Algebra Colloquium **13** : 2 (2006) 181–196

Algebra Colloquium

On Strongly Groupoid Graded Rings and the Corresponding Clifford Theorem*

Gongxiang Liu
Institute of Mathematics, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100080, China
E-mail: gxliu@amss.ac.cn

Fang Li
Department of Mathematics, Zhejiang University
Hangzhou, Zhejiang 310028, China
E-mail: fangli@zju.edu.cn

Received 6 November 2003
Revised 12 September 2005

Communicated by Nanqing Ding

Abstract. In this paper, we introduce the definition of groupoid graded rings. Group graded rings, (skew) groupoid rings, artinian semisimple rings, matrix rings and others can be regarded as special kinds of groupoid graded rings. Our main task is to classify strongly groupoid graded rings by cohomology of groupoids. Some classical results about group graded rings are generalized to groupoid graded rings. In particular, the Clifford Theorem for a strongly groupoid graded ring is given.

2000 Mathematics Subject Classification: 16W50

Keywords: groupoid graded ring, Picard groupoid, Clifford Theorem

1 Introduction and Preliminaries

Throughout this paper, we only consider the rings with identity and assume that every ring homomorphism preserves the identity unless stated otherwise. All modules will be left modules. The initial motivation to investigate groupoid graded rings comes from their connections with weak Hopf Galois extensions (see [1]). It is known that for a groupoid G, the groupoid algebra kG is a weak Hopf algebra (see [2]). In [1], the authors studied the Galois extensions over weak Hopf algebras

---

*Project supported by the Program for New Century Excellent Talents in University (No. 04-0522), the National Natural Science Foundation of China (No. 10571153) and the Natural Science Foundation of Zhejiang Province of China (No. 102028).

(also called weak Hopf Galois extensions) and proved that, under certain conditions, $\bigoplus_{e_i \in G^0} A_{e_i} \hookrightarrow A$ is kG-Galois if and only if A is strongly groupoid graded (see [1, Theorem 3.5]). This result can be viewed as a generalization of [4, Theorem 8.1.7]. Although the class of groupoid graded rings is a special kind of semigroup graded rings, it is more general than the class of group graded rings. There are some interesting examples of groupoid graded rings, which also lead the authors to this topic.

This paper is organized as follows. We give preliminaries about groupoids in the first section. Then in the second section, we will give the definition of groupoid graded rings with some examples and elementary properties. In Section 3, we mainly deal with strongly groupoid graded rings. Our goal is to classify them by the cohomology of groupoids. The Clifford Theorem for a strongly groupoid graded ring is proved in the last section.

We give the definition of groupoids and refer the reader to [6, 8] for more results.

Definition 1.1. A *groupoid* is a set G endowed with a product map $G^2 \to G$, $(x, y) \mapsto xy$, where G^2 is a subset of $G \times G$, called the set of *composable pairs*, and an inverse map $G \to G$, $x \mapsto x^{-1}$ such that the following relations are satisfied:

(i) $(x^{-1})^{-1} = x$;

(ii) $(x, y), (y, z) \in G^2 \Rightarrow (xy, z), (x, yz) \in G^2$ and $(xy)z = x(yz)$;

(iii) $(x^{-1}, x) \in G^2$, and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$;

(iv) $(x, x^{-1}) \in G^2$, and if $(z, x) \in G^2$, then $(zx)x^{-1} = z$.

For $x \in G$, $d(x) = x^{-1}x$ is the *domain* of x and $r(x) = xx^{-1}$ is its *range*. The pair (x, y) is composable if and only if the range of y is the domain of x. $G^0 = d(G) = r(G)$ is the *unit space* of G. We say that the unit space of G is *finite* if $|G^0|$, the cardinal number of G^0, is finite. Let $u \in G^0$. It is easy to see that $G^u_u = \{\sigma \in G \,|\, r(\sigma) = d(\sigma) = u\}$ is a group, called the *isotropy group* at u.

Let G and H be two groupoids and f a map from G to H. We call f a *groupoid map* or a *groupoid homomorphism* if for $(\sigma, \tau) \in G^2$, $(f(\sigma), f(\tau)) \in H^2$ and $f(\sigma\tau) = f(\sigma)f(\tau)$.

Some people use "groupoid" as a set with a binary operation. In order not to cause confusion, we call it a groupoid of binary case. Clearly, our definition of a groupoid is totally unrelated to this.

Cohomology theory for groupoids was developed by Westman in [9]. Suppose that $\mathcal{C}$ is a category. A map p from a set A onto a set A^0 such that each fiber $p^{-1}(u)$ $(u \in A^0)$ is an object of $\mathcal{C}$ will be called a $\mathcal{C}$-bundle map and A will be called a $\mathcal{C}$*-bundle*. Let A be a $\mathcal{C}$-bundle with bundle map $p : A \to A^0$. Write $A_u = p^{-1}(u)$. Let $\mathrm{Iso}(A) = \{\text{isomorphisms } \phi_{u,v} : A_v \to A_u \mid u, v \in A^0\}$, then it has a natural groupoid structure, i.e., $\phi_{u,v}$ and $\phi_{v',w}$ are composable if and only if $v' = v$, their product is $\phi_{u,v} \circ \phi_{v,w}$ and $\phi^{-1}_{u,v}$ is the isomorphism inverse of $\phi_{u,v}$. The bijection $\mathrm{id}_{u,u} \mapsto u$ identifies the unit space of $\mathrm{Iso}(A)$ with A^0. $\mathrm{Iso}(A)$ is called the isomorphism groupoid of the $\mathcal{C}$-bundle A.

Definition 1.2. Let G be a groupoid. A G*-bundle* (A, L) is a $\mathcal{C}$-bundle A together with a groupoid homomorphism $L : G \to \mathrm{Iso}(A)$ such that $L^0 : G^0 \to A^0$ is a bijection. We will often identify G^0 with A^0. When $\mathcal{C}$ is the category of abelian groups, one calls (A, L) a *G-module bundle*.

Given a G-module bundle (A, L), one can form the following cochain complex. Let us first define G^n for any natural number n. The sets G^0, $G^1 = G$ and G^2 have been defined. For $n \geq 2$, G^n is the set of n-tuples $(x_0, \dots, x_{n-1})$ such that x_i is composable with its left neighbor for $i = 1, \dots, n-1$. An n-cochain is a function f from G^n to A which satisfies the following two conditions:

(i) $p \circ f(x_0, \dots, x_{n-1}) = r(x_0)$;

(ii) if $n > 0$ and $x_i \in G^0$ for some $i = 0, \dots, n-1$, then $f(x_0, \dots, x_{n-1}) \in A^0$.

The set $C^n(G, A)$ of n-cochains is an abelian group under point-wise addition. The sequence

$$0 \to C^0(G, A) \to C^1(G, A) \to \cdots \to C^n(G, A) \xrightarrow{\delta^n} C^{n+1}(G, A) \to \cdots$$

is a cochain complex, where $\delta^0 f(x) = L(x) f \circ d(x) - f \circ r(x)$ and $\delta^n f(x_0, \dots, x_n) = L(x_0) f(x_1 \dots, x_n) + \sum_{i=1}^{n} (-1)^i f(x_0, \dots, x_{i-1}x_i, \dots, x_n) + (-1)^{n+1} f(x_0, \dots, x_{n-1})$ for $n > 0$. The groups of n-cocycles and n-coboundaries will be denoted by $Z^n(G, A)$ and $B^n(G, A)$, respectively. Thus, the n-th cohomology group is

$$H^n(G, A) = Z^n(G, A)/B^n(G, A).$$

Let R and S be two rings. An R-S-bimodule ${}_RM_S$ is *invertible* if there exists an S-R-bimodule ${}_SN_R$ such that $M \otimes_S N \cong R$ as R-bimodules and $N \otimes_R M \cong S$ as S-bimodules. In this case, we call N an *inverse module* of M. Given rings $A_1, \dots, A_n$, we define the *Picard groupoid* $G = \mathrm{Pic}(A_1, \dots, A_n)$ over $\{A_1, \dots, A_n\}$ to be the set of isomorphism classes $[{}_{A_i}P_{A_j}]$ of invertible A_i-A_j-bimodules for $i, j \in \{1, \dots, n\}$ with $([{}_{A_i}P_{A_j}], [{}_{A_l}Q_{A_m}]) \in G^2$ if and only if $A_j = A_l$.

Proposition 1.3. *If we define* $[{}_{A_i}P_{A_j}] \cdot [{}_{A_j}Q_{A_m}] = [P \otimes_{A_j} Q]$ *and* $[{}_{A_i}P_{A_j}]^{-1} = [\mathrm{Hom}_{A_j}(P, A_j)]$, *then* $G = \mathrm{Pic}(A_1, \dots, A_n)$ *is a groupoid.*

The proof of this proposition will be given in Section 3.

2 Groupoid Graded Rings

In the sequel of this paper, all unexplained definitions and notations concerning groupoids can be found in [6, 8].

Let G be a groupoid with the unit space G^0. A ring R is said to be a *graded ring of type G* if there is a family of additive subgroups of R, say $\{R_\sigma \mid \sigma \in G\}$, such that $R = \bigoplus_{\sigma \in G} R_\sigma$ and

$$R_\sigma R_\tau \begin{cases} \subseteq R_{\sigma\tau} & \text{if } (\sigma, \tau) \in G^2, \\ = 0 & \text{if } (\sigma, \tau) \notin G^2. \end{cases}$$

The elements in $\bigcup_{\sigma \in G} R_\sigma$ are called *homogeneous* elements of R, and a nonzero element $r \in R_\sigma$ is said to be *homogeneous of degree σ*, and we write $\deg r = \sigma$.

