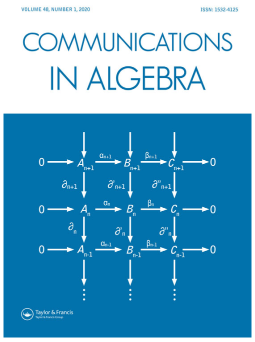

Communications in Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

On the quasitriangular structures of abelian extensions of $\mathbb{Z}_2$

Kun Zhou & Gongxiang Liu

To cite this article: Kun Zhou & Gongxiang Liu (2021): On the quasitriangular structures of abelian extensions of $\mathbb{Z}_2$, Communications in Algebra, DOI: 10.1080/00927872.2021.1929274

To link to this article: https://doi.org/10.1080/00927872.2021.1929274

Published online: 10 Jun 2021.

COMMUNICATIONS IN ALGEBRA®
https://doi.org/10.1080/00927872.2021.1929274

On the quasitriangular structures of abelian extensions of $\mathbb{Z}_2$

Kun Zhou and Gongxiang Liu

Department of Mathematics, Nanjing University, Nanjing, China

ABSTRACT
The aim of this paper is to study quasitriangular structures on a class of semisimple Hopf algebras $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ constructed through abelian extensions of $\Bbbk\mathbb{Z}_2$ by $\Bbbk^G$ for an abelian group G. We prove that there are only two forms of them and we get a complete list of all universal $\mathcal{R}$-matrices of the generalized Kac-Paljutkin algebra H_{2n^2} (see Section 2 for the definition).

ARTICLE HISTORY
Received 17 December 2020
Revised 1 May 2021
Communicated by V. A. Artamonov

KEYWORDS
Abelian extension; quasitriangular Hopf algebra

2020 MATHEMATICS SUBJECT CLASSIFICATION
16T05 (primary); 16T25 (secondary)

1. Introduction

In this paper and subsequent works, we try to determine possible quasitriangular structures on a class of semisimple Hopf algebras arising from exact factorizations of finite groups. The well-known eight-dimensional Kac-Paljutkin algebra K_8 is a very special case of them. The idea of constructing these semisimple Hopf algebras can be tracked back to G. Kac [2]: Suppose that $L = G\Gamma$ is an exact factorization of the finite group L, into its subgroups G and Γ, such that $G \cap \Gamma = 1$. Associated to this exact factorization and appropriate cohomology data σ and τ, there is a semisimple bicrossed product Hopf algebra $H = \Bbbk^G\#_{\sigma,\tau}\Bbbk\Gamma$ (see Section 2 for the definition and [3–5] for details and generalizations). The question of existence of quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\Gamma$ has been considered before. In 2011, S. Natale [7] proved that if L is almost simple, then the extension admits no quasitriangular structure. She mainly used the conclusion that full fusion subcategories of $\mathrm{Rep}D^{\omega}(G)$ are determined in [6, Theorem 5.1].

But for our purpose, we want to find more concrete quasitriangular structures rather than absence of quasitriangular structures. So comparing the Natale's viewpoint, we consider the other extreme case: the almost commutative case. That is, we assume that both G and Γ are commutative groups. If we consider the method used in [7], then we will find that it does not apply to our problems. Because the approach in [7] is based on the fact that a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\Gamma$ determines a triple (K_1, K_2, B) where K_1, K_2 are subgroups of G and $B : K_1 \times K_2 \to \Bbbk^{\times}$ is a G-invariant ω-bicharacter in our case(see [7, Corollary 2.6]). But on the other hand, we don't know which triples exactly come from the quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\Gamma$. And even if we find out which triples come from the quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\Gamma$, the ω and the G-invariant ω-bicharacter B still need to be calculated. However, it is not easy. So we choose to

CONTACT Gongxiang Liu gxliu@nju.edu.cn Department of Mathematics, Nanjing University, Nanjing, China.

use a more direct method to determine quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\Gamma$. As the start point, we further assume that Γ is just the $\mathbb{Z}_2$ in this paper.

