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On total Frobenius-Schur indicators

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Abstract. We define total Frobenius-Schur indicator for each object in a spherical fusion category $\mathcal{C}$ as a certain canonical sum of its higher indicators. The total indicators are invariants of spherical fusion categories. If $\mathcal{C}$ is the representation category of a semisimple quasi-Hopf algebra H, we prove that the total indicators are non-negative integers which satisfy a certain divisibility condition. In addition, if H is a Hopf algebra, then all the total indicators are positive. Consequently, the positivity of total indicators is a necessary condition for a quasi-Hopf algebra being gauge equivalent to a Hopf algebra. Certain twisted quantum doubles of finite groups and some examples of Tambara-Yamagami categories are discussed for the sufficiency of this positivity condition.

1. Introduction

The representation category $\text{Rep}(H)$ of a Hopf algebra H is certainly important to the understanding of the algebraic structure of H. The monoidal structure of $\text{Rep}(H)$ has also been playing important roles in other areas of mathematics and physics. For instance, the quantum invariants of knots, links or 3-manifolds constructed from certain Hopf algebras are actually determined by the monoidal structures of their representation categories [**34**].

Quasi-Hopf algebras are generalizations of Hopf algebras whose representation categories are also monoidal categories. Two quasi-Hopf algebras are said to be *gauge equivalent* if their representation categories are monoidally equivalent. For any finite-dimensional quasi-Hopf algebra K over $\mathbb{C}$, one can obtain another quasi-Hopf algebra K^F by twisting K with a gauge transformation $F \in K \otimes K$ [**15**], but K^F and K are gauge equivalent. In general, two finite-dimensional quasi-Hopf algebras K and H are gauge equivalent if, and only if, there exists a gauge transformation $F \in K \otimes K$ such that K^F and H are isomorphic as quasi-bialgebras (cf. [**7**, **26**]). However, it is generally difficult to decide the gauge equivalence of two finite-dimensional quasi-Hopf algebras if their Grothendieck rings happen to be isomorphic.

For any finite group G, and a normalized 3-cocycle ω on G with values in $\mathbb{C}^\times$, one can construct a semisimple quasi-Hopf algebra $D^\omega(G)$, called a *twisted quantum double* of G [**5**]. Dijkgraaf, Pasquier and Roche have asked the question

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whether $D^{\omega}(G)$ is gauge equivalent to some Hopf algebra in [**5**]. One can even ask a more general question: How can one determine whether a given finite-dimensional quasi-Hopf algebra K over $\mathbb{C}$ is gauge equivalent to some Hopf algebra? In view of the reconstruction theorem, the question is equivalent to ask for the existence of a *fibration*, i.e. a $\mathbb{C}$-linear, faithful and exact monoidal functor $\mathcal{F} : \mathrm{Rep}(K) \to \mathrm{Vect}_{\mathbb{C}}$, where $\mathrm{Vect}_{\mathbb{C}}$ denotes the category of finite-dimensional $\mathbb{C}$-linear spaces.

This simply stated question is generally difficult to answer. One can work out the simplest example $D^{\omega}(\mathbb{Z}_2)$, where ω is a non-trivial 3-cocycle of $\mathbb{Z}_2$. It is not completely obvious that $D^{\omega}(\mathbb{Z}_2)$ is not gauge equivalent to any Hopf algebra [**24**].

The n-th Frobenius-Schur (FS) indicator $\nu_n(V)$ of a representation V of a finite group was introduced for more than a century. It has been generalized to the representations of a semisimple (quasi-)Hopf algebra [**17**, **25**], to the primary fields of a rational conformal field theory [**3**], and more generally to the objects of a pivotal categories [**28**]. These indicators are preserved by equivalences of pivotal category [loc. cit.]. The arithmetical properties of the FS indicators also encrypt the structure of the underlying tensor categories as well as the quasi-Hopf algebras. For instance, the indicators of any complex representation of a finite group are integers.

If V is an object of a spherical fusion category $\mathcal{C}$ over $\mathbb{C}$, it has been shown that the sequence $\nu(V) := \{\nu_n(V)\}_{n\in\mathbb{N}}$ of FS indicators is a periodic sequence of cyclotomic integers. Moreover, there exists a global period N for all the higher indicator sequences [**27**]. This global period, denoted by $\mathrm{FSexp}(\mathcal{C})$, is called the Frobenius-Schur exponent of $\mathcal{C}$, and it is also an invariant of spherical fusion categories. The representation category $\mathrm{Rep}(H)$ of a semisimple quasi-Hopf algebra over $\mathbb{C}$ is a spherical fusion category [**8**], and we simply denote the Frobenius-Schur exponent of $\mathrm{Rep}(H)$ by $\mathrm{FSexp}(H)$.

The FS indicators of a representation of a semisimple Hopf algebra are not necessarily integers (see [**14**] for example). The integrality of indicators even fails for the quantum doubles of finite groups. Iovanov, Mason and Montgomery recently provided such an example of $D(G)$ for some finite group G of order 5^6 in [**10**]. However, if we consider the total indicator $\overline{\nu}(V)$ defined for any object V in a spherical fusion category $\mathcal{C}$ by

$$\overline{\nu}(V) := \sum_{n=1}^{N} \nu_n(V)$$

where $N = \mathrm{FSexp}(\mathcal{C})$, then we have the following integrality and divisibility theorem for quasi-Hopf algebras.

Theorem A. *Let H be a semisimple quasi-Hopf algebra over $\mathbb{C}$. For any finite-dimensional H-module V, $\overline{\nu}(V)$ is a non-negative integer which satisfies the divisibility*

$$\mathrm{FSexp}(H) \mid (\dim H) \cdot \overline{\nu}(V)\,.$$

In addition, for semisimple Hopf algebras, we have obtained the positivity of total indicators:

Theorem B. *Let H be a semisimple Hopf algebra $\mathbb{C}$. Then, $\overline{\nu}(V) \geq \frac{N \dim V}{\dim H}$ for all finite-dimensional H-modules V, where $N = \mathrm{FSexp}(H)$. In particular, $\overline{\nu}(V) > 0$ for any non-zero H-module V.*

Since $\mathrm{FSexp}(H) = \exp(H)$ when H is a semisimple Hopf algebra, Theorem A provides another perspective for a conjecture of Kashina [**12**, **13**]: $\exp(H)$ divides $\dim H$ for any semisimple Hopf algebra H over $\mathbb{C}$. Moreover, Theorem B yields a necessary condition for a semisimple quasi-Hopf algebra being gauge equivalent to a Hopf algebra. However, this necessary condition may not be sufficient in general. There are integral Tambara-Yamagami categories which has no fibration but their total indicators are all positive. Nevertheless, the positivity of total indicators is a necessary and sufficient for abelian twisted quantum doubles being gauge equivalent to Hopf algebras.

The organization of the paper is as follows: In Section 2, we introduce the definition of total Frobenius-Schur indicators and prove Theorem A. Section 3 is devoted to the proof of Theorem B. In Section 4, we consider twisted quantum doubles of $D^\omega(G)$ of a finite abelian group G, and show that if $D^\omega(G)$ is commutative, then $D^\omega(G)$ is gauge equivalent to a Hopf algebra if, and only if, $\bar{\nu}(V) > 0$ for all $V \in \mathrm{Rep}(D^\omega(G))$. This provides an answer to a question of Dijkgraaf, Pasquier and Roche [**5**, p69]. In Section 5, we compute the total indicators for some integral Tambara-Yamagami fusion categories, and we characterize those admitting a fibration in terms of total indicators. As a consequence, semisimple quasi-Hopf algebras with positive total indicators but not gauge equivalent to any Hopf algebra are found.

2. Total Frobenius-Schur indicators

Throughout this paper, unless stated otherwise, we will work over the field $\mathbb{C}$ of complex numbers; every monoidal category $\mathcal{C}$ in this paper is assumed to be $\mathbb{C}$-linear abelian with finite-dimensional Hom-spaces over $\mathbb{C}$ and a *strict* simple unit object $\mathbf{1}_\mathcal{C}$. All (quasi-)Hopf algebras are assumed to be semisimple and finite-dimensional over $\mathbb{C}$. We denote by $\mathrm{Rep}(H)$ the $\mathbb{C}$-linear monoidal category of finite-dimensional representations of a quasi-Hopf algebra H. The unit object of $\mathrm{Rep}(H)$, simply denoted by $\mathbf{1}_H$, is the H-module $\mathbb{C}$ equipped with the trivial H-action.

In this section, we collect some conventions, and recall some basic definitions and facts for the discussions in the remaining sections. The readers may refer to [**2**, **8**, **22**] for the basic theory of tensor categories and [**14**, **17**, **26**, **28**] for Frobenius-Schur indicators. The aim of this section is to introduce the definition of total Frobenius-Schur indicators (abbr. total indicators), and to prove Theorem A.

Let $\mathcal{C}$ be a left rigid monoidal category with tensor product $\otimes$. The left dual of $V \in \mathcal{C}$ is a triple $(V^*, \mathrm{db}, \mathrm{ev})$ in which $V^* \in \mathcal{C}$, and $\mathrm{db} : \mathbf{1} \to V \otimes V^*$ and $\mathrm{ev} : V^* \otimes V \to \mathbf{1}$ are respectively the associated dual basis and evaluation morphisms of the left dual. The left duality on $\mathcal{C}$ can be extended to a monoidal equivalence $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and hence $(-)^{**} : \mathcal{C} \to \mathcal{C}$ defines a monoidal equivalence. A pivotal structure on $\mathcal{C}$ is an isomorphism $j : \mathrm{Id} \to (-)^{**}$ of monoidal functors, and the pair $(\mathcal{C}, j)$ is a called a *pivotal category*. We will simply say that $\mathcal{C}$ is a pivotal category when the pivotal structure is understood without ambiguity.