Every nonzero element $r \in R$ has a unique expression as a sum of homogenous elements $r = \sum_{\sigma \in G} r_\sigma$ and the nonzero element r_σ in this decomposition is called a *homogeneous component* of r.

Example 2.1. (1) Clearly, every group graded ring is a groupoid graded ring.

(2) (Groupoid ring) Let A be a ring, G a groupoid with a finite unit space and $A[G]$ a free left A-module with basis G, i.e., $A[G] = \bigoplus_{\sigma \in G} A\sigma$, where $A\sigma \cong A$ as left A-modules. We give a multiplication on $A[G]$ and make it into a ring:

$$\sum_{\sigma \in G} a_\sigma \sigma \cdot \sum_{\tau \in G} b_\tau \tau = \sum_{\sigma, \tau \in G} a_\sigma b_\tau \sigma \circ \tau,$$

where $\sigma \circ \tau = \sigma\tau$ if $(\sigma, \tau) \in G^2$ and 0 otherwise. It is easy to show that $A[G]$ is a ring under the multiplication defined above. It is a graded ring of type G by setting $(A[G])_\sigma = A\sigma$.

(3) (Skew groupoid ring) Let G be a groupoid with $G^0 = \{e_1, \dots, e_n\}$ and $R = \bigcup_{i=1}^n R_i$ a ring bundle, i.e., every R_i is a ring. Suppose that there exists a map $\mu : R \to G^0$ such that $\mu(R_i) = e_i$ $(i = 1, \dots, n)$. Let

$$G * R = \{(g, r) \in G \times R \mid d(g) = \mu(r)\}.$$

A *left groupoid action* of G on R is defined to be a map $G * R \to R$ taking the pair (g, r) to $g \cdot r$ with the properties

(i) $\mu(g \cdot r) = r(g)$,
(ii) $(gh) \cdot r = g \cdot (h \cdot r)$,
(iii) $\mu(r)r = r$,
(iv) $g \cdot (r + s) = g \cdot r + g \cdot s$,
(v) $g \cdot (rs) = (g \cdot r)(g \cdot s)$,
(vi) $\sigma \cdot 1_{R_i} = 1_{R_j}$,

where $g, h, \sigma \in G$ and $r, s \in R$ with $d(\sigma) = e_i$ and $r(\sigma) = e_j$, and 1_{R_i} is the identity of R_i for $i = 1, \dots, n$. It is easy to see that the properties (iv), (v), (vi) mean that σ is a ring homomorphism from R_i to R_j. Denote $R * G = \bigoplus_{i=1}^n \bigoplus_{r(\sigma)=e_i} R_i\sigma$, where $R_i\sigma$ is a free left R_i-module with basis σ. The multiplication can be defined by $(r_i\sigma)(r_j\tau) = r_i(\sigma \cdot r_j)\sigma\tau$ if $(\sigma, \tau) \in G^2$ and 0 otherwise. $R * G$ is a groupoid graded ring with $(R * G)_\sigma = R_i\sigma$ (where we assume $r(\sigma) = e_i$).

(4) Any ring R can be considered as a groupoid graded ring. Indeed, let $1 = \sum_{i=1}^m e_i$ be a decomposition into orthogonal idempotents in $Z(R)$, the center of R. Then $R = \bigoplus_{i=1}^m Re_i$. Let $G = \{1, \dots, m\}$ and define $G^2 = \{(i, i) \mid i = 1, \dots, m\}$, $i \cdot i = i$ and $i^{-1} = i$. It is easy to see that G is a groupoid and R is a graded ring of type G with $(R)_i = Re_i$. In particular, if R is an artinian semisimple ring, then it is a groupoid graded ring.

(5) Let $A = M_n(B)$, the $n \times n$ matrices over a ring B. Suppose that G is a groupoid with $|G| = n$. Fix an ordering $\{x_1, \dots, x_n\}$ of the elements of G, and index the rows and columns of A by G. Let $\{e_{xy} \mid (x, y^{-1}) \in G^2\}$ denote the usual matrix units. Setting $A_x = \sum_{yz^{-1}=x} Be_{yz}$, we claim that if $(x, y) \in G^2$, then $A_xA_y \subseteq$

A_{xy}. In fact, $A_xA_y = \sum_{ab^{-1}=x} Be_{ab} \sum_{cd^{-1}=y} Be_{cd} \subseteq \sum_{ab^{-1}=x, cd^{-1}=y} Be_{ab}e_{cd} = \sum_{ab^{-1}=x, bd^{-1}=y} Be_{ad} \subseteq A_{xy}$ since $ad^{-1} = ab^{-1}bd^{-1} = xy$. Meanwhile, we have $A_xA_y = 0$ if $(x, y) \notin G^2$. Indeed, it is enough to show that if $e_{ab} \in A_x$, then there is no c in G such that $e_{bc} \in A_y$. Otherwise, if there is c in G such that $e_{bc} \in A_y$, then $bc^{-1} = y$ which implies $r(y) = r(b)$. Since $e_{ab} \in A_x$, $ab^{-1} = x$ and $d(x) = d(b^{-1}) = r(b) = r(y)$. Thus, $(x, y) \in G^2$. It is a contradiction. By the discussion above, $\widetilde{A} = \bigoplus_{x \in G} A_x$ is a groupoid graded ring of type G. In particular, it is easy to see that $\widetilde{A} = M_n(B)$ when G is a group (see [4, Example 6.1.7]).

Proposition 2.2. *If R is a graded ring of type G, then for $e_i \in G^0$, R_{e_i} is a subring of R and $1 \in \bigoplus_{e_i \in G^0} R_{e_i}$.*

Proof. From $R_{e_i}R_{e_i} \subseteq R_{e_i}$, it follows immediately that R_{e_i} is a subring of R. Let $1 = \sum_{\sigma \in G} r_\sigma$ be the homogeneous decomposition of $1 \in R$. Pick $\tau \in G$ and $\lambda_\tau \in R_\tau$, then $\lambda_\tau = 1 \cdot \lambda_\tau = \sum_{\sigma \in G} r_\sigma \lambda_\tau$. If $(\sigma, \tau) \notin G^2$, then $r_\sigma \lambda_\tau = 0$. If $(\sigma, \tau) \in G^2$, then $r_\sigma \lambda_\tau \in R_{\sigma\tau}$. Consequently, $r_\sigma \lambda_\tau = 0$ holds for all $\sigma \neq r(\tau)$. It follows that $r_\sigma \lambda = 0$ if $\sigma \notin G^0$ for all $\lambda \in R$. Therefore, $1 \in \bigoplus_{e_i \in G^0} R_{e_i}$. □

Let $G^0 = \{e_i \mid i \in I\}$. By Proposition 2.2, $1 = \sum_{i \in I_0} r_{e_i}$, where I_0 is a finite set.

Proposition 2.3. *Let R be a ring in the above proposition. Then $R_{e_j} = 0$ if $j \in I - I_0 = \{i \in I \mid i \notin I_0\}$ and $R_\sigma = 0$ if $r(\sigma)$ or $d(\sigma)$ belongs to $\{e_i \mid i \in I - I_0\}$.*

Proof. Let $r_{e_j} \in R_{e_j}$, where $j \in I - I_0$, then $r_{e_j} = r_{e_j} \cdot 1 = r_{e_j} \sum_{i \in I_0} r_{e_i} = 0$ by the fact that R is a groupoid graded ring. Thus, $R_{e_j} = 0$ for $j \in I - I_0$. Let $y = r_\sigma \in R_\sigma$ and assume $r(\sigma) \in \{e_i \mid i \in I - I_0\}$, then $y = 1 \cdot y = \sum_{i \in I_0} r_{e_i} r_\sigma = 0$, which implies $R_\sigma = 0$. Similarly, $R_\sigma = 0$ if $d(\sigma) \in \{e_i \mid i \in I - I_0\}$. □

Remark 2.4. (1) We always write $1 = \sum_{i \in I_0} r_{e_i} = \sum_{i \in I_0} e_i^R$. It is easy to see that e_i^R is the identity of R_{e_i} and also the left (resp., right) identity of R_σ with $r(\sigma) = e_i$ (resp., $d(\sigma) = e_i$) for $i \in I_0$. Denote $G' = \{\sigma \in G \mid d(\sigma), r(\sigma) \in \{e_i \mid i \in I_0\}\}$. By Proposition 2.3, $R = \bigoplus_{\sigma \in G} R_\sigma = \bigoplus_{\sigma \in G'} R_\sigma$. So there is no harm to assume that G has a finite unit space and $R_{e_i} \neq 0$ for $e_i \in G^0$. In this paper, we always write $G^0 = \{e_1, \ldots, e_n\}$ and assume $R_{e_i} \neq 0$ for $i = 1, \ldots, n$.

(2) In [3] and [7], the authors have given the definition for a groupoid graded ring when the groupoid is of binary case. In fact, our definition of a groupoid graded ring is a special case of that in [3]. The reason is that we can make a groupoid into a semigroup [8] as follows. Let G be a groupoid and choose $\phi \notin G$. The groupoid multiplication on G extends to a multiplication on the set $G \cup \{\phi\}$ by setting $g\phi = \phi g = \phi$ and $gh = \phi$ if $(g, h) \notin G^2$. This endows $G \cup \phi$ with a semigroup structure which is denoted as $G^{(0)}$. Let R be a groupoid graded ring of type G, then it is easy to see that it is also a semigroup graded ring of type $G^{(0)}$ by setting $R_\phi = 0$. Therefore, our definition of a groupoid graded ring is a special case of that in [3].