We find that there is a dichotomy on the forms of the quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ firstly. For convenience, we call one form trivial and the other form nontrivial. In principle, the trivial form corresponds to the bicharacters and thus is not very complicated. Secondly, we get some necessary conditions for the existence of nontrivial forms. As the main conclusion of this paper, all universal $\mathcal{R}$-matrices of Hopf algebras H_{2n^2} are given explicitly.

This paper is organized as follows. In Section 2, we recall the definition of a Hopf algebra $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and give an example of them. In Section 3, we show that there are only two possible forms of quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and give some necessary conditions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ preserving nontrivial quasitriangular structures. Using our necessary conditions, we can easily get all universal $\mathcal{R}$-matrices of Hopf algebras $H_{2n^2}(n>2)$.

Throughout the paper we work over an algebraically closed field $\Bbbk$ of characteristic 0. All Hopf algebras in this paper are finite dimensional. For the symbol δ in Section 2, we mean the classical Kronecker's symbol.

2. Abelian extensions of $\mathbb{Z}_2$

In this section, we recall the definition of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, and then we give an example of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ for guiding our further research.

2.1. The definition of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

Definition 2.1. A short exact sequence of Hopf algebras is a sequence of Hopf algebras and Hopf algebra maps

$$K \xrightarrow{\iota} H \xrightarrow{\pi} A \tag{2.1}$$

such that

(i) ι is injective,
(ii) π is surjective,
(iii) $\mathrm{Ker}(\pi)=HK^+$, K^+ is the kernel of the counit of K.

In this situation it is said that H is an extension of A by K [4, Definition 1.4]. An extension (2.1) above such that K is commutative and A is cocommutative is called *abelian*. In this paper, we only study the following special abelian extensions

$$\Bbbk^G \xrightarrow{\iota} A \xrightarrow{\pi} \Bbbk\mathbb{Z}_2,$$

where G is a finite abelian group. Abelian extensions were classified by Masuoka (see [4, Proposition 1.5]), and the above A can be expressed as $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ which is defined as follows.

Let $\mathbb{Z}_2=\{1,x\}$ be the cyclic group of order 2 and let G be a finite group. To give the description of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we need the following data

(i) $\triangleleft:\mathbb{Z}_2\to \mathrm{Aut}(G)$ is an injective group homomorphism.
(ii) $\sigma: G\to\Bbbk^\times$ is a map such that $\sigma(g\triangleleft x)=\sigma(g)$ for $g\in G$ and $\sigma(1)=1$.
(iii) $\tau:G\times G\to\Bbbk^\times$ is a unital 2-cocycle and satisfies that $\sigma(gh)\sigma(g)^{-1}\sigma(h)^{-1}=\tau(g,h)\tau(g\triangleleft x,h\triangleleft x)$ for $g,h\in G$.

The aim of (i) is to avoid making a commutative algebra (in such case all quasitriangular structures are given by bicharacters and thus is known).

Definition 2.2. [1, Section 2.2] As an algebra, the Hopf algebra $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is generated by $\{e_g\}_{g\in G}$ and x satisfying

$$e_g e_h = \delta_{g,h}e_g, xe_g = e_{g\triangleleft x}x,\ x^2 = \sum_{g\in G}\sigma(g)e_g, \quad g,h\in G.$$

The coproduct, counit and antipode are given by

$$\Delta(e_g) = \sum_{h,k\in G, hk=g} e_h\otimes e_k, \Delta(x) = \left[\sum_{g,h\in G}\tau(g,h)e_g\otimes e_h\right](x\otimes x),$$
$$\epsilon(x) = 1, \epsilon(e_g) = \delta_{g,1}1,$$
$$\mathcal{S}(x) = \sum_{g\in G}\sigma(g)^{-1}\tau(g,g^{-1})^{-1}e_{g\triangleleft x}x, \mathcal{S}(e_g) = e_{g^{-1}}, g\in G.$$

The following is an example of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and we will discuss it in next sections.

Example 2.3. Let $n\in\mathbb{N}$ and assume that w is a primitive nth root of 1 in $\Bbbk$. Then the generalized Kac-Paljutkin algebra H_{2n^2} [8, Section 2.2] belongs to $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. By definition, the data $(G, \triangleleft, \sigma, \tau)$ of H_{2n^2} is given by the following way

(i) $G = \mathbb{Z}_n\times\mathbb{Z}_n = \langle a, b|a^n = b^n = 1, ab = ba\rangle$ and $a\triangleleft x = b, b\triangleleft x = a$.
(ii) $\sigma(a^ib^j) = w^{ij}$ for $1\le i,j\le n$.
(iii) $\tau(a^ib^j, a^kb^l) = (w)^{jk}$ for $1\le i,j,k,l\le n$.

