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On the structure of tame graded basic Hopf algebras II [☆]

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ABSTRACT

In continuation of the article [G.X. Liu, On the structure of tame basic Hopf algebras, J. Algebra 299 (2006) 841–853] we classify all radically graded basic Hopf algebras of tame type over an algebraically closed field of characteristic 0.

1. Introduction

Throughout this paper k denotes an algebraically closed field and all spaces are k-spaces. By an algebra we mean a finite-dimensional associative algebra with identity element. We freely use the results, notations, and conventions of [27].

According to the fundamental result of Drozd [11], every finite-dimensional algebra exactly belongs to one of following three kinds of algebras: algebras of finite representation type, algebras of tame type and wild algebras. For the algebras of the former two kinds, a classification of indecomposable modules seems feasible. By contrast, the module category of a wild algebra, being "complicated" at least as that of any other algebra, cannot afford such a classification. Inspired by the Drozd's result, one is often interested in classifying a given kind of algebras according to their representation type. The class of finite-dimensional Hopf algebras has been considered for quite a long time. For group algebras of finite groups, the representation type of a block is governed by its defect groups. A block of a finite-dimensional group algebra is of finite representation type if and only if the corresponding defect groups are cyclic while is tame if and only if $\mathrm{char}\, k = 2$ and its defects groups are dihedral, semidihedral or generalized quaternion. See [5,6,12,22]. In the case of small quantum groups, i.e.,

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Frobenius–Lusztig kernels, the only tame one is $\mathbf{u}_q(\mathfrak{sl}_2)$ and the others are all wild [7,33,34]. The classification for finite-dimensional cocommutative Hopf algebras, i.e., finite algebraic groups, of finite representation type and tame type was given by Farnsteiner and his cooperators recently [15–19]. The representation theory of such cocommutative Hopf algebras was also studied in [13,14].

Meanwhile, basic Hopf algebras and their duals, pointed Hopf algebras, have been studied intensively by many authors. See, for example, [1,2,21]. Our intention is to classify finite-dimensional basic Hopf algebras through their representation type. In [25], the authors have classified all basic Hopf algebras of finite representation type and show that they are all monomial Hopf algebras (see [8]). For basic Hopf algebras of tame type, the following result, up to the authors's knowledge, is the best (see [24]): Let H be a basic Hopf algebra over an algebraically closed field k of characteristic different from 2, then $\operatorname{gr} H$ is tame if and only if $\operatorname{gr} H \cong k\langle x, y\rangle/I \times (kG)^*$ for some finite group G and some ideal I which is one of the following forms:

(1) $I = (x^2 - y^2, yx - ax^2, xy)$ for $0 \neq a \in k$;
(2) $I = (x^2, y^2, (xy)^m - a(yx)^m)$ for $0 \neq a \in k$ and $m \geqslant 1$;
(3) $I = (x^n - y^n, xy, yx)$ for $n \geqslant 2$;
(4) $I = (x^2, y^2, (xy)^m x - (yx)^m y)$ for $m \geqslant 1$;
(5) $I = (yx - x^2, y^2)$.

Here $\operatorname{gr} H$ denotes the radically graded algebra of H and "$\times$" is the *bosonization* defined in [26] or called *biproduct* in [29]. By this result, there are at most five classes of tame graded basic Hopf algebras. By a conclusion of Radford or Majid (see [26,29]), for an algebra Λ and a finite group G, the bosonization $\Lambda \times (kG)^*$ is a Hopf algebra if and only if Λ is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$. For an algebra $k\langle x, y\rangle/I$, the above conclusion dose not imply the existence of finite group G satisfying $k\langle x, y\rangle/I$ is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$. That is to say, for the ideals I listed in above conclusion, we do not know whether $k\langle x, y\rangle/I \times (kG)^*$ is a Hopf algebra or not! In fact, this question is formulated as an open question posted in [24] (Problem 5.1):

Problem 1.1. For a tame local graded Frobenius algebra A, give an effective method to determine whether there is a finite group G such that A is a braided Hopf algebra in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$. If such a G exists, then find all of them.

In this paper, we will solve this problem. Indeed, we will show that only the ideals $I = (x^2, y^2, (xy)^m - a(yx)^m)$ can appear. For more subtle description, see Theorems 4.9, 4.16, 5.2 and the followed remarks. Then the class of tame graded basic Hopf algebras can be classified completely.

The basic idea is simple. For a basic Hopf algebra H, we can construct its radically graded version $\operatorname{gr} H = H/J_H \oplus J_H/J_H^2 \oplus \cdots$. Then we establish the Gabriel's theorem for graded basic Hopf algebras, that is, we show that there is a Hopf surjection $kQ \to \operatorname{gr} H$ where Q is the Gabriel quiver of $\operatorname{gr} H$. By Theorem 2.3 of [21], Q is a covering quiver $\Gamma_G(W)$. We find that W consists of at most two elements and the group generated by W is automatically abelian. Then we lift the ideals I to the ideals $\widetilde{I}$ of the path algebra $k\Gamma_G(W)$ and the main difficulty is to show when $\widetilde{I}$ is a Hopf ideal.

The paper is organized as follows. The next section contains all knowledge that we need to go ahead. In particular, the works of Green and Solberg on basic Hopf algebras are recalled and the Gabriel's theorem for basic Hopf algebras is established. Some combinatorial relations, which is the key to give a criterion to determine an ideal to be a Hopf ideal, will be given in Section 3. Section 4 deals with the classification of tame graded basic Hopf algebras in the case of they are connected as algebras. Using crossed products and the results gotten in Section 4, the class of tame graded basic Hopf algebras are classified at the last section.

2. Preliminaries

In the following of this paper, we always assume that the characteristic of k is 0 unless otherwise stated.

A *quiver* is an oriented graph $Q = (Q_0, Q_1)$, where Q_0 denotes the set of vertices and Q_1 denotes the set of arrows. kQ denotes its *path algebra*. An ideal I of kQ is called *admissible* if $J^N \subset I \subset J^2$ for some $N \geqslant 2$, where J is the ideal generated by all arrows.

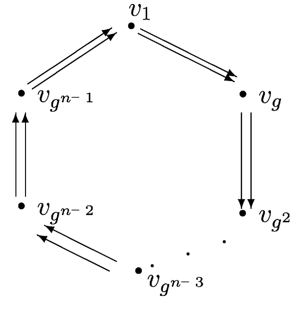
For a basic algebra A, by the Gabriel's theorem, there is a unique quiver Q_A, and an admissible ideal I of kQ_A, such that $A \cong kQ_A/I$ (see [3] and [4]). The quiver Q_A is called the *Gabriel quiver* or *Ext-quiver* of A.

Next, let us recall the definition of *covering quivers* (see [21]). Let G be a finite group and let $W = (w_1, w_2, \dots, w_n)$ be a sequence of elements of G. We say W is a *weight sequence* if, for each $g \in G$, the sequences W and $(gw_1g^{-1}, gw_2g^{-1}, \dots, gw_ng^{-1})$ are the same up to a permutation. In particular, W is closed under conjugation. Define a quiver, denoted by $\Gamma_G(W)$, as follows. The vertices of $\Gamma_G(W)$ is the set $\{v_g\}_{g\in G}$ and the arrows are given by

$$\left\{(a_i, g)' : v_{g^{-1}} \to v_{w_i g^{-1}} \;\middle|\; i = 1, 2, \dots, n,\ \ g \in G\right\}.$$

We call this quiver the covering quiver (with respect to W).

Example 2.1. Let $G = \langle g \mid g^n = 1\rangle$ and $W = (g, g)$, then the corresponding covering quiver is



We denote this quiver by $\mathbb{Z}_n(2)$.

Lemma 2.1. *Let $\Gamma_G(W)$ be a covering quiver. If the length of W is 2, then the subgroup of G generated by W is an abelian group.*

Proof. Let $W = \{g, h\}$. Since W is stable under the conjugation, $ghg^{-1} = g$ or $ghg^{-1} = h$. If $ghg^{-1} = g$, then $g = h$. If $ghg^{-1} = h$, then $gh = hg$. The lemma is clear now. □

The following conclusion (see Theorem 2.3 in [21]) states the importance of covering quivers.