In a pivotal category $(\mathcal{C}, j)$, one can define the left and right pivotal traces for any endomorphism $f : V \to V$ in $\mathcal{C}$ as

$$\underline{\mathrm{ptr}}^r(f) := \left(\mathbf{1} \xrightarrow{\mathrm{db}} V \otimes V^* \xrightarrow{f \otimes V^*} V \otimes V^* \xrightarrow{j_V \otimes V^*} V^{**} \otimes V^* \xrightarrow{\mathrm{ev}} \mathbf{1}\right),$$

$$\underline{\mathrm{ptr}}^\ell(f) := \left(\mathbf{1} \xrightarrow{\mathrm{db}} V^* \otimes V^{**} \xrightarrow{V^* \otimes j_V^{-1}} V^* \otimes V \xrightarrow{V^* \otimes f} V^* \otimes V \xrightarrow{\mathrm{ev}} \mathbf{1}\right)$$

respectively. Note that these traces are scalars as $\mathbf{1}$ is simple. A *spherical category* is a pivotal category $\mathcal{C}$ in which $\underline{\mathrm{ptr}}^r(f) = \underline{\mathrm{ptr}}^\ell(f)$ for all endomorphisms $f \in \mathcal{C}$. In this case, we simply write $\underline{\mathrm{ptr}}$ for the functions $\underline{\mathrm{ptr}}^r$ as well as $\underline{\mathrm{ptr}}^\ell$, and $d(V) := \underline{\mathrm{ptr}}(\mathrm{id}_V)$ is called the *pivotal dimension* of V. In addition, if $\mathcal{C}$ is semisimple with finitely many simple objects up to isomorphism, then $\mathcal{C}$ is called a *spherical fusion category* (cf. [**8**] for more details on fusion categories). In this case, $d(V)$ is a non-zero real number (cf. [**8**]), and the *global dimension* $\dim \mathcal{C}$ of $\mathcal{C}$ is defined as

$$\dim \mathcal{C} = \sum_{V \in \mathrm{Irr}(\mathcal{C})} d(V)^2$$

where $\mathrm{Irr}(\mathcal{C})$ denotes a complete set of non-isomorphic simple objects of $\mathcal{C}$.

By Müger [**23**], the center $\mathcal{Z}(\mathcal{C})$ is a *modular tensor category*. In particular, the associated *twist* (or *ribbon structure*) θ has finite order [**2**, **35**]. Moreover, the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ admits a two-sided adjoint $K : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$.

Let V be an object in a pivotal category $(\mathcal{C}, j)$ with associativity isomorphism Φ. We write $V^{\otimes n}$ for the n-fold tensor power of $V \in \mathcal{C}$ with rightmost parentheses. By coherence theorem, there is a unique isomorphism

$$\Phi^{(n)} : V^{\otimes(n-1)} \otimes V \to V^{\otimes n}$$

which is a composition of tensor products of id, Φ and Φ^{-1}. One can define the $\mathbb{C}$-linear operator $E_V^{(n)} : \mathcal{C}(\mathbf{1}, V^{\otimes n}) \to \mathcal{C}(\mathbf{1}, V^{\otimes n})$ by setting

$$E_V^{(n)}(f) := \Bigg(\mathbf{1} \xrightarrow{\mathrm{db}} V^* \otimes V^{**} \xrightarrow{(V^*\otimes f)\otimes V^{**}} (V^* \otimes V^{\otimes n}) \otimes V^{**} \xrightarrow{\Phi^{-1}\otimes j^{-1}} \\ ((V^* \otimes V) \otimes V^{\otimes(n-1)}) \otimes V \xrightarrow{(\mathrm{ev}\otimes V^{\otimes(n-1)})\otimes V} V^{\otimes(n-1)} \otimes V \xrightarrow{\Phi^{(n)}} V^{\otimes n}\Bigg).$$

Following [**28**, Sect. 3], the *n-th Frobenius-Schur indicator* $\nu_n(V)$ of V is defined as the scalar

$$\nu_n(V) = \mathrm{Tr}(E_V^{(n)}).$$

These indicators are proved to be invariant under pivotal equivalences, and $\nu_n(V)$ is a cyclotomic integer in $\mathbb{Q}(\zeta_n)$ where ζ_n is a primitive n-th root of unity (cf. [**28**]).

Since the antipode of a semisimple Hopf algebra H is an involution [**16**], the representation category $\mathrm{Rep}(H)$ is a spherical fusion category in which the pivotal structure is the usual canonical isomorphism $j_V : V \to V^{**}$ of finite-dimensional vector spaces. In this case, the pivotal dimension $d(V)$ of $V \in \mathrm{Rep}(H)$ is the ordinary dimension of V. More generally, for a semisimple quasi-Hopf algebra H, there is a unique (spherical) pivotal structure on $\mathrm{Rep}(H)$ such that $d(V)$ is the ordinary dimension of V for all $V \in \mathrm{Rep}(H)$ [**8**]. Moreover, $\nu_n(V)$ can be expressed in terms of the associator, the quasi-antipode and the normalized integral Λ of H (cf. [**26**, Sect. 4]). When H is a Hopf algebra, we recover the n-th Frobenius-Schur indicator formula of V introduced in [**17**]:

$$\nu_n(V) = \chi_V(\Lambda^{[n]}) \tag{2.1}$$

where χ_V is the character of H afforded by V, Λ is the normalized integral of H and $\Lambda^{[n]} = \Lambda_1\Lambda_2\cdots\Lambda_n$. Here, $x_1 \otimes \cdots \otimes x_n$ denotes the Sweedler notation of the n-fold comultiplication of $x \in H$.

The *Frobenius-Schur exponent*, denoted by $\mathrm{FSexp}(\mathcal{C})$, of a spherical category $\mathcal{C}$ (cf. [**27**]) is the least positive integer n such that $\nu_n(V) = d(V)$ for all $V \in \mathcal{C}$. If such an integer does not exist, $\mathrm{FSexp}(\mathcal{C})$ is defined to be ∞. However, the Frobenius-Schur exponent of a spherical fusion category is always finite because the following theorem proved in [**27**, Thm. 4.1 and 5.5].

THEOREM 2.1. *Let $\mathcal{C}$ be a spherical fusion category, θ the twist of $\mathcal{Z}(\mathcal{C})$ and $K : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ the two-sided adjoint of the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Then, for $V \in \mathcal{C}$,*

(i) $\nu_n(V) = \dfrac{1}{\dim \mathcal{C}} \underline{\mathrm{ptr}}\left(\theta_{K(V)}\right)$ *for all $V \in \mathcal{C}$, and*

(ii) $\mathrm{FSexp}(\mathcal{C}) = \mathrm{ord}(\theta)$. □

For a simple object $V \in \mathcal{C}$, it is known that $\nu_1(V) = 1$ if $V \cong \mathbf{1}$, and 0 otherwise. The second indicator of V can only be $0, 1, -1$ depending on whether V is self-dual or not. By Theorem 2.1, we know $\nu_N(V) = d(V)$ if $N = \mathrm{FSexp}(\mathcal{C})$. The meaning of higher indicators of V are more obscure and they are not rational integers in general (cf. [**14**, Ex. 7.5]). The theorem implies that the indicator sequence $\nu(V) := \{\nu_n(V)\}_{n\in\mathbb{N}}$ of V is periodic for any object V of a spherical fusion category $\mathcal{C}$. Moreover, $\mathrm{FSexp}(\mathcal{C})$ is the global period of all the indicator sequences of $\mathcal{C}$. The average value or the sum of these indicators over a period should also be an important invariant.

DEFINITION 2.2. *Let $\mathcal{C}$ be a spherical fusion category and $N = \mathrm{FSexp}(\mathcal{C})$. The total Frobenius-Schur indicator of $V \in \mathcal{C}$, denoted by $\bar{\nu}(V)$, is defined as*

$$\bar{\nu}(V) := \sum_{i=1}^{N} \nu_i(V)\,.$$

To prove Theorem A, we first derive a formula for the total indicator $\bar{\nu}(V)$ of an object V in a spherical fusion category $\mathcal{C}$ in terms of some data of the center $\mathcal{Z}(\mathcal{C})$. Recall from [**23**, Proposition 8.1] that the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ has a two-sided adjoint $K : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. It follows from the semisimplicity of $\mathcal{C}$, we have

$$K(V) \cong \sum_{X\in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))} \dim(\mathcal{Z}(\mathcal{C})(K(V), X))\, X$$

for $V \in \mathcal{C}$, where $\mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$ is a complete set of non-isomorphic simple objects of $\mathcal{Z}(\mathcal{C})$. For simplicity, we set

$$[U : V]_{\mathcal{C}} := \dim(\mathcal{C}(U, V)) \quad \text{for all } U, V \in \mathcal{C}\,.$$

Since K is a left adjoint of F, $\mathcal{Z}(\mathcal{C})(K(V), X) \cong \mathcal{C}(V, F(X))$ and so

$$[K(V) : X]_{\mathcal{Z}(\mathcal{C})} = [V : F(X)]_{\mathcal{C}}$$

for $X \in \mathcal{Z}(\mathcal{C})$ and $V \in \mathcal{C}$.