Among of them, if we take $n=2$ then the resulting Hopf algebra is just the well-known Kac-Paljutkin 8-dimensional algebra K_8. That's the reason why we call H_{2n^2} the generalized Kac-Paljutkin algebra.

3. Forms of universal $\mathcal{R}$-matrices

In this section, we will prove that for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ there are at most two forms of universal $\mathcal{R}$-matrices. Based on this observation, we determine all quasitriangular structures on generalized Kac-Paljutkin algebras H_{2n^2}(see Example 2.3).

3.1. Forms of universal $\mathcal{R}$-matrices

Recall that a quasitriangular Hopf algebra is a pair (H, R) where H is a Hopf algebra and $R = \sum R^{(1)}\otimes R^{(2)}$ is an invertible element in $H\otimes H$ such that

$$(\Delta\otimes\mathrm{Id})(R) = R_{13}R_{23}, (\mathrm{Id}\otimes\Delta)(R) = R_{13}R_{12}, \Delta^{op}(h)R = R\Delta(h),$$

for $h\in H$. Here by definition $R_{12} = \sum R^{(1)}\otimes R^{(2)}\otimes 1, R_{13} = \sum R^{(1)}\otimes 1\otimes R^{(2)}$ and $R_{23} = \sum 1\otimes R^{(1)}\otimes R^{(2)}$. The element R is called a universal $\mathcal{R}$-matrix of H or a quasitriangular structure on H.

To find the possible forms of universal $\mathcal{R}$-matrices, we need the following Lemmas 3.1–3.3 which will help us to check the braiding conditions. The first lemma is well-known.

Lemma 3.1. [15, Proposition 12.2.11] *Let H be a Hopf algebra and $R\in H\otimes H$. For $f\in H^*$, if we denote $l(f) := (f\otimes\mathrm{Id})(R)$ and $r(f) := (\mathrm{Id}\otimes f)(R)$, then the following statements are equivalent*

(i) $(\Delta\otimes\mathrm{Id})(R) = R_{13}R_{23}$ *and* $(\mathrm{Id}\otimes\Delta)(R) = R_{13}R_{12}$.
(ii) $l(f_1)l(f_2) = l(f_1f_2)$ *and* $r(f_1)r(f_2) = r(f_2f_1)$ *for* $f_1,f_2\in H^*$.

Lemma 3.2. *Denote the dual basis of $\{e_g, e_gx\}_{g\in G}$ by $\{E_g, X_g\}_{g\in G}$, that is, $E_g(e_h) = \delta_{g,h}, E_g(e_hx) = 0, X_g(e_h) = 0, X_g(e_hx) = \delta_{g,h}$ for $g, h \in G$. Then the following equations hold in the dual Hopf algebra $(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$:*

$$E_gE_h = E_{gh},\ \ E_gX_h = X_hE_g = 0,\ \ X_gX_h = \tau(g,h)X_{gh},\ \ g,h \in G.$$

Proof. Direct computations show that

$$E_gE_h(e_k) = E_{gh}(e_k) = \delta_{gh,k},\ \ E_gE_h(e_kx) = E_{gh}(e_kx) = 0$$

for $g, h, k \in G$. As a result, we have $E_gE_h = E_{gh}$. Similarly, one can get the last two equations. □

Let $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ as before. We need following two notions which will be used freely throughout this paper. Let

$$S := \{g \mid g \in G,\ g \triangleleft x = g\},\ \ T := \{g \mid g \in G,\ g \triangleleft x \neq g\}.$$

A very basic observation is:

Lemma 3.3. *We have $S \subseteq TT$ where $TT = \{gh \mid g, h \in T\}$.*

Proof. Clearly, for $s \in S, t \in T$, we have $ts \in T$. From the Definition 2.2 we know that the action $\triangleleft$ is injective, therefore $T \neq \emptyset$. Let $t \in T$ and it is obvious that $S = t(t^{-1}S)$ and hence $S \subseteq TT$. □