Lemma 2.2. *Let H be a finite-dimensional basic Hopf algebra over k. Then there exists a finite group G and a weight sequence $W = (w_1, w_2, \dots, w_n)$ of G, such that $H \cong k\Gamma_G(W)/I$ for an admissible ideal I.*

Let $\Gamma_G(W)$ be a covering quiver, a natural question is when there is a Hopf structure on the path algebra $k\Gamma_G(W)$. To answer this question, we need the concept *allowable kG-bimodule* which was introduced by Green and Solberg [21]. Denote V_f^d the k-space with basis the arrows from v_d to v_f for $d, f \in G$. We say a kG-bimodule structure on $k\Gamma_G(W)$ is allowable if for any $g, d, f \in G$, the following conditions hold:

(i) $g \cdot v_f = v_{fg^{-1}}$ and $v_f \cdot g = v_{g^{-1}f}$;
(ii) $g \cdot V_f^d \subset V_{fg^{-1}}^{dg^{-1}}$ and $V_f^d \cdot g \subset V_{g^{-1}f}^{g^{-1}d}$.

For any vertex v_h of $\Gamma_G(W)$ and any $x \in V_f^d$, define three maps as follows:

$$\varepsilon(v_h) = \begin{cases} 1, & \text{if } h = e, \\ 0, & \text{otherwise}, \end{cases} \qquad \varepsilon(x) = 0;$$

$$\Delta(v_h) = \sum_{g \in G} v_{hg^{-1}} \otimes v_g, \qquad \Delta(x) = \sum_{g \in G} (g \cdot x \otimes v_g + v_g \otimes x \cdot g);$$

$$S(v_h) = v_{h^{-1}}, \qquad S(x) = -f \cdot x \cdot d.$$

The following conclusion is due to Green and Solberg (see Theorem 3.3 of [21]).

Lemma 2.3. *Suppose that $k\Gamma_G(W)$ has an allowable kG-bimodule structure, then $k\Gamma_G(W)$ is a Hopf algebra with counit ε, comultiplication Δ and antipode S given above.*

Let H be a basic Hopf algebra, then its Jacobson radical J_H is a Hopf ideal (see Lemma 1.1 in [21]). Hence $H/J_H \cong (kG)^*$ for some finite group G with counit ε', comultiplication Δ' and antipode S' given in terms of the dual basis $\{p_g\}_{g \in G}$ in $(kG)^*$ in the following way:

$$\varepsilon'(p_h) = \begin{cases} 1, & \text{if } h = e, \\ 0, & \text{otherwise}; \end{cases}$$

$$\Delta'(p_h) = \sum_{g \in G} p_{hg^{-1}} \otimes p_g;$$

$$S'(v_h) = v_{h^{-1}}.$$

The set $\{p_g\}_{g \in G}$ of primitive orthogonal idempotents in H/J_H can be lifted to a set of primitive orthogonal idempotents $\{v_g\}_{g \in G}$ in H. Since H^* can act on H naturally (see [27]), kG can act on H now. Using this it follows from the action of kG on H that $g \cdot v_f = v_{fg^{-1}}$ and $v_f \cdot g = v_{g^{-1}f}$ modulo the radical. Combining Lemma 1.2 and Lemma 2.1 in [21], we have the following result.

Lemma 2.4. *Let H be a basic Hopf algebra. With the notations above, the following assertions hold.*

(a) *The counit ε for H is given by*

$$\varepsilon(v_h) = \begin{cases} 1, & \text{if } h = e, \\ 0, & \text{otherwise}, \end{cases}$$

for all $h \in G$ and $\varepsilon(J_H) = 0$;

(b) *The comultiplication for H is given by*

$$\Delta(v_h) = \sum_{g \in G} v_{hg^{-1}} \otimes v_g$$

modulo $J_{H \otimes H}$ and for $x \in v_f J_H / J_H^2 v_d$,

$$\Delta(x) = \sum_{g \in G} (g \cdot x \otimes v_g + v_g \otimes x \cdot g)$$

modulo $J_{H \otimes H}^2$.

Denote $\operatorname{gr} H = H/J_H \oplus J_H/J_H^2 \oplus \cdots$ the radically graded algebra of H. By Lemma 5.1 of [24], it is also a Hopf algebra. Now we can give the Gabriel's theorem for basic Hopf algebras, which is indeed dual to Theorem 4.5 of [28].

Lemma 2.5. *Let H be a basic Hopf algebra and $\Gamma_G(W)$ its Gabriel quiver. Then there is a Hopf algebra surjection*

$$\pi : k\Gamma_G(W) \longrightarrow \operatorname{gr} H$$

with $\operatorname{Ker}\pi$ *an admissible Hopf ideal of $k\Gamma_G(W)$.*

Proof. We use the notations above. At first, we must equip $k\Gamma_G(W)$ with a Hopf structure. By Lemma 2.3, it is enough to give an allowable kG-bimodule structure on $k\Gamma_G(W)$. Indeed, for any vertex v_f, define $g \cdot v_f = v_{fg^{-1}}$ and $v_f \cdot g = v_{g^{-1}f}$. Transporting the left and right actions of kG on $v_f J_H/J_H^2 v_d$ to the k-space with the basis of all arrows from v_d to v_f, we get the left and right actions of kG on paths of length 1. For a path $p = \alpha_n \cdots \alpha_1$ of length n, define

$$g \cdot p = (g \cdot \alpha_n) \cdots (g \cdot \alpha_1), \qquad p \cdot g = (\alpha_n \cdot g) \cdots (\alpha_1 \cdot g).$$

Thus we get an allowable kG-bimodule structure on $k\Gamma_G(W)$ now and the Hopf structure on $k\Gamma_G(W)$ is given through the way as in Lemma 2.3.

By the Gabriel's theorem, π is an algebra surjection. We only need to show that it is also a coalgebra map, i.e., $\Delta\pi = (\pi \otimes \pi)\Delta$. Set $\phi_1 = \Delta\pi$ and $\phi_2 = (\pi \otimes \pi)\Delta$. By Lemma 2.4, we have

$$\phi_1|_{k\Gamma_G(W)_0} = \phi_2|_{k\Gamma_G(W)_0}, \qquad \phi_1|_{k\Gamma_G(W)_1} = \phi_2|_{k\Gamma_G(W)_1}$$

where $k\Gamma_G(W)_0$ and $k\Gamma_G(W)_1$ denote the k-spaces spanned by all vertices and all arrows respectively. It is well known that the path algebra is indeed a tensor algebra. Using the universal property of tensor algebra, we know that every algebra morphism f from the path algebra $k\Gamma$ is determined uniquely by $f|_{k\Gamma_0}$ and $f|_{k\Gamma_1}$. Thus $\phi_1 = \phi_2$. □

Notice the difference between Lemmas 2.2 and 2.5. Lemma 2.5 tells us that the algebra isomorphism given in Lemma 2.2 can be strengthened to be a Hopf isomorphism when the basic Hopf algebra is radically graded. We also need Proposition 4.4 of [21].

Lemma 2.6. *Let $k\Gamma_G(W)$ be a Hopf algebra with Hopf structure given by an allowable kG-bimodule structure. Let $I \subset k\Gamma_G(W)$ be a Hopf ideal which is admissible. Then I is stable under left and right G-actions.*

At the end of this section, we give the definition of representation type. An algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. A is of *tame type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are finite number of A-$k[T]$-bimodules M_i which are free of finite rank as right $k[T]$-modules such that all but a finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{k[T]} k[T]/(T-\lambda)$ for $\lambda \in k$. We say that A is of *wild type* or A is a *wild* algebra if there is a finitely generated A-$k\langle X, Y\rangle$-bimodule B which is free as a right $k\langle X, Y\rangle$-module such that the functor $B\otimes_{k\langle X,Y\rangle}-$from mod-$k\langle X, Y\rangle$, the category of finitely generated $k\langle X, Y\rangle$-modules, to mod-A, the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms. See [9–12,31] for more details and in particular [20] for geometric characterization of the tameness of algebras.

3. Some combinatorial relations

For our purpose, we need to consider the following combinatorial functions:

$$H_1(m,l,t) = \sum_{0\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_l\leqslant m-l} t^{\sum_{i=1}^l m_i},$$

$$H_2(m,l,t) = \sum_{0\leqslant n_1+n_2+\cdots+n_l\leqslant m-l} t^{\sum_{i=1}^l (l+1-i)n_i},$$

$$H_3(m,l,t) = t^{m-l} \sum_{0\leqslant n_1+n_2+\cdots+n_{l-1}\leqslant m-l} t^{\sum_{i=1}^{l-1}(l-i)n_i} + \sum_{0\leqslant n_1+n_2+\cdots+n_l\leqslant m-l-1} t^{\sum_{i=1}^l (l+1-i)n_i}.$$

Here $m, l \in \mathbb{Z}^+$, $0 < l < m$, $m_1, \ldots, m_l, n_1, \ldots, n_l \in \mathbb{N}$ and t is an indeterminant.

