REPRESENTATION DIMENSION FOR HOPF ACTIONS

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Abstract. Let H be a finite-dimensional Hopf algebra and assume that both H and H^* are semisimple. The main result of this paper is to show that representation dimension is an invariant under cleft extensions of H, that is, $\text{rep.dim}(A) = \text{rep.dim}(A\#_\sigma H)$. Some of applications of this equality are also given.

1. Introduction

Auslander introduced the concept of representation dimension of a finite-dimensional algebra in [2], which was a trial to give a reasonable way of measuring homologically how far a finite-dimensional algebra is from being of finite representation type. Recently, the interest in the representation dimension has revived, and many interesting connections have been established with different problems in representation theory, as well as with other areas. For details see [9, 10, 11, 12, 15, 16, 17]. It was already proved by Auslander in [2] that a finite-dimensional algebra A is representation-finite if and only if its representation dimension rep.dim $A \leq 2$.

But in general, it is quite hard to compute the representation dimension or even to control it. Also, we don't know how to determine when two algebras have the same representation dimensions. The main result of this paper is the following conclusion:

Theorem 1.1. *Assume that H is a semisimple cosemisimple finite-dimensional Hopf algebra, then* $\text{rep.dim}(A) = \text{rep.dim}(A\#_\sigma H)$.

Here we denote the representation dimension of a finite-dimension algebra A by $\text{rep.dim}(A)$. This result indeed implies that the representation dimension is an invariant under cleft extensions for a semisimple cosemisimple finite-dimensional Hopf algebra. As a byproduct of this theorem, we shall show that for a finite group G over a field k with characteristic p, if it has only one Sylow p-subgroup P, then $\text{rep.dim}(kG) = \text{rep.dim}(kP)$.

All preliminary notions and results that are relevant for our purpose are summarized in Section 2. And, the proof of Theorem 1.1 and three applications are given in Section 3.

2. Preliminaries

Throughout of the this paper, k denotes a field of any characteristic and all algebras are finite-dimensional k-algebras. All modules are finitely generated left modules. We freely use the results, notations, and conventions of [14] for Hopf algebras.

For a finite-dimensional algebra A, we denote by mod$A$ the category of finitely generated A-modules. We denote by gl.dim$A$ the global dimension of A and by $D := \mathrm{Hom}_k(\_, k)$ the standard duality between mod$A$ and modA^{op}. If $\mathcal{C}$ is a subcategory of mod$A$, we sometimes write $C \in \mathcal{C}$ to express that C is an object of $\mathcal{C}$. We denote by add$\mathcal{C}$ the full subcategory having as objects the direct sums of indecomposable summands of objects in $\mathcal{C}$ and, if M is a module, we abbreviate add$\{M\}$ as add$M$. Let $C \in \mathcal{C}$, a map $f\colon C \to X$ is called a *right $\mathcal{C}$-approximation* of X, if $\mathrm{Hom}_A(-, C) \xrightarrow{(-,f)} \mathrm{Hom}_A(-, M) \to 0$ is exact in $\mathcal{C}$, and, f is called a *right minimal $\mathcal{C}$-approximation* of M, if it is also right minimal, i.e., $h \in \mathrm{End}_A M$ is an automorphism whenever $fh = f$ (see [5, 7]).

We denote by Gen$M$ the full subcategory having as objects those modules X such that there is an epimorphism $M_0 \to X$ with $M_0 \in \mathrm{add}M$. Recall an A-module M is called a *generator* (or a *cogenerator*) of mod$A$ if $A \in \mathrm{add}M$ (or $D(A^{op}) \in \mathrm{add}M$).

The notion of representation dimension of an algebra was introduced in [2] by Auslander, and we refer the reader to [2] for the original definition. We shall rather use the following characterization, which also proved in [2].

Definition 2.1. *The representation dimension* rep.dim A *of* A *is defined as* $\inf\{\mathrm{gl.dim}\,\mathrm{End}(M) \mid M$ *is a generator-cogenerator for* mod $A\}$ *if* A *is non-semisimple; and* rep.dim $A = 1$ *if* A *is semisimple.*

The following lemma is well-known in [4] for the calculation of global dimensions for endomorphism algebras of modules which are generator-cogenerators.

