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On quiver-theoretic description for quasitriangularity of Hopf algebras

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ABSTRACT

This paper is devoted to the study of the quasitriangularity of Hopf algebras via Hopf quiver approaches. We give a combinatorial description of the Hopf quivers whose path coalgebras give rise to coquasitriangular Hopf algebras. With a help of the quiver setting, we study general coquasitriangular pointed Hopf algebras and obtain a complete classification of the finite-dimensional ones over an algebraically closed field of characteristic 0.

1. Introduction

Quasitriangular Hopf algebras were introduced by Drinfeld, which play a crucial role in his theory of quantum groups [12]. From the viewpoint of representation theory, quasitriangular Hopf algebras are close relatives of groups and Lie algebras, as their representations form braided tensor categories. The construction and classification of such Hopf algebras have attracted much attention since they appeared. However, the general classification problem is still widely open.

As is well known, quiver methods are very useful in constructing algebras and studying their representations. Very nice quiver settings for elementary and pointed Hopf algebras have been built in various works [10,15,11,30] and shown their advantage in classifying some interesting classes of Hopf algebras as well as their representations, see, for instance, [9,14,23,7,20,19,16] and other related

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works. To us, it seems sensible to ask: is it possible to describe Drinfeld's quasitriangularity of Hopf algebras via combinatorial properties of Hopf quivers introduced by Cibils and Rosso [11]?

Our main aim is to provide such description and, by making use of it, to contribute to the classification and representation theory of quasitriangular Hopf algebras. In this paper we focus on quasitriangular elementary Hopf algebras, or more precisely the dual situation coquasitriangular pointed Hopf algebras. Recall that, for a Hopf algebra, by elementary we mean that the underlying algebra is finite-dimensional and its simple modules are 1-dimensional, while by pointed we mean that the simple comodules of the underlying coalgebra are 1-dimensional.

In some sense, what we are studying is the simplest class of quasitriangular Hopf algebras. According to their proximity to groups, quasitriangular elementary Hopf algebras should be viewed as the counterpart of finite abelian groups, as they are finite-dimensional and their simple modules are all 1-dimensional. In this point of view, the class of quasitriangular pointed Hopf algebras lies in the other extreme and so is much more complicated. Their study is left for future work.

We build a quiver framework for our object and use the combinatorial properties to study the quasitriangularity. Since the quiver approach from the coalgebraic side is more convenient and admits a wider scope (especially, including the infinite-dimensional case, see [11,30]), and the dual of quasitriangular elementary Hopf algebras are coquasitriangular pointed, we always work mainly on the latter and mention briefly the dual cases for the former. First we determine the Hopf quivers whose path coalgebras give rise to coquasitriangular Hopf algebras, then we give a Gabriel type theorem for general coquasitriangular pointed Hopf algebras, and finally we provide some examples to elucidate the quiver setting.

We also apply the constructed quiver setting to classify finite-dimensional coquasitriangular pointed Hopf algebras over an algebraically closed field of characteristic 0. First we prove that such Hopf algebras are generated by group-like and skew-primitive elements. The proof relies on the result of Andruskiewitsch, Etingof and Gelaki [1] for the cotriangular case as well as an observation motivated by our quiver setting. This partially confirms a conjecture of Andruskiewitsch and Schneider [3] which says that a finite-dimensional pointed Hopf algebra is generated by its group-like and skew-primitive elements. Then we give an explicit description of finite-dimensional coquasitriangular pointed Hopf algebras via generators and defining relations by making use of the quiver setting along with the lifting theorem of quantum linear spaces due to Andruskiewitsch and Schneider [2].

Throughout the paper, we work over a field k. Vector spaces, algebras, coalgebras, linear mappings, and unadorned $\otimes$ are over k. The readers are referred to [13,18] for general knowledge of (co)quasitriangular Hopf algebras, and to [5] for that of quivers and their applications to algebras and representation theory.

2. (Co)quasitriangular Hopf algebras and Hopf quivers

This section is devoted to some preliminaries.

2.1. A quasitriangular Hopf algebra is a pair (H, R), where H is a Hopf algebra and $R = \sum R_{(1)} \otimes R_{(2)} \in H \otimes H$ is an invertible element satisfying

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \tag{2.1}$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}, \tag{2.2}$$

$$\Delta^{o}(h) = R\Delta(h)R^{-1}, \quad \forall h \in H, \tag{2.3}$$

where Δ^o is the opposite coproduct of H, and

$$R_{12} = \sum R_{(1)} \otimes R_{(2)} \otimes 1, \qquad R_{13} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}, \qquad R_{23} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}.$$

We call R a universal R-matrix, or a quasitriangular structure, of H. A quasitriangular Hopf algebra (H, R) is called triangular, if

$$RR_{21} = 1 \otimes 1. \tag{2.4}$$

Here $R_{21} = \sum R_{(2)} \otimes R_{(1)}$.

Dually, a coquasitriangular Hopf algebra $(H, \mathcal{R})$ is a Hopf algebra H and a convolution-invertible linear function $\mathcal{R}: H \otimes H \to k$ such that

$$\mathcal{R}(ab, c) = \mathcal{R}(a, c_1)\mathcal{R}(b, c_2), \tag{2.5}$$

$$\mathcal{R}(a, bc) = \mathcal{R}(a_1, c)\mathcal{R}(a_2, b), \tag{2.6}$$

$$b_1 a_1 \mathcal{R}(a_2, b_2) = \mathcal{R}(a_1, b_1) a_2 b_2 \tag{2.7}$$

for all $a, b, c \in H$. Here and below we use the Sweedler sigma notation $\Delta(a) = a_1 \otimes a_2$ for the coproduct and $a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}$ for the result of the n-iterated application of Δ on a. The function $\mathcal{R}$ is called a universal R-form, or a coquasitriangular structure, of H. Let $\bar{\mathcal{R}}$ denote the convolution-inverse of $\mathcal{R}$. Then the last condition (2.7) is equivalent to

$$ba = \mathcal{R}(a_1, b_1) a_2 b_2 \bar{\mathcal{R}}(a_3, b_3) \tag{2.8}$$

for all $a, b \in H$. A coquasitriangular Hopf algebra $(H, \mathcal{R})$ is called cotriangular, if

$$\mathcal{R}(a_1, b_1)\mathcal{R}(b_2, a_2) = \varepsilon(a)\varepsilon(b) \tag{2.9}$$

for all $a, b \in H$.

2.2. A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t: Q_1 \to Q_0$ are two maps assigning respectively the source and the target for each arrow. A path of length $l \geqslant 1$ in the quiver Q is a finitely ordered sequence of l arrows $a_l \cdots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $1 \leqslant i \leqslant l-1$. By convention a vertex is said to be a trivial path of length 0.

The path coalgebra kQ is the k-space spanned by the paths of Q with counit and comultiplication maps defined by $\varepsilon(g) = 1$, $\Delta(g) = g \otimes g$ for each $g \in Q_0$, and for each non-trivial path $p = a_n \cdots a_1$, $\varepsilon(p) = 0$,

$$\Delta(p) = p \otimes s(a_1) + \left(\sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1\right) + t(a_n) \otimes p.$$

The length of paths gives a natural gradation to the path coalgebra. Let Q_n denote the set of paths of length n in Q, then $kQ = \bigoplus_{n \geqslant 0} kQ_n$ and $\Delta(kQ_n) \subseteq \bigoplus_{n=i+j} kQ_i \otimes kQ_j$. Clearly kQ is pointed with the set of group-likes $G(kQ) = Q_0$, and has the following coradical filtration

$$kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots.$$

Hence kQ is coradically graded.

