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Journal of Algebra 299 (2006) 841–853
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On the structure of tame graded basic Hopf algebras ☆

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Received 11 April 2005

Available online 5 October 2005

Communicated by Susan Montgomery

Abstract

In this paper, firstly, we give a complete list of tame local graded Frobenius algebras, then, by the method of bosonization and this list, we determine the structure of tame (radically) graded basic Hopf algebras completely.

1. Introduction

Throughout this paper k denotes an algebraically closed field. All spaces are k-spaces. By an algebra we mean a finite dimensional associative algebra with identity element. We freely use the results, notations, and conventions of [13].

In the view point of representation type, every finite dimensional algebra exactly belongs to one of following three types: finite representation type, tame type and wild type. Basic Hopf algebras and their duals, pointed Hopf algebras, have been studied by many authors [3,10]. Our intention is to classify finite dimensional basic Hopf algebras according to their representation types.

☆ Project supported by the Program for New Century Excellent Talents in University (No. 04-0522) and the Natural Science Foundation of Zhejiang Province of China (No. 102028).

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.08.027

In [11], the authors have classified all basic Hopf algebras of finite representation type and show that they are all monomial Hopf algebras (see [5]). This paper can be viewed as a try to classify tame basic Hopf algebras. We determine the structure of tame (radically) graded basic Hopf algebras completely (see Theorem 5.4).

We now give a short description of our method. Let H be a finite dimensional basic Hopf algebra over k. Then its radically graded algebra $\operatorname{gr} H = H/J_H \oplus J_H/J_H^2 \oplus \cdots$, where J_H is the Jacobson radical of H, is a Hopf algebra too. By a work of Radford (see [14]), we have $\operatorname{gr} H \cong R_H \times H/J_H$, where $R_H = \{a \in H \mid (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$ and $\pi : H \to H/J_H$ the canonical epimorphism. One of our results is that $\operatorname{gr} H$ and R_H have the same representation type and R_H is a local graded Frobenius algebra (see Proposition 5.3). This result help us to reduce the study of tame graded basic Hopf algebras to that of tame local graded Frobenius algebras. Fortunately, we classify all tame local graded Frobenius algebras and show that there are only five classes of local algebras which are tame graded Frobenius (see Theorem 3.1).

In Section 2, some preliminaries are given. A complete list of tame local graded Frobenius algebra is given in Section 3. The relationship between representation type of A and that of $A\#H$, when H and H^* are semisimple, is described and show that they are same in Section 4. In Section 5, Theorem 5.4 tell us that there are at most five classes algebras can play the same role as R_H in the decomposition $R_H \times H/J_H$.

2. Preliminaries

A *quiver* is an oriented graph $\Gamma = (\Gamma_0, \Gamma_1)$, where Γ_0 denotes the set of vertices and Γ_1 denotes the set of arrows. $k\Gamma$ denotes its *path algebra*. An ideal I of $k\Gamma$ is called *admissible* if $J^N \subset I \subset J^2$ for some $N \geqslant 2$, where J is the ideal generated by all arrows.

For a basic algebra A, by the Gabriel's theorem, there is a unique quiver Γ_A, and an admissible ideal I of $k\Gamma_A$, such that $A \cong k\Gamma_A/I$ (see [4]). The quiver Γ_A is called the *Ext-quiver* of A. The Ext-quiver of A also can be defined directly: Its vertices are indexed by all non-isomorphic simple A-modules and the number of arrows from simple A-module S_i to simple A-module S_j is defined to be $\dim_k \operatorname{Ext}_A^1(S_i, S_j)$. We always denote the Jacobson radical of A by J_A. Define e_i to be the primitive idempotent element of A satisfying $(A/J_A)e_i \cong S_i$, then we know that $\dim_k \operatorname{Ext}_A^1(S_i, S_j) = \dim_k e_j(J_A/J_A^2)e_i$ (see page 68 in [4]).

An algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are finite number of A-$k[T]$-bimodules M_i which are finitely generated free as right $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{k[T]} k[T]/(T - \lambda)$ for $\lambda \in k$. We say that A is of *wild type* or A is a *wild* algebra if there is a finitely generated A-$k\langle X, Y\rangle$-bimodule B which is free as a right $k\langle X, Y\rangle$-module such that the functor $B \otimes_{k\langle X,Y\rangle} -$ from mod-$k\langle X, Y\rangle$, the category of finitely generated $k\langle X, Y\rangle$-modules, to mod-A, the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms. See [8].

3. A complete list of tame local graded Frobenius algebras

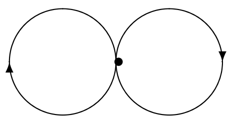
Denote the characteristic of k by char k. The main result of this section is the following.

Theorem 3.1. *Let Λ be a tame local graded Frobenius algebra. If* $\operatorname{char} k \neq 2$, *then* $\Lambda \cong k\langle x, y\rangle / I$ *where I is one of forms*:

(1) $I = (x^2 - y^2, yx - ax^2, xy)$ *for* $0 \neq a \in k$;
(2) $I = (x^2, y^2, (xy)^m - a(yx)^m)$ *for* $0 \neq a \in k$ *and* $m \geqslant 1$;
(3) $I = (x^n - y^n, xy, yx)$ *for* $n \geqslant 2$;
(4) $I = (x^2, y^2, (xy)^m x - (yx)^m y)$ *for* $m \geqslant 1$;
(5) $I = (yx - x^2, y^2)$.

We want to prove Theorem 3.1 now. Some preliminaries must be given at first. It is easy to know that a local algebra is Frobenius if and only if the dimension of its socle equals to one. In this section, Λ always denotes a local Frobenius algebra and J_Λ its Jacobson radical.

