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Super cocommutative Hopf algebras of finite representation type

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ABSTRACT

Given an algebraically closed field k of characteristic $p > 5$, we classify the finite algebraic k-supergroups whose algebras of measures are of finite representation type. Let $\mathcal{G}$ be such a supergroup and $\underline{\mathcal{G}}$ the largest ordinary algebraic k-group determined by $\mathcal{G}$. We show that both $\underline{\mathcal{G}}$ and $\mathbf{u}(\mathrm{Lie}(\mathcal{G}))$, the restricted enveloping algebra of Lie superalgebra of $\mathcal{G}$, are of finite representation type. Moreover, only some special representation-finite algebraic k-groups of dimension zero are shown to appear if $\mathcal{G} \neq \underline{\mathcal{G}}$. The structure of $\mathcal{G}$ is almost determined by $\underline{\mathcal{G}}$ and $\mathbf{u}(\mathrm{Lie}(\mathcal{G}))$. The Auslander–Reiten quivers are determined by showing that they are Nakayama algebras.

1. Introduction

According to the fundamental result of Drozd [3], every finite-dimensional algebra exactly belongs to one of the following three kinds of algebras: algebras of finite representation type, algebras of tame type and wild algebras. For the algebras of the former two kinds, a classification of indecomposable modules seems feasible. By contrast, the module category of a wild algebra, being “complicated” at least as that of any other algebra, can’t afford such a classification. Inspired by Drozd’s result, one is often interested in classifying a given kind of algebras according to their representation type.

This paper concerns the classification of (representation-finite) super cocommutative Hopf algebras over algebraically closed fields of positive characteristic. It is known that such an algebra can

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be viewed as the group algebras of a finite algebraic k-supergroup. Special cases are group algebras associated to finite algebraic k-groups, that is, finite-dimensional cocommutative Hopf algebras, as well as restricted enveloping algebras of restricted Lie superalgebras. The representation theory of both of these classes has received considerable attention. The very detailed information on the structure of representation-finite and tame cocommutative Hopf algebras, partially considered firstly by the pioneers as Hochschild [13], Feldvoss and Strade [12], Pfautsch and Voigt [19,23], etc., has been ultimately gotten by Farnsteiner and his collaborators continuous [6–8,10]. Also, the restricted Lie superalgebras of finite representation type were classified by Farnsteiner too [5]. Our final goal will be the extension of these results to arbitrary super cocommutative Hopf algebras.

There are two ways to connect a super Hopf algebra H with an ordinary Hopf algebra and both of them will be used freely in the paper. One is the Radford–Majid bosonization [17,20], which constructs from H an ordinary Hopf algebra $H \rtimes k\mathbb{Z}_2$. Another one, given by Masuoka [18], states that if H is super cocommutative, there is a unit-preserving isomorphism

$$H \cong \underline{H} \otimes \wedge(V_H)$$

as super left $\underline{H}$-module coalgebras, where $\underline{H}$ is the largest ordinary sub Hopf algebra and $V_H = P(H)_1$. These two ways will be recalled in Section 2.

So, in philosophy, one just needs to know "how to" reduce the research of representation theory of super cocommutative Hopf algebras H to that of $\underline{H}$ and V_H. Both Section 3 and Section 4 are designed to give methods of such reduction. The main result of Section 3 shows that $\mathbf{u}(P(H))$, which controls V_H essentially, has finite representation type provided that H is of finite representation type. Under assumption that H is of finite representation type, the structure of $\underline{H}$ is shown to be quite special. We will see in Section 4 that either $\underline{H}$ is semisimple or the V-uniserial group attached to it has height $\leqslant 1$. Due to the lack of Mackey decomposition for algebraic supergroups, one has to apply other methods. It turns out that the concept of complexity, which is shown effective in dealing with infinitesimal groups, is also quite useful in our case. And in Section 2, some notions and computations relevant to our purpose, particularly the concept of a path coalgebra and complexity, are summarized. Combining the results gotten in Sections 3, 4, the representation-finite finite algebraic supergroups are determined in Section 5. The representation theory of them is determined by showing that they are always Nakayama algebras in the last section, which also corrects an error stating as Theorem 4.3 in [5].

2. Preliminaries

Throughout we will be working over a field k. All spaces are k-spaces. For short, $\otimes_k$ is just denoted by $\otimes$.

2.1. Path coalgebras

Given a quiver $Q = (Q_0, Q_1)$ with Q_0 the set of vertices and Q_1 the set of arrows, denote by kQ and kQ^c, the k-space with basis the set of all paths in Q and the path coalgebra of Q, respectively. Note that they are all graded with respect to length grading. For $\alpha \in Q_1$, let $s(\alpha)$ and $t(\alpha)$ denote respectively the starting and ending vertex of α.

Recall that the comultiplication of the path coalgebra kQ^c is defined by

$$\Delta(p) = \alpha_l \cdots \alpha_1 \otimes s(\alpha_1) + \sum_{i=1}^{l-1} \alpha_l \cdots \alpha_{i+1} \otimes \alpha_i \cdots \alpha_1 + t(\alpha_l) \otimes \alpha_l \cdots \alpha_1$$

for each path $p = \alpha_l \cdots \alpha_1$ with each $\alpha_i \in Q_1$; and $\varepsilon(p) = 0$ for $l \geqslant 1$ and 1 if $l = 0$ ($l = 0$ means p is a vertex). This is a pointed coalgebra.

For a quiver Q, define

$$kQ_d := \bigoplus_{i=0}^{d-1} kQ(i)$$

where $Q(i)$ is the set of all paths of length i in Q. Our interested quiver is the simplest one, a loop $\circlearrowleft$. For any natural number n, denote the unique path of length n of $k\circlearrowleft$ by α_n. In particular, $k\circlearrowleft_{p^n}$ has a basis $1, \alpha_1, \alpha_2, \ldots, \alpha_{p^n-1}$.

2.2. Representation type

A finite-dimensional algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there is finite number of A–$k[T]$-bimodules M_i which are free of finite rank as right $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{k[T]} k[T]/(T-\lambda)$ for $\lambda \in k$. We say that A is of *wild type* or A is a *wild* algebra if there is a finitely generated A–$k\langle X, Y\rangle$-bimodule B which is free as a right $k\langle X, Y\rangle$-module such that the functor $B \otimes_{k\langle X,Y\rangle} -$ from mod-$k\langle X, Y\rangle$, the category of finitely generated $k\langle X, Y\rangle$-modules, to mod-A, the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms. See [4] for more details.

2.3. Super cocommutative Hopf algebras

We recall the two ways connecting super cocommutative Hopf algebras with usual Hopf algebras in this subsection. Let J be a Hopf algebra with bijective antipode and ${}^J_J\mathcal{YD}$ the category of the Yetter–Drinfeld modules with left J-module action and left J-comodule coaction. It naturally forms a braided monoidal category with the braiding

$$c_{M,N} : M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \sum n_0 \otimes n_{-1} \cdot m,$$

where $n \mapsto \sum n_{-1} \otimes n_0$, $N \to J \otimes N$ denotes the comodule structure, as usual. Let A be a Hopf algebra in ${}^J_J\mathcal{YD}$. In particular, A is a left J-module algebra and left J-comodule coalgebra. The Radford–Majid bosonization [17,20] gives rise to an ordinary Hopf algebra, $A \rtimes J$. As an algebra, this is the smash product $A\#J$, and it is the smash coproduct as a coalgebra. In particular, a super Hopf algebra H is a Hopf algebra in ${}^{k\mathbb{Z}_2}_{k\mathbb{Z}_2}\mathcal{YD}$ (see Section 2 in [18]) and hence we get a usual Hopf algebra $H \rtimes k\mathbb{Z}_2$. The following result is a direct consequence of Proposition 3.2 in Chapter VI of [1].

Lemma 2.1. *Assume that characteristic of k is not 2. Then H and $H \rtimes k\mathbb{Z}_2$ have the same representation type.*

For a super Hopf algebra $H = H_0 \oplus H_1$, apart from its ordinary representations, one also can consider its super representations. That is, the $\mathbb{Z}_2$-graded H-modules. Clearly, super representations of H are just the ordinary representations of the ordinary Hopf algebra $H \rtimes k\mathbb{Z}_2$. Thus,

Lemma 2.2. *Let H be a super Hopf algebra. Then the category of super H-modules is equivalent to the category of $H \rtimes k\mathbb{Z}_2$-modules.*

Combining Lemma 2.1 and Lemma 2.2, if characteristic of k is not 2, the representation type of H as an ordinary algebra is indeed the same with that of H when we consider it as a superalgebra. In this paper, we will always consider the ordinary representations except in the proof of Theorem 6.1.