Lemma 3.1. $H_1(m,l,t) = H_2(m,l,t) = H_3(m,l,t)$

Proof. It is not hard to see that $H_2(m,l,t) = H_3(m,l,t)$. Now we show that $H_1(m,l,t) = H_2(m,l,t)$. Note that

$$t^{\sum_{i=1}^l (l+1-i)n_i} = t^{n_1+(n_1+n_2)+(n_1+n_2+n_3)+\cdots+(n_1+n_2+\cdots+n_l)}.$$

Let $m_1 = n_1$, $m_2 = n_1 + n_2$, …, $m_l = n_1 + n_2 + \cdots + n_l$, we can see $H_1(m,l,t) = H_2(m,l,t)$. □

Professor Zhi-Wei Sun gives us the proof of the main result of this section.

Proposition 3.2. *$H_1(m,l,t) = 0$ for all $0 < l < m$ if and only if t is an mth primitive root of unity.*

Proof. "$\Longleftarrow$" For any $0 < l < m$, let $i_j = m_j + j$, then

$$H_1(m,l,t) = \sum_{1\leqslant i_1<i_2<\cdots<i_l\leqslant m} t^{\sum_{j=1}^l (i_j - j)} = t^{-\frac{l(l+1)}{2}} \sum_{1\leqslant i_1<i_2<\cdots<i_l\leqslant m} t^{\sum_{j=1}^l i_j}.$$

Consider the generating function

$$\prod_{r=1}^m \bigl(1+t^r x\bigr) = (1+tx)\bigl(1+t^2x\bigr)\cdots\bigl(1+t^m x\bigr),$$

where x is an indeterminant. On one hand,

$$\prod_{r=1}^m \bigl(1+t^r x\bigr) = 1 + \sum_{l=1}^m \Bigl(\sum_{1\leqslant i_1<i_2<\cdots<i_l\leqslant m} t^{\sum_{j=1}^l i_j}\Bigr) x^l.$$

On the other hand,

$$\prod_{r=1}^m \bigl(1+t^r x\bigr) = \prod_{r=1}^m \bigl(1 - t^r(-x)\bigr).$$

By using a well-known identity $y^m - 1 = \prod_{r=0}^{m-1}(y - \zeta_m^r)$ for any indeterminant y and mth primitive root of unity ζ_m, we see that

$$\prod_{r=1}^{m}\big(1+t^r x\big)=\prod_{r=1}^{m}\big(1-t^r(-x)\big)=1-(-x)^m.$$

Thus for all $0<l<m$,

$$\sum_{1\leqslant i_1<i_2<\cdots<i_l\leqslant m} t^{\sum_{j=1}^{l} i_j}=0$$

and thus $H_1(m,l,t)=0$.

"$\Longrightarrow$" Let $l=1$, the condition implies that

$$\sum_{i=0}^{m-1} t^i=0.$$

Thus $1\neq t$ is an mth root of unity. There is no harm to assume that t is a dth primitive root of unity with $d\mid m$. Consider the generating function again,

$$\prod_{s=1}^{m}\big(1+t^s x\big)=\prod_{q=0}^{\frac{m}{d}-1}\prod_{r=1}^{d}\big(1-t^{qd+r}(-x)\big).$$

Just like the proof of sufficient part, we have

$$\prod_{q=0}^{\frac{m}{d}-1}\prod_{r=1}^{d}\big(1-t^{qd+r}(-x)\big)=\prod_{q=0}^{\frac{m}{d}-1}\big(1-(-x)^d\big)=\big(1-(-x)^d\big)^{\frac{m}{d}}.$$

By the proof of sufficiency, if $d<m$, there must exist an l with $0<l<m$ such that

$$\sum_{1\leqslant i_1<i_2<\cdots<i_l\leqslant m} t^{\sum_{j=1}^{l} i_j}\neq 0$$

and thus $H_1(m,l,t)\neq 0$. It is a contradiction. So $d=m$. $\square$

4. Classification-connected case

The main result of [24] is the following result:

Lemma 4.1. *Let H be a basic Hopf algebra, then $\operatorname{gr} H$ is tame if and only if $\operatorname{gr} H\cong k\langle x,y\rangle/I\times(kG)^*$ for some finite group G and some ideal I which is one of the following forms*:

(1) $I=(x^2-y^2,\ yx-ax^2,\ xy)$ *for* $0\neq a\in k$;
(2) $I=(x^2,\ y^2,\ (xy)^m-a(yx)^m)$ *for* $0\neq a\in k$ *and* $m\geqslant 1$;
(3) $I=(x^n-y^n,\ xy,\ yx)$ *for* $n\geqslant 2$;
(4) $I=(x^2,\ y^2,\ (xy)^m x-(yx)^m y)$ *for* $m\geqslant 1$;
(5) $I=(yx-x^2,\ y^2)$.

As pointed out in the introduction, our aim is to determine which ideals I listed in Lemma 4.1 and what groups G actually make $k\langle x,y\rangle/I\times(kG)^*$ a Hopf algebra.

At first, we show that the case (5) in Lemma 4.1 will not occur.

Lemma 4.2. *$\Lambda = k\langle x, y\rangle/(yx - x^2, y^2)$ is not a local Frobenius algebra.*

Proof. Suppose it is.

Claim: $J_\Lambda^3 = (yxy) \subseteq \operatorname{soc}\Lambda$. We have that J_Λ^3 is generated by xyx and yxy by the given relations. Moreover, modulo J_Λ^4 we have that $xyx \equiv x^3 \equiv yxx \equiv y^2x \equiv 0$ and therefore $J_\Lambda^3 = (yxy)$. This implies $J_\Lambda^4 = ((yx)^2) \subseteq yJ_\Lambda^4 \subseteq J_\Lambda^5$. Thus $J_\Lambda^4 = 0$ as required.

We claim yxy must be zero now. Otherwise, assume $yxy \neq 0$ and thus $J_\Lambda^3 = (yxy) = \operatorname{soc}\Lambda$. Since $J_\Lambda^4 = 0$, we know $xyx = 0$ and $xy^2 = 0$. This means $xy \in \operatorname{soc}\Lambda$. Clearly, $xy \neq 0$ since otherwise $yxy = 0$. Since $dim_k \operatorname{soc}\Lambda = 1$, there exists non-zero $c \in k$ such that $xy = cyxy$. So we have $xy = cyxy = c^2y^2xy = 0$. It is a contradiction. This means $yxy = 0$ and thus $J_\Lambda^3 = 0$ and $J_\Lambda^2 \subseteq \operatorname{soc}\Lambda$. Therefor $\operatorname{soc}\Lambda$ is not simple, which is absurd. □

If there exists a finite group G such that $k\langle x, y\rangle/(yx - x^2, y^2) \times (kG)^*$ is a Hopf algebra, then $k\langle x, y\rangle/(yx - x^2, y^2)$ must be local Frobenius (Proposition 5.3 in [24]). This implies that the case (5) cannot appear.

So we only need to consider cases (1)–(4). In this paper, we say a basic Hopf algebra H is graded if $H \cong \operatorname{gr} H$ as Hopf algebras. Now let H be a tame graded basic Hopf algebra and assume it is connected as an algebra. In this situation, we say H is a connected tame graded basic Hopf algebra. Denote its Gabriel quiver by $\Gamma_G(W)$, which is a covering quiver by Lemma 2.2. Thus $H/J_H \cong (kG)^*$. By the assumption of H being connected, $\Gamma_G(W)$ is a connected quiver. From the definition of covering quivers, we can deduce that $G = \langle W\rangle$, the group generated by W. By Lemma 4.1, the length of W is 2. Thus G is an abelian group by Lemma 2.1. Combining these discussions, we get the next observation.

Proposition 4.3. *Let H be a connected tame graded basic Hopf algebra and $\Gamma_G(W)$ its Gabriel quiver. Then the length of W is 2, $G = \langle W\rangle$ and is abelian.*

By Lemma 2.5, the surjection $\pi : k\Gamma_G(W) \to H$ is a Hopf algebra surjection and thus $\operatorname{Ker}\pi$ is a Hopf ideal. We now lift the ideals (1)-(4) in Lemma 4.1 to the ideals of $k\Gamma_G(W)$ and we need to determine which lifting is a Hopf ideal.