Since $\dim(\mathcal{C}(U, V)) = \dim(\mathcal{C}(V, U)) = \dim(\mathcal{C}(U^*, V^*))$, we have

$$[V : U]_{\mathcal{C}} = [U : V]_{\mathcal{C}} = [U^*, V^*]_{\mathcal{C}} \text{ for all } U, V \in \mathcal{C}.$$

Thus, we have

$$K(V) \cong \sum_{X\in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : V]_{\mathcal{C}}\, X\,. \tag{2.2}$$

Let θ be the twist of $\mathcal{Z}(\mathcal{C})$. For any $X \in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))$, we define $\omega_X \in \mathbb{C}$ by the equation

$$\theta_X = \omega_X\, \mathrm{id}_X\,.$$

Note that ω_X is an N-th root of unity by Theorem 2.1.

PROPOSITION 2.3. *Let $\mathcal{C}$ be a spherical fusion categories over $\mathbb{C}$ with* $\operatorname{FSexp}(\mathcal{C}) = N$*, and $V \in \mathcal{C}$. Then*

(i) *$\bar{\nu}(V)$ is an algebraic integer invariant under pivotal equivalence, i.e. if $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ defines an equivalence of pivotal categories, then $\bar{\nu}(V) = \bar{\nu}(\mathcal{F}(V))$.*

(ii) *Moreover,*

$$\bar{\nu}(V) = \frac{N}{\dim(\mathcal{C})} \sum_{\substack{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C})) \\ \theta_X = \mathrm{id}_X}} [F(X) : V]_{\mathcal{C}}\, d(X) \tag{2.3}$$

where $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor.

PROOF. Statement (i) is an immediate consequence of the fact that $\nu_n(V)$ are algebraic integers for all n, and that both $\nu_n(V)$ and $\operatorname{FSexp}(\mathcal{C})$ are invariant under pivotal equivalences.

(ii) By Theorem 2.1, we have

$$\nu_n(V) = \frac{1}{\dim(\mathcal{C})} \underline{\operatorname{ptr}}(\theta^n_{K(V)}) = \frac{1}{\dim(\mathcal{C})} \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : V]_{\mathcal{C}}\, \omega_X^n\, d(X).$$

Therefore,

$$\begin{aligned}
\bar{\nu}(V) &= \sum_{i=1}^{N} \nu_i(V) \\
&= \frac{1}{\dim(\mathcal{C})} \sum_{i=1}^{N} \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : V]_{\mathcal{C}}\, \omega_X^i\, d(X) \\
&= \frac{1}{\dim(\mathcal{C})} \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : V]_{\mathcal{C}} \sum_{i=1}^{N} \omega_X^i\, d(X) \\
&= \frac{N}{\dim(\mathcal{C})} \sum_{\substack{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C})) \\ \omega_X = 1}} [F(X) : V]_{\mathcal{C}}\, d(X).
\end{aligned}$$

Here, the last equality follows from the fact that ω_X is an N-th root of unity. □

We can now prove Theorem A.

PROOF OF THEOREM A. Let H be a semisimple quasi-Hopf algebra. Consider the canonical pivotal structure on $\operatorname{Rep}(H)$. Then $\operatorname{Rep}(H)$ is a spherical fusion category with $d(V) = \dim V$ for all $V \in \operatorname{Rep}(H)$. In particular, $\dim(\operatorname{Rep}(H)) = \dim H$ and $d(X)$ is a non-negative integer for all $X \in \mathcal{Z}(\operatorname{Rep}(H))$. It follows from Proposition 2.3 (ii) that

$$\bar{\nu}(V) = \frac{N}{\dim H} \sum_{\substack{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C})) \\ \omega_X = 1}} [F(X) : V]_{\mathcal{C}}\, d(X)$$

is a non-negative rational number. By Proposition 2.3 (i), $\bar{\nu}(V)$ is a non-negative integer. Since

$$\frac{\bar{\nu}(V)\dim H}{N} = \sum_{\substack{X\in \mathrm{Irr}(\mathcal{Z}(\mathcal{C}))\\ \omega_X=1}} [F(X):V]_{\mathcal{C}}\, d(X) \in \mathbb{Z}\,,$$

we establish the divisibility $N \mid (\dim H)\,\bar{\nu}(V)$. □

If H is a semisimple Hopf algebra, $\exp(H) = \mathrm{ord}(\theta) = \mathrm{FSexp}(H)$ (cf. [**6**, Thm 2.5] and Theorem 2.1). Therefore, Theorem A is related to the following well-known conjecture considered by Kashina [**12**, **13**].

Conjecture 2.4. *Let H be a semisimple Hopf algebra over $\mathbb{C}$. Then the exponent of H divides* $\dim(H)$.

By the Cauchy theorem for Hopf algebras [**14**, Sect. 6] (see so [**27**, Thm. 8.4]), $\dim H$ and $\exp(H)$ have the same prime factors. Thus, if $\gcd\{\bar{\nu}(V) \mid V \in \mathrm{Irr}(H)\}$ is relatively prime to $\dim H$, then the conjecture will be proved for H. However, the Kac algebra K of dimension 8 is an example in which $\gcd\{\bar{\nu}(V) \mid V \in \mathrm{Irr}(K)\} = 4$ (see Example 3.6). Nevertheless, any upper bound of $\gcd\{\bar{\nu}(V) \mid V \in \mathrm{Irr}(H)\}$ will shed light on the conjecture.

By Theorem A, $\bar{\nu}(V) = 0$ is an extreme value and it is possible for some quasi-Hopf algebra demonstrated in the following example.

Example 2.5. Let G be a finite group and ω a normalized 3-cocycle on G with coefficients in $\mathbb{C}^\times$. Following [**26**, Sect. 7], one can construct a quasi-Hopf algebra $H(G,\omega) = (\mathbb{C}[G]^*, \Delta, \varepsilon, \phi, \alpha, \beta, S)$ where multiplication, identity, comultiplication Δ, counit ε, and antipode S are the same as the structure maps of the dual $\mathbb{C}[G]^*$ of the group algebra $\mathbb{C}[G]$, and ϕ, α, and β are given by

$$\phi = \sum_{a,b,c\in G} \omega(a,b,c)e(a)\otimes e(b)\otimes e(c), \quad \alpha = 1, \quad \text{and} \quad \beta = \sum_{a\in G} \omega(a,a^{-1},a)^{-1}e(a)\,,$$

where $\{e(x) \mid x \in G\}$ is the dual basis of G for $\mathbb{C}[G]^*$. If $\omega' \in Z^3(G,\mathbb{C}^\times)$ is cohomologous to ω, then $H(G,\omega)$ can be twisted to the quasi-bialgebra $H(G,\omega')$ by a gauge transformation (see [**15**, XV] for gauge equivalence of quasi-bialgebras). In particular, if ω is a coboundary of the 2-cochain $f : G\times G \to \mathbb{C}^\times$, then $H(G,\omega)$ can be twisted by the gauge transformation $F = \sum_{a,b\in G} f(a,b)\, e(a)\otimes e(b)$ to the ordinary bialgebra $\mathbb{C}[G]^*$.

We now consider a special case when G is of order 2. Let $G = \{1, x\}$ be an abelian group of order 2 and ω a 3-cocycle of G given by

$$\omega(a,b,c) = \begin{cases} -1 & \text{if } a=b=c=x,\\ \ \ 1 & \text{otherwise.}\end{cases}$$

Then $H = H(G,\omega)$ is a 2-dimensional commutative quasi-Hopf algebra. Let V be the nontrivial simple H-module. As computed in [**27**, Ex. 5.4], $\mathrm{FSexp}(H) = 4$ and $\nu_n(V) = \cos(\frac{n\pi}{2})$. Therefore, $\bar{\nu}(V) = 0$.

3. Proof of Theorem B

Example 2.5 shows that $\bar{\nu}(V) = 0$ for some simple module V of a semisimple quasi-Hopf algebra. However, this cannot happen for semisimple Hopf algebras which is stated in Theorem B. We will prove this theorem in this section.

Recall that the Drinfeld double $D(H)$ of a finite-dimensional Hopf algebra H is the bicrossproduct $H^{*\mathrm{cop}} \bowtie H$ (cf. [**15**, **20**]). In particular, $D(H)$ is a finite-dimensional Hopf algebra with comultiplication and multiplication of $D(H)$ described by

$$\Delta(f \bowtie a) = (f_2 \bowtie a_1) \otimes (f_1 \bowtie a_2)$$

and

$$(f \bowtie a)(g \bowtie b) = f(g(S^{-1}(a_3)?a_1)) \bowtie a_2 b$$

respectively, for $f, g \in H^*$ and $a, b \in H$. Here $a_1 \otimes a_2$ and $f_1 \otimes f_2$ are respectively the Sweedler notation for the comultiplications of $a \in H$ and $f \in H^*$ with the summation suppressed. The Hopf algebra $D(H)$ admits a canonical universal $\mathcal{R}$-matrix $R = \sum_i \varepsilon \bowtie h_i \otimes h^i \bowtie 1$ where $\{h_i\}_i$ and $\{h^i\}_i$ are dual bases of H and H^* respectively.