Any tame local algebra A must have a quiver of the form



We denote this quiver by Q. By the Gabriel's theorem (see [4]), we know $A \cong k\langle x, y\rangle / I$ for some ideal $J^2 \subseteq I \subseteq J^N$ where J is the ideal of $k\langle x, y\rangle$ generated by x, y and $N \geqslant 2$. Therefore, if A is Frobenius then $\dim_k A \geqslant 4$. There is a natural filtration on A. We denote its graded algebra associated to this filtration by $\operatorname{gr} A$. We say A is *graded* if $A \cong \operatorname{gr} A$.

For convention, we always denote the image of x, y in A by x, y too.

Lemma 3.2. *Let $\Lambda = kQ/I$ be a local graded Frobenius algebra such that J_Λ^2 is generated by x^2 and y^2. Then $xy = 0$ if and only if $I = (x^2 - y^2, yx - ax^2, xy)$ for $0 \neq a \in k$ or $xy = yx = 0$.*

Proof. It is enough to prove the necessity. By assumption, we have that Λ is spanned by $1, x, x^2, \ldots, y, y^2, \ldots$. We may write $yx = x^c w + y^d z$ where w, z are units with $w \in k[x]$, $z \in k[y]$, $c, d \geqslant 2$. Then $0 = xyx = x^{c+1} w$ and then $x^{c+1} = 0$. Since also $x^c y = 0$, it follows that $x^c \in \operatorname{soc} \Lambda$, the socle of Λ. Moreover, $0 = yxy = y^{d+1} z$ and we deduce that $y^{d+1} = 0$. Since also $xy^d = 0$, it follows that $y^d \in \operatorname{soc} \Lambda$. This shows that $yx \in \operatorname{soc} \Lambda$ since $\operatorname{soc} \Lambda$ is an ideal of Λ.

Assume $yx \neq 0$ and thus either x^c or y^d is not zero. Not loss generality, we assume $y^d \neq 0$. Since Λ is Frobenius, $\dim_k \operatorname{soc} \Lambda = 1$. It follows $x^c = by^d$ for $b \in k$. If $b = 0$, then $x^c = 0$ and $x^{c-1} \in \operatorname{soc} \Lambda$. This implies there exists a positive integer i such that $0 \neq x^i \in \operatorname{soc} \Lambda$. Using the fact that the dimension of $\operatorname{soc} \Lambda$ is one again, we have $x^i = b' y^d$ for

$0 \neq b' \in k$. Let $y' = \sqrt[d]{b'}y$ and then $x^i = y'^d$. Since $y'x \neq 0$ and $\dim_k \operatorname{soc} \Lambda = 1$, $y'x = ax^i$ for $0 \neq a \in k$. Since Λ is a graded algebra, we know $i = 2$ and $d = 2$. That is to say, there is an epimorphism $kQ/(x^2 - y'^2, y'x - ax^2, xy') \twoheadrightarrow \Lambda$. Since clearly $\dim_k kQ/(x^2 - y'^2, y'x - ax^2, xy') = 4$, $\Lambda \cong kQ/(x^2 - y'^2, y'x - ax^2, xy')$. □

Lemma 3.3. *Assume that* $\operatorname{char} k \neq 2$ *and* Λ *is a* 4*-dimensional local graded Frobenius algebra. Then* Λ *is isomorphic to one of the following algebras*:

(1) $kQ/(x^2 - y^2, yx - ax^2, xy)$ *for* $0 \neq a \in k$;
(2) $kQ/(x^2, y^2, xy - ayx)$ *for* $0 \neq a \in k$.

Proof. Let x, y be generators of J_Λ. Since $\dim_k \Lambda = 4$ and Λ is Frobenius, xy and yx belong to the socle of Λ.

(I) Assume $xy = 0$. If $yx \neq 0$, then $y^2 \neq 0$ and $x^2 \neq 0$ since $\dim_k \Lambda = 1$. Therefore, by Lemma 3.2, $\Lambda \cong kQ/(x^2 - y^2, yx - ax^2, xy)$ for $0 \neq a \in k$.

If $yx = 0$. In this case, we know that $x^2 = ay^2$ for $0 \neq a \in k$. Let $u \in k$ with $u^2 = -a$, then, by $\operatorname{char} k \neq 2$, $X = x + uy$, $Y = x - uy$ are generators. And, $X^2 = Y^2 = x^2 + u^2y^2 = x^2 - ay^2 = 0$, $XY = YX = x^2 - u^2y^2 = x^2 + ay^2$. Therefore, $\Lambda \cong k\langle X, Y\rangle/(X^2, Y^2, XY - YX)$ which is a special case of (2).

(II) Assume $xy \neq 0 \neq yx$. Then $xy = cyx$ for $0 \neq c \in k$. By $\dim_k \operatorname{soc} \Lambda = 1$, we have $x^2 = axy$ and $y^2 = bxy$. If $a = b = 0$, then $\Lambda \cong kQ/(x^2, y^2, xy - cyx)$. Otherwise, no loss generality, assume $a \neq 0$. Let $Y = x - ay$, then $xY = 0$. Therefore, we are in case (I) again. □

Lemma 3.4. *Let* Λ *be a local graded Frobenius algebra. Then*

(i) *If* Λ *is tame then* $\dim_k J_\Lambda^2/J_\Lambda^3 \leqslant 2$.
(ii) *If* $\dim_k J_\Lambda^2/J_\Lambda^3 \leqslant 1$ *then* $\dim_k \Lambda = 4$ *or* Λ *is an algebra as in Theorem* 3.1(3).

Proof. (i) If $\dim_k J_\Lambda^2/J_\Lambda^3 \geqslant 3$, then there is a homomorphic image which is wild (see (2.1) of [15]). This implies Λ is wild which contradict the assumption that Λ is tame.

