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THE QUASI-HOPF ANALOGUE OF $\mathbf{u}_q(\mathfrak{sl}_2)$

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Abstract. In [9], some quasi-Hopf algebras of dimension n^3, which can be understood as the quasi-Hopf analogues of Taft algebras, are constructed. Moreover, the quasi-Hopf analogues of generalized Taft algebras are considered in [12], where the language of the dual of a quasi-Hopf algebra is used. The Drinfeld doubles of such quasi-Hopf algebras are computed in this paper. The authors in [10] showed that the Drinfeld double of a quasi-Hopf algebra of dimension n^3 constructed in [9] is always twist equivalent to Lusztig small quantum group $\mathbf{u}_q(\mathfrak{sl}_2)$ if n is odd. Based on computations and analysis, we show that this is *not* the case if n is even. That is, the quasi-Hopf analogue $\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$ of $\mathbf{u}_q(\mathfrak{sl}_2)$ is gotten.

1. Introduction

Historically Drinfeld [8] and Jimbo [15] introduced quantum groups and Lusztig [17] found their finite-dimensional analogues, known as Frobenius–Lusztig kernels or Lusztig small quantum groups. Do we have quasi-Hopf analogues of such quantum groups? For a simple finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, Drinfeld [7, Proposition 3.16] told us that a quasi-triangular quasi-Hopf quantized enveloping algebra $U\mathfrak{g}[[h]]$ is indeed twist equivalent to the usual quantum group $U_h\mathfrak{g}$. So there is no essentially new quasi-quantum groups attached to a simple finite-dimensional Lie algebra. But, how about the restricted case? That is, do we have quasi-Hopf analogue of a Lusztig small quantum group?

The aim of the paper and following works is to find the quasi-Hopf analogues of Lusztig small quantum groups and consequently give some new examples of finite-dimensional non-semisimple quasi-triangular quasi-Hopf algebras. As a try, we want to give the quasi-Hopf analogue of $\mathbf{u}_q(\mathfrak{sl}_2)$ in this paper. Inspired by the Hopf case, one can believe that it should be the Drinfeld double of the quasi-Hopf analogue of a Taft algebra. Meanwhile, the general theory of the Drinfeld double for a quasi-Hopf algebra was already developed by Majid, Hausser–Nill and Schauenburg [11, 18, 20] and the quasi-Hopf analogues, denoted by $A(n, q)$, of Taft algebras were discovered by Gelaki [9]. So all things were prepared, and the only task is to compute them out.

But, before computation, Etingof and Gelaki [10] proved a surprised result: the double $D(A(n, q))$ is always twist equivalent to $\mathbf{u}_q(\mathfrak{sl}_2)$ if n is odd. There is a restriction on Etingof–Gelaki's result, that is, n must be odd. Is this condition necessary? Our answer is YES. As one of main results of this paper, we show that $D(A(n, q))$ is *not* twist equivalent to a Hopf algebra if n is even and consequently the quasi-Hopf analogue of $\mathbf{u}_q(\mathfrak{sl}_2)$ is gotten. We will prove the result in a general setting.

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In [12], all pointed Majid algebras $M(n,s,q)$ of finite representation type are classified. Such pointed Majid algebras are indeed the dual of the class of basic quasi-Hopf algebras $A(n,s,q)$ which can be considered as the quasi-Hopf analogues of generalized Taft algebras. Note that the quasi-Hopf algebras $A(n,s,q)$ also appeared in [1]. Maybe, the only contribution of this paper is to compute $D(A(n,s,q))$ out explicitly. The main result of paper can be described as follows.

Theorem 1.

(1) *As a quasi-Hopf algebra,* $D(A(n,s,q)) \cong \mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$.
(2) *Assume that* $n = 2^m l$ *and* $s = 2^{m'} l'$ *with* $(l,2) = (l',2) = 1$. *If* $m' < m$, *then* $D(A(n,s,q))$ *is not twist equivalent to a Hopf algebra.*

The quasi-Hopf algebra $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$ will be described in Section 2 by using generators and relations. All other preliminaries are also collected in this section. The first part of Theorem 1.1 will be proved in Section 3 and the method is by direct computation. The proof of the second part will be given in Section 4. The main idea of this section is to find some suitable representations of $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$, such that they form a subtensor category of Rep-$\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$. By using group cohomologies, we will find that the restriction of the associator to this subtensor category is not trivial.

Throughout, we work over an algebraically closed field κ of characteristic 0 and $[\frac{\bullet}{\bullet}]$ stands for the floor function. That is, for any natural numbers a, b, $[\frac{a}{b}]$ denotes the biggest integer which is not bigger than $\frac{a}{b}$. About general background knowledge, the reader is referred to [7] for quasi-Hopf algebras, to [3,16] for general theory about tensor categories, and to [12] for pointed Majid algebras.

2. Preliminaries

In this section, we recall the constructions of quasi-Hopf analogues of (generalized) Taft algebras, their dualities and the Drinfeld double of a quasi-Hopf algebra for the convenience of the reader. At last, we will introduce a new quasi-Hopf algebra $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$.

2.1. Path coalgebras and pointed Majid algebra $M(n,s,q)$.

The main aim of this subsection is to recall the definition of the pointed Majid algebra $M(n,s,q)$ constructed in [12]. To state it, the concept path coalgebra is needed.

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t:\ Q_1 \longrightarrow Q_0$ are two maps assigning, respectively, the source and the target for each arrow. A path of length $l \geq 1$ in the quiver Q is a finitely ordered sequence of l arrows $a_l \cdots a_1$, such that $s(a_{i+1}) = t(a_i)$ for $1 \leq i \leq l-1$. By convention a vertex is said to be a trivial path of length 0. For a quiver Q, the associated path coalgebra κQ is the κ-space spanned by the set of paths with counit and comultiplication maps defined by $\varepsilon(g) = 1,\ \Delta(g) = g \otimes g$ for each $g \in Q_0$, and for each non-trivial path $p = a_n \cdots a_1,\ \varepsilon(p) = 0$,

$$\Delta(a_n \cdots a_1) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p.$$

The length of paths gives a natural gradation to the path coalgebra. Let Q_n denote the set of paths of length n in Q, then $\kappa Q = \oplus_{n\geq 0}\kappa Q_n$ and $\Delta(Q_n) \subseteq \oplus_{n=i+j} Q_i \otimes Q_j$. Clearly κQ is pointed with the set of group-likes $G(Q) = Q_0$, and has the following coradical filtration:

$$\kappa Q_0 \subseteq \kappa Q_0 \oplus \kappa Q_1 \subseteq \kappa Q_0 \oplus \kappa Q_1 \oplus \kappa Q_2 \subseteq \cdots .$$

Hence κQ is coradically graded. The reader is referred to [19, Chapter 5] for general background knowledge about coradicals, and to [4–6] for more about path coalgebras together with their applications to Hopf algebras.

