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Pre-cellular algebras

Tao Wang

Supervised by Professor Gongxiang Liu

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Department of Mathematics, Nanjing University

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毕业论文题目: 张量范畴的阶和指标

数学系 院系 基础数学 专业 2012 级硕士姓名: 白志军

指导教师(姓名、职称): 刘公祥 副教授

摘要

本文首先给出了张量范畴和拟Hopf代数的定义，然后证明了拟Hopf代数的表示范畴是一个张量范畴。接下来，我们把朱永昌等关于阶和指标的定义(见参考文献[32])推广到了张量范畴，并且证明了张量范畴的阶和指标是两个不变量。进一步，我们证明了在半单Hopf代数表示范畴中关于两个不变量的一些可除性以及整性的结果。最后，我们证明了本文关于阶和指标的定义的确是朱永昌等给出的定义的推广。

关键词: 张量范畴，拟Hopf代数，阶，指标，张量等价.

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THESIS : On the Order and Index of tensor category

SPECIALIZATION : Fundamental Mathematics

POSTGRADUATE : Zhijun Bai

MENTOR : Professor Gongxiang Liu

Abstract

In this article, we first give the definitions of tensor category and quasi-Hopf algebra, and then we prove the representation category of quasi-Hopf algebras is a tensor category. Then we generalized the definitions of the order and index which given by Yong-chang Zhu etc. (see cf. [32]). to the tensor category, and we prove that the order and index of tensor category are two invariants. Furthermore, we prove various divisibility and integrality results for the two invariants about the representation category of semisimple Hopf algebras. Finally, we prove our definitions about order and index are indeed to generalize the definitions given by Yong-chang Zhu etc.(cf. [32]).

KEYWORDS : monoidal category, quasi-Hopf algebra, order, index, monoidal equivalence.

Contents

1 Introduction

We all know that tensor category is an important category in category theory. In particular, the representation category of Hopf algebras and quasi-Hopf algebras are also tensor category. In this article, we will introduce two invariants in tensor category.

In Section 2, we first give the defininition of tensor category. Furthermore, we give some examples of tensor category that we can make better sense of the definitions.

In Section 3, this section is the most important part of our article. We give the definitions of order and index in tensor category and we also prove that the order and index is two invariants of tensor category. In Section 4, we study some good properties of order and index of the representation of Hopf algebras. We define the notion of the order and the multiplicity of a module and prove that the order of a module divides its multiplicity times the dimension of the Hopf algebra. Furthermore, we study the index of imprimitivity, or briefly the index, of the matrix that represents the left multiplication by a character with respect to the canonical basis that we have in the character ring—the basis consisting of the irreducible characters. The main result of this section is a precise formula for the index in terms of central grouplike elements. Essentially, the result says that the eigenvalues of the above matrix that have the same absolute value as the degree are obtained by evaluating the character at certain central grouplike elements.

2 Preliminaries

Throughout the notes, for simplicity we will assume that the ground field k is algebraically closed unless otherwise specified, even though in many cases this assumption will not be needed.

2.1 Tensor category

Definition 2.1.1. A monoidal category is a quintuple $(\mathcal{C}, \otimes, a, 1, \iota)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called the tensor product a:

$(\bullet \otimes \bullet) \otimes \bullet \xrightarrow{\sim} \bullet \otimes (\bullet \otimes \bullet)$ is a functorial isomorphism:

$$a_{X,Y,Z}: \ (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \quad X, Y, Z \in \mathcal{C} \tag{1}$$

called the associativity constraint (or associativity isomorphism),$1 \in \mathcal{C}$ is an object of $\mathcal{C}$, and $\iota : 1 \otimes 1 \to 1$ is an isomorphism, subject to the following two axioms.

1. **The pentagon axiom**. The diagram

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W\otimes X,Y,Z}"'] \arrow[dr, "a_{W,X,Y}\otimes Id_Z"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] \\
W \otimes (X \otimes (Y \otimes Z)) & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ll, "Id_W\otimes a_{X,Y,Z}"']
\end{tikzcd} \tag{2}$$

   is commutative for all objects $W, X, Y, Z \in \mathcal{C}$.

2. **The unit axiom**.The functors L_1 and R_1 of left and right multiplication by 1 are equivalences $\mathcal{C} \to \mathcal{C}$.

The pair $(1, \iota)$ is called the unit object of $\mathcal{C}$.

We see that the set of isomorphism classes of objects in a small monoidal category indeed has a natural structure of a monoid, with multiplication $\otimes$ and unit 1. Thus, in the categorical-algebraic dictionary, monoidal categories indeed correspond to monoids (which explains their name).

Notation. Let $(\mathcal{C}, \otimes, a, 1, \iota)$ be a monoidal category. Define the isomorphism $l_X : 1 \otimes X \to X$ by the formula

$$l_X = L_1^{-1}((\iota \otimes Id) \circ a^{-1}_{1,1,X}),$$

and the isomorphism $r_X : X \otimes 1 \to X$ by the formula

$$r_X = R_1^{-1}((Id \otimes \iota) \circ a_{X,1,1}.$$

This gives rise to functorial isomorphisms $l : L_1 \to Id_{\mathcal{C}}$ and $r : R_1 \to Id_{\mathcal{C}}$. These isomorphisms are called the unit constraints or unit isomorphisms.

They provide the categorical counterpart of the unit axiom $1X = X1 = X$ of a monoid in the same sense as the associativity isomorphism provides the categorical counterpart of the associativity equation.

Proposition 2.1.2. The"triangle" diagram

$$\begin{tikzcd}
(X \otimes 1) \otimes Y \arrow[rr, "a_{X,1,Y}"] \arrow[dr, "r_X \otimes Id_Y"'] & & X \otimes (1 \otimes Y) \arrow[dl, "Id_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd} \qquad (3)$$

is commutative for all $X, Y \in \mathcal{C}$. In particular, one has $r_1 = l_1 = \iota$.

Example 2.1.3. The category **Sets** of sets is a monoidal category, where the tensor product is the Cartesian product and the unit object is a one element set; the structure morphisms a, ι, l, r are obvious.

Example 2.1.4. Any additive category is monoidal, with $\otimes$ being the direct sum functor $\oplus$, and 1 being the zero object.

Example 2.1.5. Let k be any field. The category $k - \mathbf{Vec}$ of all k-vector spaces is a monoidal category, where $\otimes = \otimes_k$, $1 = k$, and the morphisms a, ι, l, r are the obvious ones. The same is true about the category of finite dimensional vector spaces over k, denoted by $k - Vec$. We will often drop k from the notation when no confusion is possible.