(ii) Suppose now that $\dim_k J_\Lambda^2/J_\Lambda^3 \leqslant 1$. Then the dimension must be 1, since otherwise x, y would lie in $\operatorname{soc} \Lambda$ and $\operatorname{soc} \Lambda$ would not be simple. Thus x^2, y^2, xy, yx cannot be zero simultaneously.

Case (1): Either $x^2 \neq 0$ or $y^2 \neq 0$. Not loss generality, assume $x^2 \neq 0$. Therefore, $xy = ax^2$, $yx = bx^2$ and $y^2 = cx^2$ for $a, b, c \in k$ since Λ is graded and $\dim_k J_\Lambda^2/J_\Lambda^3 = 1$.

If $x^3 \neq 0$, then $xyx = ax^3 = bx^3$ and so $a = b$. This implies that $xy = yx$. That is to say Λ is commutative and Λ is a symmetric algebra. By a consequence of K. Erdmann (see Lemma III.4 of [8]), $\dim_k \Lambda = 4$ or $\Lambda \cong kQ/(x^m - y^n, xy, yx)$. Since Λ is graded, $m = n$.

If $x^3 = 0$, then $x^2y = ax^3 = xyx = 0$, $y^2x = cx^3 = xy^2 = 0$, $yx^2 = bx^3 = 0$, $yxy = bx^2y = bx(xy) = abx^3 = 0$, $y^3 = cx^2y = cax^3 = 0$. That is to say, $J_\Lambda^3 = 0$ and so $\dim_k J_\Lambda^2 = 1$. Therefore, $\dim_k \Lambda = 4$.

Case (2): Assume $x^2 = 0 = y^2$. We claim that $xy \neq 0 \neq yx$. Otherwise, x or y will belong to $\operatorname{soc} \Lambda$ and so $\dim_k \operatorname{soc} \Lambda > 1$ which is impossible since Λ is Frobenius. Therefore, $xy = ayx$ for $0 \neq a \in k$ and $\dim_k \Lambda = 4$. □

The following lemma, due to K. Erdmann, is given in [8, p. 84].

Lemma 3.5. *Let A be a tame local algebra with the quiver Q, of dimension* 5, *with $J^3 = 0$. Then $\Lambda \cong kQ/L$ where L is one of the following ideals*:

(1) (xy, yx),
(2) $(yx - x^2, xy)$,
(3) $(yx - x^2, xy - ay^2)$ *where $a \in k$ and* $0 \neq a \neq 1$,
(4) (x^2, y^2),
(5) $(yx - x^2, y^2)$.

Lemma 3.6. *Let Λ be a local graded Frobenius algebra such that xy and yx lie in J_Λ^3. Then $\Lambda \cong kQ/(x^n - y^n, xy, yx)$ for some positive integer n.*

Proof. By assumption, xy and yx belong to J_Λ^3. Since Λ is a graded algebra, $xy = yx = 0$. Since Λ is a finite dimensional algebra, there exist positive integers m_x and m_y such that $x^{m_x} = y^{m_y} = 0$ and $x^{m_x - 1} \neq 0 \neq y^{m_y - 1}$. Therefore, $x^{m_x-1}, y^{m_y-1} \in \operatorname{soc} \Lambda$. Thus $x^{m_x-1} = cy^{m_y-1}$ for $c \neq 0$. Let $y' = \sqrt[m_y-1]{c}\, y$, then $x^{m_x-1} = y'^{m_y-1}$. By Λ is a graded algebra, $m_x - 1 = m_y - 1$. This means we get our desired conclusion. □

Lemma 3.7. *Let Λ be a local graded Frobenius algebra such that $yx - x^2$ and xy lie in J_Λ^3. Then $\Lambda \cong kQ/(x^2 - y^2, yx - ax^2, xy)$ for $0 \neq a \in k$.*

Proof. Claim: *J_Λ does not have generators x', y' with $(x'y')$ and $(y'x')$ lying in J_Λ^3*. This claim was proved in [8] (see Lemma III.7 of [8]).

By assumption, $yx - x^2, xy \in J_\Lambda^3$. Since Λ is graded, we have $yx = x^2$ and $xy = 0$. We know that $yx \neq 0$, otherwise it will contradict above claim. By Lemma 3.2, we get the desired conclusion. □

Lemma 3.8. *Let Λ be a local graded Frobenius algebra such that $yx - x^2$ and $xy - ay^2$ lie in J_Λ^3 where $0 \neq a \neq 1$. Then* $\dim_k \Lambda = 4$.

Proof. By assumption and Λ is graded, $yx = x^2$ and $xy = ay^2$ for $0 \neq a \neq 1$. Therefore, $x^3 = xyx = ay^2x = ayx^2 = ax^3$ and $x^2y = axy^2 = a^2y^3 = ayxy = ax^2y$. Since $a \neq 1$, $x^3 = 0 = x^2y$. Since $a \neq 0$, $xyx = y^2x = yx^2 = xy^2 = y^3 = yxy = 0$. That is $J_\Lambda^3 = 0$ and $J_\Lambda^2 \subseteq \operatorname{soc} \Lambda$. This implies that $\dim_k \Lambda = 4$. □

Lemma 3.9. *Let Λ be a local graded Frobenius algebra such that x^2 and y^2 lie in J_Λ^3. Then $\Lambda \cong kQ/(x^2, y^2, (xy)^m - a(yx)^m)$ or $\Lambda \cong kQ/(x^2, y^2, (xy)^m x - (yx)^m y)$ for some positive integer m and $a \neq 0$.*

Proof. As before, assumption implies $x^2 = y^2 = 0$. Let m, n be the integers such that $(xy)^m \neq 0$, $(xy)^{m+1} = 0$, $(yx)^n \neq 0$ and $(yx)^{n+1} = 0$. Therefore, $(xy)^m x$, $(yx)^n y \in \operatorname{soc} \Lambda$.