2.3. According to Cibils and Rosso [11], a quiver Q is said to be a Hopf quiver if the corresponding path coalgebra kQ admits a graded Hopf algebra structure. Hopf quivers can be determined by ramification data of groups. Let G be a group, $\mathcal{C}$ the set of conjugacy classes. A ramification datum R of the group G is a formal sum $\sum_{C \in \mathcal{C}} R_C C$ of conjugacy classes with coefficients in $\mathbb{N} = \{0, 1, 2, \ldots\}$. The corresponding Hopf quiver $Q = Q(G, R)$ is defined as follows: the set of vertices Q_0 is G, and for each $x \in G$ and $c \in C$, there are R_C arrows going from x to cx. For a given Hopf quiver Q, the set of graded Hopf structures on kQ is in one-to-one correspondence with the set of kQ_0-Hopf bimodule structures on kQ_1.

The graded Hopf structures are obtained from Hopf bimodules via the quantum shuffle product of Rosso [28]. Suppose that Q is a Hopf quiver with a necessary kQ_0-Hopf bimodule structure on kQ_1. Let $p \in Q_l$ be a path. An n-thin split of it is a sequence $(p_1, \dots, p_n)$ of vertices and arrows such that the concatenation $p_n \cdots p_1$ is exactly p. These n-thin splits are in one-to-one correspondence with the n-sequences of $(n-l)$ 0's and l 1's. Denote the set of such sequences by D_l^n. Clearly $|D_l^n| = \binom{n}{l}$. For $d = (d_1, \dots, d_n) \in D_l^n$, the corresponding n-thin split is written as $dp = ((dp)_1, \dots, (dp)_n)$, in which $(dp)_i$ is a vertex if $d_i = 0$ and an arrow if $d_i = 1$. Let $\alpha = a_m \cdots a_1$ and $\beta = b_n \cdots b_1$ be paths of length m and n respectively. Assume $d \in D_m^{m+n}$ and let $\bar{d} \in D_n^{m+n}$ be the complement sequence which is obtained from d by replacing each 0 by 1 and each 1 by 0. Define an element

$$(\alpha \cdot \beta)_d = \left[(d\alpha)_{m+n}.(\bar{d}\beta)_{m+n}\right] \cdots \left[(d\alpha)_1.(\bar{d}\beta)_1\right]$$

in Q_{m+n}, where $[(d\alpha)_i.(\bar{d}\beta)_i]$ is understood as the action of kQ_0-Hopf bimodule on kQ_1 and these terms in different brackets are put together linearly by concatenation. In terms of these notations, the formula of the product of α and β is given as follows:

$$\alpha \cdot \beta = \sum_{d \in D_m^{m+n}} (\alpha \cdot \beta)_d. \tag{2.10}$$

2.4. Let H be a pointed Hopf algebra. Denote its coradical filtration by $\{H_n\}_{n=0}^{\infty}$. Define

$$\operatorname{gr}(H) = H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$$

as the corresponding (coradically) graded coalgebra. Then $\operatorname{gr}(H)$ inherits from H a coradically graded Hopf algebra structure (see, e.g., [22]). The procedure from H to $\operatorname{gr} H$ is generally called degeneration and the converse is called lifting by Andruskiewitsch and Schneider in [2].

Due to Van Oystaeyen and Zhang [30], if H is a coradically graded pointed Hopf algebra, then there exists a unique Hopf quiver $Q(H)$ such that H can be realized as a large sub-Hopf algebra of a graded Hopf structure on the path coalgebra $kQ(H)$. Here by "large" we mean that H contains the subspace $kQ(H)_0 \oplus kQ(H)_1$.

3. Coquasitriangular Hopf quivers

In this section we determine the Hopf quivers which give rise to coquasitriangular Hopf algebras. In addition, some classification results of the coquasitriangular structures on such quiver Hopf algebras are given.

3.1. Let Q be a quiver. If the corresponding path coalgebra kQ admits a graded coquasitriangular Hopf algebra structure, then we call Q a coquasitriangular Hopf quiver. First we give a combinatorial description of such class of Hopf quivers.

In the following, we need to make use of the classification of Hopf bimodules over a group algebra obtained in [10] by Cibils and Rosso. That is, for a group G, the category of kG-Hopf bimodules is equivalent to the product of the categories of usual module categories $\prod_{C \in \mathcal{C}} kZ_C$-mod, where $\mathcal{C}$ is the set of conjugacy classes and Z_C is the centralizer of one of the elements in the class $C \in \mathcal{C}$. In particular, if G is abelian, then the category of kG-Hopf bimodules is equivalent to $\prod_{g \in G} kG$-mod.

Proposition 3.1. *Let Q be a quiver. Then Q is a coquasitriangular Hopf quiver if and only if Q is a Hopf quiver of form $Q(G, R)$ where G is an abelian group and R a ramification datum.*

Proof. Assume that Q is a coquasitriangular Hopf quiver and that $(kQ, \mathcal{R})$ is a graded coquasitriangular Hopf algebra. Then in the first place Q must be a Hopf quiver, say $Q(G, R)$. Let $\mathcal{R}_0$ be the restriction of $\mathcal{R}$ to $kQ_0 \otimes kQ_0$. Note that $kQ_0 = kG$ is a group algebra and that $(kG, \mathcal{R}_0)$ is a coquasitriangular Hopf algebra. Therefore by (2.8) we have

$$hg = \mathcal{R}_0(g,h)gh\bar{\mathcal{R}}_0(g,h) = gh$$

for all $g, h \in G$. This implies that G is an abelian group.

Conversely, assume that Q is the Hopf quiver $Q(G,R)$ of some abelian group G with respect to a ramification datum R. Then we can take the kQ_0-Hopf bimodule structure on kQ_1 which corresponds to the product of a set of trivial kG-modules. This gives rise to a commutative graded Hopf structure on kQ. Apparently $(kQ, \varepsilon \otimes \varepsilon)$ is a coquasitriangular Hopf algebra. This proves that Q is a coquasitriangular Hopf quiver. □

3.2. Let Q be a coquasitriangular Hopf quiver. The proof of the previous proposition provides only a trivial coquasitriangular Hopf structure on the path coalgebra kQ. Now we study non-trivial coquasitriangular structures on kQ. Assume that $Q = Q(G,R)$. We start with the simplest case. Fix a graded Hopf structure on kQ which is determined by a given kG-Hopf bimodule structure on kQ_1. Assume that $\mathcal{R}$ is a coquasitriangular structure which concentrates at degree 0, namely, $\mathcal{R}(x,y) = 0$ for all homogeneous elements x, y unless they all lie in kQ_0. The following facts are straightforward.

Lemma 3.2. *Keep the previous assumptions. Then $\mathcal{R}$ is a bicharacter of G, that is, for all $f, g, h \in G$,*

$$\mathcal{R}(f,gh) = \mathcal{R}(f,g)\mathcal{R}(f,h), \qquad \mathcal{R}(fg,h) = \mathcal{R}(f,h)\mathcal{R}(g,h),$$
$$\mathcal{R}(f,\epsilon) = 1 = \mathcal{R}(\epsilon,f) \tag{3.1}$$

where ϵ is the unit of G. Moreover, in kQ the following equations hold:

$$\alpha g = \frac{\mathcal{R}(g,t(\alpha))}{\mathcal{R}(g,s(\alpha))} g\alpha \quad \textit{for all } g \in G,\ \alpha \in Q_1, \tag{3.2}$$

$$\beta\alpha = \frac{\mathcal{R}(t(\alpha),t(\beta))}{\mathcal{R}(s(\alpha),s(\beta))} \alpha\beta \quad \textit{for all } \alpha, \beta \in Q_1. \tag{3.3}$$

Here and below, the product notation "·" is omitted whenever it is clear according to the context.