More generally, if R is a commutative unital ring, then replacing k by R we can define monoidal categories $R - \mathbf{mod}$ of R-modules and R-mod of R-modules of finite type.

Example 2.1.6. Let G be a group. The category $\mathbf{Rep}_k(G)$ of all representations of G over k is a monoidal category, with $\otimes$ being the tensor product of representations : if for a representation V one denotes by ρ_V the corresponding map $G \to GL(V)$, then

$$\rho_{V \otimes W}(a) = \rho_V(a) \otimes Id_W + Id_V \otimes \rho_W(a)$$

The unit object in this category is the trivial representation $1 = k$. A similar statement holds for the category $Rep_k(G)$ of finite dimensional representations of G. Again, we will drop the subscript k when no confusion is possible.

As we have explained, monoidal categories are a categorification of monoids. Now we pass to categori

cation of morphisms between monoids, namely monoidal functors.

Definition 2.1.7. Let $(\mathcal{C}, \otimes, 1, a, \iota)$ and $(\mathcal{C}', \otimes', a', 1')$ be two monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J) where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and $J = \{J_{X,Y} : \ F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y) \ \mid X, Y \in \mathcal{C}\}$ is a natural isomorphism, such that $F(1)$ is isomorphic to $1'$. and the diagram

$$\begin{tikzcd} (F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "a'_{F(X,F(Y),F(Z))}"] \arrow[d, "J_{X,Y} \otimes' Id_{F(Z)}"'] & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "Id_{F(X)} \otimes' J_{Y,Z}"] \\ F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y, Z}"'] & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X, Y \otimes Z}"] \\ F((X \otimes Y) \otimes Z) \arrow[r, "F(a_{X,Y,Z})"] & F(X \otimes (Y \otimes Z)) \end{tikzcd} \tag{4}$$

is commutative for all $X, Y, Z \in \mathcal{C}$ ("the monoidal structure axiom").

A monoidal functor F is said to be an equivalence of monoidal categories if it is an equivalence of ordinary categories.

Proposition 2.1.8. For any monoidal functor $(F, J) : \mathcal{C} \to \mathcal{C}'$, the diagrams

$$\begin{tikzcd} 1' \otimes' F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\varphi \otimes' Id_{F(X)}"'] & F(X) \\ F(1) \otimes' F(X) \arrow[r, "J_{1,X}"] & F(1 \otimes X) \arrow[u, "F(l_X)"'] \end{tikzcd} \tag{5}$$

and

$$\begin{tikzcd} F(X) \otimes' 1' \arrow[r, "r'_{F(X)}"] \arrow[d, "Id_{F(X)} \otimes' \varphi"'] & F(X) \\ F(X) \otimes' F(1) \arrow[r, "J_{X,1}"] & F(X \otimes 1) \arrow[u, "F(r_X)"'] \end{tikzcd} \tag{6}$$

are commutative for all $X \in \mathcal{C}$.

Proof. This follows by applying the pentagon axiom for the quadruple of objects $X, 1, 1, Y$. More specifically, we have the following diagram:

$$
\begin{tikzcd}
((X \otimes 1) \otimes 1) \otimes Y \arrow[rrrr, "a_{X,1,1} \otimes Id"] \arrow[ddd, "a_{X \otimes 1,1,Y}"'] \arrow[drr, "r_X \otimes Id \otimes Id"'] & & & & (X \otimes (1 \otimes 1)) \otimes Y \arrow[dll, "(Id \otimes \iota) \otimes Id"] \arrow[ddd, "a_{X,1 \otimes 1,Y}"] \\
& & (X \otimes 1) \otimes Y \arrow[d, "a_{X,1,Y}"'] & & \\
& & X \otimes (1 \otimes Y) & & \\
(X \otimes 1) \otimes (1 \otimes Y) \arrow[urr, "r_X \otimes Id"] \arrow[drr, "a_{X,1,1 \otimes Y}"'] & & & & X \otimes ((1 \otimes 1) \otimes Y) \arrow[ull, "Id \otimes (\iota \otimes Id)"'] \arrow[dll, "Id \otimes a_{1,1,Y}"] \\
& & X \otimes (1 \otimes (1 \otimes Y)) \arrow[uu, "Id \otimes l_{1 \otimes Y}"] & &
\end{tikzcd}
\tag{7}
$$

To prove the proposition, it suffices to establish the commutativity of the bottom left triangle (as any object of $\mathcal{C}$ is isomorphic to one of the form $1 \otimes Y$).Since the outside pentagon is commutative (by the pentagon axiom), it suffices to establish the commutativity of the other parts of the pentagon. Now, the two quadrangles are commutative due to the functoriality of the associativity isomorphisms, the commutativity of the upper triangle is the definition of r, and the commutativity of the lower right triangle is the definition of l.
The last statement is obtained by setting $X = Y = 1$ in (6). $\square$

Proposition in the above implies that a monoidal functor can be equivalently defined as follows.

Definition 2.1.9. A monoidal functor $\mathcal{C} \to \mathcal{C}'$ is a triple (F, J, φ) which satisfies the monoidal structure axiom and above Proposition.

This is a more traditional definition of a monoidal functor.

Monoidal functors between two monoidal categories themselves form a category. Namely, one has the following notion of a morphism (or natural transformation) between two monoidal functors.

Definition 2.1.10. Let $(\mathcal{C}, \otimes, 1, a, \iota)$ and $(\mathcal{C}', \otimes', 1', a, \iota)$ be two monoidal categories, and $(F^1, J^1), (F^2, J^2)$ two monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism (or a natural transformation) of monoidal functors $\eta : (F^1, J^1) \to$

(F^2, J^2) is a natural transformation $\eta : F^1 \to F^2$ such that η_1 is an isomorphism, and the diagram

$$\begin{tikzcd} F^1(X) \otimes' F^1(Y) \arrow[r, "J^1_{X,Y}"] \arrow[d, "\eta_X \otimes' \eta_Y"'] & F^1(X \otimes Y) \arrow[d, "\eta_{X \otimes Y}"] \\ F^2(X) \otimes' F^2(Y) \arrow[r, "J^2_{X,Y}"] & F^2(X \otimes Y) \end{tikzcd} \tag{8}$$

is commutative for all $X, Y \in \mathcal{C}$.

Let us now give some examples of monoidal functors and natural transformations.

