.

COROLLARY 2.3. *Let H be a finite-dimensional basic Hopf algebra and n_H defined as above. Then*
(i) *H is of finite representation type if and only if $n_H = 0$ or $n_H = 1$;*
(ii) *If H is tame, then $n_H = 2$;*
(iii) *If $n_H \geq 3$, then H is of wild type.*

We want to take this opportunity to give two applications of Theorem 2.1. The first one is to give a new proof of Theorem 3.1 in [**42**]:

COROLLARY 2.4. [Theorem 3.1 in [**42**]] *Let H be a finite-dimensional basic Hopf algebra. Then H is of finite representation type if and only if it is a Nakayama algebra.*

PROOF. It is enough to prove the necessity since every Nakayama algebra must be of finite representation type. By the Theorem 2.1, we know that H is of finite representation type if and only if $n_H = 0$ or $n_H = 1$. When $n_H = 0$, there is no arrow in $\Gamma_G(W)$. This means H is semisimple and of course Nakayama. When $n_H = 1$, $\Gamma_G(W)$ is a disjoint union of basic cycles and H is Nakayama too (see the first paragraph of the proof of Theorem 2.1). □

The second one is to give an easy way to determine the representation type of a kind of Dinfeld doubles. Consider the basic cycle of length n (Example 2.1 (1)) and we denote this quiver by Z_n and by γ_i^m the path of length m starting at the vertex e_i $(i = 1, \ldots, n)$.

We consider the quotient algebra $\Gamma_{n,d} := kZ_n/J^d$ with $d|n$. It is a Hopf algebra with comultiplication Δ, counit ε and antipode defined as follows. We fix a primitive d-th root of unity q.

$$\Delta(e_t) = \sum_{j+l=t} e_j \otimes e_l, \quad \Delta(\gamma_t^1) = \sum_{j+l=t} e_j \otimes \gamma_l^1 + q^l \gamma_j^1 \otimes e_l,$$

$$\varepsilon(e_t) = \delta_{t0}, \;\; \varepsilon(\gamma_t^1) = 0, \;\; S(e_t) = e_{-t}, \;\; S(\gamma_t^1) = -q^{t+1}\gamma_{-t-1}^1.$$

As a Hopf algebra, $(\Gamma_{n,d})^{*cop}$ is isomorphic to the generalized Taft algebra $T_{nd}(q)$ [**37**] which as an associative algebra is generated by two elements g and x with relations

$$g^n = 1, \quad x^d = 0, \quad xg = qgx,$$

with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g,$$

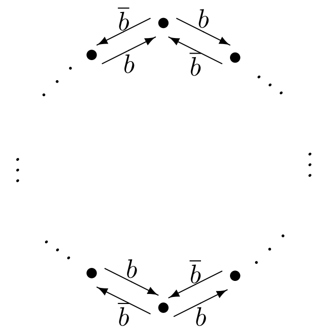
$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

$$S(g) = g^{-1}, \quad S(x) = -xg^{-1}.$$

For details, see [**19**].

In [**19**], the authors studied the representation theory of the Drinfeld Double $\mathcal{D}(\Gamma_{n,d})$ and proved the following conclusion.

LEMMA 2.5. [Theorem 2.25 in [**19**]] *The Ext-quiver of $\mathcal{D}(\Gamma_{n,d})$ has $\frac{n^2}{d}$ isolated vertices which correspond to the simple projective modules, and $\frac{n(d-1)}{2}$ copies of the quiver*



with $\frac{2n}{d}$ *vertices and* $\frac{4n}{d}$ *arrows. The relations on this quiver are* bb, $\overline{b}\overline{b}$ *and* $b\overline{b} - \overline{b}b$.

From this lemma, the authors of [**19**] find that $\mathcal{D}(\Gamma_{n,d})$ is a special biserial algebra and thus it is of finite representation type or tame type (see [**19**]). After listing all indecomposable modules of $\mathcal{D}(\Gamma_{n,d})$, they get that $\mathcal{D}(\Gamma_{n,d})$ is a tame algebra. Indeed, even without the complete list of indecomposable $\mathcal{D}(\Gamma_{n,d})$-modules, we also can prove that it is tame now.

Corollary 2.6. *$\mathcal{D}(\Gamma_{n,d})$ is a tame algebra.*

Proof. We have known that $\mathcal{D}(\Gamma_{n,d})$ is a special biserial algebra and thus it is tame or of finite representation type. Thus in order to prove that it is tame, it is enough to show that it is not of finite representation type. Note that the above quiver is a covering quiver $\Gamma_G(W)$ by setting $G =< g | g^{\frac{2n}{d}} = 1 >$ and $W = (g, g^{-1})$, and then $n_{\Gamma_G(W)} = 2$. Therefore, Theorem 2.1 gives us the desire conclusion. □

3. Classification of basic Hopf algebras of finite representation type

The classification of basic Hopf algebras of finite representation type indeed has been given by the author (with F. Li) in [**42**]. In [**42**], the conclusion is given in the language of pointed Hopf algebras. Note that the dual of pointed Hopf algebras are basic ones. For our purpose, we rewrite the result out in the language of basic Hopf algebras without proof (see Theorem 4.6 in [**42**]).

Theorem 3.1. *Let H be a finite-dimensional basic Hopf algebra. Then*

(i) *H is semisimple if and only if $H \cong (kG)^*$ for some finite group G;*

(ii) *Assume the characteristic of k is zero and H is not semisimple, then H is of finite representation type if and only if $H^* \cong A(\alpha)$ for some group datum $\alpha = (G, g, \chi, \mu)$;*

(iii) *Assume the characteristic of k is p and H is not semisimple, then H is of finite representation type if and only if there exist two natural numbers $n > 0$, $r \geq 0$, a d_0-th primitive root of unity $q \in k$ with $d_0 | n$, and $d = p^r d_0 \geq 2$ such that*

$$H^* \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

as coalgebras and

$$H^* \cong C_d(n) \#_\sigma k(G/N)$$

as Hopf algebras, where $G = G(H)$ and $N = G(C_d(n))$.

REMARK 3.2. (i) Here a group datum (for details, see [**13**]) over k is defined to be a sequence $\alpha = (G, g, \chi, \mu)$ consisting of

(1) a finite group G, with an element g in its center,

(2) a one-dimensional k-representation χ of G,

(3) an element $\mu \in k$ such that $\mu = 0$ if $o(g) = o(\chi(g))$, and if $\mu \neq 0$ then $\chi^{o(\chi(g))} = 1$.

For a group datum $\alpha = (G, g, \chi, \mu)$ over k, the corresponding Hopf algebra $A(\alpha)$ was defined in [**13**], which is generated as an algebra by x and all $h \in G$ with relations

$$x^d = \mu(1 - g^d), \quad xh = \chi(h)hx, \quad \forall\, h \in G$$

where $d = o(\chi(g))$. Its comultiplication Δ, counit ε, and antipode S are defined by

$$\Delta(x) = g \otimes x + x \otimes 1, \quad \varepsilon(x) = 0,$$

$$\Delta(h) = h \otimes h, \quad \varepsilon(h) = 1 \quad \forall\, h \in G,$$

$$S(x) = -g^{-1}x, \quad S(h) = h^{-1}, \quad \forall\, h \in G.$$

When d is a prime, the corresponding Hopf algebra $A(\alpha)$ appeared before [**13**] in [**16**].

(ii) For any quiver Γ, we define $C_d(\Gamma) := \oplus_{i=1}^{d-1} k\Gamma(i)$ for $d \geq 2$, where $\Gamma(i)$ is the set of all paths of length i in Γ. We denote the basic cycle of length n (Example 2.1 (1)) by Z_n and denote $C_d(Z_n)$ by $C_d(n)$.

For more details about this theorem, see [**42**].

4. Classification of basic Hopf algebras of tame type

For the radically graded tame basic Hopf algebras, all possible structure are determined in the author's paper [**41**]. In this section, we determine the structure of tame basic Hopf algebras (without the assumption of radical grading) completely.

