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Post-doctoral Research Report

Multiplier Hopf coquasigroups and their applications

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Fundamental Mathematics

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毕 业 论 文 题 目: 乘子 **Hopf** 余拟群及其应用

基 础 数 学 专 业 **2018** 级 博 士 后 姓 名: 杨涛

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摘 要

乘子Hopf代数将经典的Hopf代数结构推广到了未必有单位元的情形, 利用傅里叶变换构造其对偶, 解决了一类无限维Hopf代数的对偶问题, 进一步发展了Pontryagin对偶定理. 本文利用乘子Hopf代数理论的基本思想解决了一类无限维Hopf 拟群的对偶问题, 得到了乘子Hopf余拟群的概念. 在此基础之上, 进一步讨论无限维Hopf拟群与乘子Hopf余拟群的性质与双对偶定理. 最后作为傅里叶变换的应用, 本文研究了无限维余FrobeniusHopf代数的对角交叉积、对角交叉积的表示范畴与Yetter-Drinfeld范畴之间的关系, 以及乘子Hopf代数的Drinfeld扭曲理论. 全文一共分为五章.

第一章介绍了研究背景、动机以及本文的主要结果.

第二章主要给出了与本文有关的一些基本知识.

第三章首先验证了积分是一维的, 之后利用积分构造了无限维Hopf拟群的对偶, 这个对偶具有类似于乘子Hopf代数的结构, 但是不满足余结合性, 我们将它称为乘子Hopf余拟群. 我们同时考虑底代数是$*$-代数的情形, 得到类似的结论. 在本章的最后我们给出乘子Hopf余拟群的启发性例子.

在第四章中, 我们首先讨论乘子Hopf余拟群的基本性质, 例如, 局部单位元的存在性, 模自同构的存在性, 积分的唯一性等. 其次类似于经典的乘子Hopf代数与代数量子群理论, 利用积分和余积分构造了离散型乘子Hopf余拟群的对偶, 并证明此对偶是Hopf拟群. 最后证明了双对偶定理对于Hopf拟群与离散型乘子Hopf余拟群是成立的.

在第五章中, 我们利用前两章的技巧讨论了无限维余FrobeniusHopf代数对角交叉积的表示范畴, 证明了对角交叉积的表示范畴与Yetter-Drinfeld范畴同构. 同时讨论了乘子Hopf代数上的Drinfeld扭曲, 验证了在Drinfeld扭曲变换后的乘子Hopf代数的拟三角结构与对偶的性质.

关键词: 乘子Hopf余拟群; Hopf拟群; Dinfeld扭曲; 积分; Yetter-Drinfeld 范畴; 对角交叉积.

THESIS: Multiplier Hopf coquasigroups and their applications

SPECIALIZATION: Fundamental Mathematics

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Abstract

The multiplier Hopf algebra extends the classic Hopf algebra structure to the situation where there is not necessarily an identity, and uses the Fourier transform to construct its duality, solves the duality problem of a class of infinite-dimensional Hopf algebras, and further develops the Pontryagin duality theorem. This thesis uses the basic idea of the multiplier Hopf algebra theory to construct the dual of a class of infinite-dimensional Hopf quasigroups, and obtains the concept of multiplier Hopf coquasigroups. On this basis, the properties and biduality theorem of infinite dimensional Hopf quasigroups and multiplier Hopf coquasigroups are further discussed. Finally, as an application of the Fourier transform, we consider the diagonal crossed product of the infinite dimensional co Frobenius Hopf algebra, the relationship between the representation category of the diagonal crossed product and the Yetter-Drinfeld category, and the Drinfeld twist theory of multiplier Hopf algebras. This thesis is divided into five chapters.

In Chapter 1, we provide the research background, motivation and main results.

In Chapter 2, we recall some basic concepts related to this thesis.

In Chapter 3, we first verifies that the integral is one-dimensional, and then uses the integral to construct the dual of the infinite-dimensional Hopf quasigroup. This dual has a structure similar to the multiplier Hopf algebra, but does not satisfy the coassociativity. We call it multiplier Hopf coquasigroups. We also consider the case where the underlying algebra is a $*$-algebra, and get a similar conclusion. At the end of this chapter, we give our motivating example of the multiplier Hopf coquasigroup.

In Chapter 4, we first discuss the basic properties of multiplier Hopf coquasigroups, such as the existence of local units, modular automorphism, and the uniqueness of integrals et al. Secondly, similar to the classic multiplier Hopf algebra and algebraic quantum group theory, we uses integrals and cointegrals to construct the duality of a discrete multiplier Hopf coquasigroup, and prove that this duality is a Hopf quasigroup. Finally, it is proved that the biduality theorem is valid for Hopf quasigroups and discrete multiplier Hopf coquasigroups.

In Chapter 5, we use the techniques of the previous two chapters to discuss the representation category of the diagonal crossed product of an infinite dimensional co Frobenius Hopf algebra, and prove that the representation category of the diagonal crossed product is isomorphic to its Yetter-Drinfeld category. The Drinfeld twist on the multiplier Hopf algebra is also discussed, and the quasitriangular structure and duality of the new multiplier Hopf algebra after the Drinfeld twist transformation are verified.