If $(xy)^m x \neq 0 \neq (yx)^n y$, then $(yx)^n y = a(xy)^m x$ which implies $m = n$. Let $y' = ay$, $(y'x)^m y' = (xy')^m x$. That is, $\Lambda \cong kQ/(x^2, y'^2, (xy')^m x - (y'x)^m y')$.

If $(xy)^m x \neq 0$ while $(yx)^n y = 0$. Thus $0 \neq (yx)^n \in \operatorname{soc} \Lambda$. It follows that $(xy)^m x = a(yx)^n$ for $a \neq 0$. This is impossible sine $(xy)^m x$ and $(yx)^n$ have different degrees. Similarly, there is no possibility such that $(xy)^m x = 0$ while $(yx)^n y \neq 0$.

The only left case is that $(xy)^m x = (yx)^n y = 0$. Therefore, $(xy)^m$, $(yx)^n$ belong to the socle of Λ and thus $(xy)^m = a(yx)^n$ for $0 \neq a \in k$. By Λ is graded, $m = n$ and we prove this lemma. □

Lemma 3.10. *Let Λ be a local graded Frobenius algebra such that $yx - x^2$ and y^2 lie in J_Λ^3. Then $\Lambda \cong kQ/(yx - x^2, y^2)$ or* $\dim_k \Lambda = 4$.

Proof. As before, $yx = x^2$ and $y^2 = 0$. It follows that $xyx = x^3 = yx^2 = y^2x = 0$ and so $J_\Lambda^3 = (yxy)$ and $J_\Lambda^4 = 0$. Therefore, $\dim_k \Lambda \leqslant 6$.

If $\dim_k \Lambda \neq 4$, then $\dim_k \Lambda = 5$ or 6. If $\dim_k \Lambda = 6$, then $\Lambda \cong kQ/(yx - x^2, y^2)$.

Claim: $\dim_k \Lambda \neq 5$. Otherwise, $\dim_k J_\Lambda^2/J_\Lambda^3 = 1$ (if $\dim_k J_\Lambda^2/J_\Lambda^3 = 2$, then $\dim_k \operatorname{soc} \Lambda = 2$). By Lemma 3.4, we know that $\dim_k \Lambda = 4$ or $\Lambda \cong kQ/(x^n - y^n, xy, yx)$ which contradict $\dim_k \Lambda = 5$. □

Proof of Theorem 3.1. Since Λ is tame, $\dim_k J_\Lambda^2/J_\Lambda^3 \leqslant 2$ by Lemma 3.4.

If $\dim_k J_\Lambda^2/J_\Lambda^3 = 1$, Lemma 3.4 shows that Λ is one of algebras of (1)–(3) listed in this theorem.

If $\dim_k J_\Lambda^2/J_\Lambda^3 = 2$, then $\dim_k \Lambda/J_\Lambda^3 = 5$. This means that Λ/J_Λ^3 satisfies the conditions of Lemma 3.5. Therefore, Lemmas 3.6–3.10 give our desire conclusion. □

4. The relationship between the representation type of A and that of $A\#H$

In this section, our main task is to prove that the representation type of H-module algebra A is the same with that of $A\#H$ when H and H^* are semisimple.

We fix a notation at first. Let R be a ring and M, N two R-modules. If M is a direct summand of N as a R-module, then we denote it by $M|N$.

Lemma 4.1. *Let H be a semisimple Hopf algebra and A an H-module algebra. For any finitely generated $A\#H$-module X, $X|(A\#H) \otimes_A X$.*

Proof. Define $\varphi : (A\#H) \otimes_A X \to X$ by $(a\#h) \otimes x \mapsto (a\#h)x$ for $a \in A$, $h \in H$ and $x \in X$. Clearly, φ is an $A\#H$-epimorphism.

Let $0 \neq t \in \int_H^r$ and assume $\varepsilon(t) = 1$. Define $\psi : X \to (A\#H) \otimes_A X$ by $x \mapsto S(t') \otimes t''x$ for $x \in X$. Here we write $\Delta(h) = h' \otimes h''$ omitting summation symbol and index. We have, for $a \in A$, $h \in H$, $x \in X$,

$$\begin{aligned}\psi(ax) &= S(t') \otimes t''ax = S(t') \otimes (t'' \cdot a)t'''x \\ &= S(t')(t'' \cdot a) \otimes t'''x = \big(S(t'')t''' \cdot a\big)\#S(t') \otimes t''''x \\ &= \big(a\#S(t')\big) \otimes t''x = a\psi(x), \\ \psi(hx) &= S(t') \otimes t''hx = \varepsilon(h')S(t') \otimes t''h''x \\ &= h'S(h'')S(t') \otimes t''h'''x = h'S(t'h'') \otimes t''h'''x \\ &= hS(t') \otimes t''x = h\psi(x).\end{aligned}$$

Therefore, ψ is an $A\#H$-morphism. But, clearly, $\varphi\psi = \mathrm{id}_X$. Thus $X|(A\#H) \otimes_A X$. □

Proposition 4.2. *Assume H, H^* are semisimple Hopf algebras. Let A be an H-module algebra, then A is of finite representation type if and only if $A\#H$ is so.*

Proof. "*Only if part*": Let $\{B_1, \ldots, B_t\}$ be a complete set of non-isomorphic indecomposable A-modules. Suppose X is an indecomposable $A\#H$-module. Then viewing X as an A-module, we have that $X \cong \bigoplus_{j=1}^t n_j B_j$ and so $(A\#H) \otimes_A X \cong \bigoplus_{j=1}^t (n_j (A\#H) \otimes_A B_j)$. Since, by Lemma 4.1, X is an $A\#H$-summand of $(A\#H) \otimes_A X$, we have that X is an $A\#H$-summand of $(A\#H) \otimes_A B_i$ for some i. Therefore, the non-isomorphic indecomposable $A\#H$-summand of all the $(A\#H) \otimes_A B_i$ give a complete set of non-isomorphic indecomposable $A\#H$-module. Since this set is obviously finite, $A\#H$ is of finite representation type.