We now give a short description of our method which is a kind of generalization of the method used in [**41**]. Let H be a finite-dimensional basic Hopf algebra over k. Then we have a Hopf epimorphism $H \twoheadrightarrow H/J_H$ where J_H is the Jacobson radical of H. By a work of H.-J. Schneider (see [**64**]), we have $H \cong R_H \#_\sigma H/J_H$, where $R_H = \{a \in H | (id \otimes \pi)\Delta(a) = a \otimes 1\}$ and $\pi : H \to H/J_H$ the canonical epimorphism. We will show that R_H is a local Frobenius algebra. By [**40**], we know that H and R_H have the same representation type. These results help us to reduce the study of tame basic Hopf algebras to that of tame local Frobenius algebras. Fortunately, we classify all tame local Frobenius algebras and show that there are only ten classes of local algebras which are tame Frobenius (see Theorem 4.1). By this, we find one possible structure given in [**41**] will not happen and the detail will be given at the end of this section.

4.1. A complete list of tame local Frobenius algebras. Denote the characteristic of k by char$k$. The main result of this subsection is the following.

THEOREM 4.1. *Let Λ be a tame local Frobenius algebra. If char$k \neq 2$, then $\Lambda \cong k < x, y > /I$ where I is one of forms:*

(1): $I = (x^m - y^n,\ yx - ax^m,\ xy)$ for $a \in k$ and $m, n \geq 2$;

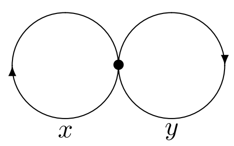
(2): $I = (x^2,\ y^2,\ (xy)^m - a(yx)^m)$ for $0 \neq a \in k$ and $m \geq 1$;

(3): $I = (x^2 - (yx)^m,\ y^2,\ (xy)^m + (yx)^m)$ for $m \geq 1$;

(4): $I = (x^2 - (yx)^m,\ y^2 - (xy)^m,\ (xy)^m + (yx)^m,\ (xy)^m x)$ *for* $m \geq 1$*;*
(5): $I = (x^2,\ y^2,\ (xy)^m x - (yx)^m y)$ *for* $m \geq 1$*;*
(6): $I = (x^2 - (yx)^{m-1} y - b(xy)^m,\ y^2,\ (xy)^m - a(yx)^m)$ *for* $a,\ b \in k$ *with* $a \neq 0$ *and* $m \geq 2$*;*
(7): $I = (x^2 - (yx)^{m-1} y - b(xy)^m,\ y^2 - (xy)^m,\ (xy)^m + (yx)^m,\ (xy)^m x)$ *for* $a,\ b \in k$ *with* $a \neq 0$ *and* $m \geq 2$*;*
(8): $I = (x^2 - (yx)^{m-1} y - f(xy)^m,\ y^2 - (xy)^{m-1} x - e(xy)^m,\ (xy)^m - a(yx)^m,\ (xy)^m x)$ *for* $a,\ e,\ f \in k$ *with* $a \neq 0$ *and* $m \geq 2$*;*
(9): $I = (x^2 - (yx)^m,\ y^2,\ (xy)^m x - a(yx)^m y)$ *for* $0 \neq a \in k$ *and* $m \geq 1$*;*
(10): $I = (x^2 - (yx)^m,\ y^2 - (xy)^m,\ (xy)^m x - a(yx)^m y,\ (xy)^{m+1})$ *for* $0 \neq a \in k$ *and* $m \geq 1$.

We want to prove Theorem 4.1 now. Some preliminaries must be given at first. It is easy to see that a local algebra is Frobenius if and only if the dimension of its socle equals to one. In this section, Λ always denotes a local Frobenius algebra and J_Λ its Jacobson radical. Recall that for any self-injective algebra Λ, we always have $soc\ {}_\Lambda\Lambda = soc\Lambda_\Lambda$ (see [**51**]). This fact will be used frequently.

Any tame local algebra A must have a quiver of the form



We denote this quiver by Q. By the Gabriel's Theorem, we know $A \cong k < x, y > /I$ for some ideal $J^2 \subseteq I \subseteq J^N$ where J is the ideal of $k < x, y >$ generated by x, y and $N \geq 2$. Therefore, if A is Frobenius then $dim_k A \geq 4$.

For convention, we always denote the image of x, y in A by x, y too.

Proposition 4.2. *All local algebras listed in Theorem 4.1 are tame local Frobenius algebras.*

Proof. By checking the dimension of the socle, it is easy to see that they are Frobenius algebras. It is known that Λ and $\Lambda/soc\Lambda$ have the same representation type. Now we can find all $\Lambda/soc\Lambda$ are images of maximal tame local algebras which given by C. Ringel [**62**]. Thus they are tame or of finite representation type. But it is known that $k < x, y > /(x, y)^2$ is tame and clearly there is a natural algebra epimorphism $\Lambda \twoheadrightarrow k < x, y > /(x, y)^2$ for any Λ in Theorem 4.1. Therefore, they are all tame. $\square$

Lemma 4.3. *Let* $\Lambda = kQ/I$ *be a local Frobenius algebra such that* J_Λ^2 *is generated by* x^2 *and* y^2*. Then* $xy = 0$ *if and only if* $I = (x^m - y^n,\ yx - ax^m, xy)$ *for* $0 \neq a \in k$ *with* $m, n \geq 2$ *or* $xy = yx = 0$*. Moreover, if* $xy = yx = 0$*, then* $I = (x^m - y^n,\ xy,\ yx)$ *for* $m, n \geq 2$*.*

Proof. It is enough to prove the necessity. By assumption, we have that Λ is spanned by $1, x, x^2, \cdots, y, y^2, \cdots$. We may write $yx = x^c w + y^d z$ where w, z are units in $k[[x]]$ and $k[[y]]$ respectively and $c, d \geq 2$. Then $0 = xyx = x^{c+1} w$ and then $x^{c+1} = 0$. Since also $x^c y = 0$, it follows that $x^c \in soc\Lambda$, the socle of Λ. Moreover, $0 = yxy = y^{d+1} z$ and we deduce that $y^{d+1} = 0$. Since also $xy^d = 0$, it follows that $y^d \in soc\Lambda$. This shows that $yx \in soc\Lambda$ since $soc\Lambda$ is an ideal of Λ.

Assume $yx \neq 0$ now. Let m, n be the maximal integers such that $x^m \neq 0,\ y^n \neq 0$ and $x^{m+1} = 0,\ y^{n+1} = 0$. Clearly, $m, n \geq 2$ and $x^m,\ y^n \in soc\Lambda$. By $dim_k soc\Lambda = 1$, there are $a, b \in k$ with $ab \neq 0$ such that $x^m = ay^n$ and $yx = bx^m$. Let $y' = \sqrt[n]{a}y$, then $x^m = y'^n$. The last statement is clear and the lemma is proved. $\square$

Lemma 4.4. *Assume that chark $\neq 2$ and Λ is a 4-dimensional local Frobenius algebra. Then Λ is isomorphic to one of the following algebras:*
(1): $kQ/(x^2 - y^2,\ yx - ax^2, xy)$ for $0 \neq a \in k$;
(2): $kQ/(x^2,\ y^2,\ xy - ayx)$ for $0 \neq a \in k$.