An algebra A is a *Nakayama algebra* if each indecomposable A-module is uniserial. The following lemma is Theorem 2.14 in Chapter IV of [1].

Lemma 2.3. *Let G be a finite group such that $|G|$ is invertible in k and A is a finite-dimensional kG-module algebra. Then $A\#kG$ is a Nakayama algebra if and only if A is so.*

Let H be a super Hopf algebra, we call H a *super Nakayama algebra* if each super indecomposable H-module is uniserial. Owing to Lemma 2.2 and Lemma 2.3, H is Nakayama if and only if it is super Nakayama. This fact will be used in the proof of Theorem 6.1.

Now let $H = H_0 \oplus H_1$ be a super cocommutative Hopf algebra over k. Define

$$\underline{H} := \Delta^{-1}(H_0 \otimes H_0).$$

This is the largest ordinary cocommutative sub Hopf algebra of H. Denote the set of primitives in H by $P(H)$ and define

$$V_H := P(H)_1$$

the vector space of odd primitives in H. Choose a totally ordered k-basis $X = (x_\lambda)_\lambda$ of V_H. Then, $x_\lambda \wedge x_\mu \wedge \cdots \wedge x_\nu (x_\lambda < x_\mu < \cdots < x_\nu)$ form a k-basis of $\wedge(V_H)$, and $x_\lambda \wedge x_\mu \wedge \cdots \wedge x_\nu \mapsto x_\lambda x_\mu \cdots x_\nu$ gives a unit-preserving super coalgebra map from $\wedge(V_H)$ to H. We collect some facts about H, which were given essentially by Masuoka in [18], as follows.

Lemma 2.4. (1) *The induced left $\underline{H}$-linear map*

$$\phi = \phi_X : \underline{H} \otimes \wedge(V_H) \longrightarrow H$$

is a unit-preserving isomorphism of super left $\underline{H}$-module coalgebra.

(2) *As an algebra, H is generated by $\underline{H}$ and V_H.*

(3) *V_H is a right $\underline{H}$-module under the conjugation $v \cdot h := \sum S(h_{(1)}) v h_{(2)}$ and*

$$vh = \sum h_{(1)} (v \cdot h_{(2)})$$

for $v \in V_H$ and $h \in \underline{H}$.

(4) *For any $u, v \in V_H$, we have $uv + vu \in P(H)$.*

Proof. (1) and the first part of (3) are Theorem 3.6 and Proposition 3.9(1) in [18] respectively. (2) is a direct consequence of Proposition 3.9(2) in [18]. Both (4) and the second part of (3) can be gotten easily by direct computations. □

Convention. Due to (1) and (2) of the above lemma, sometimes we use the notation $\underline{H}\langle V_H \rangle$ to denote the super cocommutative Hopf algebra H. This is convenient. For example, let $K \subset \underline{H}$ be a sub Hopf algebra containing $P(H)$ and K' the sub super cocommutative Hopf algebra generated by K and some $V \subset V_H$. Then we have $\underline{K'} = K$ (by (4) of the above lemma) and $V_{K'} = V$. So $K' = K\langle V \rangle$. Moreover, if $\dim_k V_H = 1$, then we will simply use the notation $\underline{H}\langle v \rangle$ instead of $\underline{H}\langle V_H \rangle$ for any non-zero element $v \in V_H$.

Let C be a (super) coalgebra, define $C^+ := \mathrm{Ker}(\varepsilon)$ as usual.

Lemma 2.5. *Let $K \subset \underline{H}$ be a sub-normal Hopf algebra containing $P(H)$ and $V \subset V_H$ a subspace of V_H. Then there is a Hopf isomorphism*

$$\underline{H}/K^+\underline{H} \cong \underline{H}\langle V \rangle / \big(K\langle V \rangle\big)^+ \underline{H}\langle V \rangle.$$

Proof. By the fact that K is normal, we have exact sequence of Hopf algebras $K \hookrightarrow \underline{H} \twoheadrightarrow \underline{H}/K^+\underline{H}$. Also, we have an obvious exact sequence $V \hookrightarrow V \twoheadrightarrow 0$. Owing to Theorem 3.13(3) in [18], the sequence $K\langle V\rangle \hookrightarrow \underline{H}\langle V\rangle \twoheadrightarrow \underline{H}/K^+\underline{H}$ is also exact. Thus the conclusion is proved. □

2.4. Complexity

Let A be an associative algebra, M an A-module with minimal projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Then the *complexity* of M is defined to be the integer

$$\mathcal{C}_A(M) := \min\bigl\{c \in \mathbb{N}_0 \cup \infty \bigm| \exists\lambda > 0\colon \dim_k P_n \leqslant \lambda n^{c-1},\ \forall n \geqslant 1\bigr\}.$$

For our purpose, we need to consider the following examples.

Example 2.6. (1) Let A be a self-injective algebra of finite representation type, then it is well known that $\mathcal{C}_A(M) \leqslant 1$ for any A-modules M.

(2) Consider the algebra $A = k[x, y]/(x^n, y^2)$ for some $n > 1$. It is a local algebra and we denote the unique simple module by k. We will show that $\mathcal{C}_A(k) = 2$. Actually, by definition the algebra $A \cong k[x]/(x^n) \otimes k[y]/(y^2)$ is the tensor product of two Nakayama algebras. Thus the minimal projective resolution $P^\bullet$ and $Q^\bullet$ of the trivial modules of the two tensor factors have growth 1. The Künneth formula then implies that $P^\bullet \otimes Q^\bullet$ is a minimal projective resolution of the trivial A-module k, so that $\mathcal{C}_A(k) = 2$.

3. Structure of $\mathbf{u}(\mathrm{Lie}(\mathcal{G}))$

Throughout this and the following sections, we assume that k is an algebraically closed field with characteristic $p > 5$. Let $\mathcal{G}$ be a finite super algebraic k-group and $H(\mathcal{G})$ be its algebra of measures. That is, $H(\mathcal{G}) = (\mathcal{O}(\mathcal{G}))^*$. Then $H(\mathcal{G})$ is a finite-dimensional super cocommutative Hopf algebra and $\mathrm{Lie}(\mathcal{G}) = P(H(\mathcal{G}))$ is a restricted Lie superalgebra. Denote by $\mathbf{u}(\mathrm{Lie}(\mathcal{G}))$ the restricted enveloping algebra of $\mathrm{Lie}(\mathcal{G})$ and note that it is a sub super Hopf algebra of $H(\mathcal{G})$. The purpose of this section is to show that $\mathbf{u}(\mathrm{Lie}(\mathcal{G}))$ is of finite representation type provided $H(\mathcal{G})$ is so.

Let $(L_0, [p])$ be a restricted Lie algebra. An element $x \in L_0$ is called to be *toral* if $x^{[p]} = x$ and *p-nilpotent* if there exists some $n \in \mathbb{N}$ such that $x^{[p]^n} = 0$. We denote by $N(L_0)$, $T(L_0)$ and $C(L_0)$, the largest nilpotent ideal, toral ideal and center of L_0. If $X \subset L_0$ is a subset, then X_p denotes the p-subalgebra of L_0 that is generated by X. See [22] for details. The following conclusion was given in [5] as Lemma 4.1 and Theorem 4.2.

Lemma 3.1. (1) *Let V be a k-vector space with exterior algebra $\wedge(V)$. If $\mathcal{C}_{\wedge(V)}(k) \leqslant 1$, then* $\dim_k V \leqslant 1$.

(2) *Let $L = L_0 \oplus L_1$ be a restricted Lie superalgebra with $L_1 \neq 0$. Then $\mathbf{u}(L)$ has finite representation type if and only if there exist a toral element $t_0 \in L_0$, a p-nilpotent element $x_0 \in L_0$, and $y \in L_1$ such that $L = L_0 \oplus ky$, $(kx_0)_p \subset [L_1, L_1]_p \subset N(L_0)$, $L_0 = N(L_0) + kt_0$, $N(L_0) = T(L_0) \oplus (kx_0)_p$.*

In the following of this paper, we fix the notations x_0 and y to denote the elements given in this lemma. We can prove our conclusion now.

Proposition 3.2. *Let $\mathcal{G}$ be a finite k-supergroup and $H(\mathcal{G})$ be its algebra of measures. If $H(\mathcal{G})$ is of finite representation type then* $\mathbf{u}(\mathrm{Lie}(\mathcal{G}))$ *is so too.*

The (1) ⇒ (2) part of the proof of Theorem 4.2 in [5] can be essentially applied to our case except some delicate points. So our proof looks like “cut and paste”. For the safety and convenience of the readers, we still write it out.