Keywords: Multiplier Hopf coquasigroup; Hopf quasigroup; Drinfeld twist; integral; Yetter-Drinfeld category; diagonal crossed product.

Chapter 1 Introduction

§1.1 Background

Hopf algebras have important connections to quantum theory, Lie algebras, knot and braid theory, operator algebras, and other areas of physics and mathematics. Since V. G. Drinfel'd gave his lecture at the 1986 International Congress of Mathematicians [15], Hopf algebras have been intensely studied in the last decades. Hopf algebras had several generalizations, such as quantum groupoids, weak Hopf algebras, quasi-Hopf algebras, multiplier Hopf algebras, Hopf (co)quasigroups, et al.

Multiplier Hopf algebras, introduced by A. Van Daele in [23], give a nice answer to the dual of a class of infinite-dimensional Hopf algebras. Roughly speaking, an algebraic quantum group is a multiplier Hopf $*$-algebra with a positive invariant functional (Haar measure), the dual of an algebraic quantum group is also an algebraic quantum group, and the dual of the dual is isomorphic to the original one, i.e., Pontryagin duality holds. The non-degenerate faithful integrals play a key role in the dual. The theory of multiplier Hopf algebra and algebraic quantum group was purely algebraic, and the main technique is so-called Fourier transform.

Just as many Lie groups have an entirely algebraic description as commutative Hopf algebras, J. Kim and S. Majid developed the corresponding theory of 'algebraic quasigroups' including the coordinate algebra $k[S^7]$ of the 7-sphere in [19]. They defined the notion of a Hopf quasigroup and showed that a theory similar to that of Hopf algebras was possible in this case. The first author in his following paper [18] developed the integral theory for Hopf (co)quasigroups, Fourier transform, and showed that a finite dimensional Hopf (co)quasigroup has a unique integration up to scalar and an invertible antipode. The dual of a finite dimensional Hopf quasigroup is a Hopf coquasigroup. Then a natural question arise: How about the dual of an infinite dimensional Hopf quasigroup? This motivates the main object of this thesis: multiplier Hopf coquasigroups, which can be considered as a generalization of multiplier Hopf algebras and Hopf coquasigroups.

Fourier transform in multiplier Hopf algebra and multiplier Hopf coquasigroup is a valuable tool to deal with infinite-dimensional case. Therefore, this thesis focus the applications of Fourier transform on the dual of Hopf quasigroups and Hopf coquasigroups.

§1.2 Main results

In this subsection, some interesting results are listed.

For an infinite dimensional Hopf quasigroup H, the left faithful integral φ is unique up to scalar.

Theorem 3.1.2 Let φ' be another left faithful integral on H, then $\varphi' = \lambda\varphi$ for some scalar $\lambda \in k$, i.e., the faithful left integral on H is unique up to scalar.

Use this faithful left integral, we construct the integral dual $\widehat{H} = \{\varphi(\cdot h) \,|\, h \in H\}$. under the assumption

$$\varphi\big((\cdot h)h'\big), \varphi\big(h'(h\cdot)\big) \in \widehat{H}, \quad \forall h, h' \in H,$$

we show that $\widehat{H}$ has a structure similar to the multiplier Hopf algebra, i.e. multiplier Hopf coquasigroup defined in Definition 3.2.11.

Theorem 3.2.12 Let (H, Δ) be an infinite dimensional Hopf quasigroup with a faithful integral φ and a bijective antipode S. Then under Assumption 3.2.3 the integral dual $(\widehat{H}, \widehat{\Delta})$ is a regular multipler Hopf coquasigroup with a faithful integral.

Next, we consider the properties of a multiplier Hopf coquasigroup A. One of the useful properties is the existence of local units:

Proposition 4.1.1 Let (A, Δ) be a regular multiplier Hopf coquasigroup with a non-zero integral φ. Given finite numbers of elements $\{a_1, a_2, \cdots, a_n\}$, there exists an element $e \in A$ such that $a_i e = a_i = e a_i$ for all i.

Using the local units, we can get

$$A = \text{span}\{(id \otimes \varphi)\big(\Delta(a)(1 \otimes b)\big) \mid a, b \in A\},$$
$$A = \text{span}\{(id \otimes \varphi)\big((1 \otimes a)\Delta(b)\big) \mid a, b \in A\},$$

where 'span' means the linear span of a set of element. This is useful in the dual construction of multiplier Hopf coquasigroup.

In order to construct the dual, we need the faithful left integrals. we show that faithful left integrals is 1-dimensional.

Theorem 4.1.4 Let φ' be another faithful left integral on (A, Δ), then $\varphi' = \lambda\varphi$ for some scalar $\lambda \in k$, i.e., the faithful left integral on A is unique up to scalar.

Using this kind of faithful left integrals, we construct the dual of discrete multiplier Hopf coquasigroup, and get another main result:

Theorem 4.2.7 Let (A, Δ) be a regular multiplier Hopf coquasigroup of discrete type with a faithful left integral φ. Then $(\widehat{A}, \widehat{\Delta})$ is a Hopf quasigroup.

