Algebra Colloquium **31** : 4 (2024) 675–688
DOI: 10.1142/S1005386724000506

Algebra Colloquium

Higher Frobenius-Schur Indicators for Semisimple Hopf Algebras in Positive Characteristic

Zhihua Wang
Department of Mathematics, Taizhou University, Taizhou, Jiangsu 225300, China
E-mail: mailzhihua@126.com

Gongxiang Liu
Department of Mathematics, Nanjing University, Nanjing 210093, China
E-mail: gxliu@nju.edu.cn

Libin Li
School of Mathematical Science, Yangzhou University, Yangzhou, Jiangsu 225002, China
E-mail: lbli@yzu.edu.cn

Received 8 February 2022
Revised 25 November 2022

Communicated by Peng Shan

Abstract. Let H be a semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. We show that the antipode S of H satisfies the equality $S^2(h) = \mathbf{u}h\mathbf{u}^{-1}$, where $h \in H$, $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$ and Λ is a nonzero integral of H. The formula of S^2 enables us to define higher Frobenius-Schur indicators for the Hopf algebra H. This generalizes the notion of higher Frobenius-Schur indicators from the case of characteristic 0 to the case of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. These indicators defined here share some properties with the ones defined over a field of characteristic 0. In particular, all these indicators are gauge invariants for the tensor category $\mathrm{Rep}(H)$ of finite-dimensional representations of H.

2020 Mathematics Subject Classification: 16T05

Keywords: S^2-formula, Frobenius-Schur indicator, gauge invariant

1 Introduction

Linchenko and Montgomery [9] generalized the classical Frobenius-Schur (FS) indicators from group-theoretic result to the setting of a semisimple involutory Hopf algebra H. They also defined higher FS indicators $\nu_n(V)$ by using an idempotent integral Λ of H, namely,

$$\nu_n(V) = \chi_V(\Lambda_{(1)} \cdots \Lambda_{(n)}) \quad \text{for } n \geq 1, \tag{1.1}$$

where χ_V is the character afforded by finite-dimensional representation V of H. The higher FS indicators were later extensively studied by Kashina et al. for semisimple

Hopf algebras over an algebraically closed field of characteristic zero [6], and by Ng and Schauenburg for semisimple quasi-Hopf algebras over the field of complex numbers [12]. The notion of higher FS indicators has been generalized to objects of a pivotal category [11, 13].

The notion of higher FS indicators for semisimple Hopf algebras over a field of positive characteristic seems not to be considered (except for those semisimple involutory Hopf algebras). For a semisimple Hopf algebra H with antipode S over a field $\Bbbk$ of positive characteristic p, it is known that $S^2(h) = uhu^{-1}$, where $h \in H$ and u is a unit of H (see [15, Theorem 5(a)]). This induces a functorial isomorphism $j_u\colon \mathrm{id} \to (-)^{**}$ of the representation category $\mathrm{Rep}(H)$. If we define the n-th FS indicator of V along the lines of [11] to be the trace of a certain $\Bbbk$-linear operator associated to the functorial isomorphism j_u, then the FS indicator defined here depends on the choice of u. Hence, it is not an invariant of the tensor category $\mathrm{Rep}(H)$. Even if, a priori, the FS indicator depends on u, with a good choice of u it may be an invariant of the tensor category $\mathrm{Rep}(H)$.

In the present paper, we consider the notion of higher FS indicators for a finite-dimensional semisimple Hopf algebra H over an algebraically closed field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. We need to point out that the Hopf algebra H here is not known to be involutory unless the characteristic p is larger than a certain number (see [3, 16]). For the antipode S of H, we first obtain a formula for S^2:

$$S^2(h) = \mathbf{u}h\mathbf{u}^{-1}, \quad \text{where } h \in H,\ \mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)},$$

which is constructed entirely in terms of a nonzero integral Λ of H. According to the formula of S^2, we have an isomorphism of H-modules

$$j_{\mathbf{u},V}:\ V \longrightarrow V^{**}, \quad j_{\mathbf{u},V}(v)(\vartheta) = \vartheta(\mathbf{u}\cdot v) \quad \text{for } v \in V,\ \vartheta \in V^*,$$

which is functorial in V. As the element $\mathbf{u}$ is not necessarily a group-like element, the functorial isomorphism $j_{\mathbf{u}}\colon \mathrm{id} \to (-)^{**}$ is not necessarily a tensor isomorphism. In other words, the category $\mathrm{Rep}(H)$ of finite-dimensional representations of H is not necessarily pivotal with respect to the structure $j_{\mathbf{u}}$. Even so, using the functorial isomorphism $j_{\mathbf{u}}$, we may still define the n-th FS indicator $\nu_n(V)$ of V to be the trace of a certain $\Bbbk$-linear operator as Ng and Schauenburg did in [11].

By comparison with the case of characteristic 0, we will see that this definition of a FS indicator in positive characteristic may be the best choice and many properties of a FS indicator are preserved in positive characteristic. Similarly to the case of characteristic 0, the n-th FS indicator $\nu_n(V)$ defined here can be entirely described in terms of the integral Λ of H and the character χ_V of V:

$$\nu_n(V) = \chi_V(\mathbf{u}^{-1}\Lambda_{(1)}\cdots\Lambda_{(n)}) \quad \text{for } n \geq 1. \tag{1.2}$$

Moreover, the formula (1.2) does not depend on the choice of the nonzero integral Λ and it recovers the original formula (1.1) when the characteristic of $\Bbbk$ is zero and Λ is idempotent.

Note that the formula (1.2) can be written as $\nu_n(V) = \chi_V(\mathbf{u}^{-1}P_n(\Lambda))$ for $n \geq 1$, where P_n is the n-th Sweedler power map of H. Clearly, the n-th Sweedler power map P_n is valid for all $n \in \mathbb{Z}$, and this motivates us to extend the n-th FS indicator from $n \geq 1$ to $n \in \mathbb{Z}$. That is, by definition, $\nu_n(V) = \chi_V(\mathbf{u}^{-1}P_n(\Lambda))$ for all $n \in \mathbb{Z}$. Similarly to the case of characteristic 0 (see [5, 6]), by replacing V with the regular representation H, we reconstruct the n-th indicator of H, a notion defined by the trace of the map $S \circ P_{n-1}$. Similarly to the case of characteristic 0 again, V and its dual V^* have the same higher FS indicators. In particular, similarly to the case of characteristic 0, the n-th FS indicator $\nu_n(V)$ defined here is a gauge invariant of the tensor category $\mathrm{Rep}(H)$ for any $n \in \mathbb{Z}$ and any finite-dimensional representation V of H. Here, a quantity $f(H)$ defined for each Hopf algebra H is called a gauge invariant if $f(H) = f(H')$ for any Hopf algebra H' with $\mathrm{Rep}(H')$ being tensor equivalent to $\mathrm{Rep}(H)$.

The paper is organized as follows: In Section 2, we give some basic results on semisimple Hopf algebras. In Section 3, we deduce a formula of S^2 by comparing two different forms of the character χ_H of the regular representation H. We investigate some properties of the element $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$ and show that the integral Λ of H is cocommutative if and only if $S^2 = \mathrm{id}$. In Section 4, we generalize the notion of higher FS indicators from characteristic 0 case to characteristic $p > \dim_{\Bbbk}(H)^{1/2}$ case, and we find that the indicators defined here share some common properties with the ones defined over a field of characteristic 0. In Section 5, we show that the n-th FS indicators are gauge invariants of the tensor category $\mathrm{Rep}(H)$.