By Proposition 4.3, for any vertex of $\Gamma_G(W)$, there are exactly two arrows going out and two arrows coming in. Denote the arrows starting from e by a and b respectively. Since x, y are generators of the Jacobson radical of $k\langle x, y\rangle/I$, we must lift x, y to linear combination of arrows. By Lemma 2.6, it is harmless to lift x and y to $\sum_{g\in G} g \cdot a$ and $\sum_{g\in G} g \cdot b$ respectively, i.e.,

$$x \mapsto X := \sum_{g\in G} g \cdot a, \qquad y \mapsto Y := \sum_{g\in G} g \cdot b.$$

Thus our task is just to determine whether the following ideals are Hopf ideals or not:

(1) $I_1(a) = (X^2 - Y^2, YX - aX^2, XY)$ for $0 \neq a \in k$;
(2) $I_2(m, a) = (X^2, Y^2, (XY)^m - a(YX)^m)$ for $0 \neq a \in k$ and $m \geqslant 1$;
(3) $I_3(n) = (X^n - Y^n, XY, YX)$ for $n \geqslant 2$;
(4) $I_4(m) = (X^2, Y^2, (XY)^m X - (YX)^m Y)$ for $m \geqslant 1$.

By Proposition 4.3, $W = (g, g)$ or $W = (g, h)$ with $g \neq h$. We discuss these two cases separately.

4.1. Case 1: $W = (g, g)$

Using the standard notations of covering quivers, $a = (a_1, e)$ and $b = (a_2, e)$. Assume that $\operatorname{ord}(g) = n$ and $\Gamma_G(W)$ is just the quiver given in Example 2.1. Since G is abelian, the action of $kG \otimes (kG)^{\mathrm{op}}$ is diagonalizable. Thus, we can assume that

$$g \cdot (a_1, e) = (a_1, g), \qquad g \cdot (a_2, e) = (a_2, g),$$
$$(a_1, e) \cdot g = q^{-1} g \cdot (a_1, e), \qquad (a_2, e) \cdot g = p^{-1} g \cdot (a_2, e)$$

for p, q are nth roots of unity. Denote v_{g^i} by v_i for simplicity.

Lemma 4.4.

$$\Delta(X) = X \otimes 1 + \left(\sum_{i=0}^{n-1} q^{-i} v_i\right) \otimes X, \qquad S(X) = -q \sum_{i=0}^{n-1} a \cdot g^i;$$
$$\Delta(Y) = Y \otimes 1 + \left(\sum_{i=0}^{n-1} p^{-i} v_i\right) \otimes Y, \qquad S(Y) = -p \sum_{i=0}^{n-1} b \cdot g^i.$$

Proof. We only prove the formulaes for X. Those for Y can be proved in the same manner.

$$\begin{aligned}
\Delta(X) &= \sum_{i=0}^{n-1} g^i \cdot X \otimes v_i + \sum_{i=0}^{n-1} g^i \cdot v_i \otimes X \cdot g^i \\
&= \sum_{i=0}^{n-1} X \otimes v_i + \sum_{i=0}^{n-1} v_i \otimes \left(\sum_{j=0}^{n-1} g^j \cdot a \cdot g^i\right) \\
&= X \otimes 1 + \sum_{i=0}^{n-1} v_i \otimes \left(\sum_{j=0}^{n-1} q^{-i} g^{i+j} \cdot a\right) \\
&= X \otimes 1 + \sum_{i=0}^{n-1} v_i \otimes q^{-i} X \\
&= X \otimes 1 + \left(\sum_{i=0}^{n-1} q^{-i} v_i\right) \otimes X
\end{aligned}$$

and

$$S(X) = S\left(\sum_{i=0}^{n-1} g^{-i} \cdot a\right) = \sum_{i=0}^{n-1} S(a) \cdot g^i = \sum_{i=0}^{n-1} -(g \cdot a) \cdot g^i = -q \sum_{i=0}^{n-1} a \cdot g^i. \qquad \square$$

For an indeterminant x, define the function $e_x := \sum_{i=0}^{n-1} x^{-i} v_i$.

Lemma 4.5. *We have the following identities*

$$X e_q = q e_q X, \qquad Y e_q = q e_q Y, \qquad X e_p = p e_p X, \qquad Y e_p = p e_p Y.$$

Proof. Note that

$$X e_q = X \sum_{i=0}^{n-1} q^{-i} v_i = \sum_{i=0}^{n-1} q^{-i} g^{-i} \cdot a$$

and

$$qe_qX = q\left(\sum_{i=0}^{n-1} q^{-i}v_i\right)X = q\sum_{i=0}^{n-1} q^{-i}g^{-(i-1)}\cdot a = \sum_{i=0}^{n-1} q^{-(i-1)}g^{-(i-1)}\cdot a.$$

Thus $Xe_q = qe_qX$. We can prove the other identities similarly. □

With the preparation, now we are ready to determine whether $I_1(a)$, $I_2(m,a)$, $I_3(n)$ and $I_4(m)$ are Hopf ideals.

Lemma 4.6. *$I_1(a)$ and $I_3(n)$ are not Hopf ideals of $k\Gamma_G(W)$.*

Proof. By Lemmas 4.4 and 4.5,

$$\begin{aligned}\Delta(XY) &= (X\otimes 1 + e_q\otimes X)(Y\otimes 1 + e_p\otimes Y)\\ &= XY\otimes 1 + pe_pX\otimes Y + e_qY\otimes X + e_{pq}\otimes XY.\end{aligned}$$

Suppose $I_1(a)$ or $I_3(n)$ is a Hopf ideal, then clearly we have

$$pe_pX\otimes Y + e_qY\otimes X = 0,$$

which is impossible. □

By Lemma 4.4 and Lemma 4.5, for any element $f(X,Y)$ generated by X, Y, we can always write uniquely $\Delta(f(X,Y))$ in the following form:

$$\begin{aligned}&f(X,Y)\otimes 1 + \big(f(X,Y)\big)_X\otimes X + \big(f(X,Y)\big)_Y\otimes Y + \big(f(X,Y)\big)_{XY}\otimes XY + \cdots\\ &\quad + \big(f(X,Y)\big)_{(XY)^i}\otimes (XY)^i + \big(f(X,Y)\big)_{(YX)^i}\otimes (YX)^i + \big(f(X,Y)\big)_{(XY)^iX}\otimes (XY)^iX\\ &\quad + \big(f(X,Y)\big)_{Y(XY)^i}\otimes Y(XY)^i + \cdots.\end{aligned}$$

In the following of this paper, we frequently use this expression without any explanation.

Lemma 4.7. *$I_4(m)$ is not a Hopf ideal of $k\Gamma_G(W)$.*

Proof. Assume that it is a Hopf ideal. It is not hard to see that

$$\big((XY)^mX - (YX)^mY\big)_X\otimes X \equiv \big((XY)^me_q + e_q(YX)^m\big)\otimes X \mod I_4(m)\otimes X.$$

Thus we have $(XY)^me_q + e_q(YX)^m \in I_4(m)$. This is absurd. □

Lemma 4.8. *$I_2(m,a)$ is a Hopf ideal if and only if $m = 1$ and $q^{-1} = a = p = -1$.*

Proof. "$\Longrightarrow$" Direct computations show that

$$\Delta\big(X^2\big) = X^2\otimes 1 + (1+q)e_qX\otimes X + e_{q^2}\otimes X^2$$

and

$$\Delta\big(Y^2\big) = Y^2\otimes 1 + (1+p)e_pY\otimes Y + e_{p^2}\otimes Y^2.$$

Thus $1+q=0=1+p$ and so

$$p=q=-1.$$

Next, we show that $m=1$. Otherwise, assume that $m>1$. In $\Delta(XY)^m$, we have the following by direct computations,

$$\big((XY)^m\big)_{XY}\otimes XY\equiv\left(e_q(YX)^{m-1}e_p+\sum_{i=1}^{m}(XY)^{m-i}e_{pq}(XY)^{i-1}\right)\otimes XY \mod I_2(m,a)\otimes XY.$$

By Lemma 4.5, $\sum_{i=1}^m (XY)^{m-i}e_{pq}(XY)^{i-1}=\sum_{i=1}^m (p^2q^2)^{m-i}e_{pq}(XY)^{m-1}$. Similarly, in $\Delta(YX)^m$,

$$\big((YX)^m\big)_{XY}\otimes XY\equiv\sum_{i=0}^{m-2}Y(XY)^{m-2-i}e_{pq}(XY)^iX\otimes XY \mod I_2(m,a)\otimes XY.$$

Thus $(e_q(YX)^{m-1}e_p+\sum_{i=1}^m(p^2q^2)^{m-i}e_{pq}(XY)^{m-1})-a\sum_{i=0}^{m-2}Y(XY)^{m-2-i}e_{pq}(XY)^iX\in I_2(m,a)$ which implies

$$\sum_{i=1}^{m}\big(p^2q^2\big)^{m-i}=0.$$

This is impossible since $p=q=-1$. Thus $m=1$. Finally, we show that $a=-1$. Indeed,

$$\Delta(XY-aYX)=(XY-aYX)\otimes 1+(p-a)e_pX\otimes Y+(1-aq)e_qY\otimes X+e_{pq}\otimes(XY-aYX).$$

So, $p-a=0=1-aq$ which implies that $a=-1$.