A dual quasi-bialgebra, or Majid bialgebra for short, is a coalgebra (M, Δ, ε) equipped with a compatible quasi-algebra structure. Namely, there exist two coalgebra homomorphisms

$$\mathrm{M} : H \otimes H \to H, \quad a \otimes b \mapsto ab, \quad \mu : \kappa \to H, \ \lambda \mapsto \lambda 1_H$$

and a convolution-invertible map $\Phi : H^{\otimes 3} \to k$ called associator, such that for all $a, b, c, d \in H$ the following equalities hold:

$$a_{(1)}(b_{(1)}c_{(1)})\Phi(a_{(2)}, b_{(2)}, c_{(2)}) = \Phi(a_{(1)}, b_{(1)}, c_{(1)})(a_{(2)}b_{(2)})c_{(2)}, \tag{2.1}$$

$$1_H a = a = a 1_H, \tag{2.2}$$

$$\begin{aligned} &\Phi(a_{(1)}, b_{(1)}, c_{(1)}d_{(1)})\Phi(a_{(2)}b_{(2)}, c_{(2)}, d_{(2)}) \\ &= \Phi(b_{(1)}, c_{(1)}, d_{(1)})\Phi(a_{(1)}, b_{(2)}c_{(2)}, d_{(2)})\Phi(a_{(3)}, b_{(1)}, c_{(3)}), \end{aligned} \tag{2.3}$$

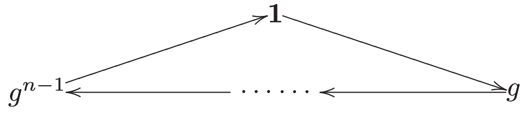
$$\Phi(a, 1_H, b) = \varepsilon(a)\varepsilon(b). \tag{2.4}$$

Here and below we use the Sweedler sigma notation $\Delta(a) = a_{(1)} \otimes a_{(2)} = a' \otimes a''$ for the comultiplication and $a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+1)}$ for the result of the n-iterated application of Δ on a. H is called a Majid algebra if, moreover, there exist a coalgebra antimorphism $S : H \to H$ and two functionals $\alpha, \beta : H \to \kappa$ such that for all $a \in H$,

$$S(a_{(1)})\alpha(a_{(2)})a_{(3)} = \alpha(a)1_H, \quad a_{(1)}\beta(a_{(2)})S(a_{(3)}) = \beta(a)1_H, \tag{2.5}$$

$$\begin{aligned} &\Phi(a_{(1)}, S(a_{(3)}), a_{(5)})\beta(a_{(2)})\alpha(a_{(4)}) \\ &= \Phi^{-1}(S(a_{(1)}), a_{(3)}, S(a_{(5)}))\alpha(a_{(2)})\beta(a_{(4)}) = \varepsilon(a). \end{aligned} \tag{2.6}$$

A Majid algebra H is said to be pointed, if the underlying coalgebra is pointed. Now we consider a very simple quiver and want to build a pointed Majid algebra structure on its path coalgebra. The quiver being considered is the following one.



As in [12], this quiver is denoted by $Q(\mathbb{Z}_n, g)$. Now let $0 \leq s \leq n-1$ be a natural number, q an n^2th primitive root of unity and $\mathbb{q} := q^n$. Let p_i^l denote the path in $Q(\mathbb{Z}_n, g)$ starting from g^i with length l. So $p_i^0 = g^i$. Let Φ_s be the 3-cocycle over $\mathbb{Z}_n$ defined by

$$\Phi_s(g^i, g^j, g^k) = \mathbb{q}^{si[\frac{j+k}{n}]}, \quad 0 \leq i, j, k \leq n-1. \tag{2.7}$$

To define $M(n,s,q)$, the definition of the Gaussian binomial coefficient is needed. For any $h \in \kappa$, define $l_h = 1 + h + \cdots + h^{l-1}$ and $l!_h = 1_h \cdots l_h$. The Gaussian binomial coefficient is defined by $\binom{l+m}{l}_h := \frac{(l+m)!_h}{l!_h m!_h}$. Let (a,b) be the greatest common divisor of two natural numbers a, b.

Now we can define the pointed Majid algebra $M(n,s,q)$. As a coalgebra, $M(n,s,q) = \oplus_{i<\frac{n^2}{(n^2,s)}} \kappa Q(\mathbb{Z}_n, g)_i$. The associator, the multiplication, the functions α, β and the antipode are given through

$$\Phi(p_i^l, p_j^m, p_k^t) = \delta_{lmt,0} \Phi_s(g^i, g^j, g^k), \tag{2.8}$$

$$p_i^l \cdot p_j^m = q^{-sjl} \mathbb{q}^{s(i+l)[\frac{m+j}{n}]} \binom{l+m}{l}_{q^{-s}} p_{i+j}^{l+m}, \tag{2.9}$$

$$\alpha(p_i^l) = \delta_{l,0} \frac{1}{\Phi_s(g^i, g^{n-i}, g^i)}, \quad \beta(p_i^l) = \delta_{l,0} 1, \tag{2.10}$$

$$S(g^i) = g^{n-i}, \quad S(p_0^1) = \mathbb{q}^{-s} p_{n-1}^1, \tag{2.11}$$

for $0 \le l, m, t < \frac{n^2}{(n^2,s)}$ and $0 \le i, j, k \le n-1$, where $\delta_{a,b}$ is the Kronecker notation, which is equal to 1 if $a = b$ and 0 otherwise.

Remark 2. For simplicity, we change the multiplication formula defined in Corollary 3.9 of [12] slightly into our formula (2.9). To recover the original formula given in Corollary 3.9 of [12] from (2.9), just substitute q by $\mathbb{q}q$.

2.2. The quasi-Hopf algebra $A(n,s,q)$. A quasi-bialgebra $(H, \mathrm{M}, \mu, \Delta, \varepsilon, \phi)$ is a κ-algebra (H, M, μ) with algebra morphisms $\Delta : H \to H \otimes H$ (the comultiplication) and $\varepsilon : H \to \kappa$ (the counit), and an invertible element $\phi \in H \otimes H \otimes H$ (also called associator), such that

$$(id \otimes \Delta)\Delta(a)\phi = \phi(\Delta \otimes id)\Delta(a), \quad a \in H, \tag{2.12}$$

$$(id \otimes id \otimes \Delta)(\phi)(\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi)(id \otimes \Delta \otimes id)(\phi)(\phi \otimes 1), \tag{2.13}$$

$$(\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta, \tag{2.14}$$

$$(id \otimes \varepsilon \otimes id)(\phi) = 1 \otimes 1. \tag{2.15}$$

We denote $\phi = \sum X^i \otimes Y^i \otimes Z^i$ and $\phi^{-1} = \sum \overline{X}^i \otimes \overline{Y}^i \otimes \overline{Z}^i$. Then a quasi-bialgebra H is called a quasi-Hopf algebra if there is a linear algebra antimorphism $S : H \to H$ (the antipode) and elements $\alpha, \beta \in H$ satisfying for all $a \in H$,

$$\sum S(a_{(1)})\alpha a_{(2)} = \alpha\varepsilon(a), \quad \sum a_{(1)}\beta S(a_{(2)}) = \beta\varepsilon(a), \tag{2.16}$$

$$\sum X^i \beta S(Y^i) \alpha Z^i = 1 = \sum S(\overline{X^i}) \alpha \overline{Y^i} \beta S(\overline{Z^i}). \tag{2.17}$$

We say that an invertible element $J \in H \otimes H$ is a *twist* of H if it satisfies $(\varepsilon \otimes id)(J) = (id \otimes \varepsilon)(J) = 1$. For a twist $J = \sum f_i \otimes g_i$ with inverse $J^{-1} = \sum \overline{f_i} \otimes \overline{g_i}$, set

$$\alpha_J := \sum S(\overline{f_i})\alpha\overline{g_i}, \quad \beta_J := \sum f_i \beta S(g_i).$$

It is well known that given a twist J of H such that β_J is invertible then one can construct a new quasi-Hopf algebra structure $H_J = (H, \Delta_J, \varepsilon, \Phi_J, S_J, \beta_J\alpha_J)$ on the algebra H, where

$$\Delta_J(a) = J\Delta(a)J^{-1}, \quad a \in H,$$
$$\Phi_J = (1 \otimes J)(id \otimes \Delta)(J)(\Delta \otimes id)(J^{-1})(J \otimes 1)^{-1}$$

and

$$S_J(a) = \beta_J S(a)\beta_J^{-1}, \quad a \in H.$$

Next we will give the definition of the quasi-Hopf algebras $A(n, s, q)$, which was also constructed in [1]. The quasi-Hopf algebras $A(q)$ constructed by Gelaki [9] turn out to be special cases of $A(n, s, q)$. The dualities of such quasi-Hopf algebras were studied in [12].