With the help of S, T, we find that

Lemma 3.4. *Let $w^1 : G\times G \to \Bbbk$, $w^2 : G\times G \to \Bbbk$, $w^3 : G\times G \to \Bbbk$, $w^4 : G\times G \to \Bbbk$ be four maps and define R as follows*

$$\begin{aligned} R := \sum_{g,h\in G} w^1(g,h)e_g \otimes e_h + \sum_{g,h\in G} w^2(g,h)e_gx \otimes e_h \\ + \sum_{g,h\in G} w^3(g,h)e_g \otimes e_hx + \sum_{g,h\in G} w^4(g,h)e_gx \otimes e_hx. \end{aligned}$$

If R satisfies $\Delta^{op}(e_g)R = R\Delta(e_g)$ for $g \in G$, then

(i) $w^2(t,g) = 0,\ \ t \in T, g \in G.$
(ii) $w^3(g,t) = 0,\ \ t \in T, g \in G.$
(iii) $w^4(s,t) = w^4(t,s) = 0,\ \ s \in S, t \in T.$

Proof. Because we have assumed that G is an abelian group, we get $\Delta^{op}(e_g) = \Delta(e_g)$. Since $\Delta^{op}(e_g)R = R\Delta(e_g)$ for $g \in G$ from the condition, we know $\Delta(e_g)R = R\Delta(e_g)$. Observe that $\{e_g, e_gx\}_{g\in G}$ is a linear basis for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and if we compare the two sides of the equation $\Delta(e_g)R = R\Delta(e_g)$ then we obtain the following equations

$$\Delta(e_g)\left[\sum_{h,k\in G} w^2(h,k)e_hx \otimes e_k\right] = \left[\sum_{h,k\in G} w^2(h,k)e_hx \otimes e_k\right]\Delta(e_g), \tag{3.1}$$

$$\Delta(e_g)\left[\sum_{h,k\in G} w^3(h,k)e_h \otimes e_kx\right] = \left[\sum_{h,k\in G} w^3(h,k)e_h \otimes e_kx\right]\Delta(e_g), \tag{3.2}$$

$$\Delta(e_g)\left[\sum_{h,k\in G} w^4(h,k)e_hx \otimes e_kx\right] = \left[\sum_{h,k\in G} w^4(h,k)e_hx \otimes e_kx\right]\Delta(e_g). \tag{3.3}$$

Firstly, we analyze Equation (3.1) as follows

$$\Delta(e_g)\left[\sum_{h,k\in G} w^2(h,k)e_hx\otimes e_k\right]=\sum_{\substack{h,k\in G\\ hk=g}} w^2(h,k)e_hx\otimes e_k, \tag{3.4}$$

$$\left[\sum_{h,k\in G} w^2(h,k)e_hx\otimes e_k\right]\Delta(e_g)=\sum_{\substack{h,k\in G\\ hk=g}} w^2(h\triangleleft x,k)e_{h\triangleleft x}x\otimes e_k. \tag{3.5}$$

Note that if $h\in T, k\in G$ such that $hk=g$, then $e_hx\otimes e_k$ will appear in (3.4) while not in (3.5). As a result, $w^2(h,k)=0$ for $h\in T, k\in G$ and thus (i) has been proved.

Similarly, for Equation (3.2), there are the following equations

$$\Delta(e_g)\left[\sum_{h,k\in G} w^3(h,k)e_h\otimes e_kx\right]=\sum_{\substack{h,k\in G\\ hk=g}} w^3(h,k)e_h\otimes e_kx, \tag{3.6}$$

$$\left[\sum_{h,k\in G} w^3(h,k)e_h\otimes e_kx\right]\Delta(e_g)=\sum_{\substack{h,k\in G\\ hk=g}} w^3(h,k\triangleleft x)e_h\otimes e_{k\triangleleft x}x. \tag{3.7}$$

Observe that if $h\in G, k\in T$ such that $hk=g$, then $e_h\otimes e_kx$ will appear in (3.6) while not in (3.7). Therefore $w^3(h,k)=0$ for $h\in G, k\in T$ and so (ii) is proved.