The preceding lemma indicates that the coquasitriangular structure $\mathcal{R}$ and the kG-Hopf bimodule structure on kQ_1 are mutually determined.

3.3. Let $Q = Q(G,R)$ be a coquasitriangular Hopf quiver. Note that Q is connected if and only if the union of the conjugacy classes with non-zero coefficients in the ramification datum R generates G. In general, let $Q(\epsilon)$ denote the connected component containing the unit $\epsilon \in G$, then $Q(\epsilon)_0$ is a normal subgroup of G and as a quiver Q is equivalent to $\bigcup_{x \in G/Q(\epsilon)_0} Q(x)$ where $Q(x)$ is identical to $Q(\epsilon)$ for all x. Similarly, coquasitriangular Hopf structures on kQ can also be decomposed as a crossed product (see, e.g., [22]) of $kQ(\epsilon)$, which inherits a sub-coquasitriangular structure of kQ, with the quotient group algebra $kG/Q(\epsilon)_0$. This enables us to work on only connected Hopf quivers.

Recall that a bicharacter $\mathcal{R}$ of a group G is said to be skew-symmetric, if $\mathcal{R}(g,h)\mathcal{R}(h,g) = 1$ for all $g, h \in G$. Now we are ready to state the main result of this section.

Theorem 3.3. *Let $Q = Q(G,R)$ be a connected coquasitriangular Hopf quiver. Then the complete list of coquasitriangular graded Hopf algebra structures $(kQ, \mathcal{R})$ with Q_0 equal to G and $\mathcal{R}$ concentrating at degree* 0 *is in one-to-one correspondence with the set of skew-symmetric bicharacters of the group G.*

Proof. First assume that $(kQ, \mathcal{R})$ is a coquasitriangular Hopf algebra with Q_0 equal to G and $\mathcal{R}$ concentrating at degree 0. Then by restricting to G, we have showed in Lemma 3.2 that $\mathcal{R}$ is a bicharacter of G satisfying (3.2)–(3.3). Assume that $R=\sum_{g\in G} R_g g$. Since $Q(G,R)$ is connected, we have $\langle g\in G \mid R_g\neq 0\rangle = G$. Suppose that $R_g\neq 0\neq R_h$, then in Q there are arrows starting from the unit ϵ of G, say $\alpha:\epsilon\to g$ and $\beta:\epsilon\to h$. Then by (2.10) and (3.2)–(3.3) we have

$$\begin{aligned}\beta\alpha &= [\beta g][\alpha]+[h\alpha][\beta] = \mathcal{R}(g,h)[g\beta][\alpha]+[h\alpha][\beta]\\ &= \mathcal{R}(g,h)\alpha\beta = \mathcal{R}(g,h)[\alpha h][\beta]+\mathcal{R}(g,h)[g\beta][\alpha]\\ &= \mathcal{R}(g,h)\mathcal{R}(h,g)[h\alpha][\beta]+\mathcal{R}(g,h)[g\beta][\alpha].\end{aligned}$$

This implies that $\mathcal{R}(g,h)\mathcal{R}(h,g)=1$. Since both g and h vary in a generating set of G, we have $\mathcal{R}(a,b)\mathcal{R}(b,a)=1$ for all $a,b\in G$.

Conversely, assume that $\mathcal{R}$ is a skew-symmetric bicharacter of G. For each arrow in Q with source ϵ, say $\alpha:\epsilon\to g$, let $h\triangleright\alpha := \mathcal{R}(g,h)\alpha$. This defines a one-dimensional left G-module on $k\alpha$. Denote by V the vector space spanned by all arrows with source ϵ and understand a G-module structure on it by defining G-action on each arrow as above. Now by the aforementioned results of Cibils and Rosso in [10, Proposition 3.3] and [11, Theorems 3.3 and 3.8], we can extend the left G-module V to a kG-Hopf bimodule on kQ_1 and this gives rise to a unique graded Hopf algebra structure on the path coalgebra kQ. We extend $\mathcal{R}$ trivially to be a function on $kQ\otimes kQ$ such that $\mathcal{R}(x,y)=0$ whenever one of the homogeneous elements x,y lies out of kQ_0. Now we claim that $(kQ,\mathcal{R})$ is a coquasitriangular Hopf algebra. The axioms (2.5)–(2.6) are direct from the definition of $\mathcal{R}$. It remains to verify (2.8). We need to show that the following equation holds

$$\beta\alpha = \mathcal{R}\big(t(a_m),t(b_n)\big)\alpha\beta\bar{\mathcal{R}}\big(s(a_1),s(b_1)\big)$$

for all paths $\alpha=a_m\cdots a_1$, $\beta=b_n\cdots b_1$. Here we use the convention: if $m=0$, then $\alpha\in Q_0$ and $t(\alpha)=\alpha=s(\alpha)$. For $\alpha,\beta\in Q_0$, the equation is obvious. For $\alpha\in Q_1$ an arrow with source ϵ and $\beta\in Q_0$, the equation follows from the definition of the action "$h\triangleright\alpha$" which corresponds to conjugation "$h\alpha h^{-1}$" in the kG-Hopf bimodule. Now let $\alpha:g\to h$ be an arbitrary arrow and $f\in G$. Note that $g^{-1}\alpha$ is an arrow from ϵ to $g^{-1}h$. Denote it by $\tilde{\alpha}$. Then we have by (3.1) and the previous case that

$$f\alpha = gf\tilde{\alpha} = g\mathcal{R}\big(g^{-1}h,f\big)\tilde{\alpha}f = \mathcal{R}(h,f)\alpha f\bar{\mathcal{R}}(g,f).$$

Finally let $\alpha=a_m\cdots a_1$, $\beta=b_n\cdots b_1$ be arbitrary paths. Then we have by the preceding cases and the product formula (2.10) that

$$\begin{aligned}\beta\alpha &= \sum_{d\in D_n^{m+n}}\big[(d\beta)_{m+n}(\bar{d}\alpha)_{m+n}\big]\cdots\big[(d\beta)_1(\bar{d}\alpha)_1\big]\\ &= \sum_{d\in D_n^{m+n}}\left[\frac{\mathcal{R}(t((\bar{d}\alpha)_{m+n}),t((d\beta)_{m+n}))}{\mathcal{R}(s((\bar{d}\alpha)_{m+n}),s((d\beta)_{m+n}))}(\bar{d}\alpha)_{m+n}(d\beta)_{m+n}\right]\\ &\quad\cdots\left[\frac{\mathcal{R}(t((\bar{d}\alpha)_1),t((d\beta)_1))}{R(s((\bar{d}\alpha)_1),s((d\beta)_1))}(\bar{d}\alpha)_1(d\beta)_1\right]\\ &= \frac{\mathcal{R}(t(a_m),t(b_n))}{\mathcal{R}(s(a_1),s(b_1))}\sum_{d\in D_n^{m+n}}\big[(\bar{d}\alpha)_{m+n}(d\beta)_{m+n}\big]\cdots\big[(\bar{d}\alpha)_1(d\beta)_1\big]\\ &= \frac{\mathcal{R}(t(a_m),t(b_n))}{\mathcal{R}(s(a_1),s(b_1))}\sum_{d\in D_m^{m+n}}\big[(d\alpha)_{m+n}(\bar{d}\beta)_{m+n}\big]\cdots\big[(d\alpha)_1(\bar{d}\beta)_1\big]\\ &= \mathcal{R}\big(t(a_m),t(b_n)\big)\alpha\beta\bar{\mathcal{R}}\big(s(a_1),s(b_1)\big).\end{aligned}$$

In the third equality we have used the fact $t((d\beta)_i) = s((d\beta)_{i+1})$ for $i = 1, \dots, m+n-1$. Now we are done. □

3.4. In the following we show that the situation "$\mathcal{R}$ concentrates at degree 0" can always be realized via restricting a general coquasitriangular structure. Let $(kQ, \mathcal{R})$ be a coquasitriangular Hopf algebra. Define $\mathcal{R}_0$ by $\mathcal{R}_0|_{kQ_0 \otimes kQ_0} := \mathcal{R}|_{kQ_0 \otimes kQ_0}$ and $\mathcal{R}_0|_{kQ_i \otimes kQ_j} := 0$ if $i + j > 0$. Clearly, $\mathcal{R}_0$ concentrates at degree 0.