Lemma 2.2. *Let* M *be a generator-cogenerator of modA and* $n \geq 3$ *be a natural number,* $\mathrm{gl.dim}(\mathrm{End}_A M) \leq n$ *if and only if for each indecomposable* A*-modules* X*, there is an exact sequence*

$$0 \to M_{n-2} \to \cdots \to M_0 \to X \to 0 \tag{2.1}$$

with $M_i \in \mathrm{add}\, M$ *for* $i = 0, 1, \cdots, n-2$*, such that* $0 \to \mathrm{Hom}_A(M, M_{n-2}) \to \cdots \to \mathrm{Hom}_A(M, M_0) \to \mathrm{Hom}_A(M, X) \to 0$ *is exact. Such an exact sequence* (2.1) *is called* an $(n-2)$-add M-resolution of X.

In this paper, we always assume that $A\#_\sigma H$ is an associative algebra. We also need the following result, which indeed implies that every $A\#_\sigma H$-module is relatively projective over A.

Lemma 2.3. ([13, Lemma 2.1]) *Let* H *be a semisimple Hopf algebra,* A *a twisted* H*-module algebra and* $\sigma \in \mathrm{Hom}_K(H \otimes H, A)$ *a 2-cocycle. Then for any* $X \in \mathrm{mod}\, A\#_\sigma H$*,* X *is isomorphic to a direct summand of* $(A\#_\sigma H) \otimes_A X$ *in* $\mathrm{mod}\, A\#_\sigma H$*, denoted it by* $X \mid (A\#_\sigma H) \otimes_A X$ *for short.*

3. Proof of Theorem 1.1 and applications

Before the proof of main conclusion, we need the following conclusions.

Lemma 3.1. *Let $X, M \in \operatorname{mod} A$ and $X = X_1 \oplus X_2 \in \operatorname{Gen} M$. If X has an n-$\operatorname{add} M$-resolution: $0 \to M_n \xrightarrow{f_n} M_{n-1} \to \cdots \to M_0 \xrightarrow{f_0} X \to 0$, then X_1 has an n-$\operatorname{add} M$-resolution: $0 \to M'_n \xrightarrow{f'_n} M'_{n-1} \to \cdots \to M'_0 \xrightarrow{f'_0} X_1 \to 0$.*

Proof. We claim that if there exists an exact sequence: $0 \to K_0 \to M_0 \xrightarrow{f_0} X \to 0$ in $\operatorname{mod} A$ with f_0 a right $\operatorname{add} M$-approximation of X, then there exists an epimorphism $f'_0 : M'_0 \to X_1$ in $\operatorname{mod} A$, which is a right $\operatorname{add} M$-approximation of X_1 and $K'_0(= \operatorname{Ker} f') \mid K_0$. Because $X_1 \in \operatorname{Gen} M$, there exists an epimorphism $f'_0 : M'_0 \to X_1$ in $\operatorname{mod} A$ which is a minimal right $\operatorname{add} M$-approximation of X_1 by ([7], Proposition 4.2). So we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K'_0 & \longrightarrow & M'_0 & \xrightarrow{f'_0} & X_1 & \longrightarrow & 0 \\
 & & \downarrow s & & \downarrow \alpha & & \downarrow i & & \\
0 & \longrightarrow & K_0 & \longrightarrow & M_0 & \xrightarrow{f_0} & X & \longrightarrow & 0 \\
 & & \downarrow t & & \downarrow \beta & & \downarrow p & & \\
0 & \longrightarrow & K'_0 & \longrightarrow & M'_0 & \xrightarrow{f'_0} & X_1 & \longrightarrow & 0
\end{array}$$

where $pi = 1_{X_1}$. The minimality of f'_0 implies that $\beta\alpha$ is an isomorphism and hence ts is also an isomorphism, which implies that s is a split monomorphism. The claim is proved. Then by using induction on n, we get the assertion easily. □

Lemma 3.2. *If there exists an exact sequence in $\operatorname{mod} A$, $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ such that $0 \to \operatorname{Hom}_A(V, N) \xrightarrow{f_*} \operatorname{Hom}_A(V, E) \xrightarrow{g_*} \operatorname{Hom}_A(V, M) \to 0$ is exact, then the sequence $0 \to \operatorname{Hom}_{M_n(A)}(FV, FN) \xrightarrow{(id \otimes_A f)_*} \operatorname{Hom}_{M_n(A)}(FV, FE) \xrightarrow{(id \otimes_A g)_*} \operatorname{Hom}_{M_n(A)}(FV, FM) \to 0$ is also exact, where $F = M_n(A) \otimes_A -$.*