Example 2.1.11. An important class of examples of monoidal functors is forgetful functors (e.g. functors of "forgetting the structure", from the categories of groups, topological spaces, etc., to the category of sets).Such functors have an obvious monoidal structure. An example important in these notes is the forgetful functor $\mathbf{Rep}_G \to \mathbf{Vec}$ from the representation category of a group to the category of vector spaces. More generally, if $H \subset G$ is a subgroup, then we have a forgetful (or restriction) functor $\mathbf{Rep}_G \to \mathbf{Vec}_H$. Still more generally, if $f : H \to G$ is a group homomorphism, then we have the pullback functor $f^* : \mathbf{Rep}_G \to \mathbf{Rep}_H$. All these functors are monoidal.

Example 2.1.12. Let $f : H \to G$ be a homomorphism of groups. Then any H–graded vector space is naturally G-graded (by pushforward of grading). Thus we have a natural monoidal functor $f_* : \mathbf{Vec}_H \to \mathbf{Vec}_G$. If G is the trivial group, then f_* is just the forgetful functor $\mathbf{Vec}_H \to \mathbf{Vec}$.

Example 2.1.13. Let S be a monoid, and $\mathcal{C} = Vec_S$, and $Id_{\mathcal{C}}$ the identity functor of $\mathcal{C}$. It is easy to see that morphisms $\eta : Id_{\mathcal{C}} \to Id_{\mathcal{C}}$ correspond to homomorphisms of monoids: $\eta : S \to k$ (where k is equipped with the multiplication operation). In particular, $\eta(s)$ may be 0 for some s, so η does not have to be an isomorphism.

2.2 Quasi-Hopf algebra

In this section we recall some definitions and results and fix notation. Throughout, k will be a fixed field and all algebras, linear spaces etc. will be over

k unadorned $\otimes$ means $\otimes_k$. For coalgebras and Hopf algebras we shall use $\sum$-natation: $\Delta(h) = \sum h_1 \otimes h_2$,etc.

Definition 2.2.1. Let H be a k-algebra, $\Delta : H \to H \otimes H$, $\varepsilon : H \to k$ two algebra homomorphisms. H is called a quasi-bialgebra if there exists an invertible $\Phi \in H \otimes H \otimes H$ such that, for all elements $h \in H$, we have :

(1.2.1) $(I \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes I)\Phi^{-1}$,

(1.2.2) $(I \otimes I \otimes \Delta)(\Phi)(\Delta \otimes I \otimes I)(\Phi) = (1 \otimes \Phi)(I \otimes \Delta \otimes I)(\Phi)(\Phi \otimes 1)$,

(1.2.3) $(\varepsilon \otimes I)(\Delta(h)) = 1 \otimes h$ and $(I \otimes \epsilon)(\Delta(h)) = h \otimes 1$,

(1.2.4) $(I \otimes \epsilon \otimes I)(\Phi) = 1 \otimes 1 \otimes 1$,

where $I = id_H$. The map Δ is called the coproduct or the comultiplication and ϵ the counit. H is called a quasi-Hopf algebra if, moreover, there exist an anti-autormorphism S of the algebra H and elements α and β of H such that, for all $h \in H$, we have :

(1.2.5) $\sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha$ and $\sum h_1 \beta S(h_2) = \varepsilon(h)\beta$,

(1.2.6) $\sum X^1 \beta S(X^2)\alpha X^3 = 1$ and $\sum S(x^1)\alpha x^2 \beta S(x^3) = 1$,

where $\Phi = \sum X^1 \otimes X^2 \otimes X^3$, $\Phi^{-1} = x^1 \otimes x^2 \otimes x^3$ (formal notation), and we used the $\sum$-notation: $\Delta(h) = \sum h_1 \otimes h_2$. In this case, S is called the antipode of H.

Notation. Every Hopf algebra with bijective antipode is a quasi-Hopf algebra with $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$.

Now we suppose that $(H, \Delta, \epsilon, \Phi)$ is a quasi-bialgebra. If U, V, W are left H-modules, define $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ by

$$a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w))$$

Then the category H-mod of left H-modules becomes a tensor category with tensor product $\otimes$ given via Δ, associativity constraints $a_{U,V,W}$, unit k as a trivial H-module and the usual left and right unit constraints.

Now we give an example of non-trivial quasi-Hopf algebra:

Example 2.2.2. Let $D(k[G])$ be the Drinfel'd double of the Hopf algebra $k[G]$ of a finite group G and $\{e_g\}_{g \in G}$ be the dual basis of the basis $\{g\}_{g \in G}$ of $k[G]$. Suppose given a normalized 3-cocycle on the group G, i.e., a function $\omega : G \times G \times G \to k \backslash \{0\}$ such that

$$\omega(x,y,z)\omega(tx,y,z)^{-1}\omega(t,xy,z)\omega(t,x,yz)^{-1}\omega(t,x,y)=1$$

for all $t,x,y,z\in G$, and such that $\omega(x,y,z)=1$ whenever $x,y,orz=1$. Consider a finite-dimensional vector space $D^\omega(G)$ with a basis $\{e_gx\}_{(g,x)\in G\times G}$. Define a product on $D^\omega(G)$ by

$$(e_gx)(e_hy)=\delta_{g,xhx^{-1}}\theta(g,x,y)e_g(xy)$$

where $\theta(g,x,y)=\omega(g,x,y)\omega(x,y,xy^{-1}gxy)\omega(x,x^{-1}gx,y)^{-1}$. It is easy to check that this product is associative and has the element $1=\sum_{g\in G}e_g1$ as a left and right unit. Then we define $\Delta: D^\omega(G)\to D^\omega(G)\otimes D^\omega(G)$ and $\epsilon: D^\omega(G)\to k$ by

$$\Delta(e_gx)=\sum_{uv=g}\gamma(x,u,v)e_ux\otimes e_x \quad and \quad \varepsilon(e_gx)=\delta_{g,1}$$

where $\gamma(x,u,v)=\omega(u,v,x)\omega(x,x^{-1}ux,x^{-1}vx)\omega(u,x,x^{-1}vx)^{-1}$. Set also

$$\Phi=\sum_{x,y,z\in G}\omega(x,y,z)^{-1}e_x\otimes e_y\otimes e_z$$

$\alpha=1$, and $\beta=\sum_{g\in G}\omega(g,g^{-1},g)e_g$. we define an anti-automorphism S of the algebra $D^\omega(G)$ by

$$S(e_gx)=\theta(g^{-1},x,x^{-1})\gamma(x,g,g^{-1})e_{x^{-1}gx}x^{-1}$$

Then $(D^\omega(G),\Delta,\epsilon,\Phi,\alpha,\beta)$ is a quasi-Hopf algebra in the sense of Definition 1.2.1.