Let n be a positive integer, $\mathbb{q}$ an nth primitive root of unity and $\kappa\mathbb{Z}_n$ the cyclic group algebra of order n. We denote a generator of $\mathbb{Z}_n$ by g_2 and define

$$1_i := \frac{1}{n}\sum_{j=0}^{n-1}(\mathbb{q}^{n-i})^j g_2^j. \tag{2.18}$$

For any $0 \le s \le n-1$ and q an nth primitive root of $\mathbb{q}$, the quasi-Hopf algebra $A(n, s, q)$ is defined as follows. As an associative algebra, it is generated by x, g_2 and satisfies the following relations:

$$g_2^n = 1, \quad x^{\frac{n^2}{(n^2,s)}} = 0, \quad g_2 x g_2^{-1} = \mathbb{q}x. \tag{2.19}$$

The associator ϕ_s, the comultiplication Δ, the counit ε, the elements α, β and the antipode S are given through

$$\phi_s = \sum_{i,j,k=0}^{n-1} \mathbb{q}^{si[\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k, \tag{2.20}$$

$$\Delta(g_2) = g_2 \otimes g_2, \quad \Delta(x) = 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i, \tag{2.21}$$

$$\alpha = g_2^{-s}, \quad \beta = 1, \tag{2.22}$$

$$S(g_2) = g_2^{-1}, \quad S(x) = -x\sum_{i=0}^{n-1} q^{s(i-n)} 1_i. \tag{2.23}$$

Lemma 3. *The algebra* $(A(n, s, q), \mathrm{M}, \mu, \Delta, \varepsilon, \phi_s, S, \alpha, \beta)$ *is a quasi-Hopf algebra and isomorphic to* $(M(n, s, q)^*, \Delta^*, \varepsilon^*, \mathrm{M}^*, \mu^*, \Phi_s^*, S^*, \alpha^*, \beta^*)$.

Proof. One can show this through direct computations. For our purpose, it is better to establish a direct isomorphism between $M(n, s, q)^*$ and $A(n, s, q)$. To prove it, we need to give a dual basis of $M(n, s, q)$. Recall $\{p_i^l | 0 \le i \le n-1, \ 0 \le l < \frac{n^2}{(n^2,s)}\}$ is a basis of $M(n, s, q)$. Let $\{(p_i^l)^* | 0 \le i \le n-1, \ 0 \le l < \frac{n^2}{(n^2,s)}\}$ be the canonical dual basis of $M(n,s,q)$. Define

$$\varphi:\ A(n,s,q)\to M(n,s,q)^*,\quad 1_i\mapsto (p_i^0)^*,\quad x\mapsto \sum_{j=0}^{n-1}(p_j^1)^*, \tag{2.24}$$

where $0\le i\le n-1$. A direct and tedious computation shows that φ gives the desired isomorphism of quasi-Hopf algebras between $M(n,s,q)^*$ and $A(n,s,q)$. □

Remark 4. Take $s=n-1$ and the resulting quasi-Hopf algebra $A(n,n-1,q)$ is indeed isomorphic the quasi-Hopf algebra $A(q)$ constructed in [9]. In this paper, $A(q)$ is denoted by $A(n,q)$ for consistence.

2.3. Drinfeld double of a quasi-Hopf algebra. The construction of the Drinfeld double of a quasi-Hopf algebra is not, at least, a trivial generalization from the Hopf to quasi-Hopf case. After all, the double of a Hopf algebra H is modelled on $H\otimes H^*$, with H and H^* as subalgebras. But if H is just a quasi-Hopf algebra, then H^* is not an associative algebra, so one losses the associative algebra structure on $H\otimes H^*$ and even expect that the double should be some kind of hybrid object. Majid [18] settled this problem at first. He gave a conceptual way to show that the double $D(H)$ is still a quasi-Hopf algebra. Hausser and Nill [11] gave a computable realization of $D(H)$ on $H\otimes H^*$. A more explicit version was gotten by Schauenburg [20]. Here we will recall the Schauenburg's construction.

Let $(H,\mathrm{M},\mu,\Delta,\varepsilon,\phi,S,\alpha,\beta)$ be a finite-dimensional quasi-Hopf algebra. Assume $\phi=\phi^{(1)}\otimes\phi^{(2)}\otimes\phi^{(3)}=\sum X^i\otimes Y^i\otimes Z^i$ and $\phi^{-1}=\phi^{(-1)}\otimes\phi^{(-2)}\otimes\phi^{(-3)}=\sum\overline{X}^i\otimes\overline{Y}^i\otimes\overline{Z}^i$. Define

$$\gamma:=\sum(S(U^i)\otimes S(T^i))(\alpha\otimes\alpha)(V^i\otimes W^i), \tag{2.25}$$

$$\mathbf{f}:=\sum(S\otimes S)(\Delta^{op}(\overline{X}^i))\cdot\gamma\cdot\Delta(\overline{Y}^i\beta S(\overline{Z}^i)), \tag{2.26}$$

$$\chi:=(\phi\otimes 1)(\Delta\otimes id\otimes id)(\phi^{-1}), \tag{2.27}$$

$$\omega:=(1\otimes 1\otimes 1\otimes\tau(\mathbf{f}^{-1}))(id\otimes\Delta\otimes S\otimes S)(\chi)(\phi\otimes 1\otimes 1), \tag{2.28}$$

where $(1\otimes\phi^{-1})(id\otimes id\otimes\Delta)(\phi)=\sum T^i\otimes U^i\otimes V^i\otimes W^i$ and τ is the flip map, i.e., $\tau(a\otimes b)=b\otimes a$.

As a linear space, $D(H)=H\otimes H^*$ and we write $h\bowtie\psi:=h\otimes\psi\in D(H)$. There are two canonical actions, denoted by $\rightharpoonup$, $\leftharpoonup$, of H on H^*. By definition, for any $a,b\in H$ and $\psi\in H^*$

$$\begin{aligned}\rightharpoonup:\ \ H\otimes H^*\longrightarrow H^*,\quad (a\rightharpoonup\psi)(b)=\psi(ba),\\ \leftharpoonup:\ \ H^*\otimes H\longrightarrow H^*,\quad (\psi\leftharpoonup a)(b)=\psi(ab).\end{aligned}$$

Define a map $\mathbf{T}:\ H^*\to D(H)$ by

$$\mathbf{T}(\psi)=\phi^{(1)}_{(2)}\bowtie S(\phi^{(2)})\alpha\phi^{(3)}\rightharpoonup\psi\leftharpoonup\phi^{(1)}_{(1)}. \tag{2.29}$$

With such preparations, $D(H)$ can be described as the following form (see Theorems 6.3 and 9.3 in [20]).

Theorem 5. *Let H be a finite-dimensional quasi-Hopf algebra. The quasi-Hopf structure on $D(H) = H \otimes H^*$, which contains H as a subquasi-Hopf algebra through the embedding $h \hookrightarrow h \bowtie \varepsilon$, is determined by*

(1) *As an associative algebra, it is generated by H and $\mathbf{T}(H^*)$ and multiplication rule is*

$$(g \bowtie \varphi)(h \bowtie \psi) = gh_{(1)(2)}\omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \psi \leftharpoonup \omega^{(1)})(\omega^{(4)}S(h_{(2)}) \rightharpoonup \varphi \leftharpoonup h_{(1)(1)}\omega^{(2)}), \quad (\star)$$

and the comultiplication is given by

$$\Delta_D(\mathbf{T}(\psi)) = \tilde{\phi}^{(2)}\mathbf{T}(\psi_{(1)} \leftharpoonup \tilde{\phi}^{(1)})\phi^{(-1)}\phi^{(1)} \otimes \tilde{\phi}^{(3)}\phi^{(-3)}\mathbf{T}(\phi^{(3)} \rightharpoonup \psi_{(2)} \leftharpoonup \phi^{(-2)})\phi^{(2)}, \quad (\star\star)$$

for $g, h \in H$ and $\varphi, \psi \in H^*$, where $\tilde{\phi}$ denote another copy of ϕ.

