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Support varieties and representation types for basic classical Lie superalgebras

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A B S T R A C T

Let κ be an algebraically closed field of characteristic $p>3$ and $\mathfrak{g}$ a restricted Lie superalgebra over κ. We introduce the definition of restricted cohomology for $\mathfrak{g}$ and show its cohomology ring is finitely generated provided $\mathfrak{g}$ is a basic classical Lie superalgebra. As a consequence, we show that the restricted enveloping algebra of a basic classical Lie superalgebra $\mathfrak{g}$ is always wild except $\mathfrak{g}=\mathfrak{sl}_2$ or $\mathfrak{g}=\mathfrak{osp}(1|2)$ or $\mathfrak{g}=\mathbf{C}(2)$.

1. Introduction

1.1. As generalizations and deep continuations of classical Lie theory, Lie superalgebras, supergroups and their representation theory over the field of complex numbers $\mathbb{C}$ have been studied extensively since the classification of finite-dimensional complex simple Lie superalgebras by Kac [20]. More on supergroups, supergeometry and supersymmetric theory can be found in [12,26]. In recent years, there has been increasing interest in modular representation theory of algebraic supergroups. Especially, the modular representations of $GL(m|n)$, $Q(n)$ and ortho-symplectic supergroups have been initiated by Brundan, Kleshchev, Kujawa [6–9,21], and Shu and Wang [32]. A systematic research of modular Lie superalgebras has been started [36,37]. In [36], the super version of the celebrated Kac–Weisfeiler Property is shown to hold for the basic classical Lie superalgebras, which by definition admit an even nondegenerate supersymmetric bilinear form and whose even subalgebras are reduc-

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tive. Actually, the modular representation theory of supergroups and Lie superalgebras not only is of intrinsic interest in its own right, but also has found remarkable applications to classical mathematics. See [32] for some historical remarks.

Support varieties were introduced in the pioneering work of Alperin [1] and Carlson [10,11] nearly 30 years ago as a method to study complexes and resolutions of modules over group algebras. They open an algebro–geometric gate to linear representations of finite groups. Since then such ideas have been extended to restricted Lie algebras [18], Steenrod algebra [30], infinitesimal group schemes [35], arbitrary finite-dimensional cocommutative Hopf algebras [19] and even to finite-dimensional algebras [33]. See [34] for a nice survey on the theory of support varieties.

1.2. Up to now, we are lack of this algebro–geometric tool for modular Lie superalgebras, perhaps due to the representation theory of simple Lie superalgebras over $\mathbb{C}$ is already very difficult and remains to be better understood. Recently, such tools were introduced for Lie superalgebras over $\mathbb{C}$ in [5] by using so-called relative cohomology. It seems that the methods used in [5] cannot be applied to positive characteristic case directly. The main aim of this paper is to establish a kind of definition for a support variety, which is suitable for our purpose, and give an application. At first, we realize that for any restricted Lie superalgebra $\mathfrak{g}$ one can relate it with an ordinary Hopf algebra $\mathbf{u}(\mathfrak{g}) \rtimes \kappa\mathbb{Z}_2$ possessing equivalent representation theory as $\mathbf{u}(\mathfrak{g})$. So we can pass from “super world” to the “usual world” without losing information. Using this ordinary Hopf algebra, we can define its cohomology algebra naturally.

It is known that support varieties can be defined once the finite generation of cohomology is established, which is hard to prove in general. In this paper, we prove this finite generation property for the class of basic classical Lie superalgebras. It consists of several infinite series and 3 exceptional ones. We divide our proof into two different cases: $\mathfrak{g} \neq \mathbf{A}(1,1)$ or $\mathfrak{g} = \mathbf{A}(1,1)$. In the first case, we give a two-step filtration to reduce $\mathbf{u}(\mathfrak{g})$ to a familiar algebra whose cohomology ring is known and each of filtration involves a convergent spectral sequence. We find some permanent cycles in such spectral sequences and apply a lemma cited from [27] to conclude finite generation. To give the filtration, a new kind of PBW basis is developed. We put the case $\mathfrak{g} = \mathbf{A}(1,1)$ in a bigger context, in which all $\mathbf{u}(\mathfrak{g})$ are equipped with a nice filtration similar to the coradical filtration of a coalgebra. Through this one-step filtration, we can reduce $\mathbf{u}(\mathbf{A}(1,1))$ to a familiar algebra already. Then the same idea developed in the first case can be applied.

One central question in the modern representation theory of algebras is the determination of the representation type. By Drozd’s fundamental trichotomy [13], finite-dimensional algebras over an algebraically closed field may be subdivided into the disjoint classes of representation finite, tame and wild algebras. As an application of support varieties we built, we will prove all $\mathbf{u}(\mathfrak{g})$ are wild with only three exceptions: $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{osp}(1|2), \mathbf{C}(2)$. The case $\mathbf{C}(2)$ is conjectured to be wild and we have known $\mathbf{u}(\mathfrak{sl}_2)$ and $\mathbf{u}(\mathfrak{osp}(1|2))$ are tame.

The paper is organized as follows. All subsidiary results to prove the finite generation of cohomology algebras are builded in Section 2. Especially, a new kind of PBW basis suitable for our purpose and some filtrations are given. Section 3 is to give the proof of finite generation. The definition of a support variety is given in Section 4. Moreover, its connections with complexity and representation type are established. As the final conclusion of this section, the representation type of any $\mathbf{u}(\mathfrak{g})$ is determined except the case $\mathbf{C}(2)$, which is conjectured to be a wild algebra.

As pointed out by the referee to the author, Irfan Bagci independently posted a paper [2] covering similar results on finite-generation of cohomology for restricted Lie superalgebras.

2. Preliminaries

Throughout of this paper, κ is an algebraically closed field of characteristic $p \neq 0$ and $p > 3$ is always assumed unless stated otherwise. All spaces are κ-spaces. All modules are left modules. For a superalgebra $A = A_0 \oplus A_1$, all commutators considered in paper are *graded commutators*, that is, $[x, y] = xy - (-1)^{\alpha\beta} yx$ for $x \in A_\alpha$, $y \in A_\beta$ with $\alpha, \beta \in \{0, 1\}$.

2.1. Hopf algebras in Yetter–Drinfeld categories

Let J be a Hopf algebra with bijective antipode and ${}^J_J\mathcal{YD}$ the category of the Yetter–Drinfeld modules with left J-module action and left J-comodule coaction. It is naturally forms a braided monoidal category with the braiding

$$c_{M,N}: M \otimes N \to N \otimes M, \quad m \otimes n \mapsto \sum m_{-1} \cdot n \otimes m_0,$$

where $m \mapsto \sum m_{-1} \otimes m_0$, $M \to J \otimes M$ denotes the comodule structure, as usual. Let A be a *braided Hopf algebra* in ${}^J_J\mathcal{YD}$. By definition, it is an algebra as well as coalgebra in ${}^J_J\mathcal{YD}$ such that its comultiplication and counit are algebra morphism, and such that the identity morphism has a convolution inverse in ${}^J_J\mathcal{YD}$. When we say that the comultiplication $\Delta: A \to A \otimes A$ should be an algebra morphism, the braiding defined as above arises in the definition of the algebra structure of $A \otimes A$ and so A is not an ordinary Hopf algebra in general. Through the Radford–Majid bosonization [25,31], it gives rise to an ordinary Hopf algebra $A \rtimes J$. As an algebra, this is the smash product $A \# J$, and it is the smash coproduct as a coalgebra.

Lemma 2.1. *Let J be a Hopf algebra with bijective antipode and A a braided Hopf algebra in ${}^J_J\mathcal{YD}$. Then the cohomology ring $\mathrm{H}^*(A, \kappa) := \bigoplus_{i \geqslant 0} \mathrm{Ext}^i_A(\kappa, \kappa)$ is a braided graded commutative algebra in ${}^J_J\mathcal{YD}$.*

Proof. By Theorem 3.12 in [27], the Hochschild cohomology ring

$$\mathrm{HH}^*(A, \kappa) := \bigoplus_{i \geqslant 0} \mathrm{Ext}^i_{A \otimes A^{op}}(A, \kappa)$$

is a braided graded commutative algebra in ${}^J_J\mathcal{YD}$. By the standard bar resolution for computing these extension groups, one can see that $\mathrm{Ext}^i_A(\kappa, \kappa) \cong \mathrm{Ext}^i_{A \otimes A^{op}}(A, \kappa)$ for $i \geqslant 0$ (see also Subsection 2.4 in [27]). The proof is complete. □

2.2. Cohomology of restricted Lie superalgebras

We fix some notions at first. By definition, a superalgebra is nothing but a $\mathbb{Z}_2$-graded algebra. By forgetting the grading we may consider any superalgebra A as a usual algebra and this algebra will be denoted by $|A|$. For any two $\mathbb{Z}_2$-graded vector spaces $V = V_0 \oplus V_1$, $W = W_0 \oplus W_1$, we use $\mathrm{Hom}_\kappa(V, W)$ to represent the set of all linear maps from V to W and $\underline{\mathrm{Hom}}_\kappa(V, W)$ to denote that of all even linear maps. By definition, $\underline{\mathrm{Hom}}_\kappa(V, W) = \{f \in \mathrm{Hom}_\kappa(V, W) \mid f(V_i) \subseteq W_i,\ i = 0, 1\}$.

Now let $A = A_0 \oplus A_1$ be a superalgebra. Then there is a natural action of $\mathbb{Z}_2 = \langle g \mid g^2 = 1\rangle$ on A given by

$$g \cdot a = a, \qquad g \cdot b = -b, \quad \text{for } a \in A_0,\ b \in A_1.$$

Note that this definition makes sense as stated only for homogeneous elements, it should be interpreted via linearity in the general case. Thus A is a $\kappa\mathbb{Z}_2$-module algebra (for definition, see Section 4.1 in [29]) and the smash product $A \# \kappa\mathbb{Z}_2$ is a usual algebra. We use A-smod to denote the category of all finitely generated left A-supermodules with even homomorphisms and $A \# \kappa\mathbb{Z}_2$-mod the usual finitely generated left $A \# \kappa\mathbb{Z}_2$-modules category.

Lemma 2.2. *Let A be a superalgebra. Then A-*smod *is equivalent to $A \# \kappa\mathbb{Z}_2$-*mod.

Proof. Let $M = M_0 \oplus M_1$ be an A-supermodule and $g \in \mathbb{Z}_2$ the generator of $\mathbb{Z}_2$. Through assigning $g \cdot m_0 := m_0$, $g \cdot m_1 := -m_1$ for $m_0 \in M_0$, $m_1 \in M_1$, M is a $\kappa\mathbb{Z}_2$-module. Now just define the action of $A \# \kappa\mathbb{Z}_2$ on M through $(a \otimes g) \cdot m := a \cdot (g \cdot m)$ for $a \in A$ and $m \in M$. To show it is indeed an $A \# \kappa\mathbb{Z}_2$-module, one need verify the equality

$$(1 \otimes g)(a \otimes 1) \cdot m = \big((g \cdot a) \otimes g\big) \cdot m, \tag{$*$}$$

for $a \in A$ and $m \in M$. It is not hard to see that this is equivalent to the fact $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$.

Conversely, let M be an $A \# \kappa\mathbb{Z}_2$-module. Since the characteristic of κ is not equal to 2, $\kappa\mathbb{Z}_2$ is semisimple. Therefore, $M = M_0 \oplus M_1$ with $M_0 = \{m \in M \mid g \cdot m = m\}$ and $M_1 = \{m \in M \mid g \cdot m = -m\}$. Also, the equality $(*)$ implies that $A_i M_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}_2$. Thus M is an A-supermodule.

At last, it is clear that $\underline{\mathrm{Hom}}_A(-,-) = \mathrm{Hom}_{A\#\kappa\mathbb{Z}_2}(-,-)$. The lemma is proved. □

Now we specialize this simple observation to the case of restricted enveloping algebras of restricted Lie superalgebras.