Then, we consider the biduality theorem of Hopf quasigroup and discrete multiplier Hopf coquasigroup, and get the following two results:

Theorem 4.3.1 Let (H, Δ) be a Hopf quasigroup, and $(\widehat{H}, \widehat{\Delta})$ be the dual mutiplier Hopf coquasigroup of discrete type. For $h \in H$ and $f \in \widehat{H}$, we set $\Gamma(h)(f) = f(h)$. Then $\Gamma(h) \in \widehat{\widehat{H}}$ for all $h \in H$. Moreover, Γ is an isomorphism between the Hopf quasigroups (H, Δ) and $(\widehat{\widehat{H}}, \widehat{\widehat{\Delta}})$.

Theorem 4.3.2 Let (A, Δ) be a discrete multiplier Hopf coquasigroup, and $(\widehat{A}, \widehat{\Delta})$ be the dual Hopf quasigroup. For $a \in A$ and $w \in \widehat{A}$, we set $\Gamma(a)(w) = w(a)$. Then $\Gamma(a) \in \widehat{\widehat{A}}$ for all $a \in A$. Moreover, Γ is an isomorphism between the multiplier Hopf coquasigroup (A, Δ) and $(\widehat{\widehat{A}}, \widehat{\widehat{\Delta}})$.

In the last chapter, we apply the techniques introduced in multiplier Hopf algebra and multiplier Hopf coquasigroup theories to some special cases. First, we construct the diagonal crossed product $\bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha, \beta)$ for an infinite dimensional coFrobenius Hopf algebra H with $\alpha, \beta \in Aut(H)$.

Proposition 5.1.1 Let H be a coFrobenius Hopf algebra with its dual multiplier Hopf algebra $\widehat{H}$. Then $\mathcal{A} = \bigoplus_{(\alpha,\beta)\in G} \mathcal{A}_{(\alpha,\beta)} = \bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha, \beta)$ is a quasitriangular G-cograded multiplier Hopf algebra with the following strucrures:

- For any $(\alpha, \beta) \in G$, $\mathcal{A}_{(\alpha,\beta)}$ has the multiplication given by

$$(p \bowtie h)(q \bowtie l) = p\big(\alpha(h_{(1)}) \blacktriangleright q \blacktriangleleft S^{-1}\beta(h_{(3)})\big) \bowtie h_{(2)}l$$

  for $p, q \in \widehat{H}$ and $h, l \in H$.

- The comultiplication on $\mathcal{A}$ is given by:

$$\Delta_{(\alpha,\beta),(\gamma,\delta)} : \mathcal{A}_{(\alpha,\beta)*(\gamma,\delta)} \longrightarrow M(\mathcal{A}_{(\alpha,\beta)} \otimes \mathcal{A}_{(\gamma,\delta)}),$$
$$\Delta_{(\alpha,\beta),(\gamma,\delta)}(p \bowtie h) = \Delta^{cop}(p)(\gamma \otimes \gamma^{-1}\beta\gamma)\Delta(h).$$

- The counit $\varepsilon_{\mathcal{A}}$ on $\mathcal{A}_{(\iota,\iota)} = D(H)$ is the counit on the Drinfel'd double of H.

- For any $(\alpha, \beta) \in G$, the antipode is given by

$$S : \mathcal{A}_{(\alpha,\beta)} \longrightarrow \mathcal{A}_{(\alpha,\beta)^{-1}},$$
$$S_{(\alpha,\beta)}(p \bowtie h) = T(\alpha\beta S(h) \otimes S^{-1}(p)) \text{ in } \mathcal{A}_{(\alpha,\beta)^{-1}} = \mathcal{A}_{(\alpha^{-1},\alpha\beta^{-1}\alpha^{-1})}.$$

- A crossing action $\xi : G \longrightarrow Aut(\mathcal{A})$ is given by

$$\xi^{(\gamma,\delta)}_{(\alpha,\beta)} : \mathcal{A}_{(\gamma,\delta)} \longrightarrow \mathcal{A}_{(\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}} = \mathcal{A}_{(\alpha\gamma\alpha^{-1},\alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})},$$
$$\xi^{(\gamma,\delta)}_{(\alpha,\beta)}(p \bowtie h) = p \circ \beta\alpha^{-1} \bowtie \alpha\gamma^{-1}\beta^{-1}\gamma(h).$$

- A generalized R-matrix is given by

$$R = \sum_{(\alpha,\beta),(\gamma,\delta)\in G} R_{(\alpha,\beta),(\gamma,\delta)} = \sum_{(\alpha,\beta),(\gamma,\delta)\in G} \varepsilon \bowtie \beta^{-1}(u) \otimes v \bowtie 1.$$

Then we consider the epresentation category of the diagonal crossed product, and show it is isomorphic to the Yetter-Drinfeld category.

Theorem 5.1.5 For a coFrobenius Hopf algebra H,

$${}_H\mathcal{YD}^H(\alpha, \beta) \cong {}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}. \tag{1.1}$$

Theorem 5.1.7 For a coFrobenius Hopf algebra H and its G-cograded multiplier Hopf algebra $\mathcal{A} = \bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha, \beta)$, $Rep(\mathcal{A})$ and $\mathcal{YD}(H)$ are isomorphic as braided T-categories over G.