Proof. The result seems known and we give a short proof for safety. Notice that $FV \cong V^{(n^2)}$ as right A-modules, and so we have an exact sequence $0 \to \operatorname{Hom}_A(FV, N) \xrightarrow{f_*} \operatorname{Hom}_A(FV, E) \xrightarrow{g_*} \operatorname{Hom}_A(FV, M) \to 0$. Since $\operatorname{Hom}_{M_n(A)}(FV, F-) \cong \operatorname{Hom}_A(V, F-) \cong \operatorname{Hom}_A(FV, -)$, we have the following commutative diagram:

$$\begin{array}{ccccc}
0 \to \operatorname{Hom}_{M_n(A)}(FV, FN) & \xrightarrow{(id \otimes_A f)_*} & \operatorname{Hom}_{M_n(A)}(FV, FE) & \xrightarrow{(id \otimes_A g)_*} & \operatorname{Hom}_{M_n(A)}(FV, FM) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 \to \operatorname{Hom}_A(FV, N) & \xrightarrow{f_*} & \operatorname{Hom}_A(FV, E) & \xrightarrow{g_*} & \operatorname{Hom}_A(FV, M) \to 0.
\end{array}$$

The exactness of lower row implies that the top row is exact. Thus we complete our proof. □

As a consequence of Lemma 3.2, we get the following result which is crucial in the proof of Theorem 1.1.

Proposition 3.3. *For a finite-dimensional semisimple Hopf algebra H, a twisted H-module algebra A, and a 2-cocycle $\sigma \in \operatorname{Hom}_k(H \otimes H, A)$. Suppose that there exists an exact sequence in $\operatorname{mod} A$, $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$ such that $0 \to$*

$\mathrm{Hom}_A(V,N) \overset{f_*}{\to} \mathrm{Hom}_A(V,E) \overset{g_*}{\to} \mathrm{Hom}_A(V,M) \to 0$ *is exact, then we have the following exact sequence*

$$0 \to \mathrm{Hom}_{A\#_\sigma H}(GV,GN) \xrightarrow{f'} \mathrm{Hom}_{A\#_\sigma H}(GV,GE) \xrightarrow{g'} \mathrm{Hom}_{A\#_\sigma H}(GV,GM) \to 0$$

where $G = A\#_\sigma H \otimes_A -$, $f' = (Gf)_*$ *and* $g' = (Gg)_*$.

Proof. Since $A\#_\sigma H$ is a free right A-module, there exists an exact sequence in $\mathrm{mod} A\#_\sigma H$:

$$0 \longrightarrow A\#_\sigma H \otimes_A N \xrightarrow{id\otimes_A f} A\#_\sigma H \otimes_A E \xrightarrow{id\otimes_A g} A\#_\sigma H \otimes_A M \longrightarrow 0 \ .$$

We need only to show that $g' = \mathrm{Hom}_{A\#_\sigma H}(A\#_\sigma H \otimes_A V, id \otimes g)$ is an epimorphism. Indeed, for any nonzero homorphism $\varphi \in \mathrm{Hom}_{A\#_\sigma H}(G(V), G(M))$, then $id \otimes \varphi \in \mathrm{Hom}_{A\#_\sigma H\#H^*}(A\#_\sigma H\#H^* \otimes_{A\#_\sigma H} G(V), A\#_\sigma H\#H^* \otimes_{A\#_\sigma H} G(M)) \cong \mathrm{Hom}_{M_n(A)}(M_n(A)\otimes_A V, M_n(A)\otimes_A M)$ by noting $A\#_\sigma H\#H^* \cong M_n(A)$ (see Section 9.4 in [14]). By Lemma 3.2, there exists a $\phi \in \mathrm{Hom}_{A\#_\sigma H\#H^*}(A\#_\sigma H\#H^* \otimes_{A\#_\sigma H} G(V), A\#_\sigma H\#H^* \otimes_{A\#_\sigma H} G(E))$, such that $id \otimes \varphi = (id \otimes_{A\#_\sigma H\#H^*} (id \otimes g))\phi$.