2 Preliminaries

Throughout this paper, H is a finite-dimensional semisimple Hopf algebra over an algebraically closed field $\Bbbk$ of characteristic p. We need to add a special condition $p > \dim_{\Bbbk}(H)^{1/2}$ so as to make sure that $\dim_{\Bbbk} V \neq 0$ for any simple H-module V. We stress that all results presented here are also valid for the case of characteristic 0, although we only deal with the case of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$.

As a Hopf algebra, H has a counit ε, antipode S, multiplication m and comultiplication Δ. The comultiplication $\Delta(a)$ will be written as $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for $a \in H$, where we omit the summation sign. We denote by Λ and λ the left and right integrals of H and H^*, respectively, so that $\lambda(\Lambda) = 1$. Since the semisimple Hopf algebra H is unimodular, the left and right integrals of H are the same. We refer to [10] for basic theory of Hopf algebras.

If V is a finite-dimensional H-module, then V is also called a representation of H via the algebra homomorphism $\rho_V : H \to \mathrm{End}_{\Bbbk}(V)$ given by $\rho_V(h)(v) = h \cdot v$ for $h \in H$ and $v \in V$. We will make no distinction between the two notions. The character of V is the map $\chi_V : H \to \Bbbk$ given by $\chi_V(h) = \mathrm{tr}(\rho_V(h))$ for $h \in H$. The $\Bbbk$-linear dual space V^* is also an H-module via $(h \cdot \vartheta)(v) := \vartheta(S(h) \cdot v)$ for $h \in H$, $\vartheta \in V^*$ and $v \in V$. In particular, the dual module V^* has the character $\chi_{V^*} = \chi_V \circ S$. The category $\mathrm{Rep}(H)$ of finite-dimensional representations of H is a semisimple tensor category, where the monoidal structure stems from the comultiplication Δ.

Recall that the dual Hopf algebra H^* has an H-bimodule structure given by

$$(a \rightharpoonup f)(b) = f(ba), \quad (f \leftharpoonup a)(b) = f(ab) \quad \text{for } a, b \in H,\ f \in H^*.$$

In addition, $(H^*, \leftharpoonup)$ and $(\rightharpoonup, H^*)$ are free H-modules generated by λ, that is to say, $H^* = \lambda \leftharpoonup H$ and $H^* = H \rightharpoonup \lambda$ (see [15, Corollary 2(b)]). This provides an associative and non-degenerate bilinear form $H \times H \to \Bbbk$ given by $a \times b \mapsto \lambda(ab)$ for $a, b \in H$. Moreover, the pair (H, λ) is a Frobenius algebra with the Frobenius homomorphism λ satisfying the equality (see [15, Eq. (1)])

$$a = \lambda(a\Lambda_{(1)})S(\Lambda_{(2)}) = \lambda(S(\Lambda_{(2)})a)\Lambda_{(1)} \quad \text{for } a \in H. \tag{2.1}$$

The pair $\Lambda_{(1)} \otimes S(\Lambda_{(2)})$ satisfying (2.1) is called the dual basis of H with respect to the Frobenius homomorphism λ.

The semisimplicity of H shows that there exists a unit u in the Hopf algebra H such that $S^2(a) = uau^{-1}$ for $a \in H$ (see [15, Theorem 5(a)]). Since the right integral λ of H^* satisfies $\lambda(ab) = \lambda(S^2(b)a)$ for all $a, b \in H$ (see [15, Theorem 3(a)]), the Hopf algebra H is a symmetric algebra with a symmetric bilinear form $H \times H \to \Bbbk$ given by $a \times b \mapsto \lambda(uab) = (\lambda \leftharpoonup u)(ab) = (u \rightharpoonup \lambda)(ab)$, where $\lambda \leftharpoonup u = u \rightharpoonup \lambda$ holds because $\lambda(au) = \lambda(S^2(u)a) = \lambda(ua)$ for all $a \in H$. Using (2.1) we may see that the pair $\Lambda_{(1)} \otimes u^{-1}S(\Lambda_{(2)})$ is a dual basis of H with respect to $\lambda \leftharpoonup u$ $(= u \rightharpoonup \lambda)$ (see also [2, Lemma 1.4(2)]). The symmetry of the Frobenius homomorphism $\lambda \leftharpoonup u$ $(= u \rightharpoonup \lambda)$ means

$$\Lambda_{(1)} \otimes u^{-1}S(\Lambda_{(2)}) = u^{-1}S(\Lambda_{(2)}) \otimes \Lambda_{(1)}. \tag{2.2}$$

By Wedderburn's theorem, the semisimple Hopf algebra H is isomorphic to a direct sum of full matrix algebras over $\Bbbk$, namely, $H \cong \bigoplus_{i\in I} M_{d_i}(\Bbbk)$. Let e_i be the idempotent of H satisfying $He_i \cong M_{d_i}(\Bbbk)$. Then $\{e_i\}_{i\in I}$ forms a complete set of central primitive idempotents of H. Let V_i be a simple left module (unique up to isomorphism) over the matrix algebra $M_{d_i}(\Bbbk)$. Then $\dim_{\Bbbk}(V_i) = d_i$ and $\{V_i\}_{i\in I}$ forms a complete set of simple left H-modules up to isomorphism. The left regular representation H has the decomposition $H \cong \bigoplus_{i\in I} V_i^{\oplus d_i}$ as H-modules, so the character χ_H of the left regular representation H is equal to $\sum_{i\in I} d_i\chi_i$, where each χ_i is the character of V_i.

For any simple H-module V_i and any $\varphi \in \mathrm{End}_{\Bbbk}(V_i)$, we use the dual basis $\Lambda_{(1)} \otimes u^{-1}S(\Lambda_{(2)})$ with respect to the Frobenius homomorphism $\lambda \leftharpoonup u$ to define the map $\mathcal{I}(\varphi) \in \mathrm{End}_{\Bbbk}(V_i)$ by $\mathcal{I}(\varphi)(v) = \Lambda_{(1)}\varphi(u^{-1}S(\Lambda_{(2)})v)$ for $v \in V_i$. Note that $\mathcal{I}(\varphi)$ lies in $\mathrm{End}_H(V_i) \cong \Bbbk$. There exists a unique element $c_i \in \Bbbk$ such that

$$\mathcal{I}(\varphi) = c_i \operatorname{tr}(\varphi) \operatorname{id}_{V_i} \quad \text{for all } \varphi \in \mathrm{End}_k(V_i).$$

Such an element c_i, depending only on the isomorphism class of V_i, is called the Schur element associated to V_i (see [4, Theorem 7.2.1]). Since H is semisimple, it follows from [4, Theorem 7.2.6] that the Schur element $c_i \neq 0$ in $\Bbbk$ and the Frobenius homomorphism $\lambda \leftharpoonup u$ can be written explicitly as follows:

$$\lambda \leftharpoonup u = u \rightharpoonup \lambda = \sum_{i\in I} \frac{1}{c_i}\chi_i. \tag{2.3}$$

3 A Formula for the Square of Antipodes

In this section, we will provide a formula for S^2 by virtue of a nonzero integral Λ of H. Then we study some properties of the element $\mathbf{u} := S(\Lambda_{(2)})\Lambda_{(1)}$. In particular, we will give a sufficient and necessary condition for $S^2 = \mathrm{id}$ via the integral Λ.

Let u be a unit of H satisfying $S^2(a) = uau^{-1}$ for all $a \in H$. We fix a left integral Λ of H and a right integral λ of H^* such that $\lambda(\Lambda) = 1$. Denote by $\{V_i\}_{i\in I}$ the set of all simple left H-modules up to isomorphism. For each V_i we write c_i for the Schur element of V_i associated to the dual basis $\Lambda_{(1)} \otimes u^{-1}S(\Lambda_{(2)})$ of H with respect to the Frobenius homomorphism $\lambda \leftharpoonup u$. We denote by $\{e_i\}_{i\in I}$ the set of all central primitive idempotents of H. First we establish a relationship between the elements u and $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$.