(2) *The associator ϕ_D, the counit ε_D, elements α_D, β_D and the antipode S_D are given by*

$$\phi_D = \phi \bowtie \varepsilon, \varepsilon_D(\mathbf{T}(\psi)) = \psi(\phi^{(1)}S(\phi^{(2)})\alpha\phi^{(3)}), \quad (2.30)$$

$$\alpha_D = \alpha \bowtie \varepsilon, \quad \beta_D = \beta \bowtie \varepsilon, \quad (2.31)$$

$$S_D(\mathbf{T}(\psi)) = \mathbf{f}^{(2)}\mathbf{T}(\mathbf{f}^{(-2)} \rightharpoonup S^{-1}(\psi) \leftharpoonup \mathbf{f}^{(1)})\mathbf{f}^{(-1)}, \quad (2.32)$$

for $\psi \in H^*$.

Remark 6. (1) By formula $(\star)$, $1 \bowtie \varepsilon$ is the unit element of $D(H)$. Moreover, as a special case of this formula, we also have

$$(1 \bowtie \varphi)(h \bowtie \varepsilon) = h_{(1)(2)} \bowtie S(h_{(2)}) \rightharpoonup \varphi \leftharpoonup h_{(1)(1)}, \quad (2.33)$$

for $h \in H$ and $\varphi \in H^*$.

(2) In the process of our computations, we find that there are some misprints in [20] and [11]. Especially, there are misprints in the expression of the element $\mathbf{f}$ given both in [20] and [11], the element χ given in [20] and the comultiplication formula given in [20]. The correct versions are (2.26), (2.27) and $(\star\star)$.

2.4. The quasi-Hopf algebra $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$. The quasi-Hopf algebra $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$ is defined as follows. As an associative algebra, it is generated by four elements g_1, g_2, x, y satisfying

$$g_1^n = g_2^{2s}, \quad g_2^n = 1, \quad g_1g_2 = g_2g_1, \quad x^{\frac{n^2}{(n^2,s)}} = y^{\frac{n^2}{(n^2,s)}} = 0, \quad (2.34)$$

$$g_1xg_1^{-1} = \mathrm{q}^{-s}q^{2s}x, \quad g_2xg_2^{-1} = \mathrm{q}x, \quad (2.35)$$

$$g_1yg_1^{-1} = \mathrm{q}^{s}q^{-2s}y, \quad g_2yg_2^{-1} = \mathrm{q}^{-1}y, \quad (2.36)$$

$$yx - q^sxy = 1 - g_1g_2^s. \quad (2.37)$$

Define

$$1_i := \frac{1}{n}\sum_{j=0}^{n-1}(\mathbb{q}^{n-i})^j g_2^j. \tag{2.38}$$

The associator ϕ_s, the comultiplication Δ, the counit ε, the elements α, β and the antipode S are given through

$$\phi_s = \sum_{i,j,k=0}^{n-1} \mathbb{q}^{si[\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k, \tag{2.39}$$

$$\Delta(g_1) = g_1 \otimes g_1, \quad \Delta(g_2) = g_2 \otimes g_2, \tag{2.40}$$

$$\Delta(x) = 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i, \tag{2.41}$$

$$\Delta(y) = y \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i + g_1 \otimes y 1_0, \tag{2.42}$$

$$\varepsilon(g_1) = \varepsilon(g_2) = 1, \quad \varepsilon(x) = \varepsilon(y) = 0, \tag{2.43}$$

$$\alpha = g_2^{-s}, \quad \beta = 1, \tag{2.44}$$

$$S(g_1) = g_1^{-1}, \quad S(g_2) = g_2^{-1}, \tag{2.45}$$

$$S(x) = -x \sum_{i=0}^{n-1} q^{s(i-n)} 1_i, \quad S(y) = -g_1^{-1} g_2^{-s} y \sum_{i=0}^{n-1} q^{s(n-i)'} 1_i, \tag{2.46}$$

where for any integer $i \in \mathbb{N}$, we denote by i' the remainder of division of i by n.

Lemma 7. $\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$ *is a quasi-Hopf algebra.*

Proof. We will show that $D(A(n, s, q))$ is isomorphic to $\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$ and thus $\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$ is a quasi-Hopf algebra. Moreover, it is quasi-triangular. Of course, one can show the result by a direct computation. Here we only check the equality $\Delta(y)\Delta(x) - q^s\Delta(x)\Delta(y) = \Delta(1) - \Delta(g_1)\Delta(g_2^s)$. Indeed,

$$\begin{aligned}
\Delta(y)\Delta(x) &= \left(y \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i + g_1 \otimes y 1_0\right) \\
&\quad \times \left(1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i\right) \\
&= y \otimes \sum_{i=1}^{n-1} q^{si} 1_i x + y g_2^s \otimes 1_0 x + yx \otimes 1 + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i x \\
&\quad + g_1 g_2^s x \otimes y \sum_{i=1}^{n-1} q^{-si} 1_i + g_1 g_2^s \otimes y 1_0 x + g_1 x \otimes y 1_0
\end{aligned}$$

$$= y \otimes \sum_{i=1}^{n-1} q^{si} 1_i x + y g_2^s \otimes 1_0 x + yx \otimes 1 + g_1 g_2^s \otimes yx$$
$$+ g_1 g_2^s x \otimes y \sum_{i=1}^{n-1} q^{-si} 1_i + g_1 x \otimes y 1_0,$$

and

$$\begin{aligned} q^s \Delta(x)\Delta(y) &= q^s \left(1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i \right) \\ &\quad \times \left(y \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i + g_1 \otimes y 1_0 \right) \\ &= q^s [y \otimes \sum_{i=1}^{n-1} q^{s(i-1)} 1_i x + g_1 g_2^s \otimes xy \sum_{i=1}^{n-1} 1_i + g_2^s y \otimes q^{s(n-1)} 1_0 x \\ &\quad + g_1 g_2^s \otimes xy 1_0 + xy \otimes 1 + x g_1 g_2^s \otimes y \sum_{i=1}^{n-1} q^{-s(i-1)} 1_i \\ &\quad + x g_1 \otimes q^{-s(n-1)} y 1_0] \\ &= q^s [y \otimes \sum_{i=1}^{n-1} q^{s(i-1)} 1_i x + g_1 g_2^s \otimes xy + q^{-s} y g_2^s \otimes 1_0 x \\ &\quad + xy \otimes 1 + q^{-s} g_1 g_2^s x \otimes y \sum_{i=1}^{n-1} q^{-si} 1_i + q^{-s} g_1 x \otimes y 1_0]. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta(y)\Delta(x) - q^s \Delta(x)\Delta(y) &= (yx - q^s xy) \otimes 1 + g_1 g_2^s \otimes (yx - q^s xy) \\ &= (1 - g_1 g_2^s) \otimes 1 + g_1 g_2^s \otimes (1 - g_1 g_2^s) \\ &= 1 \otimes 1 - g_1 g_2^s \otimes g_1 g_2^s \\ &= \Delta(1) - \Delta(g_1)\Delta(g_2^s). \end{aligned}$$

□

Lemma 8. *The dimension of* $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$ *is equal to* $(\frac{n^3}{(n^2,s)})^2$.