Definition 2.3. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called a *restricted Lie superalgebra*, if there is a pth map $\mathfrak{g}_0 \to \mathfrak{g}_0$, denoted as ${}^{[p]}$, satisfying

(a) $(cx)^{[p]} = c^p x^{[p]}$ for all $c \in k$ and $x \in \mathfrak{g}_0$,
(b) $[x^{[p]}, y] = (adx)^p(y)$ for all $x \in \mathfrak{g}_0$ and $y \in \mathfrak{g}$,
(c) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$ for all $x, y \in \mathfrak{g}_0$ where $is_i(x, y)$ is the coefficient of λ^{i-1} in $(ad(\lambda x + y))^{p-1}(x)$.

In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module by the adjoint action of the even subalgebra. All the Lie (super)algebras in this paper will be assumed to be restricted. For a restricted Lie superalgebra $\mathfrak{g}$, $U(\mathfrak{g})$ is denoted to be its universal enveloping algebra and $\mathbf{u}(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} \mid x \in \mathfrak{g}_0)$ its restricted enveloping algebra. The following is a consequence of PBW theorem for $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$.

Lemma 2.4. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and $x_1, \ldots, x_s$ a basis of $\mathfrak{g}_1$, $y_1, \ldots, y_t$ a basis of $\mathfrak{g}_0$. Then*:

(1) *$U(\mathfrak{g})$ has a basis*

$$\big\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} \bigm| b_i \in \mathbb{N},\ 0 \leqslant a_j \leqslant 1 \text{ for all } i, j\big\}.$$

(2) *$\mathbf{u}(\mathfrak{g})$ has a basis*

$$\big\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} \bigm| 0 \leqslant b_i \leqslant p - 1,\ 0 \leqslant a_j \leqslant 1 \text{ for all } i, j\big\}.$$

The following proposition gives new kinds of PBW basis, which are suitable for our purpose.

Proposition 2.5. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and $x_1, \ldots, x_s$ a basis of $\mathfrak{g}_1$ in which we assume $[x_i, x_i] = 0$ for $i \leqslant s_1$ and $z_j := [x_j, x_j] \neq 0$ for $s_1 < j \leqslant s$. Assume that $z_{s_1+1}, \ldots, z_s$ are linear independent and denote the subspace of $\mathfrak{g}_0$ spanned by them by V. Let W be a subspace of $\mathfrak{g}_0$ such that $\mathfrak{g}_0 = W \oplus V$ and $y_1, \ldots, y_{t_1}$ be a basis of W. Then*:

(1) *$U(\mathfrak{g})$ has a basis consisting of*

$$x_1^{a_1} \cdots x_{s_1}^{a_s} x_{s_1+1}^{b_1} \cdots x_s^{b_{s-s_1}} y_1^{c_1} \cdots y_{t_1}^{c_{t_1}}$$

where $0 \leqslant a_i \leqslant 1$, $b_j, c_k \in \mathbb{N}$ for all i, j, k.

(2) $\mathbf{u}(\mathfrak{g})$ *has a basis consisting of*

$$x_1^{a_1} \cdots x_{s_1}^{a_s} x_{s_1+1}^{b_1} \cdots x_s^{b_{s-s_1}} y_1^{c_1} \cdots y_{t_1}^{c_{t_1}}$$

where $0 \leqslant a_i \leqslant 1$, $0 \leqslant b_j \leqslant 2p-1$, $0 \leqslant c_k \leqslant p-1$ *for all* i, j, k.

Proof. We only prove (2) since (1) can be proved similarly. By assumption the set $\{z_i, y_j \mid s_1 < i \leqslant s, 0 \leqslant j \leqslant t_1\}$ is a basis of $\mathfrak{g}_0$. Owing to Lemma 2.4(2),

$$\left\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_{t_1+s-s_1}^{b_{t_1+s-s_1}} \;\middle|\; 0 \leqslant b_i \leqslant p-1,\ 0 \leqslant a_j \leqslant 1 \text{ for all } i, j\right\}$$

is a basis of $\mathbf{u}(\mathfrak{g})$ where we set $y_{t_1+i} := z_{s_1+i}$ $(1 \leqslant i \leqslant s - s_1)$ for consistence. By the proof of the PBW theorem, there is no any restriction on the order of elements we choose and thus the following elements also form a basis of $\mathbf{u}(\mathfrak{g})$:

$$x_1^{a_1} \cdots x_{s_1}^{a_{s_1}} x_{s_1+1}^{a_{s_1+1}} z_{s_1+1}^{b_{s_1+1}} \cdots x_s^{a_s} z_s^{b_s} y_1^{b_1} \cdots y_{t_1}^{b_{t_1}} \tag{2.1}$$

where $0 \leqslant b_i \leqslant p-1$, $0 \leqslant a_j \leqslant 1$ for all i, j. Since

$$z_i = [x_i, x_i] = 2x_i^2$$

in $\mathbf{u}(\mathfrak{g})$ for $s_1 + 1 \leqslant i \leqslant s$, the κ-span of the set $\{x_i^{a_i} z_i^{b_i} \mid 0 \leqslant a_i \leqslant 1,\ 0 \leqslant b_i \leqslant p-1\}$ is equal to the κ-span of the set $\{x_i^{m_i} \mid 0 \leqslant m_i \leqslant 2p-1\}$. So we can abbreviate elements of (2.1) and get the ones described in the proposition. The conclusion is proved. □

Both $U(\mathfrak{g})$ and $\mathbf{u}(\mathfrak{g})$ are super cocommutative Hopf algebras. Explicitly, for any $x \in \mathfrak{g}$, their comultiplications Δ and the antipodes S are defined in the same way with the usual (restricted) enveloping algebras:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad S(x) = -x.$$

Thus they are braided Hopf algebras in ${}^{\kappa\mathbb{Z}_2}_{\kappa\mathbb{Z}_2}\mathcal{YD}$. In particular, $\mathbf{u}(\mathfrak{g}) \rtimes \kappa\mathbb{Z}_2$ is an ordinary Hopf algebra (see [4] for an alternate interpretation). In order to emphasize its algebra structure, $\mathbf{u}(\mathfrak{g}) \rtimes \kappa\mathbb{Z}_2$ is rewritten by $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$. Let M, N be two $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$-modules and $P_\bullet \to M$ be a projective resolution of M. Define

$$\mathrm{H}^i_{\mathbf{u}(\mathfrak{g})}(M, N) := \mathrm{Ext}^i_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(M, N) = \mathrm{H}^i\big(\mathrm{Hom}_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(P_\bullet, N)\big),$$
$$\mathrm{H}^i\big(\mathbf{u}(\mathfrak{g}), M\big) := \mathrm{Ext}^i_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(\kappa, M)$$

and

$$\mathrm{H}^i\big(\mathbf{u}(\mathfrak{g}), \kappa\big) := \mathrm{Ext}^i_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(\kappa, \kappa)$$

for $i \geqslant 0$, where κ is the trivial $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$-module with the action gotten through the counit $\varepsilon : \mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2 \to \kappa$.

Remark 2.6. By Lemma 2.2, this is equivalent to say that we consider the restricted cohomology of a restricted Lie superalgebra $\mathfrak{g}$ exactly in the category $\mathbf{u}(\mathfrak{g})$-smod. That is, we only consider even homomorphisms. This is totally different with the relative cohomology defined in [5], where the authors indeed bring all homomorphisms into consideration.

For any coalgebra C, we denote $\operatorname{Ker}\varepsilon$ by C^+ as usual. Also, as a usual algebra $|\mathbf{u}(\mathfrak{g})|$ has its usual cohomology $\mathrm{H}^i(|\mathbf{u}(\mathfrak{g})|, N)$ for any $|\mathbf{u}(\mathfrak{g})|$-module N. For any Hopf algebra H and H-module M, we define $M^H := \{m \in M \mid h \cdot m = \varepsilon(h)m,\ \text{for all } h \in H\}$.

Lemma 2.7. *Let N be a $\mathbf{u}(\mathfrak{g})$-supermodule. Then for any natural number i,*

$$\mathrm{H}^i\big(\mathbf{u}(\mathfrak{g}), N\big) \cong \mathrm{H}^i\big(\big|\mathbf{u}(\mathfrak{g})\big|, N\big)^{\kappa\mathbb{Z}_2}.$$

Proof. At first, we prove the conclusion in the case $N = \kappa$. Note that $|\mathbf{u}(\mathfrak{g})|^+$ is the augmentation ideal of $|\mathbf{u}(\mathfrak{g})|$. Now consider the bar resolution of κ

$$\cdots \to \big|\mathbf{u}(\mathfrak{g})\big| \otimes \big(\big|\mathbf{u}(\mathfrak{g})\big|^+\big)^{\otimes 2} \xrightarrow{d_2} \big|\mathbf{u}(\mathfrak{g})\big| \otimes \big|\mathbf{u}(\mathfrak{g})\big|^+ \xrightarrow{d_1} \big|\mathbf{u}(\mathfrak{g})\big| \xrightarrow{\varepsilon} \kappa \to 0, \tag{2.2}$$

where $d_i(a_0 \otimes \cdots \otimes a_i) = \sum_{j=0}^{i-1}(-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i$. Thus every differential map d_i is indeed an even homomorphism. Applying $\operatorname{Hom}_{|\mathbf{u}(\mathfrak{g})|}(-,\kappa)$, one gets

$$0 \to \operatorname{Hom}_\kappa(\kappa,\kappa) \xrightarrow{\delta_0} \operatorname{Hom}_\kappa\big(\big|\mathbf{u}(\mathfrak{g})^+\big|,\kappa\big) \xrightarrow{\delta_1} \operatorname{Hom}_\kappa\big(\big|\mathbf{u}(\mathfrak{g})^+\big|^{\otimes 2},\kappa\big) \xrightarrow{\delta_2} \cdots, \tag{2.3}$$

where $\delta_i = d_i^*$. By definition, $\mathrm{H}^i(|\mathbf{u}(\mathfrak{g})|,\kappa) = \operatorname{Ker}\delta_i / \operatorname{Im}\delta_{i-1}$. Meanwhile, $\mathrm{H}^i(\mathbf{u}(\mathfrak{g}),\kappa)$ is exactly the ith cohomology of the following complex

$$0 \to \underline{\operatorname{Hom}}_\kappa(\kappa,\kappa) \xrightarrow{\delta_0} \underline{\operatorname{Hom}}_\kappa\big(\mathbf{u}(\mathfrak{g})^+,\kappa\big) \xrightarrow{\delta_1} \underline{\operatorname{Hom}}_\kappa\big(\big(\mathbf{u}(\mathfrak{g})^+\big)^{\otimes 2},\kappa\big) \xrightarrow{\delta_2} \cdots.$$

Here Lemma 2.2 is applied and see, say Section 2.1 in [23] for explanation of this sequence. Since $\operatorname{Hom}_\kappa((\mathbf{u}(\mathfrak{g})^+)^{\otimes i},\kappa)^{\kappa\mathbb{Z}_2} = \operatorname{Hom}_{\kappa\mathbb{Z}_2}((\mathbf{u}(\mathfrak{g})^+)^{\otimes i},\kappa) = \underline{\operatorname{Hom}}_\kappa((\mathbf{u}(\mathfrak{g})^+)^{\otimes i},\kappa)$, $\mathrm{H}^i(\mathbf{u}(\mathfrak{g}),\kappa) \cong \mathrm{H}^i(|\mathbf{u}(\mathfrak{g})|,\kappa)^{\kappa\mathbb{Z}_2}$.

In general, for any $\mathbf{u}(\mathfrak{g})$-supermodule N, one can apply $\operatorname{Hom}_{|\mathbf{u}(\mathfrak{g})|}(-,N)$ to (2.2) to get a similar complex like (2.3). Using totally the same argument as κ, one can get the desired conclusion.