Proof. Let $x,\ y$ be generators of J_Λ. Since $dim_k\Lambda = 4$ and Λ is Frobenius, xy and yx belong to the socle of Λ.
(I): Assume $xy = 0$. If $yx \neq 0$, then $y^2 \neq 0$ and $x^2 \neq 0$ since $dim_k soc\Lambda = 1$. Therefore, by above lemma, we can find $m = 2,\ n = 2$ since otherwise the dimension of Λ will bigger than 4. Thus, $\Lambda \cong kQ/(x^2 - y^2,\ yx - ax^2, xy)$ for $0 \neq a \in k$.
If $yx = 0$. In this case, we know that $x^2 = ay^2$ for $0 \neq a \in k$. Let $u \in k$ with $u^2 = -a$, then, by char$k \neq 2$, $X = x + uy$, $Y = x - uy$ are generators. And, $X^2 = Y^2 = x^2 + u^2y^2 = x^2 - ay^2 = 0$, $XY = YX = x^2 - u^2y^2 = x^2 + ay^2$. Therefore, $\Lambda \cong k < X, Y > /(X^2,\ Y^2,\ XY - YX)$ which is a special case of (2).
(II): Assume $xy \neq 0 \neq yx$. Then $xy = cyx$ for $0 \neq c \in k$. By $dim_k soc\Lambda = 1$, we have $x^2 = axy$ and $y^2 = bxy$. If $a = b = 0$, then $\Lambda \cong kQ/(x^2,\ y^2,\ xy - cyx)$. Otherwise, no loss generality, assume $a \neq 0$. Let $Y = x - ay$, then $xY = 0$. Therefore we are in case (I) again. $\square$

Lemma 4.5. *Let Λ be a local Frobenius algebra. Then*
(i): If Λ is tame then $dim_k J_\Lambda^2/J_\Lambda^3 \leq 2$.
(ii): If chark $\neq 2$ and $dim_k J_\Lambda^2/J_\Lambda^3 \leq 1$ then $dim_k\Lambda = 4$ or Λ is an algebra as in Theorem 4.1 (1).

Proof. (i) If $dim_k J_\Lambda^2/J_\Lambda^3 \geq 3$, then there is a homomorphic image which is wild (see (2.1) of [**62**]). This implies Λ is wild which contradict the assumption that Λ is tame.
(ii) Suppose now that $dim_k J_\Lambda^2/J_\Lambda^3 \leq 1$. Then the dimension must be 1, since otherwise x, y would lie in $soc\Lambda$ and $soc\Lambda$ would not be simple. By this, we know that $dim_k \Lambda/J_\Lambda^3 = 4$.
Case (1): If Λ/J_Λ^3 is Frobenius, then by Lemma 4.4 we have

$$\Lambda/J_\Lambda^3 \cong kQ/(x^2 - y^2,\ yx - ax^2, xy) \ \text{ or } \ \Lambda/J_\Lambda^3 \cong kQ/(x^2,\ y^2,\ xy - ayx)$$

for $a \neq 0$.
If $\Lambda/J_\Lambda^3 \cong kQ/(x^2 - y^2,\ yx - ax^2, xy)$, then $xy,\ yx - ax^2,\ x^2 - y^2 \in J_\Lambda^3$. By $xy \in J_\Lambda^3$, $x^2y, yxy, xy^2, xyx \in J_\Lambda^4$. By $x^2 - y^2 \in J_\Lambda^3$, $x^3 - xy^2 \in J_\Lambda^4$ and thus $x^3 \in J_\Lambda^4$. Using $x^2 - y^2 \in J_\Lambda^3$ again, we can find $x^3 - y^2x \in J_\Lambda^4$ and thus $y^2x \in J_\Lambda^4$. Similarly, by $yx^2 - ax^3 \in J_\Lambda^4$ and $x^2y - y^3 \in J_\Lambda^4$, we have $yx^2, y^3 \in J_\Lambda^4$. Therefore, $J_\Lambda^3 \subseteq J_\Lambda^4$ and thus $J_\Lambda^3 = 0$. This implies $dim_k\Lambda = 4$.
If $\Lambda/J_\Lambda^3 \cong kQ/(x^2,\ y^2,\ xy - ayx)$, we have $x^2,\ y^2,\ xy - ayx \in J_\Lambda^3$. By this, it is easy to show that $xy^2,\ x^2y,\ yx^2,\ y^2x,\ x^3,\ y^3,\ xyx,\ yxy \in J_\Lambda^4$. Thus $J_\Lambda^3 \subseteq J_\Lambda^4$ and so $J_\Lambda^3 = 0$. This also means that $dim_k\Lambda = 4$.
Case (2): Assume now Λ/J_Λ^3 is not Frobenius. Therefore, $dim_k soc(\Lambda/J_\Lambda^3) \geq 2$. This implies $x \in soc(\Lambda/J_\Lambda^3)$ or $y \in soc(\Lambda/J_\Lambda^3)$. Not loss generality, assume $x \in soc(\Lambda/J_\Lambda^3)$. Thus we have $x^2,\ xy,\ yx \in J_\Lambda^3$. This means J_Λ^2 is generated by y^2 and

thus $xy = uy^l$ where u is a unit of $k[[y]]$ and $l \geq 3$. Let $x' = x - uy^{l-1}$ and we have $x'y = 0$. By Lemma 4.3, we know that $yx' = 0$ or $I = (x'^m - y^n,\ yx' - ax'^m,\ x'y)$.

If $I = (x'^m - y^n,\ yx' - ax'^m,\ x'y)$, the proof is done. In the case of $yx' = 0$, we write $x'^2 = vy^s$ for u a unit of $k[[y]]$ and $s \geq 3$. Thus $x'^3 = vy^s x' = 0$ and so $x'^2 \in soc\Lambda$. Take m to be the maximal integer such that $y^m \neq 0$ and $y^{m+1} = 0$. Therefore, $y^m \in soc\Lambda$ and thus $x'^2 = ay^m$. Let $y' = \lambda y$. Take a suitable λ, we have $x'^2 = y'^m$ and $\Lambda \cong kQ/I$ for $I = (x'^2 - y'^m,\ x'y',\ y'x')$. This is a special case of Theorem 4.1 (1). □

The following lemma is given in [**18**] (page 84).

LEMMA 4.6. *Let A be a tame local algebra with the quiver Q, of dimension 5, with $J^3 = 0$. Then $\Lambda \cong kQ/L$ where L is one of the following ideals:*
(1): $(xy,\ yx)$;
(2): $(yx - x^2,\ xy)$;
(3): $(yx - x^2,\ xy - ay^2)$ where $a \in k$ and $0 \neq a \neq 1$;
(4): $(x^2,\ y^2)$;
(5): $(yx - x^2,\ y^2)$.

LEMMA 4.7. *Let Λ be a local Frobenius algebra such that xy and yx lie in J^3_Λ. Assume $chark \neq 2$, then $dim_k\Lambda = 4$ or Λ is an algebra as in Theorem 4.1 (1).*

PROOF. By Lemma 4.4, we may assume $dim_k\Lambda > 4$. The algebra has a basis of the form
$$\{1,\ x, \ldots, x^s,\ y, \ldots, y^t\}.$$
If $xy = 0$, then by Lemma 4.3, Λ is of the form given in Theorem 4.1 (1).

We consider now the case when $xy \neq 0 \neq yx$. Let p be as large as possible such that xy and $yx \in J^p_\Lambda$. Clearly, $p \geq 3$. Then one of them does not lie in J^{p+1}_Λ. Write $xy \equiv x^p u + y^p v$ and $yx \equiv x^p w + y^p z$ (modulo J^{p+1}_Λ) where $u,\ v,\ w,\ z \in k$. We may replace $x,\ y$ by $x',\ y'$ where $x' = x - y^{p-1}z$ and $y' = y - x^{p-1}u$. Then we have new relations $xy \equiv cy^p$ and $yx \equiv dx^p$ for $c,\ d \in k$. Moreover, one of them, c say, is non-zero. Now, J^{p+1}_Λ is generated by x^{p+1} and y^{p+1}, and $cy^{p+1} \equiv yxy \equiv dx^p y \equiv dcx^{p-1}y^p \in J^{p+2}_\Lambda$ and already $J^{p+1}_\Lambda = (x^{p+1})$.