"*If part*": By the proof of necessity, we know that $(A\#H)\#H^*$ is of finite representation type too. By Blattner–Montgomery Duality Theorem (see Section 9.4 in [13]), $(A\#H)\#H^* \cong M_n(A)$ which is Morita equivalent to A. Thus A is of finite representation type. □

Let Λ be a finite dimensional algebra. We define a category called *generic category* (since its close relation with tame algebras and thus with *generic modules* (see [6])) whose objects are all Λ-$k[T]$-bimodules which are finitely generated free as right $k[T]$-modules and whose morphisms are Λ-$k[T]$-morphisms. We denote this category by $GC(\Lambda)$.

Lemma 4.3. *Let $X \in GC(\Lambda)$. X is indecomposable in $GC(\Lambda)$ if and only if $X \otimes_{k[T]} k[T]/(T - \lambda)$ is indecomposable as a Λ-$k[T]/(T - \lambda)$-bimodule for $\lambda \in k$.*

Proof. "*If part*": Clearly, if X is decomposable, then $X \otimes_{k[T]} k[T]/(T - \lambda)$ decomposable too.

"*Only if part*": Assume $X \otimes_{k[T]} k[T]/(T - \lambda) = N_1 \oplus N_2$ for Λ-$k[T]/(T - \lambda)$-bimodules N_1 and N_2. Then

$$\begin{aligned}X &\cong X \otimes_{k[T]} k[T]/(T - \lambda) \otimes_{k[T]/(T-\lambda)} k[T] \\ &= \big(N_1 \otimes_{k[T]/(T-\lambda)} k[T]\big) \oplus \big(N_2 \otimes_{k[T]/(T-\lambda)} k[T]\big)\end{aligned}$$

where $k[T]$ is a $k[T]/(T-\lambda)$-module since $k[T]/(T-\lambda) \cong k$ as algebras. Since, by assumption, X is indecomposable, $N_1 = 0$ or $N_2 = 0$. That is, $X \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable. □

Proposition 4.4. *Assume H, H^* are semisimple Hopf algebras. Let A be an H-module algebra, then A is tame if and only if $A\#H$ is tame.*

Proof. "*Only if part*": We will prove that $A\#H$ is tame through the definition of tame algebras. Clearly, $A\#H$ is not finite representation type. Otherwise, A is finite representation type by Proposition 4.2. Let d be a positive integer and X an indecomposable $(A\#H)$-module. Assume $\dim_k X = d$. By Lemma 4.1, $X | (A\#H) \otimes_A X$. We denote X by X_A when we consider X as an A-module through the canonical way. Assume $X_A = X_1 \oplus \cdots \oplus X_m$ as A-modules. Thus, for some $i \in \{1, \ldots, m\}$, $X | (A\#H) \otimes_A X_i$.

Since the dimension of X is finite, the dimension of X_i is finite too. Since A is a tame algebra, there are finite number of A-$k[T]$-bimodules M_j $(j = 1, \ldots, n)$ which are free as right $k[T]$-modules such that all but finite number of indecomposable A-modules of dimension not bigger than d are isomorphic to $M_j \otimes_{k[T]} k[T]/(T-\lambda)$ for some j and some $\lambda \in k$. In order to prove that $A\#H$ is a tame algebra, there is no harm in assuming that $X_i \cong M_{i_j} \otimes_{k[T]} k[T]/(T-\lambda)$ for some $i_j \in \{1, \ldots, n\}$ and some $\lambda \in k$. Clearly, $(A\#H) \otimes_A M_{i_j} \in GC(A\#H)$.

We decompose $(A\#H) \otimes_A M_{i_j}$ into direct sum of indecomposable objects in $GC(A\#H)$, $(A\#H)_A \otimes M_{i_j} = \bigoplus_{k \in I_j} M_{i_j}^k$ for a finite index set I_j. By Lemma 4.3, $M_{i_j}^k \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable as an $A\#H$-$k[T]/(T-\lambda)$-bimodule too. This is equivalent to saying that $M_{i_j}^k \otimes_{k[T]} k[T]/(T-\lambda)$ is indecomposable as an $A\#H$-module since $k[T]/(T-\lambda) \cong k$. By discussions above, $X | (A\#H) \otimes_A X_i = (A\#H) \otimes_A M_{i_j} \otimes_{k[T]} k[T]/(T-\lambda) = \bigoplus_k M_{i_j}^k \otimes_{k[T]} k[T]/(T-\lambda)$. This implies, $X \cong M_{i_j}^k \otimes_{k[T]} k[T]/(T-\lambda)$ for some $i_j \in \{1, \ldots, n\}$, $k \in I_j$ and $\lambda \in k$. Since the set $\{M_{i_j}^k\}_{i_j \in \{1,\ldots,n\}, k \in I_j}$ is finite, $A\#H$ is tame too.

"*If part*": Similar to the proof of the "If part" of Proposition 4.2. But for completeness, we write it out. By the proof of necessity, we know that $(A\#H)\#H^*$ is tame. By Blattner–Montgomery Duality Theorem (see Section 9.4 in [13]), $(A\#H)\#H^* \cong M_n(A)$ which is Morita equivalent to A. Thus A is tame. □

Theorem 4.5. *Assume H, H^* are semisimple Hopf algebras. Let A be an H-module algebra, then A and $A\#H$ have the same representation type.*

Proof. Recall Drozd's celebrated tame-and-wild theorem states that a finite dimensional algebra, which is not finite representation type, is either tame or wild (see [7]). Thus by Propositions 4.2 and 4.4, we deduce that A is wild if and only if $A\#H$ is wild. Summarizing Propositions 4.2, 4.4 and this conclusion, we get the desired result. □

Remark 4.6. By Drozd's definition of tameness, finite representation type is indeed a special case of tame type. Thus his original tame-and-wild theorem tell us that every finite dimensional algebra is either tame or wild. For our purpose, we make a slight modification of this famous theorem in the form given in the proof of above theorem.