Proposition 3.4. *Let $(kQ, \mathcal{R})$ be a coquasitriangular Hopf algebra and let $\mathcal{R}_0$ be defined as above. Then $(kQ, \mathcal{R}_0)$ is coquasitriangular too.*

Proof. We only show Eq. (2.7) for $\mathcal{R}_0$ since (2.5) and (2.6) are obvious. Let a, b be two paths in kQ of lengths m, n respectively. Then by the assumption that $(kQ, \mathcal{R})$ is coquasitriangular we have

$$b_1 a_1 \mathcal{R}(a_2, b_2) = \mathcal{R}(a_1, b_1) a_2 b_2.$$

Denote the linear combination of all paths of length $m+n$ appearing in the left (resp. right) side of the above equality by $L_{m+n}(a, b)$ (resp. $R_{m+n}(a, b)$). Therefore,

$$L_{m+n}(a, b) = R_{m+n}(a, b).$$

Note that $L_{m+n}(a, b) = b_1 a_1 \mathcal{R}_0(a_2, b_2)$ and $R_{m+n}(a, b) = \mathcal{R}_0(a_1, b_1) a_2 b_2$. This implies the desired equation (2.7) and the proof is completed. □

3.5. Now the following consequences are immediate.

Corollary 3.5. *Let $Q = Q(G, R)$ be a coquasitriangular Hopf quiver and $(kQ, \mathcal{R})$ a graded coquasitriangular Hopf algebra.*

(1) *If Q is connected, then $(kQ, \mathcal{R}_0)$ is a cotriangular Hopf algebra.*
(2) *Denote by $\mathcal{R}(\epsilon)$ the restriction of $\mathcal{R}$ on the connected component $Q(\epsilon)$ of Q containing the unit element. Then the sub-coquasitriangular Hopf algebra $(kQ(\epsilon), \mathcal{R}_0(\epsilon))$ is cotriangular.*

Here, it is worthy to note that for graded coquasitriangular Hopf algebras $(kQ, \mathcal{R})$ as above, the "connectedness" of Q leads to the "cotriangularity" of kQ. Accordingly, a general graded coquasitriangular Hopf algebra $(kQ, \mathcal{R})$ is a crossed product of a cotriangular one with a group algebra. Finally, we remark that there may be many coquasitriangular structures of kQ not concentrating at degree 0. Such examples will be given in Subsection 5.1. A complete classification of the set of coquasitriangular structures on a general kQ is not known yet.

4. Quiver setting for coquasitriangular pointed Hopf algebras

The aim of this section is to provide a quiver setting for general coquasitriangular pointed Hopf algebras. In particular, a Gabriel type theorem is obtained.

4.1. Assume that $(H, \mathcal{R})$ is a coquasitriangular pointed Hopf algebra. Let $\mathrm{gr}(H)$ be its coradically graded version as mentioned in Subsection 2.4 and let $\mathrm{gr}(\mathcal{R}) : \mathrm{gr}(H) \otimes \mathrm{gr}(H) \to k$ be the function defined by (for homogeneous elements $g, h \in \mathrm{gr}(H)$)

$$\mathrm{gr}(\mathcal{R})(g, h) = \begin{cases} \mathcal{R}(g, h), & \text{if } g, h \in H_0; \\ 0, & \text{otherwise.} \end{cases}$$

Our first observation is that the degeneration $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ of $(H, \mathcal{R})$ does not lose the coquasitriangularity.

Lemma 4.1. *Assume that $(H, \mathcal{R})$ is a coquasitriangular pointed Hopf algebra. Then its graded version $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is still a coquasitriangular pointed Hopf algebra.*

Proof. Denote the multiplication notation in $\mathrm{gr}(H)$ by $\circ$. At first, we check Eq. (2.5). Take any three homogeneous elements $a, b, c \in \mathrm{gr}(H)$. If $a \notin H_0$ or $b \notin H_0$, then

$$\mathrm{gr}(\mathcal{R})(a \circ b, c) = 0 = \mathrm{gr}(\mathcal{R})(a, c_1)\, \mathrm{gr}(\mathcal{R})(a, c_2)$$

by the definition of $\mathrm{gr}(\mathcal{R})$. If $a, b \in H_0$ and $c \notin H_0$, then clearly either $c_1 \notin H_0$ or $c_2 \notin H_0$. Also, by the definition of $\mathrm{gr}(\mathcal{R})$, Eq. (2.5) is right since both sides are zero. We only need to prove the case $a, b, c \in H_0$, but this is clear since H_0 is a sub-Hopf algebra of H and hence coquasitriangular. Eq. (2.6) can be proved similarly.

Next let us show Eq. (2.7). By the Gabriel type theorem for graded pointed Hopf algebras [30, Theorem 4.5] and pointed coalgebras [8, Theorem 2.1], there exists a unique Hopf quiver Q such that $\mathrm{gr}(H)$ is a sub-Hopf algebra of kQ and H is sub-coalgebra of kQ. By this, we can consider the elements of $\mathrm{gr}(H)$ and H as linear combinations of paths. Take $a, b \in H$ and for simplicity we may assume that a, b are paths of length m, n respectively. Since H is coquasitriangular, one has

$$b_1 a_1 \mathcal{R}(a_2, b_2) = \mathcal{R}(a_1, b_1) a_2 b_2.$$

Similar to the proof of Proposition 3.4, we denote the linear combination of all paths of length $m + n$ appearing in the left (resp. right) side of the above equality by $L_{m+n}(a, b)$ (resp. $R_{m+n}(a, b)$). So,

$$L_{m+n}(a, b) = R_{m+n}(a, b).$$

Meanwhile, it is not hard to see that we always have

$$L_{m+n}(a, b) = b_1 \circ a_1\, \mathrm{gr}(\mathcal{R})(a_2, b_2), \qquad R_{m+n}(a, b) = \mathrm{gr}(\mathcal{R})(a_1, b_1) a_2 \circ b_2.$$

Now the proof is finished. □

4.2. Our next observation is that the Gabriel type theorem for pointed Hopf algebras (see Subsection 2.4) given in [30, Theorem 4.5] can be restricted to the coquasitriangular situation.

Theorem 4.2. *Suppose that $(H, \mathcal{R})$ is a coquasitriangular pointed Hopf algebra. Then there exists a unique coquasitriangular Hopf quiver Q such that $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is isomorphic to a large sub-coquasitriangular Hopf structure of $(kQ, \mathfrak{R})$.*

Proof. Let G denote the set of group-like elements of H. Then the coradical H_0 of H is the group algebra kG. By restricting the function $\mathcal{R}$, we have a sub-coquasitriangular Hopf algebra $(kG, \mathcal{R})$. This implies that G is an abelian group and $\mathcal{R}$ is a bicharacter of it. By the Gabriel type theorem for general pointed Hopf algebras, there exists a unique Hopf quiver $Q = Q(G, R)$ such that $\mathrm{gr}(H)$ is isomorphic to a large sub-Hopf algebra of some graded Hopf structure on kQ determined by the kG-Hopf bimodule H_1/H_0. Note that $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is a graded coquasitriangular Hopf algebra with $\mathrm{gr}(\mathcal{R})$ concentrating at degree zero. By the same argument as in the proof of Theorem 3.3, we can show that the kG-Hopf bimodule H_1/H_0 is completely determined by the bicharacter $\mathrm{gr}(\mathcal{R})$ and so the associated graded Hopf structure on kQ is coquasitriangular. Let $\mathfrak{R}$ be the trivial extension of $\mathrm{gr}(\mathcal{R})$ to the whole $kQ \otimes kQ$, then $(kQ, \mathfrak{R})$ is coquasitriangular and apparently $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is isomorphic to a large sub-structure of $(kQ, \mathfrak{R})$. This completes the proof. □

4.3. According to a generalization of the Cartier–Gabriel decomposition theorem due to Montgomery [23, Theorem 3.2], the study of general coquasitriangular pointed Hopf algebras can be reduced to the connected case, namely their quivers (in the sense of Theorem 4.2) are connected. This can also be seen intuitively by the fact that general Hopf quivers consisting of copies of identical connected components.