If $d \neq 0$, then similarly $x^{p+1} \in J^{p+2}_\Lambda$ and thus $J^{p+1}_\Lambda = 0$. So $J^p_\Lambda \subseteq soc\Lambda$. We assumed that $0 \neq xy \in J^p_\Lambda$ and it follows that $J^p_\Lambda = soc\Lambda$. We have $x^p = ay^p$ for $0 \neq a \in k$ and also $xy = cy^p$ and $yx = dx^p$ with $cd \neq 0$. Replace now $x,\ y$ by $x' = x - cy^{p-1}$ and y. Then $x'y = 0$ and thus we can use Lemma 4.3 again.

Suppose now that $d = 0$. Then $yx \in J^{p+1}_\Lambda = (x^{p+1})$. Note that $yx \neq 0$ and consequently $yx = x^m u$ for $m \geq p+1$, where u is a unit of $k[[x]]$. Replace y by $y' = y - x^{m-1}u$, then $y'x = 0$. We also can use Lemma 4.3 again. □

LEMMA 4.8. *Let Λ be a local Frobenius algebra such that $yx - x^2$ and xy lie in J^3_Λ. Then $\Lambda \cong kQ/(x^m - y^n,\ yx - ax^m,\ xy)$ for $0 \neq a \in k$.*

PROOF. Claim: J_Λ *does not have generators x', y' with $(x'y')$ and $(y'x')$ lying in J^3_Λ*. This claim was proved in [**18**] (see Lemma III.7 of [**18**]).

We have that $x^3 \equiv xyx$ (modulo J^4_Λ). But $xyx \in J^4_\Lambda$ and thus $x^3 \in J^4_\Lambda$. So J^3_Λ is generated by y^3. So we may have $xy = y^s u$ where u is a unit in $k[[y]]$ and $s \geq 3$. Let $x' = x - y^{s-1}u$, then we have $x'y = 0$. By the claim, $yx' \neq 0$ and thus Lemma 4.3 is applied. □

Lemma 4.9. *Let Λ be a local Frobenius algebra such that $yx - x^2$ and $xy - ay^2$ lie in J_Λ^3 where $0 \neq a \neq 1$. Then $dim_k\Lambda = 4$.*

Proof. By assumption, we have $x^3 \equiv xyx \equiv ay^2x \equiv ayx^2 \equiv ax^3$ (modulo J_Λ^4) and $x^2y \equiv axy^2 \equiv a^2y^3 \equiv ayxy \equiv ax^2y$ (modulo J_Λ^4). Since $a \neq 1$, $x^3 \in J_\Lambda^4$ and $x^2y \in J_\Lambda^4$. Since $a \neq 0$, it follows that all the other monomials occurring lie in J_Λ^4. This means that $J_\Lambda^3 \subseteq J_\Lambda^4$ and thus $J_\Lambda^3 = 0$. So, $J_\Lambda^2 \subseteq soc\Lambda$. By Λ is Frobenius, $dim_k\Lambda = 4$. $\square$

Lemma 4.10. *Let Λ be a local Frobenius algebra such that x^2 and y^2 lie in J_Λ^3. Assume chark $\neq 2$, then Λ is isomorphic to one of algebras in Theorem 4.1.*

Proof. By Lemma 4.4 and Lemma 4.5, we can assume that $dim_k\Lambda > 4$ and $dim_k J_\Lambda^2/J_\Lambda^3 = 2$. Thus xy and yx are generators of J_Λ^2 which are independent.

Case (1): Assume x^2 and y^2 lie in $soc\Lambda$. Let $m \geq 1$ be the integer such that $(xy)^m \neq 0$ and $(xy)^{m+1} = 0$. We claim that $soc\Lambda = ((xy)^m)$ or $soc\Lambda = ((xy)^m x)$. Indeed, since $y^2 \in soc\Lambda$, we always have $(xy)^m y = 0$. Thus if $(xy)^m x = 0$ then $0 \neq (xy)^m \in soc\Lambda$. By $dim_k soc\Lambda = 1$, $soc\Lambda = ((xy)^m)$. Otherwise, $(xy)^m x \neq 0$. By $(xy)^{m+1} = 0$ and $(xy)^m x^2 = 0$, $soc\Lambda = ((xy)^m x)$. Thus the claim is proved.

If $soc\Lambda = ((xy)^m)$, then there exists $0 \neq a \in k$ such that $(yx)^m = a(xy)^m$. By x^2 and y^2 lie in $soc\Lambda$, we have $x^2 = b(xy)^m$ and $y^2 = c(xy)^m$ for $b,\ c \in k$. If $a \neq -1$, let $d = \frac{b}{1+a}$, $e = \frac{c}{1+a}$ and $x' = x - dy(xy)^{m-1}$, $y' = y - e(xy)^{m-1}y$, then we have $x'^2 = 0$ and $y'^2 = 0$. Thus it is isomorphic to the one of algebras in Theorem 4.1 (2). If $a = -1$, then consider $b,\ c$. If $b = c = 0$, it is isomorphic to the one of algebras in Theorem 4.1 (2). If one of $b,\ c$ is zero while the other is not zero, say $b \neq 0$ and $c = 0$. Let $x' = \lambda x$ and $y' = \mu y$ for $\lambda,\ \mu \in k$. By suitable choice of $\lambda,\ \mu$, we can assume $x'^2 = (x'y')^m$ and $y'^2 = 0$. Thus the algebra has the form as in Theorem 4.1 (3). Similarly, if $bc \neq 0$ then we can show the algebra has the form as in Theorem 4.1 (4).

If $soc\Lambda = ((xy)^m x)$, then there exists $0 \neq a \in k$ such that $(yx)^m y = a(xy)^m x$. By x^2 and y^2 lie in $soc\Lambda$, we have $x^2 = b(xy)^m x$ and $y^2 = cy(xy)^m$ for $b,\ c \in k$. Let $x' = b(xy)^m - x$ and $y' = c(yx)^m - y$. Then we can find $x'^2 = 0$ and $y'^2 = 0$. Moreover, clearly we can take a to be 1. Thus the algebra has the form as in Theorem 4.1 (5).

Case (2): Otherwise, choose $n,\ l$ such that $x^2 \in J_\Lambda^n - J_\Lambda^{n+1}$ and $y^2 \in J_\Lambda^l$ with $l \geq n$. Take also n as large as possible with respect to these conditions.

Claim: $J_\Lambda^{n+1} \subseteq soc\Lambda$. This claim was proved in [**18**] (see Lemma III.10 of [**18**]). Now we consider two possibilities: $soc\Lambda$ is an even power of J_Λ or $soc\Lambda$ is an odd power of J_Λ.

(i) If $soc\Lambda$ is an even power of J_Λ, then $soc\Lambda = J_\Lambda^{2m}$. By the hypothesis at the beginning, $J_\Lambda^n \nsubseteq soc\Lambda$. So $J_\Lambda^{n+1} \neq 0$ and thus $J_\Lambda^{n+1} = soc\Lambda$. Therefore, $n + 1 = 2m$. Clearly, $soc\Lambda = ((xy)^m)$ and $(yx)^m = a(xy)^m$ for $0 \neq a \in k$. Then $x^2 = c(xy)^{m-1}x + d(yx)^{m-1}y + f(xy)^m$ and $(c,\ d) \neq (0,\ 0)$. Without loss of generality, we can assume $c = 0$ since otherwise we can replace x by $x' = x - c(xy)^{m-1}$.

Now consider y^2. By the hypothesis, the element lies in $soc_2\Lambda$. If $y^2 \in soc\Lambda$, then we show similarly as in Case (1) that Λ is an algebra in Theorem 4.1 (6), (7). Otherwise, $y^2 = a(xy)^{m-1}x + b(yx)^{m-1}y + e(xy)^m$ and $(a,\ b) \neq (0,\ 0)$. Similarly, we can assume $b = 0$. Thus $a \neq 0$.

We have now $x^2 = d(yx)^{m-1}y + f(xy)^m$ and $y^2 = a(xy)^{m-1}x + e(xy)^m$. Let $x' = \lambda x$ and $y' = \mu y$. By a suitable choice of $\lambda,\ \mu$, we can assume $a = 1 = d$. This is an algebra in Theorem 4.1 (8).