Proof. Let G be the group in $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$ generated by g_1, g_2 and clearly $|G| = n^2$. To show the result, it is enough to show that the set $T = \{g x^i y^j | g \in G, 0 \le i < \frac{n^2}{(n^2,s)}, 0 \le j < \frac{n^2}{(n^2,s)}\}$ is a basis of $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$. We use the Diamond Lemma [2] to prove this and terminologies together with notions appeared in [2] will be used freely here.

Define a partial order on words generated by x, y, g ($g \in G$) through the following way: (1) all elements in G are minimal and $y > x > g$; (2) if two words A, B have the same length, they are ordered by lexicographic order; (3) $A < B$ if the word A is of smaller length than the word B. Through this, we can find that the set T consists of all irreducible words. By Theorem 1.2 in [2], the set T will be a basis of $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$ if we can show that all ambiguities are resolvable. In our case, the situation is simple since we only have ten relations. For short, let $m = \frac{n^2}{(n^2,s)}$. We only consider the following two typical ambiguities: $(\sigma_{y^{m-1}y}, \sigma_{yx}, y^{m-1}, y, x)$ and $(\sigma_{yx}, \sigma_{xg}, y, x, g)$. To resolve the first one relative to $\leq$, we must study the element

$$r_{1\sigma_{y^{m-1}y}x}(y^{m-1}yx) - r_{y^{m-1}\sigma_{yx}1}(y^{m-1}yx) = 0x - y^{m-1}(q^s xy + (1 - g_1 g_2^s)).$$

To further reduce the term $q^s y^{m-1}xy$, the following identity, which can be proved inductively, is needed.

$$y^t x = q^{ts} xy^t + \sum_{i=0}^{t-1} q^{si} y^{t-1} - q^{2s(t-1)} \sum_{i=0}^{t-1} q^{-is} g_1 g_2^s y^{t-1}, \quad t \in \mathbb{N}.$$

From this identity, the term $y^{m-1}(q^s xy + (1 - g_1 g_2^s))$ is reduced to

$$q^{ms} xy^m + \sum_{i=0}^{m-1} q^{si} y^{m-1} - q^{2s(t-1)} \sum_{i=0}^{m-1} q^{-is} g_1 g_2^s y^{m-1},$$

which is zero since q^s is an mth root of unity. To resolve the second one we must consider the element

$$r_{1\sigma_{yx}g}(yxg) - r_{y\sigma_{xg}1}(yxg) = (q^s xy + (1 - g_1 g_2^s))g - yc_x gx.$$

Here we assume that $xg = c_x gx$ for $c_x \in \kappa^*$. Similarly, assume $yg = c_y gy$ for some $c_y \in \kappa^*$. The element $q^s xyg$ is further reduced to $q^s c_x c_y gxy$, $yc_x gx$ is reduced to $c_x c_y gyx$ and further to $c_x c_y g(q^s xy + (1 - g_1 g_2^s))$. In conclusion, $(q^s xy + (1 - g_1 g_s))g - yc_x gx$ is reduced 0 too. □

Corollary 9. *Let $\mathbf{u}^+, \mathbf{u}^-$ and $\mathbf{u}^0$ be the subalgebras of $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$ generated by x, y and g_1, g_2, respectively. Then $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$ has a triangular decomposition*

$$\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2) = \mathbf{u}^- \mathbf{u}^0 \mathbf{u}^+.$$

Proof. Using the same method as in the proof of Lemma 8, we have that $T' = \{x^i g y^j | g \in G, 0 \leq i < \frac{n^2}{(n^2,s)}, 0 \leq j < \frac{n^2}{(n^2,s)}\}$ is also a basis of $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$. □

3. The Drinfeld double of $A(n, s, q)$

The main result of this section is the following result.

Theorem 10. *The Drinfeld double $D(A(n, s, q))$ of $A(n, s, q)$ is isomorphic to the quasi-Hopf algebra $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$. That is,*

$$D(A(n, s, q)) \cong \mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2).$$

To show it, we need to understand better $D(A(n, s, q))$. Recall that for any integer $i \in \mathbb{N}$, we denote by i' the remainder of division of i by n. The following lemma is useful in our computations.

Lemma 11. *For any two natural numbers i, j, we always have*

$$\left[\frac{i+j'}{n}\right] = \left[\frac{i+j}{n}\right] - \left[\frac{j}{n}\right]. \tag{3.1}$$

Proof.

$$\begin{aligned}\left[\frac{i+j'}{n}\right] &= \left[\frac{i+j-\left[\frac{j}{n}\right]n}{n}\right] \\ &= \left[\frac{i+j}{n}\right] - \left[\frac{j}{n}\right].\end{aligned}$$

□

The formula (3.1) will be used frequently without explanation. Recall that the associator of $A(n, s, q)$ is defined to be

$$\phi_s = \sum_{i,j,k=0}^{n-1} \mathrm{q}^{si[\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k.$$

The next lemma will give the explicit formalism of the elements γ, $\mathbf{f}$, χ and ω in such case. Throughout this section, ϕ_s is denoted by ϕ for short when this is no confusion.

Lemma 12. *For the quasi-Hopf algebra $A(n, s, q)$, we have*

$$\gamma = \sum_{j,k=0}^{n-1} \mathrm{q}^{s(j+k)[\frac{j+k}{n}]+sk[\frac{n-j}{n}]-s(j+2k)} 1_j \otimes 1_k, \tag{3.2}$$

$$\mathbf{f} = \sum_{j,k=0}^{n-1} \mathrm{q}^{s(j+k)[\frac{j+k}{n}]+sk[\frac{n-j}{n}]-sk} 1_j \otimes 1_k, \tag{3.3}$$

$$\chi = \sum_{i_1,i_2,j,k=0}^{n-1} \mathrm{q}^{si_1[\frac{i_2+j}{n}]-s(i_1+i_2)[\frac{j+k}{n}]} 1_{i_1} \otimes 1_{i_2} \otimes 1_j \otimes 1_k \tag{3.4}$$

and

$$\omega = \sum_{i_1,i_2,i_3,i_4,i_5=0}^{n-1} \mathrm{q}^{si_5-s(\Sigma_{t=1}^{5} i_t)[\frac{i_4+i_5}{n}]+si_1[\frac{i_2+i_3+i_4}{n}]-si_5[\frac{n-i_4}{n}]}$$
$$1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}).$$

Proof. Note that the definitions of such elements were given in equations (2.25)–(2.28). The results are followed by direct computations. □

Once the element ω is known, the multiplication rules of $D(A(n, s, q))$ can be determined according to formula $(\star)$.

Proposition 13. *In $D(A(n,s,q))$, we have the following relations:*

$$(g_2 \bowtie \varepsilon)^n = 1 \bowtie \varepsilon, \quad (x \bowtie \varepsilon)^{\frac{n^2}{(n^2,s)}} = 0, \tag{3.5}$$

$$(g_2 \bowtie \varepsilon)(x \bowtie \varepsilon)(g_2 \bowtie \varepsilon)^{-1} = \mathbb{q}(x \bowtie \varepsilon), \tag{3.6}$$

$$(1 \bowtie g)(g_2 \bowtie \varepsilon) = (g_2 \bowtie \varepsilon)(1 \bowtie g), \quad \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g\right)^n = g_2^{2s} \bowtie \varepsilon, \tag{3.7}$$

$$(1 \bowtie p_0^1)^{\frac{n^2}{(n^2,s)}} = 0, \quad (g_2 \bowtie \varepsilon)(1 \bowtie p_0^1)(g_2 \bowtie \varepsilon)^{-1} = \mathbb{q}^{-1}(1 \bowtie p_0^1), \tag{3.8}$$

$$\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g\right)(x \bowtie \varepsilon)\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g\right)^{-1} = \mathbb{q}^{-s} q^{2s}(x \bowtie \varepsilon), \tag{3.9}$$

$$\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g\right)(1 \bowtie p_0^1)\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g\right)^{-1} = \mathbb{q}^{s} q^{-2s}(1 \bowtie p_0^1), \tag{3.10}$$

and

$$\begin{aligned}\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1\right)(x \bowtie \varepsilon) - q^s (x \bowtie \varepsilon)\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1\right)\\ = (1 \bowtie \varepsilon) - \left(g_2^s \sum_{i=0}^{n-1} q^{si} 1_i \bowtie g\right).\end{aligned}$$