As pointed out by the referee, one can give an alternate proof of this fact without using bar resolution as follows. Since $\mathbf{u}(\mathfrak{g})$ is a normal subalgebra of $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ and $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2 // \mathbf{u}(\mathfrak{g}) \cong \kappa\mathbb{Z}_2$, there is a Lyndon–Hochschild–Serre type spectral sequence

$$E_2^{i,j} = \mathrm{H}^i\big(\kappa\mathbb{Z}_2, \mathrm{H}^j\big(\big|\mathbf{u}(\mathfrak{g})\big|, N\big)\big) \quad\Rightarrow\quad \mathrm{H}^{i+j}\big(\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2, N\big)$$

where N is a $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$-module. For details of such spectral sequence and related notions, see [3]. Now since $\kappa\mathbb{Z}_2$ is semisimple, $E_2^{i,j} = 0$ for all $i > 0$. The spectral sequence collapses to yield for all $j \in \mathbb{N}$ the isomorphisms $\mathrm{H}^j(\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2, N) \cong E_2^{0,j} \cong \mathrm{H}^j(|\mathbf{u}(\mathfrak{g})|, N)^{\kappa\mathbb{Z}_2}$. $\square$

The following result is a direct consequence of Lemma 2.1 by noting that $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ is an ordinary Hopf algebra.

Corollary 2.8. *Let M be a $\mathbf{u}(\mathfrak{g})$-supermodule. Then under cup product, $\mathrm{H}^{ev}(\mathbf{u}(\mathfrak{g}),\kappa) := \bigoplus_{i \geqslant 0} \mathrm{H}^{2i}(\mathbf{u}(\mathfrak{g}),\kappa)$ is a commutative algebra and $\mathrm{H}^*(\mathbf{u}(\mathfrak{g}), M) := \bigoplus_{i \geqslant 0} \mathrm{H}^i(\mathbf{u}(\mathfrak{g}), M)$ is an $\mathrm{H}^{ev}(\mathbf{u}(\mathfrak{g}),\kappa)$-module.*

2.3. Basic classical Lie superalgebras

Definition 2.9. A Lie superalgebra is a *basic classical Lie superalgebra* if it admits an even nondegenerate supersymmetric bilinear form and its even subalgebra is reductive.

In the following, we only deal with basic classical Lie superalgebras unless we state otherwise. We recall the list of basic classical Lie superalgebra (see [20,36]). They are four infinite series $\mathbf{A}(m,n)$, $\mathbf{B}(m,n)$, $\mathbf{C}(n)$, $\mathbf{D}(m,n)$ and three exceptional versions $\mathbf{D}(2,1;\alpha)$, $\mathbf{G}(3)$, $\mathbf{F}(4)$ for $\alpha \in \kappa\backslash\{0,-1\}$. One merit of a basic classical Lie superalgebra $\mathfrak{g}$ is that it admits a nice root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

such that

(i) $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$;
(ii) $\dim_\kappa \mathfrak{g}_\alpha = 1$ for $\alpha \in \Phi$ except for $\mathbf{A}(1,1)$;
(iii) Except for $\mathbf{A}(1,1)$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq 0$ if and only if $\alpha, \beta, \alpha+\beta \in \Phi$.

See Section 2.5.3 in [20] for details by noting we still can do such decompositions in positive characteristic case. In order to discriminate different root in characteristic p case, we always assume $p > 3$. Also, we fix a root decomposition just as described in Section 2.5.4 in [20] from now on. Φ is called a *root supersystem* of $\mathfrak{g}$. Clearly, $\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 is the root system of $\mathfrak{g}_0$ and Φ_1 is the system of weights of the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$. Φ_0 is called the *even system* and Φ_1 the *odd system*. Define

$$\Phi_{11} := \bigl\{\alpha \in \Phi_1 \bigm| [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0\bigr\}, \qquad \Phi_{12} := \bigl\{\alpha \in \Phi_1 \bigm| [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \neq 0\bigr\}.$$

Clearly, $2\Phi_{12} := \{2\alpha \mid \alpha \in \Phi_{12}\}$ is a subset of Φ_0. Define

$$\Phi_0^\circ := \Phi_0 - 2\Phi_{12}.$$

By observing the root supersystem of $\mathbf{B}(m,n)$, $\Phi_{12} \neq \phi$ in general.

Lemma 2.10. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Then for any $\alpha \in \Phi_0$ and $x \in \mathfrak{g}_\alpha$, $x^p = 0$ in $\mathbf{u}(\mathfrak{g})$.*

Proof. This should be known, but the author cannot find suitable reference. So we give a short proof here. It is known that the even part $\mathfrak{g}_0$ of a basic classical Lie superalgebra $\mathfrak{g}$ is a direct sum of some Lie algebras of types $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{G}_2$ and the one-dimensional abelian Lie algebra κ. Therefore there is no harm to assume that $\mathfrak{g}_0$ is a simple Lie algebra of type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$ or $\mathbf{G}_2$. So $\mathfrak{g}_0$ is generated by $\mathfrak{sl}_2$-triples $\{e_i, f_i, h_i \mid i \in I\}$. By observing that all root strings for such simple Lie algebras have length at most four, $ad(x)^p(e_i) = ad(x)^p(f_i) = ad(x)^p(h_i) = 0$ for $i \in I$. By the definition of restricted Lie algebra, $x^{[p]}$ lies in the center of $\mathfrak{g}_0$ and so $x^{[p]} = 0$, which implies $x^p = 0$ in $\mathbf{u}(\mathfrak{g})$ too. □

There is a filtration on $\mathbf{u}(\mathfrak{g})$ with degrees

$$\deg_1(\mathfrak{h}) = 0, \qquad \deg_1\Bigl(\bigoplus_{\alpha\in\Phi_1} \mathfrak{g}_\alpha\Bigr) = 1, \qquad \deg_1\Bigl(\bigoplus_{\alpha\in\Phi_0} \mathfrak{g}_\alpha\Bigr) = 2.$$

The associated graded algebra is denoted by $\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))$. It is still a super cocommutative Hopf algebra. It is not hard to see that there is a natural projection from $\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))$ to $\mathbf{u}(\mathfrak{h})$ and thus there is a subsuperalgebra $R_{\mathfrak{g}}$ such that

$$\mathrm{Gr}^1\big(\mathbf{u}(\mathfrak{g})\big) = R_{\mathfrak{g}} \# \mathbf{u}(\mathfrak{h}).$$

Actually, $R_{\mathfrak{g}}$ is the graded subalgebra generated by $\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$.

For any set S, its cardinal number is denoted by $S^\#$. Assume that $\mathfrak{g} \neq \mathbf{A}(1,1)$. Then by property (ii) of the root space decomposition, up to scalers there is a unique nonzero element x_α belonging to $\mathfrak{g}_\alpha$.

Lemma 2.11. *Assume that $\mathfrak{g} \neq \mathbf{A}(1,1)$ and let x_α be defined as above. Then the graded algebra $R_{\mathfrak{g}}$ has the following PBW basis consisting of elements*

$$x_{\alpha_1}^{a_1}\cdots x_{\alpha_r}^{a_r} x_{\beta_1}^{b_1}\cdots x_{\beta_s}^{b_s} x_{\gamma_1}^{c_1}\cdots x_{\gamma_t}^{c_t} \tag{2.4}$$

where $\alpha_i \in \Phi_{11}$, $\beta_j \in \Phi_{12}$, $\gamma_k \in \Phi_0^\circ$, $r = \Phi_{11}^\#$, $s = \Phi_{12}^\#$, $t = \Phi_0^\# - \Phi_{12}^\#$ and $0 \leqslant a_i \leqslant 1$, $0 \leqslant b_j \leqslant 2p-1$, $0 \leqslant c_k \leqslant p-1$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, $1 \leqslant k \leqslant t$.

Proof. Under the grading Gr^1, one can see that

$$\Big[\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha, \bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\Big] \subseteq \bigoplus_{\alpha\in\Phi_0}\mathfrak{g}_\alpha.$$

So to show the conclusion, we can assume that $\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$ is a Lie subsuperalgebra of $\mathfrak{g}$. Being living in different root spaces, $\{[x_{\beta_j}, x_{\beta_j}] \mid 1 \leqslant j \leqslant s\}$ are linear independent. Note that the choice of $\gamma_k \in \Phi_0^\circ$ guarantees that those x_{γ_k} span a subspace of $\mathfrak{g}_0$ complementary to the span of $\{[x_{\beta_j}, x_{\beta_j}] \mid 1 \leqslant j \leqslant s\}$. So Proposition 2.5 can be applied and thus the set of elements in (2.4) forms a basis of $\mathbf{u}(\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha)$. Clearly such elements are homogeneous in $R_{\mathfrak{g}}$ and so they also give a basis of $R_{\mathfrak{g}}$. $\square$

Throughout the following of this subsection, we always assume that $\mathfrak{g} \neq \mathbf{A}(1,1)$. In order to reduce $R_{\mathfrak{g}}$ to a familiar algebra, we introduce another kind of filtration on $R_{\mathfrak{g}}$. To attack it, the degree of an element in (2.4) is defined to be

$$\deg_2\big(x_{\alpha_1}^{a_1}\cdots x_{\gamma_s}^{c_t}\big) = (a_1,\ldots,a_r,b_1,\ldots,b_s,c_1,\ldots,c_t) \in \mathbb{N}^{\Phi^\# - \Phi_{12}^\#}$$

and totally order the elements (2.4) lexicographically by setting

$$(1,0,\ldots,0) > \cdots > (0,1,\ldots,0) > \cdots > (0,0,\ldots,1).$$

For convenience and consistence, we set $\alpha_{r+i} := \beta_i$ $(1 \leqslant i \leqslant s)$ and $\alpha_{r+s+i} := \gamma_i$ $(1 \leqslant i \leqslant t)$.

Lemma 2.12. *Under the total order defined above, for all $i < j$,*

$$\deg_2\big([x_{\alpha_i}, x_{\alpha_j}]\big) < \deg_2(x_{\alpha_i}x_{\alpha_j})$$

unless $[x_{\alpha_i}, x_{\alpha_j}] = 0$.

Proof. It is not hard to see that any $x \in \bigoplus_{\alpha\in\Phi_0}$ actually lies in the center of $R_{\mathfrak{g}}$. So to show the lemma, one can assume that both x_{α_i} and x_{α_i} are odd elements and $[x_{\alpha_i}, x_{\alpha_j}] \neq 0$. Now $[x_{\alpha_i}, x_{\alpha_j}]$ lies in $\mathfrak{g}_0$ automatically and thus either $[x_{\alpha_i}, x_{\alpha_j}] = cx_{\alpha_l}$ for $l > j$ and $0 \neq c \in \kappa$ or $[x_{\alpha_i}, x_{\alpha_j}] = d[x_{\alpha_k}, x_{\alpha_k}]$ for some odd element with $[x_{\alpha_k}, x_{\alpha_k}] \neq 0$ and $0 \neq d \in \kappa$. In the first case, the conclusion is clear. In the second case, we still need to consider two cases: $[x_{\alpha_i}, x_{\alpha_i}] = [x_{\alpha_j}, x_{\alpha_j}] = 0$ or either of them is not zero. Also, the first case implies that $j < k$ by the PBW basis we choose and thus the conclusion is proved. By property (iii) of the root space decomposition, $\alpha_i + \alpha_j$ is still a root and it is equals to $2\alpha_k$ by assumption. Comparing with the root supersystem listed in Section 2.5.4 in [20], this is happened only in the case $[x_{\alpha_i}, x_{\alpha_i}] = [x_{\alpha_j}, x_{\alpha_j}] = 0$. □

By Lemma 2.12, the above ordering induces a filtration on $R_{\mathfrak{g}}$. The associated graded algebra is denoted by $\mathrm{Gr}^2(R_{\mathfrak{g}})$. It is generated by $\{x_{\alpha_i} \mid 1 \leqslant i \leqslant \Phi^\# - \Phi^\#_{12}\}$ with relations

$$[x_{\alpha_i}, x_{\alpha_j}] = 0 \quad \text{for } i \neq j, \qquad x_{\alpha_i}^{N_i} = 0 \tag{2.5}$$

where

$$N_i = \begin{cases} 2, & 0 \leqslant i \leqslant \Phi^\#_{11}, \\ 2p, & \Phi^\#_{11} + 1 \leqslant i \leqslant \Phi^\#_{11} + \Phi^\#_{12}, \\ p, & \Phi^\#_{11} + \Phi^\#_{12} + 1 \leqslant i \leqslant \Phi^\# - \Phi^\#_{12}. \end{cases}$$

Note that $\mathrm{Gr}^2(R_{\mathfrak{g}})$ inherits the action of $\mathbf{u}(\mathfrak{h})$ from that on $R_{\mathfrak{g}}$ naturally, define

$$\mathrm{Gr}^2\big(\mathbf{u}(\mathfrak{g})\big) := \mathrm{Gr}^2(R_{\mathfrak{g}}) \,\#\, \mathbf{u}(\mathfrak{h}).$$

2.4. Spectral sequences and finite generation

We will see in the next section that there are some convergent spectral sequences associated to the filtrations given in Subsection 2.3. The following lemma, which is essentially used in this paper, is given in [27] as its Lemma 2.5. Recall that an element $a \in E_r^{p,q}$ is called a *permanent cycle* if $d_i(a) = 0$ for all $i \geqslant r$.