3 The order and index of tensor category

In this section we always assume that $\mathcal{C}$ is a tensor product. Now we will introduce the most important concepts of my article.

3.1 Order

Definition 3.1.1. For any $V\in Ob(\mathcal{C})$, we define:

$$\mathrm{ord}(V):=\min\{n\mid I \text{ is the direct summand of } V^{\otimes m}\}.$$

Proposition 3.1.2. If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal equivalence, then we have ord(V) = ord(F(V)), i.e. , Order is an invariant of tensor equivalence.

Proof. Firstly, we proof ord(F(V)) $\leq$ ord(V).
From the definition of monoidal equivalence, for any $V \in \mathcal{C}$, we have $F(V^{\otimes m}) \simeq (F(V))^{\otimes m}$. If $\mathrm{ord}(V) := m$, so there is $W \in \mathcal{C}$ such that $V^{\otimes m} \simeq I \oplus W$, and then we have

$$(F(V))^{\otimes m} \simeq F(V^{\otimes m}) \simeq F(I \oplus W) \simeq F(I) \oplus F(W) \simeq I' \oplus F(W).$$

From this, ord(F(V)) $\leq$ ord(V).
Secondly, we proof ord(V) $\leq$ ord(F(V)). There is a monoidal equivalence $G : \mathcal{D} \to \mathcal{C}$ such that $FG \approx 1_{\mathcal{D}}, GF \approx 1_{\mathcal{C}}$. If $\mathrm{ord}(V) := n$, we have $F(V^{\otimes n}) \simeq I' \oplus U$,

$$V^{\otimes n} \simeq G((F(V))^{\otimes n}) \simeq G(I') \oplus G(U) \simeq I \oplus G(U).$$

so $\mathrm{ord}(V) \leq \mathrm{ord}(F(V))$.
From the above, we know that $\mathrm{ord}(V) = \mathrm{ord}(F(V))$.

□

3.2 Index

In this subsection, we further let any tensor category be the semisimple tensor category. we assume that $\{V_i\}_{i=1}^n$ are all the simple objects of $\mathcal{C}$.

Definition 3.2.1. $K_0(\mathcal{C}) := \bigoplus_{i=1}^n \mathbb{Z}V_i$.

Notation.
- If we define $V_i \cdot V_j := [V_i \otimes V_j]$, then it is obvious that $K_0(\mathcal{C})$ is a ring.
- If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal equivalence, then F induces a ring isomorphism from $K_0(\mathcal{C})$ to $K_0(\mathcal{D})$.

As $\mathcal{C}$ is semisimple, for $V \in \mathcal{C}$, then we have

$$V \cdot V_j = \sum_{i=1}^n a_{ij} V_i,$$

a_{ij} is the multiplicity of V_i in the decomposition of $V \otimes V_j$ into simple objects-clearly a nonnegative integer. We let A be the matrix $(a_{ij})_{n\times n}$. For convenience, we let A be a indecomposable matrix (see definition 3.0.7), and from Perron-Frobenius theorem (see Thm. 3.0.11 and Thm. 3.0.12). we have :

Definition 3.2.2. We define the index of V by

$$\mathrm{ind}(V) := \mid \{\mu \mid \mu \text{ is an eigenvalue of } A \text{ with } \mid \mu \mid = \lambda\} \mid,$$

where λ is the biggest eigenvalue of A.

Proposition 3.2.3. If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal equivalence, then we have $\mathrm{ind}(V) = \mathrm{ind}(F(V))$. i.e. Index is an invariant of tensor equivalence.

Proof. If we let

$$V \cdot (V_1, \cdots, V_n) = (V_1, \cdots, V_n)A, \quad A = (a_{ij})_{n\times n}.$$

$$F(V) \cdot (F(V_1), \cdots, F(V_n)) = (F(V_1), \cdots F(V_n))B, \quad B = (b_{ij})_{n\otimes n}$$

we just need to prove that $A = B$. For V_j, $(1 \leq j \leq n)$, we have

$$F(V \otimes V_j) = F(a_{1j}V_1 + \cdots + a_{nj}V_n) = a_{1j}F(V_1) + \cdots + a_{nj}F(V_n).$$

$$F(V) \otimes F(V_j) = b_{1j}F(V_1) + \cdots + b_{nj}F(V_n).$$

As $F(V \otimes V_j) \simeq F(V) \otimes F(V_j)$, and $K_0(\mathcal{C}) \simeq K_0(\mathcal{D})$. so we get $a_{ij} = b_{ij}$ for all $1 \leq i, j \leq n$. i.e. $A = B$. $\square$

4 Some properties in representation category of semisimple Hopf algebras

We know that the representation category of Hopf algebras is absolutely tensor category. In this section we will research the representation category of semisimple Hopf algebras.

In this part, we assume from now on that K is an algebraically closed field of characteristic zero and that H is a semisimple Hopf algebra over K with an integral Λ satisfying $\varepsilon(\Lambda) = 1$. Firstly, we let $(V^{\otimes m})^H := \{v \in V^{\otimes m} \mid h \cdot v = \epsilon(h)v\}$.

Definition 4.0.4. Suppose that V is an H-module. The smallest natural number m such that $V^{\otimes m}$ contains a nonzero invariant subspace $(V^{\otimes m})^H$ is called the order of V and is denoted by $\mathrm{ord}(V)$. If m is the order of V, then the dimension of this invariant subspace is called the multiplicity of V and is denoted by $\mathrm{mult}(V) := \dim((V^{\otimes m})^H)$.

Notation. It is easy to see that this definition is equivalent to the order which we defined in the above section.

The notion of the order of a module generalizes the notion of the order of an element in the theory of finite groups. To see this, let G be a finite group and consider the ring K^G of functions on G, which is isomorphic to the dual group ring $K[G]^*$. Since this is a commutative Hopf algebra, all its simple modules are one-dimensional, and their characters are given by evaluating a function at a fixed element of the group. This sets up a one-to-one correspondence between the elements of the group and the simple K^G-module under which the product of two group elements corresponds to the tensor product of the modules. Since all these modules are one-dimensional, a tensor power contains a nonzero invariant subspace if and only if it is trivial, which means that the corresponding element of the group is the unit element. Therefore, the order of the module in the sense of the above definition coincides with the order of the element in the sense of group theory. In the case of a general semisimple Hopf algebra H, we see by the same reasoning that, if V is one-dimensional, and therefore determined by its character $\gamma : H \to K$, the order of V in the sense of the above definition coincides with the order of γ in the group $G(H^*)$ of grouplike elements of the dual Hopf algebra H^*.