"$\Longleftarrow$" By the proof of necessity,

$$\Delta\big(I_2(1,-1)\big)\subset I_2(1,-1)\otimes k\Gamma_G(W)+k\Gamma_G(W)\otimes I_2(1,-1).$$

We only need to show that $S(I_2(1,-1))\subset I_2(1,-1)$ and $\varepsilon(I_2(1,-1))=0$. The verification of $\varepsilon(I_2(1,-1))=0$ is trivial, and by Lemma 4.4,

$$\begin{aligned}S(XY+YX)&=S(Y)S(X)+S(X)S(Y)\\&=\sum_{i=0}^{n-1}\big(b\cdot g^i\big)\sum_{i=0}^{n-1}\big(a\cdot g^i\big)+\sum_{i=0}^{n-1}\big(a\cdot g^i\big)\sum_{i=0}^{n-1}\big(b\cdot g^i\big)\\&=\sum_{i=0}^{n-1}\big(b\cdot g^{i-1}\big)\big(a\cdot g^i\big)+\sum_{i=0}^{n-1}\big(a\cdot g^{i-1}\big)\big(b\cdot g^i\big)\\&=-\sum_{i=0}^{n-1}v_{-i+2}(XY+YX)v_{-i}.\end{aligned}$$

That is $S(I_2(1,-1))\subset I_2(1,-1)$. □

Recall the quiver $\mathbb{Z}_n(2)$ given in Example 2.1. Summarizing the previous arguments, we get the main result for Case 1.

Theorem 4.9. *Let H be a connected tame graded basic Hopf algebra and $\Gamma_G(W)$ its Gabriel quiver. If $W = (g, g)$, then as a Hopf algebra,*

$$H \cong k\mathbb{Z}_n(2)/\big(X^2, Y^2, XY + YX\big)$$

for some even n. Here $X = \sum_{i=0}^{n-1} g^i \cdot (a_1, e)$ and $Y = \sum_{i=0}^{n-1} g^i \cdot (a_2, e)$.

Remark 4.10. Note that p, q are nth roots of unity. By Lemma 4.8, $p = q = -1$ and thus n must be an even. That is the reason why n is assumed to be even in the above theorem. Conversely, for any cyclic group $G = \langle g \mid g^n = 1\rangle$ with n an even, define the allowable kG-bimodule on $k\mathbb{Z}_n(2)$ just as that given at the beginning of this subsection. Then $k\mathbb{Z}_n(2)/(X^2, Y^2, XY + YX)$ is a Hopf algebra by setting $p = q = -1$. Notice that this indeed gives the answer to Problem 1.1 posted in Section 1 in this case.

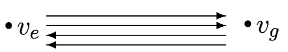
Example 4.1 *(Book algebras).* Let q be an nth primitive root of unity and m a positive integer satisfying $(m, n) = 1$. Let $H = \mathbf{h}(q, m) = k\langle y, x, g\rangle/(x^n, y^n, g^n - 1, gx - qxg, gy - q^m yg, xy - yx)$ with comultiplication, antipode and counit given by

$$\Delta(x) = x \otimes g + 1 \otimes x, \qquad \Delta(y) = y \otimes 1 + g^m \otimes y, \qquad \Delta(g) = g \otimes g,$$

$$S(x) = -xg^{-1}, \qquad S(y) = -g^{-m} y, \qquad S(g) = g^{-1}, \qquad \varepsilon(x) = \varepsilon(y) = 0, \qquad \varepsilon(g) = 1.$$

It is a Hopf algebra and called book algebra in [1]. It is a basic algebra since $\mathbf{h}(q, m)/J_{\mathbf{h}(q,m)}$ is a commutative semisimple algebra. By Example 5.2 in [24], only $\mathbf{h}(-1, 1)$ is tame and the others are wild.

Taking $n = 2$ in Example 2.1, $\mathbb{Z}_2(2)$ is the following quiver:



The allowable $k\mathbb{Z}_2$-bimodule structure on $k\mathbb{Z}_2(2)$ is given by

$$g \cdot v_e = v_e \cdot g = v_g, \qquad g \cdot v_g = v_g \cdot g = v_e,$$

$$g \cdot (a_1, e) = (a_1, g) = -(a_1, e) \cdot g, \qquad g \cdot (a_2, e) = (a_2, g) = -(a_2, e) \cdot g.$$

Define $\varphi : k\mathbb{Z}_2/(X^2, Y^2, XY + YX) \to \mathbf{h}(-1, 1)$ by

$$v_e \mapsto \frac{1}{2}(1 + g), \qquad v_g \mapsto \frac{1}{2}(1 - g), \qquad (a_1, e) \mapsto xg\frac{1}{2}(1 + g), \qquad (a_2, e) \mapsto y\frac{1}{2}(1 + g),$$

$$(a_1, g) \mapsto xg\frac{1}{2}(1 - g), \qquad (a_2, g) \mapsto y\frac{1}{2}(1 - g).$$

It is straightforward to show that φ is an isomorphism of Hopf algebras, i.e.,

$$k\mathbb{Z}_2/\big(X^2, Y^2, XY + YX\big) \cong \mathbf{h}(-1, 1).$$

4.2. *Case 2: $W=(g,h)$ with $g \neq h$*

Fix the covering quiver $\Gamma_G(W)$. Using the standard notations of covering quivers, we can assume that $w_1 = g$ and $w_2 = h$. Just like in the case 1, we can assume that

$$g \cdot (a_i, e) = (a_i, g), \qquad h \cdot (a_i, e) = (a_i, h),$$
$$(a_i, e) \cdot g = q_i^{-1} g \cdot (a_i, e), \qquad (a_i, e) \cdot h = p_i^{-1} h \cdot (a_i, e)$$

for $i = 1, 2$ and $p_i^{\operatorname{ord}(h)} = q_i^{\operatorname{ord}(g)} = 1$. Abbreviate $v_{g^i h^j}$ as v_{ij} for simplicity. For two indeterminants x, y, define the function $e_{x,y} := \sum_{g^i h^j \in G} x^{-i} y^{-j} v_{ij}$. The proof of the following is identical to that of Lemmas 4.4 and 4.5, so we state it directly.

Lemma 4.11.

$$\Delta(X) = X \otimes 1 + e_{q_1,p_1} \otimes X, \qquad \Delta(Y) = Y \otimes 1 + e_{q_2,p_2} \otimes Y,$$
$$X e_{q_1,p_1} = q_1 e_{q_1,p_1} X, \qquad X e_{q_2,p_2} = q_2 e_{q_2,p_2} X.$$
$$Y e_{q_1,p_1} = p_1 e_{q_1,p_1} Y, \qquad Y e_{q_2,p_2} = p_2 e_{q_2,p_2} Y.$$

It is also easy to see that Lemmas 4.6 and 4.7 are still true in this case by using the same method.

Lemma 4.12. *$I_1(a)$, $I_3(n)$ and $I_4(m)$ are not Hopf ideals of $k\Gamma_G(W)$.*

It remains to determine when $I_2(m,a)$ is a Hopf ideal.

Lemma 4.13. *If $I_2(m,a)$ is a Hopf ideal, then $q_1 = p_2 = -1$ and $a = (-1)^{m-1} q_2^m = (-1)^{m-1} p_1^{-m}$.*

Proof. It follows by direct computations that

$$\Delta\big(X^2\big) = X^2 \otimes 1 + (1+q_1) e_{q_1,p_1} X \otimes X + e_{q_1^2,p_1^2} \otimes X^2$$

and

$$\Delta\big(Y^2\big) = Y^2 \otimes 1 + (1+p_2) e_{q_2,p_2} Y \otimes Y + e_{q_2^2,p_2^2} \otimes Y^2.$$

Thus $1 + q_1 = 0 = 1 + p_2$ and so $q_1 = p_2 = -1$.