Proof of Proposition 3.2. For simplicity, denote the restricted Lie superalgebra Lie($\mathscr{G}$) by $L = L_0 + L_1$. If $L_1 = 0$, the conclusion can be proved easily. In fact, Example 2.6(1) implies that $\mathcal{C}_{H(\mathscr{G})}(k) \leqslant 1$ where we consider k as the trivial $H(\mathscr{G})$-module through the map $\varepsilon : H(\mathscr{G}) \to k$. Since $H(\mathscr{G})$ is a free $\mathbf{u}(L)$-module, $\mathcal{C}_{\mathbf{u}(L)}(k) \leqslant 1$. Consequently, $\mathbf{u}(L)$ has finite representation type by Theorem 2.4 in [12]. Therefore, one can assume that $L_1 \neq 0$. Our goal is to show that L indeed has the structure as described in Lemma 3.1(2). We divide the task into several steps.

(a) *There exist a toral element $t_0 \in L_0$, a p-nilpotent element $x_0 \in L_0$ such that $L_0 = N(L_0) + kt_0$, $N(L_0) = T(L_0) \oplus (kx_0)_p$.* Owing to the discussion in the above paragraph, $\mathbf{u}(L_0)$ has finite representation type and thus Theorem 4.3 in [6] implies the desired result.

(b) *Let $T := T(L_0) + kt_0$. Then T is a maximal torus of L_0 and there exists at most one root α relative to T. The corresponding root space $(L_0)_\alpha$ has dimension* 1. Totally the same with the part (1) $\Rightarrow$ (2) (c) of the proof of Theorem 4.2 in [5].

In the following of the proof, we decompose the T-module L_1 into its weight spaces and write $L_1 = \bigoplus_{\lambda \in \mathscr{W}} (L_1)_\lambda$, where $\mathscr{W} \subset T^*$ is the set of weights of L_1 relative to T.

(c) *Let $\beta \in \mathscr{W} \backslash \{0, \frac{1}{2}\alpha\}$. Then $\dim_k (L_1)_\beta = 1$, $[(L_1)_\beta, (L_1)_\beta] = 0$, and $\mathscr{W} \subset \{0, \frac{1}{2}\alpha, \beta, -\beta\}$ or $\mathscr{W} \subset \{0, \frac{1}{2}\alpha, \beta, \alpha - \beta\}$.* See the part (1) $\Rightarrow$ (2) (d) of the proof of Theorem 4.2 in [5].

(d) *Suppose that $\mathscr{W} \backslash \{0, \frac{1}{2}\alpha\} \neq \emptyset$. Then there exists $\gamma \in \mathscr{W} \backslash \{0, \frac{1}{2}\alpha\}$ such that $[(L_0)_\alpha, (L_1)_\gamma] = 0$.* See the part (1) $\Rightarrow$ (2) (e) of the proof of Theorem 4.2 in [5].

(e) *If $\mathscr{W} \backslash \{0, \frac{1}{2}\alpha\} \neq \emptyset$, then $L = T(L_0) \oplus ky_1$, where $ky_1 = L_1$, and $[y_1, y_1] = 0$.*

By (d), there exists $\gamma \in \mathscr{W} \backslash \{0, \frac{1}{2}\alpha\}$ such that $[(L_0)_\alpha, (L_1)_\gamma] = 0$. According to (b), $(L_0)_\alpha = kx_\alpha$, for some $x_\alpha \in L_0$, such that $N(L_0) = C(L_0) + kx_\alpha$. Since $N(L_0) = C(L_0) + kx_0$ (by the structure of L_0 described in (a)), it follows that $x_0 = x_a + z$ for $z \in C(L_0) \subset (L_0)_0$. This implies $[x_0, (L_1)_\gamma] = [z, (L_1)_\gamma] \subset (L_1)_\gamma$. As x_0 is p-nilpotent and $\dim_k (L_1)_\gamma = 1$, $[x_0, (L_1)_\gamma] = 0$. Consider the restricted Lie superalgebra $\mathscr{L} := (kx_0)_p \oplus (L_1)_\gamma$, then its restricted enveloping algebra is $\mathbf{u}(k(x_0)_p) \otimes \wedge((L_1)_\gamma)$. Also, since $H(\mathscr{G})$ is projective over $\mathbf{u}(\mathscr{L})$, $\mathcal{C}_{\mathbf{u}(\mathscr{L})}(k) \leqslant 1$. Then the Künneth formula implies $\mathbf{u}(\mathscr{L}) = k$ and so $x_0 = 0$. Therefore, $N(L_0) = T(L_0) = L_0$.

From (c) we now obtain $\mathscr{W} \subset \{0, \gamma, -\gamma\}$ and $L = T(L_0) \oplus (L_1)_0 \oplus (L_1)_\gamma \oplus (L_1)_{-\gamma}$. Let $X := \operatorname{Ker} \gamma$, it is a p-ideal of L_0. Applying the same computation used in the part (1) $\Rightarrow$ (2) (f) of the proof of Theorem 4.2 in [5], X satisfies the relations $[X, L_1] = 0$ and $[L_1, L_1] \subset X$. Let $\mathscr{L} := L/X$. Consider the quotient super Hopf algebra $H(\mathscr{G})/(X)$, and of course it is of finite representation type. Owing to the fact that $H(\mathscr{G})$ is faithfully flat over $\mathbf{u}(\mathscr{L})$ (in fact, $H(\mathscr{G})$ is free over $\mathbf{u}(\mathscr{L})$), $\mathbf{u}(\mathscr{L}) \cap (X) = X$. Then $\mathbf{u}(\mathscr{L})$ is a sub super Hopf algebra of $H(\mathscr{G})/(X)$ and so $\mathcal{C}_{\mathbf{u}(\mathscr{L})}(k) \leqslant 1$. Therefore, we also have $\mathcal{C}_{\wedge \mathscr{L}_1}(k) \leqslant 1$, and Lemma 3.1(1) yields $\dim_k L_1 = \dim_k \mathscr{L}_1 \leqslant 1$.

In view of the result in (e), we shall hence forth assume the $\frac{1}{2}\alpha$ and 0 are the only weights of L_1 relative to T.

(f) $[N(L_0), L_1] = 0$. See the part (1) $\Rightarrow$ (2) (g) of the proof of Theorem 4.2 in [5].

(g) $[L_1, L_1]_p \subset N(L_0)$ *and* $\dim_k L_1 = 1$.

It follows from (f) that $[N(L_0), [L_1, L_1]] = 0$, proving that $[L_1, L_1]$ is contained in the centralizer of $C_{L_0}(N(L_0))$ which equals to $N(L_0)$ by (a). Thus $[L_1, L_1]_p \subset N(L_0)$. Let $\mathscr{L} := L/N(L_0)$. Also, consider the quotient $H(\mathscr{G})/(N(L_0))$ and it has finite representation type. Similar to the proof of part (e), $\mathbf{u}(\mathscr{L})$ is a sub super Hopf algebra of $H(\mathscr{G})/(N(L_0))$ and so $\mathcal{C}_{\mathbf{u}(\mathscr{L})}(k) \leqslant 1$. Therefore, we also have $\mathcal{C}_{\wedge \mathscr{L}_1}(k) \leqslant 1$, and Lemma 3.1(1) yields $1 \leqslant \dim_k L_1 = \dim_k \mathscr{L}_1 \leqslant 1$.

(h) $(kx_0)_p \subset [L_1, L_1]_p$.

By (g), $L_1 = ky$ for some $0 \neq y \in L_1$. Put $v := [y, y]$. Owing to (f) and (g), $[L_1, L_1]_p = (kv)_p$ is an ideal of L. Let $\mathscr{L} := L/(kv)_p$ and consider the quotient $H(\mathscr{G})/((kv)_p)$. Using the methods developed in the proof of (e) again, one has $\mathcal{C}_{\mathbf{u}(\mathscr{L})}(k) \leqslant 1$. Denote the natural projection $L \to \mathscr{L}$ by π. Since $[\mathscr{L}_1, \mathscr{L}_1] = 0$ and $(kx_0)_p$ operates trivially on L_1, it follows that $\mathbf{u}((k\pi(x_0))_p \oplus \mathscr{L}_1)$ is isomorphic to $\mathbf{u}((k\pi(x_0))_p) \otimes k[X]/(X^2)$. By the fact that $\mathcal{C}_{\mathbf{u}(\mathscr{L})}(k) \leqslant 1$ and $\mathbf{u}(\mathscr{L})$ is projective over $\mathbf{u}((k\pi(x_0))_p \oplus \mathscr{L}_1)$, $\mathcal{C}_{\mathbf{u}((k\pi(x_0))_p \oplus \mathscr{L}_1)}(k) \leqslant 1$. The Künneth formula implies that $\mathbf{u}((k\pi(x_0))_p) = k$ and consequently $(kx_0)_p \subset [L_1, L_1]_p$.