To proceed further, we will need some properties of symmetric polynomials. Consider the polynomial ring $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ in n variables. Recall that, for a nonnegative integer k, the k-th elementary symmetric polynomial $e_k = e_k(x_1, \ldots, x_n)$ is defined as

$$e_k(x_1, \ldots, x_n) := \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k}$$

In particular, we have $e_0 = 1, e_1 = x_1 + x_2 + \ldots + x_n, e_n = x_1 x_2 \ldots x_n$, and $e_k = 0$ for $k > n$. The fundamental theorem on symmetric polynomials asserts that every symmetric polynomial with integer coefficients can be expressed as a polynomial with integer coefficients in the elementary symmetric polynomials. This holds in particular for the power sums

$$s_k = s_k(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^k$$

Conversely, the power sums have the property that every symmetric polynomial can be expressed as a polynomial with rational coefficients in the power sums. The polynomials that achieve this for the elementary symmetric polynomials are the so-called (fractional) Newton polynomials Q_n, which are defined via the following $n \times n$-determinant:

$$Q_n(x_1, \ldots, x_n) := \frac{(-1)^n}{n!} \begin{vmatrix} x_1 & 1 & 0 & 0 & \ldots & 0 \\ x_2 & x_1 & 2 & 0 & \ldots & 0 \\ x_3 & x_2 & x_1 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ x_{n-1} & x_{n-2} & x_{n-3} & x_{n-4} & \ldots & n-1 \\ x_n & x_{n-1} & x_{n-2} & x_{n-3} & \ldots & x_1 \end{vmatrix}$$

The polynomials $Q_1, \ldots, Q_{n-1}$, which involve only fewer variables, can of course also be considered as elements of $\mathbb{Q}[x_1, x_2, \ldots, x_n]$. The formula that expresses the elementary symmetric polynomials in terms of the power sums is known as Newton's formula (cf. [18] , P. 110) :

Lemma 4.0.5. For $n \geq 1$ and $j = 1, \ldots, n$, we have $e_j = (-1)^j Q_j(s_1, s_2, \ldots, s_n)$.

With the help of this formula, we can now prove the following theorem:

Theorem 4.0.6. The order of a nonzero H-module V is a finite number. It is not larger than $\dim(H)$ and divides $\dim(H)\mathrm{mult}(V)$.

Proof. Firstly, for any $\varphi \in H^*$, the trace of the left multiplication

$$L_\varphi : H^* \to H^*, \quad \psi \mapsto \varphi\psi$$

is $n\varphi(\Lambda)$, where $n = \dim(H)$. Let $\chi \in H^*$ be the character of V, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the not necessarily distinct eigenvalues of L_χ. If $m := \mathrm{ord}(V) \leq n$, we have for $k = 1, \ldots, m-1$ that

$$s_k(\lambda_1, \ldots, \lambda_n) = tr(L_{\chi^k}) = n\chi^k(\Lambda) = 0$$

since $V^{\otimes k}$ does not contain a nonzero invariant subspace. For $k = m$, we get similarly that $s_m(\lambda_1, \ldots, \lambda_n) = n\mathrm{mult}(V)$. By Newton's formula, we therefore have

$$e_m(\lambda_1, \ldots, \lambda_n) = \frac{1}{m!} \begin{vmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & m-1 \\ s_m & 0 & 0 & 0 & \ldots & 0 \end{vmatrix} = \frac{(-1)^{m-1}}{m} s_m(\lambda_1, \ldots, \lambda_n)$$

Now note that the eigenvalues $\lambda_1, \ldots, \lambda_n$, and therefore the elementary symmetric functions $e_k \lambda_1, \ldots, \lambda_n$, are algebraic integers. To see this, consider the left multiplication by χ not on the whole dual Hopf algebra H^*, but only on the character ring $Ch(H)$, Its matrix representation with respect to the basis consisting of the irreducible characters has integer entries, and therefore the Cayley-Hamilton theorem implies that it satisfies a monic polynomial with integer coefficients, namely its characteristic polynomial. Evaluating this on the unit of the character ring, we see that χ itself satisfies this polynomial, and therefore also L_χ satisfies this polynomial. Since $\lambda_1, \ldots, \lambda_n$ are roots of this polynomial, they must be algebraic integers. This shows that the fraction

$$\frac{n\mathrm{mult}(V)}{m} = \frac{1}{m} s_m(\lambda_1, \ldots, \lambda_n) = (-1)^{m-1} e_m(\lambda_1, \ldots, \lambda_n)$$

is an algebraic integer, which therefore must be an integer (cf. [18], P.91). This proves the divisibility assertion.

It still remains to be shown that the order is finite and bounded by n. So suppose that this is not the case. The reasoning above then shows that the power sums $s_1, \ldots, s_n$ of the eigenvalues are zero, and therefore, by Newton's formula, the elementary symmetric functions $e_1, \ldots, e_n$ of the eigenvalues are zero. From this, we conclude as follows that all eigenvalues $\lambda_1, \ldots, \lambda_n$ are zero: First, we have $e_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \lambda_2 \ldots \lambda_n = 0$, which implies that $\lambda_i = 0$ for some i. Considering the next elementary symmetric function, we have

$$e_{n-1}(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdot \ldots \cdot \lambda_{i-1} \lambda_{i+1} \cdot \ldots \cdot \lambda_n = 0$$

since all the other summands in the definition of e_{n-1} vanish. This implies that we have $\lambda_j = 0$ for another $j \neq i$. Proceeding in this way, we arrive at the assertion that all eigenvalues of L_χ are zero. But this is not the case, as a nonzero integral of H^* is an eigenvector for L_χ corresponding to the eigenvalue $\dim(V)$. $\square$

Suppose that $\chi_1, \ldots, \chi_k$ are the distinct irreducible characters of the semisimple Hopf algebra H under consideration, and that $V_1, \ldots, V_k$ are simple H-modules of dimension $n_1, \ldots, n_k$ corresponding to these characters. We can assume that $V_1 = K$, the base field considered as a trivial H-module, with character $\chi_1 = \varepsilon$, the counit. If V is an arbitrary H-module with character χ, we have already used in the proof of Theorem (3.0.1) that the matrix representation of the left multiplication by χ on the character ring $Ch(H)$ with respect to the basis $\chi_1, \ldots, \chi_k$ has nonnegative integer entries: If

$$\chi\chi_j = \sum_{i=1}^{k} a_{ij}\chi_i$$

then a_{ij} is just the multiplicity of V_i in the decomposition of $V \otimes V_j$ into simple modules clearly a nonnegative integer.