For Equation (3.3), we obtain the following equations

$$\Delta(e_g)\left[\sum_{h,k\in G} w^4(h,k)e_hx\otimes e_kx\right]=\sum_{\substack{h,k\in G\\ hk=g}} w^4(h,k)e_hx\otimes e_kx, \tag{3.8}$$

$$\left[\sum_{h,k\in G} w^4(h,k)e_hx\otimes e_kx\right]\Delta(e_g)=\sum_{\substack{h,k\in G\\ hk=g}} w^4(h\triangleleft x,k\triangleleft x)e_{h\triangleleft x}\otimes e_{k\triangleleft x}x. \tag{3.9}$$

Note that if $h\in S, k\in T$, then $e_hx\otimes e_kx$ and $e_kx\otimes e_hx$ will appear in (3.8) and not in (3.9). This implies that $w^4(h,k)=0$ for $h\in S, k\in T$. Similarly, one can find that $w^4(h,k)=0$ for $h\in T, k\in S$. Therefore (iii) has been proved. □

Lemma 3.5. *Let R be the element given in Lemma 3.4 and assume that $(\Delta\otimes\mathrm{Id})(R)=R_{13}R_{23}$, $(\mathrm{Id}\otimes\Delta)(R)=R_{13}R_{12}$. Then the following equations hold*

(i) $w^2(s_1,s_2)=w^3(s_1,s_2)=w^4(s_1,s_2)=0,\ s_1,s_2\in S.$
(ii) $w^1(g,t_2)w^4(t_1,t_2)=0,\ g\in G,\ t_1,t_2\in T.$
(iii) $w^1(t_1,g)w^4(t_1,t_2)=0,\ g\in G,\ t_1,t_2\in T.$

Proof. We have known $l(X_g)l(X_h)=l(X_gX_h)$ for $g,h\in G$ due to Lemma 3.1. Let $s\in S$ and we can find $t_1,t_2\in T$ such that $t_1t_2=s$ because of Lemma 3.3 and hence the following equation holds

$$l(X_{t_1}X_{t_2})=\tau(t_1,t_2)l(X_{t_1t_2})=\tau(t_1,t_2)\left[\sum_{g\in G} w^2(t_1t_2,g)e_g+\sum_{s\in S} w^4(t_1t_2,s)e_sx\right].$$

At the same time,

$$\begin{aligned} l(X_{t_1})l(X_{t_2}) &= \left(\sum_{t\in T} w^4(t_1,t)e_tx\right)\left(\sum_{t\in T} w^4(t_2,t)e_tx\right)\\ &=\sum_{t\in T} w^4(t_1,t)w^4(t_2,t\triangleleft x)e_tx^2\\ &=\sum_{t\in T} w^4(t_1,t)w^4(t_2,t\triangleleft x)\sigma(t)e_t. \end{aligned}$$

Since $l(X_{t_1})l(X_{t_2}) = l(X_{t_1}X_{t_2})$, we get that $w^4(s,s') = w^2(s,s') = 0$ for $s' \in S$ and thus $w^4(s,s') = w^2(s,s') = 0$ for $s,s' \in S$. Similarly by $r(X_{t_1})r(X_{t_2}) = r(X_{t_2}X_{t_1})$, one can get that $w^3(s,s') = 0$ for $s,s' \in S$. Therefore, (i) is proved.

It remains to show (ii) and (iii). We have known $l(E_g)l(X_{t_1}) = 0$ due to Lemma 3.2. However, a direct computation shows that $l(E_g)l(X_{t_1}) = \sum_{t\in T} w^1(g,t)w^4(t_1,t)e_tx$. Therefore $w^1(g,t)$ $w^4(t_1,t) = 0$ for $g \in G, t_1, t \in T$. Similarly, by $r(E_g)r(X_{t_1}) = 0$ we get that $w^1(t,g)w^4(t,t_1) = 0$ for $g \in G, t_1, t \in T$. These are exactly (ii), (iii). □

The following proposition shows that universal $\mathcal{R}$-matrices of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ has only two possible forms.