It should be well-known to experts that $\mathrm{Rep}(D(H))$ and $\mathrm{Rep}(D(H^*))$ are equivalent braided monoidal categories. For the sake of completeness, we provide a proof of the statement in terms of gauge equivalence for subsequent discussion.

LEMMA 3.1. *Let H be a finite-dimensional Hopf algebra over any field $\Bbbk$, not necessarily semisimple. Then $D(H)$ and $D(H^*)$ are gauge equivalent quasi-triangular Hopf algebras via the gauge transformation $R \in D(H^*) \otimes D(H^*)$, which is the universal $\mathcal{R}$-matrix of $D(H^*)$, and the algebra isomorphism*

$$\sigma : \ D(H) \to D(H^*), \quad f \bowtie a \mapsto (1 \bowtie f)(a \bowtie \varepsilon).$$

Here, H^{**} is identified with H as Hopf algebras under the natural isomorphism $j : H \to H^{**}$ of vector spaces.

PROOF. Since $\Delta^{\mathrm{op}}_{D(H^*)}(x) = R\Delta_{D(H^*)}(x)R^{-1}$ for all $x \in D(H^*)$, it suffices to show that $\sigma : D(H) \to D(H^*)^{\mathrm{cop}}$ is a bialgebra homomorphism, but this is straightforward verification. □

The algebra isomorphism $\sigma : D(H) \to D(H^*)$ defines a braided monoidal equivalence $(\mathcal{F}_\sigma, \xi, \mathrm{id}) : \mathrm{Rep}(D(H^*)) \to \mathrm{Rep}(D(H))$ (cf. [**15**, XIII.3.2]). Here, $\mathcal{F}_\sigma : \mathrm{Rep}(D(H^*)) \to \mathrm{Rep}(D(H))$ is the $\mathbb{C}$-linear functor with $\mathcal{F}_\sigma(V)$ defined as the $D(H)$-module via the isomorphism σ for $V \in \mathrm{Rep}(D(H^*))$, and $\mathcal{F}_\sigma : \mathrm{Hom}_{D(H^*)}(V, W) \to \mathrm{Hom}_{D(H)}(\mathcal{F}_\sigma V, \mathcal{F}_\sigma W)$ the identity function. The coherence isomorphism $\xi : \mathcal{F}_\sigma(V) \otimes \mathcal{F}_\sigma(W) \to \mathcal{F}_\sigma(V \otimes W)$ is the left action of R^{-1}, and $\mathcal{F}_\sigma(\mathbf{1}_{D(H^*)}) = \mathbf{1}_{D(H)}$.

The Hopf algebra H^* can be considered as a subalgebra of $D(H)$ and $D(H^*)$ via the embeddings $i_1 : H^* \to D(H^*)$ and $i_2 : H^* \to D(H)$ defined by

$$i_1(f) = 1 \bowtie f, \quad i_2(f) = f \bowtie 1$$

respectively, for $f \in H^*$. It follows immediately from the definition of σ that we have the commutative diagram of algebra maps:

$$\begin{array}{ccc} D(H) & \xrightarrow{\ \sigma\ } & D(H^*) \\ {\scriptstyle i_2}\uparrow & \nearrow{\scriptstyle i_1} & \\ H^* & & \end{array} \qquad (3.1)$$

This implies the following lemma for the two pairs of induction and restriction functors.

LEMMA 3.2. *Let* $\operatorname{Res}_{H^*}^{D(H^*)}$ *and* $\operatorname{Res}_{H^*}^{D(H)}$ *be the restriction functors along the embeddings* i_1 *and* i_2 *respectively, and* $\operatorname{Ind}_{H^*}^{D(H^*)}$ *and* $\operatorname{Ind}_{H^*}^{D(H)}$ *the associated induction functors. Then we have*

$$\mathcal{F}_\sigma \circ \operatorname{Ind}_{H^*}^{D(H^*)} = \operatorname{Ind}_{H^*}^{D(H)} \quad \text{and} \quad \operatorname{Res}_{H^*}^{D(H)} \circ \mathcal{F}_\sigma = \operatorname{Res}_{H^*}^{D(H^*)}. \quad \square \tag{3.2}$$

The preceding lemmas hold for any finite-dimensional Hopf algebras. We now turn to semisimple Hopf algebras H. In this case, $\mathcal{C} = \operatorname{Rep}(H)$ is a spherical fusion category, and the *right* center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is equivalent to the $\operatorname{Rep}(D(H))$ as braided monoidal category. Equipped with the canonical pivotal structure, $\operatorname{Rep}(D(H))$ is a modular tensor category with the twist θ given by the action of the Drinfeld element of $D(H)$. The forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is the restriction $\operatorname{Res}_H^{D(H)}$ which has a left adjoint $\operatorname{Ind}_H^{D(H)} : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$. Thus, by the uniqueness of adjoint functor, $K : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is equivalent to $\operatorname{Ind}_H^{D(H)}$. If V is a self-dual H-module, then so is $K(V)$. This observation even holds for spherical fusion categories.

LEMMA 3.3. *Let* $\mathcal{C}$ *be a spherical fusion category,* $\mathcal{Z}(\mathcal{C})$ *the center of* $\mathcal{C}$ *and* $K : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ *the left adjoint of the forgetful functor* $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$*. Then* $K(V)$ *is a self-dual object of* $\mathcal{Z}(\mathcal{C})$ *whenever* $V \in \mathcal{C}$ *is self-dual. In particular,* $K(\mathbf{1}_\mathcal{C})$ *is self-dual. Moreover, if* θ *is the twist of* $\mathcal{Z}(\mathcal{C})$*, then* $\theta_{K(\mathbf{1}_\mathcal{C})} = \operatorname{id}_{K(\mathbf{1}_\mathcal{C})}$.

PROOF. Note that for $X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))$, X^* is isomorphic to a unique object of $\operatorname{Irr}(\mathcal{Z}(\mathcal{C}))$. Let $V \in \mathcal{C}$ be self-dual. It follows from (2.2) that

$$\begin{aligned}
K(V)^* &\cong \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : V]_\mathcal{C}\, X^* \\
&\cong \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X^*) : V]_\mathcal{C}\, X \\
&\cong \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X)^* : V]_\mathcal{C}\, X \\
&\cong \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X)^* : V^*]_\mathcal{C}\, X \\
&\cong \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : V]_\mathcal{C}\, X \\
&\cong K(V).
\end{aligned}$$

Here, the third isomorphism is a consequence of the fact that the forgetful functor $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ defines a monoidal functor, and the last isomorphism follows from the remark preceding Proposition 2.3. Since $\mathbf{1}_\mathcal{C}$ is self-dual, and so is $K(\mathbf{1}_\mathcal{C})$.

By [**29**, Prop. 2.8 (iii)], $[F(X) : \mathbf{1}_\mathcal{C}]_\mathcal{C} \neq 0$ implies $\omega_X = 1$. Therefore,

$$\theta_{K(\mathbf{1}_\mathcal{C})} = \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : \mathbf{1}_\mathcal{C}]_\mathcal{C}\, \theta_X = \sum_{X \in \operatorname{Irr}(\mathcal{Z}(\mathcal{C}))} [F(X) : \mathbf{1}_\mathcal{C}]_\mathcal{C}\, \operatorname{id}_X = \operatorname{id}_{K(\mathbf{1}_\mathcal{C})}.$$

This proves the last assertion. □

We can now prove Theorem B.

Proof of Theorem B. Let $\mathcal{C} = \mathrm{Rep}(H)$. In view of Proposition 2.3 (ii), it suffices to show the inequality

$$\sum_{\substack{X\in \mathrm{Irr}(D(H))\\ \theta_X=\mathrm{id}_X}} [\mathrm{Res}_H^{D(H)} X : V]_{\mathcal{C}} \dim(X) \geq \dim(V)$$

for all $V \in \mathcal{C}$. This is equivalent to show that $[\mathrm{Res}_H^{D(H)} W : V]_{\mathcal{C}} > 0$ for some $D(H)$-module W satisfying $\theta_W = \mathrm{id}_W$.

Recall that H^* is also a semisimple Hopf algebra [**16**]. By Lemma 3.3, $Y = \mathrm{Ind}_{H^*}^{D(H^*)}(\mathbf{1}_{H^*})$ is a self-dual $D(H^*)$-module and $\theta_Y = \mathrm{id}_Y$. Moreover,

$$Y \cong \sum_{U\in \mathrm{Irr}(D(H^*))} [U : \mathbf{1}_{H^*}]_{\mathcal{C}'} U \tag{3.3}$$

where $\mathcal{C}' = \mathrm{Rep}(H^*)$. Since $\mathrm{Rep}(D(H^*))$ and $\mathrm{Rep}(D(H))$ are spherical categories equipped with their canonical pivotal structures, $\mathcal{F}_\sigma : \mathrm{Rep}(D(H^*)) \to \mathrm{Rep}(D(H))$ defines an equivalence of braided monoidal categories as well as pivotal categories. Therefore, by [**9**, Prop. 6.2], we have

$$\mathrm{id}_{\mathcal{F}_\sigma Y} = \mathcal{F}_\sigma(\mathrm{id}_Y) = \mathcal{F}_\sigma(\theta_Y) = \theta_{\mathcal{F}_\sigma Y}\,.$$