For a coquasitriangular pointed Hopf algebra $(H, \mathcal{R})$, let $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R})) \hookrightarrow (kQ, \mathfrak{R})$ be an embedding as in Theorem 4.2. By $\mathrm{gr}(H)(\epsilon)$ we denote the image in the connected component $kQ(\epsilon)$. This is the principal block of $\mathrm{gr}(H)$ which is a sub-coquasitriangular Hopf algebra with structure $\mathrm{gr}(\mathcal{R})(\epsilon)$ obtained by the obvious restriction. The other blocks can be obtained by multiplying group-like elements. Then $\mathrm{gr}(H)$ can be recovered as a crossed product of $\mathrm{gr}(H)(\epsilon)$ by a group algebra. The coquasitriangular structure can also be recovered by extending that of $\mathrm{gr}(H)(\epsilon)$ via the formulae (2.5)–(2.6).

Now we have the following directly from Theorem 4.2 and Corollary 3.5.

Corollary 4.3. *Let $(H, \mathcal{R})$ be a coquasitriangular pointed Hopf algebra and $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ its degeneration as before. Then $(\mathrm{gr}(H)(\epsilon), \mathrm{gr}(\mathcal{R})(\epsilon))$ is cotriangular and $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is a crossed product of $(\mathrm{gr}(H)(\epsilon), \mathrm{gr}(\mathcal{R})(\epsilon))$ by a group algebra. In particular, if the quiver of H is connected, then $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is cotriangular.*

It is interesting to note that coquasitriangular pointed Hopf algebras can always be obtained as lifting of cotriangular ones and crossed product by group algebras if disconnected.

4.4. We include briefly a dual quiver setting for quasitriangular elementary Hopf algebras. In this case, we should use the path algebras of finite quivers (i.e., the set consisting of vertices and arrows is finite) and their admissible quotients to construct algebras.

Let Q be a finite quiver. The associated path algebra, denoted by kQ^a, has the same underlying vector space as the path coalgebra and the multiplication is defined by concatenation of paths. In fact, the path algebra kQ^a is the graded dual of the path coalgebra kQ. The path algebra kQ^a admits a graded Hopf structure if and only if Q is a Hopf quiver, see [10,15].

For a quasitriangular elementary Hopf algebra (H, R), we can consider the graded version $(\mathrm{gr}(H), \mathrm{gr}(R))$ induced by its chain of Jacobson radical. With this, a quiver setting for such class of Hopf algebras is given in the following. We state the dual of Proposition 3.1, Theorems 3.3 and 4.2, Corollary 4.3 without further explanation.

Theorem 4.4. *Let Q be a finite quiver.*

(1) *Then the path algebra kQ^a admits a graded quasitriangular Hopf structure if and only if Q is the Hopf quiver $Q(G, S)$ of some abelian group G with respect to a ramification datum S.*
(2) *Assume that $Q = Q(G, S)$ is a connected Hopf quiver with G abelian. Then the set of graded quasitriangular Hopf algebras (kQ^a, R) with $kQ_0^a = (kG)^*$ and $R \in kQ_0^a \otimes kQ_0^a$ is in one-to-one correspondence with the set of skew-symmetric bicharacters of G.*

Theorem 4.5. *Let (H, R) be a quasitriangular elementary Hopf algebra and $(\mathrm{gr}(H), \mathrm{gr}(R))$ its radically graded version. Then there exists a unique Hopf quiver $Q(H) = Q(G, S)$ with G abelian such that $\mathrm{gr}(H) \cong kQ(H)^a/I$ as graded quasitriangular Hopf algebras, where I is an admissible Hopf ideal.*

Corollary 4.6. *Let (H, R) be a quasitriangular elementary Hopf algebra and $(\mathrm{gr}(H), \mathrm{gr}(R))$ its radically graded version. Then the principal block of $\mathrm{gr}(H)$ is a quotient Hopf algebra and is triangular. The quasitriangular Hopf algebra $(\mathrm{gr}(H), \mathrm{gr}(R))$ can be presented as a crossed product of its principal block by the dual of a group algebra in the sense of Schneider* (*see Theorem* 2.2 *in* [29]).

5. Examples

In this section, we construct some examples via the quiver setting. For the convenience of the exposition, we assume in this section that the ground field k is the field of complex numbers.

5.1. Let $\mathbb{Z}_n = \langle g \mid g^n = 1\rangle$ denote the finite cyclic group of order n. Consider the (coquasitriangular) Hopf quiver $Q(\mathbb{Z}_n, g)$. It is a basic cycle of length n. For each integer i modulo n, let a_i denote the arrow $g^i \to g^{i+1}$. Let $\mathcal{R}$ be a skew-symmetric bicharacter of $\mathbb{Z}_n$. Then it is completely determined by the value $\lambda = \mathcal{R}(g, g)$ which should satisfy the equation

$$\lambda^2 = 1$$

by skew-symmetry. In general, we have $\mathcal{R}(g^i, g^j) = \lambda^{ij}$.

If n is odd, then λ must be 1. Therefore, there is only one skew-symmetric bicharacter, and hence only one graded coquasitriangular Hopf structure concentrating at degree 0 which is isomorphic to $k[x] \otimes k\mathbb{Z}_n$ as algebra.

If n is even, then $\lambda = \pm 1$. The case when $\lambda = 1$ is known. When $\lambda = -1$, then we can associate to the bicharacter a $k\mathbb{Z}_n$-Hopf bimodule on $kQ(\mathbb{Z}_n, g)_1$ as follows:

$$g a_i = a_{i+1}, \qquad a_i g = -a_{i+1}$$

for all integer i modulo n. This gives rise to a cotriangular Hopf algebra on $kQ(\mathbb{Z}_n, g)$ with path multiplication given by

$$p_i^l p_j^m = \begin{cases} 0, & \text{if } l, m \text{ are odd;} \\ (-1)^{im} p_{i+j}^{l+m}, & \text{otherwise.} \end{cases} \tag{5.1}$$

Here the notation p_i^l means the path of length l with source g^i. In particular, if we consider the sub-Hopf algebra generated by vertices and arrows, we get the generalized Taft algebra $C_2(n, -1)$ as denoted in [7]. It can be presented by generators g and x with relations

$$g^n = 1, \qquad x^2 = 0, \qquad gx = -xg.$$

Note that if $n = 2$ then it is exactly the well-known Sweedler's 4-dimensional Hopf algebra. Such Hopf algebras appeared in the works of Radford [26,27] in which their quasitriangular structures were classified.