Let R be a graded ring of type G. An R-module M is said to be a *graded left R-module of type G* if there is a family $\{M_\sigma \mid \sigma \in G\}$ of additive subgroups of M

such that $M = \bigoplus_{\sigma\in G} M_\sigma$ and

$$R_\sigma M_\tau \begin{cases} \subseteq M_{\sigma\tau} & \text{if } (\sigma,\tau)\in G^2,\\ = 0 & \text{if } (\sigma,\tau)\notin G^2. \end{cases}$$

Elements in $h(M) = \bigcup_{\sigma\in G} M_\sigma$ are called *homogeneous elements* of M, and a nonzero element $m \in M_\sigma$ is said to be *homogeneous of degree* σ, and we write $\deg m = \sigma$. Any nonzero element $m \in M$ has a unique expression as a sum of homogenous elements $m = \sum_{\sigma\in G} m_\sigma$ and the nonzero element m_σ in this decomposition is called a *homogeneous component* of m.

A submodule N of M is a *graded submodule* if $N = \bigoplus_{\sigma\in G}(N \cap M_\sigma)$. Consider a graded R-module M and denote $M^i = \bigoplus_{d(\sigma)=e_i} M_\sigma$ for $i \in I = \{1,\dots,n\}$. An R-linear map f from M to another graded module N is said to be a *graded morphism of degree* $(\tau_i)_{i\in I}$ with $r(\tau_i) = e_i$ if $f(M_\sigma) \subseteq N_{\sigma\tau_i}$ for $M_\sigma \subseteq M^i$ and $i \in I$. In particular, let R and S be two graded rings of type G, a ring homomorphism $f : R \to S$ is called a *groupoid graded ring map* if $f(R_\sigma) \subseteq S_\sigma$ for $\sigma \in G$. If N is a graded submodule of M, then M/N can be made into a graded module by putting $(M/N)_\sigma = (M_\sigma + N)/N$. With this definition, the canonical R-linear projection $M \to M/N$ is a graded morphism of degree $(e_i)_{i\in I}$. The category R-gr$_G$ (or briefly, R-gr if there is no ambiguity about G) consists of graded R-modules of type G and all graded morphisms of degree $(e_i)_{i\in I}$. If $M, N \in R$-gr, then we write $\mathrm{Hom}_{R\text{-gr}}(M,N)$ for the graded morphisms of degree $(e_i)_{i\in I}$ from M to N. For any $f \in \mathrm{Hom}_{R\text{-gr}}(M,N)$, it is easy to see that both $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are in R-gr.

3 Strongly Groupoid Graded Rings

A graded ring $R = \bigoplus_{\sigma\in G} R_\sigma$ of type G is said to be a *strongly graded ring of type* G if $R_\sigma R_\tau = R_{\sigma\tau}$ for $(\sigma,\tau)\in G^2$.

Lemma 3.1. *Let R be a graded ring of type G, then the following conditions are equivalent*:

(i) *R is strongly graded.*
(ii) *$R_x R_{x^{-1}} = R_{r(x)}$ for all $x \in G$.*
(iii) *$R_{x^{-1}} R_x = R_{d(x)}$ for all $x \in G$.*

Proof. "(i)⇒(ii)" and "(i)⇒(iii)" are easy. We only prove "(ii)⇒(i)" since "(iii)⇒(i)" is similar. Assume $(x,y)\in G^2$, note that there is a left identity of R_{xy} in $R_{r(x)}$ by Remark 2.4. Then $R_{xy} = R_{r(x)}R_{xy} = R_x R_{x^{-1}} R_{xy} \subseteq R_x R_{x^{-1}xy} = R_x R_y \subseteq R_{xy}$. □

Example 3.2. (1) Clearly, groupoid graded rings given in (2)–(4) of Example 2.1 are strongly graded. The groupoid graded ring $\widetilde{A}$ constructed in (5) of Example 2.1 is also strongly graded. Indeed, it is enough to prove the condition (iii) in Lemma 3.1. By the definition of $\widetilde{A}$, we have $(\widetilde{A})_{d(x)} = A_{d(x)} = \sum_{yz^{-1}=d(x)} Be_{yz}$. From $yz^{-1} = d(x)$, we know $y = yz^{-1}z = d(x)z = z$, so $y = z$. Thus, $A_{d(x)} = \sum_{yy^{-1}=d(x)} Be_{yy}$ and $(x,y)\in G^2$. But $e_{yy} = e_{y,xy}e_{xy,y} \in A_{x^{-1}}A_x$. Therefore, $A_{d(x)} \subseteq A_{x^{-1}}A_x$, which implies $A_{d(x)} = A_{x^{-1}}A_x$ and the condition (iii) in Lemma 3.1 is satisfied.

(2) Let R be a strongly graded ring of type G. An ideal I of R is called a *graded ideal* if it is graded as a left R-module. Clearly, R/I is a strongly graded ring with $(R/I)_\sigma = (R_\sigma + I)/I$.

(3) Let R be a strongly graded ring of type G, and $G' \subseteq G$ a subgroupoid of G. Then $R^{G'} = \bigoplus_{\sigma \in G'} R_\sigma$ is a strongly graded ring of type G'.

Recall $1 = \sum_{i\in I} e_i^R$. By Remark 2.4(1), e_i^R is the identity of R_{e_i} and also the left (resp., right) identity of R_σ with $r(\sigma) = e_i$ (resp., $d(\sigma) = e_i$). Let $x \in G$ with $d(x) = e_i$ and $r(x) = e_j$. It is clear that $R_x R_{x^{-1}} = R_{r(x)} = R_{e_j}$ if and only if $e_j^R \in R_x R_{x^{-1}}$, and $R_{x^{-1}} R_x = R_{d(x)} = R_{e_i}$ if and only if $e_i^R \in R_{x^{-1}} R_x$. Combining this observation and Lemma 3.1, we have:

Corollary 3.3. *The graded ring R of type G is strongly graded if and only if $e_i^R \in R_x R_{x^{-1}}$ for all $i \in I$ and x with $r(x) = e_i$, and if and only if $e_i^R \in R_{x^{-1}} R_x$ for all $i \in I$ and x with $d(x) = e_i$.*

Theorem 3.4. *If R is a strongly graded ring of type G, then R_σ is finitely generated and projective in ${}_{R_{r(\sigma)}}\mathcal{M}$ and also in $\mathcal{M}_{R_{d(\sigma)}}$ for all $\sigma \in G$.*

Proof. Assume $d(\sigma) = e_i$ and $r(\sigma) = e_j$. From $R_{\sigma^{-1}} R_\sigma = R_{d(\sigma)} = R_{e_i}$, it follows that $e_i^R = \sum_k u_k v_k$ with $u_k \in R_{\sigma^{-1}}$ and $v_k \in R_\sigma$. Let $L_\sigma = \sum_k R_{e_j} v_k$. Thus, $R_{\sigma^{-1}} L_\sigma = R_{\sigma^{-1}} \sum_k R_{e_j} v_k = R_{e_i}$ since $e_i^R \in R_{\sigma^{-1}} L_\sigma$. Then $R_\sigma R_{\sigma^{-1}} L_\sigma = R_\sigma R_{e_i} = R_\sigma$. Meanwhile, $R_\sigma R_{\sigma^{-1}} L_\sigma = R_{e_j} L_\sigma = L_\sigma$. Thus, $L_\sigma = R_\sigma$ and R_σ is finitely generated in ${}_{R_{r(\sigma)}}\mathcal{M}$. Since $e_i^R = \sum_k u_k v_k$, the R_{e_j}-morphisms from R_σ to R_{e_j} via right multiplication by u_k satisfy $r = \sum u_k(r) v_k$ for each $r \in R_\sigma$. Therefore, by the well-known "Basis Lemma" about finitely generated projective modules, R_σ is projective in ${}_{R_{r(\sigma)}}\mathcal{M}$. Similarly, we can prove that R_σ is finitely generated and projective in $\mathcal{M}_{R_{d(\sigma)}}$. □

We call a graded R-module M *strongly graded* if $R_\sigma M_\tau = M_{\sigma\tau}$ for $(\sigma, \tau) \in G^2$. The next theorem gives the relationship between strongly graded rings and strongly graded modules.

Theorem 3.5. *If R is a graded ring of type G, then the following statements are equivalent*:

(i) *R is strongly graded.*

(ii) *Every graded R-module M is strongly graded.*

Proof. "(ii)⇒(i)" is easy. We now prove "(i)⇒(ii)". If $(\sigma, \tau) \in G^2$, then $M_{\sigma\tau} = R_{r(\sigma)} M_{\sigma\tau} = R_\sigma R_{\sigma^{-1}} M_{\sigma\tau} \subseteq R_\sigma M_{\sigma^{-1}\sigma\tau} = R_\sigma M_\tau$. So M is strongly graded. □

Proposition 3.6. *Let R be a strongly graded ring of type G with $G^0 = \{e_1, \dots, e_n\}$. For every $(\sigma, \tau) \in G^2$, the canonical morphism $\varphi_\sigma : R_\sigma \otimes_{R_{d(\sigma)}} R_\tau \to R_{\sigma\tau}$ given by $r_\sigma \otimes r_\tau \mapsto r_\sigma r_\tau$ is an isomorphism of $R_{r(\sigma)}$-$R_{d(\tau)}$-bimodules. In particular, R_σ is an invertible $R_{r(\sigma)}$-$R_{d(\sigma)}$-bimodule.*