Proposition 3.1. *With the notions above, we have* $\mathbf{u} = u\sum_{i\in I} \dim_{\Bbbk}(V_i)c_ie_i$, *which is a unit of* H.

Proof. Note that each central primitive idempotent e_i acts as the identity on V_i and annihilates V_j for $j \neq i$. It follows that $\chi_j(e_i) = \dim_{\Bbbk}(V_i)$ if $i = j$ and 0 otherwise. By (2.3), $\chi_i(a) = \chi_i(ae_i) = \sum_{j\in I} \frac{1}{c_j}\chi_j(c_iae_i) = (u \rightharpoonup \lambda)(c_iae_i) = (uc_ie_i \rightharpoonup \lambda)(a)$. Thus $\chi_i = uc_ie_i \rightharpoonup \lambda$, and hence

$$\chi_H = \sum_{i\in I} \dim_{\Bbbk}(V_i)\chi_i = u\sum_{i\in I} \dim_{\Bbbk}(V_i)c_ie_i \rightharpoonup \lambda. \tag{3.1}$$

For any map $\varphi \in \mathrm{End}_{\Bbbk}(H)$, the trace of φ is $\mathrm{tr}(\varphi) = \lambda(\varphi(S(\Lambda_{(2)}))\Lambda_{(1)})$ (see [15, Theorem 2]). Taking into account that $\varphi = L_a$, where L_a is the left multiplication operator of H by a, we get $\chi_H(a) = \mathrm{tr}(L_a) = \lambda(aS(\Lambda_{(2)})\Lambda_{(1)}) = (S(\Lambda_{(2)})\Lambda_{(1)} \rightharpoonup \lambda)(a)$. This implies $\chi_H = S(\Lambda_{(2)})\Lambda_{(1)} \rightharpoonup \lambda$. Comparing it with (3.1) and using the non-degeneracy of the Frobenius homomorphism λ, we have

$$S(\Lambda_{(2)})\Lambda_{(1)} = u\sum_{i\in I} \dim_{\Bbbk}(V_i)c_ie_i.$$

Since $p > \dim_{\Bbbk}(H)^{1/2}$, it follows that $p^2 > \dim_{\Bbbk}(H) = \sum_{i\in I} \dim_{\Bbbk}(V_i)^2 \geq \dim_{\Bbbk}(V_i)^2$. Hence, $p > \dim_{\Bbbk}(V_i)$ and $\dim_{\Bbbk}(V_i) \neq 0$ in $\Bbbk$ for any $i \in I$. Thus, the element u is the same as $S(\Lambda_{(2)})\Lambda_{(1)}$ up to a central unit $\sum_{i\in I} \dim_{\Bbbk}(V_i)c_ie_i$. □

Remark 3.2. Proposition 3.1 also holds if the field $\Bbbk$ has characteristic 0. In this case, $S^2 = \mathrm{id}$ (see [7] or [8]), implying that $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)} = S(\Lambda_{(2)})S^2(\Lambda_{(1)}) = S(S(\Lambda_{(1)})\Lambda_{(2)}) = \varepsilon(\Lambda)$.

Proposition 3.1 gives a formula for S^2, namely,

$$S^2(a) = \mathbf{u}a\mathbf{u}^{-1} \quad \text{for } a \in H,$$

where $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$. In what follows, we will replace u with $\mathbf{u}$. In this case, the equality (2.2) turns out to be

$$\Lambda_{(1)} \otimes \mathbf{u}^{-1}S(\Lambda_{(2)}) = \mathbf{u}^{-1}S(\Lambda_{(2)}) \otimes \Lambda_{(1)}, \tag{3.2}$$

which is the dual basis of H with respect to the Frobenius homomorphism $\lambda \leftharpoonup \mathbf{u}$. The Schur element associated to the simple H-module V_i under the new dual basis $\Lambda_{(1)} \otimes \mathbf{u}^{-1}S(\Lambda_{(2)})$ with respect to the Frobenius homomorphism $\lambda \leftharpoonup \mathbf{u}$ is $\frac{1}{\dim_{\Bbbk}(V_i)}$. Therefore, the equality (2.3) turns out to be

$$\lambda \leftharpoonup \mathbf{u} = \mathbf{u} \rightharpoonup \lambda = \sum_{i\in I} \dim_{\Bbbk}(V_i)\chi_i = \chi_H. \tag{3.3}$$

By applying [2, Theorem 1.5] and (3.2), we obtain the expression of each central primitive idempotent e_i of H as follows:

$$e_i = \dim_{\Bbbk}(V_i)\chi_i(\Lambda_{(1)})\mathbf{u}^{-1}S(\Lambda_{(2)}) = \dim_{\Bbbk}(V_i)\chi_i(\mathbf{u}^{-1}S(\Lambda_{(2)}))\Lambda_{(1)}. \tag{3.4}$$

Let $g \in G(H)$ and $\alpha \in \mathrm{Alg}(H, k)$ be the modular elements of H and H^*, respectively. Recall that the Radford formula of S^4 has the form (see [14, Proposition 6]) $S^4(a) = \alpha^{-1} \rightharpoonup (gag^{-1}) \leftharpoonup \alpha$. Since H is unimodular, i.e., $\alpha = \varepsilon$, the Radford formula of S^4 now becomes $S^4(a) = gag^{-1}$. The distinguished group-like element g and integral Λ of H satisfy the following useful equality (see [15, Theorem 3(d)]):

$$\Lambda_{(2)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes S^2(\Lambda_{(2)})g. \tag{3.5}$$

After these preparations, we give some properties of the element $\mathbf{u}$.

Proposition 3.3. *The element $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$ satisfies the following properties:*

(1) $\mathbf{u} = \chi_H(\Lambda_{(1)})S(\Lambda_{(2)})$.
(2) $\Lambda_{(1)}\mathbf{u}^{-1}S(\Lambda_{(2)}) = 1$.
(3) $\lambda(e_i) = \dim_{\Bbbk}(V_i)\chi_i(\mathbf{u}^{-1})$.
(4) $\mathbf{u}S(\mathbf{u}) = S(\mathbf{u})\mathbf{u} = \varepsilon(\Lambda)\sum_{i\in I}\frac{\dim_{\Bbbk}(V_i)^2}{\lambda(e_i)}e_i$.
(5) $S(\mathbf{u}^{-1})\mathbf{u} = \mathbf{u}S(\mathbf{u}^{-1})$, *which is the distinguished group-like element g of H.*

Proof. (1) It follows from (3.4) that $e_i\mathbf{u} = \dim_{\Bbbk}(V_i)\chi_i(\Lambda_{(1)})S(\Lambda_{(2)})$. Thus, we see that $\mathbf{u} = \sum_{i\in I} e_i\mathbf{u} = \sum_{i\in I}\dim_{\Bbbk}(V_i)\chi_i(\Lambda_{(1)})S(\Lambda_{(2)}) = \chi_H(\Lambda_{(1)})S(\Lambda_{(2)})$.

(2) Since $\Lambda_{(1)} \otimes \mathbf{u}^{-1}S(\Lambda_{(2)}) = \mathbf{u}^{-1}S(\Lambda_{(2)}) \otimes \Lambda_{(1)}$ by (3.2), we obtain the desired result by multiplying the tensor factors together.