5. Tame graded basic Hopf algebras

The main aim of this section is to describe the structure of tame graded basic Hopf algebras (see Theorem 5.4).

Let H be a finite dimensional Hopf algebra and J_H its Jacobson radical. Denote $\mathrm{gr}\,H$ its radically graded algebra, i.e., $\mathrm{gr}\,H = H/J_H \oplus J_H/J_H^2 \oplus \cdots \oplus J_H^{m-1}$ if $J_H^m = 0$. Recall that a Hopf algebra H is called *graded* if (i) $H = \bigoplus H(n)$ as vector space; (ii) $H(i)H(j) \subset H(i+j)$; (iii) $\Delta(H(n)) \subset \sum_{i=0}^{n} H(i) \otimes H(n-i)$ and $S(H(n)) \subset H(n)$. The following lemma, which is indeed dual to Lemma 5.2.8 in [13], will help us to determine when $\mathrm{gr}\,H$ is a graded Hopf algebra.

Lemma 5.1. $\mathrm{gr}\,H = H/J_H \oplus J_H/J_H^2 \oplus \cdots \oplus J_H^{m-1}$ *is a graded Hopf algebra if and only if J_H is a Hopf ideal.*

Proof. "*Only if part*": By assumption, H/J_H is a Hopf algebra and so J_H is a Hopf ideal.

"*If part*": Clearly, it is graded as an algebra. Since $S(J_H^i) \subset J_H^i$ for $i \in \{0, 1, \ldots, m-1\}$, $S(J_H^i/J_H^{i+1}) \subset J_H^i/J_H^{i+1}$.

So it is enough to prove that it is also graded as a coalgebra. Since J_H is a Hopf ideal, Δ induce an algebra morphism $\Delta : H/J_H \to H \otimes H/(H \otimes J_H + J_H \otimes H) \cong H/J_H \otimes H/J_H$. That is, $\Delta(H/J_H) \subset H/J_H \otimes H/J_H$.

It is easy to see that we have the exact sequence of vector spaces

$$0 \to \big(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H\big)\big/\big(J_H^2 \otimes H + H \otimes J_H^2\big)$$
$$\to H/J_H^2 \otimes H/J_H^2 \to (H \otimes H)\big/\big(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H\big) \to 0.$$

Therefore, a vector space basis for $(H \otimes H)/(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H)$ is given by $e_i \otimes e_j, e_i \otimes a, a \otimes e_i$ where $\{e_i\}$ is a basis for H/J_H and a runs through a basis for J_H/J_H^2. This implies the vector basis for $(H \otimes H)/(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H)$ lies in $H/J_H \otimes H/J_H + H/J_H \otimes J_H/J_H^2 + J_H/J_H^2 \otimes H/J_H$. Then clearly the elements of the form $e_i \otimes a, a \otimes e_i$ form a vector space basis for $H \otimes J_H + J_H \otimes H$ in $(H \otimes H)/(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H)$. That is, the vector basis for $(H \otimes J_H + J_H \otimes H)/(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H)$ lies in $H/J_H \otimes J_H/J_H^2 + J_H/J_H^2 \otimes H/J_H$ and thus

$$(H \otimes J_H + J_H \otimes H)\big/\big(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H\big)$$
$$\cong H/J_H \otimes J_H/J_H^2 + J_H/J_H^2 \otimes H/J_H.$$

Therefore,

$$\Delta\big(J_H/J_H^2\big) \subset (H \otimes J_H + J_H \otimes H)\big/\big(H \otimes J_H^2 + J_H \otimes J_H + J_H^2 \otimes H\big)$$
$$\cong H/J_H \otimes J_H/J_H^2 + J_H/J_H^2 \otimes H/J_H.$$

Similar to the case of J_H/J_H^2, we can show that for any $i \in \{2, \ldots, m-1\}$ the vector basis for $(H \otimes H)/(H \otimes J_H^i + J_H \otimes J_H^{i-1} + \cdots + J_H^i \otimes H)$ lies in $H/J_H \otimes H/J_H + H/J_H \otimes$

$J_H/J_H^2+\cdots+H/J_H\otimes J_H^{i-1}/J_H^i+J_H/J_H^2\otimes H/J_H+\cdots+J_H/J_H^2\otimes J_H^{i-2}/J_H^{i-1}+\cdots+J_H^{i-1}/J_H^i\otimes H/J_H$ and

$$\begin{aligned}&\bigl(H\otimes J_H^{i-1}+J_H\otimes J_H^{i-2}+\cdots+J_H^{i-1}\otimes H\bigr)/\bigl(H\otimes J_H^i+J_H\otimes J_H^{i-1}+\cdots+J_H^i\otimes H\bigr)\\&\quad\cong H/J_H\otimes J_H^{i-1}/J_H^i+J_H/J_H^2\otimes J_H^{i-2}/J_H^{i-1}+\cdots+J_H^{i-1}/J_H^i\otimes H/J_H.\end{aligned}$$

Thus $\Delta(J_H^{i-1}/J_H^i)\subset(H\otimes J_H^{i-1}+J_H\otimes J_H^{i-2}+\cdots+J_H^{i-1}\otimes H)/(H\otimes J_H^i+J_H\otimes J_H^{i-1}+\cdots+J_H^i\otimes H)$ which is isomorphic to $H/J_H\otimes J_H^{i-1}/J_H^i+J_H/J_H^2\otimes J_H^{i-2}/J_H^{i-1}+\cdots+J_H^{i-1}/J_H^i\otimes H/J_H$.