Proof. It is enough to prove that the canonical morphism φ_σ is bijective since it is clearly an $R_{r(\sigma)}$-$R_{d(\tau)}$-bimodule map. It is surjective by $R_\sigma R_\tau = R_{\sigma\tau}$. Next we show that φ_σ is injective. Assume $d(\sigma) = r(\tau) = e_i$. We construct an R-module map $\Phi = \bigoplus_{d(\alpha)=e_i} \varphi_\alpha$ from $(\bigoplus_{d(\alpha)=e_i} R_\alpha) \otimes_{R_{e_i}} R_\tau = \bigoplus_{d(\alpha)=e_i}(R_\alpha \otimes_{R_{e_i}} R_\tau)$ to $\bigoplus_{d(\alpha)=e_i} R_{\alpha\tau}$. Clearly, $\bigoplus_{d(\alpha)=e_i}(R_\alpha \otimes_{R_{e_i}} R_\tau)$ and $\bigoplus_{d(\alpha)=e_i} R_{\alpha\tau}$ are left R-modules via multiplication and Φ is an R-module map. Moreover, $\bigoplus_{d(\alpha)=e_i}(R_\alpha \otimes_{R_{e_i}} R_\tau)$ is a graded R-module by setting $(\bigoplus_{d(\alpha)=e_i}(R_\alpha \otimes_{R_{e_i}} R_\tau))_\beta = R_\alpha \otimes_{R_{e_i}} R_\tau$ if there is a unique $\alpha \in G$ such that $\beta = \alpha\tau$ and $(\bigoplus_{d(\alpha)=e_i}(R_\alpha \otimes_{R_{e_i}} R_\tau))_\beta = 0$ otherwise. Clearly, Φ is a graded morphism of degree (e_j). So $K = \operatorname{Ker}\Phi$ is a graded submodule of $\bigoplus_{d(\alpha)=e_i}(R_\alpha \otimes_{R_{e_i}} R_\tau)$. Let $K_\tau = (\operatorname{Ker}\Phi)_\tau = \operatorname{Ker}(\varphi_{e_i})$, where $\varphi_{e_i} : R_{e_i} \otimes_{R_{e_i}} R_\tau \to R_\tau$ is an isomorphism. So $K_\tau = 0$. Also, $K_{\alpha\tau} = R_\alpha K_\tau = 0$ for $d(\alpha) = e_i$ since K is a strongly graded R-module by Theorem 3.5. Thus, $K = 0$ and Φ is injective, which implies that φ_σ is injective. □

Let S be a ring, we denote the opposite ring of S by S^{op}, i.e., S^{op} has the same underlying additive group as S while its multiplication is defined by the rule $x \circ y = yx$ for $x, y \in S^{op}$. Let P and P' be A-B-bimodules, and $f : P \to P'$ an A-B-bimodule map. In the sequel, we always say that f is a bimodule map from P to P' rather than an A-B-bimodule map if there is no ambiguity about A and B. Similarly, for $P \in \mathcal{M}_R$ and $Q \in {}_R\mathcal{M}$, we sometimes denote the tensor product $P \otimes_R Q$ by $P \otimes Q$ if there is no ambiguity about the ring R.

Lemma 3.7. *Let A, B be two rings and P an invertible A-B-bimodule. Then*

(1) $A \cong \operatorname{End}_B(P_B)$ *and* $B \cong \operatorname{End}_A({}_AP)^{op}$.

(2) $Z(A) \cong \operatorname{End}({}_AP_B) \cong Z(B)$, *where $Z(A)$ and $Z(B)$ denote the centers of A and B, respectively.*

(3) *P is a left* (*resp., right*) *finitely generated projective A-module* (*resp., B-module*).

(4) *The duality $P^* = \operatorname{Hom}_B(P_B, B)$ is an invertible B-A-bimodule, $P \otimes_B P^* \cong A$ as A-bimodules and $P^* \otimes_A P \cong B$ as B-bimodules.*

Proof. This lemma can be proven in the same way as in the proof of [5, Lemma I.3.11]. □

Proof of Proposition 1.3. By Lemma 3.7, we know that $P^* = \operatorname{Hom}_{A_j}(P, A_j)$ is an inverse module of ${}_{A_i}P_{A_j}$. Firstly, we prove that if M is an invertible S-R-bimodule and N, N' are inverse modules of M, then $N \cong N'$ as R-S-bimodules. In fact, by the definition of invertible bimodules, we have

$$N \otimes_S M \xrightarrow{\cong} R, \quad M \otimes_R N' \xrightarrow{\cong} S.$$

So $N \cong N \otimes S \cong N \otimes M \otimes N' \cong R \otimes N' \cong N'$. For all P in $\operatorname{Pic}(A_1, \ldots, A_n)$, $(P^*)^*$ and P are two inverse modules of P^*, so $(P^*)^* \cong P$, i.e., $([P]^{-1})^{-1} = [P]$. Thus, the condition (i) in Definition 1.1 is satisfied.

For simplicity, we replace ${}_{A_i}P_{A_j}$ by ${}_iP_j$. Next we prove that the multiplication on $G = \operatorname{Pic}(A_1, \ldots, A_n)$ is well defined. In order to do it, we need to prove that (1) ${}_iP_j \otimes {}_jP_m$ is an invertible A_i-A_m-bimodule; (2) Assuming $f : {}_iP_j \cong {}_iP'_j$ and $g : {}_jP_m \cong {}_jP'_m$, we have ${}_iP_j \otimes {}_jP_m \cong {}_iP'_j \otimes {}_jP'_m$ for $i, j, m \in \{1, \ldots, n\}$. In fact, by Lemma 3.7,

$${}_iP_j \otimes {}_jP_m \otimes {}_mP^*_j \otimes {}_jP^*_i \cong {}_iP_j \otimes A_j \otimes {}_jP^*_i \cong {}_iP_j \otimes {}_jP^*_i \cong A_i.$$

Similarly, ${}_mP_j^* \otimes {}_jP_i^* \otimes {}_iP_j \otimes {}_jP_m \cong A_m$. So (1) is proved. We prove (2) now. It is easy to see that $f \otimes g : {}_iP_j \otimes {}_jP_m \to {}_iP'_j \otimes {}_jP'_m$ is a surjective A_i-A_m-bimodule map. By Lemma 3.7(3), we know that ${}_iP_j$ (resp., ${}_jP_m$) is a projective right (resp., left) A_j-module, and thus a right (resp., left) flat module. So $f \otimes 1$, $1 \otimes g$ are injective and $f \otimes g = (f \otimes 1)(1 \otimes g)$ is injective too. As for the conditions (ii)–(iv) in Definition 1.1, we can prove them easily. □

Let G be a groupoid with a finite unit space $\{e_1, \ldots, e_n\}$, and

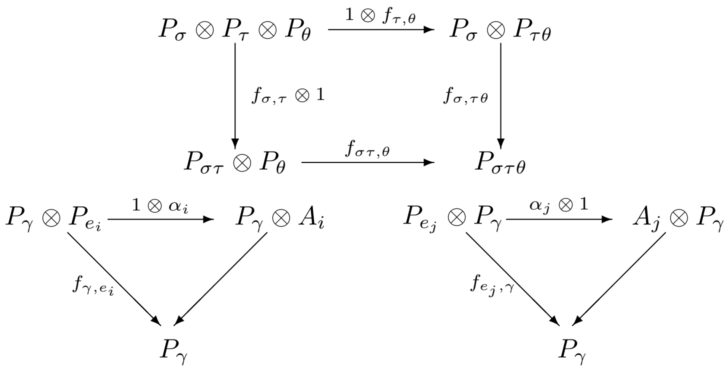
$$\Phi : G \to \mathrm{Pic}(A_1, \ldots, A_n)$$

a groupoid map with $\Phi(e_i) = [A_i]$. Put $\Phi(\sigma) = [P_\sigma]$ for all $\sigma \in G$.

Proposition 3.8. *If $d(\sigma) = e_i$ and $r(\sigma) = e_j$, then P_σ is an invertible A_j-A_i-bimodule.*

Proof. For $(e_j, \sigma) \in G^2$, we have $(\Phi(e_j), \Phi(\sigma)) \in \mathrm{Pic}(A_1, \ldots, A_n)^2$, i.e., $\Phi(e_j) = [A_j]$ and ${}_{A_l}\Phi(\sigma)_{A_m}$ can be composed for $l, m \in \{1, \ldots, n\}$. So $A_l = A_j$. Similarly, $A_m = A_i$. Thus, P_σ is an invertible A_j-A_i-bimodule. □

By a *factor set associated to* Φ, we mean a family $f = \{f_{\sigma,\tau} \mid (\sigma, \tau) \in G^2\}$, where $f_{\sigma,\tau} : P_\sigma \otimes P_\tau \to P_{\sigma\tau}$ is an isomorphism of bimodules for $(\sigma, \tau) \in G^2$ such that the following diagrams



are commutative for every $(\sigma, \tau, \theta) \in G^3$, $\gamma \in G$ with $d(\gamma) = e_i$ and $r(\gamma) = e_j$, where $\alpha_i : P_{e_i} \to A_i$ is a given A_i-bimodule isomorphism for $i = 1, \ldots, n$.

We denote by $F_S(\Phi)$ the set of factor sets associated to Φ. If $f \in F_S(\Phi)$, then we write $(A_1, \ldots, A_n)\langle f, \Phi, G\rangle = \bigoplus_{\sigma \in G} P_\sigma$ with multiplication defined by the rule that for all $\sigma, \tau \in G$, $x \in P_\sigma$ and $y \in P_\tau$,

$$x \cdot y = \begin{cases} f_{\sigma,\tau}(x, y) & \text{if } (\sigma, \tau) \in G^2, \\ 0 & \text{if } (\sigma, \tau) \notin G^2. \end{cases}$$

Proposition 3.9.