Proof. Formulas (3.5) and (3.6) are clear since $A(n,s,q)$ is a subquasi-Hopf algebra of its double. The first part of formula (3.7) and the second part of formula (3.8) are direct consequences of formula (2.33). For the second part of (3.7), one has

$$\begin{aligned}(1 \bowtie g)(1 \bowtie g) &= \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup g \leftharpoonup \omega^{(1)})(\omega^{(4)} \rightharpoonup g \leftharpoonup \omega^{(2)})\\ &= \sum_{i_3=0}^{n-1} \mathbb{q}^{-s[\frac{(n-1)'+(n-1)'}{n}]} 1_{i_3} \bowtie g^2 = g_2^{-s} \bowtie g^2.\end{aligned}$$

Inductively, one has

$$(1 \bowtie g)^i = g_2^{-s(i-1)} \bowtie g^i \tag{3.11}$$

for $1 \le i \le n$. Thus $(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)^n = (\sum_{i=0}^{n-1} q^{si} 1_i \bowtie \varepsilon)^n (1 \bowtie g)^n = (\sum_{i=0}^{n-1} \mathbb{q}^{si} 1_i \bowtie \varepsilon)(1 \bowtie g)^n = g_2^{2s} \bowtie \varepsilon$.

Since the multiplication of $M(n,s,q)$ is not associative in general, we need two notions. For any algebra (maybe not associative) A, let $X \in A$. Define

$$X^{\overrightarrow{l}} =: \overbrace{(\cdots(X \cdot X) \cdot X) \cdots)}^{l}, \quad X^{\overleftarrow{l}} =: \overbrace{(\cdots(X \cdot (X \cdot X)) \cdots)}^{l}.$$

For the first part of (3.8), we have

$$\begin{aligned}(1\bowtie p_0^1)(1\bowtie p_0^1) &= \omega^{(3)}\bowtie(\omega^{(5)}\rightharpoonup p_0^1\leftharpoonup\omega^{(1)})(\omega^{(4)}\rightharpoonup p_0^1\leftharpoonup\omega^{(2)})\\ &= \sum_{i_3=0}^{n-1}\mathfrak{q}^{s[\frac{1+i_3}{n}]}1_{i_3}\bowtie(p_0^1)^{\overrightarrow{2}}\\ &= \sum_{i_3=0}^{n-1}\mathfrak{q}^{s[\frac{1+i_3}{n}]}1_{i_3}\bowtie(p_0^1)^{\overleftarrow{2}}.\end{aligned}$$

Inductively, one can prove the following equalities:

$$(1\bowtie p_0^1)^{\overrightarrow{l}} = g_2^{s[\frac{l}{n}]}\sum_{i_3=0}^{n-1}\mathfrak{q}^{s[\frac{1+i_3}{n}]+s[\frac{2+i_3}{n}]+\cdots+s[\frac{l'-1+i_3}{n}]}1_{i_3}\bowtie(p_0^1)^{\overleftarrow{l}}, \tag{3.12}$$

$$(1\bowtie p_0^1)^{\overleftarrow{l}} = \mathfrak{q}^{sl'[\frac{l}{n}]}g_2^{s[\frac{l}{n}]}\sum_{i_3=0}^{n-1}\mathfrak{q}^{s[\frac{1+i_3}{n}]+s[\frac{2+i_3}{n}]+\cdots+s[\frac{l'-1+i_3}{n}]}1_{i_3}\bowtie(p_0^1)^{\overrightarrow{l}}. \tag{3.13}$$

Comparing with Lemma 3.6 in [12], we indeed have $(1\bowtie p_0^1)^{\overrightarrow{l}} = (1\bowtie p_0^1)^{\overleftarrow{l}}$ and $(1\bowtie p_0^1)^{\frac{n^2}{(n^2,s)}} = 0$. Now let us prove the formula (3.9).

$$\begin{aligned}&\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)(x\bowtie\varepsilon)\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)^{-1}\\ &= \left(\sum_{i=0}^{n-1}q^{si}1_i x_{(1)(2)}\omega^{(3)}\bowtie(\omega^{(5)}\rightharpoonup\varepsilon\leftharpoonup\omega^{(1)}\right)(\omega^{(4)}S(x_{(2)})\rightharpoonup g\leftharpoonup x_{(1)(1)}\omega^{(2)}))\\ &\quad\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)^{-1}\\ &= \left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie\varepsilon\right)\left(\sum_{j=1}^{n-1}1_i x\bowtie q^{-(n-1)s}g+1_0x\bowtie q^sg\right)\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)^{-1}\\ &= \left(x\sum_{j=0}^{n-2}q^{s(j+1)}1_j\bowtie q^{-(n-1)s}g+x1_{n-1}\bowtie q^sg\right)\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)^{-1}\\ &= \left(x\sum_{j=0}^{n-2}\mathfrak{q}^{-s}q^{2s}q^{sj}1_j\bowtie g+x\mathfrak{q}^{-s}q^{2s}q^{(n-1)s}1_{n-1}\bowtie g\right)\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)^{-1}\\ &= \mathfrak{q}^{-s}q^{2s}(x\bowtie\varepsilon)\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)\left(\sum_{i=0}^{n-1}q^{si}1_i\bowtie g\right)^{-1}\\ &= \mathfrak{q}^{-s}q^{2s}(x\bowtie\varepsilon).\end{aligned}$$

For (3.10), note that $(\sum_{i=0}^{n-1} q^{si}1_i \bowtie g)^{-1} = (g_2^s \sum_{i=0}^{n-1} q^{-si}1_i \bowtie g^{n-1})$.

$$\begin{aligned}
&\left(\sum_{i=0}^{n-1} q^{si}1_i \bowtie g\right)(1 \bowtie p_0^1)\left(\sum_{i=0}^{n-1} q^{si}1_i \bowtie g\right)^{-1} \\
&= \left(\sum_{i=0}^{n-1} q^{si}1_i\omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup p_0^1 \leftharpoonup \omega^{(1)})(\omega^{(4)} \rightharpoonup g \leftharpoonup \omega^{(2)})\right)\left(\sum_{i=0}^{n-1} q^{si}1_i \bowtie g\right)^{-1} \\
&= \left(\sum_{i=0}^{n-1} \mathbb{q}^s q^{si}1_i \bowtie (p_0^1 g)\right)\left(g_2^s \sum_{i=0}^{n-1} q^{-si}1_i \bowtie g^{n-1}\right) \\
&= \left(\sum_{i=0}^{n-1} \mathbb{q}^s q^{si}1_i \bowtie \varepsilon\right) \\
&\quad \times \left(\sum_{i_3=0}^{n-1} \mathbb{q}^s \mathbb{q}^{s-s(1+i_3)+s(n-1)[\frac{n+1+i_3}{n}]+s[\frac{2}{n}]} q^{-s(1+i_3)'} 1_{i_3} g_2^s \bowtie g^{n-1}(p_0^1 g)\right) \\
&= \left(\sum_{i=0}^{n-1} \mathbb{q}^s q^{si}1_i \bowtie \varepsilon\right)\left(\sum_{i_3=0}^{n-1} \mathbb{q}^s \mathbb{q}^{-s(1+i_3)+s[\frac{2}{n}]} q^{-s(1+i_3)} 1_{i_3} g_2^s \bowtie g^{n-1}(p_0^1 g)\right) \\
&= \left(\sum_{i=0}^{n-1} \mathbb{q}^s q^{si}1_i \bowtie \varepsilon\right)\left(\sum_{i_3=0}^{n-1} \mathbb{q}^{s[\frac{2}{n}]} q^{-s(1+i_3)} 1_{i_3} \bowtie g^{n-1}(p_0^1 g)\right) \\
&= \sum_{i=0}^{n-1} \mathbb{q}^s q^{-s} \mathbb{q}^{s[\frac{2}{n}]} 1_i \bowtie g^{n-1}(p_0^1 g) \\
&= \sum_{i=0}^{n-1} \mathbb{q}^s q^{-s} \mathbb{q}^{s[\frac{2}{n}]} 1_i \bowtie q^{-s} \mathbb{q}^{-s[\frac{2}{n}]} p_0^1 \\
&= \mathbb{q}^s q^{-2s} 1 \bowtie p_0^1,
\end{aligned}$$

where the second last equality are gotten from (2.9).