Proposition 3.6. *Let R be the element given in Lemma 3.4 and assume that it is a universal $\mathcal{R}$-matrix of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Then R must belong to one of the following two cases:*

(i) $R = \sum_{g,h\in G} w^1(g,h)e_g \otimes e_h$;

(ii) $R = \sum_{s_1,s_2\in S} w^1(s_1,s_2)e_{s_1} \otimes e_{s_2} + \sum_{s\in S, t\in T} w^2(s,t)e_sx \otimes e_t + \sum_{t\in T, s\in S} w^3(t,s)e_t \otimes e_sx + \sum_{t_1,t_2\in T} w^4(t_1,t_2)e_{t_1}x \otimes e_{t_2}x$.

Proof. Owing to Lemmas 3.4 and 3.5, we can assume that R has the following form:

$$R = \sum_{g,h\in G} w^1(g,h)e_g \otimes e_h + \sum_{s\in S, t\in T} w^2(s,t)e_sx \otimes e_t + \sum_{t\in T, s\in S} w^3(t,s)e_t \otimes e_sx + \sum_{t_1,t_2\in T} w^4(t_1,t_2)e_{t_1}x \otimes e_{t_2}x.$$

If $w^4(t_1,t_2) = 0$ for all $t_1,t_2 \in T$, then $l(X_{t_1}) = l(X_{t_2}) = 0$. Using Lemma 3.2 we know that $l(X_{t_1})l(X_{t_2}) = l(X_{t_1}X_{t_2})$ and as a result $l(X_{t_1}X_{t_2}) = 0$ for all $t_1,t_2 \in T$. For $s \in S$, we can take $t_1,t_2 \in T$ such that $s = t_1t_2$. Hence, we have that $l(X_{t_1}X_{t_2}) = \tau(t_1,t_2)(\sum_{t\in T} w^2(s,t)e_t) = 0$ which implies that $w^2(s,t) = 0$ for $s \in S$, $t \in T$. Similarly, by $r(X_{t_1}) = r(X_{t_2}) = 0$ and $r(X_{t_2}X_{t_1}) = \sum_{t\in T}\tau(t_2,t_1)w^3(t,s)e_t$, we have $w^3(t,s) = 0$ for $s \in S, t \in T$. Since $w^2(s,t) = w^3(t,s) = 0$ for $s \in S, t \in T$, we know that $R = \sum_{g,h\in G} w^1(g,h)e_g \otimes e_h$ and therefore we get the first case.

If there are $t_0, t_0' \in T$ such that $w^4(t_0,t_0') \neq 0$, then we will show that $w^1(t,g) = w^1(g,t) = 0$ for all $g \in G, t \in T$. For any $g \in G$, we have $w^1(g,t_0')w^4(t_0,t_0') = 0$ by (ii) of Lemma 3.5 and as a result $w^1(g,t_0') = 0$. Since R is invertible and $(e_t \otimes e_{t_0'})R = w^4(t,t_0')e_tx \otimes e_{t_0'}x$, we know that $w^4(t,t_0') \neq 0$ for $t \in T$. Next, we use (ii) and (iii) of Lemma 3.5 repeatedly. We have $w^1(t,g)w^4(t,t_0') = 0$ due to (iii) of Lemma 3.5. Thus $w^1(t,g) = 0$ for $t \in T, g \in G$. Since R is invertible and $(e_{t_1} \otimes e_{t_2})R = w^4(t_1,t_2)e_{t_1}x \otimes e_{t_2}x$ for $t_1,t_2 \in T$, we get that $w^4(t_1,t_2) \neq 0$ for $t_1,t_2 \in T$. Because $w^1(g,t)w^4(t_1,t) = 0$ by (ii) of Lemma 3.5, we know that $w^1(g,t) = 0$ for $g \in G, t \in T$ and hence we get the second case. □

Remark 3.7. For simple, we will call a universal $\mathcal{R}$-matrix R in case (i) (resp. case (ii)) of Proposition 3.6 by a trivial (resp. non-trivial) quasitriangular structure.

3.2. Universal $\mathcal{R}$-matrices of H_{2n^2}

To determine all universal $\mathcal{R}$-matrices of H_{2n^2}, we give necessary conditions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ preserving a non-trivial quasitriangular structure firstly. For any finite set X, we use $|X|$ to denote the number of elements in X.