(1) $(A_1, \ldots, A_n)\langle f, \Phi, G\rangle = \bigoplus_{\sigma \in G} P_\sigma$ *is a strongly graded ring of type G with identity and it contains a subring isomorphic to A_i $(i = 1, \ldots, n)$. In fact, $P_{e_i} \cong A_i$.*

(2) *Conversely, if* $R = \bigoplus_{\sigma\in G} R_\sigma$ *is a strongly graded ring of type* G*, then there exists a groupoid homomorphism* $\Phi : G \to \mathrm{Pic}(R_{e_1}, \dots, R_{e_n})$ *with* $\Phi(e_i) = [R_{e_i}]$ *and a factor set* $f \in F_S(\Phi)$ *such that* $R \cong (R_{e_1}, \dots, R_{e_n})\langle f, \Phi, G\rangle$.

Proof. (1) Clearly, $(A_1, \dots, A_n)\langle f, \Phi, G\rangle$ is a ring and strongly graded by G. Since $f_{e_i,e_i} : P_{e_i} \otimes_{A_{e_i}} P_{e_i} \to P_{e_i}$ is an A_i-bimodule isomorphism, P_{e_i} is a subring of $(A_1, \dots, A_n)\langle f, \Phi, G\rangle$. By $\Phi(e_i) = [A_i] = [P_{e_i}]$, we have $P_{e_i} \cong A_i$ as A_i-bimodules.

Next we prove that $P_{e_i} \cong A_i$ as rings and $(A_1, \dots, A_n)\langle f, \Phi, G\rangle$ has an identity. Denote $A_i\langle f, \Phi, G\rangle = \bigoplus_{d(\sigma)=r(\sigma)=e_i} P_\sigma$. Obviously, it is a strongly *group* graded ring of type $G_{e_i}^{e_i}$, the isotropy group at e_i. Thus, the conclusions about strongly group graded rings can be applied to $A_i\langle f, \Phi, G\rangle$. In particular, by [5, Proposition I.3.13], $P_{e_i} \cong A_i$ as rings and $A_i\langle f, \Phi, G\rangle$ has an identity. Denote this identity by e_i^A. Clearly, $P_{e_i} = A_i e_i^A = e_i^A A_i$. We claim that $\sum_{i=1}^n e_i^A$ is the identity of $(A_1, \dots, A_n)\langle f, \Phi, G\rangle$. Indeed, for $\tau \in G$ and $x \in P_\tau$, assume $d(\tau) = e_i$ and $r(\tau) = e_j$. By the definition of $f_{e_j,\tau}$, we know that there exist $a_l \in A_j$ and $y_l \in P_\tau$ $(l = 1, \dots, t)$ such that

$$\begin{aligned} x &= f_{e_j,\tau}\big(\textstyle\sum_{l=1}^t a_l \otimes y_l\big) = \textstyle\sum_{l=1}^t f_{e_j,\tau}(a_l \otimes y_l) \\ &= \textstyle\sum_{l=1}^t f_{e_j,\tau}(e_j^A \otimes a_l y_l) = f_{e_j,\tau}(e_j^A \otimes y), \end{aligned}$$

where $y = \sum_{l=1}^t a_l y_l$. Since $\sum_{i=1}^n e_i^A \cdot x = e_j^A \cdot x$, we have

$$\begin{aligned} e_j^A \cdot x &= f_{e_j,\tau}(e_j^A \otimes x) = f_{e_j,\tau}(e_j^A \otimes f_{e_j,\tau}(e_j^A \otimes y)) \\ &= f_{e_j,\tau}(f_{e_j,e_j}(e_j^A \otimes e_j^A) \otimes y) = f_{e_j,\tau}(e_j^A \otimes y) = x. \end{aligned}$$

(2) Let $R = \bigoplus_{\sigma\in G} R_\sigma$ be a strongly graded ring of type G. By Proposition 3.6, R_σ is an invertible $R_{r(\sigma)}$-$R_{d(\sigma)}$-bimodule for $\sigma \in G$. Define $\Phi : G \to \mathrm{Pic}(R_{e_1}, \dots, R_{e_n})$ by $\sigma \mapsto [R_\sigma]$. Clearly, $\Phi(e_i) = [R_{e_i}]$ for $i = 1, \dots, n$ and Φ is a groupoid map. We now define a factor set associated to Φ:

$$f_{\sigma,\tau} : R_\sigma \otimes R_\tau \to R_{\sigma\tau}, \quad x \otimes y \mapsto xy$$

for all $(\sigma, \tau) \in G^2$, $x \in R_\sigma$ and $y \in R_\tau$. By Proposition 3.6, $f_{\sigma,\tau}$ is an $R_{r(\sigma)}$-$R_{d(\tau)}$-bimodule isomorphism. Clearly, the diagrams in the definition of factor set associated to Φ commute by setting $\alpha_i = \mathrm{id}_{R_{e_i}}$, and $R \cong (R_{e_1}, \dots, R_{e_n})\langle f, \Phi, G\rangle$. □

Let Φ, $\Phi' : G \to \mathrm{Pic}(A_1, \dots, A_n)$ be two groupoid homomorphisms with $\Phi(e_i) = \Phi'(e_i) = [A_i]$ for $i = 1, \dots, n$, and $f \in F_S(\Phi)$ and $f' \in F_S(\Phi')$ be factor sets associated to Φ and Φ', respectively. Put $\Phi(\sigma) = [P_\sigma]$ and $\Phi'(\sigma) = [P'_\sigma]$ for $\sigma \in G$.

A *morphism* from f to f' is a family $a = (a_\sigma)_{\sigma\in G}$, where $a_\sigma : P_\sigma \to P'_\sigma$ is a bimodule homomorphism such that $a_{\sigma\tau} \circ f_{\sigma,\tau} = f'_{\sigma,\tau} \circ (a_\sigma \otimes a_\tau)$ for every $(\sigma, \tau) \in G^2$.

Proposition 3.10.

(1) *The map* $a = \bigoplus_{\sigma\in G} a_\sigma : (A_1, \dots, A_n)\langle f, \Phi, G\rangle \to (A_1, \dots, A_n)\langle f', \Phi', G\rangle$ *is a graded ring homomorphism which may not preserve the identity.*

(2) *If every* a_{e_i} $(i = 1, \dots, n)$ *is surjective, then* $a(1) = 1$*. Moreover,* a *is an isomorphism if and only if* a_σ *is an isomorphism for every* $\sigma \in G$.

Proof. (1) For $x \in P_\sigma$ and $y \in P_\tau$, if $(\sigma, \tau) \in G^2$, then $a(xy) = a(f_{\sigma,\tau}(x \otimes y)) = a_{\sigma\tau}(f_{\sigma,\tau}(x \otimes y)) = f'_{\sigma,\tau}(a_\sigma(x) \otimes a_\tau(y)) = a_\sigma(x) \cdot a_\tau(y) = a(x) \cdot a(y)$. On the other hand, if $(\sigma, \tau) \notin G^2$, then $a(xy) = a(0) = 0 = a(x)a(y)$ for $x \in P_\sigma$ and $y \in P_\tau$. Therefore, a is a ring homomorphism.

(2) Just as in the proof of Proposition 3.9, $A_i\langle f, \Phi, G\rangle = \bigoplus_{d(\sigma)=r(\sigma)=e_i} P_\sigma$ is a strongly group graded ring for every $i = 1, \dots, n$. Clearly, if a is surjective, then $a|_{A_i\langle f,\Phi,G\rangle} : A_i\langle f, \Phi, G\rangle \to A_i\langle f', \Phi', G\rangle$ is surjective too. So by [5, Proposition I.3.14], $a(e_i^A) = e_i^{A'}$, where e_i^A and $e_i^{A'}$ are the identities of $A_i\langle f, \Phi, G\rangle$ and $A_i\langle f', \Phi', G\rangle$, respectively. Note that in the proof of Proposition 3.9, we proved $1_{(A_1,\dots,A_n)\langle f,\Phi,G\rangle} = \sum_{i=1}^n e_i^A$ and so $1_{(A_1,\dots,A_n)\langle f',\Phi',G\rangle} = \sum_{i=1}^n e_i^{A'}$. Thus, $a(1) = 1$. The last statement is obvious. □

Remark 3.11. From Proposition 3.10, we know that the ring $(A_1, \dots, A_n)\langle f, \Phi, G\rangle$ is independent of the choice of the family $(P_\sigma)_{\sigma\in G}$ satisfying $\Phi(\sigma) = [P_\sigma]$. Indeed, if $\Phi(\sigma) = [P'_\sigma]$, then there exists a bimodule isomorphism $a_\sigma : P'_\sigma \to P_\sigma$. Denote $f'_{\sigma,\tau} = a_{\sigma\tau}^{-1} \circ f_{\sigma,\tau} \circ (a_\sigma \otimes a_\tau)$ for all $(\sigma, \tau) \in G^2$, where $f = (f_{\sigma,\tau})_{(\sigma,\tau)\in G^2} \in F_S(\Phi)$. Then clearly $f' = (f'_{\sigma,\tau})_{(\sigma,\tau)\in G^2}$ is a factor set. By Proposition 3.10, we have $(A_1, \dots, A_n)\langle f, \Phi, G\rangle \cong (A_1, \dots, A_n)\langle f', \Phi, G\rangle$.