Now the only task is to prove the last formula in this proposition. Note that

$$\begin{aligned}
(\Delta \otimes id)\Delta(x) = {} & 1 \otimes 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes g_2^s \otimes 1_0 x + 1 \otimes \sum_{i=1}^{n-1} 1_i x \otimes \sum_{i=0}^{n-1} q^{-si}1_i \\
& + g_2^s \otimes 1_0 x \otimes \sum_{i=0}^{n-1} q^{-si}1_i + x \otimes \sum_{i=0}^{n-1} q^{-si}1_i \otimes \sum_{i=0}^{n-1} q^{-si}1_i.
\end{aligned}$$

By applying above five items $1 \otimes 1 \otimes \sum_{i=1}^{n-1} 1_i x$, $g_2^s \otimes g_2^s \otimes 1_0 x$, $1 \otimes \sum_{i=1}^{n-1} 1_i x \otimes \sum_{i=0}^{n-1} q^{-si}1_i$, $g_2^s \otimes 1_0 x \otimes \sum_{i=0}^{n-1} q^{-si}1_i$ and $x \otimes \sum_{i=0}^{n-1} q^{-si}1_i \otimes \sum_{i=0}^{n-1} q^{-si}1_i$ into the following formula:

$$(1 \bowtie p_0^1)(x \bowtie \varepsilon) = x_{(1)(2)} \bowtie S(x_2) \rightharpoonup p_0^1 \leftharpoonup x_{(1)(1)},$$

we get

$$(1 \bowtie p_0^1)(x \bowtie \varepsilon) = \sum_{i=1}^{n-1} 1_i x \bowtie p_0^1 + q^s 1_0 x \bowtie p_0^1 - g_2^s \bowtie g + \sum_{i=0}^{n-1} q^{-si} 1_i \bowtie \varepsilon.$$

Multiplying the element $\sum_{i=0}^{n-1} q^{si} 1_i \bowtie \varepsilon$ to above equality, we have

$$\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1\right)(x \bowtie \varepsilon) = q^s (x \bowtie \varepsilon)\left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1\right) + (1 \bowtie \varepsilon) - \left(g_2^s \sum_{i=0}^{n-1} q^{si} 1_i \bowtie g\right). \qquad \square$$

Next, let us determine the comultiplication, the counit, the antipode and the elements α, β of $D(A(n,s,q))$. From now on, sometimes, we denote $h \bowtie \varphi \in D(A(n,s,q))$ by $h\varphi$ for short.

Proposition 14. *In $D(A(n,s,q))$, we have*

$$\Delta(g_2) = g_2 \otimes g_2, \tag{3.14}$$

$$\Delta(x) = 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i, \tag{3.15}$$

$$\Delta\left(\sum_{i=0}^{n-1} q^{si} 1_i g\right) = \sum_{i=0}^{n-1} q^{si} 1_i g \otimes \sum_{i=0}^{n-1} q^{si} 1_i g, \tag{3.16}$$

$$\Delta\left(\sum_{i=0}^{n-1} q^{si} 1_i p_0^1\right) = \sum_{i=0}^{n-1} q^{si} 1_i p_0^1 \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_2^s \sum_{i=0}^{n-1} q^{si} 1_i g \otimes \sum_{i=0}^{n-2} q^{si} 1_i p_0^1 + \sum_{i=0}^{n-1} q^{si} 1_i g \otimes q^{s(n-1)} 1_{n-1} p_0^1, \tag{3.17}$$

$$\varepsilon(g_2) = 1, \quad \varepsilon(x) = 0, \tag{3.18}$$

$$\varepsilon\left(\sum_{i=0}^{n-1} q^{si} 1_i g\right) = 1, \quad \varepsilon\left(\sum_{i=0}^{n-1} q^{si} 1_i p_0^1\right) = 0, \tag{3.19}$$

$$S(g_2) = g_2^{-1}, \quad S(x) = -x \sum_{i=0}^{n-1} q^{s(i-n)} 1_i, \tag{3.20}$$

$$S\left(\sum_{i=0}^{n-1} q^{si} 1_i g\right) = \left(\sum_{i=0}^{n-1} q^{si} 1_i g\right)^{-1}, \tag{3.21}$$

$$S\left(\sum_{i=0}^{n-1} q^{si} 1_i p_0^1\right) = -q^{(n-1)s} g_2^{-s} \left(\sum_{i=0}^{n-1} q^{si} g\right)^{-1} p_0^1, \tag{3.22}$$

$$\alpha = g_2^{-s} \bowtie \varepsilon, \quad \beta = 1 \bowtie \varepsilon. \tag{3.23}$$

Proof. Formulas (3.14),(3.15), (3.19) and (3.21) are clear since $D(A(n,s,q))$ contains $A(n,s,q)$ as a subquasi-Hopf algebra. By (2.29), one can verify directly that

$$\mathbf{T}(g)=g_2^s\bowtie g,\quad \mathbf{T}(p_0^1)=1\bowtie p_0^1.$$

Using the comultiplication formula ($\star\star$), we have

$$\Delta(\mathbf{T}(g))=\sum_{i,j=0}^{n-1}\mathrm{q}^{s[\frac{i+j}{n}]}(1_i\otimes 1_j)(\mathbf{T}(g)\otimes\mathbf{T}(g)), \tag{3.24}$$

$$\Delta(\mathbf{T}(p_0^1))=\sum_{i,j=0}^{n-1}\mathrm{q}^{s[\frac{i+j}{n}]}1_i\mathbf{T}(p_0^1)\otimes 1_j + \sum_{i,j=0}^{n-1}\mathrm{q}^{s[\frac{i+j}{n}]-si[\frac{1+j}{n}]}1_i\otimes 1_j\mathbf{T}(p_0^1). \tag{3.25}$$

From (3.25), we now know that $\sum_{i=0}^{n-1}q^{si}1_i\mathbf{T}(g)$ is a group-like element. Since $\mathbf{T}(g)=g_2^s\bowtie g$ and g_2 is group-like, $\sum_{i=0}^{n-1}q^{si}1_ig$ is a group-like element. Therefore, (3.16) is proved. As Δ is an algebra morphism,

$$\Delta\left(\sum_{i=0}^{n-1}q^{si}1_ip_0^1\right)=\Delta\left(\sum_{i=0}^{n-1}q^{si}1_i\right)\Delta(p_0^1)=\Delta\left(\sum_{i=0}^{n-1}q^{si}1_i\right)\Delta(\mathbf{T}(p_0^1)).$$

Using (3.25) directly, one can get the formula (3.17). Once the comultiplication rule is determined, the counit is clear now. Also, by the definition of the Drinfeld double, we know that $\alpha=g_2^{-s}\bowtie\varepsilon$, $\beta=1\bowtie\varepsilon$. From this and the comultiplication formulas (3.16) and (3.17), one can verify that (3.22) and (3.23) are the desired formulas for the antipode. ☐