Proposition 3.8. *If there is a non-trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, then*

(i) $|S| = |T|$;
(ii) *there is* $b \in S$ *such that* $b^2 = 1$ *and* $t \triangleleft x = tb$ *for* $t \in T$;
(iii) $|G| = 4m$ *for some* $m \in \mathbb{N}^+$;

Proof. Assume that R is a non-trivial quasitriangular structure on $\Bbbk^G \#_{\sigma,\tau} \Bbbk \mathbb{Z}_2$, then we have $l(E_{t_1})l(E_{t_2}) = \sum_{s \in S} w^3(t_1, s)w^3(t_2, s)\sigma(s)e_s$ for $t_1, t_2 \in T$. In this situation, we claim that $TT = S$. In fact, suppose that there are $t_1, t_2 \in T$ satisfying $t_1 t_2 \in T$. Then it is easy to see that $l(E_{t_1 t_2}) = \sum_{s \in S} w^3(t_1 t_2, s)e_s x$ which contradicts to the fact $l(E_{t_1})l(E_{t_2}) = l(E_{t_1 t_2})$ (Lemma 3.1). Thus, we have $TT = S$. Take a $t \in T$. We get that $tT \subseteq S$ and thus $|T| \leq |S|$. Since $tS \subseteq T$, $|T| \geq |S|$. As a result, we have $|T| = |S|$ and thus (i) has been proved. Next, we will show (ii). Take a $t_0 \in T$, then we have $T = t_0 S$. Let $t_0 \triangleleft x = t_1$ and denote $b = t_0^{-1} t_1$, then we have $b \in S$ by $TT = S$. Since $t_0 S \subseteq T$ and $(t_0 s) \triangleleft x = (t_0 s)b$, we have $t \triangleleft x = tb$ for $t \in T$. It is easy to know that $b^2 = 1$ since $\triangleleft x$ is a group automorphism with order 2 and thus (ii) has been proved. Now let's show (iii). By definition, S is a subgroup of G and $b^2 = 1$. Therefore, we know that $2 \mid |S|$. Since (HTML translation failed) and $|S| = |T|$, we can see that $|G| = 4m$ for some $m \in \mathbb{N}^+$. This completes the proof of (iii). □

Corollary 3.9. *If* $|G|$ *is an odd number or there are* $t_1, t_2 \in T$ *such that* $t_1^{-1}(t_1 \triangleleft x) \neq t_2^{-1}(t_2 \triangleleft x)$, *then* $\Bbbk^G \#_{\sigma,\tau} \Bbbk \mathbb{Z}_2$ *has no non-trivial quasitriangular structure.*

Proof. If $\Bbbk^G \#_{\sigma,\tau} \Bbbk \mathbb{Z}_2$ has a non-trivial quasitriangular structure, then we obtain that $|G|$ is an even number and there is $b \in S$ such that $t \triangleleft x = tb$ for $t \in T$ by (ii), (iii) of Proposition 3.8. Therefore $t^{-1}(t \triangleleft x) \equiv b$ for $t \in T$ and we have completed the proof. □

The following proposition determine all possible trivial quasitriangular structures.

Proposition 3.10. *R is a trivial quasitriangular structure on* $\Bbbk^G \#_{\sigma,\tau} \Bbbk \mathbb{Z}_2$ *if and only if*

(i) $R = \sum_{g,h \in G} w(g,h)e_g \otimes e_h$ *for some bicharacter* w *on* G;
(ii) $w(g \triangleleft x, h \triangleleft x) = w(g,h)\eta(g,h)$ *where* $\eta(g,h) = \tau(g,h)\tau(h,g)^{-1}$ *for* $g, h \in G$.

Proof. We can assume that $R = \sum_{g,h \in G} w(g,h)e_g \otimes e_h$ is a trivial quasitriangular structure on it. Owing to $(\Delta \otimes \mathrm{Id})(R) = R_{13}R_{23}$ and $(\mathrm{Id} \otimes \Delta)(R) = R_{13}R_{12}$, we know (i). Expanding $\Delta^{op}(x)R = R\Delta(x)$, one can get (ii). □

The following proposition is an application of above results and we get all universal $\mathcal{R}$-matrices of $H_{2n^2} (n \geq 3)$.