In the end, we consider Drinfeld twists of multiplier Hopf algebras, and determines how the integral changes under a Drinfeld twist in multiplier Hopf algebras case.

Theorem 5.2.11 Let (A, Δ) be a quasitriangular multiplier Hopf algebra with generalized R-matrix $\mathcal{R}$. Then (A^J, Δ^J) is also quasitriangular, and the quasitriangular structure given by

$$\mathcal{R}^J = J_{21}^{-1} \mathcal{R} J.$$

Theorem 5.2.13 Let (A, Δ) be a counimodular algebraic quantum group with a non-zero left (resp. right) integral φ (resp. ψ) and J be a Drinfeld twist. Then the elements $\varphi^J = u_J \rightharpoonup \varphi$ and $\psi^J = \psi \leftharpoonup u_J^{-1}$ are non-zero left and right integrals on (A^J, Δ^J) respectively.

Chapter 2 Preliminaries

All spaces we considered are over a fixed field k. Let A be an (associative) algebra. We do not assume that A has a unit, but we do require that the product, seen as a bilinear form, is non-degenerated. This means that, whenever $a \in A$ and $ab = 0$ for all $b \in A$ or $ba = 0$ for all $b \in A$, we must have that $a = 0$. Then we can consider the multiplier algebra $M(A)$ of A. Recall that $M(A)$ is characterized as the largest algebra with identity containing A as an essential two-sided ideal. In particularly, we still have that, whenever $a \in M(A)$ and $ab = 0$ for all $b \in A$ or $ba = 0$ for all $b \in A$, again $a = 0$. Furthermore, we consider the tensor algebra $A \otimes A$. It is still non-degenerated and we have its multiplier algebra $M(A \otimes A)$. There are natural imbeddings

$$A \otimes A \subseteq M(A) \otimes M(A) \subseteq M(A \otimes A).$$

In generally, when A has no identity, these two inclusions are stict. If A already has an identity, the product is obviously non-degenerate and $M(A) = A$ and $M(A \otimes A) = A \otimes A$. More details about the concept of the multiplier algebra of an algebra, we refer to [23].

Let A and B be non-degenerate algebras, if homomorphism $f : A \longrightarrow M(B)$ is non-degenerated (i.e., $f(A)B = B$ and $Bf(A) = B$), then has a unique extension to a homomorphism $M(A) \longrightarrow M(B)$, we also denote it f.

§2.1 Multiplier Hopf algebras

Now, we recall the definition of a multiplier Hopf algebra (see [23] for details). A comultiplication on algebra A is a homomorphism $\Delta : A \longrightarrow M(A \otimes A)$ such that $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\Delta(b)$ belong to $A \otimes A$ for all $a, b \in A$. We require Δ to be coassociative in the sense that

$$(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$$

for all $a, b, c \in A$ (where ι denotes the identity map).

A pair (A, Δ) of an algebra A with a non-degenerate product and a comultiplication Δ on A is called a *multiplier Hopf algebra*, if the linear map T_1, T_2 defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b), \qquad T_2(a \otimes b) = (a \otimes 1)\Delta(b) \tag{2.1}$$

are bijective.

The bijectivity of the above two maps is equivalent to the existence of a counit and an antipode S satisfying (and defined by)

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab, \qquad m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)b, \tag{2.2}$$

$$(\iota \otimes \varepsilon)((a \otimes 1)\Delta(b)) = ab, \qquad m(\iota \otimes S)((a \otimes 1)\Delta(b)) = \varepsilon(b)a, \tag{2.3}$$

where $\varepsilon : A \longrightarrow k$ is a homomorphism, $S : A \longrightarrow M(A)$ is an anti-homomorphism and m is the multiplication map, considered as a linear map from $A \otimes A$ to A and extended to $M(A) \otimes A$ and $A \otimes M(A)$.

A multiplier Hopf algebra (A, Δ) is called *regular* if (A, Δ^{cop}) is also a multiplier Hopf algebra, where Δ^{cop} denotes the co-opposite comultiplication defined as $\Delta^{cop} = \tau \circ \Delta$ with τ the usual flip map from $A \otimes A$ to itself (and extended to $M(A \otimes A)$). In this case, $\Delta(a)(b \otimes 1), (1 \otimes a)\Delta(b) \in A \otimes A$ for all $a, b \in A$. By Proposition 2.9 in [24], a multiplier Hopf algebra (A, Δ) is regular if and only if the antipode S is bijective from A to A.

Throughout this paper we freely use the coalgebra, Hopf algebra and multiplier Hopf algebra terminology introduced in [4, 12, 21, 23, 24]. We will use the adapted Sweedler notation (see [25]) for multiplier Hopf algebras, e.g., write $a_{(1)} \otimes a_{(2)}b$ for $\Delta(a)(1 \otimes b)$ and $ab_{(1)} \otimes b_{(2)}$ for $(a \otimes 1)\Delta(b)$.

§2.1.1 Algebraic quantum groups and their dualities

Assume in what follows that (A, Δ) is a regular multiplier Hopf algebra. A linear functional φ on A is called *left invariant* if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ in $M(A)$ for all $a \in A$. A non-zero left invariant functional φ is called a *left integral* on A. A right integral ψ can be defined similarly.