Proof of Theorem 10. Define a map

$$\Psi:\ \mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)\to D(A(n,s,q)),\quad g_1\mapsto\sum_{i=0}^{n-1}q^{si}1_ig,\ g_2\mapsto g_2,$$
$$x\mapsto x,\quad y\mapsto\sum_{i=0}^{n-1}q^{si}1_ip_0^1.$$

By Proposition 13, it is an algebra morphism. It is also surjective by Theorem 5 (1). By Lemma 8, we have $\dim\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)=\dim D(A(n,s,q))$. Therefore, Φ is a bijection. To show the result, it is enough to show that it is a coalgebra morphism. This is a direct consequence of Proposition 14 by noting that in the formula (3.17), $\sum_{i=0}^{n-2}q^{si}1_ip_0^1=(\sum_{i=0}^{n-1}q^{si}1_ip_0^1)\sum_{j=1}^{n-1}1_j$ and $q^{s(n-1)}1_{n-1}p_0^1=(\sum_{i=0}^{n-1}q^{si}1_ip_0^1)1_0$. The theorem is proved. ☐

4. Twist equivalence

We give a sufficient condition to determine when $D(A(n,s,q))$ is not trivial, i.e., not twist equivalent to a Hopf algebra.

Theorem 15. *Assume that* $n = 2^m l$ *and* $s = 2^{m'} l'$ *with* $(l, 2) = (l', 2) = 1$. *If* $m' < m$, *then* $D(A(n, s, q))$ *is not twist equivalent to a Hopf algebra.*

Proof. By Theorem 10, we can identify $D(A(n, s, q))$ with $\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$. We construct an 1-dimensional representation for $\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$ through the following algebra morphism

$$\rho: \ \mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2) \to \kappa, \quad g_1 \mapsto -1, \quad g_2 \mapsto (-1)^{\frac{1}{(s,n)}}, \quad x \mapsto 0, \quad y \mapsto 0.$$

One can check directly that ρ is well-defined. Denote this representation by X. Let Rep-$\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$ be the representation category of $\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$. It is a tensor category.

Let $\langle X \rangle$ be the subtensor category generated by X. Explicitly, define

$$X^{\vec{\otimes} i} =: \overbrace{(\cdots(X \otimes X) \otimes X) \cdots)}^{i}.$$

Then the objects of $\langle X \rangle$ are direct sums of elements being in $\{X^{\vec{\otimes} i} | 0 \le i < (2s, n)\}$. Now assume that $\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$ is twist equivalent to a Hopf algebra. By the general principle of Tannaka–Krein duality (see, e.g., [3]), there is a fibre functor from the category Rep-$\mathrm{Q}_s\, \mathbf{u}_q(\mathfrak{sl}_2)$ to the category of κ-spaces. Thus its restriction to $\langle X \rangle$ is still a fibre functor. This implies the restriction of ϕ_s to $\langle X \rangle$ should be gotten from a 3-coboundary of $\mathbb{Z}_{(2s,n)}$. It is not hard to see that

$$\begin{aligned}
\phi_s|_{\langle X \rangle} &= \sum_{i,j,k=0}^{(2s,n)} \mathrm{q}^{s\frac{ni}{(2s,n)}\left[\frac{\frac{nj}{(2s,n)} + \frac{nk}{(2s,n)}}{n}\right]} 1_{\frac{ni}{(2s,n)}} \otimes 1_{\frac{nj}{(2s,n)}} \otimes 1_{\frac{nk}{(2s,n)}} \\
&= \sum_{i,j,k=0}^{(2s,n)} (-1)^{i[\frac{j+k}{(2s,n)}]} 1_{\frac{ni}{(2s,n)}} \otimes 1_{\frac{nj}{(2s,n)}} \otimes 1_{\frac{nk}{(2s,n)}}.
\end{aligned}$$

By the general theory of group cohomology, it is known that the 3-cocycle

$$f(g^i_{(2s,n)}, g^j_{(2s,n)}, g^k_{(2s,n)}) = (-1)^{i[\frac{j+k}{(2s,n)}]}, \quad 0 \le i, j, k < (2s, n)$$

is not a 3-coboundary where $g_{(2s,n)}$ denoting a generator of $\mathbb{Z}_{(2s,n)}$. That is a contradiction. □

Corollary 16. *If* n *is even and* s *is odd, then* $D(A(n, s, q))$ *is not twist equivalent to a Hopf algebra.*

Corollary 17. *The quasi-triangular quasi-Hopf algebra* $D(A(n, q))$ *is twist equivalent to* $\mathbf{u}_q(\mathfrak{sl}_2)$ *if and only if* n *is odd.*

Proof. As pointed out in Remark 4, $A(n, q) = A(n, n-1, q)$ and thus the "only if" part is just a direct consequence of Corollary 16. The sufficiency is prove in [10] by using conceptual way. Here we give another proof. Let $n = 2m + 1$ and construct $1^{n^2}_i := \frac{1}{n^2} \sum_{j=0}^{n^2-1} q^{-ij} (g_1^{m+1})^{ij}$. Define

$$J := \sum_{i,j=0}^{n^2-1} q^{i(j-j')} 1^{n^2}_i \otimes 1^{n^2}_j.$$

One can verify that $\phi_{J^{-1}} = (1\otimes J^{-1})(id\otimes\Delta)(J^{-1})\phi_{n-1}(\Delta\otimes id)(J)(J\otimes 1) = 1\otimes 1\otimes 1$. Thus $D(A(n,q))_{J^{-1}}$ is a Hopf algebra, which is $\mathbf{u}_q(\mathfrak{sl}_2)$ obviously. □

Remark 18.

(1) In particular, the quasi-triangular quasi-Hopf algebra $\mathrm{Q}_{n-1}\,\mathbf{u}_q(\mathfrak{sl}_2)$ is not twist equivalent to a Hopf algebra if n is even. In this case, we denote it by $\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$. Its representation theory will be given elsewhere [13].

(2) Theorem 15 implies that Rep-$\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$ is not monoidal equivalent to Rep-$\mathbf{u}_q(\mathfrak{sl}_2)$ or Rep-$\mathfrak{U}_q(\mathfrak{sl}_2)$. Here, $\mathfrak{U}_q(\mathfrak{sl}_2)$ is called a restricted quantum universal enveloping algebra, [21]. But is it possible that Rep-$\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$ is equivalent to Rep-$\mathbf{u}_q(\mathfrak{sl}_2)$ or Rep-$\mathfrak{U}_q(\mathfrak{sl}_2)$ as abelian categories? This is also not right since both $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$ have Steinberg modules, i.e., simple projective modules, while $\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$ has no such modules (see Remark 2.17 (3) in [13]). However, the quasi-Hopf algebra $\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$ has a more "symmetric" structure than $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$: all blocks of $\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$ have the same dimension and are Morita equivalent to each other. Moreover, the basic algebra of $\mathrm{Q}\,\mathbf{u}_q(\mathfrak{sl}_2)$ can be equipped with a Hopf algebra structure while there is no any Hopf structure on the basic algebras of $\mathbf{u}_q(\mathfrak{sl}_2)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)$. See Section 2 in [13].

(3) The subtensor category $\langle X\rangle$ constructed in the proof of Theorem 15 can be realized as the representation category of the following quasi-Hopf algebra. Let χ be the character of X and define $I := \bigcap_{i=0}^{(2s,n)} \operatorname{Ker}\chi^i$. I is a Hopf ideal of $\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)$. Then $\langle X\rangle$ is isomorphic to Rep-$\mathrm{Q}_s\,\mathbf{u}_q(\mathfrak{sl}_2)/I$.

(4) It is expected that the methods used in the present paper can be generalized to arbitrary simple Lie algebra. This is indeed the case and we realize it in [14].

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