Now we determine all the coquasitriangular structures on $C_2(n, -1)$. Clearly it has $\{g^i x^l \mid i = 0, 1, \dots n-1,\ l = 0, 1\}$ as a basis. Let $\mathcal{R}$ be a coquasitriangular structure. Then we already know $\mathcal{R}(g^i, g^j) = (-1)^{ij}$. By

$$0 = \mathcal{R}\big(g, x^2\big) = \mathcal{R}(g, x)\mathcal{R}(g, x)$$

we have $\mathcal{R}(g, x) = 0$. Then by (2.5)–(2.6) it follows that $\mathcal{R}(g^i, g^j x) = 0$. Similarly, we have $\mathcal{R}(g^i x, g^j) = 0$. By using (2.8) we have

$$0 = x^2 = \mathcal{R}(x, x)1 - \bar{\mathcal{R}}(x, x)g^2.$$

On the other hand, by applying $\mathcal{R}\bar{\mathcal{R}} = \varepsilon \otimes \varepsilon$ to (x, x), we can show that $\mathcal{R}(x, x) = \bar{\mathcal{R}}(x, x)$. That means,

$$\mathcal{R}(x, x)\big(1 - g^2\big) = 0.$$

This forces $\mathcal{R}(x,x)=0$ if $n>2$. Similarly by (2.5)–(2.6) we have in this case $\mathcal{R}(g^i x, g^j x)=0$. That is, the coquasitriangular structure on $C_2(n,-1)$ $(n>2)$ is unique which concentrates at degree 0. If $n=2$, then $\mathcal{R}(x,x)$ can take any value. Assume $\mathcal{R}(x,x)=\nu$. Then by (2.5)–(2.6) it is direct to deduce that

$$\mathcal{R}(gx,x)=-\nu, \qquad \mathcal{R}(x,gx)=\nu, \qquad \mathcal{R}(gx,gx)=\nu.$$

Of course this is known from Radford's calculation [26] by the fact that the Sweedler's 4-dimensional Hopf algebra is self-dual. By direct calculation, it is easy to show that such coquasitriangular structures of $C_2(2,-1)$ can be extended trivially, i.e. set $\mathcal{R}(u,v)=0$ if one of the homogeneous element u, v has degree $\geqslant 2$, to be those of the graded Hopf algebra on the whole $kQ(\mathbb{Z}_2, g)$. These are the promised examples of coquasitriangular structures not concentrating at degree 0 on path Hopf algebras.

We remark that the dual of $C_2(n,-1)$ can be presented by $kQ(\mathbb{Z}_n, g)^a/J^2$ where J is the ideal generated by the set of arrows. Therefore the elementary Hopf algebra $kQ(\mathbb{Z}_n, g)^a/J^2$ $(n>2)$ has a unique quasitriangular structure, while $kQ(\mathbb{Z}_2, g)^a/J^2$ has a 1-parameter family of quasitriangular structures. This approach via path algebras was used previously by Cibils in [9] for the quiver $Q(\mathbb{Z}_n, g)$.

5.2. Now we consider the infinite cyclic group $Z=\langle g\rangle$ and the Hopf quiver $Q(Z,g)$ which is a linear chain. The group Z has two skew-symmetric bicharacters, namely, $\mathcal{R}(g^i, g^j)=(\pm 1)^{ij}$. For the trivial bicharacter, the corresponding coquasitriangular Hopf algebra on $kQ(Z,g)$ is isomorphic to $k[x]\otimes kZ$ as algebra. For the non-trivial bicharacter, the coquasitriangular Hopf algebra on $kQ(Z,g)$ has multiplication formula similar to (5.1). The sub-Hopf algebra generated by vertices and arrows can be presented by generators g, g^{-1} and x with relations

$$gg^{-1}=1=g^{-1}g, \qquad x^2=0, \qquad gx=-xg.$$

By arguments in the same manner as Subsection 5.1, one can show that it has a unique coquasitriangular structure concentrating at degree 0.

As a direct consequence of the previous two examples, we can see that quantized enveloping algebras and small quantum groups are not coquasitriangular since their sub-algebras $U_q(sl_2)^{\geqslant 0}$ and $u_q(sl_2)^{\geqslant 0}$ have $Q(Z,g)$ and $Q(\mathbb{Z}_n, g)$ respectively as their quivers (see, for instance, [21,17]) but with different algebra structures from the coquasitriangular ones we have just classified.

5.3. Note that basic cycles and the linear chain are minimal connected Hopf quivers and are basic ingredients of the general. The previous examples classify all the possible coquasitriangular structures on these building blocks of general coquasitriangular Hopf quivers. Next we will consider those examples which are compatible gluing of them.

Firstly we consider the Hopf quiver $Q=Q(\mathbb{Z}_n, mg)$. It is a multi-cycle. To avoid the trivial case, we assume $n=2l$, $m>1$ and $\mathcal{R}$ is the non-trivial skew-symmetric bicharacter of $\mathbb{Z}_n$. Let x_i $(i=1,\dots,m)$ denote the arrows with source 1. Then we have the following multiplication formulae in $(kQ,\mathcal{R})$:

$$x_i^2=0, \qquad x_i x_j=-x_j x_i, \qquad gx_i=-x_i g.$$

The sub-Hopf algebra, denoted by $E(n,m)$, generated by vertices and arrows can be presented by generators g and x_i $(i=1,\dots,m)$ with additional relations $g^n=1$. Therefore the algebra has a PBW type basis

$$\left\{g^i x_1^{\sigma_1}\cdots x_m^{\sigma_m} \;\middle|\; i=0,1,\dots,n-1;\ \sigma_j=0,1\right\}.$$

It is not difficult to determine all the coquasitriangular structures. Let $\mathcal{R}$ be a coquasitriangular structure. As in Subsection 5.1, we have

$$\mathcal{R}(g^i, g^j) = (-1)^{ij}, \qquad \mathcal{R}(g^i, g^j x_k) = 0 = \mathcal{R}(g^i x_k, g^j)$$

and

$$\mathcal{R}(x_i, x_j)(1 - g^2) = 0.$$

If $n > 2$, then we have $\mathcal{R}(x_i, x_j) = 0$, and further $\mathcal{R}(g^u x_i, g^v x_j) = 0$ for all i and j. Therefore $E(n, m)$ has only one coquasitriangular structure which is the trivial extension of the bicharacter. If $n = 2$, then the Hopf algebra $E(2, m)$ is known as Nichols' Hopf algebra [24] and its set of quasitriangular structures was classified by Panaite and Van Oystaeyen in [25]. By the previous equations we can show that its set of coquasitriangular structures is in one-to-one correspondence with the set of the matrices $(\mathcal{R}(x_i, x_j))_{m\times m}$. By the fact that $E(2, m)$ is self-dual, this coincides with the result of Panaite and Van Oystaeyen.

5.4. Finally we consider the Hopf quiver $Q = Q(G, g + h)$ where G is the abelian group $\langle g\rangle \times \langle h\rangle$. Assume that the order of g and h are m and n respectively. A skew-symmetric bicharacter $\mathcal{R}$ of G is determined by three roots of unity $\mathcal{R}(g, g), \mathcal{R}(h, h), \mathcal{R}(g, h)$. As before, we have the equation

$$\mathcal{R}(g, g)^2 = 1 = \mathcal{R}(h, h)^2.$$

Let (m, n) denote the greatest common divisor of m and n. Then the order of $\mathcal{R}(g, h)$ in the multiplicative group $k^* = k \setminus \{0\}$ should satisfy

$$\operatorname{ord}\mathcal{R}(g, h) | (m, n).$$

In order to have more interesting examples, we assume further that m and n are even. Take the bicharacter $\mathcal{R}$ is such that

$$\mathcal{R}(g, g) = -1 = \mathcal{R}(h, h).$$

Let q denote $\mathcal{R}(g, h)$. Consider the sub-coquasitriangular Hopf algebra, denoted by $H(m, n, q)$, of $(kQ, \mathcal{R})$ generated by the set of vertices and arrows. Let $x : 1 \to g$ and $y : 1 \to h$ be the arrows with source 1. Then $H(m, n, q)$ is generated by g, h, x, y satisfying the following relations

$$g^m = 1 = h^n, \qquad x^2 = 0 = y^2, \qquad gx = -xg, \qquad hy = -yh,$$
$$hx = qxh, \qquad yg = qgy, \qquad yx = qxy.$$

As before, it is not hard to determine the complete list of coquasitriangular structures on $H(m, n, q)$. We do not repeat the details.