In particular, $W = \mathcal{F}_\sigma Y$ is a self-dual $D(H)$-module satisfying $\theta_W = \mathrm{id}_W$. Since $\mathrm{Ind}_H^{D(H)}$ is a left adjoint of $\mathrm{Res}_H^{D(H)}$, we have

$$[V : \mathrm{Res}_H^{D(H)} W]_{\mathcal{C}} = [\mathrm{Ind}_H^{D(H)} V : W]_{\mathcal{Z}(\mathcal{C})}\,.$$

However, by Lemma 3.2, $W = \mathrm{Ind}_{H^*}^{D(H)} \mathbf{1}_{H^*}$. Therefore,

$$\begin{aligned}
\mathrm{Hom}_{D(H)}\left(\mathrm{Ind}_H^{D(H)} V, W\right) &= \mathrm{Hom}_{D(H)}\left(\mathrm{Ind}_H^{D(H)} V, \mathrm{Ind}_{H^*}^{D(H)} \mathbf{1}_{H^*}\right)\\
&\cong \mathrm{Hom}_{D(H)}\left(\left(\mathrm{Ind}_{H^*}^{D(H)} \mathbf{1}_{H^*}\right)^* \otimes \left(\mathrm{Ind}_H^{D(H)} V\right), \mathbf{1}_{D(H)}\right)\\
&\cong \mathrm{Hom}_{D(H)}\left(\left(\mathrm{Ind}_{H^*}^{D(H)} \mathbf{1}_{H^*}\right) \otimes \left(\mathrm{Ind}_H^{D(H)} V\right), \mathbf{1}_{D(H)}\right)\,.
\end{aligned}$$

By [**4**, Thm. 8 (1)], we have

$$\left(\mathrm{Ind}_{H^*}^{D(H)} \mathbf{1}_{H^*}\right) \otimes \left(\mathrm{Ind}_H^{D(H)} V\right) \cong D(H)^{\dim V}$$

as $D(H)$-modules. Thus,

$$[\mathrm{Ind}_H^{D(H)} V : W]_{\mathcal{Z}(\mathcal{C})} = [D(H)^{\dim V} : \mathbf{1}_{D(H)}]_{\mathcal{Z}(\mathcal{C})} = \dim V,$$

and so $[V : \mathrm{Res}_H^{D(H)} W]_{\mathcal{C}} = \dim V$. This completes the proof. □

Definition 3.4. A quasi-Hopf algebra H is said to be *genuine* if H is not gauge equivalent to any ordinary Hopf algebra.

Theorem B provides a sufficient condition for a semisimple quasi-Hopf algebra being genuine.

Corollary 3.5. *Let H be a semisimple quasi-Hopf algebra. If there exists $V \in \mathrm{Rep}(H)$ such that $\bar{\nu}(V) = 0$, then H is a genuine quasi-Hopf algebra.*

PROOF. Assume contrary. Then H is gauge equivalent to a Hopf algebra H'. There exists a monoidal equivalence $\mathcal{F} : \mathrm{Rep}(H) \to \mathrm{Rep}(H')$ (see [**26**, Thm. 2.2]). Thus, $\mathcal{F}$ preserves their canonical pivotal structures (cf. [**28**]). By Proposition 2.3 (i) and Theorem B, we have

$$\bar{\nu}(V) = \bar{\nu}(\mathcal{F}(V)) > 0$$

for all $V \in \mathrm{Rep}(H)$. □

The following example shows that the existence of vanishing total indicators can also be a necessary condition for a genuine quasi-Hopf algebra.

EXAMPLE 3.6. There are exactly four gauge inequivalent 8-dimensional quasi-Hopf algebras with five simple objects $\{a_1, a_2, a_3, a_4, m\}$ and fusion rules given by

$$\{a_1, a_2, a_3, a_4\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \quad m^2 \cong \sum_{i=1}^{4} a_i \quad \text{and} \quad ma_i \cong a_i m \cong m$$

for $i = 1, 2, 3, 4$. Their categories of representations are Tambara-Yamagami categories [**32**]. They are $\mathbb{C}[Q], \mathbb{C}[D]$, the 8-dimensional Kac algebra K and its twisted version K_u (see [**26**, Sec.6] for details), where Q and D are, respectively, the quaternion group and the dihedral group of order 8. We can assume a_1 to be the unit object. Then $\nu_n(a_1) = 1$ for all $n \geq 1$. The objects a_2, a_3 and a_4 are 1-dimensional representations of these algebras, and their orders are 2. The indicator $\nu_n(a_i) = 1$ if n is even, and 0 otherwise, for $i = 2, 3, 4$. Thus $\bar{\nu}(a_i) > 0$ for $i = 1, 2, 3, 4$. By [**26**, Sec. 6], we find the following table:

	$\nu_2(m)$	$\nu_3(m)$	$\nu_4(m)$	$\nu_5(m)$	$\nu_6(m)$	$\nu_7(m)$	$\nu_8(m)$	$\bar{\nu}(m)$
K	1	0	0	0	1	0	2	4
K_u	-1	0	0	0	-1	0	2	0
$\mathbb{C}[D]$	1	0	2	0	1	0	2	3
$\mathbb{C}[Q]$	-1	0	2	0	-1	0	2	1

In particular, the Frobenius-Schur exponents of $K, K_u, \mathbb{C}[D], \mathbb{C}[Q]$ are $8, 8, 4, 4$ respectively. The quasi-Hopf algebra K_u is the only one in the list for which $\bar{\nu}(m) = 0$. Therefore, K_u is a genuine quasi-Hopf algebra. It is well-known that $K, \mathbb{C}[D], \mathbb{C}[Q]$ are all the noncommutative semisimple Hopf algebras of dimension 8 (cf. [**19**]). Their corresponding indicators $\nu_n(m)$ can be computed using the formula (2.1).

4. Twisted quantum doubles

The main result of this section is to show that existence of vanishing total indicators is also a necessary condition for an *abelian* twisted quantum double, considered in [**24**], being genuine.

The *twisted quantum double* $D^\omega(G)$ of G relative to a normalized 3-cocycle $\omega : G \times G \times G \to \mathbb{C}^\times$ is the semisimple quasi-Hopf algebra with underlying vector space $\mathbb{C}[G]^* \otimes \mathbb{C}[G]$ in which multiplication, comultiplication Δ, associator ϕ, counit

ε and quasi-antipode (S, α, β) are given by

$$(e(g) \otimes x)(e(h) \otimes y) = \theta_g(x, y)\delta_{g^x,h} e(g) \otimes xy,$$
$$\Delta(e(g) \otimes x) = \sum_{hk=g} \gamma_x(h, k) e(h) \otimes x \otimes e(k) \otimes x,$$
$$\phi = \sum_{g,h,k \in G} \omega(g, h, k)^{-1} e(g) \otimes 1 \otimes e(h) \otimes 1 \otimes e(k) \otimes 1,$$
$$\varepsilon(e(g) \otimes x) = \delta_{g,1}, \quad \alpha = 1, \quad \beta = \sum_{g \in G} \omega(g, g^{-1}, g) e(g) \otimes 1,$$
$$S(e(g) \otimes x) = \theta_{g^{-1}}(x, x^{-1})^{-1} \gamma_x(g, g^{-1})^{-1} e(x^{-1}g^{-1}x) \otimes x^{-1},$$

where $\{e(g) | g \in G\}$ is the dual basis of $\{g | g \in G\}$, $\delta_{g,1}$ is the Kronecker delta, $g^x = x^{-1}gx$, and

$$\theta_g(x, y) = \frac{\omega(g, x, y)\omega(x, y, (xy)^{-1}gxy)}{\omega(x, x^{-1}gx, y)},$$
$$\gamma_g(x, y) = \frac{\omega(x, y, g)\omega(g, g^{-1}xg, g^{-1}yg)}{\omega(x, g, g^{-1}yg)}$$

for any $x, y, g \in G$ (cf. [**5**, **15**]).

As in [**24**], we denote by Γ^ω the set of all group-like elements of $D^\omega(G)$ and call ω *abelian* if $D^\omega(G)$ is a commutative algebra. This can only happen when G is abelian and Γ^ω spans $D^\omega(G)$. The following theorem characterizes those abelian twisted doubles of finite groups which are genuine quasi-Hopf algebras, and it provides an answer for a question of Dijkgraaf, Pasquier and Roche posed in [**5**, p69].

THEOREM 4.1. *Let G be a finite abelian group, and ω a normalized 3-cocycle of G such that $D^\omega(G)$ is a commutative algebra. Then $D^\omega(G)$ is a genuine quasi-Hopf algebra if, and only if, there exists $V \in \mathrm{Rep}(D^\omega(G))$ such that $\bar{\nu}(V) = 0$.*

To prove this theorem, we need some preparations. Recall that an Eilenberg-MacLane (EM) 3-cocycle of G is a pair (ω, c) where $\omega \in Z^3(G, \mathbb{C}^\times)$ and c a 2-cochain of G satisfying the following conditions:

$$\frac{c(xy, z)}{c(x, z)c(y, z)} \frac{\omega(x, z, y)}{\omega(x, y, z)\omega(z, x, y)} = 1 = \frac{c(x, yz)}{c(x, y)c(x, z)} \frac{\omega(x, y, z)\omega(y, z, x)}{\omega(y, x, z)} \tag{4.1}$$

for all $x, y, z \in G$. The EM 3-cocycle (ω, c) is a coboundary if there exists a 2-cochain h of G such that

$$\omega = \delta h \quad \text{and} \quad c(x, y) = \frac{h(x, y)}{h(y, x)}.$$

The EM cohomology group $\mathrm{H}^3_{ab}(G, \mathbb{C}^\times)$ is then defined by

$$\mathrm{H}^3_{ab}(G, \mathbb{C}^\times) = \frac{Z^3_{ab}(G, \mathbb{C}^\times)}{B^3_{ab}(G, \mathbb{C}^\times)}$$

where $Z^3_{ab}(G, \mathbb{C}^\times)$ and $B^3_{ab}(G, \mathbb{C}^\times)$ are respectively the abelian groups of EM 3-cocycles and 3-coboundaries.