Definition 4.0.7. The $k \times k$-matrix $A = (a_{ij})_{i,j=1,\ldots,k}$ is called decomposable if it is possible to find a decomposition $I_k = M \cup N$ of $I_k = \{1, \ldots, k\}$ into disjoint nonempty sets M and N such that $a_{ij} = 0$ whenever $i \in M$ and j_N, otherwise, it is called indecomposable.

Notation. Any power A^m of a decomposable matrix is still decomposable, since its matrix elements are

$$\sum_{i_1,\ldots,i_{m-1}}^{k} a_{ii_1}a_{i_1i_2}\ldots a_{i_{m-1}j}$$

and these are zero whenever $i \in M$ and $j \in N$.

Suppose that V is an H-module. The set of all elements of H that act on V identically as zero is called the annihilator of V. This is obviously a two-sided ideal, but not necessarily a Hopf ideal. But if we denote by J the intersection of the annihilators of all the tensor powers $V^{\otimes m}$ of V, including the trivial module K for $m = 0$, we get the largest Hopf ideal contained in the annihilator:

Lemma 4.0.8. *J is a Hopf ideal of H. Every other Hopf ideal of H that is contained in the annihilator of V is contained in J.*

For a proof of this lemma, we refer to (cf [30] Thm. 1, p. 125). Note that the usage of the notion of a Hopf algebra in [30] differs from our usage. To adopt the proof, one has to use the fact that in a finite-dimensional Hopf algebra a bi-ideal is a Hopf ideal(see cf [31] Lem. 6, p. 331).

Notation. A a simple H-module can be embedded into a tensor power of V if and only if it is annihilated by J, this also shows that if a simple module can be embedded into a tensor power of V, its dual can also be embedded into a tensor power of V.

Now, let as before χ be the character and I be the annihilator of V, and denote by A the matrix representation of the left multiplication by χ on the character ring $Ch(H)$ with respect to the basis consisting of the irreducible characters $\chi_1, \dots, \chi_k$.

Proposition 4.0.9. *The following statements are equivalent:*
1. I does not contain a nonzero Hopf ideal.
2. A is indecomposable.

Proof. The statement is correct if V is zero, even if H is one-dimensional, so let us assume that V is nonzero. We first show that the first statement implies the second. Let us assume that A is decomposable, and choose a corresponding decomposition $I_k = M \cup N$. Since we saw above that powers of a decomposable matrix are still decomposable, we then have for $i \in M$ and $j \in N$ that V_i never appears as a direct summand of $V^{\otimes m} \otimes V_j$. On the other hand, it follows from the above lemma that the intersection of the annihilators of the tensor powers of V are zero, which means that every simple module appears as a constituent of some $V^{\otimes m}$. Suppose now that V_i appears as a constituent of $V^{\otimes r}$ and that V_j^* appears as a constituent of $V^{\otimes s}$, which means that there are injective H-linear maps from V_i to $V^{\otimes r}$ and from V_j^* to $V^{\otimes s}$, which we indicate by hooked arrows. By Schur's lemma, the trivial module K appears as a constituent of $V_j^* \otimes V_j$:

$$K \hookrightarrow V_j^* \otimes V_j \hookrightarrow V^{\otimes s} \otimes V_j$$

This implies that V_i appears as a constituent of $V^{\otimes(r+s)} \otimes V_j$:

$$V_i \hookrightarrow V^{\otimes r} \otimes K \hookrightarrow V^{\otimes r} \otimes V^{\otimes s} \otimes V_j = V^{\otimes(r+s)} \otimes V_j$$

We have therefore reached a contradiction.

Let us next prove that the second statement implies the first; so assume that A is indecomposable. Define the sets

$$M := \{i \leq k \mid V_i \text{ cannot be embedded into } V^{\otimes m} \text{ for any } m\}$$

$$N := \{j \leq k \mid V_j \text{ can be embedded into } V^{\otimes m} \text{ for some } m\}$$

If $j \in N$, choose m such that V_j can be embedded into $V^{\otimes m}$. V_i then appears in $V \otimes V_j$ with multiplicity a_{ij}, and therefore V_i appears in $V^{\otimes(m+1)}$ at least with multiplicity a_{ij}. Therefore, if $i \in M$, we have $a_{ij} = 0$. Since A is indecomposable and N is not empty, as it contains the indices corresponding to the simple constituents of the nonzero module V , M must be empty. This means that every simple module appears as a constituent of some tensor power of V , and therefore the intersection of the annihilators of all these tensor powers must be zero. But by the preceding lemma, this is the largest Hopf ideal contained in I, and the assertion follows. □

We can now prove that the order of a module is bounded by the dimension of the character ring:

Corollary 4.0.10. For a nonzero H-module V , we have :

$$\operatorname{ord}(V) \leq \dim(Ch(H)).$$

Proof. As above, let J be the intersection of the annihilators of all the tensor powers of V . Since this is a Hopf ideal, H/J is a Hopf algebra, and V can also be considered as a module over this algebra. If χ' denotes the character of V as a module over H/J, then the matrix A that represents the left multiplication by χ' with respect to the basis consisting of the irreducible characters of H/J is, by the preceding proposition, indecomposable. If $l :=$

$\dim(Ch(H/J)) \leq \dim(Ch(H))$, then one of the powers $A, A^2, \ldots, A^l$ has a nonzero (1, 1)-component (cf.[19], P. 397). If A^m is this matrix, this means that $V^{\otimes m} \otimes K$ contains the trivial module with nonzero multiplicity, i.e. , $V^{\otimes m}$ contains a nonzero invariant submodule. $\square$

For the following, we need to recall the main result of the theory of non-negative matrices, namely the Perron-Frobenius theorem (cf. [19], P.398). We divide the theorem into two parts, the first part saying the following:

Theorem 4.0.11. (Part 1) Suppose that A is an indecomposable square matrix with nonnegative real entries. Then A has a positive eigenvalue λ, called the Perron- Frobenius eigenvalue, with the property that $\mid \mu \mid \leq \lambda$ for every other eigenvalue μ. The algebraic multiplicity of λ is one, i.e. λ is a simple root of the characteristic polynomial. The corresponding eigenvector, which is therefore unique up to scalar multiples, can be chosen to have positive components. Such an eigenvector is then called a Perron-Frobenius eigenvector.