(ii) Otherwise, $soc\Lambda$ is an odd power of J_Λ. Similarly, we have $soc\Lambda = J_\Lambda^{n+1} = ((xy)^m x)$ and $(xy)^m x = a(yx)^m y$ for $a \neq 0$. By assumption, $x^2 = b(xy)^m + c(yx)^m + f(xy)^m x$. As before, we can assume $b = 0$. Also, we consider y^2. If $y^2 \in soc\Lambda$, then by the discussion of Case (1), the algebra is isomorphic to one of algebras in Theorem 4.1 (9). If not, we have $y^2 = d(xy)^m + e(yx)^m + g(xy)^m x$ for $d,\ e,\ g \in k$. Similarly, we can assume that $e = 0$ and $d \neq 0$. As in Case (1), we also can assume $f = g = 0$. So now we have $x^2 = c(yx)^m$ and $y^2 = d(xy)^m$. Similarly, let $x' = \lambda x$ and $y' = \mu y$ and choose suitable $\lambda,\ \mu$, we may assume $c = d = 1$. Thus it is an algebra in Theorem 4.1 (10). □

Lemma 4.11. *There is no local Frobenius algebra Λ such that Λ/J_Λ^3 satisfies Lemma 4.6 (5).*

Proof. Suppose such algebra exists.

Claim: $J_\Lambda^3 = (xyx) \subseteq soc\Lambda$. We have that J_Λ^3 is generated by xyx and yxy by the given relations. Moreover, modulo J_Λ^4 we have that $xyx \equiv x^3 \equiv yxx \equiv y^2x \equiv 0$ and therefore $J_\Lambda^3 = (yxy)$. This implies $J_\Lambda^4 = ((yx)^2) \subseteq yJ_\Lambda^4 \subseteq J_\Lambda^5$. Thus $J_\Lambda^4 = 0$ as required.

We claim yxy must be zero now. Otherwise, assume $yxy \neq 0$ and thus $J_\Lambda^3 = (yxy) = soc\Lambda$. Since $J_\Lambda^4 = 0$, we know $xyx = 0$ and $xy^2 = 0$. This means $xy \in soc\Lambda$. Clearly, $xy \neq 0$ since otherwise $yxy = 0$. Since $dim_k soc\Lambda = 1$, there exists non-zero $c \in k$ such that $xy = cyxy$. So we have $xy = cyxy = c^2y^2xy = 0$. It's a contradiction. This means $yxy = 0$ and thus $J_\Lambda^3 = 0$ and $J_\Lambda^2 \subseteq soc\Lambda$. Therefor $soc\Lambda$ is not simple, which is absurd. □

Proof of Theorem 4.1: Since Λ is tame, $dim_k J_\Lambda^2/J_\Lambda^3 \leq 2$ by Lemma 4.5.

If $dim_k J_\Lambda^2/J_\Lambda^3 = 1$, Lemma 4.5 shows that Λ is one of algebras of this list.

If $dim_k J_\Lambda^2/J_\Lambda^3 = 2$, then $dim_k \Lambda/J_\Lambda^3 = 5$. This means that Λ/J_Λ^3 satisfies the conditions of Lemma 4.6. Therefore, Lemma 4.7-4.11 give our desired conclusion.

4.2. Tame basic Hopf algebras. The main aim of this subsection is to describe the structure of tame basic Hopf algebras (see Theorem 4.15).

Let H be a basic Hopf algebra and J_H is its Jacobson radical. Recall $H/J_H \cong (kG)^*$ for some finite group G. In this section, we always assume char$k \neq 2$ and char$k \nmid |G|$. Thus kG is always semisimple.

Denote H/J_H by $\overline{H}$. Now we have a Hopf algebra epimorphism

$$H \twoheadrightarrow \overline{H}.$$

By a result which given by H.-J. Schneider [**64**], there is an algebra R_H such that

$$H \cong R_H \#_\sigma \overline{H}.$$

Lemma 4.12. *R_H is a local algebra.*

Proof. For any finite-dimensional algebra A, we write $grA = A/J_A \oplus J_A/J_A^2 \oplus \cdots$. By [**61**] and [**44**], there is an algebra R_{grH}, which is a graded braided Hopf algebra in ${}^{\overline{H}}_{\overline{H}}\mathcal{YD}$, such that $grH \cong R_{grH} \# \overline{H}$. Hence R_{grH} is Frobenius by [**26**] and is local since the degree 0 part is k. Thus R_{grH} is a local Frobenius algebra.

By Blattner-Montgomery Duality Theorem (see Section 9.4 in [**49**]), we have

$$(R_H\#_\sigma\overline{H})\#(\overline{H})^*\cong M_n(R_H),$$
$$(R_{grH}\#\overline{H})\#(\overline{H})^*\cong M_n(R_{grH})$$

where $n=dim_k\overline{H}$. Note that $(\overline{H})^*$ is a group algebra now, thus we have $J_{(R_H\#_\sigma\overline{H})\#(\overline{H})^*}=(J_{R_H\#_\sigma\overline{H}})\#(\overline{H})^*$. This means we have the following isomorphism

$$grM_n(R_H)\cong(gr(R_H\#_\sigma\overline{H}))\#(\overline{H})^*.$$

Thus,

$$M_n(grR_H)\cong grM_n(R_H)\cong(grH)\#(\overline{H})^*\cong(R_{grH}\#\overline{H})\#(\overline{H})^*\cong M_n(R_{grH}).$$

So we have $M_n(grR_H)\cong M_n(R_{grH})$ and thus $grR_H\cong R_{grH}$. By R_{grH} is local, grR_H and thus R_H is local. □

LEMMA 4.13. *R_H is a Frobenius algebra.*

PROOF. By the Lemma 4.12, it is enough to show that R_H is self-injective since any basic self-injective algebra must be Frobenius. By Blattner-Montgomery Duality Theorem, we need only show $H\#(\overline{H})^*$ is self-injective. Let P be a projective $H\#(\overline{H})^*$-module, we need to show that P is also injective.

For $H\#(\overline{H})^*$-modules $M,\ N$, let $i:\ M\hookrightarrow N$ and $h:\ M\to P$ be two $H\#(\overline{H})^*$-module morphisms such that i is injective. In order to prove that P is injective as an $H\#(\overline{H})^*$-module, it is enough to find an $\widetilde{f}\in Hom_{H\#(\overline{H})^*}(N,P)$ satisfying $h=\widetilde{f}i$. It is known that $H\#(\overline{H})^*$ is a free H-module. Thus P is also a projective H-module. By H is Frobenius, P is injective as an H-module. Thus there exists an H-morphism f such that $h=fi$. Define $\widetilde{f}(n)=\sum S(t_1)\cdot f(t_2\cdot n)$ for $n\in N$, where t is a non-zero right integral with $\varepsilon(t)=1$. Then $\widetilde{f}$ is $H\#(\overline{H})^*$-linear by [**15**] and satisfies $h=\widetilde{f}i$. □

The following lemma is proved in [**40**] (see Theorem 2.6 in [**40**]).

LEMMA 4.14. *Let A be a finite-dimensional algebra and H a finite-dimensional Hopf algebra. If H and H^* are semisimple, then $A\#_\sigma H$ and A have the same representation type.*

The next conclusion will give us all possible structures of tame basic Hopf algebras.