Using the notation introduced before Lemma 4.7, we can see that

$$(XY)_X^m \otimes X \equiv e_{q_1,p_1} (YX)^{m-1} Y \otimes X \mod I_2(m,a) \otimes X$$

and

$$\begin{aligned}(YX)_X^m \otimes X &\equiv (YX)^{m-1} Y e_{q_1,p_1} \otimes X \\ &= p_1^m q_1^{m-1} e_{q_1,p_1} (YX)^{m-1} Y \otimes X \mod I_2(m,a) \otimes X.\end{aligned}$$

Similarly,

$$(XY)^m_Y \otimes Y \equiv (XY)^{m-1} X e_{q_2,p_2} \otimes Y$$
$$= q_2^m p_2^{m-1} e_{q_2,p_2} (XY)^{m-1} X \otimes Y \mod I_2(m,a) \otimes Y$$

and

$$(YX)^m_Y \otimes Y \equiv e_{q_2,p_2} (XY)^{m-1} X \otimes Y \mod I_2(m,a) \otimes X.$$

Thus $q_2^m p_2^{m-1} - a = 1 - a p_1^m q_1^{m-1} = 0$ which implies that $a = (-1)^{m-1} q_2^m = (-1)^{m-1} p_1^{-m}$. □

In the following, we need to use the functions defined at the beginning of Section 3.

Lemma 4.14. *Let* $0 < l < m$, *if* $q_1 = p_2 = -1$ *and* $a = (-1)^{m-1} q_2^m = (-1)^{m-1} p_1^{-m}$, *then*

(1)

$$(XY)^m_{(XY)^l} - a(YX)^m_{(XY)^l} \equiv H_2(m,l,p_1q_2) e_{(q_1q_2)^l,(p_1p_2)^l} (XY)^{m-l} + (-p_1)^{l-m} H_3(m,l,p_1q_2) e_{(q_1q_2)^l,(p_1p_2)^l} (YX)^{m-l} \mod I_2(m,a);$$

(2)

$$(XY)^m_{(YX)^l} - a(YX)^m_{(YX)^l} \equiv -a H_2(m,l,p_1q_2) e_{(q_1q_2)^l,(p_1p_2)^l} (YX)^{m-l} - a(-q_2)^{l-m} H_3(m,l,p_1q_2) e_{(q_1q_2)^l,(p_1p_2)^l} (XY)^{m-l} \mod I_2(m,a);$$

(3)

$$(XY)^m_{Y(XY)^l} - a(YX)^m_{Y(XY)^l} \equiv \left((-1)^{m-1} q_2^m - a\right) \sum_{0 \leqslant n_1+n_2+\cdots+n_l \leqslant m-l-1} (p_1q_2)^{\sum_{i=1}^l (l+1-i)n_i} \times e_{q_1^l q_2^{l+1}, p_1^l p_2^{l+1}} X(YX)^{m-l-1} \mod I_2(m,a);$$

(4)

$$(XY)^m_{(XY)^l X} - a(YX)^m_{(XY)^l X} \equiv \left(1 - a(-1)^{m-1} p_1^m\right) \sum_{0 \leqslant n_1+n_2+\cdots+n_l \leqslant m-l-1} (p_1q_2)^{\sum_{i=1}^l (l+1-i)n_i} \times e_{q_1^{l+1} q_2^l, p_1^{l+1} p_2^l} (YX)^{m-l-1} Y \mod I_2(m,a).$$

Proof. We only prove (1) because the others can be proved similarly. Since $X^2, Y^2 \in I_2(m,a)$, up to modulo $I_2(m,a)$, X and Y should appear alternately in the left items in $(XY)^m_{(XY)^l}$. Thus there are two possibilities: starting with X or with Y. By this observation, the items starting with X are just

$$\sum_{0 \leqslant n_1+n_2+\cdots+n_l \leqslant m-l} (XY)^{n_1} e_{q_1q_2,p_1p_2} (XY)^{n_2} e_{q_1q_2,p_1p_2} \cdots (XY)^{n_l} e_{q_1q_2,p_1p_2} (XY)^{n_{l+1}}.$$

By iterated application of Lemma 4.11, this item equals to

$$\sum_{0\leqslant n_1+n_2+\cdots+n_l\leqslant m-l}(p_1q_2)^{n_1}(p_1q_2)^{n_1+n_2}\cdots(p_1q_2)^{n_1+n_2+\cdots+n_l}e_{(q_1q_2)^l,(p_1p_2)^l}(XY)^{m-l}$$

and thus equals to

$$H_2(m,l,p_1q_2)e_{(q_1q_2)^l,(p_1p_2)^l}(XY)^{m-l}.$$

Similarly, the items starting with Y are just

$$\sum e_{q_1,p_1}(YX)^{n_1}e_{q_1q_2,p_1p_2}(YX)^{n_2}e_{q_1q_2,p_1p_2}\cdots(YX)^{n_{l-1}}e_{q_1q_2,p_1p_2}(YX)^{n_l}e_{q_2,p_2}$$

which equals to

$$(q_2p_2)^{m-l}\sum_{n_1+n_2+\cdots+n_{l-1}\leqslant m-l}(p_1q_2)^{n_1}(p_1q_2)^{n_1+n_2}\cdots(p_1q_2)^{n_1+n_2+\cdots+n_{l-1}}e_{(q_1q_2)^l,(p_1p_2)^l}(XY)^{m-l}.$$

Meanwhile, all items in $(YX)^m_{(XY)^l}$ start from Y:

$$Y\sum(XY)^{n_1}e_{q_1q_2,p_1p_2}(XY)^{n_2}e_{q_1q_2,p_1p_2}\cdots(XY)^{n_l}e_{q_1q_2,p_1p_2}(XY)^{n_{l+1}}X$$

which equals to

$$(p_1p_2)^l\sum_{0\leqslant n_1+n_2+\cdots+n_l\leqslant m-l-1}(p_1q_2)^{n_1}(p_1q_2)^{n_1+n_2}\cdots(p_1q_2)^{n_1+n_2+\cdots+n_l}e_{(q_1q_2)^l,(p_1p_2)^l}(YX)^{m-l}.$$

Note that $q_1=p_2=-1$ and $a=(-1)^{m-1}p_1^{-m}$,

$$\begin{aligned}&(q_2p_2)^{m-l}\sum_{n_1+n_2+\cdots+n_{l-1}\leqslant m-l}(p_1q_2)^{n_1}(p_1q_2)^{n_1+n_2}\cdots(p_1q_2)^{n_1+n_2+\cdots+n_{l-1}}\\&\quad=(-q_2)^{m-l}\sum_{n_1+n_2+\cdots+n_{l-1}\leqslant m-l}(p_1q_2)^{\sum_{i=1}^{l-1}(l-i)n_i}\end{aligned}\qquad(\star)$$

and

$$\begin{aligned}&-a(p_1p_2)^l\sum_{0\leqslant n_1+n_2+\cdots+n_l\leqslant m-l-1}(p_1q_2)^{n_1}(p_1q_2)^{n_1+n_2}\cdots(p_1q_2)^{n_1+n_2+\cdots+n_l}\\&\quad=(-p_1)^{l-m}\sum_{n_1+n_2+\cdots+n_l\leqslant m-l-1}(p_1q_2)^{\sum_{i=1}^{l}(l+1-i)n_i}.\end{aligned}\qquad(*)$$

By the definition of $H_3(m,l,t)$, we see that

$$(\star)-(*)=(-p_1)^{l-m}H_3(m,l,p_1q_2)$$

and (1) is proved. □

Proposition 4.15. *$I_2(m,a)$ is a Hopf ideal if and only if*

(1) $q_1 = p_2 = -1$ *and* $a = (-1)^{m-1}q_2^m = (-1)^{m-1}p_1^{-m}$;
(2) p_1q_2 *is an mth primitive root of unity.*

Proof. "$\Longrightarrow$" (1) is just Lemma 4.13. By Lemma 4.14, $H_2(m,l,p_1q_2) = 0$ for all $0 < l < m$. Lemma 3.1 and Proposition 3.2 give us the desired conclusion.