For $(\omega, c) \in Z^3_{ab}(G, \mathbb{C}^\times)$, one can assign the function $t(x) := c(x, x)$, called its trace. The trace t of an EM 3-cocycle of G is a quadratic function on G, that means

(i) $t(x^a) = t(x)^{a^2}$ for $a \in \mathbb{Z}$, and

(ii) $b_t(x,y) := \frac{t(xy)}{t(x)t(y)}$ defines a bicharacter of G.

The following theorem of Eilenberg and Mac Lane is essential to the discussion in this section (cf. [**18**, Thm. 3] and [**11**, §7]).

THEOREM 4.2 (Eilenberg-MacLane). *Let $Q(G,\mathbb{C}^\times)$ be the abelian group of all quadratic functions from G to $\mathbb{C}^\times$. The map assigning to each 3-cocycle its trace induces an isomorphism*

$$\mathrm{H}^3_{ab}(G,\mathbb{C}^\times) \xrightarrow{\cong} Q(G,\mathbb{C}^\times). \quad \square$$

REMARK 4.3. Recall the example of quasi-Hopf algebra $H(G,\omega)$ defined in Example 2.5 for a normalized 3-cocycle ω on a finite group G. Let $\mathcal{C} = \mathrm{Rep}(H(G,\omega))$. Then $\mathrm{Irr}(\mathcal{C})$ consists of a set of 1-dimensional representations V_x indexed by G. Moreover, a pair (ω,c) is an EM 3-cocycle of a finite abelian group G if, and only if, $\mathrm{Rep}(H(G,\omega))$ is a braided spherical fusion category with the braiding given by

$$c_{V_x,V_y} := \left(V_x \otimes V_y \xrightarrow{c(x,y)} V_y \otimes V_x\right)$$

for $x,y \in G$.

For simplicity, we say that a quadratic function $t : G \to \mathbb{C}^\times$ of a finite abelian group G can be obtained from a bicharacter b of G if $t(x) = b(x,x)$ for all $x \in G$. We denote by $K(G,\mathbb{C}^\times)$ the set of all quadratic functions which can be obtained from some bicharacters of G.

LEMMA 4.4. *Let G be a finite abelian group and (ω,c) an EM 3-cocycle of G with trace t. Then:*

(i) *ω is a coboundary if, and only if, $t \in K(G,\mathbb{C}^\times)$.*

(ii) *ω^2 is a 3-coboundary of G.*

PROOF. Let us denote the cohomology class represented by the EM 3-cocycle (ω,c) of G as $[(\omega,c)]$. Then ω is a coboundary if, and only if, $[(\omega,c)] = [(1,b)]$ for some 2-cochain b of G. By (4.1), b is a bicharacter of G and $t(x) = c(x,x) = b(x,x)$ for all $x \in G$. Therefore, $t \in K(G,\mathbb{C}^\times)$. Conversely, suppose there exists a bicharacter b on G such that $b(x,x) = t(x)$ for all $x \in G$. Since $(1,b)$ is an EM 3-cocycle which has the same trace as (ω,c), $[(\omega,c)] = [(1,b)]$ by Theorem 4.2. In particular, ω is a coboundary. This proves statement (i).

(ii) By (4.1), $b(x,y) := c(x,y)c(y,x)$ defines a bicharacter of G, and the trace of the EM 3-cocycle (ω^2,c^2) can be obtained from b. It follows from (i) that ω^2 is a coboundary. $\square$

LEMMA 4.5. *Let G be a finite abelian group and (ω,c) an EM 3-cocycle of G. Then ω is a coboundary if, and only if, $\bar{\nu}(V) > 0$ for all simple $H(G,\omega)$-module V. In this case, $H(G,\omega)$ is gauge equivalent to the Hopf algebra $\mathbb{C}[G]^*$.*

PROOF. In view of Example 2.5, if ω is a coboundary, then $H(G,\omega)$ can be twisted to the bialgebra $\mathbb{C}[G]^*$ by a gauge transformation. It follows from Corollary 3.5 that $\bar{\nu}(V) > 0$ for all simple $H(G,\omega)$-modules V. Conversely, we assume the positivity of total indicators. Let t be the trace of the EM 3-cocycle (ω,c). By Lemma 4.4(i), it is equivalent to show that $t \in K(G,\mathbb{C}^\times)$. Since G is a direct sum of its cyclic subgroups, by [**24**, Lem. 6.2(i)], $t \in K(G,\mathbb{C}^\times)$ if, and only if, $t_C \in K(C,\mathbb{C}^\times)$ for each cyclic summand C of G, where t_C denotes the restriction of t on C. Note that the restriction (ω_C, c_C) of (ω, c) on C is an EM 3-cocycle of C and its trace is equal to t_C. Therefore, by Lemma 4.4(i), it is enough to prove that ω_C is a coboundary for each cyclic subgroup C of G.

By Lemma 4.4(ii), ω^2 is a coboundary. We may simply assume $\omega^2 = 1$ as $\bar{\nu}(V)$ are preserved by gauge equivalence of semisimple quasi-Hopf algebras. A complete set of non-isomorphic simple $H(G,\omega)$-modules is given by $\mathrm{Irr}(H(G,\omega)) = \{V_x | x \in G\}$. By [**26**, Prop. 7.1],

$$\nu_n(V_x) = \delta_{x^n,1} \prod_{r=1}^{n-1} \omega(x, x^r, x). \tag{4.2}$$

The equation implies that $\nu_n(V_x) = 0$ if $\ell \nmid n$ where $\ell = \mathrm{ord}(x)$. Since $\omega^2 = 1$, $\omega(x, x^r, x) = \pm 1$ for all $r \in \mathbb{Z}$. By (4.2), $\nu_\ell(V_x) = \pm 1$ and

$$\nu_{k\ell}(V_x) = \nu_\ell(V_x)^k .$$

for all positive integer k. We claim that $\nu_\ell(V_x) = 1$. Suppose not. Then $\nu_\ell(V_x) = -1$. Let $N = \mathrm{FSexp}(H(G,\omega))$. Then $\nu_N(V_x) = \dim V_x = 1$. Therefore, N/ℓ is even and so $\bar{\nu}(V_x) = 0$ which contradicts the assumption of the positivity of total indicators.

We now have

$$1 = \nu_\ell(V_x) = \prod_{r=1}^{\ell-1} \omega(x, x^r, x) .$$

Note that right hand side is also the ℓ-th indicator of V_x considered as an $H(C, \omega_C)$-module where C is the cyclic subgroup generated by x. Since $\nu_\ell(V_x)$ is invariant under gauge transformations, the product depends only on the cohomology class of ω_C in $H^3(C, \mathbb{C}^\times)$, which is a cyclic group of order ℓ. A generating 3-cocycle ϕ, as described in [**21**], is defined by

$$\phi(x^i, x^j, x^k) = q^{\bar{i}(\bar{j}+\bar{k}-\overline{j+k})/\ell}$$

where $\bar{i}$ denotes the least non-negative residue of i modulo ℓ, and q a primitive ℓ-th root of unity. Thus, ω_C is cohomologous to ϕ^a for some non-negative integer $0 \le a \le \ell - 1$, and

$$1 = \prod_{r=1}^{\ell-1} q^{a(r+1-\overline{r+1})/\ell} = q^a .$$

This implies $a = 0$ and so ω_C is a coboundary. This completes the proof of the lemma. □

Now we can proof Theorem 4.1

Proof Theorem 4.1. By [**24**, Cor. 3.6], $D^\omega(G)$ is spanned by the set Γ^ω of group-like elements when $D^\omega(G)$ is commutative. In particular, Γ^ω is a finite abelian group which fits into an exact sequence of abelian groups

$$1 \to \hat{G} \to \Gamma^\omega \to G \to 1$$

determined by G and ω. Moreover, $D^\omega(G)$ is isomorphic to $H(\Gamma^\omega, \omega')$ as quasi-bialgebras, where $\omega' \in Z^3(\Gamma^\omega, \mathbb{C}^\times)$ is the inflation of ω^{-1} along the above map $\Gamma^\omega \to G$ [**24**, Sect. 9]. The braiding of $\mathrm{Rep}(D^\omega(G))$ determines an EM 3-cocycle (ω', c) of Γ^ω. By Lemma 4.5, the existence of a vanishing total indicator if, and only if, ω' is a non-trivial 3-cocycle of Γ^ω. This is equivalent to that $H(\Gamma^\omega, \omega')$, or equivalently $D^\omega(G)$, is a genuine quasi-Hopf algebra. □

5. Tambara-Yamagami categories

In this section, we study the positivity of total indicators for *integral* Tambara-Yamagami categories associated with elementary p-groups [**32**]. By [**8**], these fusion categories are monoidally equivalent to the categories of representations of semisimple quasi-Hopf algebras. In [**33**], a necessary and sufficient condition is obtained for an integral Tambara-Yamagami category admitting a fibration, i.e. it is monoidally equivalent to the category of representations of a semisimple Hopf algebra. By Corollary 3.5, positivity of total indicators is a necessary condition for the existence of a fibration, but not sufficient in general. We will demonstrate the sufficiency of positivity of total indicators for these integral Tambara-Yamagami categories in this section.