Now we can the index of A, to be

$$\mathrm{ind}(A) := \mid \{\mu \mid \mu \text{ is an eigenvalue of } A \text{ with } \mid \mu \mid = \lambda\} \mid$$

the number of eigenvalues for which the above inequality is actually an equality. If $\zeta \in \mathbb{C}$ is a primitive $\mathrm{ind}(A)$-th root of unity, then the second part of the Perron-Frobenius theorem can be formulated in the following way :

Theorem 4.0.12. (Part 2) There is a diagonal matrix D whose diagonal entries are $\mathrm{ind}(A)$-th roots of unity such that $DAD_{-1} = \zeta A$.

In particular, since A is similar to ζA, $\zeta\mu$ is an eigenvalue of A whenever μ is, showing that the eigenvalues μ of A that satisfy $\mid \mu \mid = \lambda$ are exactly the numbers of the form $\mu = \zeta^m \lambda$. Furthermore, if x is an eigenvector of D corresponding to the eigenvalue ζ^m, then Ax is an eigenvector of D corresponding to the eigenvalue ζ^{m+1}, so that A shifts the eigenspaces of D around cyclicly, although, since it is not necessarily invertible, it does not always induce an isomorphism between these eigenspaces.

Suppose now that V is an H-module with character χ. We have already seen in the above Lemma that the annihilator of V contains a unique largest Hopf ideal J, namely the intersection of the annihilators of the tensor powers of V. As in the above Corollary, we consider V as a module over the quotient Hopf algebra H/J, and it follows from the above Proposition that the matrix representation A of the left multiplication by the character of V on the character ring $Ch(H/J)$ with respect to the basis that consists of the irreducible characters of H/J is indecomposable.

Notation. If V is an H-module, then V is a faithful $G(H/J)$-module : If $g \in G(H/J)$ is a grouplike element of the quotient Hopf algebra H/J that acts as the identity on V , then it also acts as the identity on every tensor power of V , which implies that $g - 1 \in J/J$, so that $g = 1$.

We now consider the subgroup of $G(H/J)$ consisting of those elements that act on V as a scalar multiple of the identity:

Definition 4.0.13. $G_V := \{g \in G(H/J) \mid \exists \xi \in K, \forall v \in V : g.v = \xi v\}$

Using this group, we can give the following formula for the index :

Theorem 4.0.14. The group G_V is cyclic and contained in the center $Z(H/J)$ of H/J. Its order is equal to the index of A:

$$\mathrm{ind}(A) = \mid G_V \mid$$

Proof. As before, by replacing H by H/J, we can assume that J is zero. For $g \in G_V$, there is by definition a unique scalar $\xi \in K$ such that $g.v = \xi v$ for every $v \in V$. In this way, we get a group homomorphism :

$$G_V \to K^\times, g \mapsto \xi$$

into the group of nonzero elements $K^\times$ of K, which, as we saw above, is injective. This shows that G_V is isomorphic to a finite subgroup of $K^\times$ and therefore cyclic. To see that it is central, note that, for $g \in G_V$ and $h \in H$, the elements gh and hg act in the same way on every tensor power of V, so that $gh - hg \in J$ and therefore $gh = hg$.

It remains to show the index formula. Using a bar to denote the character of the dual module, we have the identity :

$$\sum_{i=1}^{k} \chi\chi_i \otimes \overline{\chi_i} = \sum_{i=1}^{i} \chi_i \otimes \overline{\chi_i}\chi$$

(cf. [20],P. 211). Evaluating the second tensorand on an element $g \in G_V$ that acts on V by multiplication with $\xi \in K$, we get

$$\chi \sum_{i=1}^{k} \chi_i(g^{-1})\chi_i = \xi \mathrm{dim}(V) \sum_{i=1}^{k} \chi_i(g^{-1})\chi_i$$

which means that $\sum_{i=1}^{k} \chi_i(g^{-1})\chi_i$ is an eigenvector for the left multiplication by χ corresponding to the eigenvalue $\xi\mathrm{dim}(V)$. Obviously, the absolute value of this eigenvalue is $\mathrm{dim}(V)$, so that, if ζ is the primitive $\mathrm{ind}(A)$-th root of unity appearing in the second part of the Perron-Frobenius theorem, the discussion there shows that ξ is a power of ζ. Since ξ and g have the same order, the order of g divides $\mathrm{ind}(A)$, and if we choose for g a generator of the cyclic group G_V, we get that $\mid G_V \mid$ divides $\mathrm{ind}(A)$.

The more difficult part is to establish the converse. For this, let $\chi' \in Ch(H)$ be an eigenvector for the left multiplication by χ corresponding to the eigenvalue $\zeta\mathrm{dim}(V)$. Note that χ' is not necessarily the character of any module. By the Perron-Frobenius theorem, such an eigenvector is unique up to scalar multiples. Now, for any $\chi'' \in Ch(H)$, $\chi'\chi''$ is also an eigenvector corresponding to this eigenvalue, and therefore there is a number $\gamma(\chi'') \in K$ such that $\chi'\chi'' = \gamma(\chi'')\chi'$. It is clear that

$$\gamma : Ch(H) \to K$$

is an algebra homomorphism. Since $Ch(H)$ is semisimple (cf. [21], P.55), this shows that $K\chi'$ is a one-dimensional two-sided ideal of $Ch(H)$; in particular, χ' is central and we have

$$\zeta \mathrm{dim}(V)\chi' = \chi\chi' = \chi'\chi = \gamma(\chi)\chi'$$

so that $\gamma(\chi) = \zeta\mathrm{dim}(V)$. Raising this equation to the m-th power, we get $\gamma(\chi^m) = \zeta^m \mathrm{dim}(V^{\otimes m})$. By decomposing $V^{\otimes m}$ into simple modules, we get an equation of the form $\chi^m = \sum_{i=1}^{k} k_i\chi_i$ for some nonnegative integers k_i.

Applying γ, this equation becomes

$$\sum_{i=1}^{k} \zeta^m \dim(V^{\otimes m}) = \gamma(\chi^m) = \sum_{i=1}^{k} k_i \gamma(\chi_i)$$

Since $\chi_i \chi' = \gamma(\chi_i)\chi'$, we have that $\gamma(\chi_i)$ is an eigenvalue of the left multiplication by χ_i. But we know that the absolute value of $\gamma(\chi_i)$ is bounded by $n_i = \dim(V_i)$. The above equality can therefore only hold if we have $\zeta^m k_i n_i = k_i \gamma(\chi_i)$ for all $i = 1, \ldots, k$. This shows that we have $\gamma(\chi_i) = \zeta^m n_i$ if $k_i \neq 0$, i.e. , if V_i appears as a constituent of $V^{\otimes m}$. In particular, the numbers m for which V_i appears as a constituent of $V^{\otimes m}$ cannot be arbitrary, but can only appear in an $\mathrm{ind}(A)$-arithmetic progression.