Lemma 2.13.

(1) *Let $E_1^{p,q} \Rightarrow E_\infty^{p+q}$ be a multiplicative spectral sequence of κ-algebras concentrated in the half plane $p+q \geqslant 0$, and let $A^{*,*}$ be a bigraded commutative κ-algebra concentrated in even (total) degrees. Assume that there exists a bigraded map of algebras $\varphi : A^{*,*} \to E_1^{*,*}$ such that*
   (i) *φ makes $E_1^{*,*}$ into a Noetherian $A^{*,*}$-module, and*
   (ii) *the image of $A^{*,*}$ in $E_1^{*,*}$ consists of permanent cycles.*
   Then E_∞^* is a Noetherian module over $\mathrm{Tot}(A^{*,*})$.
(2) *Let $\tilde{E}_1^{p,q} \Rightarrow \tilde{E}_\infty^{p+q}$ be a spectral sequence that is a bigraded module over the spectral sequence $E^{*,*}$. Assume that $\tilde{E}_1^{*,*}$ is a Noetherian module over $A^{*,*}$ where $A^{*,*}$ acts on $\tilde{E}_1^{*,*}$ via the map φ. Then $\tilde{E}_\infty^*$ is a finitely generated E_∞^*-module.*

3. Finite generation

The following conclusion is one of main results of this paper.

Theorem 3.1. *Let $\mathfrak{g}$ be one of basic classical Lie superalgebras over κ and $\mathbf{u}(\mathfrak{g})$ its restricted enveloping algebra. Then*:

(1) *the algebra $\mathrm{H}^*(\mathbf{u}(\mathfrak{g}), \kappa) := \bigoplus_{i \geqslant 0} \mathrm{H}^i(\mathbf{u}(\mathfrak{g}), \kappa)$ is finitely generated.*
(2) *$\mathrm{H}^*(\mathbf{u}(\mathfrak{g}), M)$ is a finitely generated module over $\mathrm{H}^*(\mathbf{u}(\mathfrak{g}), \kappa)$ for M a finitely generated $\mathbf{u}(\mathfrak{g})$-supermodule.*

We will divide the proof into two cases: $\mathfrak{g} \neq \mathbf{A}(1,1)$ or $\mathfrak{g} = \mathbf{A}(1,1)$. The basic idea of the proof is to modify the procedure developed in [27] into our cases by applying preliminary results gotten in Section 2. Firstly, $\mathfrak{g} \neq \mathbf{A}(1,1)$ is assumed until Subsection 3.4.

3.1. Cohomology of $\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))$

The algebraic structure of $\mathrm{Gr}^2(R_{\mathfrak{g}})$ has been described clearly in (2.5). Recall that we denote the usual algebra of superalgebra A by $|A|$. For continuation, we write the algebraic structure of $|\mathrm{Gr}^2(R_{\mathfrak{g}})|$ again as follows: it is generated by $\{x_{\alpha_i} \mid 1 \leqslant i \leqslant \Phi^\# - \Phi^\#_{12}\}$ with relations

$$x_{\alpha_i}x_{\alpha_j} = \begin{cases} -x_{\alpha_j}x_{\alpha_i}, & 1 \leqslant i < j \leqslant \Phi^\#_1, \\ x_{\alpha_j}x_{\alpha_i}, & 1 \leqslant i < j \text{ and } j > \Phi^\#_1, \end{cases} \qquad x_{\alpha_i}^{N_i} = 0 \tag{3.1}$$

where

$$N_i = \begin{cases} 2, & 0 \leqslant i \leqslant \Phi^\#_{11}, \\ 2p, & \Phi^\#_{11} + 1 \leqslant i \leqslant \Phi^\#_{11} + \Phi^\#_{12}, \\ p, & \Phi^\#_{11} + \Phi^\#_{12} + 1 \leqslant i \leqslant \Phi^\# - \Phi^\#_{12}. \end{cases}$$

The algebra $|\mathrm{Gr}^2(R_{\mathfrak{g}})|$ is a special case of so-called *quantum complete intersection algebras*: Let N be positive integer, and for each $i \in \{1, \dots, N\}$, N_i be an integer greater than 1. Let $q_{ij} \in \kappa^* = \kappa \backslash \{0\}$ for $1 \leqslant i < j \leqslant N$. Define S to be the κ-algebra generated by $x_1, \dots, x_N$ subject to the relations

$$x_i x_j = q_{ij} x_j x_i \quad \text{for all } i < j \quad \text{and} \quad x_i^{N_i} = 0 \quad \text{for all } i. \tag{3.2}$$

S is called a quantum complete intersection algebra. For such S, its cohomology ring $\mathrm{H}^*(S, \kappa) = \bigoplus_{i \geqslant 0} \mathrm{Ext}^i_S(\kappa, \kappa)$ was determined in Section 4 of [27]. For completeness and consistence of the paper, let us sketch it.

Let $K_\bullet$ be the following complex of free S-modules. For each N-tuple $(a_1, \dots, a_N)$ of nonnegative integers, let $\Psi(a_1, \dots, a_N)$ be a free generator in degree $a_1 + \cdots + a_N$. Then define $K_n = \bigoplus_{a_1 + \cdots + a_N = n} S\Psi(a_1, \dots, a_N)$. For each $i \in \{1, \dots, N\}$, let $\sigma_i, \tau_i : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$\sigma_i(a) = \begin{cases} 1, & a \text{ is odd}, \\ N_i - 1, & a \text{ is even}, \end{cases}$$

and $\tau_i(a) = \sum_{j=1}^a \sigma_i(a)$ for $a \geqslant 1$, $\tau(0) = 0$. Let

$$d_i\big(\Psi(a_1, \dots, a_N)\big) = \left(\prod_{l<i} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)}\right) x_i^{\sigma_i(a_i)} \Psi(a_1, \dots, a_i - 1, \dots, a_N)$$

if $a_i > 0$, and $d_i(\Psi(a_1, \dots, a_N)) = 0$ if $a_i = 0$. Extend each d_i to an S-module homomorphism and set

$$d = d_1 + \cdots + d_N.$$

It is shown in Section 4 of [27] that $(K_\bullet, d)$ is a resolution of κ.

From this resolution, one can compute $\mathrm{Ext}^i_S(\kappa, \kappa)$. Applying $\mathrm{Hom}_S(-, \kappa)$ to $K_\bullet$, the induced differential d^* is the zero map (since $x_i^{\sigma_i(a_i)}$ is always in the augmentation ideal) and thus the cohomology is just the complex $\mathrm{Hom}_S(K_\bullet, \kappa)$. Now let $\xi_i \in \mathrm{Hom}_S(K_2, \kappa)$, $\eta_i \in \mathrm{Hom}_S(K_1, \kappa)$ be the functions dual to $\Psi(0, \dots, 2, \dots, 0)$ (the 2 in the ith place) and $\Psi(0, \dots, 1, \dots, 0)$ (the 1 in the ith place) respectively. The following conclusion is Theorem 4.1 in [27].

Lemma 3.2. *The algebra* $\mathrm{H}^*(S,\kappa)$ *is generated by* ξ_i, η_i $(1 \leqslant i \leqslant N)$ *with* $\deg \xi_i = 2$ *and* $\deg \eta_i = 1$, *subject to the relations*

$$\xi_i\xi_j = q_{ji}^{N_iN_j}\xi_j\xi_i, \qquad \eta_i\xi_j = q_{ji}^{N_j}\xi_j\eta_i, \qquad \eta_i\eta_j = -q_{ji}\eta_j\eta_i$$

where $q_{ij} = q_{ji}^{-1}$ *if* $i > j$.

For any two nonnegative integers m, n, define an algebra $\bigwedge(m|n)$ as follows. It is generated by $\eta_1, \dots, \eta_{m+n}$ with relations

$$\eta_i\eta_j = \begin{cases} \eta_j\eta_i, & 1 \leqslant i < j \leqslant m, \\ -\eta_j\eta_i, & 1 \leqslant i < j \text{ and } j > m, \end{cases} \qquad \eta_i^2 = 0.$$

Proposition 3.3. *Let* $\mathfrak{g}$ *be a basic classical Lie superalgebra different from* $\mathbf{A}(1,1)$ *and* Φ *its root supersystem. Then*

$$\mathrm{H}^*\big(\big|\mathrm{Gr}^2(R_{\mathfrak{g}})\big|, \kappa\big) \cong \kappa[\xi_1, \dots, \xi_{m+n}] \otimes \bigwedge(m|n)$$

where $m = \Phi_1^\#$, $n = \Phi_0^\# - \Phi_{12}^\#$ *and* $\deg \xi_i = 2$, $\deg \eta_i = 1$.

Proof. It is a direct consequence of Lemma 3.2 and the definition of $|\mathrm{Gr}^2(R_{\mathfrak{g}})|$. □

Proposition 3.4. *Let* $\mathfrak{g}$ *be a basic classical Lie superalgebra different from* $\mathbf{A}(1,1)$. *Fix notions as above. Then*:

(1) $\mathrm{H}^*(|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa) \cong \mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)^{\mathbf{u}(\mathfrak{h})}$ *where the action of* $\mathbf{u}(\mathfrak{h})$ *on* $\mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ *is given through*

$$h \cdot \xi_i = -N_i\alpha_i(h)\xi_i, \qquad h \cdot \eta_i = -\alpha_i(h)\eta_i, \tag{3.3}$$

for $1 \leqslant i \leqslant \Phi^\# - \Phi_{12}^\#$ *and* $h \in \mathbf{u}(\mathfrak{h})$.