Let $A_1, \dots, A_n$ be rings. Denote by $\mathrm{Iso}(Z(A_i), Z(A_j))$ the set of ring isomorphisms $f : Z(A_i) \to Z(A_j)$. Let

$$\mathrm{Iso}(Z(A_i))_{i=1,\dots,n} = \{f \in \mathrm{Iso}(Z(A_i), Z(A_j)) \mid i, j \in \{1, \dots, n\}\},$$

which has a natural groupoid structure, i.e., $f : Z(A_i) \to Z(A_j)$ and $g : Z(A_l) \to Z(A_m)$ are composable if and only if $j = l$ for $i, j, l, m \in \{1, \dots, n\}$, and in this case, the product is $g \circ f$ and f^{-1} is the inverse of f.

Lemma 3.12. *There exists a groupoid homomorphism*

$$\theta : \mathrm{Pic}(A_1, \dots, A_n) \to \mathrm{Iso}(Z(A_i))_{i=1,\dots,n}.$$

Proof. We replace ${}_{A_i}P_{A_j}$ by ${}_iP_j$. Let $[{}_iP_j] \in \mathrm{Pic}(A_1, \dots, A_n)$ and $c \in Z(A_j)$. We have an A_i-A_j-bimodule homomorphism ${}_iP_j \to {}_iP_j$, $x \mapsto xc$. Since $Z(A_i) \cong \mathrm{End}({}_iP_j)$ by Lemma 3.7(2), there exists a unique element $\alpha_{ij}^P(c) \in Z(A_i)$ such that $\alpha_{ij}^P(c)x = xc$ for all $x \in {}_iP_j$. We claim that $\alpha_{ij}^P : Z(A_j) \to Z(A_i)$ is a ring isomorphism. Clearly, α_{ij}^P is a map. It is injective by the uniqueness of $\alpha_{ij}^P(c)$. We prove that it is also surjective. For all $a_i \in Z(A_i)$, the map ${}_iP_j \to {}_iP_j$ given by $x \mapsto a_ix$ is an A_i-A_j-bimodule homomorphism. By Lemma 3.7(2) again, there exists $c_j \in Z(A_j)$ such that $a_ix = xc_j$, which means that $\alpha_{ij}^P(c_j) = a_i$ and α_{ij}^P is surjective. Clearly, for all $c, d \in Z(A_j)$ and $x \in {}_iP_j$, we have $\alpha_{ij}^P(c + d) = \alpha_{ij}^P(c) + \alpha_{ij}^P(d)$ and

$$\alpha_{ij}^P(cd)x = xcd = (xc)d = (\alpha_{ij}^P(c)x)y = \alpha_{ij}^P(d)\alpha_{ij}^P(c)x = \alpha_{ij}^P(c)\alpha_{ij}^P(d)x,$$

which means $\alpha_{ij}^P(cd) = \alpha_{ij}^P(c)\alpha_{ij}^P(d)$. So α_{ij}^P is a ring isomorphism.

Define $\theta : \mathrm{Pic}(A_1, \dots, A_n) \to \mathrm{Iso}(Z(A_i))_{i=1,\dots,n}$ by $[{}_iP_j] \mapsto \alpha_{ij}^P$. We show that it is well defined. Let $\varphi : {}_iP_j \cong {}_iQ_j$, we need to show that for all $y \in {}_iQ_j$ and

$c \in Z(A_j)$, it holds $\alpha_{ij}^P(c)y = yc$. In fact, there is $x \in {}_iP_j$ such that $\varphi(x) = y$ since φ is bijective. So

$$\alpha_{ij}^P(c)y = \alpha_{ij}^P(c)\varphi(x) = \varphi(\alpha_{ij}^P(c)x) = \varphi(xc) = \varphi(x)c = yc.$$

Next we show that θ is a groupoid homomorphism. Equivalently, it is needed to show that for all $[{}_iP_j]$, $[{}_jQ_l] \in \mathrm{Pic}(A_1, \ldots, A_n)$, we have $\alpha_{il}^{P\otimes Q} = \alpha_{ij}^P \circ \alpha_{jl}^Q$. Indeed, for all $c \in Z(A_l)$, $p \in P$ and $q \in Q$, we have

$$\alpha_{il}^{P\otimes Q}(c)(p \otimes q) = p \otimes qc$$

and

$$\alpha_{ij}^P \circ \alpha_{jl}^Q(c)(p \otimes q) = \alpha_{ij}^P(\alpha_{jl}^Q(c))p \otimes q = p\alpha_{jl}^Q(c) \otimes q = p \otimes \alpha_{jl}^Q(c)q = p \otimes qc.$$

Thus, $\alpha_{il}^{P\otimes Q} = \alpha_{ij}^P \circ \alpha_{jl}^Q$ and θ is a groupoid homomorphism. □

Let G be a groupoid with $G^0 = \{e_1, \ldots, e_n\}$ and $\Phi : G \to \mathrm{Pic}(A_1, \ldots, A_n)$ a groupoid homomorphism with $\Phi(e_i) = [A_i]$. Put $\Phi(\sigma) = [P_\sigma]$. By Lemma 3.12, we have groupoid homomorphisms

$$G \xrightarrow{\Phi} \mathrm{Pic}(A_1, \ldots, A_n) \xrightarrow{\theta} \mathrm{Iso}(Z(A_i))_{i=1,\ldots,n}.$$

Hence, Φ gives an action of G on $\bigcup_{i=1}^n U(Z(A_i))$, where

$$U(Z(A_i)) = \{z \in Z(A_i) \mid z \text{ is a unit}\}.$$

Explicitly, for $\sigma \in G$ with $d(\sigma) = e_i$, $r(\sigma) = e_j$, $x \in P_\sigma$ and $c \in U(Z(A_i))$, we have $\sigma(c)x = xc$. Recall the definition of a G-module bundle in Definition 1.2. Let $A = \bigcup_{i=1}^n U(Z(A_i))$ and

$$L : G \to \mathrm{Iso}(A) = \{\text{isomorphisms } f : U(Z(A_i)) \to U(Z(A_j)) \mid i, j = 1, \ldots n\}$$

be given by $\sigma \mapsto L(\sigma)$, where $L(\sigma)$ is defined by $L(\sigma)(c)x = xc$ for $x \in P_\sigma$, $c \in U(Z(A_i))$ and $\sigma \in G$ with $d(\sigma) = e_i$ and $r(\sigma) = e_j$. We often write $L(\sigma)c$ by $\sigma(c)$. Just as the proof of Lemma 3.12, $L(\sigma)$ is an isomorphism from $U(Z(A_i))$ to $U(Z(A_j))$ as abelian groups. We claim that L is a groupoid homomorphism. In fact, we know that $\theta([{}_iP_j]) = \alpha_{ij}^P$ is a groupoid homomorphism from $\mathrm{Pic}(A_1, \ldots, A_n)$ to $\mathrm{Iso}(Z(A_i))_{i=1,\ldots,n}$ by Lemma 3.12 and its proof. If we define $\theta' : \mathrm{Pic}(A_1, \ldots, A_n) \to \mathrm{Iso}(A)$ by $[{}_iP_j] \mapsto \alpha_{ij}^P|_{U(Z(A_j))}$, then it is easy to see that θ' is a groupoid homomorphism and $L = \theta'\Phi$. Obviously,

$$L^0 : G^0 \to A^0 = \{1_{A_1}, \ldots, 1_{A_n}\}, \quad e_i \mapsto 1_{A_i}$$

is bijective, where 1_{A_i} is the identity of A_i for $i = 1, \ldots, n$. Define $p : A \to A^0$ by $p(U(Z(A_i))) = 1_{A_i}$. So (A, L) is a G-module bundle.

Let us consider the group $Z^2(G, A)$. An element $q \in Z^2(G, A)$ is a function $q : G^2 \to A$, $q_{\sigma,\tau} := q(\sigma, \tau)$ such that for all $(\sigma, \tau) \in G^2$ and $(\sigma, \tau, \theta) \in G^3$,

(∗1) $p \circ q_{\sigma,\tau} = r(\sigma)$;
(∗2) if σ or τ belongs to G^0, then $q_{\sigma,\tau} \in A^0$;
(∗3) $q_{\sigma,\tau\theta}\sigma(q_{\tau,\theta}) = q_{\sigma\tau,\theta}q_{\sigma,\tau}$.

Also, we can consider the group $B^2(G, A)$, i.e., the group of all functions $q : G^2 \to A$, $q_{\sigma,\tau} := q(\sigma, \tau)$, $(\sigma, \tau) \in G^2$ for which there is a function $d : G \to A$, $d_\sigma := d(\sigma)$ such that

(⋆1) $p \circ d_\sigma = r(\sigma)$;
(⋆2) if $\sigma \in G^0$, then $d_\sigma \in A^0$;
(⋆3) $q_{\sigma,\tau} = d_{\sigma\tau}d_\sigma^{-1}\sigma(d_\tau)^{-1}$.