Since J, the intersection of the annihilators of all the tensor powers of V , is zero, every simple module V_i appears as a constituent of some tensor power $V^{\otimes m_i}$, and we therefore have $\gamma(\chi_i) = \zeta^{m_i} n_i$ for some number m_i, which is unique modulo $\mathrm{ind}(A)$. Note that we can assume that $m_i = 1$ whenever V_i is a constituent of V . Now define

$$g := \sum_{i=1}^{k} \zeta_{m_i} e_i$$

where $e_i \in Z(H)$ is the centrally primitive idempotent corresponding to V_i. This is obviously a central element of order $\mathrm{ind}(A)$ that satisfies $\gamma(\chi_i) = \chi_i(g)$ for all $i = 1, \ldots, k$. We claim that g is grouplike. For this, note that by construction g acts on $V^{\otimes m}$ by multiplication with ζ^m. This implies that both $\Delta(g)$ and $g \otimes g$ act on $V^{\otimes m} \otimes V^{\otimes l}$ by multiplication with ζ^{m+l}, so that $\Delta(g) - g \otimes g$ annihilates this module. Now the annihilator of the $H \otimes H$-module $V^{\otimes m} \otimes V^{\otimes l}$ is the sum $I_1 \otimes H + H \otimes I_2$, where I_1 is the annihilator of $V^{\otimes m}$ and I_2 is the annihilator of $V^{\otimes l}$. This shows that the intersection of all these annihilators is zero, so that in particular $\Delta(g) - g \otimes g = 0$, i.e. g is grouplike. Since g acts on V by multiplication with ζ we have that g is an element of G_V of the order $\mathrm{ind}(A)$, so that conversely $\mathrm{ind}(A)$ divide $\mid G_V \mid$. Furthermore, this shows that g generates G_V. $\square$

Notation. This proof also shows what the diagonal matrix D that appears in the second part of the Perron-Frobenius theorem is in the present case : In the situation and with the notation of the proof, we have $D = diag(\zeta^{m_1}, \dots, \zeta^{m_k})$, so that D is the matrix representation of the map

$$Ch(H) \to Ch(H), \varphi \mapsto (g \to \varphi)$$

with respect to the basis consisting of the irreducible characters. Here, the action appearing in this expression is defined by $(g \to \varphi)(h) = \varphi(hg)$. This holds since we have $g \to \chi = \zeta\chi$ and therefore

$$g \to (\chi\varphi) = (g \to \chi)(g \to \varphi) = \zeta\chi(g \to \varphi)$$

If V is simple, then every central grouplike element of H/J, i.e. , every grouplike element that is central in H/J, acts on V by multiplication with a scalar. Therefore, $G_V = G(H/J) \cap Z(H/J)$ is exactly the set of central grouplike elements, and we get the following corollary:

Corollary 4.0.15. Suppose that V is simple. Then the group $G(H/J) \cap Z(H/J)$ of central grouplike elements of H/J is cyclic, and its order is equal to the index of A:

$$\text{ind}(A) = \mid G(H/J) \cap Z(H/J) \mid$$

As another consequence, we record the following relation between the three invariants that we have studied:

Proposition 4.0.16. $\text{ind}(A)$ divides $\exp(H)$ and $\text{ord}(V)$.

Proof. As before, let J be the largest Hopf ideal contained in the annihilator of V. That $\text{ind}(A) = \exp(G_V)$ divedes $\exp(H)$ follows from the fact that the exponent of Hopf subalgebras and quotients divide the exponent of the larger object. To see that $\text{ind}(A)$ divides $\text{ord}(V)$, note first that the order of V as a module over H is the same as its order as a module over H/J, which implies that we can, as before, assume that J is zero. By the second part of the Perron-Frobenius theorem, we now have a diagonal matrix D such that $DAD^{-1} = \zeta A$, where ζ is a primitive $\text{ind}(A)$-th root of unity. If m is the order of V , we know, for every irreducible character χ_i at least $\text{mult}(V)$ times, which shows that the diagonal entries of A^m are all strictly positive integers. By comparing the diagonal components on both sides of the equation $DA^mD^{-1} = \zeta^m A^m$, we see that $\zeta^m = 1$, which shows that $\text{ind}(A)$ divides $m = \text{ord}(V)$. □

Now, we have the follow important proposition which will state that the order and index in this section are indeed special situation of tensor category.

Proposition 4.0.17. The order and index for semisimple Hopf algebra are equivalent to the definition of tensor category.

Proof. Firstly, we have state that the representation category is obvious tensor category. Let $\mathcal{C} := Rep(H)$, then the unit object I in $\mathcal{C}$ is indeed field k. If m is the smallest number such that $(V^{\otimes m})^H \neq 0$, it means that k is the direct summand of $V^{\otimes m}$. So we have prove that Order is equivalent to the definition of tensor category. Secondly,

$$K_0(\mathcal{C}) \simeq Ch(H), \quad via\, V_i \mapsto \chi_i.$$

V_i is any simple H-module with χ_i the corresponding character, it is obviously isomorphism of ring. From the definition of index, we can easily get that the Index is also equivalent to the definition of tensor category. $\square$

5 Examples

In this section, we will use the order to prove a simple proposition.

Proposition 5.0.18. *$Rep(k[\mathbb{Z}_n])$ is not tensor equivalent to $Rep(k[\mathbb{Z}_{n+1}])$.*

Proof. Let V be a simple $k[\mathbb{Z}_n]$-module and g is a generator of $k[\mathbb{Z}_n]$. It is easy to see that V is one-dimensional. If $V := kv$, then $g \cdot v = \zeta v$, from $g^n = 1$, we have $\zeta^n = 1$ i.e. ζ is a n-th primitive root of unit. So we can easily get $\text{ord}(V) = \text{ord}(\zeta) = n$. If

$$F : Rep(k[\mathbb{Z}_n]) \to Rep(k[\mathbb{Z}_{n+1}])$$

is an equivalence of tensor category, then for simple $k[\mathbb{Z}_n]$-module, $F(V)$ is also a simple $k[\mathbb{Z}_{n+1}]$-module. So we have $\text{ord}(F(V)) = n + 1$, $\text{ord}(V) \neq \text{ord}(F(V))$. It is a contradiction. $\square$

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