(3) Since $e_i = \dim_{\Bbbk}(V_i)\chi_i(\Lambda_{(1)})\mathbf{u}^{-1}S(\Lambda_{(2)})$, it follows that

$$e_i = \mathbf{u}e_i\mathbf{u}^{-1} = \dim_{\Bbbk}(V_i)\chi_i(\Lambda_{(1)})S(\Lambda_{(2)})\mathbf{u}^{-1}.$$

Hence, $\lambda(e_i) = \dim_{\Bbbk}(V_i)\chi_i(\Lambda_{(1)})\lambda(S(\Lambda_{(2)})\mathbf{u}^{-1}) = \dim_{\Bbbk}(V_i)\chi_i(\mathbf{u}^{-1})$, where the last equality follows from (2.1).

(4) For any $a \in H$, $S^3(a) = S(S^2(a)) = S(\mathbf{u}a\mathbf{u}^{-1}) = S(\mathbf{u}^{-1})S(a)S(\mathbf{u})$, and $S^3(a) = S^2(S(a)) = \mathbf{u}S(a)\mathbf{u}^{-1}$. It follows that $S(\mathbf{u})\mathbf{u}$ is a central unit of H. The equality $\mathbf{u}S(\mathbf{u}) = S(\mathbf{u})\mathbf{u}$ holds because $S(\mathbf{u}) = S(S^2(\mathbf{u})) = S^2(S(\mathbf{u})) = \mathbf{u}S(\mathbf{u})\mathbf{u}^{-1}$. For the central unit $\mathbf{u}S(\mathbf{u})$, we suppose that $\mathbf{u}S(\mathbf{u}) = \sum_{i\in I} k_ie_i$, where each scalar $k_i \neq 0$ in $\Bbbk$. Then $e_i\mathbf{u}^{-1} = \frac{1}{k_i}e_iS(\mathbf{u})$. We have

$$\begin{aligned}
\lambda(e_i) &= (\mathbf{u}^{-1} \rightharpoonup \chi_H)(e_i) = \chi_H(e_i\mathbf{u}^{-1}) = \frac{1}{k_i}\chi_H(e_iS(\mathbf{u}))\\
&= \frac{\dim_{\Bbbk}(V_i)}{k_i}\chi_i(e_iS(\mathbf{u})) = \frac{\dim_{\Bbbk}(V_i)}{k_i}\chi_i(S(\mathbf{u}))\\
&= \frac{\dim_{\Bbbk}(V_i)}{k_i}(\chi_i \circ S)(\mathbf{u}) = \frac{\dim_{\Bbbk}(V_i)}{k_i}(\chi_i \circ S)(S(\Lambda_{(2)})\Lambda_{(1)})\\
&= \frac{\dim_{\Bbbk}(V_i)}{k_i}(\chi_i \circ S)(\Lambda_{(1)}S(\Lambda_{(2)})) = \frac{\dim_{\Bbbk}(V_i)^2\varepsilon(\Lambda)}{k_i} \neq 0.
\end{aligned}$$

It follows that $k_i = \frac{\dim_{\Bbbk}(V_i)^2\varepsilon(\Lambda)}{\lambda(e_i)}$ and $\mathbf{u}S(\mathbf{u}) = \sum_{i\in I} k_ie_i = \varepsilon(\Lambda)\sum_{i\in I}\frac{\dim_{\Bbbk}(V_i)^2}{\lambda(e_i)}e_i$.

(5) Note that $\Lambda_{(2)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes S^2(\Lambda_{(2)})g$ by (3.5). Applying $S \otimes \mathrm{id}$ to both sides of this equality and multiplying the tensor factors together, we have $\mathbf{u} = S(\mathbf{u})g$ or $g = S(\mathbf{u}^{-1})\mathbf{u}$. □

Corollary 3.4. *For any central primitive idempotent e_i of H, $\lambda(e_i) = \lambda(S(e_i))$.*

Proof. We define $S(e_i) = e_{i^*}$ for some $i^* \in I$. Then $V_i^* \cong V_{i^*}$, or equivalently, $\chi_i \circ S = \chi_{i^*}$ (see [2, Lemma 1.8]). By Proposition 3.3(3),

$$\lambda(S(e_i)) = \lambda(e_{i^*}) = \dim_{\Bbbk}(V_{i^*})\chi_{i^*}(\mathbf{u}^{-1}) = \dim_{\Bbbk}(V_i)\chi_i(S(\mathbf{u}^{-1})).$$

Since $\mathbf{u}S(\mathbf{u}) = \varepsilon(\Lambda)\sum_{i\in I}\frac{\dim_{\Bbbk}(V_i)^2}{\lambda(e_i)}e_i$, we get $S(\mathbf{u}^{-1}) = \mathbf{u}\frac{1}{\varepsilon(\Lambda)}\sum_{i\in I}\frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2}e_i$. Thus,

$$\begin{aligned}\lambda(S(e_i)) &= \dim_{\Bbbk}(V_i)\chi_i(S(\mathbf{u}^{-1}))\\ &= \frac{\lambda(e_i)}{\varepsilon(\Lambda)\dim_{\Bbbk}(V_i)}\chi_i(\mathbf{u}) = \frac{\lambda(e_i)}{\varepsilon(\Lambda)\dim_{\Bbbk}(V_i)}\chi_i(\Lambda_{(1)}S(\Lambda_{(2)})) = \lambda(e_i).\end{aligned}$$

This completes the proof. □

If the field $\Bbbk$ has characteristic 0, then the antipode S of H satisfies $S^2 = \mathrm{id}$ (see [7] or [8]). This further implies that the integral Λ of H is cocommutative (see [7, Proposition 2(b)]). The following result shows that Λ being cocommutative is equivalent to $S^2 = \mathrm{id}$ when the characteristic of $\Bbbk$ is larger than $\dim_{\Bbbk}(H)^{1/2}$.

Proposition 3.5. *Let H be a finite-dimensional semisimple Hopf algebra over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. The following statements are equivalent:*

(1) *The nonzero integral Λ of H is cocommutative.*
(2) *The nonzero integral λ of H^* is cocommutative.*
(3) $S^2 = \mathrm{id}$.

Proof. It can be seen from [15, Corollary 5] that (2) and (3) are equivalent. We next show that (1) and (3) are equivalent. If Λ is cocommutative, then $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)} = S(\Lambda_{(1)})\Lambda_{(2)} = \varepsilon(\Lambda)$. It follows from $S^2(a) = \mathbf{u}a\mathbf{u}^{-1}$ that $S^2 = \mathrm{id}$. Conversely, if $S^2 = \mathrm{id}$, then $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)} = S(\Lambda_{(2)})S^2(\Lambda_{(1)}) = S(S(\Lambda_{(1)})\Lambda_{(2)}) = \varepsilon(\Lambda)$. By Proposition 3.3, we have $g = S(\mathbf{u}^{-1})\mathbf{u} = 1$. Since $\Lambda_{(2)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes S^2(\Lambda_{(2)})g$ by (3.5), it follows that $\Lambda_{(2)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes \Lambda_{(2)}$. This completes the proof. □

4 Higher FS Indicators

For the field $\Bbbk$ of characteristic 0, the n-th FS indicators of finite-dimensional representations of semisimple Hopf algebras have been studied in [6]. In this section, we will generalize these indicators from the case of characteristic 0 to the case of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$ and describe them via a nonzero integral Λ of H. Let us begin with the following preparations.