(2) *Define* $\mathrm{H}^*(\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g})), \kappa) := \mathrm{H}^*(\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g})) \# \kappa\mathbb{Z}_2, \kappa)$. *Then* $\mathrm{H}^*(\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g})), \kappa) \cong \mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)^{\mathbf{u}(\mathfrak{h}) \otimes \kappa\mathbb{Z}_2}$ *where the action of* $\kappa\mathbb{Z}_2 = \kappa\langle g \mid g^2 = 1\rangle$ *on* $\mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ *is given through*

$$g \cdot \xi_i = \xi_i, \qquad g \cdot \eta_i = \begin{cases} -\eta_i, & i \leqslant \Phi_1^\#, \\ \eta_i, & i > \Phi_1^\#. \end{cases} \tag{3.4}$$

Proof. (1) To give the action $\mathbf{u}(\mathfrak{h})$ on $\mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$, we explain ξ_i, η_i and $h \in \mathbf{u}(\mathfrak{h})$ as chain maps $K_\bullet \to K_\bullet$. Then action is given by forming the commutators of compositions of these chain maps. In fact, ξ_i, η_i has been explained as chain maps in [27] and they are described as follows:

$$\xi_i\big(\Psi(a_1, \dots, a_N)\big) = \prod_{i<l} q_{il}^{N_i\tau_l(a_l)}\Psi(a_1, \dots, a_i - 2, \dots, a_N),$$

$$\eta_i\big(\Psi(a_1, \dots, a_N)\big) = cx_i^{\sigma_i(a_i)-1}\Psi(a_1, \dots, a_i - 1, \dots, a_N)$$

where $c = \prod_{l<i} q_{li}^{(\sigma_i(a_i)-1)\tau_l(a_l)} \prod_{i<l} (-1)^{a_l} q_{il}^{\tau_l(a_l)}$ and $N = \Phi^\# - \Phi_{12}^\#$. Now let h be an element in $\mathbf{u}(\mathfrak{h})$. Then $h \cdot \Psi(0, \dots, 1, \dots, 0)$ (the 1 is in the ith place) should equal to $\alpha_i(h)\Psi(0, \dots, 1, \dots, 0)$ (since one can regard $\Psi(0, \dots, 1, \dots, 0)$ as the generator x_{α_i}). Extend it to higher items and one can verify directly the following extension of $\mathbf{u}(\mathfrak{h})$ on $K_\bullet$ indeed commutes with the differentials:

$$h \cdot \Psi(a_1, \ldots, a_N) = \sum_{l=1}^{N} \tau_l(a_l)\alpha_l(h)\Psi(a_1, \ldots, a_N)$$

for $h \in \mathbf{u}(\mathfrak{h})$ and $a_1, \ldots, a_N \geqslant 0$. Then the induced action of $\mathbf{u}(\mathfrak{h})$ on generators ξ_i, η_i is given by

$$h \cdot \xi_i = h\xi_i - \xi_i h = -N_i\alpha_i(h)\xi_i, \qquad h \cdot \eta_i = h\eta_i - \eta_i h = -\alpha_i(h)\eta_i$$

for $h \in \mathbf{u}(\mathfrak{h})$.

As $\mathbf{u}(\mathfrak{h})$ is a commutative semisimple algebra, we indeed have

$$\mathrm{Ext}^i_{|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|}(\kappa, \kappa) = \mathrm{Ext}^i_{|\mathrm{Gr}^2(R_\mathfrak{g})|\#\mathbf{u}(\mathfrak{h})}(\kappa, \kappa) \cong \mathrm{Ext}^i_{|\mathrm{Gr}^2(R_\mathfrak{g})|}(\kappa, \kappa)^{\mathbf{u}(\mathfrak{h})}$$

for $i \geqslant 0$ (one can prove this fact similarly by applying the methods used in the proof of Lemma 2.7). Thus $\mathrm{H}^*(|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa) \cong \mathrm{H}^*(|\mathrm{Gr}^2(R_\mathfrak{g})|, \kappa)^{\mathbf{u}(\mathfrak{h})}$ now.

(2) By Lemma 2.7 and (1),

$$\mathrm{H}^*\big(\mathrm{Gr}^2\big(\mathbf{u}(\mathfrak{g})\big), \kappa\big) \cong \mathrm{H}^*\big(\big|\mathrm{Gr}^2\big(\mathbf{u}(\mathfrak{g})\big)\big|, \kappa\big)^{\kappa\mathbb{Z}_2} \cong \mathrm{H}^*\big(\big|\mathrm{Gr}^2(R_\mathfrak{g})\big|, \kappa\big)^{\mathbf{u}(\mathfrak{h})\otimes\kappa\mathbb{Z}_2}.$$

Similar to (1), the following action of $\kappa\mathbb{Z}_2$ on $K_\bullet$ commutes with the differentials:

$$g \cdot \Psi(a_1, \ldots, a_N) = \prod_{l=1}^{\Phi_1^\#}(-1)^{\tau_l(a_l)}\Psi(a_1, \ldots, a_N).$$

This induces the action

$$g \cdot \xi_i = g\xi_i g^{-1} = \begin{cases} (-1)^{N_i}\xi_i, & i \leqslant \Phi_1^\#, \\ \xi_i, & i > \Phi_1^\#, \end{cases} \qquad g \cdot \eta_i = g\eta_i g^{-1} = \begin{cases} -\eta_i, & i \leqslant \Phi_1^\#, \\ \eta_i, & i > \Phi_1^\#. \end{cases}$$

By the definition of N_i in (3.1), it is an even when $i \leqslant \Phi_1^\#$. □

Remark 3.5. The actions of $\mathbf{u}(\mathfrak{h})$ and $\mathbf{u}(\mathfrak{h}) \# \kappa\mathbb{Z}_2$ on the cohomology rings given in Proposition 3.4 are intrinsic, and do not depend on the particular resolution used to compute such cohomology rings. Indeed, the actions of $\mathbf{u}(\mathfrak{h})$ and $\mathbf{u}(\mathfrak{h}) \# \kappa\mathbb{Z}_2$ on cohomology rings are induced by the Hopf-actions of $\mathbf{u}(\mathfrak{h})$ and $\mathbf{u}(\mathfrak{h}) \# \kappa\mathbb{Z}_2$ on $\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))$, respectively. This fits into a general context described as Theorem 4.3.1 in [14].

3.2. *Cohomology of* $\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))$

For a basic classical Lie superalgebra $\mathfrak{g}$, its enveloping algebra is denoted by $U(\mathfrak{g})$. As the case of $\mathbf{u}(\mathfrak{g})$, define

$$\deg_1(\mathfrak{h}) := 0, \qquad \deg_1\bigg(\bigoplus_{\alpha\in\Phi_1} \mathfrak{g}_\alpha\bigg) := 1, \qquad \deg_1\bigg(\bigoplus_{\alpha\in\Phi_0} \mathfrak{g}_\alpha\bigg) := 2.$$

Then we will get a filtration on $U(\mathfrak{g})$ and associated graded algebra

$$\mathrm{Gr}^1\big(U(\mathfrak{g})\big) = \tilde{R}_\mathfrak{g} \# U(\mathfrak{h})$$

similarly.

Lemma 3.6. *Assume that* $\mathfrak{g} \neq \mathbf{A}(1,1)$ *and let* x_α *be defined as in Lemma* 2.11. *Then the graded algebra* $\tilde{R}_\mathfrak{g}$ *has the following PBW basis consisting of elements*

$$x_{\alpha_1}^{a_1} \cdots x_{\alpha_r}^{a_r} x_{\beta_1}^{b_1} \cdots x_{\beta_s}^{b_s} x_{\gamma_1}^{c_1} \cdots x_{\gamma_t}^{c_t} \tag{3.5}$$

where $\alpha_i \in \Phi_{11}$, $\beta_j \in \Phi_{12}$, $\gamma_k \in \Phi_0^\circ$, $r = \Phi_{11}^\#$, $s = \Phi_{12}^\#$, $t = \Phi_0^\# - \Phi_{12}^\#$ *and* $0 \leqslant a_i \leqslant 1$, $b_j, c_k \in \mathbb{N}$ *for* $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$, $1 \leqslant k \leqslant t$.

Proof. Similar to that of Lemma 2.11. $\square$

Also, we set $\alpha_{r+i} := \beta_i$ $(1 \leqslant i \leqslant s)$ and $\alpha_{r+s+i} := \gamma_i$ $(1 \leqslant i \leqslant t)$ for convenience and consistence. Clearly,

$$R_\mathfrak{g} \cong \tilde{R}_\mathfrak{g} / \big(x_{\alpha_i}^{N_i},\ 1 \leqslant i \leqslant \Phi^\# - \Phi_{12}^\#\big)$$

where N_i is defined the same as in (2.5). Define $N := \Phi^\# - \Phi_{12}^\#$ and for any $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{N}^N$ with $0 \leqslant a_i \leqslant 1$ $(1 \leqslant i \leqslant \Phi_{11}^\#)$, denote the corresponding PBW basis element $x_{\alpha_1}^{a_1} \cdots x_{\alpha_N}^{a_N}$ by $\mathbf{x}^\mathbf{a}$ for short.

Our next aim is to give some elements of $\mathrm{H}^2(|R_\mathfrak{g}|, \kappa)$. Recall $|\tilde{R}_\mathfrak{g}|^+$ is the augmentation ideal of $|\tilde{R}_\mathfrak{g}|$. Now for each $i \in \{1, \ldots, N\}$, define $\tilde{\xi}_{\alpha_i} : |\tilde{R}_\mathfrak{g}|^+ \otimes |\tilde{R}_\mathfrak{g}|^+ \to \kappa$ by

$$\tilde{\xi}_{\alpha_i}\big(\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}\big) = c_{\alpha_i}$$

where c_{α_i} is the coefficient of $x_{\alpha_i}^{N_i}$ in the product $\mathbf{x}^\mathbf{a}\mathbf{x}^\mathbf{b}$ as a linear combination of PBW basis elements. By its definition, $\tilde{\xi}_{\alpha_i}$ is associative on $|\tilde{R}_\mathfrak{g}|^+$ and thus it may be extended to a normalized two-cocycle on $|\tilde{R}_\mathfrak{g}|$. We next show that $\tilde{\xi}_{\alpha_i}$ factors through the quotient map $\pi : |\tilde{R}_\mathfrak{g}| \to |R_\mathfrak{g}|$ to give a nonzero two-cocycle on $|R_\mathfrak{g}|$. To attack this, we need show the $\tilde{\xi}_{\alpha_i}(\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}) = 0$ whenever $\mathbf{x}^\mathbf{a}$ or $\mathbf{x}^\mathbf{b}$ is in the kernel of the quotient map π. Suppose $\mathbf{x}^\mathbf{a} \in \operatorname{Ker}\pi$, which implies that $a_j \geqslant N_j$ for some j. By the proof of Lemma 2.10, $x_{\alpha_j}^{N_j}$ lies in the center of $U(\mathfrak{g})$ and so $\mathbf{x}^\mathbf{a} = x_{\alpha_j}^{N_j}\mathbf{x}^\mathbf{c}$ for some $\mathbf{c} \in \mathbb{N}^N$. Then $\tilde{\xi}_{\alpha_i}(\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}) = \tilde{\xi}_{\alpha_i}(x_{\alpha_j}^{N_j}\mathbf{x}^\mathbf{c}, \mathbf{x}^\mathbf{b})$ is the coefficient of $x_{\alpha_i}^{N_i}$ in the product $x_{\alpha_j}^{N_j}\mathbf{x}^\mathbf{c}\mathbf{x}^\mathbf{b}$. It is zero now: If $j = i$, then since $\mathbf{x}^\mathbf{b} \in |\tilde{R}_\mathfrak{g}|^+$, this product cannot have a nonzero coefficient for $x_{\alpha_i}^{N_i}$. If $j \neq i$, the same conclusion is true since $x_{\alpha_j}^{N_j}$ is always a factor of $\mathbf{x}^\mathbf{a}\mathbf{x}^\mathbf{b}$. One can show the result similarly in the case $\mathbf{x}^\mathbf{b} \in \operatorname{Ker}\pi$.

Choose the section $\tilde{-} : |R_\mathfrak{g}| \to |\tilde{R}_\mathfrak{g}|$ of π which just sent the PBW basis elements in $R_\mathfrak{g}$, given in Lemma 2.11, to the same elements in $\tilde{R}_\mathfrak{g}$, described in Lemma 3.6. Since $\tilde{\xi}_{\alpha_i}$ factors through $\pi : |\tilde{R}_\mathfrak{g}| \to |R_\mathfrak{g}|$, we may define $\hat{\xi}_{\alpha_i} : |R_\mathfrak{g}|^+ \otimes |R_\mathfrak{g}|^+ \to \kappa$ by

$$\hat{\xi}_{\alpha_i}\big(\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}\big) := \tilde{\xi}_{\alpha_i}\big(\tilde{\mathbf{x}}^\mathbf{a}, \tilde{\mathbf{x}}^\mathbf{b}\big)$$

where $\tilde{\mathbf{x}}^\mathbf{a}, \tilde{\mathbf{x}}^\mathbf{b}$ are defined via the section $\tilde{-}$.

Proposition 3.7. *The set* $\{\hat{\xi}_{\alpha_i} \mid i = 1, \ldots, N\}$ *represents a linear independent subset of* $\mathrm{H}^2(|R_\mathfrak{g}|, \kappa)$.