In general, left and right integrals are unique up to a scalar if they exist. And if a left integral φ exists, a right integral also exists, namely $\varphi \circ S$. Although may be different, left and right integrals are related. For a left integral φ, there is a unique group-like element (modular element) $\delta \in M(A)$ such that $\varphi(S(a)) = \varphi(a\delta)$ for all $a \in A$.

For an algebraic quantum group (A, Δ) with integrals, define $\widehat{A}$ as the space of linear functionals on A of the form $\varphi(\cdot a)$, where $a \in A$. Then $\widehat{A}$ can be made into a regular multiplier Hopf algebra with a product(resp. coproduct $\widehat{\Delta}$ on $\widehat{A}$) dual to the coproduct Δ on A (resp. product of A). It is called the dual of (A, Δ). The various objects associated with $(\widehat{A}, \widehat{\Delta})$ are denoted as for (A, Δ) but with a hat. However, we use ε and S also for the counit and antipode on the dual. The dual $(\widehat{A}, \widehat{\Delta})$ also has integrals, i.e., the dual is also an algebraic quantum group. A right integral $\widehat{\psi}$ on $\widehat{A}$ is defined by $\widehat{\psi}(\varphi(\cdot a)) = \varepsilon(a)$ for all $a \in A$. Repeating the procedure, i.e., taking the dual of $(\widehat{A}, \widehat{\Delta})$, we can get $\widehat{\widehat{A}} \cong A$ (see Theorem (Biduality) 4.12 in [24]).

From Definition 1.5 in [6], an algebraic quantum group A is called counimodular, if the dual multiplier Hopf algebra $\widehat{A}$ is unimodular integral, i.e., $\widehat{\delta} = 1$ in $M(A)$. For a counimodular algebraic quantum group, we have $\varphi(ab) = \varphi(bS^2(a))$ for all $a, b \in A$ (see Proposition 1.6 in [6]).

Start with two regular multiplier Hopf algebras A and B together with a non-degenerate bilinear map $\langle \cdot, \cdot \rangle$ from $A \times B$ to K satisfying certain properties. The main property is the comultiplication in A is dual to the product in B and vice versa. For more details, see [13].

For $a \in A$ and $b \in B$, we can define multipliers $a \blacktriangleright b, b \blacktriangleleft a \in M(B)$ and $b \blacktriangleright a, a \blacktriangleleft b \in M(A)$ in the following way. For $a' \in A$ and $b' \in B$, we have: $(b \blacktriangleright a)a' = \sum \langle a_{(2)}, b \rangle a_{(1)} a'$, $(a \blacktriangleright b)b' = \sum \langle a, b_{(2)} \rangle b_{(1)} b'$, $(a \blacktriangleleft b)a' = \sum \langle a_{(1)}, b \rangle a_{(2)} a'$ and $(b \blacktriangleleft a)b' = \sum \langle a, b_{(1)} \rangle b_{(2)} b'$. The regularity conditions on the dual paring $\langle , \rangle$ say that the multipliers $b \blacktriangleright a$ and $a \blacktriangleleft b$ in $M(A)$ (resp. $a \blacktriangleright b$ and $b \blacktriangleleft a$ in $M(B)$) actually belong to A (resp. B). For more details, see [8].

We mention that $\langle S(a), b \rangle = \langle a, S(b) \rangle$, $\langle 1_{M(A)}, b \rangle = \varepsilon(b)$ and $\langle a, 1_{M(B)} \rangle = \varepsilon(b)$. Sometimes without confusion we denote the unit $1_{M(A)}$ of $M(A)$ by 1. We also use bilinear forms on the tensor products in the following way

$$\langle a \otimes a', b \otimes b' \rangle = \langle a, b \rangle \langle a', b' \rangle, \quad \langle b \otimes a, a' \otimes b' \rangle = \langle a', b \rangle \langle a, b' \rangle$$

for all $a, a' \in A$ and $b, b' \in B$. These bilinear forms are non-degenerate and can be extended in a natural way to the multiplier algebra at one side.

§2.1.2 Multiplier Hopf T-coalgebras and their quasitriangular structures

Let (A, Δ) be a multiplier Hopf algebra and G a group with unit e. Assume that there is a family of (non-trivial) subalgebras $\{A_p\}_{p\in G}$ of A so that

(1) $A = \bigoplus_{p\in G} A_p$ with $A_pA_q = 0$ whenever $p, q \in G$ and $p \neq q$.

(2) $\Delta(A_{pq})(1 \otimes A_q) = A_p \otimes A_q$ and $(A_p \otimes 1)\Delta(A_{pq}) = A_p \otimes A_q$ for all $p, q \in G$.

Then (A, Δ) is called a *G-cograded multiplier Hopf algebra*(see [**?**, 10]).