Let H be a finite-dimensional semisimple Hopf algebra over the field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$ with a nonzero integral Λ and $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$. Applying $\Delta_{n-1} \otimes \mathrm{id}$ to both sides of the equality $\Lambda_{(2)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes S^2(\Lambda_{(2)})g$ (see (3.5)), we have $\Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)} \otimes \Lambda_{(1)} = \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(n-1)} \otimes S^2(\Lambda_{(n)})g$. Since $g = \mathbf{u}S(\mathbf{u}^{-1})$ and $S^2(\Lambda_{(n)}) = \mathbf{u}\Lambda_{(n)}\mathbf{u}^{-1}$, the above equality induces the following equality:

$$\Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)} \otimes \mathbf{u}^{-1}\Lambda_{(1)} = \Lambda_{(1)} \otimes \cdots \otimes \Lambda_{(n-1)} \otimes \Lambda_{(n)}S(\mathbf{u}^{-1}). \tag{4.1}$$

Note that the category $\mathrm{Rep}(H)$ of finite-dimensional representations of H is a semisimple tensor category. Let $j_{\mathbf{u}}\colon \mathrm{id} \to (-)^{**}$ be a natural isomorphism between the identity functor and the functor of taking the second dual. It is completely determined by a collection of H-module isomorphisms

$$j_{\mathbf{u},V}:\ V \longrightarrow V^{**}, \quad j_{\mathbf{u},V}(v)(\vartheta) = \vartheta(\mathbf{u}v) \quad \text{for } v \in V,\ \vartheta \in V^*.$$

The inverse of $j_{\mathbf{u},V}$ is $j_{\mathbf{u},V}^{-1}\colon V^{**} \to V$ given by $\alpha \mapsto j_{\mathbf{u},V}^{-1}(\alpha)$, where $j_{\mathbf{u},V}^{-1}(\alpha) \in V$ satisfies the equality $\vartheta(j_{\mathbf{u},V}^{-1}(\alpha)) = \alpha(S^{-1}(\mathbf{u}^{-1})\vartheta)$ for $\vartheta \in V^*$. Since $S^2(h) = \mathbf{u}h\mathbf{u}^{-1}$ and $\mathbf{u}$ is not known to be a group-like element, the natural isomorphism $j_{\mathbf{u}}$ is not necessarily a tensor isomorphism. Although the representation category $\mathrm{Rep}(H)$ with respect to the structure $j_{\mathbf{u}}$ is not necessarily pivotal, we may still define higher FS indicators for any finite-dimensional representation of H using the structure $j_{\mathbf{u}}$ of $\mathrm{Rep}(H)$.

We denote by $V^{\otimes n}$ the n-th tensor power of V, where $V^{\otimes 0}$ is the trivial H-module $\Bbbk$. For any natural number $n \geq 1$, we define the following $\Bbbk$-linear map:

$$E_V^n:\ \mathrm{Hom}_H(\Bbbk, V^{\otimes n}) \longrightarrow \mathrm{Hom}_H(\Bbbk, V^{\otimes n}), \quad f \longmapsto E_V^n(f),$$

where $E_V^n(f)$ is an H-module morphism from $\Bbbk$ to $V^{\otimes n}$ given by

$$\begin{aligned} E_V^n(f):\ \Bbbk &\xrightarrow{\mathrm{coev}_{V^*}} V^* \otimes V^{**} = V^* \otimes \Bbbk \otimes V^{**} \xrightarrow{\mathrm{id}\otimes f \otimes \mathrm{id}} V^* \otimes V^{\otimes n} \otimes V^{**} \\ &\xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}} V^{\otimes(n-1)} \otimes V^{**} \xrightarrow{\mathrm{id}\otimes j_{\mathbf{u},V}^{-1}} V^{\otimes n}. \end{aligned}$$

Here the maps coev_{V^*} and ev_V are the usual coevaluation morphism of V^* and evaluation morphism of V, respectively. If we set $f(1) = \sum v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$, the above definition of $E_V^n(f)$ shows that

$$E_V^n(f)(1) = \sum v_2 \otimes \cdots \otimes v_n \otimes \mathbf{u}^{-1}v_1. \tag{4.2}$$

Similarly to [11], we define the n-th FS indicator of V to be the trace of the linear operator E_V^n as follows:

Definition 4.1. Let H be a finite-dimensional semisimple Hopf algebra over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. For any finite-dimensional representation V of H, the *n-th FS indicator* of V is defined by $\nu_n(V) = \mathrm{tr}(E_V^n)$ for $n \geq 1$.

Similarly to the characteristic 0 case, the n-th FS indicator of V defined above can also be described by a nonzero integral Λ of H:

Theorem 4.2. *Let Λ be a nonzero integral of H and $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$. Suppose that χ_V is the character of a finite-dimensional representation V of H. Then we have $\nu_n(V) = \chi_V(\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)})$ for $n \geq 1$.*

Proof. We first show that the equality $\nu_n(V) = \chi_V(\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)})$ holds for an idempotent integral Λ. Suppose that α is the following $\Bbbk$-linear map:

$$\alpha:\ V^{\otimes n} \longrightarrow V^{\otimes n}, \quad v_1 \otimes v_2 \otimes \cdots \otimes v_n \longmapsto v_2 \otimes \cdots \otimes v_n \otimes v_1,$$

and $\delta = \alpha \circ (\mathbf{u}^{-1}\Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)})$. We have

$$\begin{aligned}\delta(v_1 \otimes v_2 \otimes \cdots \otimes v_n) &= \alpha(\mathbf{u}^{-1}\Lambda_{(1)}v_1 \otimes \Lambda_{(2)}v_2 \otimes \cdots \otimes \Lambda_{(n)}v_n) \\ &= \Lambda_{(2)}v_2 \otimes \cdots \otimes \Lambda_{(n)}v_n \otimes \mathbf{u}^{-1}\Lambda_{(1)}v_1 \\ &= \Lambda_{(1)}v_2 \otimes \cdots \otimes \Lambda_{(n-1)}v_n \otimes \Lambda_{(n)}S(\mathbf{u}^{-1})v_1 \quad \text{(by (4.1))} \\ &= \Lambda \cdot (v_2 \otimes \cdots \otimes v_n \otimes S(\mathbf{u}^{-1})v_1). \end{aligned} \tag{4.3}$$

This shows $\delta(V^{\otimes n}) \subseteq \Lambda \cdot V^{\otimes n} = (V^{\otimes n})^H$. Note that the map

$$\Phi : \ \mathrm{Hom}_H(\Bbbk, V^{\otimes n}) \longrightarrow (V^{\otimes n})^H, \quad f \longmapsto f(1)$$

is an H-module isomorphism. We claim that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_H(\Bbbk, V^{\otimes n}) & \xrightarrow{E_V^n} & \mathrm{Hom}_H(\Bbbk, V^{\otimes n}) \\ \Phi \downarrow & & \downarrow \Phi \\ (V^{\otimes n})^H & \xrightarrow{\delta} & (V^{\otimes n})^H \end{array}$$

Indeed, for any $f \in \mathrm{Hom}_H(\Bbbk, V^{\otimes n})$, we suppose that $f(1) = \sum v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$. It follows from $f(1) = f(\Lambda \cdot 1) = \Lambda \cdot f(1)$ that

$$\sum v_1 \otimes \cdots \otimes v_n = \sum \Lambda_{(1)}v_1 \otimes \cdots \otimes \Lambda_{(n)}v_n. \tag{4.4}$$

On the one hand, by (4.3) we have

$$(\delta \circ \Phi)(f) = \delta(f(1)) = \delta\big(\sum v_1 \otimes \cdots \otimes v_n\big) = \Lambda \cdot \big(\sum v_2 \otimes \cdots \otimes v_n \otimes S(\mathbf{u}^{-1})v_1\big).$$