THEOREM 4.15. *Let H be a basic Hopf algebra. Assume $chark\neq 2$ and dim_kH/J_H is invertible in k, then H is tame if and only if $H\cong k<x,y>/I\#_\sigma(kG)^*$ for some finite group G and some ideal I which is one of the following forms:*

(1): $I=(x^m-y^n,\ yx-ax^m,\ xy)$ for $a\in k$ and $m,n\geq 2$;
(2): $I=(x^2,\ y^2,\ (xy)^m-a(yx)^m)$ for $0\neq a\in k$ and $m\geq 1$;
(3): $I=(x^2-(yx)^m,\ y^2,\ (xy)^m+(yx)^m,)$ for $m\geq 1$;
(4): $I=(x^2-(yx)^m,\ y^2-(xy)^m,\ (xy)^m+(yx)^m,\ (xy)^mx)$ for $m\geq 1$;
(5): $I=(x^2,\ y^2,\ (xy)^mx-a(yx)^my)$ for $0\neq a\in k$ and $m\geq 1$;
(6): $I=(x^2-(yx)^{m-1}y-b(xy)^m,\ y^2,\ (xy)^m-a(yx)^m)$ for $a,\ b\in k$ with $a\neq 0$ and $m\geq 2$;
(7): $I=(x^2-(yx)^{m-1}y-b(xy)^m,\ y^2-(xy)^m,\ (xy)^m+(yx)^m,\ (xy)^mx)$ for $a,\ b\in k$ with $a\neq 0$ and $m\geq 2$;

(8): $I = (x^2-(yx)^{m-1}y-f(xy)^m,\ y^2-(xy)^{m-1}x-e(xy)^m,\ (xy)^m-a(yx)^m,\ (xy)^m x)$ *for* $a,\ e\ f \in k$ *with* $a \neq 0$ *and* $m \geq 2$*;*

(9): $I = (x^2-(yx)^m,\ y^2,\ (xy)^m x - a(yx)^m y)$ *for* $0 \neq a \in k$ *and* $m \geq 1$*;*

(10): $I = (x^2-(yx)^m,\ y^2-(xy)^m,\ (xy)^m x - a(yx)^m y,\ (xy)^{m+1})$ *for* $0 \neq a \in k$ *and* $m \geq 1$.

PROOF. "Only if part: " On one hand, by Lemma 4.12, Lemma 4.13 and Lemma 4.14, R_H is a tame local Frobenius algebra. On the other hand, H/J_H is a commutative semisimple Hopf algebra and thus $H/J_H \cong (kG)^*$ for some finite group. Therefore, by Theorem 4.1 and $H \cong R_H \#_\sigma H/J_H$, we get the desired conclusion.

"If part: " By Proposition 4.2, we know that $k < x, y > /I$ is a tame algebra. So the sufficiency is gotten from Lemma 4.14. □

REMARK 4.16. (1) In order to apply Lemma 4.14, we need kG to be semisimple and thus the hypothesis $\mathrm{char} k \nmid |G|$, posed at the beginning of this subsection, is needed. Note that at the end of proof of Lemma 4.13, this hypothesis was also used to guarantee the existence of the right integral t satisfying $\varepsilon(t) = 1$.

(2) By a conclusion of Radford or Majid (see [**61**][**44**]), if Λ is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$ for some finite group G, then we can form the bosonization $\Lambda \times (kG)^*$ which is a Hopf algebra. For a tame local Frobenius algebra A, above theorem dose not imply the existence of finite group G satisfying A is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$.

PROBLEM 4.1. *For a tame local Frobenius algebra A, give an effective method to determine that whether there is a finite group G satisfying A is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$. If such a G exists, then find all of them.*

A similar problem was given in [**41**] ([**41**], Problem 5.1). For a tame local radically graded Frobenius algebra, this problem has been solved by the author with his co-workers. The details will appear elsewhere.

EXAMPLE 4.1. **(Tensor products of Taft algebras)** Let $T_{n^2}(q),\ T_{m^2}(q')$ be two Taft algebras. Direct computation shows that

$$T_{n^2}(q) \otimes_k T_{m^2}(q') \cong k < x, y > /I \# k(\mathbb{Z}_n \times \mathbb{Z}_m)$$

where $I = (x^n,\ y^m,\ xy - yx)$. Thus by Theorem 4.15, $T_{n^2}(q) \otimes_k T_{m^2}(q')$ is tame if and only if $m = n = 2$.

EXAMPLE 4.2. **(Book Algebras)** Let q be a n-th primitive root of unity and m a positive integer satisfying $(m, n) = 1$. Let $H = \mathbf{h}(q, m) = k < y, x, g > /(x^n, y^n, g^n - 1, gx - qxg, gy - q^m yg, xy - yx)$ and with comultiplication, antipode and counit given by

$$\Delta(x) = x \otimes g + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + g^m \otimes y, \quad \Delta(g) = g \otimes g$$

$$S(x) = -xg^{-1},\ \ S(y) = -g^{-m}y,\ \ S(g) = g^{-1},\ \ \varepsilon(x) = \varepsilon(y) = 0,\ \ \varepsilon(g) = 1.$$

It is a Hopf algebra and called book algebra. As in [**41**], we have

$$\boldsymbol{h}(q, m) \cong k < x, y > /I \# k\mathbb{Z}_n$$

where $I = (x^n,\ y^n,\ xy - q^m yx)$. Thus by Theorem 4.15, $\mathbf{h}(q, m)$ is tame if and only if $n = 2$. In this case, q must equal to -1 and $m = 1$. Thus only $\mathbf{h}(-1, 1)$ is tame and the others are all wild.

Example 4.3. **(The dual of Frobenius-Lusztig kernel)** Let p be an odd number and q a p-th primitive root of unity. By definition, the Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2)$ is an associative algebra generated by E, F, K with relations

$$K^p = 1,\ \ E^p = 0,\ \ F^p = 0,\ \ KE = q^2EK,\ \ KF = q^{-2}FK,\ \ EF-FE = \frac{K-K^{p-1}}{q-q^{-1}}.$$

Its comultiplication, counit and antipode are defined by

$$\Delta(E) = 1\otimes E + E\otimes K,\ \ \Delta(F) = K^{-1}\otimes F + F\otimes 1,\ \ \Delta(K) = K\otimes K;$$
$$\varepsilon(E) = \varepsilon(F) = 0,\ \ \varepsilon(K) = 1;$$
$$S(E) = -q^2K^{-1}E,\ \ S(F) = -KF,\ \ S(K) = K^{-1}.$$

It is a pointed Hopf algebra and thus $\mathbf{u}_q(\mathfrak{sl}_2)^*$ is a basic Hopf algebra. We now give the Hopf structure of $\mathbf{u}_q(\mathfrak{sl}_2)^*$ explicitly.

It is known that $\mathbf{u}_q(\mathfrak{sl}_2)$ has a basis $\{K^lE^iF^j|0\leq l,i,j\leq p-1\}$ and thus $dim_k\mathbf{u}_q(\mathfrak{sl}_2) = p^3$. We denote by $(K^lE^iF^j)^*$ the element of $\mathbf{u}_q(\mathfrak{sl}_2)^*$ which sent $K^lE^iF^j$ to 1 and the other element in the above basis to 0.

Let

$$a = \sum_{i=0}^{p-1} q^i(K^i)^* + \sum_{i=0}^{p-1} q^i(K^iEF)^*,\ \ b = \sum_{i=0}^{p-1} q^i(K^iE)^*,$$
$$c = \sum_{i=0}^{p-1} q^{-i}(K^iF)^*,\qquad d = \sum_{i=0}^{p-1} q^{-i}(K^i)^*.$$

By direct computations, the following relations hold.

$$ba = qab,\ \ db = qbd,\ \ ca = qac.\ dc = qcd,\ \ bc = cb,$$
$$ad - da = (q^{-1}-q)bc,\ \ da - qbc = 1,\ \ d^p = 1,\ \ c^p = b^p = 0.$$

For example, let us check the relation $bc = cb$ and the other relations can be checked similarly. By definition, $bc = \sum_{i,j} q^{i-j}(K^iE)^*(K^jF)^*$. In order to make $(K^iE)^*(K^jF)^*(K^lE^mF^n) \neq 0$, we must have $m = n = 1$. But

$$\Delta(K^lEF) = K^{l-1}\otimes K^lEF + q^2K^{l-1}E\otimes K^{l+1}F + K^lF\otimes K^lE + K^lEF\otimes K^l.$$

This implies if $(K^iE)^*(K^jF)^* \neq 0$ then $j = i+2$. Thus

$$bc = \sum_{i,j} q^{i-j}(K^iE)^*(K^jF)^* = \sum_{l=0}^{p-1} q^{-2}q^2(K^lEF)^* = \sum_{l=0}^{p-1}(K^lEF)^*.$$

Similarly, we can show $cb = \sum_{l=0}^{p-1}(K^lEF)^*$ also.