"$\Longleftarrow$" Using Lemma 3.1 and Proposition 3.2 again, $H_2(m,l,p_1q_2) = H_3(m,l,p_1q_2) = 0$. Then Lemmas 4.13 and 4.14 imply

$$\Delta\big(I_2(m,a)\big) \subset I_2(m,a) \otimes k\Gamma_G(W) + k\Gamma_G(W) \otimes I_2(m,a).$$

By almost the same proof as in Lemma 4.8, we can show that

$$\varepsilon\big(I_2(m,a)\big) = 0,\ S\big(I_2(m,a)\big) \subset I_2(m,a). \qquad \square$$

Theorem 4.16. *Let H be a connected tame graded basic Hopf algebra and $\Gamma_G(W)$ its Gabriel quiver. If $W = (g,h)$ with $g \neq h$, then under the assumption before Lemma* 4.11*,*

$$H \cong k\Gamma_G(W)/\big(X^2, Y^2, (XY)^m - (-1)^{m-1}q_2^m(YX)^m\big)$$

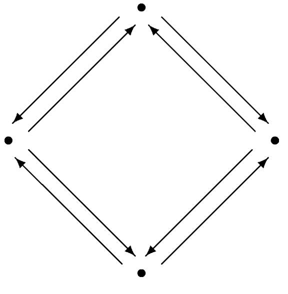
as Hopf algebras for some $m > 0$.

Remark 4.17. (1) Lemma 4.8 and Proposition 4.15 indeed give us the method to construct all possible connected tame graded basic Hopf algebras and thus give us some new examples of finite quantum groups.

(2) If $I_2(m,a)$ is Hopf ideal of $k\Gamma_G(W)$ for some m, then m is factor of $l.c.m(\mathrm{ord}(g), \mathrm{ord}(h))$, i.e., $m \mid l.c.m(\mathrm{ord}(g), \mathrm{ord}(h))$. The reason is that $(p_1q_2)^{l.c.m(\mathrm{ord}(g),\mathrm{ord}(h))} = 1$ and p_1q_2 is an mth primitive root of unity. Conversely, assume that G is an abelian group generated by g, h $(g \neq h)$ with $m \mid l.c.m(\mathrm{ord}(g), \mathrm{ord}(h))$. Define the allowable kG-bimodule on $k\Gamma_G((g,h))$ through the way given at the beginning of this subsection. Let $q_1 = p_2 = -1$. By a suitable choice of p_1, q_2, we can assume that p_1q_2 is an mth primitive root of unity. Now $k\Gamma_G(W)/(X^2, Y^2, (XY)^m - (-1)^{m-1}q_2^m(YX)^m)$ is an Hopf algebra. Notice that this is also give the answer to Problem 1.1 in this case.

At the end of this subsection, we recall a familiar example.

Example 4.2 *(Tensor products of Taft algebras).* Let $T_{n^2}(q), T_{m^2}(q')$ be two Taft algebras. It is known that $T_{n^2}(q) \otimes_k T_{m^2}(q')$ is tame if and only if $m = n = 2$ (see Example 5.1 in [24]). Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle g, h \mid g^2 = h^2 = 1,\ gh = hg\rangle$ and the covering quiver $\Gamma_G((g,h))$ is the following graph:



Through

$$(a_1,e)\cdot g=-g\cdot(a_1,e),\qquad (a_2,e)\cdot g=g\cdot(a_2,e),$$
$$(a_1,e)\cdot h=h\cdot(a_1,e),\qquad (a_2,e)\cdot h=-h\cdot(a_2,e),$$

$k\Gamma_G((g,h))$ is a Hopf algebra. Define $\varphi : k\Gamma_G((g,h))/(X^2,Y^2,XY-YX)\to T_{2^2}(-1)\otimes T_{2^2}(-1)$ by

$$\begin{aligned}
v_e &\mapsto \frac{1}{2}(1+g)\frac{1}{2}(1+h), & v_g &\mapsto \frac{1}{2}(1-g)\frac{1}{2}(1+h),\\
v_h &\mapsto \frac{1}{2}(1+g)\frac{1}{2}(1-h), & v_{gh} &\mapsto \frac{1}{2}(1-g)\frac{1}{2}(1-h),\\
(a_1,e) &\mapsto xg\frac{1}{2}(1+g)\frac{1}{2}(1+h), & (a_2,e) &\mapsto yh\frac{1}{2}(1+g)\frac{1}{2}(1+h),\\
(a_1,g) &\mapsto xg\frac{1}{2}(1-g)\frac{1}{2}(1+h), & (a_2,g) &\mapsto yh\frac{1}{2}(1-g)\frac{1}{2}(1+h),\\
(a_1,h) &\mapsto xg\frac{1}{2}(1+g)\frac{1}{2}(1-h), & (a_2,h) &\mapsto yh\frac{1}{2}(1+g)\frac{1}{2}(1-h),\\
(a_1,gh) &\mapsto xg\frac{1}{2}(1-g)\frac{1}{2}(1-h), & (a_2,gh) &\mapsto yh\frac{1}{2}(1-g)\frac{1}{2}(1-h).
\end{aligned}$$

It is straightforward to show that φ induces an isomorphism of Hopf algebras, i.e.,

$$T_{2^2}(-1)\otimes T_{2^2}(-1)\cong k\Gamma_G\big((g,h)\big)/\big(X^2,Y^2,XY-YX\big)$$

as Hopf algebras.

5. Classification-general case

Let H be a graded basic Hopf algebra and $\Gamma_G(W)$ its covering quiver. Let $N\subset G$ be the subgroup generated by W. It is known that k is an H-module through the counit map $\varepsilon : H\to k$. We say a block of H is the *principle block* if k, as a simple H-module, belongs to this block. We denote this block by H_0.

Proposition 5.1.

(1) *N is a normal subgroup of G;*
(2) *As an algebra,*

$$H\cong H_0\oplus H_0\oplus\cdots\oplus H_0$$

for $|G/N|$ copies of H_0;
(3) *H_0 is a Hopf algebra and is a Hopf quotient of H.*

Proof. (1) is obvious since W is stable under the conjugation.

Now we prove (2). By the Gabriel theorem for Hopf algebras (Lemma 2.5), there is a Hopf algebra isomorphism

$$k\Gamma_G(W)/I\cong H$$

with I an admissible ideal of $k\Gamma_G(W)$. By the proof of Lemma 2.5, the Hopf structure on $k\Gamma_G(W)$ is given by an allowable kG-bimodule. Denote the connected component of $\Gamma_G(W)$ containing v_e by $\Gamma_G(W)^\circ$. By the definition of covering quivers, every connected component of $\Gamma_G(W)$ is indeed

$g \cdot \Gamma_G(W)^\circ$ for some $g \in G$. It is easy to see that $g \cdot \Gamma_G(W)^\circ = \Gamma_G(W)^\circ$ if and only if $g \in N$. Thus there are $|G/N|$ numbers of connected components.

Let $I^\circ := k\Gamma_G(W)^\circ \cap I$ and thus $H_0 \cong k\Gamma_G(W)^\circ/I^\circ$. Using Lemma 2.6, I is stable under G-action. By the definition of allowable kG-bimodule, $k(g \cdot \Gamma_G(W)^\circ) \cap I$ is exactly $g \cdot I^\circ$. This fact implies any block of H must equal to $g \cdot H_0$ and thus is isomorphic to H_0. (2) is proved.

At last, let us prove (3). For $h \in G$, it is known that $\Delta(v_h) = \sum_{g\in G} v_g \otimes v_{g^{-1}h}$. This implies $\sum_{g\notin N} kv_g$ generates a Hopf ideal of $k\Gamma_G(W)$. Thus

$$H_0 \cong k\Gamma_G(W)/\Bigl(I, \sum_{g\notin N} kv_g\Bigr)$$

is a Hopf algebra which clearly is a Hopf quotient of H. □

The structure of tame graded basic Hopf algebras can be determined now. For a Hopf algebra H, let H^* denote its dual.

Theorem 5.2. *Let H be a tame graded basic Hopf algebra and $\Gamma_G(W)$ its Gabriel quiver. Denote by H_0 the principle block of H and $\Gamma_G(W)^\circ$ the connected component of $\Gamma_G(W)$ containing v_e. Let $N \subset G$ be the subgroup generated by W.*

(1) *If $W = (g, g)$ for some $g \in G$, then as an algebra,*

$$H \cong H_0 \oplus H_0 \oplus \cdots \oplus H_0$$

for $|G/N|$ copies of H_0 and

$$H \cong \bigl(k(G/N)\bigr)^* \#_\sigma \bigl(k\Gamma_G(W)^\circ/\bigl(X^2, Y^2, XY + YX\bigr)\bigr)$$

as Hopf algebras where $X = \sum_{t\in N} t \cdot (a_1, e)$ and $Y = \sum_{t\in N} t \cdot (a_2, e)$.