By the results getting in (a)–(h), L has the structure described in sufficiency's part of Lemma 3.1(2). Thus $\mathbf{u}(L)$ has finite representation type. □

4. Structure of $\underline{H}(\mathscr{G})$

Recall from Section 3, for a super cocommutative Hopf algebra H we denote its largest cocommutative sub Hopf algebra by $\underline{H}$. Let $\mathscr{G}$ be a finite algebraic k-supergroup and $H(\mathscr{G})$ be its algebra of measures. Denote by $\underline{\mathscr{G}}$ the largest ordinary algebraic k-group of $\mathscr{G}$, i.e., by definition its algebra of measures $H(\underline{\mathscr{G}})$ is $\underline{H}(\mathscr{G})$. That is, $H(\underline{\mathscr{G}}) = \underline{H}(\mathscr{G})$. *Throughout this section, we always assume that* $\mathscr{G} \neq \underline{\mathscr{G}}$. The task of this section is to analyze the structure of $\underline{\mathscr{G}}$.

Proposition 4.1. *Assume that $H(\mathscr{G})$ has finite representation type. Then $\underline{H}(\mathscr{G})$ has finite representation type too.*

Proof. Denote $H(\mathscr{G})$ by H for simplicity. Owing to Proposition 3.2 and Lemma 3.1(2), $\dim_k \mathrm{Lie}(\mathscr{G})_1 = 1$. So there exists $0 \neq y \in V_H$ (recall from Section 2, V_H was defined to be $P(H)_1$) such that $V_H = ky$. By Lemma 2.4(3), V_H is right $\underline{H}$-module. Thus, there exists an algebra map

$$\chi : \underline{H} \longrightarrow k$$

such that $y \cdot h = \chi(h) y$ for $h \in \underline{H}$. Let $\alpha \in \mathrm{Aut}(\underline{H})$ be the algebra automorphism determined by χ, that is, $\alpha(h) := (id * \chi)(h) = \sum h_{(1)} \chi(h_{(2)})$ for $h \in \underline{H}$. By Lemma 2.4(3), we always have for $h \in \underline{H}$

$$yh = \alpha(h) y. \tag{4.1}$$

Thus

$$H = \underline{H} \oplus \underline{H}\, y$$

as $\underline{H}$-bimodules and Lemma 3.1(a) in Chapter VI of [1] implies that $\underline{H}$ is of finite representation type too. □

In the following of this section, we always assume that $H(\mathscr{G})$ has finite representation type. By the proof of this proposition there exists $0 \neq y \in V_H$ such that $H = \underline{H}\langle y \rangle$ (see Convention after Lemma 2.4). It is known that any ordinary finite algebraic k-group $\mathscr{H}$ can be decomposed into a semidirect product $\mathscr{H} = \mathscr{H}^\circ \rtimes \mathscr{H}_{red}$ with a constant group $\mathscr{H}_{red}$ and a normal infinitesimal subgroup $\mathscr{H}^\circ$. In particular,

$$\underline{\mathscr{G}} = \underline{\mathscr{G}}^\circ \rtimes \underline{\mathscr{G}}_{red}.$$

With such notations,

Lemma 4.2. *$H(\underline{\mathscr{G}}_{red})$ is always semisimple.*

Proof. At first, assume that $[y, y] = 0$. If $H(\underline{\mathscr{G}}_{red})$ is not semisimple, then there exists $g \in \underline{\mathscr{G}}_{red}$ of order p. Since the automorphism group of ky is the multiplicative group $k^\times$, the cyclic group $C_p := \langle g \rangle$ operates trivially on ky. As a result, the subalgebra

$$H(C_p)\langle y \rangle \cong k[x, y]/\big(x^p, y^2\big)$$

and thus $\mathcal{C}_{H(C_p)\langle y \rangle}(k) = 2$ by Example 2.6(2). By the fact that $\mathcal{C}_{H(\mathscr{G})}(k) \leqslant 1$ and $H(\mathscr{G})$ is projective over $H(C_p)\langle y \rangle$, $\mathcal{C}_{H(C_p)\langle y \rangle}(k) \leqslant 1$. It is a contradiction.

Next, assume that $[y, y] \neq 0$. Also, if $H(\underline{\mathscr{G}}_{red})$ is not semisimple, then similarly to the above proof one can find $g \in \underline{\mathscr{G}}_{red}$ of order p such that the cyclic group $C_p = \langle g \rangle$ commutes with y. In the following, let $L := \mathrm{Lie}(\mathscr{G})$ and so $[L_1, L_1]_p \subset N(L_0)$ (by Lemma 3.1(2) and Proposition 3.2). Since $\dim_k L_1 = \dim_k(ky) = 1$, $[[L_1, L_1]_p, L_1] = 0$. Now consider the quotient $H(\underline{\mathscr{G}})\langle y \rangle / ([L_1, L_1])$, it contains a sub super Hopf algebra generated by g and y (we identify g, y with their images in $H(\underline{\mathscr{G}})\langle y\rangle/([L_1, L_1])$). Note that $y \notin ([L_1, L_1])$ by $[[L_1, L_1]_p, L_1] = 0$. As an algebra, this sub super Hopf algebra is isomorphic to $k[x, y]/(x^p, y^2)$. So we also have $\mathcal{C}_{k[x,y]/(x^p,y^2)}(k) = 2$. A contradiction. □

Proposition 4.3. *If $[y, y] = 0$, then $\underline{H}$ is semisimple.*

Proof. By Proposition 3.2, $\mathbf{u}(L)$ has finite representation type and thus it has the structure given in Lemma 3.1(2). The assumption implies that $x_0 = 0$ and so $\mathbf{u}(L_0)$ is semisimple. Thus $\underline{\mathscr{G}}^\circ$ does not contain a copy of ${}_p\alpha_k$, the Frobenius kernel of the additive group α_k. Then Chapter IV, Section 3, (3.7) in [2] implies that $\underline{\mathscr{G}}^\circ$ is multiplicative.

Now, by Lemma 4.2, $\underline{H} = H(\underline{\mathscr{G}}^\circ)\#H(\underline{\mathscr{G}}_{red})$ is semisimple too since, for example, gl.dim $\underline{H} =$ gl.dim $H(\underline{\mathscr{G}}^\circ)$ by Theorem 1.1 in [16]. □

Let $\mathscr{M}(\underline{\mathscr{G}}^\circ)$ be the largest multiplicative center of $\underline{\mathscr{G}}^\circ$. By definition, it is the largest multiplicative normal subgroup of $\underline{\mathscr{G}}^\circ$. Similarly, one can define $\mathscr{M}(\mathscr{G}^\circ)$ to be the largest multiplicative normal subgroup of $\mathscr{G}^\circ$. It is not hard to see that they are indeed the same. It seems better to use notation $\mathscr{M}(\mathscr{G}^\circ)$ directly. For consistency and our convenience, we choose the notation $\mathscr{M}(\underline{\mathscr{G}}^\circ)$. Moreover, let ${}_p\mathscr{W}(n)_k$ be the infinitesimal group corresponding to the restricted enveloping algebra $\mathbf{u}(L_n)$ of the n-dimensional p-nilpotent abelian restricted Lie algebra $L_n := \bigoplus_{i=0}^{n-1} kx^{[p]^i}$ with $x^{[p]^n} = 0$. It is the Frobenius kernel of the nth *Witt group* $\mathscr{W}(n)_k$ (see Chapter V in [2]). Denote the nth Frobenius kernel of the multiplicative group μ_k by ${}_{p^n}\mu_k$.

Proposition 4.4. *If $[y, y] \neq 0$, then either $\underline{\mathscr{G}}^\circ$ is multiplicative or*

$$\underline{\mathscr{G}}^\circ/\mathscr{M}\big(\underline{\mathscr{G}}^\circ\big) \cong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$$

for some $m, n \in \mathbb{N}$.