By $da - qbc = 1$ and $d^p = 1$, we have $a = d^{-1}(1+qbc)$. It is straightforward to show that the algebra, which is generated by a, b, c, d with above relations, has dimension p^3. Thus algebra is just $\mathbf{u}_q(\mathfrak{sl}_2)^*$. The comultiplication, counit and the antipode are given as follows.

$$\Delta(a) = a\otimes a + b\otimes c,\qquad \Delta(b) = a\otimes b + b\otimes d;$$
$$\Delta(c) = c\otimes a + d\otimes c,\qquad \Delta(d) = c\otimes b + d\otimes d;$$
$$\varepsilon(a) = \varepsilon(d) = 1,\qquad \varepsilon(b) = \varepsilon(c) = 0;$$
$$S(a) = d,\ \ S(b) = -qb,\ \ S(c) = -q^{-1}c,\ \ S(d) = a.$$

Clearly, $J_{\mathbf{u}_q(\mathfrak{sl}_2)^*} = (b,c)$ and $\mathbf{u}_q(\mathfrak{sl}_2)^*/(b,c) \cong k\mathbb{Z}_p$. Thus

$$\mathbf{u}_q(\mathfrak{sl}_2)^* \cong R_{\mathbf{u}_q(\mathfrak{sl}_2)^*}\#_\sigma k\mathbb{Z}_p$$

where by definition $R_{\mathbf{u}_q(\mathfrak{sl}_2)^*} = \{x \in \mathbf{u}_q(\mathfrak{sl}_2)^* | (id \otimes \pi)\Delta(x) = x \otimes 1\}$. Here $\pi : \mathbf{u}_q(\mathfrak{sl}_2)^* \to \mathbf{u}_q(\mathfrak{sl}_2)^*/(b, c)$ is the canonical map. Thus it is easy to see that $dc,\ d^{-1}b \in R_{\mathbf{u}_q(\mathfrak{sl}_2)^*}$ which generate $R_{\mathbf{u}_q(\mathfrak{sl}_2)^*}$ and satisfy the following relations

$$(dc)^p = 0, \ \ (d^{-1}b)^p = 0, \ \ dc \cdot d^{-1}b = q^2(d^{-1}b) \cdot dc.$$

Denote dc by x and $d^{-1}b$ by y, we have

$$R_{\mathbf{u}_q(\mathfrak{sl}_2)^*} \cong k < x, y > /I$$

where $I = (x^p,\ y^p,\ xy - q^2yx)$ which is not an algebra in Theorem 4.1. Thus, by Theorem 4.15, $\mathbf{u}_q(\mathfrak{sl}_2)^*$ is wild.

At last, I want to take this chance to give an addendum to [**41**]. In Section 3 of [**41**], we give the following conclusion (see Theorem 4.1 in [**41**]).

"Let Λ be a tame local graded Frobenius algebra. If char$k \neq 2$, then $\Lambda \cong k < x, y > /I$ where I is one of forms:
(1): $I = (x^2 - y^2,\ yx - ax^2,\ xy)$ for $0 \neq a \in k$;
(2): $I = (x^2,\ y^2,\ (xy)^m - a(yx)^m)$ for $0 \neq a \in k$ and $m \geq 1$;
(3): $I = (x^n - y^n,\ xy,\ yx)$ for $n \geq 2$;
(4): $I = (x^2,\ y^2,\ (xy)^m x - (yx)^m y)$ for $m \geq 1$;
(5): $I = (yx - x^2,\ y^2)$."

By the Lemma 4.11, we know that the case (5) of above conclusion will not appear. Thus the better form of Theorem 4.1 of [**41**] is the following.

Theorem *Let Λ be a tame local graded Frobenius algebra. If char$k \neq 2$, then $\Lambda \cong k < x, y > /I$ where I is one of forms:*
(1): $I = (x^2 - y^2,\ yx - ax^2,\ xy)$ for $0 \neq a \in k$;
(2): $I = (x^2,\ y^2,\ (xy)^m - a(yx)^m)$ for $0 \neq a \in k$ and $m \geq 1$;
(3): $I = (x^n - y^n,\ xy,\ yx)$ for $n \geq 2$;
(4): $I = (x^2,\ y^2,\ (xy)^m x - (yx)^m y)$ for $m \geq 1$.

Of course, Theorem 5.4 of [**41**] should be changed accordingly. That is, delete the case (5) of Theorem 5.4 in [**41**].

Acknowledgement

I would like to express my hearty thanks to Professor Nanhua Xi for his encouragements. The work in Section 4 was finished when the author visit the Oxford University under the financial support from the Leverhulme Trust through the Academic Interchange Network Algebras, Representations and Applications. I am grateful to Professors K. Erdmann and A. Henke for stimulation conversations. I thank their hospitality from the university and the financial support from the Leverhulme Trust. I also want to thank my supervisor, Professor Fang Li, for his helpful comments. I am very grateful to the organizers for their excellent work.

I also would like to thank the referee for very valuable comments.

References

[1] N. Andruskiewitsch, About finite-dimensional Hopf Algebras, Contemp.Math 294 (2002), 1-57.