On the other hand, we have

$$\begin{aligned}(\Phi \circ E_V^n)(f) = E_V^n(f)(1) &= \sum v_2 \otimes \cdots \otimes v_n \otimes \mathbf{u}^{-1}v_1 \quad \text{(by (4.2))} \\ &= \sum \Lambda_{(2)}v_2 \otimes \cdots \otimes \Lambda_{(n)}v_n \otimes \mathbf{u}^{-1}\Lambda_{(1)}v_1 \quad \text{(by (4.4))} \\ &= \sum \Lambda_{(1)}v_2 \otimes \cdots \otimes \Lambda_{(n-1)}v_n \otimes \Lambda_{(n)}S(\mathbf{u}^{-1})v_1 \quad \text{(by (4.1))} \\ &= \Lambda \cdot \big(\sum v_2 \otimes \cdots \otimes v_n \otimes S(\mathbf{u}^{-1})v_1\big).\end{aligned}$$

We obtain $\delta \circ \Phi = \Phi \circ E_V^n$, or equivalently, $E_V^n = \Phi^{-1} \circ \delta \circ \Phi$. It follows that

$$\begin{aligned}\nu_n(V) = \mathrm{tr}(E_V^n) &= \mathrm{tr}_{V^{\otimes n}}(\delta) \\ &= \mathrm{tr}_{V^{\otimes n}}(\alpha \circ (\mathbf{u}^{-1}\Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)})) \\ &= \mathrm{tr}_V(\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)}) \\ &= \chi_V(\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)}),\end{aligned}$$

where the equality $\mathrm{tr}_{V^{\otimes n}}(\alpha \circ (\mathbf{u}^{-1}\Lambda_{(1)} \otimes \Lambda_{(2)} \otimes \cdots \otimes \Lambda_{(n)})) = \mathrm{tr}_V(\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)})$ follows from [6, Lemma 2.3]. We have shown that $\nu_n(V) = \chi_V(\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)})$, where Λ is idempotent. Since $\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)}$ does not depend on the choice of the nonzero integral Λ, the equality $\nu_n(V) = \chi_V(\mathbf{u}^{-1}\Lambda_{(1)} \cdots \Lambda_{(n)})$ holds for any nonzero integral Λ of H. $\square$

Remark 4.3. If the field $\Bbbk$ has characteristic 0 and Λ is idempotent, then we get $\mathbf{u} = \varepsilon(\Lambda) = 1$. In this case, the n-th FS indicator of V is $\chi_V(\Lambda_{(1)} \cdots \Lambda_{(n)})$, which is the one defined in [6, Definition 2.3].

In the rest of this section, we will extend the n-th FS indicator $\nu_n(V)$ of V from $n \geq 1$ to $n \in \mathbb{Z}$. Recall that the n-th Sweedler power map $P_n \colon H \to H$ is defined by

$$P_n(a) = \begin{cases} a_{(1)} \cdots a_{(n)} & \text{for } n \geq 1, \\ \varepsilon(a) & \text{for } n = 0, \\ S(a_{(1)}) \cdots S(a_{(-n)}) & \text{for } n \leq -1. \end{cases}$$

From the n-th Sweedler power map P_n of H, we see that $\nu_n(V) = \chi_V(\mathbf{u}^{-1} P_n(\Lambda))$ for $n \geq 1$. However, this expression is well-defined for any integer n. Thus, we may extend this formula from $n \geq 1$ to any integer n, stated as follows.

Definition 4.4. Let H be a finite-dimensional semisimple Hopf algebra over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. For any finite-dimensional representation V of H and $n \in \mathbb{Z}$, the *n-th FS indicator* of V is defined by $\nu_n(V) = \chi_V(\mathbf{u}^{-1} P_n(\Lambda))$, where $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$.

Remark 4.5. (1) Note that $S(\Lambda) = \Lambda$. The n-th FS indicator of V can be written as

$$\nu_n(V) = \begin{cases} \chi_V(\mathbf{u}^{-1} \Lambda_{(1)} \cdots \Lambda_{(n)}) & \text{for } n \geq 1, \\ \chi_V(\mathbf{u}^{-1} \varepsilon(\Lambda)) & \text{for } n = 0, \\ \chi_V(\mathbf{u}^{-1} \Lambda_{(-n)} \cdots \Lambda_{(1)}) & \text{for } n \leq -1. \end{cases}$$

(2) By Proposition 3.3(4) we have

$$\mathbf{u}^{-1} S(\mathbf{u}^{-1}) = \sum_{i \in I} \frac{\lambda(e_i)}{\varepsilon(\Lambda) \dim_{\Bbbk}(V_i)^2} e_i \in Z(H).$$

It follows that

$$\begin{aligned} \nu_0(V) &= \varepsilon(\Lambda)\chi_V(\mathbf{u}^{-1}) = \varepsilon(\Lambda)\chi_V(\mathbf{u}^{-1} S(\mathbf{u}^{-1}) S(\mathbf{u})) \\ &= \sum_{i \in I} \frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2} \chi_V(e_i S(\mathbf{u})) = \sum_{i \in I} \frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2} \chi_V(e_i S(\Lambda_{(1)}) S^2(\Lambda_{(2)})) \\ &= \sum_{i \in I} \frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2} \chi_V(e_i S^2(\Lambda_{(2)}) S(\Lambda_{(1)})) = \varepsilon(\Lambda) \sum_{i \in I} \frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2} \chi_V(e_i). \end{aligned}$$

(3) $\nu_{-1}(V) = \nu_1(V) = \chi_V(\mathbf{u}^{-1}\Lambda) = \chi_V\big(\frac{\Lambda}{\varepsilon(\Lambda)}\big)$.

(4) By [17, Proposition 3.1], $\Lambda_{(1)}\Lambda_{(2)}$ and $\Lambda_{(2)}\Lambda_{(1)}$ are both central elements of H, and they are determined by the values that the characters χ_i for all $i \in I$ take on them. It follows from $\chi_i(\Lambda_{(1)}\Lambda_{(2)}) = \chi_i(\Lambda_{(2)}\Lambda_{(1)})$ that $\Lambda_{(1)}\Lambda_{(2)} = \Lambda_{(2)}\Lambda_{(1)}$. Therefore, $\nu_{-2}(V) = \nu_2(V)$.

The higher FS indicators of any simple module V_i can be described as follows:

Proposition 4.6. *For any $n \in \mathbb{Z}$ and any simple module V_i with the character χ_i,*

$$\nu_n(V_i) = \frac{\chi_i(P_n(\Lambda))\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2}.$$

Proof. Since $P_n(\Lambda) \in Z(H)$ for any $n \in \mathbb{Z}$ (see [17, Proposition 3.1]), it follows that $P_n(\Lambda) = \sum_{i\in I} \frac{\chi_i(P_n(\Lambda))}{\dim_{\Bbbk}(V_i)} e_i$. The n-th FS indicator of V_i is

$$\nu_n(V_i) = \chi_i(\mathbf{u}^{-1}P_n(\Lambda)) = \frac{\chi_i(P_n(\Lambda))}{\dim_{\Bbbk}(V_i)}\chi_i(\mathbf{u}^{-1}) = \frac{\chi_i(P_n(\Lambda))\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2},$$

where the last equality follows from Proposition 3.3(3). □

For any semisimple Hopf algebra over a field $\Bbbk$ of characteristic 0, the finite-dimensional representation V and its dual V^* have the same n-th FS indicators for all $n \geq 1$ (see [6, Section 2.3]). The following result shows that this result also holds for the n-th FS indicators defined for the Hopf algebra H over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$.