Let $B = A\#_\sigma H$ for short. Of course, now we have

$$id \otimes_{B\#H^*\#H} id \otimes \varphi = (id \otimes_{B\#H^*\#H} id \otimes_{B\#H^*} (id \otimes g))(id \otimes_{B\#H^*\#H} \phi) \quad (\star).$$

Since $B\#H^*\#H \cong B \otimes (H^*\#H)$ (see also Section 9.4 in [14]), $B|B\#H^*\#H$ as a B-B-bimodule. Thus,

$$B\#_\sigma H^*\#H \otimes_B G(V) = G(V) \oplus V',$$

$$B\#_\sigma H^*\#H \otimes_B G(E) = G(E) \oplus E',$$

$$B\#_\sigma H^*\#H \otimes_B G(M) = G(M) \oplus M',$$

for some B-modules V', E', M'. Hence

$$id \otimes_{B\#H^*\#H} id \otimes \varphi = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi' \end{pmatrix}, \quad id \otimes_{B\#H^*\#H} \phi = \begin{pmatrix} \phi_1 & * \\ * & \phi_2 \end{pmatrix},$$

$$id \otimes_{B\#H^*\#H} id \otimes_{B\#H^*} (id \otimes g) = \begin{pmatrix} id \otimes g & 0 \\ 0 & * \end{pmatrix},$$

where $\varphi' \in \mathrm{Hom}_B(V', M'), \phi_1 \in \mathrm{Hom}_B(G(V), G(E))$, and $\phi_2 \in \mathrm{Hom}_B(V', E')$. Thus, by the equality $(\star)$ we have that $\varphi = (id \otimes g)\phi_1$, and we obtain our claim. □

Proof of Theorem 1.1. By ([13], Proposition 2.2), the assertion holds provided either rep.dim(A) or rep.dim$(A\#_\sigma H)$ is at most two.

Now suppose that rep.dim$(A)= n(> 2)$ and M is a generator-cogenerator for $\mathrm{mod}\, A$ such that $\mathrm{gl.dim}(\mathrm{End}_A M) = n$. Also, for short, we take $B = A\#_\sigma H$. By $A \oplus D(A^{op}) \in \mathrm{add}\, M$, we have $B \cong B \otimes_A A \in \mathrm{add}(B \otimes_A M)$, and $D(B^{op}) \cong B \otimes_A D(A^{op}) \in \mathrm{add}(B \otimes_A M)$. So $B \otimes_A M$ is a generator-cogenerator for mod B. Let $X \in \mathrm{mod}\, B$ be any indecomposable module. Then by Lemma 2.2, as an A-module X has an $(n-2)$- add$M$-resolution:

$$0 \to M_{n-2} \overset{f_{n-2}}{\longrightarrow} M_{n-3} \to \cdots \to M_0 \overset{f_0}{\longrightarrow} X \to 0. \qquad (3.1)$$

By Proposition 3.3, the following sequence is an $(n-2)$- add$(B \otimes_A M)$-resolution for $B \otimes_A X$:

$$0 \to B \otimes_A M_{n-2} \overset{\mathrm{id}\otimes_A f_{n-2}}{\longrightarrow} \cdots \to B \otimes_A M_0 \overset{\mathrm{id}\otimes_A f_0}{\longrightarrow} B \otimes_A X \to 0. \tag{3.2}$$

By Lemma 2.3 and Lemma 3.1, X has an $(n-2)$-add$(B \otimes_A M)$-resolution. Thus by Lemma 2.2, we have that rep.dim $B \leq$ gl.dim $\mathrm{End}_B(B \otimes M) \leq$ rep.dim(A). From the above steps, we have that rep.dim($B\#H^*) \leq$ rep.dim B. By Blattner-Montgomery Duality Theorem (see Section 9.4 in [14]), $B\#H^* = (A\#_\sigma H)\#H^* \cong M_n(A)$ and A is Morita equivalent to $M_n(A)$, we have that

$$\text{rep.dim}(A) = \text{rep.dim}(M_n(A)) = \text{rep.dim}(B\#H^*) \leq \text{rep.dim}\, B \leq \text{rep.dim}(A).$$

Thus, we have rep.dim(A)=rep.dim B. The theorem is proved. □

Remark 3.4. At the first glance, it seems that $A\#_\sigma H$ is isomorphic to a direct sum of copies of ${}_AA_A$ as A-A-bimodule and thus our result will be trivial. But this is far from true. Actually, we even *do not* have $A|A\#_\sigma H$ as A-A-bimodules. For example, let H' be any finite-dimensional Hopf algebra which is not of finite representation type (e.g. $\mathbf{u}_q(sl_2)$). Assume that we have $H'|(H'\#(H')^*)$ as H'-H'-bimodules. Using Blattner-Montgomery Duality Theorem again, $(H'\#(H')^*)$ is a simple algebra and thus of finite representation type. But this will imply H' is of finite representation type too (see Chapter VI, Lemma 3.1 (a) in [6]) which is a contradiction.