Theorem 3.13. *Let G be a groupoid with $G^0 = \{e_0, \dots, e_n\}$ and $\Phi : G \to \mathrm{Pic}(A_1, \dots, A_n)$ a groupoid homomorphism with $\Phi(e_i) = [A_i]$ for $i = 1, \dots, n$. Put $\Phi(\sigma) = [P_\sigma]$ for $\sigma \in G$.*

(1) *If $f = (f_{\sigma,\tau})_{(\sigma,\tau)\in G^2} \in F_S(\Phi)$ and $q = (q_{\sigma,\tau}) \in Z^2(G, A)$, then $qf = (q_{\sigma,\tau}f_{\sigma,\tau})_{(\sigma,\tau)\in G^2}$ is a factor set associated to Φ.*

(2) *Conversely, if $f, g \in F_S(\Phi)$, then there exists $q \in Z^2(G, A)$ such that $g = qf$.*

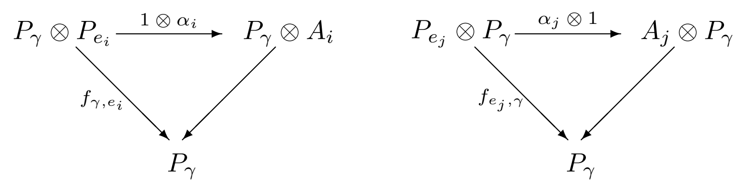
Proof. (1) Let $g_{\sigma,\tau} = q_{\sigma,\tau}f_{\sigma,\tau}$. Note that $\mathrm{Im}(f_{\sigma,\tau}) \in P_{\sigma\tau}$, and if we assume $r(\sigma) = e_j$, then $q_{\sigma,\tau} \in U(Z(A_j))$ by (∗1). Thus, $q_{\sigma,\tau}f_{\sigma,\tau}$ can be defined for all $(\sigma, \tau) \in G^2$. We first show $g_{\sigma\tau,\theta} \circ (g_{\sigma,\tau} \otimes 1) = g_{\sigma,\tau\theta} \circ (1 \otimes g_{\tau,\theta})$ for $(\sigma, \tau, \theta) \in G^3$. Indeed, if $x \in P_\sigma$, $y \in P_\tau$ and $z \in P_\theta$, then we have

$$\big(g_{\sigma\tau,\theta} \circ (g_{\sigma,\tau} \otimes 1)\big)(x \otimes y \otimes z) = (q_{\sigma\tau,\theta}q_{\sigma,\tau})\big(f_{\sigma\tau,\theta} \circ (f_{\sigma,\tau} \otimes 1)\big)(x \otimes y \otimes z)$$

and

$$\begin{aligned}
&\big(g_{\sigma,\tau\theta} \circ (1 \otimes g_{\tau,\theta})\big)(x \otimes y \otimes z)\\
&= g_{\sigma,\tau\theta}\big(x \otimes g_{\tau,\theta}(y \otimes z)\big)\\
&= g_{\sigma,\tau\theta}\big(x \otimes q_{\tau,\theta}f_{\tau,\theta}(y \otimes z)\big)\\
&= g_{\sigma,\tau\theta}\big(xq_{\tau,\theta} \otimes f_{\tau,\theta}(y \otimes z)\big)\\
&= g_{\sigma,\tau\theta}\big(\sigma(q_{\tau,\theta})x \otimes f_{\tau,\theta}(y \otimes z)\big)\\
&= q_{\sigma,\tau\theta}\sigma(q_{\tau,\theta})\big(f_{\sigma,\tau\theta} \circ (1 \otimes f_{\tau,\theta})\big)(x \otimes y \otimes z).
\end{aligned}$$

By (∗3), since $f = (f_{\sigma,\tau})$ is a factor set, we have $g_{\sigma\tau,\theta} \circ (g_{\sigma,\tau} \otimes 1) = g_{\sigma,\tau\theta} \circ (1 \otimes g_{\tau,\theta})$. Therefore, in order to prove that qf is a factor set associated to Φ, we only need to show that the following diagrams are commutative for $\gamma \in G$ with $r(\gamma) = e_j$ and $d(\gamma) = e_i$:



Here $\alpha_i : P_{e_i} \to A_i$ is a given A_i-bimodule isomorphism for $i \in \{1, \dots, n\}$. In fact, note that $e_i, e_j \in G^0$, so by (∗2), we have $q_{\gamma,e_i}, q_{e_j,\gamma} \in A^0$. By (∗1), we know $q_{\gamma,e_i} = q_{e_j,\gamma} = 1_{A_j}$. Thus, $g_{\gamma,e_i} = q_{\gamma,e_i}f_{\gamma,e_i} = 1_{A_j}f_{\gamma,e_i} = f_{\gamma,e_i}$. Similarly,

$g_{e_j,\gamma} = f_{e_j,\gamma}$. Hence, the above diagrams commute since f is a factor set. Therefore, qf is a factor set associated to Φ.

(2) If $f, g \in F_S(\Phi)$, $f = (f_{\sigma,\tau})_{(\sigma,\tau)\in G^2}$ and $g = (g_{\sigma,\tau})_{(\sigma,\tau)\in G^2}$, we put $h_{\sigma,\tau} = g_{\sigma,\tau} \circ f_{\sigma,\tau}^{-1}$. Clearly, $h_{\sigma,\tau}$ is an A_i-A_j-bimodule automorphism of $P_{\sigma\tau}$ if we assume $r(\sigma) = e_i$ and $d(\tau) = e_j$. By Lemma 3.7, $h_{\sigma,\tau}$ is the multiplication by an element $q_{\sigma,\tau} \in U(Z(A_i))$. Hence, $(*1)$ is satisfied and $g_{\sigma,\tau} = q_{\sigma,\tau} f_{\sigma,\tau}$. Since $g_{\sigma\tau,\theta} \circ (g_{\sigma,\tau} \otimes 1) = g_{\sigma,\tau\theta} \circ (1 \otimes g_{\tau,\theta})$ for $(\sigma, \tau, \theta) \in G^3$, $(*3)$ is satisfied. Now $(*2)$ is deduced from the commutativity of the diagrams in the proof of the statement (1). Therefore, $q = (q_{\sigma,\tau}) \in Z^2(G, A)$. □

Definition 3.14. A graded ring isomorphism

$$\psi : (A_1, \dots , A_n)\langle f, \Phi, G\rangle \to (A_1, \dots , A_n)\langle g, \Phi, G\rangle$$

is called a *graded ring* $(A_1, \dots , A_n)$*-isomorphism* if $\psi|_{P_\sigma}$ is a bimodule homomorphism for all $P_\sigma \in (A_1, \dots , A_n)\langle f, \Phi, G\rangle$.

Theorem 3.15. *Let $q \in Z^2(G, A)$. Then $q \in B^2(G, A)$ if and only if there exists a graded ring $(A_1, \dots , A_n)$-isomorphism*

$$(A_1, \dots , A_n)\langle f, \Phi, G\rangle \to (A_1, \dots , A_n)\langle qf, \Phi, G\rangle.$$

Proof. Suppose $q \in B^2(G, A)$. Then there exists a function $d : G \to A$, $d_\sigma := d(\sigma)$ such that $q_{\sigma,\tau} = d_{\sigma\tau} d_\sigma^{-1} \sigma(d_\tau)^{-1}$ for $(\sigma, \tau) \in G^2$. We define

$$\alpha : (A_1, \dots , A_n)\langle f, \Phi, G\rangle \to (A_1, \dots , A_n)\langle qf, \Phi, G\rangle$$

by $\alpha(x) = d_\sigma x$ for $x \in P_\sigma$. Obviously, $\alpha(0) = 0$. Firstly, we prove that α is a graded ring isomorphism. Clearly, α is bijective since $d_\sigma \in \bigcup_{i=1}^n U(Z(A_i))$. For all $x \in P_\sigma$, $y \in P_\tau$ and $(\sigma, \tau) \in G^2$, we have $\alpha(xy) = d_{\sigma\tau} f_{\sigma,\tau}(x\otimes y) = q_{\sigma,\tau} d_\sigma \sigma(d_\tau) f_{\sigma,\tau}(x\otimes y) = q_{\sigma,\tau} f_{\sigma,\tau}(d_\sigma x \otimes d_\tau y) = \alpha(x)\alpha(y)$. If $(\sigma, \tau) \notin G^2$, then $\alpha(xy) = \alpha(0) = 0 = \alpha(x)\alpha(y)$ since $\alpha(x) \in P_\sigma$ and $\alpha(y) \in P_\tau$. So α is a graded ring isomorphism.

Next, for all $\sigma \in G$, $d(\sigma) = e_i$, $r(\sigma) = e_j$, $x \in P_\sigma$, $a \in A_j$ and $b \in A_i$, we have $\alpha(axb) = d_\sigma(axb) = a(d_\sigma x)b = a\alpha(x)b$. So $\alpha|_{P_\sigma}$ is an A_j-A_i-bimodule homomorphism. Thus, α is a graded ring $(A_1, \dots , A_n)$-isomorphism.

Conversely, suppose that there exists a graded ring $(A_1, \dots , A_n)$-isomorphism $\alpha : (A_1, \dots , A_n)\langle f, \Phi, G\rangle \to (A_1, \dots , A_n)\langle qf, \Phi, G\rangle$, where $q \in Z^2(G, A)$. Then we can write $\alpha = \bigoplus_{\sigma\in G} \alpha_\sigma$, where α_σ is a bimodule automorphism of P_σ. Assume $d(\sigma) = e_i$ and $r(\sigma) = e_j$. By Lemma 3.7(2), there exists $d_\sigma \in U(Z(A_j))$ such that $\alpha_\sigma(x) = d_\sigma x$ for $x \in P_\sigma$, which implies $(\star1)$. Because α commutes with multiplication, for $x \in P_\sigma$, $y \in P_\tau$ and $(\sigma, \tau) \in G^2$, we obtain

$$q_{\sigma,\tau} d_\sigma \sigma(d_\tau) f_{\sigma,\tau}(x \otimes y) = d_{\sigma\tau} f_{\sigma,\tau}(x \otimes y).$$

Hence, $q_{\sigma,\tau} d_\sigma \sigma(d_\tau) = d_{\sigma\tau}$, which implies $q_{\sigma,\tau} = d_{\sigma\tau} d_\sigma^{-1} \sigma(d_\tau)^{-1}$. Thus, $(\star3)$ is satisfied. As for $(\star2)$, it is enough to prove $d(e_i) = 1_{A_i}$ for $i = 1, \dots , n$. In fact, let $1_{P_{e_i}}$ be the identity of P_{e_i}. Since $P_{e_i} = A_i \cdot 1_{P_{e_i}} = 1_{P_{e_i}} \cdot A_i$ and α_{e_i} is an

A_i-bimodule homomorphism preserving the identity, α_{e_i} is the identity on P_{e_i}. By the uniqueness of $d(e_i)$, we have $d(e_i) = 1_{A_i}$. So $q \in B^2(G, A)$. □

Corollary 3.16. *Let $f \in F_S(\Phi)$. Then the map $q \mapsto qf$ from $Z^2(G, A)$ to $F_S(\Phi)$ is bijective.*

Let $G^0 = \{e_1, \ldots, e_n\}$. If $\Phi : G \to \mathrm{Pic}(A_1, \ldots, A_n)$ with $\Phi(e_i) = [A_i]$ is a groupoid homomorphism, then we can consider the set of strongly graded rings $\mathcal{S} = \{(A_1, \ldots, A_n)\langle f, \Phi, G\rangle\}_{f \in F_S(\Phi)}$. Denote by $C(A, \Phi)$ the set $\mathcal{S}/\approx$, where for $D, D' \in \mathcal{S}$, $D \approx D'$ if and only if D and D' are graded ring $(A_1, \ldots, A_n)$-isomorphic.