Proposition 4.7. *Let H be a finite-dimensional semisimple Hopf algebra over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. Let V be a finite-dimensional representation of H with the dual V^*. We have $\nu_n(V) = \nu_n(V^*)$ for all $n \in \mathbb{Z}$.*

Proof. Since $S(\Lambda) = \Lambda$, we have $S(P_n(\Lambda)) = P_n(\Lambda)$ for any $n \in \mathbb{Z}$. For the case $n \geq 1$, the n-th FS indicator of V^* is

$$\begin{aligned}\nu_n(V^*) &= (\chi_{V^*})(\mathbf{u}^{-1}P_n(\Lambda)) = (\chi_V \circ S)(\mathbf{u}^{-1}P_n(\Lambda))\\ &= \chi_V(\Lambda_{(1)}\cdots\Lambda_{(n)}S(\mathbf{u}^{-1})) = \chi_V(\Lambda_{(2)}\cdots\Lambda_{(n)}\mathbf{u}^{-1}\Lambda_{(1)}) \quad \text{(by (4.1))}\\ &= \chi_V(\mathbf{u}^{-1}\Lambda_{(1)}\Lambda_{(2)}\cdots\Lambda_{(n)}) = \nu_n(V).\end{aligned}$$

For the case $n \leq -1$, the n-th FS indicator of V^* is

$$\begin{aligned}\nu_n(V^*) &= (\chi_{V^*})(\mathbf{u}^{-1}P_n(\Lambda)) = (\chi_V \circ S)(\mathbf{u}^{-1}P_n(\Lambda))\\ &= \chi_V(\Lambda_{(-n)}\cdots\Lambda_{(1)}S(\mathbf{u}^{-1})) = \chi_V(S(\mathbf{u}^{-1})\Lambda_{(-n)}\cdots\Lambda_{(1)})\\ &= \chi_V(S(\mathbf{u}^{-1})\mathbf{u}^{-1}\Lambda_{(1)}S(\mathbf{u})\Lambda_{(-n)}\cdots\Lambda_{(2)}) \quad \text{(by (4.1))}\\ &= \chi_V(\Lambda_{(1)}\mathbf{u}^{-1}\Lambda_{(-n)}\cdots\Lambda_{(2)}) = \chi_V(\mathbf{u}^{-1}\Lambda_{(-n)}\cdots\Lambda_{(1)}) = \nu_n(V).\end{aligned}$$

For the case $n = 0$, we set $S(e_i) = e_{i^*}$ for any $i \in I$. Then $*$ is a permutation of I, $V_{i^*} \cong V_i^*$ and $\lambda(e_{i^*}) = \lambda(e_i)$ by Corollary 3.4. We have

$$\begin{aligned}\nu_0(V^*) &= \varepsilon(\Lambda)\sum_{i\in I}\frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2}\chi_V(S(e_i)) \quad \text{(by Remark 4.5(2))}\\ &= \varepsilon(\Lambda)\sum_{i\in I}\frac{\lambda(e_{i^*})}{\dim_{\Bbbk}(V_{i^*})^2}\chi_V(e_{i^*}) = \varepsilon(\Lambda)\sum_{i\in I}\frac{\lambda(e_i)}{\dim_{\Bbbk}(V_i)^2}\chi_V(e_i) = \nu_0(V).\end{aligned}$$

This completes the proof. □

Kashina et al. have shown in [6, Proposition 2.5] that the n-th FS indictor of the regular representation of a semisimple Hopf algebra over a field of characteristic 0 can be described as $\mathrm{tr}(S \circ P_{n-1})$ for $n \geq 1$. The following result shows that this formula also holds for the n-th FS indicators defined for the Hopf algebra H over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$.

Proposition 4.8. *Let H be a finite-dimensional semisimple Hopf algebra over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. For any $n \in \mathbb{Z}$, the n-th FS indictor of the regular representation of H can be written as $\nu_n(H) = \operatorname{tr}(S \circ P_{n-1})$, where P_{n-1} is the $(n-1)$-th Sweedler power map of H.*

Proof. We choose a left integral Λ of H and a right integral λ of H^* such that $\lambda(\Lambda) = 1$. For any $n \in \mathbb{Z}$, by Radford's trace formula [15, Theorem 2], we have

$$\begin{aligned}\operatorname{tr}(S \circ P_{n-1}) &= \operatorname{tr}(P_{n-1} \circ S) = \lambda(S(\Lambda_{(2)})(P_{n-1} \circ S)(\Lambda_{(1)})) \\ &= \lambda(S(\Lambda_{(2)})P_{n-1}(S(\Lambda_{(1)}))) = \lambda(\Lambda_{(1)}P_{n-1}(\Lambda_{(2)})) \\ &= \lambda(P_n(\Lambda)) = \chi_H(\mathbf{u}^{-1}P_n(\Lambda)) \text{ (by (3.3))} = \nu_n(H).\end{aligned}$$

□

5 Gauge Invariants

In this section, we will show that the n-th FS indicator $\nu_n(V)$ defined in Section 4 is a gauge invariant of the tensor category $\operatorname{Rep}(H)$ for any $n \in \mathbb{Z}$ and any finite-dimensional representation V of the semisimple Hopf algebra H.

Recall from [1] that a (normalized) twist for a semisimple Hopf algebra H is an invertible element $J \in H \otimes H$ that satisfies $(\varepsilon \otimes \mathrm{id})(J) = (\mathrm{id} \otimes \varepsilon)(J) = 1$ and

$$(\Delta \otimes \mathrm{id})(J)(J \otimes 1) = (\mathrm{id} \otimes \Delta)(J)(1 \otimes J).$$

We write $J = J^{(1)} \otimes J^{(2)}$ and $J^{-1} = J^{-(1)} \otimes J^{-(2)}$, where the summation is understood.

Given a twist J for H, one can define a new Hopf algebra H^J with the same algebra structure and counit as H, for which the comultiplication Δ^J and antipode S^J are given, respectively, by

$$\Delta^J(a) = J^{-1}\Delta(a)J \quad \text{and} \quad S^J(a) = Q_J^{-1}S(a)Q_J$$

for $a \in H$, where $Q_J = S(J^{(1)})J^{(2)}$, which is an invertible element of H with the inverse $Q_J^{-1} = J^{-(1)}S(J^{-(2)})$. With the notions above, we have the following result:

Proposition 5.1. *Let H be a finite-dimensional semisimple Hopf algebra over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$ and V a finite-dimensional representation of H. The n-th FS indicator $\nu_n(V)$ of V is invariant under twisting for any $n \in \mathbb{Z}$.*

Proof. Let Λ be a nonzero integral of H and J a normalized twist for H. It follows from [17, Theorem 3.4] that $P_n^J(\Lambda) = P_n(\Lambda)$, where P_n^J and P_n are the n-th Sweedler power maps of H^J and H, respectively. Moreover, $P_n(\Lambda)$ is a central element of H (see [17, Proposition 3.1]). Since $\Delta^J(\Lambda) = Q_J^{-1}\Lambda_{(1)} \otimes \Lambda_{(2)}Q_J$, it follows that

$$\mathbf{u}^J := S^J(\Lambda_{(2)}Q_J)Q_J^{-1}\Lambda_{(1)} = Q_J^{-1}S(Q_J)\mathbf{u}, \tag{5.1}$$

where $\mathbf{u} = S(\Lambda_{(2)})\Lambda_{(1)}$. For an H-module V with the character χ_V, we define V^J to be the same as V as a $\Bbbk$-linear space, but thought of as an H^J-module. Then the character of V^J is also χ_V. For any $n \in \mathbb{Z}$, we have

$$\begin{aligned}
\nu_n(V^J) &= \chi_V((\mathbf{u}^J)^{-1}P_n^J(\Lambda)) = \chi_V(\mathbf{u}^{-1}S(Q_J^{-1})Q_J P_n^J(\Lambda)) \quad \text{(by (5.1))}\\
&= \chi_V(\mathbf{u}^{-1}S(Q_J^{-1})Q_J P_n(\Lambda))\\
&= \chi_V(\mathbf{u}^{-1}S^2(J^{-(2)})S(J^{-(1)})S(J^{(1)})J^{(2)}P_n(\Lambda))\\
&= \chi_V(J^{-(2)}\mathbf{u}^{-1}S(J^{-(1)})S(J^{(1)})J^{(2)}P_n(\Lambda))\\
&= \chi_V(\mathbf{u}^{-1}S(J^{-(1)})S(J^{(1)})J^{(2)}P_n(\Lambda)J^{-(2)})\\
&= \chi_V(\mathbf{u}^{-1}S(J^{-(1)})S(J^{(1)})J^{(2)}J^{-(2)}P_n(\Lambda))\\
&= \chi_V(\mathbf{u}^{-1}P_n(\Lambda)) = \nu_n(V).
\end{aligned}$$