Proof. At first, let us show that every $\hat{\xi}_{\alpha_i}$ is a 2-cocycle. For this, it is enough to show that it is associative, that is, for any three PBW basis elements $\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}, \mathbf{x}^\mathbf{c}$, we have $\hat{\xi}_{\alpha_i}(\mathbf{x}^\mathbf{a}\mathbf{x}^\mathbf{b}, \mathbf{x}^\mathbf{c}) = \hat{\xi}_{\alpha_i}(\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}\mathbf{x}^\mathbf{c})$. Since π is an algebra homomorphism, we have $\tilde{\mathbf{x}}^\mathbf{a}\tilde{\mathbf{x}}^\mathbf{b} = \widetilde{\mathbf{x}^\mathbf{a}\mathbf{x}^\mathbf{b}} + y$ and $\tilde{\mathbf{x}}^\mathbf{b}\tilde{\mathbf{x}}^\mathbf{c} = \widetilde{\mathbf{x}^\mathbf{b}\mathbf{x}^\mathbf{c}} + z$ for $y, z \in \operatorname{Ker}\pi$. Therefore,

$$\begin{aligned}\hat{\xi}_{\alpha_i}(\mathbf{x^a x^b}, \mathbf{x^c}) &= \tilde{\xi}_{\alpha_i}(\widetilde{\mathbf{x^a x^b}}, \tilde{\mathbf{x}}^{\mathbf{c}}) \\ &= \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}\tilde{\mathbf{x}}^{\mathbf{b}} - y, \tilde{\mathbf{x}}^{\mathbf{c}}) = \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}\tilde{\mathbf{x}}^{\mathbf{b}}, \tilde{\mathbf{x}}^{\mathbf{c}}) \\ &= \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}, \tilde{\mathbf{x}}^{\mathbf{b}}\tilde{\mathbf{x}}^{\mathbf{c}}) \\ &= \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}, \widetilde{\mathbf{x^b x^c}} + z) = \tilde{\xi}_{\alpha_i}(\tilde{\mathbf{x}}^{\mathbf{a}}, \widetilde{\mathbf{x^b x^c}}) \\ &= \hat{\xi}_{\alpha_i}(\mathbf{x^a}, \mathbf{x^b x^c}).\end{aligned}$$

Next, let us show that they are linear independent in $\mathrm{H}^2(|R_{\mathfrak{g}}|, \kappa)$. It is equivalent to show that for any linear combination $f = \sum_{i=1}^{N} c_i \hat{\xi}_{\alpha_i}$, if it is a coboundary then every $c_i = 0$. Assume that $f = \partial h$ for some $h : |R_{\mathfrak{g}}|^+ \to \kappa$. Then

$$c_i = f(x_{\alpha_i}, x_{\alpha_i}^{N_i - 1}) = \partial h(x_{\alpha_i}, x_{\alpha_i}^{N_i - 1}) = -h(x_{\alpha_i}^{N_i}) = 0$$

since $x_{\alpha_i}^{N_i} = 0$ in $|R_{\mathfrak{g}}|$ by Lemma 2.10. □

See Section 6 in [28] for the definitions of such elements in the case of pointed Hopf algebras. We are now in the position to prove the following theorem.

Theorem 3.8. *The algebra* $\mathrm{H}^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$ *is finitely generated. If* M *is a finitely generated* $|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|$*-module, then* $\mathrm{H}^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, M)$ *is a finitely generated module over* $\mathrm{H}^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$.

Proof. By Lemma 2.12, there is a filtration on $|R_{\mathfrak{g}}|$ and results a graded algebra $|\mathrm{Gr}^2(R_{\mathfrak{g}})|$. As the filtration is finite, there is a convergent spectral sequence associated to the filtration by 5.4.1 in [38]:

$$E_1^{s,t} = \mathrm{H}^{s+t}(\mathrm{Gr}^2_{(s)}(|R_{\mathfrak{g}}|), \kappa) \quad \Rightarrow \quad \mathrm{H}^{s+t}(|R_{\mathfrak{g}}|, \kappa). \tag{3.6}$$

Since $\mathbf{u}(\mathfrak{h})$ is a semisimple algebra, the fixed-point functor $(-)^{\mathbf{u}(\mathfrak{h})}$ is exact, and thus we further get a spectral sequence converging to the cohomology of $|R_{\mathfrak{g}} \# \mathbf{u}(\mathfrak{h})| = |\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|$:

$$\mathrm{H}^{s+t}(\mathrm{Gr}^2_{(s)}(|R_{\mathfrak{g}}|), \kappa)^{\mathbf{u}(\mathfrak{h})} \quad \Rightarrow \quad \mathrm{H}^{s+t}(|R_{\mathfrak{g}}|, \kappa)^{\mathbf{u}(\mathfrak{h})} \cong \mathrm{H}^{s+t}(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa), \tag{3.7}$$

where the isomorphism "≅" can be proved similarly just as in the proof of Proposition 3.4. Using similar method given in Section 3.1 in [22], we can replace κ by M in (3.6), (3.7) to get convergent spectral sequences with coefficients in M.

By Proposition 3.7, we have some elements $\hat{\xi}_{\alpha_i}$ in $\mathrm{H}^2(|R_{\mathfrak{g}}|, \kappa)$. We wish to related the functions $\hat{\xi}_{\alpha_i}$ to elements on the E_1-page of the spectral sequence (3.6). In fact, one can copy the arguments stating before Lemma 5.1 in [27] and can assume that $\hat{\xi}_{\alpha_i} \in E_1^{c,2-c} \cong \mathrm{H}^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ for some $c \in \mathbb{N}$. Since $\hat{\xi}_{\alpha_i} \in \mathrm{H}^2(|R_{\mathfrak{g}}|, \kappa)$, they are permanent cycles. Now, by Proposition 3.3, $\mathrm{H}^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ is finitely generated over the ξ_i for $1 \leqslant i \leqslant N = \Phi^\# - \Phi^\#_{12}$.

Claim 1. *In* $\mathrm{H}^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$, $\xi_i = \hat{\xi}_{\alpha_i}$. (The proof of this claim is the same with that of Lemma 5.1 in [27] and thus is omitted.)

Let $B^{*,*}$ be the bigraded subalgebra of $E_1^{*,*}$ generated by the elements ξ_i. By Claim 1, $B^{*,*}$ consists of permanent cycles. Let $A^{*,*}$ be the subalgebra of $B^{*,*}$ generated by ξ_i^p where p is the characteristic of κ. By (3.3) and (3.4) in Proposition 3.4, ξ_i^p is invariant under the action of $\mathbf{u}(\mathfrak{h}) \otimes \kappa\mathbb{Z}_2$. Therefore, $A^{*,*}$ is a subalgebra of $\mathrm{H}^*(\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g})), \kappa)$. Lemma 2.1 implies that $A^{*,*}$ is commutative since it is concentrated in even (total) degrees.

Claim 2. *$A^{*,*}$ satisfies the hypotheses of Lemma* 2.13. To show it, it is enough to show that $E_1^{*,*}$ is a finitely generated module over $A^{*,*}$. Proposition 3.3 implies $E_1^{*,*} \cong \mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ is generated by ξ_i and η_i where $\eta_i^2 = 0$. Hence $E_1^{*,*}$ is a finitely generated module over $B^{*,*}$ which is clearly a finitely generated module over $A^{*,*}$. Therefore, the claim is proved.

Thus Lemma 2.13(1) is applied and so $\mathrm{H}^*(|R_{\mathfrak{g}}|, \kappa)$ is a Noetherian $\mathrm{Tot}(A^{*,*})$-module. Moreover, the action of $\mathbf{u}(\mathfrak{h})$ on $\mathrm{H}^*(|R_{\mathfrak{g}}|, \kappa)$ is compatible with the action on $A^{*,*}$, since the spectral sequence (3.6) is compatible with the action of $\mathbf{u}(\mathfrak{h})$. Therefore, $\mathrm{H}^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa) \cong \mathrm{H}^*(|R_{\mathfrak{g}}|, \kappa)^{\mathbf{u}(\mathfrak{h})}$ is a Noetherian $\mathrm{Tot}(A^{*,*})$-module. Now, $\mathrm{Tot}(A^{*,*})$ is finitely generated since $A^{*,*}$ is just the polynomial algebra generated by ξ_i^p. We conclude that $\mathrm{H}^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$ is finitely generated.

The second statement of the this theorem follows by a direct application of Lemma 2.13(2) provided that we can show the same statement for $|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|$. In fact, it is known that the statement is true for quantum complete intersection algebras (see [27]). In our case, this implies that the statement is true for $|\mathrm{Gr}^2(R_{\mathfrak{g}})|$. Therefore for any finitely-generated $|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|$-module V, $\mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, V)$ is a Noetherian $\mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$-module. By above discussions, we know that $\mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ is a Noetherian $\mathrm{H}^*(|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa) \cong \mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)^{\mathbf{u}(\mathfrak{h})}$-module. Thus $\mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, V)$ is a finitely generated $\mathrm{H}^*(|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa)$-module. So the $\mathrm{H}^*(|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|, \kappa)$-submodule $\mathrm{H}^*(|\mathrm{Gr}^2(\mathbf{u}(\mathfrak{g}))|, V) \cong \mathrm{H}^*(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, V)^{\mathbf{u}(\mathfrak{h})}$ is also Noetherian. □

Next result is a direct consequence of Theorem 3.8 and Lemma 2.7.

Corollary 3.9. *The algebra $\mathrm{H}^*(\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g})), \kappa)$ is finitely generated. If M is a finitely generated $\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))$-supermodule, then $\mathrm{H}^*(\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g})), M)$ is a finitely generated module over $\mathrm{H}^*(\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g})), \kappa)$.*

3.3. Cohomology of $\mathbf{u}(\mathfrak{g})$

In this subsection, we will give the proof of Theorem 3.1 provided $\mathfrak{g} \neq \mathbf{A}(1,1)$. Similar to Subsection 3.2, we have convergent spectral sequences associated the first kind of filtration given before Lemma 2.11:

$$E_1^{s,t} = \mathrm{H}^{s+t}\big(\mathrm{Gr}^1_{(s)}\big(\big|\mathbf{u}(\mathfrak{g})\big|\big), \kappa\big) \quad \Rightarrow \quad \mathrm{H}^{s+t}\big(\big|\mathbf{u}(\mathfrak{g})\big|, \kappa\big), \tag{3.8}$$

$$\mathrm{H}^{s+t}\big(\mathrm{Gr}^1_{(s)}\big(\big|\mathbf{u}(\mathfrak{g})\big|\big), M\big) \quad \Rightarrow \quad \mathrm{H}^{s+t}\big(\big|\mathbf{u}(\mathfrak{g})\big|, M\big), \tag{3.9}$$

for any $|\mathbf{u}(\mathfrak{g})|$-module M.

Previously, we identify the element $\xi_i \in \mathrm{H}^2(|\mathrm{Gr}^2(R_{\mathfrak{g}})|, \kappa)$ with the element $\hat{\xi}_{\alpha_i} \in \mathrm{H}^2(|R_{\mathfrak{g}}|, \kappa)$. From this, we know that ξ_i is a permanent cycle and $\mathrm{H}^*(|R_{\mathfrak{g}} \# \mathbf{u}(\mathfrak{h})|, \kappa) = \mathrm{H}^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$ is finitely generated over the subalgebra generated by all $\hat{\xi}_{\alpha_i}^p$. So our next aim is to find an element $f_{\alpha_i} \in \mathrm{H}^*(|\mathbf{u}(\mathfrak{g})|, \kappa)$ which can be identified with $\hat{\xi}_{\alpha_i}^p$. If so, $\hat{\xi}_{\alpha_i}^p$ will be permanent cycles and Lemma 2.13 can be applied.