So $\operatorname{gr}H$ is also graded as a coalgebra. □

Recall one result, due to Green and Solberg (see [10]), states that the Jacobson radical of a basic Hopf algebra is a Hopf ideal. So, we have the following corollary.

Corollary 5.2. *Let H be a basic Hopf algebra, then* $\operatorname{gr}H$ *is a graded Hopf algebra.*

Let H, H_0 be Hopf algebras and $\pi: H\to H_0$ and $\iota: H_0\to H$ Hopf homomorphisms. Assume that $\pi\iota=\mathrm{id}_{H_0}$, so that π is surjective and ι is injective. Define

$$R_H:=H^{co\pi}=\bigl\{h\in H\mid(\mathrm{id}\otimes\pi)\Delta(h)=h\otimes 1\bigr\}.$$

By a Radford's conclusion (see Theorem 3 of [14]),

$$H\cong R_H\times H_0\quad\text{as Hopf algebras}$$

where "×" was called *biproduct* in [14] and *bosonization* in [12]. Knowledge about biproduct or bosonization can be found in [3,12,14]. We list some facts, which we will not give any proof, from [3,12,14]:

(Fact i) R_H is a braided Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$, the category of Yetter–Drinfeld modules over H_0 (see [3]);
(Fact ii) As a linear space, $R_H\times H_0=R_H\otimes H_0$. Its multiplication and comultiplication are usual smash product and smash coproduct respectively;
(Fact iii) $R_H=\{h'\iota\pi S(h'')\mid h\in H\}$.

In the following of this section, H always denotes a basic Hopf algebra. Thus $\operatorname{gr}H$, by above corollary, is a graded Hopf algebra. Clearly, $H/J_H=\operatorname{gr}H(0)$ is a Hopf subalgebra of $\operatorname{gr}H$ and there is a natural Hopf algebra epimorphism $\pi:\operatorname{gr}H\to H/J_H$ with a retraction of the inclusion. We can then apply above discussion. Let $R_H=\{h\in H\mid(\mathrm{id}\otimes\pi)\Delta(h)=h\otimes 1\}$, then $\operatorname{gr}H$ can be reconstruct from R_H and H/J_H as a bosonization

$$\operatorname{gr}H\cong R_H\times H/J_H.$$

Proposition 5.3. *Let H be a basic Hopf algebra over k and R_H constructed above. Assume* $\dim_k H/J_H$ *is invertible in k, then*

(i) *R_H and* $\operatorname{gr} H$ *have the same representation type*;
(ii) *R_H is a local graded Frobenius algebra.*

Proof. (i) By the (Fact ii) of bosonization, $\operatorname{gr} H \cong R_H \# H/J_H$ as algebras. Since H/J_H is a commutative semisimple Hopf algebra, $H/J_H \cong (kG)^*$ for some finite group. By assumption and the well-known Maschke Theorem, $(H/J_H)^* \cong kG$ is also semisimple. Thus (i) is a direct consequence of Theorem 4.5.

(ii) At first, we show that R_H is also graded as an algebra. In fact, on one hand, it is easy to see that if $h \in \operatorname{gr} H(n)$ then $h'\iota\pi S(h'')$ belongs to $\operatorname{gr} H(n)$ too. On the other hand, by (Fact iii) of bosonization, $R_H = \{h'\iota\pi S(h'') \mid h \in H\}$. Therefore, R_H is graded and $R_H(n) = H(n) \cap R$.

Secondly, let us prove that R_H is a local algebra. In order to prove this, it is enough to prove that $R_H(0) = k1$. We have known that $R_H(0) = R_H \cap \operatorname{gr} H(0) = R_H \cap H/J_H$. Let $x \in R_H \cap H/J_H$, then $\Delta(x) = \Delta(\pi(x)) = (\pi \otimes \pi)\Delta(x) = (\pi \otimes \mathrm{id})(\mathrm{id} \otimes \pi)\Delta(x) = \pi(x) \otimes 1$. Therefore, $x = (\varepsilon \otimes \mathrm{id})\Delta(x) = \varepsilon(\pi(x))1 = \varepsilon(x)1$. That is, $x \in k1$ and so $R_H \cap H/J_H = k1$. Therefore, R_H is a local algebra.

The only left task is to show that R_H is a Frobenius algebra. This fact can be gotten directly as long as we note that any finite dimensional braided Hopf algebra is Frobenius (see 5.6 and Remark 5.8 of [9]) and (Fact i) of bosonization. □

The next conclusion will give us the structure of tame (radically) graded basic Hopf algebras.

Theorem 5.4. *Let H be a basic Hopf algebra. Assume* $\operatorname{char} k \neq 2$ *and* $\dim_k H/J_H$ *is invertible in k, then* $\operatorname{gr} H$ *is tame if and only if* $\operatorname{gr} H \cong k\langle x, y\rangle/I \times (kG)^*$ *for some finite group G and some ideal I which is one of the following forms*:

(1) $I = (x^2 - y^2, yx - ax^2, xy)$ *for* $0 \neq a \in k$;
(2) $I = (x^2, y^2, (xy)^m - a(yx)^m)$ *for* $0 \neq a \in k$ *and* $m \geqslant 1$;
(3) $I = (x^n - y^n, xy, yx)$ *for* $n \geqslant 2$;
(4) $I = (x^2, y^2, (xy)^m x - (yx)^m y)$ *for* $m \geqslant 1$;
(5) $I = (yx - x^2, y^2)$.

Proof. "*Only if part*": On one hand, by (i), (ii) of Proposition 5.3, R_H is a tame local graded Frobenius algebra. On the other hand, H/J_H is a commutative semisimple Hopf algebra and thus $H/J_H \cong (kG)^*$ for some finite group. Therefore, by Theorem 3.1 and $\operatorname{gr} H \cong R_H \times H/J_H$, we get the desire conclusion.