Corollary 3.17. *Let $f \in F_S(\Phi)$. Then the map $\overline{q} \mapsto qf$ from $H^2(G, A)$ to $C(A, \Phi)$ is bijective.*

4 Clifford Theorem

The Clifford Theorem for a strongly group graded ring can be found in [5, Theorem I.3.33]. In this section, we will give the version of Clifford Theorem for a strongly groupoid graded ring.

Lemma 4.1. *Let R be a strongly graded ring of type G with $G^0 = \{e_1, \ldots, e_n\}$. If M is a simple left R_{e_i}-module, then for every $\sigma \in G$ with $d(\sigma) = e_i$ and $r(\sigma) = e_j$, $R_\sigma \otimes_{R_{e_i}} M$ is a simple left R_{e_j}-module.*

Proof. Let $N \subseteq R_\sigma \otimes_{R_{e_i}} M$ be a left R_{e_j}-module. By Lemma 3.7, $R_{\sigma^{-1}}$ is a projective right R_{e_j}-module and thus a flat right R_{e_j}-module. So

$$R_{\sigma^{-1}} \otimes_{R_{e_j}} N \subseteq R_{\sigma^{-1}} \otimes_{R_{e_j}} R_\sigma \otimes_{R_{e_i}} M \cong R_{e_i} \otimes_{R_{e_i}} M \cong M.$$

Thus, $R_{\sigma^{-1}} \otimes_{R_{e_j}} N = M$ or 0. The latter case entails $N = 0$ while the former case yields $N = R_\sigma \otimes_{R_{e_i}} M$. □

Let R be a groupoid graded ring of type G and M a graded left R-module. Let $G^0 = \{e_1, \ldots, e_n\}$, $G_{e_i}^{e_i}$ be the isotropy group at e_i, $R_i^i = \bigoplus_{d(\sigma)=r(\sigma)=e_i} R_\sigma$ and $M_i^i = \bigoplus_{d(\sigma)=r(\sigma)=e_i} M_\sigma$. Clearly, R_i^i is a group graded ring of type $G_{e_i}^{e^i}$ and M_i^i a graded left R_i^i-module.

In Section 2, we have used R-gr to denote the category of graded modules of type G. The forgetful functor $U : R\text{-gr} \to {}_R\mathcal{M}$ associates to $M \in R$-gr the underlying ungraded R-module. We will write $U(M) = \underline{M}$.

Theorem 4.2. (Clifford Theorem) *Let G be a groupoid with $G^0 = \{e_1, \ldots, e_n\}$ and R a strongly graded ring of type G. If $\underline{M_i^i}$ is a simple R_i^i-module for some $i \in \{1, \ldots, n\}$, then*

(1) *$\underline{M_i^i}$ contains a simple R_{e_i}-submodule W such that ${}_{R_{e_i}}M_i^i = \sum_{r(\sigma)=d(\sigma)=e_i} R_\sigma W$. In particular, M_i^i is a semisimple R_{e_i}-module.*

(2) *Put $H = \{\sigma \in G_{e_i}^{e_i} \mid R_\sigma \otimes_{R_{e_i}} W \cong W\} = \{\sigma \in G_{e_i}^{e_i} \mid R_\sigma W = W\}$ and let M_W^i be the sum of all R_{e_i}-submodules X of $\underline{M_i^i}$ such that $X \cong W$. Let*

$R_i^H = \bigoplus_{\sigma\in H} R_\sigma$ (R_i^H is clearly a subring of R_i^i). Then H is a subgroup of $G_{e_i}^{e_i}$ and M_W^i is a simple R_i^H-module such that $M_i^i \cong R_i^i \otimes_{R_i^H} M_W^i$.

Proof. (1) Clearly, $0 \neq M_{e_i}$ is an R_{e_i}-submodule of M_i^i. We claim that M_{e_i} is a simple R_{e_i}-module. Indeed, let $0 \neq N \subseteq M_{e_i}$ be an R_{e_i}-submodule of M_{e_i}. Therefore, $R_i^i N = M_i^i$ since $\underline{M_i^i}$ is a simple R_i^i-module. So $N = (R_i^i N)_{e_i} = (M_i^i)_{e_i} = M_{e_i}$, which means that $\overline{M_{e_i}}$ is a simple R_{e_i}-module. Clearly, $M_i^i = R_i^i M_{e_i}$ entails that $M_i^i = \sum_{r(\sigma)=d(\sigma)=e_i} R_\sigma M_{e_i}$. By Lemma 4.1, $R_\sigma \otimes_{R_{e_i}} M_{e_i}$ is a simple R_{e_i}-module. The canonical multiplication map $R_\sigma \otimes_{R_{e_i}} M_{e_i} \to R_\sigma M_{e_i}$ is an isomorphism of R_{e_i}-modules since R_σ is invertible, and so $R_\sigma M_{e_i} \neq 0$. Thus, $R_\sigma M_{e_i}$ is a simple R_{e_i}-module and ${}_{R_{e_i}}M_i^i$ is a semisimple R_{e_i}-module.

(2) It is easy to see that H is a subgroup of $G_{e_i}^{e_i}$. Next we prove that M_W^i is a simple R_i^H-module. By (1), $M_W^i = \sum_{\sigma\in H} R_\sigma M_{e_i}$ as an R_{e_i}-module. Let $0 \neq N \subseteq M_W^i$ be an R_i^H-submodule of M_W^i. So N is a semisimple R_{e_i}-module by (1) and $N = \sum R_{\sigma'} M_{e_i}$ for some $\sigma' \in H$. Hence, there exists $\tau \in H$ such that $R_\tau M_{e_i} \subseteq N$. Since N is an R_i^H-module, for $\sigma \in H$, $R_\sigma M_{e_i} = R_{\sigma\tau^{-1}} R_\tau M_{e_i} \subseteq R_{\sigma\tau^{-1}} N \subseteq N$. Thus, $M_W^i \subseteq N$ and $N = M_W^i$. Finally, we show $M_i^i \cong R_i^i \otimes_{R_i^H} M_W^i$. In fact,

$$\begin{aligned} R_i^i \otimes_{R_i^H} M_W^i &= R_i^i \otimes_{R_i^H} \left(\textstyle\bigoplus_{\sigma\in H} R_\sigma M_{e_i}\right) \cong R_i^i \otimes_{R_i^H} \textstyle\bigoplus_{\sigma\in H} (R_\sigma \otimes_{R_{e_i}} M_{e_i}) \\ &\cong R_i^i \otimes_{R_i^H} \left(\textstyle\bigoplus_{\sigma\in H} R_\sigma\right) \otimes_{R_{e_i}} M_{e_i} = R_i^i \otimes_{R_i^H} \otimes R_i^H \otimes_{R_{e_i}} M_{e_i} \\ &\cong R_i^i \otimes_{R_{e_i}} M_{e_i} = \textstyle\bigoplus_{\sigma\in G_{e_i}^{e_i}} R_\sigma \otimes_{R_{e_i}} M_{e_i} \\ &\cong \textstyle\bigoplus_{\sigma\in G_{e_i}^{e_i}} R_\sigma M_{e_i} = M_i^i, \end{aligned}$$

which completes the proof. □

Acknowledgement. The authors are grateful to the referee(s) who made valuable comments on previous versions of this paper.

References

[1] S. Caenepeel, E. De Groot, Galois theory for weak Hopf algebras (http://arXiv:math.RA/0406186).

[2] L. Kadison, D. Nikshych, Frobenius extensions and weak Hopf algebra, *J. Algebra* **244** (2001) 312–342.

[3] A.V. Kelarev, On groupoid graded rings, *J. Algebra* **178** (1995) 391–399.

[4] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conf. Series in Math., No. 82, Amer. Math. Soc., Provindence, RI, 1993.

[5] C. Năstăsescu, F. van Oystaeyen, *Graded Ring Theory*, North-Holland Mathematical Library, 28, North-Holland, Amsterdam-New York, 1982.

[6] J. Renault, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Mathematics, 793, Springer-Verlag, New York, 1980.

[7] G. Schiffels, *Graduierte Ringe und Moduln*, Bonn. Math. Schr. No. 11, 1960.

[8] A. Weinstein, A.C. da Silva, *Geometric Models for Noncommutative Algebras*, Amer. Math. Soc., Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, 1999.

[9] J. Westman, *Groupoid Theory in Algebra, Topology and Analysis*, University of California at Irvine, 1971.