To show this conclusion, one preparation is needed. By Theorem 2.7 in [10], $\underline{\mathscr{G}}^\circ/\mathscr{M}(\underline{\mathscr{G}}^\circ) \cong \mathscr{U} \rtimes {}_{p^m}\mu_k$ with a V-uniserial normal subgroup $\mathscr{U}$. All V-uniserial groups are classified in [9] and they are described as ${}_p\mathscr{W}(n)_k$, $\mathscr{U}_{n,d}$ and $\mathscr{U}_{n,d}^j$ respectively (see Theorem 1 in [9] for details). Due to the complexity of such the groups, the Hopf structures of the algebras of measures of them are not very clear. Incidentally, the author with his collaborators [14] realized that such Hopf structures can be described through the path coalgebra over a loop. In fact, the coordinate rings of such groups are denoted as $L(n, d)$ in [14]. By definition, for any $0 \leqslant d \leqslant n$, $L(n, d)$ is defined to be the Hopf algebra over $k \circlearrowleft_{p^n}^c$ (see Section 2.1 for the notations) with relations:

$$\alpha_{p^i}\alpha_{p^j} = \alpha_{p^j}\alpha_{p^i}, \quad \text{for } 0 \leqslant i, j \leqslant n-1; \tag{4.2}$$

$$\alpha_{p^i}^p = 0, \quad \text{for } i < d; \tag{4.3}$$

$$\alpha_{p^i}^p = \alpha_{p^{i-d}}, \quad \text{for } i \geqslant d. \tag{4.4}$$

Lemma 4.5. (1) *Any one of $H({}_p\mathscr{W}(n)_k)$, $H(\mathscr{U}_{n,d})$ and $H(\mathscr{U}_{n,d}^j)$ is isomorphic to $(L(n', d'))^*$ for some n', d'. In particular, $H({}_p\mathscr{W}(n)_k) \cong (L(n, n))^*$.*

(2) *As an algebra, there is a canonical isomorphism $(L(n, d))^* \cong k[x]/(x^{p^n})$, and under such isomorphism $x^{p^{n-d}}, x^{p^{n-d+1}}, \ldots, x^{p^{n-1}}$ is a basis of the space of primitive elements of $L(n, d)^*$.*

Proof. (1) is indeed a direct consequence of the proof of Theorem 5.1 in [14].

Denote the dual basis of $k \circlearrowleft^{c}_{p^n}$ by $\{\alpha_i^*\}_{0\leqslant i<p^n-1}$. That is,

$$\alpha_i^*(\alpha_j)=\begin{cases}1 & i=j,\\ 0 & i\neq j.\end{cases}$$

Define a map $((L(n,d))^*)^* \to k[x]/(x^{p^n})$ through $\alpha_i^* \mapsto x^i$ for $0\leqslant i<p^n-1$, and it is straightforward to show this is an isomorphism of algebras. Consider this isomorphism as an identity for short. By the relations defined through (4.3) and (4.4), one can see that

$$\Delta(x)=1\otimes x+x\otimes 1+\sum_{i=1}^{p-1}\big(x^{p^d}\big)^i\otimes\big(x^{p^d}\big)^{p-i}+\text{higher items}.$$

Here "higher items" are items such as $x^j\otimes x^l$ with $j+l>p^{d+1}$. Therefore,

$$\Delta\big(x^{p^{n-d}}\big)=\big(\Delta(x)\big)^{p^{n-d}}=1\otimes x^{p^{n-d}}+x^{p^{n-d}}\otimes 1,$$

and so $\{x^{p^{n-d}},x^{p^{n-d+1}},\dots,x^{p^{n-1}}\}\subset P((L(n,d))^*)$. To attack that it is indeed a basis, it is enough to show that $L(n,d)$ is indeed generated by d elements. In fact, if we write $n=md+i$ for $0\leqslant i<d$, then relation (4.4) shows us that

$$\alpha_{p^{(m-1)d+(i+1)}},\alpha_{p^{(m-1)d+(i+2)}},\dots,\alpha_{p^{(md+i)}}$$

can generate the whole $L(n,d)$. □

Proof of Proposition 4.4. Clearly, to prove the result, there is no harm to assume that $\underline{\mathscr{G}}=\underline{\mathscr{G}}^\circ$. Moreover, one can even assume that $x_0\neq 0$ since otherwise $H(\underline{\mathscr{G}})$ will be semisimple by the proof of Proposition 4.3. Consider the quotient

$$\underline{H}(\mathscr{G})\langle y\rangle/\big(H\big(\mathscr{M}(\underline{\mathscr{G}})\big)^+\big).$$

We claim that $y\notin (H(\mathscr{M}(\underline{\mathscr{G}}))^+)$. Otherwise, $2y^2=[y,y]\in (H(\mathscr{M}(\underline{\mathscr{G}}))^+)\cap\mathbf{u}(L_0)$. By the fact that $H(\mathscr{G})$ is faithfully flat over $H(\underline{\mathscr{G}})$, $(H(\mathscr{M}(\underline{\mathscr{G}}))^+)\cap\mathbf{u}(L_0)=H(\mathscr{M}(\underline{\mathscr{G}}))^+H(\underline{\mathscr{G}})\cap\mathbf{u}(L_0)$ which is contained in $\mathbf{u}(T(L_0)+kt)$ (see notations in Lemma 3.1(2)) by the definition of $\mathscr{M}(\underline{\mathscr{G}})$. By $x_0\in k[y,y]_p$, $x_0\in\mathbf{u}(T(L_0)+kt)$ which is impossible. Thus

$$\underline{H}(\mathscr{G})\langle y\rangle/\big(H\big(\mathscr{M}(\underline{\mathscr{G}})\big)^+\big)\cong\big(\underline{H}(\mathscr{G})/H\big(\mathscr{M}(\underline{\mathscr{G}})\big)^+\underline{H}(\mathscr{G})\big)\langle y\rangle$$

which, by Theorem 2.7 in [10], is isomorphic to $(H(\mathscr{U}\rtimes{}_{p^m}\mu_k))\langle y\rangle$ for some V-uniserial group $\mathscr{U}$. By Lemma 4.5(1),

$$H(\mathscr{U}\rtimes{}_{p^m}\mu_k)\cong H(\mathscr{U})\#(k\mathbb{Z}_{p^m})^*\cong\big(L(n,d)\big)^*\#(k\mathbb{Z}_{p^m})^*$$

for some n,d with $d\leqslant n$. Owing to Lemma 4.5(2), $\{x^{p^{n-d}},x^{p^{n-d+1}},\dots,x^{p^{n-1}}\}$ is a basis of the space of primitive elements of $L(n,d)^*$. Therefore, $P(L(n,d)^*)=(kx^{p^{n-d}})_p$. Denote the Lie algebra of $(L(n,d))^*\#(k\mathbb{Z}_{p^m})^*$ by L_0. Consequently,

$$L_0=\big(kx^{p^{n-d}}\big)_p+kt$$

with a toral element t which does not commute with $x^{p^{n-d}}$. Now $N(L_0) = (kx^{p^{n-d}})_p$. So, by Proposition 3.2 and Lemma 3.1(2),

$$[ky, ky]_p = \left(kx^{p^{n-d}}\right)_p.$$

Also, by the proof of Proposition 3.2 (part (f)), $[x, y] = 0$. So, as an algebra,

$$(*) \quad H(\mathscr{U})\langle y\rangle \cong k[x, y]/\left(x^{p^n}, y^2 - x^{p^{n-d}}\right).$$

Forming the quotient super Hopf algebra $H(\mathscr{U} \rtimes {}_{p^m}\mu_k)\langle y\rangle/(x^{p^{n-d}})$, it contains $H(\mathscr{U})\langle y\rangle/(x^{p^{n-d}})$ as a sub super Hopf algebra. By the fact that $H(\mathscr{U} \rtimes {}_{p^m}\mu_k)\langle y\rangle/(x^{p^{n-d}})$ has finite representation type, $\mathcal{C}_{H(\mathscr{U})\langle y\rangle/(x^{p^{n-d}})}(k) \leqslant 1$. Thus $(*)$ implies that $\mathcal{C}_{k[x,y]/(x^{p^{n-d}}, y^2)}(k) \leqslant 1$. Owing to Example 2.6(2), this is possible only in the case $n = d$. Thus, $H(\mathscr{U}) \cong (L(n, n))^*$ and by Lemma 4.5(1), $\mathscr{U} \cong {}_p\mathscr{W}(n)_k$ as desire. □

5. Representation-finite supergroups of dimension zero

Combining the conclusions gotten in Sections 3, 4, we will determine the structure of representation-finite supergroups of dimension zero in this section. The following conclusion is a direct consequence of the proof of Proposition 2.2(1) in [15].