Proposition 3.11. *All universal* $\mathcal{R}$*-matrices of* $H_{2n^2} (n \geq 3)$ *are given by*

$$R = \sum_{1 \leq i,j,k,l \leq n} \alpha^{ik+jl} \beta^{il+jk} e_{a^i b^j} \otimes e_{a^k b^l}$$

for some $\alpha, \beta \in \Bbbk$ *satisfying* $\alpha^n = \beta^n = 1$.

Proof. Since $n \geq 3$, we know $a^{-1}(a \triangleleft x) \neq b^{-1}(b \triangleleft x)$. Therefore H_{2n^2} has no non-trivial quasitriangular structure by Corollary 3.9. Assume that $R = \sum_{g,h \in G} w(g,h)e_g \otimes e_h$ is a trivial quasitriangular structure on H_{2n^2}, then w is a bicharacter on G and it satisfies the following equations by Proposition 3.10

$$\begin{aligned} w(a,a)^n = 1, &\quad w(a,b)^n = 1, \\ w(b,a) = w(a,b), &\quad w(b,b) = w(a,a). \end{aligned} \tag{3.10}$$

Let $w(a,a) := \alpha, w(a,b) := \beta$ and using the above series of Equation (3.10), we get what we want. □

Remark 3.12. If $n=2$, then H_8 is the 8-dimensional Kac-Paljutkin algebra K_8. All possible quasitriangular structures on K_8 were given in [16]. In Proposition 3.11, we only consider the case $n \geq 3$ and thus does not imply possible quasitriangular structures on K_8. But one can get that through using our argument directly. In fact, trivial quasitriangular structures on K_8 can be given by Proposition 3.11 too in which we only need to set the parameter $n=2$. Non-trivial quasitriangular structures on K_8 can be completely determined by using the (ii) in Lemma 3.1 and the equation $\Delta^{op}(x)R = R\Delta(x)$.

Funding

National Natural Science Foundation of China NSFC 11722016.

References

[1] Abella, A. (2019). Some advances about the existence of compact involutions in semisimple Hopf algebras. *São Paulo J. Math. Sci.* 13(2):628–651. DOI: 10.1007/s40863-018-0109-9.
[2] Kac, G. I. (1968). Group extensions which are ring groups. *Mat. Sb. (N.S.)*. 76:473–496.
[3] Masuoka, A. (2000). Extensions of Hopf algebras and Lie bialgebras. *Trans. Amer. Math. Soc*. 352(8): 3837–3879. DOI: 10.1090/S0002-9947-00-02394-1.
[4] Masuoka, A. (2002). Hopf algebra extensions and cohomology, New directions in Hopf algebras. *Math. Sci. Res. Inst. Publ.*, Vol. 43. Cambridge: Cambridge Univ. Press, p. 167–209.
[5] Masuoka, A. (2003). Cohomology and coquasi-bialgebra extensions associated to a matched pair of bialgebras. *Adv. Math.* 173(2):262–315. DOI: 10.1016/S0001-8708(02)00053-1.
[6] Naidu, D., Nikshych, D., Witherspoon, S. (2009). Fusion subcategories of representation categories of twisted quantum doubles of finite groups. *Int. Math. Res. Not. IMRN.* (22):4183–4219.
[7] Natale, S. (2011). On quasitriangular structures in Hopf algebras arising from exact group factorizations. *Comm. Algebra*. 39(12):4763–4775. DOI: 10.1080/00927872.2011.617625.
[8] Pansera, D. (2019). A class of semisimple Hopf algebras acting on quantum polynomial algebras, Rings, modules and codes, Contemp. Math., Vol. 727. Providence, RI: Amer. Math. Soc., 303–316.
[9] Radford, D. E. (2012). *Hopf Algebras, world scientific, series on knots and everything*, Vol. 49. Hackensack, NJ: World Scientific Publishing Co. Pte. Ltd., xxii + p. 559.
[10] Suzuki, S. (1998). A family of braided cosemisimple Hopf algebras of finite dimension. *Tsukuba J. Math.* 22(1):1–29. DOI: 10.21099/tkbjm/1496163467.