6. Classification of finite-dimensional coquasitriangular pointed Hopf algebras over an algebraically closed field of characteristic 0

From now on our ground field k is assumed to be algebraically close of characteristic 0. The aim of this section is to give a complete classification of finite-dimensional coquasitriangular pointed Hopf algebras over k.

6.1. Firstly we consider the generation problem of finite-dimensional coquasitriangular pointed Hopf algebras. In [1] Andruskiewitsch, Etingof and Gelaki proved that finite-dimensional cotriangular pointed Hopf algebras are generated by their group-like and skew-primitive elements. It turns out that, with our observation Corollary 4.3, one can extend their result to more general case in a fairly straightforward manner.

Proposition 6.1. *Suppose that H is a finite-dimensional coquasitriangular pointed Hopf algebra over k. Then as an algebra H is generated by its group-like and skew-primitive elements.*

Proof. To prove that H is generated by its group-like and skew-primitive elements, it suffices to prove this is the case for its coradically graded version $\mathrm{gr}\,H$ by Lemma 2.2 of [2]. According to Corollary 4.3, the Hopf algebra $\mathrm{gr}\,H$ is coquasitriangular and is a crossed product of a cotriangular one by a group algebra. Note that the cotriangular one is the principal block of $\mathrm{gr}\,H$, while other blocks are obtained by multiplying group-like elements. Now by applying the theorem of Andruskiewitsch, Etingof and Gelaki [1, Theorem 6.1], we can say that $\mathrm{gr}\,H$ is generated by its group-likes and the skew-primitives of its principal block. The proposition follows immediately from this. The proof is completed. □

6.2. Next we study a general finite-dimensional coquasitriangular pointed Hopf algebras $(H, \mathcal{R})$ via its quiver setting. We may assume without loss of generality that H is connected. Let G denote its group of group-like elements. The quiver Q of H is assumed to be $Q(G, R)$ with ramification datum $R = \sum_{i=1}^{t} R_i g_i$. Here the R_i's are assumed to be positive integers if $t \geqslant 1$.

Proposition 6.2. *Keep the above assumptions and notations. If G is the unit group $\{1\}$, then $H \cong k$. When $G \neq \{1\}$, we have*:

(1) *$t \geqslant 1$ and the order of g_i is even for all i. In particular the order of G is even.*
(2) *$\mathcal{R}$ is a skew-symmetric bicharacter of the group G (by restriction) and $\mathcal{R}(g_i, g_i) = -1$ for all i.*
(3) *$(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is isomorphic to the sub-coquasitriangular Hopf algebra of $(kQ, \mathfrak{R})$ generated by vertices and arrows, where $\mathfrak{R}$ is obtained by trivial extension of the bicharacter $\mathcal{R}$ of G.*

Proof. By Theorem 4.2, we can view $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ as a large sub-coquasitriangular Hopf algebra of $(kQ, \mathfrak{R})$. Therefore $\mathrm{gr}(H)$ contains the sub-Hopf algebra generated by vertices and arrows of Q.

If $G = \{1\}$, and suppose that there was an arrow x in Q. Then by the multiplication formula (2.10) it is easy to verify that the sub-Hopf algebra generated by x is actually isomorphic to the polynomial algebra in one variable which is of course infinite-dimensional. This is absurd since we assume the dimension of H, hence of $\mathrm{gr}(H)$, is finite. Therefore the quiver Q is a single vertex and so H must be k.

In the following we assume that $G \neq \{1\}$. Since the quiver Q is assumed to be connected, there are arrows and therefore the ramification datum R is not 0. It follows that $t \geqslant 1$. By Corollary 4.3, $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ is cotriangular, therefore $\mathrm{gr}(\mathcal{R})$ is a skew-symmetric bicharacter of G which is the restriction of $\mathcal{R}$. This implies that, by restriction, $\mathcal{R}$ is a skew-symmetric character of G. By the assumption $R = \sum_{i=1}^{t} R_i g_i$, we know that the quiver Q contains the sub-Hopf quivers $Q(\langle g_i \rangle, g_i)$ which are basic cycles. By Subsection 5.1, the sub-coquasitriangular Hopf algebra of $kQ(\langle g_i \rangle, g_i)$ generated by vertices and arrows is finite-dimensional if and only if $\mathcal{R}(g_i, g_i) = -1$. This forces that the order of g_i must be even. It is immediate by Lagrange's theorem that the order of G is even. Now we have proved (1) and (2). For (3), just note that the space of $\mathrm{gr}(H)$ spanned by group-likes and skew-primitives is equivalent to the space spanned by vertices and arrows of Q. Now by Proposition 6.1 $\mathrm{gr}(H)$ is generated by its group-like and skew-primitive elements, hence it is contained in the sub-algebra of kQ generated by vertices and arrows. Along with the first paragraph, we have (3). We are done. □

6.3. As a direct consequence of this, we give an explicit description for coradically graded finite-dimensional coquasitriangular pointed Hopf algebras via generators and defining relations.

Corollary 6.3. *Let H be a coradically graded finite-dimensional pointed Hopf algebra with coquasitriangular structure $\mathcal{R}$. Assume that it is connected and its group of group-likes G is not the unit group. Then there exists a generating set $\{g_i \mid i = 1, \dots, t\}$ of the group G and for each g_i there is a set of $(1, g_i)$-primitive elements $\{x_{i,i'} \mid i' = 1, \dots, R_i\}$ such that H is generated by $\{g_i, x_{i,i'} \mid i = 1, \dots, t;\ i' = 1, \dots, R_i\}$ with defining relations*

$$\text{the defining relations of } G \text{ by the } g_i's, \tag{6.1}$$

$$x_{j,j'} g_i = \mathcal{R}(g_i, g_j) g_i x_{j,j'}, \quad \forall i, j, j', \tag{6.2}$$

$$x_{i,i'}^2 = 0, \quad \forall i, i', \tag{6.3}$$

$$x_{j,j'} x_{i,i'} = \mathcal{R}(g_i, g_j) x_{i,i'} x_{j,j'}, \quad \forall i, i', j, j'. \tag{6.4}$$

Moreover, H has a PBW type basis

$$\left\{ g x_{1,1}^{\sigma_{1,1}} \cdots x_{1,R_1}^{\sigma_{1,R_1}} \cdots x_{t,1}^{\sigma_{t,1}} \cdots x_{t,R_t}^{\sigma_{t,R_t}} \mid g \in G,\ \sigma_{i,i'} = 0, 1 \right\}.$$

Proof. Note that Proposition 3.4 can be established for our case directly, and so there is no harm to assume that $\mathcal{R}$ concentrates at degree 0. By Proposition 6.2, there is a Hopf quiver $Q = Q(G, R)$ with $R = \sum_{i=1}^t R_i g_i$ such that H is isomorphic to the sub-Hopf algebra of the coquasitriangular Hopf algebra kQ, determined by the bicharacter $\mathcal{R}$ as in Theorem 3.3, generated by vertices and arrows. Denote by $\{x_{i,i'} \mid i' = 1, \dots, R_i\}$ the set of arrows with source 1 and target g_i which are $(1, g_i)$-primitive elements. Since any arrow of Q can be obtained as the ones with source 1 multiplied by group-like elements, hence H is generated by $\{g_i, x_{i,i'} \mid i = 1, \dots, t;\ i' = 1, \dots, R_i\}$. Just as the arguments in Section 5, it is easy to verify that they satisfy the relations (6.1)–(6.4). So as a linear space H is spanned by the set

$$\left\{ g x_{1,1}^{\sigma_{1,1}} \cdots x_{1,R_1}^{\sigma_{1,R_1}} \cdots x_{t,1}^{\sigma_{t,1}} \cdots x_{t,R_t}^{\sigma_{t,R_t}} \mid g \in G,\ \sigma_{i,i'} = 0, 1 \right\}.$$