Let A be a finite abelian group. Tambara and Yamagami [**32**] classified fusion categories with a complete set of simple objects $A \sqcup \{m\}$ satisfying the fusion rules

$$a \otimes b \cong ab, \quad a \otimes m \cong m \cong m \otimes a, \quad m \otimes m \cong \bigoplus_{x \in A} x \quad (a, b \in A). \tag{5.1}$$

They also showed that such categories are parametrized by pairs (χ, τ) where χ is a non-degenerate symmetric bicharacter of A, and τ is a square root of $|A|^{-1}$. The corresponding fusion category, denoted by $\mathcal{TY}(A, \chi, \tau)$, can be described more precisely in the following definition.

Definition 5.1. $\mathcal{TY}(A, \chi, \tau)$ is a skeletal fusion category over $\mathbb{C}$ with the set of simple objects $S := A \sqcup \{m\}$. Hom-sets between elements of S are given by

$$\operatorname{Hom}(s, s') = \begin{cases} \mathbb{C} & \text{if } s = s', \\ 0 & \text{otherwise} \end{cases}$$

with $\mathrm{id}_s = 1 \in \mathbb{C}$. The compositions of morphisms are obvious one. Tensor products of elements of S are given by (5.1) (but with $\cong$ replaced by $=$). The unit object $\mathbf{1}$ is strict and equal to the identity $e \in A$. The associativity constraint Φ is determined by

$$\begin{aligned} \Phi_{a,m,b} &= \chi(a, b)\, \mathrm{id}_m : \ m \to m, \\ \Phi_{m,a,m} &= (\chi(a, x)\delta_{x,y}\, \mathrm{id}_x)_{x,y} : \bigoplus_{x \in A} x \to \bigoplus_{y \in A} y, \\ \Phi_{m,m,m} &= (\tau\chi(x, y)^{-1}\, \mathrm{id}_m)_{x,y} : \bigoplus_{x \in A} m \to \bigoplus_{y \in A} m, \end{aligned}$$

where $a, b \in A$, and the other $\Phi_{s,t,u}$ $(s, t, u \in S)$ are identity morphisms. Duality of $\mathcal{TY}(A, \chi, \tau)$ is described as follows: For $a \in A$, $a^* := a^{-1}$ with $\mathrm{ev}_a, \mathrm{db}_a$ being identity id_e. For the object m, $m^* := m$ with morphisms $\mathrm{ev}_m = \tau^{-1}\pi : m \otimes m \to \mathbf{1}$ and $\mathrm{db}_m = \iota : \mathbf{1} \to m \otimes m$ where $\pi : m \otimes m \to \mathbf{1}$ and $\iota : \mathbf{1} \to m \otimes m$ are respectively the canonical projection and embedding satisfying $\pi\iota = \mathrm{id}_{\mathbf{1}}$

Since the Frobenius-Perron dimension of $\mathcal{TY}(A, \chi, \tau)$ is an integer, $\mathcal{TY}(A, \chi, \tau)$ is pseudo-unitary by [**8**]. In particular, there exists a unique spherical pivotal structure j for which the pivotal dimension $d(V)$ of an object V is equal to its Frobenius-Perron dimension, i.e.

$$d(a) = 1 \quad \text{for} \quad a \in A, \quad \text{and} \quad d(m) = \sqrt{|A|}\,.$$

One can verify directly that $j_a = \mathrm{id}_a$ for $a \in A$, and $j_m = \mathrm{sgn}(\tau)\,\mathrm{id}_m$, where $\mathrm{sgn}(\tau)$ means the sign of the real number τ. We will always assume $\mathcal{TY}(A, \chi, \tau)$ is a spherical fusion category relative to this canonical pivotal structure j.

By [**32**], two Tambara-Yamagami categories $\mathcal{TY}(A, \chi, \tau)$ and $\mathcal{TY}(A', \chi', \tau')$ are monoidally equivalent if, and only if, $\tau = \tau'$ and there exists a group isomorphism $\sigma : A \to A'$ satisfying $\chi(a, b) = \chi'(\sigma(a), \sigma(b))$ for all $a, b \in A$ (i.e. (A, χ) and (A', χ') are *isometric*).

We now turn to the case of elementary p-groups V of order p^r, where p is a prime. Then V is an r-dimensional vector space over the finite field $\mathbb{F}_p$ of order p. A non-degenerate symmetric bilinear form $B : V \times V \to \mathbb{F}_p$ on V determines a non-degenerate symmetric bicharacter χ_B defined by $\chi_B(a, b) = \exp(2\pi i B(a, b))$; moreover, the assignment $B \to \chi_B$ is a one-to-one correspondence. Two bilinear forms B and B' respectively on the $\mathbb{F}_p$-spaces V and V' are said to be *isometric* if there exists an isomorphism $\sigma : V \to V'$ of $\mathbb{F}_p$-spaces such that $B'(\sigma(a), \sigma(b)) = B(a, b)$ for all $a, b \in V_1$. It is easy to see that two bilinear forms B, B' are isometric if, and only if, χ_B, $\chi_{B'}$ are isometric bicharacters. Moreover, any bilinear form B on a $\mathbb{F}_p$-space is uniquely determined by its *Gram matrix* $[B(v_i, v_j)]_{ij}$ relative to a basis $\{v_i\}_i$ of V. In particular, we will denote by B_0 the bilinear form on $\mathbb{F}_p^r$ whose Gram matrix relative to the standard basis is the identity.

REMARK 5.2. The Tambara-Yamagami category $\mathcal{TY}(A, \chi, \tau)$ is integral if, and only if, $d(m) = \sqrt{|A|}$ is an integer, or equivalently $|A|$ is a square. In this case, by [**8**], the fusion category is monoidally equivalent the representation category of a semisimple quasi-Hopf algebra over $\mathbb{C}$. If V is an elementary p-group of order p^r, then $\mathcal{TY}(V, \chi, \tau)$ is integral if, and only if, r is even.

5.1. Characteristic two. We will show in this subsection that a Tambara-Yamagami category associated with an elementary 2-group of square order admits a fibration if, and only if, all its total indicators are positive.

Recall that all the non-degenerate alternating bilinear forms on $V = \mathbb{F}_2^r$ with r even are isometric. Any non-degenerate symmetric bilinear form on V, which is not alternating, is isometric to B_0. In particular, there are exactly two isometric classes of non-degenerate symmetric bilinear on $\mathbb{F}_2^r$ when r is even. For odd r, every non-degenerate symmetric bilinear form is isometric to B_0 (see, for example, [**1**]).

Using this classification of symmetric bilinear forms, the indicators of the object m in these Tambara-Yamagami categories have been obtained by Shimizu [**31**, Thm. 6.3]: For any non-degenerate symmetric bilinear form B on $\mathbb{F}_2^r$, the n-th indicator $\nu_n(m)$ of the simple object m in $\mathcal{TY}(\mathbb{F}_2^r, \chi_B, \tau)$ is zero if n odd. Moreover,

(i) if B is not alternating, then

$$\nu_{2k}(m) = \mathrm{sgn}(\tau)^k \left(\frac{1+i}{\sqrt{2}}\right)^{rk} \left(\frac{1+i^{-k}}{\sqrt{2}}\right)^{r}. \tag{5.2}$$

(ii) If r is even and B is alternating, then

$$\nu_{2k}(m) = \begin{cases} \mathrm{sgn}(\tau) & \text{if } k \text{ is odd,} \\ 2^{r/2} & \text{if } k \text{ is even.} \end{cases} \tag{5.3}$$

We can now compute the total indicators for the integral Tambara-Yamagami categories associated with an elementary 2-group of square order.

PROPOSITION 5.3. *Let* $V = \mathbb{F}_2^{2\ell}$, B *a non-degenerate symmetric bilinear form on* V, τ *a square root of* $|V|^{-1}$, *and* $\mathcal{C} = \mathcal{TY}(V, \chi_B, \tau)$.

(i) *If* B *is not alternating, then* $\mathrm{FSexp}(\mathcal{C}) = 8$ *and* $\bar{\nu}(m) = 2\,\mathrm{sgn}(\tau) + 2^\ell$.

(ii) *If* B *is alternating, then* $\mathrm{FSexp}(\mathcal{C}) = 4$ *and* $\bar{\nu}(m) = \mathrm{sgn}(\tau) + 2^\ell$.

In particular, $\bar{\nu}(m) = 0$ *if, and only if,* B *is not alternating,* $\ell = 1$ *and* $\mathrm{sgn}(\tau) = -1$.

PROOF. Note that for all $a \in V$, $\nu_n(a) = 1 = d(a)$ if n is even, and 0 otherwise.

(i) If B is not alternating, then, by (5.2), we have

$$\nu_2(m) = \mathrm{sgn}(\tau), \quad \nu_4(m) = 0, \quad \nu_6(m) = \mathrm{sgn}(\tau), \;\; \nu_8(m) = 2^\ell = d(m).$$

Therefore, $\mathrm{FSexp}(\mathcal{C}) = 8$ and $\bar{\nu}(m) = \sum_{k=1}^4 \nu_{2k}(m) = 2\,\mathrm{sgn}(\tau) + 2^\ell$.