[2] N. Andruskiewitsch, I. Angiono, On Nichols Algebras with Generic Braiding, to appear in "Modules and Comodules", Proceedings of a conference dedicated to Robert Wisbauer, arXiv:math/070392v2.
[3] N. Andruskiewitsch, P. Etingof, S. Gelaki, Triangular Hopf Algebras with the Chevalley Property, Michigan Math. J. 49 (2001), no. 2, 277–298.
[4] N. Andruskiewitsch, S. Natale, Counting Arguments for Hopf Algebras of Low Dimension, Tsukuba Math J. 25 (2001), no. 1, 187-201.
[5] N. Andruskiewitsch, H. -J. Schneider, A Characterization of Quantum Groups, J. Reine Angew. Math 577 (2004), 81-104.
[6] N. Andruskiewitsch, H. -J. Schneider, Pointed Hopf Algebras, in "New direction in Hopf algebras", 1-68, Math. Sci. Res. Inst. Publ. 43, Cambridge Univ. Press, Cambridge, 2002.
[7] N. Andruskiewitsch, H. -J. Schneider, Finite Quantum Groups and Cartan Matrices, Adv. Math 154 (2000), 1-45.
[8] N. Andruskiewitsch, H. -J. Schneider, Hopf Algebras of Order p^2 and Braided Hopf Algebras of Order p, J. Algebra 199 (1998), 430-454.
[9] N. Andruskiewitsch, H. -J. Schneider, Lifting of Quantum Linear Spaces and Pointed Hopf Algebras of Order p^3, J. Algebra 209 (1998), 659-691.
[10] N. Andruskiewitsch, H. -J. Schneider, On the Classification of Finite-dimensional Pointed Hopf Algebras, to appear in Ann. Math.
[11] M. Auslander, I. Reiten, S. Smalϕ, Representation Theory of Artin Algebras, Cambridge University Press, 1995.
[12] M. Beattie, S. Dascalescu, Hopf Algebras of Dimension 14, J. LMS. II 69 (2001), no. 1, 65-78.
[13] Xiao-Wu Chen, Hua-Lin Huang, Yu Ye, Pu Zhang, Monomial Hopf Algebras, J. Algebra 275 (2004), 212-232.
[14] C. Cibils, M. Rosso, Hopf Quiver, J. Algebra 254 (2002), 241-251.
[15] M. Cohen, D. Fishman, Hopf Algebra Actions, J. Algebra 100 (1986), 363-379.
[16] S. Dascalescu, Pointed Hopf Algebras with Large Coradical, Comm. Alg 27 (1999), no. 10, 4821-4826.
[17] Yu. A. Drozd, Tame and Wild Matrix Problems, Representations and Quadratic Forms, Inst.Math., Acad.Sciences.Ukrainian SSR, Kiev 1979, 39-74. Amer.Math.Soc. Transl. 128 (1986), 31-55.
[18] K. Erdmann, Blocks of Tame Representation Type and Related Algebras, LNM 1428, Springer-Verlag, 1990.
[19] K. Erdmann, E. L. Green, N. Snashall, R. Taillefer, Representation Theory of the Drinfeld Double of a Family of Hopf Algebras, J. Pure Appl. Algebra 204 (2006), no. 2, 413-454.
[20] P. Etingof, S. Gelaki, On Finite-dimensional Semisimple and Cosemisimple Hopf Algebras in Positive Characteristic, International Mathematics Research Notices 16 (1998), 851-864.
[21] P. Etingof, S. Gelaki, The Classification of Triangular Semisimple and Cosemisimple Hopf Algebras over an Algebraically Closed Field, International Mathematics Research Notices 5 (1997), 223-234.
[22] P. Etingof, S. Gelaki, Semisimple Hopf Algebras of Dimension pq Are Trivial, J. Algebra 210 (1998), no. 2, 664-669.
[23] P. Etingof, S. Gelaki, On Hopf Algebras of Dimension pq, J. Algebra 276 (2004), no. 1, 399-406.
[24] P. Etingof, S. Gelaki, On Cotriangular Hopf Algebras, Amer. J. Math 123 (2001), no 4, 699-713.
[25] P.Etingof, D. Nikshych, V. Ostrik, On Fusion Categories, Ann. Math 162 (2005), 581-642.
[26] D. Fischman, S. Montgomery, H. -J. Schneider, Frobenius Extensions of Subalgebras of Hopf Algebras, Trans. AMS 349 (1997), 4857-4895.
[27] C. Galindo, S. Natale, Normal Hopf Subalgebras in Cocycle Deformations of Finite Groups, to appear in Manuscripta Mathematica. arXiv: 0708.3407v3.
[28] C. Galindo, S. Natale, Simple Hopf Algebras and Deformations of Finite Groups, Math. Res. Lett 14 (2007), 943-954.
[29] G. Garcia, C. Vay, Hopf Algebras of Dimension 16, arXiv: 0712.0405v1.
[30] S. Gelaki, S. Westreich, On Semisimple Hopf Algebras of dimension pq, Proc. AMS 128 (2000), no. 1, 39-47.
[31] M. Gra$\widetilde{n}$a, On Nichols Algebras of Low Dimension, in "New Trends in Hopf Algebra Theory", Contemp. Math 267 (2000), 111-134.

[32] E. Green, $\emptyset$. Solberg, Basic Hopf Algebras and Quantum Groups, Math. Z 229 (1998), 45-76.
[33] I. Heckenberger, Examples of Finite Dimensional Rank 2 Nichols Algebras of Diagonal Type, Compos. Math 143 (2007), no. 1, 165-190.
[34] I. Heckenberger, Rank 2 Nichols Algebras with Finite Arithmetic Root System, published in Algebras and Representation Theory, DOI 10.1007/s10468-007-9060-7.
[35] I. Heckenberger, The Wyle Groupoid of a Nichols Algebra of Diagonal Type, Invent. Math 164 (2006), 175-188.
[36] I. Heckenberger, Classification of Arithmetic Root Systems, arXiv: math/0605795v1.
[37] H. L. Huang, H. X. Chen, P. Zhang, Generalized Taft Algebras, Alg. Colloq. 11 (2004), no. 3, 313-320.
[38] H. Krause, Stable Equivalence Preserves Representation Type, Comment. Math. Helv 72 (1997), 266-284.
[39] R. G. Larson, D. E. Radford, Finite Dimensional Cosemisimple Hopf Algebras in Characteristic 0 Are Semisimple. J. Algebra 117 (1988), 267-289.
[40] F. Li, M. Zhang, Invariant Properties of Representations under Cleft Extensions, Sciences in China 50 (2007), no. 1, 121-131.
[41] G. Liu, On the Structure of Tame Graded Basic Hopf Algebras, J. Algebra 299 (2006), 841-853.
[42] G. Liu, F. Li, Pointed Hopf Algebras of Finite Corepresentation Type and Their Classification, Proc. AMS 135 (2007), no. 3, 649-657.
[43] D. -M. Lu, Q. -S. Wu and J. J. Zhang, Homological Integral of Hopf Algebras, Trans. AMS, 359 (2007), no. 10, 4945-4975.
[44] S. Majid, Crossed Products by Braided Groups and Bosonization, J. Algebra 163 (1994), 165-190.
[45] A. Masuoka, Self Dual Hopf Algebras of Dimension p^3 Obtained by Extension, J. Algerba 178 (1995), 791-806.
[46] A. Masuoka, Semisimple Hopf Algebras of Dimension $2p$, Comm. Algebra 23 (1995), 1931-1940.
[47] A. Masuoka, Calculations of Some Groups of Hopf Extensions, J. Algebra, 191 (1997), 568-588.
[48] A. Masouka, The p^n Theorem for Semisimple Hopf Algebras, Proc. AMS 124(1996), no. 3, 735-737.
[49] S. Montgomery, Hopf Algebras and Their Actions on Rings. **CBMS**, Lecture in Math.; Providence, RI, (1993); Vol. 82.
[50] S. Montgomery, Classifying Finite Dimensional Semisimple Hopf Algebras, Contemp. Math 229 (1998), 265-279.
[51] T. Nakayama, On Frobenusean Algebras I and II, Ann. of Math 40 (1939), 611-633 and Ann. of Math 42 (1941), 1-21.
[52] S. Natale, On Semisimple Hopf Algerbas of Dimension pq^2, J. Algebra 221 (1999), 242-278.
[53] S. Natale, On Semisimple Hopf Algebras of Dimension pq^2 II, Algebra Representation Theory, 2003.
[54] S. Natale, Semisolvability of Semismple Hopf Algebras of Low Dimension, Mem. AMS 186 (2007).
[55] S-H. Ng, Non-semisimple Hopf Algebras of Dimension p^2, J. Algebra 255 (2002), no. 1, 182-197.
[56] S-H. Ng, Hopf Algebras of Dimension pq, J. Algebra 276 (2004), no. 1, 399-406.
[57] S-H. Ng, Hopf Algebras of Dimension pq II, J. Algebra, In Press. arXiv: 0704.2428v1
[58] S-H. Ng, Hopf Algebras of Dimension $2p$, Proc. AMS 133 (2005), no. 8, 2237-2242.
[59] D. Nikshych, Non Group-theoretical Semisimple Hopf Algebras from Group Actions on Fusion Categories, arXiv: 0712.0585v1.
[60] Fred van Oystaeyen, Pu Zhang, Quiver Hopf Algebras, J. Algebra 280 (2004), 577-589.
[61] D. Radford, The Structure of Hopf Algebras with a Projection, J. Algebra 92 (1985), 322-347.
[62] C. M. Ringel, The Representation Type of Local Algebras, In Representation of Algebras, LNM 488, Springer-Verlag (1975), 282-305.
[63] C. M. Ringel, Tame Algebras and Integral Quadratic Forms, LNM 1099, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[64] H. -J. Schneider, Normal Basis and Transitivity of Crossed Products for Hopf Algebras, J. Algebra 152 (1992), 289-312.

[65] Y. Zhu, Hopf Algebras of Prime Dimension, International Mathematical Research Notices 1 (1994), 53-59.

Department of Mathematics, Nanjing University, Nanjing 210093, China
E-mail address: gxliu@nju.edu.cn