For each $i \in \{1, \ldots, \Phi^\# - \Phi_{12}^\#\}$, let α_i be the corresponding root. For our purpose, we choose a PBW basis of $U(\mathfrak{g})$, described as in Proposition 2.5(1), with requirements: $s_1 = \Phi_{11}^\#$, $s = \Phi_1^\#$, $x_i = x_{\alpha_i}$ for $1 \leqslant i \leqslant s$ and $y_j = x_{\alpha_{s+j}}$ for $1 \leqslant j \leqslant \Phi_0^\# - \Phi_{12}^\#$ where x_{α_k} is defined before Lemma 2.11. Roughly speaking, we just want the PBW basis elements given in Lemma 3.6 to be still PBW basis elements in the following discussions. We choose a PBW basis for $\mathbf{u}(\mathfrak{g})$ with the same requirements as $U(\mathfrak{g})$. Such PBW basis will be fixed from now on until the end of this subsection.

Define a κ-linear function $\tilde{f}_{\alpha_i} : (|U(\mathfrak{g})|^+)^{2p} \to \kappa$ as follows. Let $r_1, \ldots, r_{2p}$ be PBW basis elements. If all of them have no factors belonging to $U(\mathfrak{h})$, then

$$\tilde{f}_{\alpha_i}(r_1 \otimes \cdots \otimes r_{2p}) := c_{12}c_{34} \cdots c_{2p-1,2p}$$

where c_{ij} is the coefficient of $x_{\alpha_i}^{N_i}$ in the product $r_i r_j$ as a linear combination of PBW basis elements. And set $\tilde{f}_{\alpha_i}$ to be zero whenever there is an r_i which contains a factor living in $U(\mathfrak{h})$.

Similar to Subsection 3.2, we will show that $\tilde{f}_{\alpha_i}$ factors through the quotient $\pi : U(\mathfrak{g}) \to \mathbf{u}(\mathfrak{g})$ to give a map $(|\mathbf{u}(\mathfrak{g})|^+)^{2p} \to \kappa$. Note that by the definition of $\tilde{f}_{\alpha_i}$, it is always 0 whenever the elements of $U(\mathfrak{h})$ appear in a PBW basis element. So we need only to consider the PBW basis elements totally the same as that of $\tilde{R}_{\mathfrak{g}}$. So we can apply the same arguments designed for $\tilde{\xi}_{\alpha_i}$ to $\tilde{f}_{\alpha_i}$ and show that it indeed factors through the quotient map $\pi : U(\mathfrak{g}) \to \mathbf{u}(\mathfrak{g})$. Also, we choose a section $\widetilde{-} : \mathbf{u}(\mathfrak{g}) \to U(\mathfrak{g})$ of the quotient map π. Then define $f_{\alpha_i} : (|\mathbf{u}(\mathfrak{g})|^+)^{2p} \to \kappa$ by setting

$$f_{\alpha_i}(r_1 \otimes \cdots \otimes r_{2p}) := \tilde{f}_{\alpha_i}(\tilde{r}_1 \otimes \cdots \otimes \tilde{r}_{2p})$$

for PBW basis elements $r_1, \ldots, r_{2p} \in \mathbf{u}(\mathfrak{g})$.

Proposition 3.10. *The set* $\{f_{\alpha_i} \mid i=1, \ldots, \Phi^\# - \Phi^\#_{12}\}$ *represents a linear independent subset of* $\mathrm{H}^{2p}(|\mathbf{u}(\mathfrak{g})|, \kappa)$.

Proof. The proof is similar to that of Lemma 6.2 in [27] and Proposition 3.7. For completeness, we write it out.

Firstly, we show that $\tilde{f}_{\alpha_i}$ is a $2p$-cocycle on $|U(\mathfrak{g})|$. Let $r_0, \ldots, r_{2p} \in |U(\mathfrak{g})|^+$ be PBW basis elements without factors coming from $U(\mathfrak{h})$. Then

$$\partial(\tilde{f}_{\alpha_i})(r_0 \otimes \cdots \otimes r_{2p}) = \sum_{j=0}^{2p-1} (-1)^{i+1} \tilde{f}_{\alpha_i}(r_0 \otimes \cdots \otimes r_j r_{j+1} \otimes \cdots \otimes r_{2p}).$$

By the definition of $\tilde{f}_{\alpha_i}$, the first two terms cancel and similarly for all other terms. So $\partial(\tilde{f}_{\alpha_i}) = 0$.

Now we verify that f_{α_i} is a $2p$-cocycle. Also, let $r_0, \ldots, r_{2p} \in |\mathbf{u}(\mathfrak{g})|^+$ be PBW basis elements. Then

$$\partial(f_{\alpha_i})(r_0 \otimes \cdots \otimes r_{2p}) = \sum_{j=0}^{2p-1} (-1)^{i+1} f_{\alpha_i}(r_0 \otimes \cdots \otimes r_j r_{j+1} \otimes \cdots \otimes r_{2p}).$$

Using the same methods as in the proof of Proposition 3.6, we have

$$\begin{aligned} f_{\alpha_i}(r_0 r_1 \otimes r_2 \otimes \cdots \otimes r_{2p}) &= \tilde{f}_{\alpha_i}(\widetilde{r_0 r_1} \otimes \tilde{r}_2 \otimes \cdots \otimes \tilde{r}_{2p}) \\ &= \tilde{f}_{\alpha_i}(\tilde{r}_0 \tilde{r}_1 \otimes \tilde{r}_2 \otimes \cdots \otimes \tilde{r}_{2p}) \\ &= \tilde{f}_{\alpha_i}(\tilde{r}_0 \otimes \tilde{r}_1 \tilde{r}_2 \otimes \cdots \otimes \tilde{r}_{2p}) \\ &= \tilde{f}_{\alpha_i}(\tilde{r}_0 \otimes \widetilde{r_1 r_2} \otimes \cdots \otimes \tilde{r}_{2p}) \\ &= f_{\alpha_i}(r_0 \otimes r_1 r_2 \otimes \cdots \otimes r_{2p}). \end{aligned}$$

Similarly, we have

$$f_{\alpha_i}(r_0 \otimes \cdots \otimes r_j r_{j+1} \otimes \cdots \otimes r_{2p}) = f_{\alpha_i}(r_0 \otimes \cdots \otimes r_{j+1} r_{j+2} \otimes \cdots \otimes r_{2p})$$

for $j = 0, \ldots, 2p-2$. So $\partial(f_{\alpha_i}) = 0$.

Now assume that $\sum_i c_i f_{\alpha_i} = \partial h$ for some $h \in \mathrm{Hom}_\kappa((|\mathbf{u}(\mathfrak{g})|^+)^{\otimes 2p-1}, \kappa)$. Then for each i,

$$\begin{aligned} c_i &= \Big(\sum_j c_j f_{\alpha_j}\Big)\big(x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1} \otimes \cdots \otimes x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1}\big) \\ &= (\partial h)\big(x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1} \otimes \cdots \otimes x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1}\big) \end{aligned}$$

$$= \sum \pm h\big(x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1} \otimes \cdots \otimes x_{\alpha_i}^{N_i} \otimes \cdots \otimes x_{\alpha_i} \otimes x_{\alpha_i}^{N_i-1}\big)$$
$$= 0$$

since $x_{\alpha_i}^{N_i} = 0$ in $\mathbf{u}(\mathfrak{g})$ by Lemma 2.10. □

Proof of Theorem 3.1 in case $\mathfrak{g} \neq \mathbf{A}(1,1)$. The functions f_{α_i} correspond to their counterpart $\hat{\xi}_{\alpha_i}^p$ defined on $|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|$, in the E_1-page of the spectral sequence (3.8), by observing that they are the same functions at the level of chain complex (2.3) where we need replace $|\mathbf{u}(\mathfrak{g})|^+$ by $|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|^+$. Thus Proposition 3.10 implies that the function $\hat{\xi}_{\alpha_i}^p$ is a permanent cycle. Now we have known that $E_1^{*,*} \cong \mathrm{H}^*(|\mathrm{Gr}^1(\mathbf{u}(\mathfrak{g}))|, \kappa)$ is finitely generated over the subalgebra $A^{*,*}$ generated by all $\hat{\xi}_{\alpha_i}^p$ (see the proof of Theorem 3.8). Thus $A^{*,*}$ satisfies the conditions of Lemma 2.13 and thus $\mathrm{H}^*(|\mathbf{u}(\mathfrak{g})|, \kappa)$ is a Noetherian $\mathrm{Tot}(A^{*,*})$-module and thus finitely generated. By Lemma 2.7, the first part of Theorem 3.1 is proved. The second part can be prove similarly by applying Lemma 2.13(2) and Lemma 2.7. □

3.4. *The case* $\mathfrak{g} = \mathbf{A}(1,1)$

We deal with the case $\mathfrak{g} = \mathbf{A}(1,1)$ in a bigger context: Those basic classical Lie superalgebras with Φ_{12} being empty. By the descriptions of root supersystems given in Section 2.5.4 in [20], this includes all basic classical Lie superalgebras except $\mathbf{B}(m,n)$ and $\mathbf{G}(3)$. For such Lie superalgebras, we have a nice filtration on them.

We give a notion at first. For a coalgebra C and $D \subseteq C$ a subcoalgebra of C, define

$$\bigwedge^0 D := D, \qquad \bigwedge^1 D := \Delta^{-1}(C \otimes D + D \otimes C),$$
$$\bigwedge^i D := \bigwedge^1\Big(\bigwedge^{i-1} D\Big) = \Delta^{-1}\Big(C \otimes \bigwedge^{i-1} D + \bigwedge^{i-1} D \otimes C\Big)$$

for $i \geqslant 2$. If D contains the coradical C_0 of C, by definition C_0 is the sum of all simple subcoalgebras of C, then $D \subseteq \bigwedge D \subseteq \bigwedge^2 D \subseteq \cdots$ will give a filtration of C. See Chapter 5 in [29] for details.

Let $\mathfrak{g}$ be a basic classical Lie superalgebra with $\Phi_{12} = \phi$. Then $\mathbf{u}(\mathfrak{g})$ is a finite-dimensional super cocommutative Hopf algebra and its coradical is κ. Define

$$F^i\mathbf{u}(\mathfrak{g}) := \bigwedge^i \mathbf{u}(\mathfrak{h})$$

for $i \geqslant 0$ and then this gives a filtration of $\mathbf{u}(\mathfrak{g})$. The associated graded algebra is denoted by $\mathrm{gr}(\mathbf{u}(\mathfrak{g}))$. It is a superalgebra naturally. For any $\alpha \in \Phi$, we fix a basis b_α of $\mathfrak{g}_\alpha$. By taking the union of such b_α, we get a basis of $\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Denote this basis by $\{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}\}$ and assume that $x_i \in \bigoplus_{\alpha\in\Phi_1} \mathfrak{g}_\alpha$ for $1 \leqslant i \leqslant m$ while $x_i \notin \bigoplus_{\alpha\in\Phi_1} \mathfrak{g}_\alpha$ for $m < i \leqslant m+n$.