We extend the Sweedler notation for a comultiplication in the following way: for any $p, q \in G$, $a \in A_{pq}$ and $a' \in A_q$, we write

$$\Delta_{p,q}(a)(1 \otimes a') = a_{(1,p)} \otimes a_{(2,q)}a'.$$

Let A be a G-cograded multiplier Hopf algebra, then A has the form $A = \bigoplus_{p\in G} A_p$. Assume that there is a group homomorphism $\pi : G \longrightarrow Aut(A)$. We call π an *admissible* action of G on A if also the following requirements hold

(1) $\Delta(\pi_p(a)) = (\pi_p \otimes \pi_p)\Delta(a)$ for all $a \in A$.

(2) $\pi_p(A_q) = A_{\rho_p(q)}$, where ρ is an action of the group G on itself.

(3) $\pi_{\rho_p(q)} = \pi_{pqp^{-1}}$.
This means that the map π takes care of ρ not being the adjoint action. If ρ is the adjoint action, π is called a *crossing*.

A group-cograded multiplier Hopf algebra $A = \bigoplus_{p\in G} A_p$ is said to be a *multiplier Hopf T-coalgebra* provided it is endowed with a crossing π such that each π_q preserves the comultiplication and the counit, i.e., for all $p, q, r \in G$,

$$(\pi_q \otimes \pi_q)\Delta_{p,r} = \Delta_{qpq^{-1},qrq^{-1}}\pi_q, \quad \varepsilon\pi_q = \varepsilon,$$

and π is multiplicative in the sense that $\pi_{pq} = \pi_p\pi_q$ for all $p, q \in G$. It can be considered as generalization of crossed Hopf G-coalgebra introduced in [**?**].

Let A be a multiplier Hopf T-coalgebra, then we can construct a new regular multiplier Hopf algebra on A by deforming the comultiplication while the algebra structure on A is kept (see Theorem 3.11 in [9]). The comultiplication deformation of A depends on the crossing π in the following way: for all $a \in A$ and $a' \in A_q$,

$$\widetilde{\Delta}(a)(1 \otimes a') = (\pi_{q^{-1}} \otimes \iota)(\Delta(a)(1 \otimes a')).$$

Recall from [10], a G-cograded multiplier Hopf algebra with a crossing action π is called *quasitriangular* if there is a multiplier $R = \sum_{p,q \in G} R_{p,q}$ with $R_{p,q} \in M(A_p \otimes A_q)$ such that

$$\begin{aligned} (\pi_p \otimes \pi_p)(R) = R, &\qquad R\Delta(a) = (\widetilde{\Delta})^{cop}(a)R, \\ (\widetilde{\Delta} \otimes \iota)(R) = R_{13}R_{23}, &\qquad (\iota \otimes \Delta)(R) = R_{13}R_{12}. \end{aligned}$$

for all $p \in G$ and $a \in A$. Sometimes we call R a generalized R-matrix.

§2.2 Hopf (co)quasigroups

Recall from [19] a *Hopf quasigroup* is possibly non-associative but unital algbera H equipped with algebra homomorphisms $\Delta : H \longrightarrow H \otimes H$, $\varepsilon : H \longrightarrow k$ forming a coassociative coalgebra and a map $S : H \longrightarrow H$ such that

$$\begin{aligned} m(id \otimes m)(S \otimes id \otimes id)(\Delta \otimes id) = \varepsilon \otimes id = m(id \otimes m)(id \otimes S \otimes id)(\Delta \otimes id), \\ m(m \otimes id)(id \otimes S \otimes id)(id \otimes \Delta) = id \otimes \varepsilon = m(m \otimes id)(id \otimes id \otimes S)(id \otimes \Delta). \end{aligned}$$

These two equations can be written more explicitly as: for all $g, h \in H$,

$$\sum S(h_{(1)})(h_{(2)}g) = \sum h_{(1)}\big(S(h_{(2)})g\big) = \sum \big(gS(h_{(1)})\big)h_{(2)} = \sum(gh_{(1)})S(h_{(2)}) = \varepsilon(h)g,$$

where we write $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ and for brevity, we shall omit the summation signs.

The Hopf quasigroup H is called *flexible* if

$$h_{(1)}(gh_{(2)}) = (h_{(1)}g)h_{(2)}, \qquad \forall g, h \in H, \tag{2.4}$$

and *alternative* if also

$$h_{(1)}(h_{(2)}g) = (h_{(1)}h_{(2)})g, \qquad h(g_{(1)}g_{(2)}) = (hg_{(1)})g_{(2)}, \qquad \forall g, h \in H. \tag{2.5}$$

H is called *Moufang* if

$$h_{(1)}\big(g(h_{(2)}f)\big) = \big((h_{(1)}g)h_{(2)}\big)f, \qquad \forall h, g, f \in H. \tag{2.6}$$

It was proved that the antipode S is antimultiplicative and anticomultiplicative, i.e., for all $g, h \in H$,

$$S(gh) = S(h)S(g), \qquad \Delta(Sh) = S(h_{(2)}) \otimes S(h_{(1)}).$$

Moreover, if H is cocommutative flexible Hopf quasigroup, then $S^2 = id$ and for all $g, h \in H$,

$$h_{(1)}\big(gS(h_{(2)})\big) = (h_{(1)}g)S(h_{(2)}).$$

Dually, we can obtain a Hopf coquasigroup by reversing the arrows on each map in Hopf quasigroup.