Lemma 5.1. *Let H be a semisimple Hopf algebra and A a finite-dimensional twisted H-module algebra such that $A\#_\sigma H$ exists. Then $A\#_\sigma H$ is of finite representation type if A is so.*

The next result, which given as Theorem 3.3 in [11], is also needed.

Lemma 5.2. *Let $\mathscr{H}$ be an infinitesimal group such that $\mathscr{H}/\mathscr{M}(\mathscr{H}) \cong {}_p\mathscr{W}(n)_k$. Then $\mathscr{H}$ is commutative and $\mathscr{H} \cong {}_p\mathscr{W}(n)_k \times \mathscr{M}(\mathscr{H})$.*

Theorem 5.3. *Let $\mathscr{G}$ be a finite algebraic k-supergroup with $\mathscr{G} \neq \underline{\mathscr{G}}$ and $H(\mathscr{G})$ be its algebra of measures. Then the following are equivalent*:

(1) *$H(\mathscr{G})$ has finite representation type.*

(2) *$\mathbf{u}(\mathrm{Lie}(\mathscr{G}))$ has finite representation type and either $H(\underline{\mathscr{G}})$ is semisimple or*

$$\underline{\mathscr{G}}^\circ/\mathscr{M}\left(\underline{\mathscr{G}}^\circ\right) \cong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$$

for some $m, n \in \mathbb{N}$.

Proof. "(1) ⇒ (2)" By Proposition 3.2, $\mathbf{u}(\mathrm{Lie}(\mathscr{G}))$ has finite representation type. Thus there is $0 \neq y \in V_{H(\mathscr{G})}$ such that $H(\mathscr{G}) = H(\underline{\mathscr{G}})\langle y\rangle$. If $[y, y] = 0$, $H(\underline{\mathscr{G}})$ is semisimple by Proposition 4.3. Otherwise, $[y, y] \neq 0$. In this case, if $\underline{\mathscr{G}}^\circ/\mathscr{M}(\underline{\mathscr{G}}^\circ) \ncong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$, then Proposition 4.4 implies that $\underline{\mathscr{G}}^\circ$ is multiplicative. So due to an application of Lemma 4.2, $H(\underline{\mathscr{G}})$ is semisimple.

"(2) ⇒ (1)" At first, assume that $H(\underline{\mathscr{G}})$ is semisimple. Since $H(\mathscr{G})$ is a super Hopf algebra, it is a Hopf algebra in the category ${}^{\mathbb{Z}_2}_{\mathbb{Z}_2}\mathscr{Y}\mathscr{D}$. So $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$ is a usual Hopf algebra. Lemma 2.4(3) implies that $\mathbf{u}(\mathrm{Lie}(\mathscr{G})) \rtimes k\mathbb{Z}_2$ is a normal sub Hopf algebra and so we have a Hopf surjection

$$H(\mathscr{G}) \rtimes k\mathbb{Z}_2 \twoheadrightarrow \left(H(\mathscr{G}) \rtimes k\mathbb{Z}_2\right)/\left(\mathbf{u}\left(\mathrm{Lie}(\mathscr{G})\right) \rtimes k\mathbb{Z}_2\right)^+\left(H(\mathscr{G}) \rtimes k\mathbb{Z}_2\right).$$

Owing to Theorem 8.4.6 in [21], $H(\mathscr{G}) \rtimes k\mathbb{Z}_2 \cong (\mathbf{u}(\mathrm{Lie}(\mathscr{G})) \rtimes k\mathbb{Z}_2)\#_\sigma((H(\mathscr{G}) \rtimes k\mathbb{Z}_2)/(\mathbf{u}(\mathrm{Lie}(\mathscr{G})) \rtimes k\mathbb{Z}_2)^+(H(\mathscr{G}) \rtimes k\mathbb{Z}_2))$. By assumption, $(H(\mathscr{G}) \rtimes k\mathbb{Z}_2)/(\mathbf{u}(\mathrm{Lie}(\mathscr{G})) \rtimes k\mathbb{Z}_2)^+(H(\mathscr{G}) \rtimes k\mathbb{Z}_2)$ is semisimple. Note that $\mathbf{u}(\mathrm{Lie}(\mathscr{G})) \rtimes k\mathbb{Z}_2$ has finite representation type (by Lemma 2.1) and by Lemma 5.1, $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$ and so $H(\mathscr{G})$ (by using Lemma 2.1 again) has finite representation type.

Next, assume $\underline{\mathscr{G}}^\circ/\mathscr{M}(\underline{\mathscr{G}}^\circ) \cong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$. Let $\mathscr{N}(\underline{\mathscr{G}})$ be the nilpotent radical of $\underline{\mathscr{G}}$. So assumption and Lemma 5.2 show that both

$$H(\mathscr{N}(\underline{\mathscr{G}}))/\mathbf{u}(L_0)^+H(\mathscr{N}(\underline{\mathscr{G}})) \quad \text{and} \quad H(\underline{\mathscr{G}})/H(\mathscr{N}(\underline{\mathscr{G}}))^+H(\underline{\mathscr{G}})$$

are semisimple. By Lemma 2.5, both

$$H(\mathscr{G})/\big(H(\mathscr{N}(\underline{\mathscr{G}}))\langle y\rangle\big)^+H(\mathscr{G}) \quad \text{and} \quad H(\mathscr{N}(\underline{\mathscr{G}}))\langle y\rangle/\mathbf{u}(L)^+H(\mathscr{N}(\underline{\mathscr{G}}))\langle y\rangle$$

are semisimple. By the fact that $\mathbf{u}(L)$ is of finite representation type and applying the same methods used in the above paragraph twice, $H(\mathscr{G})$ has finite representation type. □

6. The Auslander–Reiten quiver

Recall that an algebra A is a Nakayama algebra if each indecomposable A-module is uniserial. According to Theorem 2.1 in Chapter VI of [1], every Nakayama algebra has finite representation type. The main result of this section is to show that the converse is also true for super cocommutative Hopf algebras and the Auslander–Reiten quivers of representation-finite super cocommutative Hopf algebras can be deduced by this result right now.

Theorem 6.1. *Let $\mathscr{G}$ be a finite algebraic k-supergroup with $\mathscr{G} \neq \underline{\mathscr{G}}$ and $H(\mathscr{G})$ be its algebra of measures. If $H(\mathscr{G})$ is of finite representation type, then it is a Nakayama algebra.*

To show it, we begin with some observations. By the proof of Proposition 4.1, $H(\mathscr{G}) = H(\underline{\mathscr{G}})\langle y\rangle$ which is isomorphic to $H(\underline{\mathscr{G}}^\circ)\#H(\underline{\mathscr{G}}_{red})\langle y\rangle \cong H(\underline{\mathscr{G}}^\circ)\langle y\rangle\#H(\underline{\mathscr{G}}_{red})$. Owing to Lemma 4.2, $H(\underline{\mathscr{G}}_{red})$ is always semisimple. Thus Lemma 2.3 implies that $H(\underline{\mathscr{G}}^\circ)\langle y\rangle\#H(\underline{\mathscr{G}}_{red})$ is a Nakayama algebra if and only if $H(\underline{\mathscr{G}}^\circ)\langle y\rangle$ is so. Therefore, to show the theorem one can assume that

$$\underline{\mathscr{G}} = \underline{\mathscr{G}}^\circ.$$

Under such an assumption, we have

Lemma 6.2. *If $H(\underline{\mathscr{G}})$ is semisimple, then $H(\mathscr{G})$ is a Nakayama algebra.*

Proof. By Nagata's theorem (Chapter IV, §3, 3.6), $H(\underline{\mathscr{G}})$ is commutative. Thus $H(\underline{\mathscr{G}})$ decomposes into a direct sum

$$H(\underline{\mathscr{G}}) = \bigoplus_\gamma k_\gamma$$

of one-dimensional modules. Hence, we obtain

$$H(\mathscr{G}) \cong \bigoplus_\gamma H(\mathscr{G}) \otimes_{H(\underline{\mathscr{G}})} k_\gamma,$$

a direct sum of projective $H(\mathscr{G})$-modules. Consequently, the dimension of each projective indecomposable $H(\mathscr{G})$-module is bounded by 2, forcing all these modules to be uniserial. Note that $H(\mathscr{G})$ is a Frobenius algebra, all projective modules are injective and vice versa. As a result, $H(\mathscr{G})$ is a Nakayama algebra. □

In the following, we always assume that $\underline{\mathscr{G}} = \underline{\mathscr{G}}^\circ$ unless stated otherwise. Using Theorem 5.3 and the above lemma, we only need to consider the case $\underline{\mathscr{G}}/\mathscr{M}(\underline{\mathscr{G}}) \cong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$ for some $m, n \in \mathbb{N}$.

