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A NOTE ON THE GLOBAL DIMENSION OF SMASH PRODUCTS#

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Let H be a finite dimensional Hopf algebra over a field k, and A an H-module algebra. If H and H^* are semisimple, then we prove that $gl.dim(A\#H) = gl.dim(A)$. The relationship between this result and Kaplansky's Fifth Conjecture is discussed.

Key Words: Global dimension; Kaplansky's fifth conjecture; Smash product.

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1. INTRODUCTION

Throughout this article, all modules are left modules and all spaces are k-spaces for a fixed field k. By gl.dim(R) we denote the left global dimension of a ring R. H always denotes a finite-dimensional Hopf algebra and A an H-module algebra.

The global dimension for skew group algebras, or more generally, for smash products and crossed products was discussed by several authors, for example in Lorenz and Lorenz (1995), Shilin (2002), and Yi (1994). When H is semisimple, the inequality $\text{gl.dim}(A\#H) \leq \text{gl.dim}(A)$ was shown in Shilin (2002) and also in Lorenz and Lorenz (1995) in a more general form. Under some special conditions, the author of Shilin (2002) proved that $\text{gl.dim}(A\#H) = \text{gl.dim}(A)$ (see Theorem 2.2 of Shilin, 2002). In this article, by using the Blattner-Montgomery duality, we prove the following conclusion.

Theorem 1.1. *If H and H^* are semisimple, then* $\text{gl.dim}(A\#H) = \text{gl.dim}(A)$.

This result is related to Kaplansky's Fifth Conjecture, which says that the antipode S of a semisimple Hopf algebra satisfies $S^2 = id$.

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Proposition 1.2. *Let H be a finite dimensional semisimple Hopf algebra with antipode S over a field k with* $\dim(H) \neq 0$ *in k. Then* $\text{gl.dim}(A\#H) = \text{gl.dim}(A)$ *for any H-module algebra A if and only if $S^2 = id$.*

Some corollaries are also given. Standard notations about Hopf algebras are used freely. For example, $\Delta(h) = \Sigma h_1 \otimes h_2$ for $h \in H$.

2. PROOFS AND COROLLARIES

Firstly we note a version of Maschke's theorem.

Lemma 2.1. *Let H be a finite dimensional semisimple Hopf algebra, and P a left $A\#H$-module. Then P is a projective $A\#H$-module if and only if P is a projective A-module.*

Proof. The "only if" part can be gotten directly since $A\#H$ is a free A-module. Next, let's consider the sufficiency. For $A\#H$-modules M, N, let $g : M \to N$ and $h : P \to N$ be two $A\#H$-module morphisms such that g is surjective. In order to prove that P is projective as an $A\#H$-module, it is enough to find an $\tilde{f} \in Hom_{A\#H}(P, N)$ satisfying $h = g\tilde{f}$. Since P is projective as an A-module, there is an $f \in Hom_A(P, N)$ such that $h = gf$, where we consider $A\#H$-modules as A-modules in the natural way. Define $\tilde{f}(p) = \sum S(t_1) \cdot f(t_2 \cdot p)$ for $p \in P$, where t is a nonzero right integral with $\varepsilon(t) \neq 0$. Then $\tilde{f}$ is $A\#H$-linear by Cohen and Fishman (1986, Proposition 2). □

Proof of Theorem 1.1. Claim: $\text{gl.dim}(A\#H) \leq \text{gl.dim}(A)$. This claim was proved in Shilin (2002) and also in Lorenz and Lorenz (1995). For completeness, we give a more direct proof by using the above lemma. Clearly, it is harmless to assume the left global dimension of A is finite, say n. For any $A\#H$-module N, consider any one of its projective resolutions. By Lemma 2.1, it is also a projective resolution for N as an A-module. Therefore its nth syzygy is projective as an A-module and thus as an $A\#H$-module, by Lemma 2.1. This implies $\text{p.dim}(N) \leq n$ and $\text{gl.dim}(A\#H) \leq \text{gl.dim}(A)$.

Note that $A\#H$ is an H^*-module algebra via $f \cdot (a\#h) = a\#(f \rightharpoonup h)$, where $f \rightharpoonup h = \Sigma f(h_2)h_1$ for $a\#h \in A\#H$ and $f \in H^*$. Thus, by the above claim, $\text{gl.dim}((A\#H)\#H^*) \leq \text{gl.dim}(A\#H)$. By the Blattner-Montgomery Duality Theorem (see Section 9.4 in Montgomery, 1993), when H is finite dimensional, $(A\#H)\#H^*$ is Morita equivalent to A. Therefore, $\text{gl.dim}(A) = \text{gl.dim}((A\#H)\#H^*)$. Thus,

$$\text{gl.dim}(A) = \text{gl.dim}((A\#H)\#H^*) \leq \text{gl.dim}(A\#H) \leq \text{gl.dim}(A)$$

This means that $\text{gl.dim}(A\#H) = \text{gl.dim}(A)$. □

The following corollary follows directly from Theorem 1.1.

Corollary 2.2. *If H and H^* are semisimple, then*

(1) *$A\#H$ is semisimple if and only if A is.*
(2) *$A\#H$ is hereditary if and only if A is.* □

Proof of Proposition 1.2. The "if" part: Recall that H and H^* are semisimple if $S^2 = id$ and char(k) does not divide dim(H) (Corollary 3.5 in Schneider, 1995). Thus, sufficiency follows from Theorem 1.1.

The "only if" part: Since H^* is an H-module algebra via $a \cdot f = \sum f_2(a) f_1$ for $a \in H$, $f \in H^*$, $\text{gl.dim}(H^*\#H) = \text{gl.dim}(H^*)$. But, by Corollary 9.4.3 in Montgomery (1993), $H^*\#H \cong \text{End}_k(H^*)$, which is a semisimple algebra. Therefore, $0 = \text{gl.dim}(H^*\#H) = \text{gl.dim}(H^*)$. This means that H^* is semisimple. By Corollary 3.2 of Etingof and Gelaki (1998), $S^2 = \text{id}$. □

When $\text{char}\, k = 0$, R. Larson and D. Radford proved Kaplansky's Fifth Conjecture (see Larson and Radford, 1988a,b). Y. Sommerhäuser also proved this conjecture when $\text{char}\, k > m^{m-4}$, where $m = 2(\dim(H))^2$ (see Theorem 4.2 of Sommerhäuser, 1998). P. Etingof and S. Gelaki made an improvement of Y. Sommerhäuser's result and proved the conjecture when the $\text{char}\, k > d^{\varphi(d)/2}$, where $2 < d = \dim H$ (here φ is the Euler function, see Theorem 4.2 of Etingof and Gelaki, 1998). Note that in these cases, the characteristic of k does not divide dim(H). Thus Proposition 1.2 implies

Corollary 2.3. *Let H be a d-dimensional semisimple Hopf algebra, $d > 2$, over a field k with* $\text{char}\, k = 0$ *or* $\text{char}\, k > d^{\varphi(d)/2}$, *then* $\text{gl.dim}(A\#H) = \text{gl.dim}(A)$. □

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