Lemma 3.11. $\mathrm{gr}(\mathbf{u}(\mathfrak{g})) \cong S_{\mathfrak{g}} \# \mathbf{u}(\mathfrak{h})$ *where* $S_{\mathfrak{g}}$ *is generated by* $x_1, \ldots, x_{m+n}$ *with relations*

$$x_i x_j = \begin{cases} -x_j x_i, & 1 \leqslant i < j \leqslant m, \\ x_j x_i, & 1 \leqslant i < j,\ j > m, \end{cases} \qquad x_i^{n_i} = 0, \tag{3.10}$$

where

$$n_i = \begin{cases} 2, & 1 \leqslant i \leqslant m, \\ p, & m < i \leqslant m+n. \end{cases}$$

Proof. Here the action of $\mathbf{u}(\mathfrak{h})$ on $S_{\mathfrak{g}}$ is gotten through extending the actions of $\mathfrak{h}$ on $\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$ naturally. By the definition of the coproduct of $\mathbf{u}(\mathfrak{g})$, $\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\subset\bigwedge^1\mathbf{u}(\mathfrak{h})$. So $[x_i,x_j]\in\bigwedge^1\mathbf{u}(\mathfrak{h})$. This implies we have

$$[x_i,x_j]=0$$

in $\mathrm{gr}(\mathbf{u}(\mathfrak{g}))$. It is direct to show that every $x_i^{n_i}$ is still a primitive element and so $x_i^{n_i}\in\bigwedge^1\mathbf{u}(\mathfrak{h})$ too. Therefore, $x_i^{n_i}=0$ in $\mathrm{gr}(\mathbf{u}(\mathfrak{g}))$. Now all relations in (3.10) are fulfilled. By comparing the dimensions, we indeed get the desire isomorphism. □

Proof of Theorem 3.1 in case $\Phi_{12}=\phi$. Since $\mathbf{u}(\mathfrak{g})$ is finite-dimensional, then the filtration $F^0\mathbf{u}(\mathfrak{g})\subset F^1\mathbf{u}(\mathfrak{g})\subset\cdots$ is finite, that is, there is $n\in\mathbb{N}$ such that $F^n\mathbf{u}(\mathfrak{g})=\mathbf{u}(\mathfrak{g})$. So have a convergent spectral sequence

$$E_1^{s,t}=\mathrm{H}^{s+t}\big(\mathrm{gr}_{(s)}\big(\big|\mathbf{u}(\mathfrak{g})\big|\big),\kappa\big)\quad\Rightarrow\quad\mathrm{H}^{s+t}\big(\big|\mathbf{u}(\mathfrak{g})\big|,\kappa\big).\tag{3.11}$$

By Lemma 3.11, $|\mathrm{gr}\,\mathbf{u}(\mathfrak{g})|\cong|S_{\mathfrak{g}}\#\mathbf{u}(\mathfrak{h})|$. Now it is clear $|S_{\mathfrak{g}}|$ is a quantum complete intersection algebra (see the second paragraph of Subsection 3.1). Thus it cohomology algebra is clear by Lemma 3.2. Actually, similar to Proposition 3.3, we have

$$\mathrm{H}^*\big(|S_{\mathfrak{g}}|,\kappa\big)\cong k[\xi_1,\ldots,\xi_{m+n}]\otimes\bigwedge(m|n)$$

with $m=\Phi_1^\#$, $n=\Phi_0^\#$ and $\deg\xi_i=2$, $\deg\eta_i=1$. Also one can get that $\xi_i^p\in\mathrm{H}^*(|S_{\mathfrak{g}}|,\kappa)^{\mathbf{u}(\mathfrak{h})}\cong\mathrm{H}^*(|\mathrm{gr}(\mathbf{u}(\mathfrak{g}))|,\kappa)$. By applying the same discussions used in the proof of Claim 2 in that of Theorem 3.8, $E_1^{*,*}$ is finitely generated over the subalgebra generated by all ξ_i^p. So Lemma 2.13(1) can be applied if we can show all ξ_i^p are permanent cycles. In fact, we can define $f_i\in\mathrm{H}^{2p}(|\mathbf{u}(\mathfrak{g})|,\kappa)$ through the same way as that of f_{α_i} (see Proposition 3.10) and get f_i corresponds to its counterpart ξ_i^p defined on $|\mathrm{gr}(\mathbf{u}(\mathfrak{g}))|$. Therefore, every ξ_i^p is a permanent cycle and thus $\mathrm{H}^*(|\mathbf{u}(\mathfrak{g})|,\kappa)$ is a finitely generated algebra. Using Lemma 2.7, we know that $\mathrm{H}^*(\mathbf{u}(\mathfrak{g}),\kappa)$ is also finitely generated as an algebra.

Using the same way as in the last paragraph of the proof of Theorem 3.8, the second part of the theorem can be proved by noting that $|S_{\mathfrak{g}}|$ is also a quantum complete intersection algebra. □

Remark 3.12. To show the theorem, we cannot apply the filtration developed in this subsection to Lie superalgebras $\mathbf{B}(m,n)$, $\mathbf{G}(3)$ directly since otherwise more nilpotent elements will be created. On the contrary, the two kinds of filtration given in Section 2 can be applied to $\mathbf{A}(1,1)$ and indeed $\mathrm{Gr}^2(\mathbf{A}(1,1))=\mathrm{gr}(\mathbf{A}(1,1))$. But in the case of $\mathfrak{g}=\mathbf{A}(1,1)$, it is possible that $\dim_\kappa\mathfrak{g}_\alpha\geqslant 2$ and so the notation x_α has no meaning now. Therefore, if we want to deal with all basic classical Lie superalgebras in a unified way (that is, by using two kinds of filtration), the notations and descriptions will be too delicate to grasp the main line.

4. Support varieties and representation type of Lie superalgebras

In this section, we will recall the definition of the support variety of a module and give its relation with the complexity of this module. As a consequence, we will prove all $\mathbf{u}(\mathfrak{g})$ are wild with only three exceptions: $\mathfrak{g}=\mathfrak{sl}_2,\mathfrak{osp}(1|2),\mathbf{C}(2)$.

Let $\mathfrak{g}$ be a basic classical Lie superalgebra and N a finitely generated left $\mathbf{u}(\mathfrak{g})$-supermodule. By Corollary 2.8 and Theorem 3.1, $\mathrm{H}^{ev}(\mathbf{u}(\mathfrak{g}),\kappa)$ is a finitely generated commutative algebra and $\mathrm{H}^*(\mathbf{u}(\mathfrak{g}),N)$ is a finitely generated $\mathrm{H}^{ev}(\mathbf{u}(\mathfrak{g}),\kappa)$-module. In particular, for any finitely generated $\mathbf{u}(\mathfrak{g})$-supermodule M, $\mathrm{Ext}^*_{\mathbf{u}(\mathfrak{g})}(M,M):=\bigoplus_{i\geqslant 0}\mathrm{H}^i_{\mathbf{u}(\mathfrak{g})}(M,M)\cong\bigoplus_{i\geqslant 0}\mathrm{H}^i(\mathbf{u}(\mathfrak{g}),M^*\otimes M)$ is finitely generated over $\mathrm{H}^{ev}(\mathbf{u}(\mathfrak{g}),\kappa)$ where M^* is the dual $\mathbf{u}(\mathfrak{g})$-module of M. Let I_M be the annihilator of action of $\mathrm{H}^{ev}(\mathbf{u}(\mathfrak{g}),\kappa)$ on $\mathrm{Ext}^*_{\mathbf{u}(\mathfrak{g})}(M,M)$. The *cohomological support variety* of M is defined to be

$$\mathcal{V}_{\mathbf{u}(\mathfrak{g})}(M) := Z(I_M) \subset \mathrm{Maxspec}\big(\mathrm{H}^{ev}\big(\mathbf{u}(\mathfrak{g}),\kappa\big)\big).$$

Note that we can regard M as a $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$-module by Lemma 2.2.

Let A be an associative algebra, M an A-module with minimal projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Then the *complexity* of M is defined to be the integer

$$\mathrm{C}_A(M) := \min\big\{c \in \mathbb{N}_0 \cup \infty \,\big|\, \exists \lambda > 0\colon \dim_k P_n \leqslant \lambda n^{c-1},\ \forall n \geqslant 1\big\}.$$

Lemma 4.1. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra and M a finitely generated left $\mathbf{u}(\mathfrak{g})$-supermodule. Then*

$$\dim \mathcal{V}_{\mathbf{u}(\mathfrak{g})}(M) = \mathrm{C}_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(M).$$

Proof. By definition, $\mathrm{H}^{ev}(\mathbf{u}(\mathfrak{g}),\kappa) = \bigoplus_{i\geqslant 0} \mathrm{Ext}^{2i}_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(\kappa,\kappa)$ and now $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ is an ordinary finite-dimensional Hopf algebra. So this lemma is just a corollary of Proposition 2.3 in [17]. □

Recall the finite-dimensional associative algebras over an algebraically closed field κ can be divided into three classes (see [13]): A finite-dimensional algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are finite number of A-$\kappa[T]$-bimodules M_i which are free of finite rank as right $\kappa[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{\kappa[T]} \kappa[T]/(T-\lambda)$ for $\lambda \in k$. We say that A is of *wild type* or A is a *wild* algebra if there is a finitely generated A-$\kappa\langle X,Y\rangle$-bimodule B which is free as a right $\kappa\langle X,Y\rangle$-module such that the functor $B \otimes_{\kappa\langle X,Y\rangle} -$ from $\kappa\langle X,Y\rangle$-mod, the category of finitely generated $\kappa\langle X,Y\rangle$-modules, to A-mod, the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms.

The following result is a special case of Theorem 4.5 in [24].

Lemma 4.2. *Let A be a superalgebra and assume that characteristic of κ is not 2. Then $|A|$ and $A \# \kappa\mathbb{Z}_2$ have the same representation type.*

Remark 4.3. For a finite-dimensional superalgebra A, one also can define its representation type in the super world, that is, in the category of supermodules with even homomorphisms. By Lemma 4.2 and Lemma 2.2, the representation type of $|A|$ as an ordinary algebra is indeed the same with that of A when we consider it as a superalgebra. So to consider the representation type of a superalgebra A, it is enough to consider that of its underline algebra $|A|$.

The following conclusion is also needed.

Lemma 4.4. *If there is a finite-dimensional $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$-module M such that $\mathrm{C}_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(M) \geqslant 3$, then $\mathbf{u}(\mathfrak{g}) \# \kappa\mathbb{Z}_2$ is wild.*

Proof. Let H be an arbitrary finite-dimensional Hopf algebra such that $\mathrm{C}_H(N) \geqslant 3$ for some H-module N. Then Theorem 3.1 in [17] implies that H is wild provided $\mathrm{H}^*(H,\kappa)$ is finitely generated and $\mathrm{H}^*(H,N')$ is a Noetherian module over $\mathrm{H}^*(H,\kappa)$ for any finite-dimensional H-module N'. So the lemma is proved due to our Theorem 3.1. □

Theorem 4.5. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical Lie superalgebra over κ. Then $|\mathbf{u}(\mathfrak{g})|$ is wild except $\mathfrak{g} = \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{osp}(1|2)$ or $\mathfrak{g} = \mathbf{C}(2)$. Both $|\mathbf{u}(\mathfrak{sl}_2)|$ and $|\mathbf{u}(\mathfrak{ops}(1|2))|$ are tame.*

Proof. The proof is base on the estimation of the number $\mathrm{C}_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(\kappa)$. By Proposition 2.1 in [17], we have

$$\mathrm{C}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) \leqslant \mathrm{C}_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(\kappa).$$

Owing to (1.4) in [18], $\mathcal{V}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa)$ can be identified with

$$\mathscr{V}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) := \{x \in \mathfrak{g}_0 \mid x^{[p]} = 0\} \cup \{0\}.$$

Now we have known that $\mathfrak{g}_0$ is a direct sum of simple Lie algebras of type $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{D}_n$, $\mathbf{G}_2$ or κ. By Lemma 2.10,

$$\dim \mathcal{V}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) = \dim \mathscr{V}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) \geqslant 3$$

except $\mathfrak{g}_0 = \mathfrak{sl}_2$ or $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \kappa$. Thus Lemma 4.1 implies that $\mathrm{C}_{\mathbf{u}(\mathfrak{g})\#\kappa\mathbb{Z}_2}(\kappa) \geqslant \mathrm{C}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) = \dim \mathcal{V}_{\mathbf{u}(\mathfrak{g}_0)}(\kappa) \geqslant 3$ unless $\mathfrak{g}_0 = \mathfrak{sl}_2$ or $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \kappa$. The latter only appear in the case $\mathfrak{g} = \mathbf{C}(2)$. So now it is not hard to see that in the rest list of basic classical Lie superalgebras only $\mathfrak{sl}_2$ and $\mathfrak{osp}(1|2)$ satisfy its even part is $\mathfrak{sl}_2$. By applying Lemma 4.4, the first part of theorem is proved.

For the second part, it is known that $\mathbf{u}(\mathfrak{sl}_2)$ is tame (see for example [15]). The algebra $|\mathbf{u}(\mathfrak{osp}(1|2))|$ is proved to be a tame algebra by Farnsteiner in the example in Section 4 of [16]. □

Conjecture 4.6. *The algebra $|\mathbf{u}(\mathbf{C}(2))|$ is a wild algebra.*

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