A *Hopf coquasigroup* is a unital associative algebra A equipped with counital algebra homomorphism $\Delta : A \longrightarrow A \otimes A$, $\varepsilon : A \longrightarrow k$ and a linear map $S : A \longrightarrow A$ such that for all $a \in A$,

$$(m \otimes id)(S \otimes id \otimes id)(id \otimes \Delta)\Delta = 1 \otimes id = (m \otimes id)(id \otimes S \otimes id)(id \otimes \Delta)\Delta,$$
$$(id \otimes m)(id \otimes S \otimes id)(\Delta \otimes id)\Delta = id \otimes 1 = (id \otimes m)(id \otimes id \otimes S)(\Delta \otimes id)\Delta.$$

In other word,

$$S(a_{(1)})a_{(2)(1)} \otimes a_{(2)(2)} = 1 \otimes a = a_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)},$$
$$a_{(1)(1)} \otimes S(a_{(1)(2)})a_{(2)} = a \otimes 1 = a_{(1)(1)} \otimes a_{(1)(2)}S(a_{(2)}).$$

A Hopf coquasigroup is *flexible* if

$$a_{(1)}a_{(2)(2)} \otimes a_{(2)(1)} = a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)}, \qquad \forall a \in A, \tag{2.7}$$

and *alternative* if also

$$a_{(1)}a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)}a_{(1)(2)} \otimes a_{(2)}, \tag{2.8}$$

$$a_{(1)} \otimes a_{(2)(1)}a_{(2)(2)} = a_{(1)(1)} \otimes a_{(1)(2)}a_{(2)}, \qquad \forall a \in A. \tag{2.9}$$

A is called *Moufang* if

$$a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}, \qquad \forall a \in A, \tag{2.10}$$

The term 'counital' here means

$$(id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta.$$

However, Δ is not assumed to be coassociative.

It was shown in [19] Proposition 5.2 that: Let A be a Hopf coquasigroup, then

(1) $m(S \otimes id)\Delta = \mu\varepsilon = m(id \otimes S)\Delta$.

(2) S is antimultiplicative $S(ab) = S(b)S(a)$ for all $a, b \in A$.

(3) S is anticomultiplicative $\Delta S(a) = S(a_{(2)}) \otimes S(a_{(1)})$ for all $a \in A$.

Hence a Hopf coquasigroup is Hopf algebra iff it is coassociative.

Chapter 3　Hopf quasigroups with faithful integrals

For an infinite dimensional Hopf quasigroup, if the faithful integral exists, then it is unique up to scalar. Base on the faithful integrals, we construct the integral dual of a class of infinite dimensional Hopf quasigroups, and show that the integral dual has a similar structure to Hopf coquasigroup, which is a regular multiplier Hopf coquasigroup with a faithful integral.

§3.1　Integrals on Hopf quasigroups

Let H be a finite dimensional Hopf quasigroup and $H^* = Hom(H, k)$ be the dual space with natural Hopf coquasigruoup structure given by

$$\langle ab, h\rangle = \langle a, h_{(1)}\rangle\langle b, h_{(2)}\rangle, \qquad \langle \Delta(a), h\otimes g\rangle = \langle a, hg\rangle,$$
$$\langle 1, h\rangle = \varepsilon(h), \qquad \langle a, 1\rangle = \varepsilon(a), \qquad \langle S(a), h\rangle = \langle a, S(h)\rangle.$$

Then there is a natural question: For an infinite dimensional Hopf quasigroup H, how about its dual?

Recall from [18], a left (resp. right) integral on H is a nonzero element $\varphi \in H^*$ (resp. $\psi \in H^*$) such that

$$(id\otimes\varphi)\Delta(h) = \varphi(h)1_H \quad \big(\text{resp.}(\psi\otimes id)\Delta(h) = \psi(h)1_H\big), \qquad \forall h\in H.$$

And from Lemma 3.3 in [18], we have that $\varphi\circ S$ is a right integral on H.

Lemma 3.1.1 [[18], Lemma 3.4, 3.8] Let φ (resp. ψ) be a left (resp. right) integral on H, then for $h, g\in H$

$$h_{(1)}\varphi\big(h_{(2)}S(g)\big) = \varphi\big(hS(g_{(1)})\big)g_{(2)}, \qquad h_{(1)}\varphi(gh_{(2)}) = S(g_{(1)})\varphi(g_{(2)}h). \quad (3.1)$$
$$\psi\big(S(g)h_{(1)}\big)h_{(2)} = \psi\big(S(g_{(2)})h\big)g_{(1)}, \qquad \psi(g_{(1)}h)g_{(2)} = \psi(gh_{(1)})S(h_{(2)}). \quad (3.2)$$

Proof From the proof of Lemma 3.4 and 3.8 in [18], we can easily check the above equations also hold in infinite dimensional case. □

In the following, we will constuct the 'integral dual' of a class of infinite dimensional Hopf quasigroup. Let H be an infinite dimensional Hopf quasigroup with a bijective antipode and a faithful left integral, i.e., $\varphi(gh)=0, \forall h\in H \Rightarrow g=0$ and $\varphi(gh)=0, \forall g\in H \Rightarrow h=0$.