Lemma 6.3. *If $\underline{\mathscr{G}}/\mathscr{M}(\underline{\mathscr{G}}) \cong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$ for some $m, n \in \mathbb{N}$, then $H(\mathscr{M}(\underline{\mathscr{G}}))$ commutes with y.*

Proof. If not, there exists an element $h \in H(\mathscr{M}(\underline{\mathscr{G}}))$ such that $hy \neq yh$. By the proof of Proposition 4.1, there is a character $\chi : H(\underline{\mathscr{G}}) \to k$ such that $y \cdot h = \chi(h)y$. Thus assumption implies that $\chi(h) \neq \varepsilon(h)$. So

$$0 \neq \big(\chi(h) - \varepsilon(h)\big)y = y \cdot \big(h - \varepsilon(h)1\big) \in \big(H\big(\mathscr{M}(\underline{\mathscr{G}})\big)^+\big)$$

the ideal generated by $H(\mathscr{M}(\underline{\mathscr{G}}))^+$. Thus $y \in (H(\mathscr{M}(\underline{\mathscr{G}}))^+)$ which is impossible by the proof of Proposition 4.4. □

Denote by $\mathscr{B}_0(H(\mathscr{G}))$ the block of $H(\mathscr{G})$ containing the trivial module k.

Lemma 6.4. *If $\underline{\mathscr{G}}/\mathscr{M}(\underline{\mathscr{G}}) \cong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$ for some $m, n \in \mathbb{N}$, then:*

(1) *As an algebra, $H(\mathscr{G})/(H(\mathscr{M}(\underline{\mathscr{G}}))^+) \cong k[y]/(y^{2p^n})\#(k\mathbb{Z}_{p^m})^*$ which is a Nakayama algebra.*

(2) *The canonical projection $\pi : H(\mathscr{G}) \to H(\mathscr{G})/(H(\mathscr{M}(\underline{\mathscr{G}}))^+)$ induces an isomorphism $\mathscr{B}_0(H(\mathscr{G})) \cong H(\mathscr{G})/(H(\mathscr{M}(\underline{\mathscr{G}}))^+)$.*

Proof. (1) By the proof of Proposition 4.4,

$$H(\underline{\mathscr{G}})/H\big(\mathscr{M}(\underline{\mathscr{G}})\big)^+H(\underline{\mathscr{G}}) \cong \big(L(n,n)\big)^*\#(k\mathbb{Z}_{p^m})^* \cong k[x]/\big(x^{p^n}\big)\#(k\mathbb{Z}_{p^m})^*$$

and $[ky, ky]_p = (kx)_p$. Thus it is harmless to assume that $y^2 = x$ and so

$$H(\mathscr{G})/\big(H\big(\mathscr{M}(\underline{\mathscr{G}})\big)^+\big) \cong \big(L(n,n)\big)^*\langle y\rangle\#(k\mathbb{Z}_{p^m})^* \cong k[y]/\big(y^{2p^n}\big)\#(k\mathbb{Z}_{p^m})^*.$$

Since ky is invariant under the action of $(k\mathbb{Z}_{p^m})^*$, the Jacobson radical $J_{k[y]/(y^{2p^n})\#(k\mathbb{Z}_{p^m})^*}$ equals to $(ky)\#(k\mathbb{Z}_{p^m})^*$. And so

$$k[y]/\big(y^{2p^n}\big)\#(k\mathbb{Z}_{p^m})^*/J_{k[y]/(y^{2p^n})\#(k\mathbb{Z}_{p^m})^*} \cong (k\mathbb{Z}_{p^m})^*$$

and

$$J_{k[y]/(y^{2p^n})\#(k\mathbb{Z}_{p^m})^*}/J^2_{k[y]/(y^{2p^n})\#(k\mathbb{Z}_{p^m})^*} \cong (k\mathbb{Z}_{p^m})^*.$$

From this, Gabriel's quiver of $k[y]/(y^{2p^n})\#(k\mathbb{Z}_{p^m})^*$ is a basic cycle with $\dim_k(k\mathbb{Z}_{p^m})^*$ vertices. Thus it is Nakayama.

(2) According to (1), $H(\mathscr{G})/(H(\mathscr{M}(\underline{\mathscr{G}}))^+)$ is connected by noting that x (and so y) does not commute with $(k\mathbb{Z}_{p^m})^*$. It follows that the restriction $\pi : \mathscr{B}_0(H(\mathscr{G})) \to H(\mathscr{G})/(H(\mathscr{M}(\underline{\mathscr{G}}))^+)$ of the canonical projection maps the primitive central idempotent of $\mathscr{B}_0(H(\mathscr{G}))$ onto the identity. Consequently, π is surjective. Since the ideal $(H(\mathscr{M}(\underline{\mathscr{G}}))^+) = H(\mathscr{G})H(\mathscr{M}(\underline{\mathscr{G}}))^+$ (by Lemma 6.3) is indeed generated by central idempotents not belonging to $\mathscr{B}_0(H(\mathscr{G}))$, the map π is also injective, and our assertion follows. □

Let H be an ordinary Hopf algebra and M, N two H-modules. One can equip the tensor product $M \otimes N$ with an H-module structure through the comultiplication $\Delta : H \to H \otimes H$ and make $\mathrm{Hom}_k(M, N)$ to be an H-module by $(h \cdot f)(m) := \sum h_{(1)} f(S(h_{(2)})m)$ for $f \in \mathrm{Hom}_k(M, N)$ and $h \in H$. In the case of H is a super Hopf algebra, one also can do the same constructions by using supermodules. The following result is the counter part of Corollary 2.5(1) in [11] in super case.

Lemma 6.5. *Assume that $\underline{\mathscr{G}}/\mathscr{M}(\underline{\mathscr{G}}) \cong {}_p\mathscr{W}(n)_k \rtimes {}_{p^m}\mu_k$ for some $m, n \in \mathbb{N}$. Let $\mathscr{B}$ be a block of $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$ and S, T be two simple modules belonging to $\mathscr{B}$. Then there exists a character $\gamma : H(\mathscr{G}) \rtimes k\mathbb{Z}_2 \to k$ such that $T \cong k_\gamma \otimes S$.*

Proof. Note that $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$ is an ordinary Hopf algebra. Consider the $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$-module $\mathrm{Hom}_k(S, T)$. By Lemma 6.3, $H(\mathscr{M}(\underline{\mathscr{G}}))$ lies in the center of $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$. By S, T belonging to the same block, $H(\mathscr{M}(\underline{\mathscr{G}}))$ operates on S and T via the same character and so acts trivially on $\mathrm{Hom}_k(S, T)$. Hence $\mathrm{Hom}_k(S, T)$ is an $H(\mathscr{G}) \rtimes k\mathbb{Z}_2/(H(\mathscr{M}(\underline{\mathscr{G}}))^+)$-module, which is a basic algebra by Lemma 6.4(1). Therefore, $\mathrm{Hom}_k(S, T)$ contains a 1-dimensional submodule k_γ, defined by a character γ of $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$. Let ψ be a non-zero element of k_γ and consider

$$\hat{\psi} : k_\gamma \otimes S \to T, \qquad \psi \otimes x \mapsto \psi(x).$$

Now, for $h \in H(\mathscr{G}) \rtimes k\mathbb{Z}_2$,

$$\begin{aligned}\hat{\psi}\big(h \cdot (\psi \otimes x)\big) &= \sum \hat{\psi}(h_{(1)} \cdot \psi \otimes h_{(2)} \cdot x) \\ &= \sum (h_{(1)} \cdot \psi)(h_{(2)} \cdot x) = \sum h_{(1)}\psi\big(S(h_{(2)})h_{(3)} \cdot x\big) \\ &= h\psi(x) = h \cdot \hat{\psi}(\psi \otimes x).\end{aligned}$$

Consequently, $\hat{\psi}$ is, as a non-zero $H(\mathscr{G}) \rtimes k\mathbb{Z}_2$-linear map between two simple modules, an isomorphism. □

We are now in the position to prove Theorem 6.1.

Proof of Theorem 6.1. The proof is easy now. In fact, Lemmas 2.2, 2.3 and 6.5 ensure that the arguments using in [10] (see the "(2) ⇒ (3)" part of the proof of Theorem 2.7 in [10]) can be applied to our case directly. □

Let $\mathbf{A}_l$ be the type A quiver of length l. For more information on quivers and the definition of the stable Auslander–Reiten quiver $\Gamma_s(\Lambda)$ of a self-injective algebra Λ the reader may consult [1].