(2) *With the notations given in Section 4.2. If $W = (g, h)$ for some $g, h \in G$ and $g \neq h$, then as an algebra,*

$$H \cong H_0 \oplus H_0 \oplus \cdots \oplus H_0$$

for $|G/N|$ copies of H_0 and

$$H \cong \bigl(k(G/N)\bigr)^* \#_\sigma \bigl(k\Gamma_G(W)^\circ/\bigl(X^2, Y^2, (XY)^m + (-q_2)^m (YX)^m\bigr)\bigr)$$

as Hopf algebras for some $m \in \mathbb{N}$ and $q_2 \in k$ where $X = \sum_{h\in N} h \cdot (a_1, e)$ and $Y = \sum_{h\in N} h \cdot (a_2, e)$.

Proof. Proposition 5.1 tells us that we have a Hopf epimorphism

$$\pi : H \to H_0.$$

By a result of Schneider [30],

$$H \cong H^{co\pi} \#_\sigma H_0$$

where $H^{co\pi} = \{a \in H \mid (id \otimes \pi)\Delta(a) = a \otimes 1\}$. It is not hard to see that $H^{co\pi} = (k(G/N))^*$. Now the theorem follows directly by Proposition 5.1, Theorem 4.9 and Theorem 4.16. □

Remark 5.3. (1) We can answer Problem 1.1 now. By this theorem, only some special ideals of $\{(x^2, y^2, (xy)^m - a(yx)^m) \mid 0 \neq a \in k, m \geqslant 1\}$ can appear and if one of them appears, then G is necessary and sufficient to contain a normal subgroup N satisfying the conditions given in Remark 4.10 or Remark 4.17(2).

(2) Almost all of computations of this paper are based on a basic and simple observation, that is, the action of $kG \otimes (kG)^{\mathrm{op}}$ is diagonalizable (see paragraphs before Lemma 4.4 and Lemma 4.11) when G is a finite abelian group. This is a direct consequence of the assumption that k is an algebraically closed field of characteristic zero. Of course, if the characteristic of k is big enough to make kG to be semisimple, then our main results can also be established. Through developing a suitable lifting method (see [1,2] for lifting of pointed Hopf algebras), it is hopeful to get the classification of all tame basic Hopf algebras over an algebraically closed field k of characteristic zero at last. In general, the classification of tame basic Hopf algebras (even radically graded) over an algebraically closed field of *positive* characteristic is still an open and interesting question.

(3) It is known that finite-dimensional Hopf algebras are Frobenius algebras and of course they are selfinjective. The classification of selfinjective algebras according to their representation type over an algebraically closed field has been researched for a long time. For the current stage of this subject, see the survey article [32]. The same question for tensor product algebras, which are essential for Hopf algebras, has also been investigated. In particular, all tame tensor product algebras of non-trivial basic algebras over an algebraically closed field are completely described [23].

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References

[1] N. Andruskiewitsch, H.-J. Schneider, Pointed Hopf algebras, in: New Direction in Hopf Algebras, in: Math. Sci. Res. Inst. Publ., vol. 43, Cambridge University Press, Cambridge, 2002, pp. 1–68.
[2] N. Andruskiewitsch, H.-J. Schneider, On the classification of finite-dimensional pointed Hopf algebras, Ann. of Math., in press.
[3] I. Assem, D. Simson, A. Skowronski, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Math. Soc. Stud. Texts, vol. 65, Cambridge University Press, 2006.
[4] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Stud. Adv. Math., vol. 36, Cambridge University Press, 1995.
[5] D. Benson, Representation and Cohomology, I, Cambridge Stud. Adv. Math., vol. 30, Cambridge University Press, 1991.
[6] V. Bondarenko, Y. Drozd, Representation type of finite groups, J. Soviet Math. 20 (1982) 2515–2528.
[7] C. Cibils, Half-quantum groups at roots of unity, path algebras, and representation type, Int. Math. Res. Not. IMRN 12 (1997) 541–553.
[8] Xiao-Wu Chen, Hua-Lin Huang, Yu Ye, Pu Zhang, Monomial Hopf algebras, J. Algebra 275 (2004) 212–232.
[9] W.W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. 63 (1991) 241–264.
[10] P. Dowbor, A. Skowronski, On the representation type of locally bounded categories, Tsukuba J. Math. 10 (1) (1986) 63–72.
[11] Y. Drozd, Tame and wild matrix problems, in: Representations and Quadratic Forms, Inst. Math. Acad. Sciences, Ukrainian SSR, Kiev, 1979, pp. 39–74; Amer. Math. Soc. Transl. 128 (1986) 31–55.
[12] K. Erdmann, Blocks of Tame Representation Type and Related Algebras, Lecture Notes in Math., vol. 1428, Springer-Verlag, 1990.
[13] R. Farnsteiner, Auslander–Reiten components for Lie algebras of reductive groups, Adv. Math. 155 (2000) 49–83.
[14] R. Farnsteiner, On the Auslander–Reiten quiver of an infinitesimal group, Nagoya Math. J. 160 (2000) 103–121.
[15] R. Farnsteiner, Polyhedral groups, Mckey quivers and the finite algebraic groups with tame principal blocks, Invent. Math. 166 (2006) 27–94.
[16] R. Farnsteiner, A. Skowronski, Classification of restricted Lie algebras with tame principal block, J. Reine Angew. Math. 546 (2002) 1–45.
[17] R. Farnsteiner, A. Skowronski, Galois actions and blocks of tame infinitesimal group schemes, Trans. Amer. Math. Soc. 359 (12) (2007) 5867–5898.
[18] R. Farnsteiner, D. Voigt, On cocommutative Hopf algebras of finite representation type, Adv. Math. 155 (2000) 1–22.
[19] R. Farnsteiner, D. Voigt, On infinitesimal groups of tame representation type, Math. Z. 244 (2003) 479–513.
[20] P. Gabriel, L.A. Nazarova, A.V. Roiter, V.V. Sergeichuk, D. Vossieck, Tame and wild subspace problems, Ukrainian Math. J. 45 (3) (1993) 313–352, Amer. Math. Soc. Transl. 45 (3) (1993) 335–372.
[21] E. Green, Ø. Solberg, Basic Hopf algebras and quantum groups, Math. Z. 229 (1998) 45–76.
[22] D. Higman, Indecomposable representation at characteristic p, Duke Math. J. 21 (1954) 377–381.

[23] Z. Leszczynski, A. Skowronski, Tame tensor products of algebras, Colloq. Math. 98 (1) (2003) 125–145.
[24] G.X. Liu, On the structure of tame basic Hopf algebras, J. Algebra 299 (2006) 841–853.
[25] G.X. Liu, F. Li, Pointed Hopf algebras of finite corepresentation type and their classification, Proc. Amer. Math. Soc. 135 (3) (2007) 649–657.
[26] S. Majid, Crossed products by braided groups and bosonization, J. Algebra 163 (1994) 165–190.
[27] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Reg. Conf. Ser. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.
[28] F. van Oystaeyen, P. Zhang, Quiver Hopf algebras, J. Algebra 280 (2004) 577–589.
[29] D. Radford, The structure of Hopf algebras with a projection, J. Algebra 92 (1985) 322–347.
1
[30] H.-J. Schneider, Normal basis and transitivity of crossed products for Hopf algebras, J. Algebra 1152 (1992) 289–312.
[31] D. Simson, A. Skowroski, Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras, London Math. Soc. Stud. Texts, vol. 72, Cambridge University Press, 2007.
[32] A. Skowronski, Selfinjective algebras: Finite and tame type, in: Trends in Representation Theory of Algebras and Related Topics, in: Contemp. Math., vol. 406, Amer. Math. Soc., 2006, pp. 169–238.
[33] R. Suter, Modules over $U_q(\mathfrak{sl}_2)$, Comm. Math. Phys. 163 (1994) 359–393.
[34] J. Xiao, Finite-dimensional representations of $U_t(\mathfrak{sl}_2)$ at roots of unity, Canad. J. Math. 49 (4) (1997) 772–787.