This completes the proof. □

We are now ready to state the main result, which says that higher FS indicators are gauge invariants of the tensor category $\mathrm{Rep}(H)$.

Theorem 5.2. *Let H and H' be two finite-dimensional semisimple Hopf algebras over a field $\Bbbk$ of characteristic $p > \dim_{\Bbbk}(H)^{1/2}$. If $\mathcal{F}\colon \mathrm{Rep}(H) \to \mathrm{Rep}(H')$ is an equivalence of tensor categories, then $\nu_n(V) = \nu_n(\mathcal{F}(V))$ for any $n \in \mathbb{Z}$ and any finite-dimensional representation V of H.*

Proof. Since the $\Bbbk$-linear equivalence $\mathcal{F}\colon \mathrm{Rep}(H) \to \mathrm{Rep}(H')$ is a tensor equivalence, it follows from [12, Theorem 2.2] that H and H' are gauge equivalent in the sense that there exists a twist J of H such that H' is isomorphic to H^J as bialgebras. Let $\sigma\colon H' \to H^J$ be such an isomorphism. Then σ is automatically a Hopf algebra isomorphism. The isomorphism σ induces a $\Bbbk$-linear equivalence $(-)^\sigma\colon \mathrm{Rep}(H) \to \mathrm{Rep}(H')$ as follows: for any finite-dimensional H-module V, $V^\sigma = V$ as a $\Bbbk$-linear space with the H'-module action given by $h'v = \sigma(h')v$ for $h' \in H'$, $v \in V$, and $f^\sigma = f$ for any morphism f in $\mathrm{Rep}(H)$. Moreover, the equivalence $\mathcal{F}$ is naturally isomorphic to the $\Bbbk$-linear equivalence $(-)^\sigma$ (see [5, Theorem 1.1]). Therefore,

$$\nu_n(\mathcal{F}(V)) = \nu_n(V^\sigma).$$

Let Λ' be a nonzero integral of H' and S' the antipode of H'. Note that the map $\sigma\colon H' \to H^J$ is a Hopf algebra isomorphism. It follows that $\sigma(\Lambda') = \Lambda$, which is a nonzero integral of H^J and $\sigma(P'_n(\Lambda')) = P_n^J(\Lambda)$, where P'_n and P_n^J are the n-th Sweedler power maps of H' and H^J, respectively. In particular,

$$\sigma((\mathbf{u}')^{-1}P'_n(\Lambda')) = (\mathbf{u}^J)^{-1}P_n^J(\Lambda),$$

where $\mathbf{u}' = S'(\Lambda'_{(2)})\Lambda'_{(1)}$ and $\mathbf{u}^J = S^J(\Lambda_{\langle 2\rangle})\Lambda_{\langle 1\rangle}$. We have

$$\begin{aligned}
\nu_n(V^\sigma) &= \chi_{V^\sigma}((\mathbf{u}')^{-1}P'_n(\Lambda'))\\
&= \chi_{V^J}(\sigma((\mathbf{u}')^{-1}P'_n(\Lambda')))\\
&= \chi_{V^J}((\mathbf{u}^J)^{-1}P_n^J(\Lambda))\\
&= \nu_n(V^J) = \nu_n(V),
\end{aligned}$$

where the last equality follows by Proposition 5.1. We conclude that for any $n \in \mathbb{Z}$ and any finite-dimensional representation V of H, $\nu_n(\mathcal{F}(V)) = \nu_n(V)$. □

Acknowledgements. The authors would like to thank Professor Juan Cuadra for pointing out a mistake in a previous version of the manuscript. The second author was supported by National Natural Science Foundation of China (Grant No. 12271243). The third author was supported by National Natural Science Foundation of China (Grant No. 12371041).

References

[1] E. Aljadeff, P. Etingof, S. Gelaki, D. Nikshych, On twisting of finite-dimensional Hopf algebras, *J. Algebra* **256** (2) (2002) 484–501.
[2] Y. Doi, Group-like algebras and their representations, *Comm. Algebra* **38** (7) (2010) 2635–2655.
[3] P. Etingof, S. Gelaki, On finite-dimensional semisimple and cosemisimple Hopf algebras in positive characteristic, *Internat. Math. Res. Notices* **1998** (16) 851–864.
[4] M. Geck, G. Pfeiffer, *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*, Oxford University Press, New York, 2000.
[5] Y. Kashina, S. Montgomery, S.-H. Ng, On the trace of the antipode and higher indicators, *Israel J. Math.* **188** (2012) 57–89.
[6] Y. Kashina, Y. Sommerhäuser, Y. Zhu, *On Higher Frobenius-Schur Indicators*, Mem. Amer. Math. Soc., Vol. 181, no. 855, Amer. Math. Soc., Providence, RI, 2006.
[7] R.G. Larson, D.E. Radford, Semisimple cosemisimple Hopf algebras, *Amer. J. Math.* **109** (1) (1988) 187–195.
[8] R.G. Larson, D.E. Radford, Finite-dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, *J. Algebra* **117** (2) (1988) 267–289.
[9] V. Linchenko, S. Montgomery, A Frobenius-Schur theorem for Hopf algebras, *Algebr. Represent. Theory* **3** (4) (2000) 347–355.
[10] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Series in Math., 82, Amer. Math. Soc., Providence, RI, 1993.
[11] S.-H. Ng, P. Schauenburg, Higher Frobenius-Schur indicators for pivotal categories, in: *Hopf Algebras and Generalizations*, Contemp. Math., 441, Amer. Math. Soc., Providence, RI, 2007, pp. 63–90.
[12] S.-H. Ng, P. Schauenburg, Central invariants and higher indicators for semisimple quasi-Hopf algebras, *Trans. Amer. Math. Soc.* **360** (4) (2008) 1839–1860.
[13] S.-H. Ng, P. Schauenburg, Congruence subgroups and generalized Frobenius-Schur indicators, *Comm. Math. Phys.* **300** (1) (2011) 1–46.
[14] D.E. Radford, The order of the antipode of a finite dimensional Hopf algebra is finite, *Amer. J. Math.* **98** (2) (1976) 333–355.
[15] D.E. Radford, The trace function and Hopf algebras, *J. Algebra* **163** (3) (1994) 583–622.
[16] Y. Sommerhäuser, On Kaplansky's fifth conjecture, *J. Algebra* **204** (1) (1998) 202–224.
[17] Z. Wang, G.X. Liu, L.B. Li, Invariants from the Sweedler power maps on integrals, *J. Algebra* **606** (2022) 590–612.