"*If part*": By Theorem 3.1, we know that $k\langle x, y\rangle/I$ is a tame algebra. So the desire conclusion is got from (i) of Proposition 5.3. □

By a conclusion of Radford or Majid (see [12,14]), if Λ is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$ for some finite group G, then we can form the bosonization $\Lambda \times (kG)^*$ which is a Hopf algebra. For a tame local graded Frobenius algebra A, above theorem dose not imply the existence of finite group G satisfying A is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$. Indeed, at present, we do not know whether such a group exists or not.

Problem 5.1. *For a tame local graded Frobenius algebra A, give an effective method to determine that whether there is a finite group G satisfying A is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$. If such a G exists, then find all of them.*

The following two examples show that for some special tame local graded Frobenius algebras, such G exist. Thus there do exist tame graded basic Hopf algebras.

Example 5.1. Let q be a nth primitive root of unity. Note that this implies that $\operatorname{char} k \nmid n$. Recall that the Taft algebra $T_{n^2}(q)$ is a Hopf algebra generated by elements g and x, with relations

$$g^n = 1, \qquad x^n = 0, \qquad xg = qgx$$

with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = 1 \otimes x + x \otimes g,$$
$$\varepsilon(g) = 1, \qquad \varepsilon(x) = 0,$$
$$S(g) = g^{-1}, \qquad S(x) = -xg^{-1}.$$

It is a basic Hopf algebra since $T_{n^2}(q)/J_{T_{n^2}(q)}$ is isomorphic to the cyclic group algebra kZ_n which is a commutative semisimple algebra.

We consider $H = T_{2^2}(-1) \otimes T_{2^2}(-1)$. We denote the generators of the first Taft algebra by x, g and that of latter by y, h. Clearly, as a vector space,

$$\begin{aligned} H = {} & \operatorname{span}\{1 \otimes 1, 1 \otimes h, g \otimes 1, g \otimes h\} \\ & \oplus \operatorname{span}\{1 \otimes y, 1 \otimes yh, g \otimes y, g \otimes yh, x \otimes 1, xg \otimes 1, x \otimes h, xg \otimes h\} \\ & \oplus \operatorname{span}\{x \otimes y, x \otimes yh, xg \otimes y, xg \otimes yh\} \end{aligned}$$

where $\operatorname{span}\{S\}$ means the vector space with basis the set S. It is straightforward to show that

$$\begin{aligned} J_H &= \operatorname{span}\{1 \otimes y, 1 \otimes yh, g \otimes y, g \otimes yh, x \otimes 1, xg \otimes 1, x \otimes h, xg \otimes h\} \\ &\quad \oplus \operatorname{span}\{x \otimes y, x \otimes yh, xg \otimes y, xg \otimes yh\}, \\ H/J_H &= \operatorname{span}\{1 \otimes 1, 1 \otimes h, g \otimes 1, g \otimes h\}, \\ J_H/J_H^2 &= \operatorname{span}\{1 \otimes y, 1 \otimes yh, g \otimes y, g \otimes yh, x \otimes 1, xg \otimes 1, x \otimes h, xg \otimes h\}, \\ J_H^2 &= \operatorname{span}\{x \otimes y, x \otimes yh, xg \otimes y, xg \otimes yh\} \end{aligned}$$

and H is indeed a radically graded Hopf algebra. Thus $H \cong R_H \times H/J_H$. Therefore, $H/J_H \cong k(Z_2 \times Z_2)$ and $\dim_k(H/J_H) = 4$. This implies $\dim_k R_H = 4$ since $\dim_k H = 16$. By the definition of R_H, we know that $xg \otimes 1, 1 \otimes yh \in R_H$. But clearly

$$\dim_k\bigl(k\langle xg \otimes 1, 1 \otimes yh\rangle \big/ \bigl((xg \otimes 1)^2, (1 \otimes yh)^2, (xg \otimes 1)(1 \otimes yh) - (1 \otimes yh)(xg \otimes 1)\bigr)\bigr) = 4.$$

Thus $R_H \cong k\langle X, Y\rangle/(X^2, Y^2, XY - YX)$ which is tame by Theorem 3.1. This means H is tame too.

Example 5.2 *(Book algebras).* Let q be a nth primitive root of unity and m a positive integer satisfying $(m, n) = 1$. Let $H = \mathbf{h}(q, m) = k\langle y, x, g\rangle/(x^n, y^n, g^n - 1, gx - qxg, gy - q^m yg, xy - yx)$ and with comultiplication, antipode and counit given by

$$\Delta(x) = x \otimes g + 1 \otimes x, \qquad \Delta(y) = y \otimes 1 + g^m \otimes y, \qquad \Delta(g) = g \otimes g,$$
$$S(x) = -xg^{-1}, \qquad S(y) = -g^{-m} y, \qquad S(g) = g^{-1},$$
$$\varepsilon(x) = \varepsilon(y) = 0, \qquad \varepsilon(g) = 1.$$

It is a Hopf algebra and called book algebra. It is a basic algebra since $\mathbf{h}(q, m)/J_{\mathbf{h}(q,m)}$ is a commutative semisimple algebra. For more knowledge about book algebras, see [2]. It is straightforward to show that $R_H = k\langle x', y\rangle/(x'^n, y^n, x'y - q^m yx')$ where $x' = xg^{-1}$ (see 1.4.2 of [1]). In particular, if $n = 2, q = -1, m = 1$, then $R_H \cong k\langle X, Y\rangle/(X^2, Y^2, XY + YX)$, which is tame by Theorem 3.1. Thus $\mathbf{h}(-1, 1)$ is tame.

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