Let H be a Hopf algebra. Recall $A \subset B$ is called a (right) H-extension, if B is a right H-comodule algebra with structure map ρ satisfying $B^{coH} = A$. Here B^{coH} is defined as the subcomodule $\{b \in B : \rho(b) = b \otimes 1\}$. An H-extension $A \subset B$ is called H-cleft if there exists a right H-comodule map $\gamma : H \to B$ which is convolution invertible. Doi and Takeuchi proved that $A \subset B$ is H-cleft if and only if $B \cong A\#_\sigma H$ (Theorem 7.2.2 in [14]). This indeed imply the following result.

Corollary 3.5. *Let H be a simisimple cosemisimple Hopf algebra and $A \subset B$ be H-cleft, then* rep.dim A = rep.dim B.

Let k be a field of characteristic p. Let G be a finite group and $N \subset G$ a normal subgroup of G. So we may write $kG = kN\#_\sigma k(G/N)$, a crossed product of kN with the quotient group algebra $k(G/N)$. Since clearly $k(G/N)$ is semisimple when N is a normal Sylow p-subgroup, we have the following corollary.

Corollary 3.6. *If P is a normal Sylow p-subgroup of of G, then* rep.dim(kG) = rep.dim(kP).

Remark 3.7. One direction in Corollary 3.6, that is, rep.dim$(kG) \geq$ rep.dim(kP) is indeed a special case of Proposition 4.1 in [18]. Our method, relied heavily on the speciality of crossed product, is quite different from that used in [18]. In fact, we do not know how to show Proposition 3.3 by applying the methods developed in [18] only.

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References

[1] Alperin, J. L., Local representation theory, Cambridge University Press, Cambridge, 1986.
[2] Auslander, M., Reprensentation dimension of artin algebras, Math. Notes, Queen Mary College, London, 1971.
[3] Anderson, F. W. Fuller, K. R., Rings and categories of modules, Springer, 1992.
[4] Auslander, M., Representation theory of Artin algebras I., Commun. Algebra, (1974), 1(3): 177C268.
[5] Auslander, M. Reiten, I.: Representation theory of artin algebras IV: Invariants given by almost split sequences, Comm. Algebra, 1977, 5: 443-581.
[6] M. Auslander, I. Reiten, S. Smalϕ, Representation theory of artin algebras, Cambridge Studies in Advanced Mathematics, 36. Cambridge University Press, 1995.
[7] Auslander, M., Smalϕ, S.O., Preprojective modules over Artin algebras, J. Algebra, 1980, 66: 61-22.
[8] Blattner R. J., Montgomery, S., Crossed products and inner actions of Hopf algebras, Trans. Amer. Math. Soc., 1986, 298: 671-711.
[9] Doi, Y., Takeuchi, M., Cleft comodule algebras for a bialgebra, Commun. Algebra, 1986, 14: 801-818.
[10] Erdmann, K., Holm, T. O. Iyama, et.al, Radical embeddings and representation dimension, Adv. Math. 2004, 185: 159-177.
[11] Igusa, K., Todorov, G., On the finitistic global dimension conjecture for artin algebras, in: Representations of Algebras and Related Topics, in: Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, 201-204.
[12] Iyama O., Finiteness of representation dimension, Proc. Amer. Math. Soc. 2003, 131: 1011-1014.
[13] Li, F., Zhang, M. M., Invariant properties of representations under cleft extensions, Science in China, 2007, 50(1): 121-131.
[14] Montgomery, S., Hopf algebras and their actions on rings. **CBMS**, Lecture in Math.; Providence, RI, (1993); Vol. 82.
[15] Rouquier, R., Representation dimension of exterior algebras, Invent. math., 2006, 165: 357-367.
[16] Xi, C.C., On the representation dimension of finite-dimensional algebras, J. Algebra, 2000, 226: 332-346.
[17] Xi, C.C., Representation dimension and quasi-hereditary algebras, Adv. Math., 2002, 168: 193-212.
[18] Xi, C.C., Adjoint functors and representation dimensions. Acta Math. Sin. (Engl. Ser.), 2006, 22(2), 625-640.

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