Note that this set is linearly independent. This can be verified by straightforward computation using the multiplication formula (2.10). On the other hand, by a standard application of Bergman's diamond lemma [6], the algebra generated by $\{g_i, x_{i,i'} \mid i = 1, \dots, t;\ i' = 1, \dots, R_i\}$ with defining relations (6.1)–(6.4) has the aforementioned set as a basis. It follows that (6.1)–(6.4) are sufficient defining relations for H. Now the proof is finished. □

6.4. With a help of the lifting theorem of Andruskiewitsch and Schneider [2, Theorem 5.5] (see also [4]), now we are ready to classify finite-dimensional coquasitriangular pointed Hopf algebras.

Theorem 6.4. *Let $(H, \mathcal{R})$ be a finite-dimensional coquasitriangular pointed Hopf algebra. Assume that it is connected and its group of group-likes G is not the unit group. Then there exist a generating set $\{g_i \mid i = 1, \dots, t\}$ of the group G and for each g_i a set of $(1, g_i)$-primitive elements $\{x_{i,i'} \mid i' = 1, \dots, R_i\}$ such that H is generated by $\{g_i, x_{i,i'} \mid i = 1, \dots, t;\ i' = 1, \dots, R_i\}$ with defining relations*

$$\text{the defining relations of } G \text{ by the } g_i's, \tag{6.5}$$

$$x_{j,j'} g_i = \mathcal{R}(g_i, g_j) g_i x_{j,j'}, \quad \forall i, j, j', \tag{6.6}$$

$$x_{i,i'}^2 = \mu_{i,i'}\left(1 - g_i^2\right), \quad \forall i, i', \tag{6.7}$$

$$x_{j,j'} x_{i,i'} - \mathcal{R}(g_i, g_j) x_{i,i'} x_{j,j'} = \lambda_{i,i',j,j'}(1 - g_i g_j), \quad \forall \left(i, i'\right) \neq \left(j, j'\right). \tag{6.8}$$

Here $\mu_{i,i'}$ and $\lambda_{i,i',j,j'}$ are some appropriate constants in the field k. Moreover, H has a PBW type basis

$$\left\{ g x_{1,1}^{\sigma_{1,1}} \cdots x_{1,R_1}^{\sigma_{1,R_1}} \cdots x_{t,1}^{\sigma_{t,1}} \cdots x_{t,R_t}^{\sigma_{t,R_t}} \mid g \in G,\ \sigma_{i,i'} = 0, 1 \right\}.$$

Proof. By assumption $(\mathrm{gr}(H), \mathrm{gr}(\mathcal{R}))$ satisfies the condition of Corollary 6.3, hence it can be presented by generators with relations as given there. Now the theorem follows from the lifting theorem of Andruskiewitsch and Schneider. For the convenience of the reader, we include a detailed proof.

Clearly, the coradical H_0 is the sub-Hopf algebra generated by $\{g_i \mid i = 1, \dots, t\}$. By Lemma 6.1 in [4], there are elements $\{x_{i,i'} \mid i' = 1, \dots, R_i\} \in H$ corresponding to $\{x_{i,i'} \mid i' = 1, \dots, R_i\} \in \mathrm{gr}(H)$ and they satisfy

$$\Delta(x_{i,i'}) = g_i \otimes x_{i,i'} + x_{i,i'} \otimes 1, \qquad x_{j,j'} g_i = \mathcal{R}(g_i, g_j) g_i x_{j,j'}$$

for any i, i', j, j'. Since $\mathrm{gr}(H)$ is generated by group-like elements and skew-primitive elements, we know H is indeed generated by $\{g_i, x_{i,i'} \mid i = 1, \dots, t;\ i' = 1, \dots, R_i\}$. For any i, i', j, j', direct computations show that the element $x_{i,i'}^2$ is a $(1, g_i^2)$-primitive element and $x_{j,j'} x_{i,i'} - \mathcal{R}(g_i, g_j) x_{i,i'} x_{j,j'}$ is a $(1, g_i g_j)$-primitive element.

Claim. *For any $i, j \in \{1, \dots, t\}$, there are no non-trivial $(1, g_i^2)$-primitive elements and no non-trivial $(1, g_i g_j)$-primitive elements.*

Here the trivial $(1, g)$-primitive elements are defined as the elements belonging to the space spanned by $1 - g$. Otherwise, assume y_i is a non-trivial $(1, g_i^2)$-primitive element. By $g_i x_{i,i'} g_i^{-1} = \mathcal{R}(g_i, g_i) x_{i,i'} = -x_{i,i'}$, $g_i^2 x_{i,i'} g_i^{-2} = \mathcal{R}(g_i, g_i^2) x_{i,i'} = x_{i,i'}$. Thus $\mathcal{R}(g_i, g_i^2) = 1$ and so

$$\mathcal{R}\big(g_i^2, g_i\big)\mathcal{R}\big(g_i^2, g_i\big) = \mathcal{R}\big(g_i^2, g_i^2\big) = 1.$$

Note that we always have $g_i^2 y_i g_i^{-2} = \mathcal{R}(g_i^2, g_i^2) y_i = -y_i$ which implies $\mathcal{R}(g_i^2, g_i^2) = 1$. It's a contradiction. Similarly, assume that z is a non-trivial $(1, g_i g_j)$-primitive element, then

$$\begin{aligned} g_i g_j z (g_i g_j)^{-1} &= \mathcal{R}(g_i g_j, g_j) \mathcal{R}(g_i g_j, g_i) z \\ &= \mathcal{R}(g_j, g_j) \mathcal{R}(g_i, g_i) \mathcal{R}(g_i, g_j) \mathcal{R}(g_j, g_i) z \\ &= (-1)^2 z = z. \end{aligned}$$

Here we have used the fact that $\mathcal{R}$ is skew-symmetric. Thus this also implies that $\mathcal{R}(g_i g_j, g_i g_j) = 1$. This contradicts to $\mathcal{R}(g_i g_j, g_i g_j) = -1$ by Proposition 6.2(2).

Therefore, by the claim above, there are $\mu_{i,i'}, \lambda_{i,i',j,j'} \in k$ such that

$$x_{i,i'}^2 = \mu_{i,i'}\big(1 - g_i^2\big), \qquad x_{j,j'} x_{i,i'} - \mathcal{R}(g_i, g_j) x_{i,i'} x_{j,j'} = \lambda_{i,i',j,j'}(1 - g_i g_j).$$

Consider a Hopf algebra A which is generated by group-like elements $\{g_i \mid i = 1, \dots, t\}$ and for each g_i a set of $(1, g_i)$-primitive elements $\{x_{i,i'} \mid i' = 1, \dots, R_i\}$ and assume it satisfies the relations (6.5)–(6.8). The preceding discussions imply that there is a surjective Hopf algebra map from A to H. By comparing the dimensions, this surjective map is indeed an isomorphism. □

We remark that, by carrying out the same computational process as the examples in Subsections 5.3 and 5.4, it is not difficult to determine all the possible coquasitriangular structures for any given coquasitriangular pointed Hopf algebra as classified in the previous theorem. This is left for the interested reader.

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