Corollary 6.6. *Let $\mathscr{G}$ be a finite algebraic super k-group, $H(\mathscr{G})$ its algebra of measures and $\mathscr{B} \subset H(\mathscr{G})$ a block. If $H(\mathscr{G})$ has finite representation type, then*

$$\Gamma_s(\mathscr{B}) \cong \mathbb{Z}\mathbf{A}_{(l-1)}/\big(\tau^n\big)$$

for l the Loewy length of $\mathscr{B}$ and n the number of simple $H(\mathscr{G})$-modules belonging to $\mathscr{B}$.

Proof. A direct consequence of Theorem 6.1 and general result stated in page 253 of [1]. □

We use the following example to explain the results we have gotten.

Example 6.7. Assume that $p = 3$ (although now $p < 5$, it is not essential for this example). Let $L = L_0 \oplus L_1$ be a Lie superalgebra with $L_0 = kx + kt + kt_1$ and $L_1 = ky$ with relations

$$[t, x] = [t, y] = [t, t_1] = 0, \qquad [t_1, x] = x + t,$$

$$[t_1, y] = 2y, \qquad [x, y] = 0, \qquad [y, y] = x + t.$$

The p-mapping is given by

$$t^{[p]} = t, \qquad t_1^{[p]} = t_1, \qquad x^{[p]} = 0.$$

By Lemma 3.1(2), $\mathbf{u}(L)$ has finite representation type. Let $e_0 := 1 - t^2$, $e_1 := 2t + 2t^2$, $e_2 := t + 2t^2$, then

$$\mathbf{u}(L) = \mathbf{u}(L)e_0 \oplus \mathbf{u}(L)e_1 \oplus \mathbf{u}(L)e_2$$

is the block decomposition of $\mathbf{u}(L)$.

For $\mathbf{u}(L)e_0$, by $te_0 = t - t^3 = 0$, it is isomorphic to $k\{y, t_1\}/(y^6, t_1^3 - t_1, t_1 y - y t_1 - 2y)$. Note the subalgebra generated by t_1 is isomorphic to $(k\mathbb{Z}_3)^*$ and the subalgebra generated by y is a $(k\mathbb{Z}_3)^*$-module algebra through the action $t_1 \cdot y := [t_1, y] = 2y$. Thus $\mathbf{u}(L)e_0 \cong k[y]/(y^6)\#(k\mathbb{Z}_3)^*$. It is not hard to see that the group algebra of the largest multiplicative center is the algebra generated by t and thus we indeed have $\mathbf{u}(L)/(t) \cong \mathbf{u}(L)e_0$. All facts stated in Lemma 6.4 are verified in this case.

For $\mathbf{u}(L)e_1$, by $te_1 = e_1$, it is isomorphic to $k\{x, y, t_1\}/(y^2 - x - 1, x^3, t_1^3 - t_1, t_1 y - y t_1 - 2y)$. We will show that it is a super simple algebra. This is equivalent to show that $\mathbf{u}(L)e_1\#k\mathbb{Z}_2$ is simple. Indeed, denote the generator of $\mathbb{Z}_2$ by g and define

$$I_{2,-1} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad I_{1,2} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As usual, let I_2 be the 2×2 identity matrix. Consider the map

$$\phi : \mathbf{u}(L)e_1\#k\mathbb{Z}_2 \to \mathbf{M}_6(k)$$

by sending

$$t_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 2I_2 \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 0 & I_{1,2} & 0 \\ 0 & 0 & I_{1,2} \\ I_{1,2} & 0 & 0 \end{pmatrix},$$

$$x \mapsto \begin{pmatrix} -I_2 & 0 & I_2 \\ I_2 & -I_2 & 0 \\ 0 & I_2 & -I_2 \end{pmatrix}, \qquad g \mapsto \begin{pmatrix} I_{2,-1} & 0 & 0 \\ 0 & I_{2,-1} & 0 \\ 0 & 0 & I_{2,-1} \end{pmatrix}.$$

By direct computations, one can show that ϕ is an algebra isomorphism.

Similarly, $\mathbf{u}(L)e_2\#k\mathbb{Z}_2 \cong \mathbf{M}_6(k)$ too. Thus $\mathbf{u}(L)$ is a Nakayama algebra.

We end this section with the following remarks.

Remark 6.8. (1) It is known that for an infinitesimal group $\underline{\mathscr{G}}^\circ$, $H(\underline{\mathscr{G}}^\circ)$ has finite representation type if and only if it is a Nakayama algebra (see Theorem 2.7 [10]). But for a constant group G, kG may be not a Nakayama algebra even if kG is of finite representation type. That is, for a representation-finite finite algebraic k-group $\underline{\mathscr{G}}$, $H(\underline{\mathscr{G}})$ may not be a Nakayama algebra. In contrast with the ordinary finite algebraic groups, Theorem 6.1 tells us that the phenomenon will not appear in super case.

(2) Theorem 6.1 also corrects a mistake of [5]. As one of the main results of [5] (Theorem 4.3), the author showed that the representation-finite restricted enveloping algebras of Lie superalgebras are not always Nakayama algebras. The author's proof seems not correct, I think.

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References

[1] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36, Cambridge University Press, 1995.
[2] M. Demazure, P. Gabriel, Groupes algébriques, Paries, North-Holland Publishing Co., Amsterdam, 1970.
[3] Y. Drozd, Tame and wild matrix problems, in: Representations and Quadratic Forms, Inst. Math. Acad. Sciences Ukrainian SSR, Kiev, 1979, pp. 39–74; Amer. Math. Soc. Transl. 128 (1986) 31–55.
[4] K. Erdmann, Blocks of Tame Representation Type and Related Algebras, Lecture Notes in Math., vol. 1428, Springer-Verlag, 1990.
[5] R. Farnsteiner, Note on Frobenius extensions and restricted Lie superalgebras, J. Pure Appl. Algebra 108 (3) (1996) 241–256.
[6] R. Farnsteiner, Periodicity and representation type of modular Lie algebras, J. Reine Angew. Math. 464 (1995) 47–65.
[7] R. Farnsteiner, A. Skowronski, Classification of restricted Lie algebras with tame principal block, J. Reine Angew. Math. 546 (2002) 1–45.
[8] R. Farnsteiner, Polyhedral groups, Mckey quivers and the finite algebraic groups with tame principal blocks, Invent. Math. 166 (2006) 27–94.
[9] R. Farnsteiner, G. Röhrle, D. Voigt, Infinitesimal unipotent group schemes of complexity 1, Colloq. Math. 89 (2) (2001) 179–192.
[10] R. Farnsteiner, D. Voigt, On cocommutative Hopf algebras of finite representation type, Adv. Math. 155 (2000) 1–22.
[11] R. Farnsteiner, D. Voigt, Modules of solvable infinitesimal groups and the structures of representation-finite cocommutative Hopf algebras, Math. Proc. Cambridge Philos. Soc. 127 (1999) 441–459.
[12] J. Feldvoss, H. Strade, Restricted Lie algebras with bounded cohomology and related classes of algebras, Manuscripta Math. 74 (1992) 47–67.
[13] G. Hochschild, Representations of restricted Lie algebras of characteristic p, Proc. Amer. Math. Soc. 5 (1954) 603–605.
[14] H.-L. Huang, G. Liu, Y. Ye, Commutative Hopf structures over a loop, arXiv:1002.0405.
[15] F. Li, M. Zhang, Invariant properties of representations under Cleft extensions, Sci. China 50 (1) (2007) 121–131.
[16] G. Liu, A note on the global dimension of smash products, Comm. Algebra 33 (2005) 2625–2627.
[17] S. Majid, Crossed products by braided groups and bosonization, J. Algebra 163 (1994) 165–190.
[18] A. Masuoka, The fundamental correspondences in super affine groups and super formal groups, J. Pure Appl. Algebra 202 (1–3) (2005) 284–312.
[19] W. Pfautsch, D. Voigt, The representation-finite algebraic groups of dimension zero, C. R. Acad. Sci. Paris 306 (1988) 685–689.
[20] D. Radford, The structure of Hopf algebras with a projection, J. Algebra 92 (1985) 322–347.
[21] H.-J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras, J. Algebra 152 (2) (1992) 289–312.
[22] H. Strade, R. Farnsteiner, Modular Lie Algebras and Their Representations, Marcel Dekker Monographs, vol. 116, Marcel Dekker, New York, 1988.
[23] D. Voigt, Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen, Lecture Notes in Math., vol. 592, Springer-Verlag, Berlin, New York, 1977.