(i) If B is alternating, then, by (5.3), we have

$$\nu_2(m) = \mathrm{sgn}(\tau) \quad \text{and} \quad \nu_4(m) = 2^\ell = d(m).$$

Therefore, $\mathrm{FSexp}(\mathcal{C}) = 4$ and $\bar{\nu}(m) = \sum_{k=1}^2 \nu_{2k}(m) = \mathrm{sgn}(\tau) + 2^\ell$.

The last statement follows directly from statement (i) and (ii). □

COROLLARY 5.4. *Let* $\mathcal{C}$ *be a Tambara-Yamagami category associated with an elementary 2-group* V *of order* $2^{2\ell}$. *Then* $\mathcal{C}$ *is monoidally equivalent to* $\mathrm{Rep}(H)$ *for some semisimple Hopf algebra* H *if, and only if, all its total indicators are positive.*

PROOF. By [**33**, Prop 5.5], $\mathcal{C}$ has a fibration if, and only if, $\mathcal{C}$ is not monoidally equivalent to $\mathcal{TY}(\mathbb{F}_2^2, B_0, -2)$. It follows from Proposition 5.3 that the simple object m in $\mathcal{C}$ satisfies $\bar{\nu}(m) > 0$ if, and only if, $\mathcal{C}$ is not monoidally equivalent to $\mathcal{TY}(\mathbb{F}_2^2, B_0, -2)$. This proves the corollary. □

5.2. Odd characteristic. In contrast to the characteristic two case, positivity of total indicators may not be sufficient for the existence of fibration of an integral Tambara-Yamagami category associated with an elementary p-group for odd p. We will demonstrate this fact by computing the total indicators. By [**30**, Ch. 4], there are exactly two isometric classes of non-degenerate symmetric bilinear form on an r-dimensional $\mathbb{F}_p$-space. They are represented by B_0 and B_1 whose Gram matrix relative to the standard basis is given by

$$\left[\begin{array}{c|c} I_{r-1} & 0 \\ \hline 0 & u \end{array}\right]$$

where u can be any fixed quadratic nonresidue in $\mathbb{F}_p$, and I_{r-1} denotes the identity matrix of rank $r-1$. In particular, a non-degenerate symmetric bilinear form B on $\mathbb{F}_p^r$ is determined by its *discriminant* $\det(B)$ in $\mathbb{F}_p^\times/(\mathbb{F}_p^\times)^2$, where

$$\det B = \det\left([B(e_i, e_j)]_{ij}\right) \in \mathbb{F}_p^\times .$$

Therefore, the isometric class of a bilinear form B is uniquely determined by the Legendre symbol $\left(\frac{\det B}{p}\right)$ which is 1 if $\det B$ is a quadratic residue, and -1 otherwise. We can now compute the total indicators using the following formula obtained by

Shimizu [**31**, Thm. 6.1]: The n-th indicator of m in $\mathcal{TY}(\mathbb{F}_p^r, \chi_B, \tau)$ for any non-degenerate symmetric bilinear form B on $\mathbb{F}_p^r$ is given by

$$(5.4)\quad \nu_n(m) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \operatorname{sgn}(\tau)^k \varepsilon_p^{r(k+1)} \left(\frac{-k}{p}\right)^r \left(\frac{-2}{p}\right)^{r(k+1)} \left(\frac{\det B}{p}\right)^{k+1} & \text{if } n = 2k \text{ and } p \nmid k, \\ \operatorname{sgn}(\tau)^k \varepsilon_p^{rk} \left(\frac{-2}{p}\right)^{rk} \left(\frac{\det B}{p}\right)^k \sqrt{p^r} & \text{if } n = 2k \text{ and } p \mid k, \end{cases}$$

where $\varepsilon_p = \sqrt{\left(\frac{-1}{p}\right)}$.

Proposition 5.5. *Let $V = \mathbb{F}_p^{2\ell}$, χ_B the bicharacter associated with a non-degenerate bilinear form B on V, τ a square root of $|V|^{-1}$, and $\mathcal{C} = \mathcal{TY}(V, \chi_B, \tau)$.*

(i) *If $\left(\frac{-1}{p}\right)^\ell \left(\frac{\det B}{p}\right) \operatorname{sgn}(\tau) = 1$, then* $\operatorname{FSexp}(\mathcal{C}) = 2p$ *and*

$$\bar{\nu}(m) = p^\ell + (p-1)\operatorname{sgn}(\tau)\,.$$

(ii) *If $\left(\frac{-1}{p}\right)^\ell \left(\frac{\det B}{p}\right) \operatorname{sgn}(\tau) = -1$, then* $\operatorname{FSexp}(\mathcal{C}) = 4p$ *and $\bar{\nu}(m) = 0$.*

Proof. By [**31**, Thm. 3.2], $\nu_n(a) = \delta_{na,0}$ for $a \in V$. Therefore, $p \mid \operatorname{FSexp}(\mathcal{C})$. To determine $\operatorname{FSexp}(\mathcal{C})$, it is enough to consider the values $\nu_n(m)$ with $n = 2k$ and $p \mid k$ by virtue of (5.4). Note that

$$\varepsilon_p^{2\ell}\left(\frac{\det B}{p}\right) = \left(\frac{-1}{p}\right)^\ell \left(\frac{\det B}{p}\right) = \epsilon \operatorname{sgn}(\tau)$$

for some $\epsilon = \pm 1$. Therefore, (5.4) becomes

$$(5.5)\qquad \nu_n(m) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \operatorname{sgn}(\tau)\epsilon^{k+1} & \text{if } n = 2k \text{ and } p \nmid k, \\ \epsilon^k p^\ell & \text{if } n = 2k \text{ and } p \mid k, \end{cases}$$

If $\epsilon = 1$, then $\nu_{2p}(m) = p^\ell = d(m)$ and so $\operatorname{FSexp}(\mathcal{C}) = 2p$. Moreover,

$$\bar{\nu}(m) = p^\ell + \sum_{k=1}^{p-1} \nu_{2k}(m) = p^\ell + \operatorname{sgn}(\tau)(p-1)\,.$$

If $\epsilon = -1$, then $\nu_{2p}(m) = -p^\ell$ and $\nu_{4p}(m) = p^\ell$. Thus, $\operatorname{FSexp}(\mathcal{C}) = 4p$ and

$$\bar{\nu}(m) = \sum_{k=1}^{2p} \nu_{2k}(m) = 0\,. \quad \square$$

Corollary 5.6. *Let B be a non-degenerate symmetric bilinear form on $V = \mathbb{F}_p^{2\ell}$ and $\tau = \pm p^{-\ell}$. Then $\mathcal{C} = \mathcal{TY}(V, \chi_B, \tau)$ admits a fibration if, and only if, $\bar{\nu}(s) \geq d(s)$ for all simple object $s \in \mathcal{C}$.*

Proof. By the preceding proposition, for $s \in V$, $\bar{\nu}(s) \geq 2 > d(s)$. In particular, the inequality holds for all $s \in V$ automatically. Therefore, we only need to consider the simple object m.

By [**33**, Prop. 4.1], $\mathcal{C}$ admits a fibration if, and only if, $\tau = p^{-\ell}$ and B is hyperbolic, i.e. the Gram matrix of B relative to some basis of $\mathbb{F}_p^{2\ell}$ is of the form $\left[\begin{array}{c|c} 0 & I_\ell \\ \hline I_\ell & 0 \end{array}\right]$, or equivalently, $\left(\frac{\det B}{p}\right) = \left(\frac{-1}{p}\right)^\ell$.

If $\mathcal{C}$ admits a fibration, then $\left(\frac{\det B}{p}\right)\left(\frac{-1}{p}\right)^{\ell} = 1 = \mathrm{sgn}(\tau)$ and so $\bar{\nu}(m) = p^{\ell} + p - 1 > p^{\ell} = d(m)$ by Proposition 5.5. Conversely, if $\bar{\nu}(m) \geq p^{\ell}$, then, by Proposition 5.5, $\mathrm{sgn}(\tau) = 1$ and $\left(\frac{\det B}{p}\right)\left(\frac{-1}{p}\right)^{\ell} = 1$, or equivalently $\left(\frac{\det B}{p}\right) = \left(\frac{-1}{p}\right)^{\ell}$. Therefore, $\mathcal{C}$ admits a fibration by the preceding paragraph. □

REMARK 5.7. The corollary implies that there exists a genuine semisimple quasi-Hopf algebra of which the total indicators of its representations are all positive. Let B be the bilinear form on $\mathbb{F}_p^{2\ell}$ whose Gram matrix relative to the standard basis is $\left[\begin{array}{c|c} I_{2\ell-1} & 0 \\ \hline 0 & (-1)^{\ell+1} \end{array}\right]$. Then $\left(\frac{\det B}{p}\right)\left(\frac{-1}{p}\right)^{\ell} = -1$. If we take $\mathrm{sgn}(\tau) = -1$ or $\tau = -p^{\ell}$, then, by Corollary 5.6 and Proposition 5.5, $\mathcal{TY}(\mathbb{F}_p^{2\ell}, \chi_B, \tau)$ is monoidally equivalent to $\mathrm{Rep}(H)$ of a *genuine* quasi-Hopf algebra H with positive total indicators. Therefore, in general, the existence of vanishing total indicators is not a necessary condition for a semisimple quasi-Hopf algebra being genuine.

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