First, we show that for the infinite dimensional Hopf quasigroup, the faithful left integral is unique up to scalar.

Theorem 3.1.2 Let φ' be another faithful left integral on H, then $\varphi'=\lambda\varphi$ for some scalar $\lambda\in k$, i.e., the faithful left integral on H is unique up to scalar.

Proof From Lemma 3.1.1, we have $h_{(1)}\varphi(gh_{(2)})=S(g_{(1)})\varphi(g_{(2)}h)$ for all $h,g\in H$. Apply φ' to both expressions in this equation. Because $\varphi'\circ S$ is a right integral, the right hand side will give

$$\varphi'(Sg_{(1)})\varphi(g_{(2)}h)=\varphi(\varphi'S(g_{(1)})g_{(2)}h)=\varphi(\varphi'S(g)1_H\cdot h)=\varphi'S(g)\varphi(h).$$

For the left hand side,

$$\varphi'(h_{(1)}\varphi(gh_{(2)}))=\varphi'(h_{(1)})\varphi(gh_{(2)})=\varphi(g\varphi'(h_{(1)})h_{(2)})=\varphi(g\delta_h),$$

where $\delta_h=\varphi'(h_{(1)})h_{(2)}$. Therefore, $\varphi'S(g)\varphi(h)=\varphi(g\delta_h)$ for all $h,g\in H$.

We claim that there is an element $\delta\in H$ such that $\delta_h=\varphi(h)\delta$ for all $h\in H$. Indeed, for any $h'\in H$

$$\begin{aligned}\varphi(g\varphi(h')\delta_h) &= \varphi(h')\varphi(g\delta_h)=\varphi(h')\varphi'S(g)\varphi(h)\\ &= \varphi(h)\varphi'S(g)\varphi(h')=\varphi(h)\varphi(g\delta_{h'})\\ &= \varphi(g\varphi(h)\delta_{h'}),\end{aligned}$$

then $\varphi(h')\delta_h=\varphi(h)\delta_{h'}$ for all $h,h'\in H$, since φ is faithful. Choose an $h'\in H$ such that $\varphi(h')=1$ and denote $\delta=\delta_{h'}$, then $\delta_h=\varphi(h)\delta$.

If we apply ε, we get

$$\begin{aligned}\varphi(h)\varepsilon(\delta) &= \varepsilon(\delta_h)=\varepsilon(\varphi'(h_{(1)})h_{(2)})\\ &= \varphi'(h_{(1)})\varepsilon(h_{(2)})=\varphi'(h_{(1)}\varepsilon(h_{(2)}))\end{aligned}$$

$$= \quad \varphi'(h)$$

for all $h \in H$ and with $\lambda = \varepsilon(\delta)$, we find the desired result. $\square$

Remark Similarly, the right faithful integral on H is unique up to scalar. However, it is a pity that the non-zero faithful integrals do not always exist in infinite dimensional case, even for the special infinite dimensional Hopf algebra case.

Proposition 3.1.3 There is a unique group-like element $\delta \in H$ such that for all $h \in H$

(1) $(\varphi \otimes id)\Delta(h) = \varphi(h)\delta$.

(2) $\varphi S(a) = \varphi(a\delta)$.

Furthermore, if the antipode S is bijective, then $(id \otimes \psi)\Delta(h) = \psi(h)\delta^{-1}$.

Proof From the proof of Proposition 3.1.2, $\varphi(h)\delta = \delta_h = \varphi'(h_{(1)})h_{(2)}$ and $\varphi' S(g)\varphi(h) = \varphi(g\delta_h)$, we take $\varphi' = \varphi$ and get a element $\delta \in H$ such that $(\varphi \otimes id)\Delta(h) = \varphi(h)\delta$ and $\varphi S(h) = \varphi(h\delta)$. This gives the first part of (1) and (2).

If we apply ε and Δ on the first equation, we find $\varepsilon(\delta) = 1$ and $\Delta(\delta) = \delta \otimes \delta$. by Proposition 4.2 (1) in [19] $S(\delta)\delta = 1 = \delta S(\delta)$, then $S(\delta) = \delta^{-1}$. Hence δ is a group-like element.

Because S flips the coproduct and if we let $\psi = \varphi \circ S$, we get

$$\begin{aligned}
(id \otimes \psi)\Delta(h) &= S^{-1}(S \otimes \psi)\Delta(h) = S^{-1}(S \otimes \varphi \circ S)\Delta(h) \\
&= S^{-1}(id \otimes \varphi)(S \otimes S)\Delta(h) = S^{-1}(id \otimes \varphi)\Delta^{cop}(S(h)) \\
&= S^{-1}(\varphi \otimes id)\Delta(S(h)) \overset{(1)}{=} S^{-1}(\varphi(S(h))\delta) \\
&= \psi(h)\delta^{-1}.
\end{aligned}$$

This completes the proof. $\square$

Remark (1) The square S^2 leaves the coproduct invariant, it follows that the composition $\varphi \circ S^2$ of the faithful left integral φ with S^2 will again a faithful left integral. By the uniqueness of faithful left integrals, there is a number $\tau \in k$ such that $\varphi \circ S^2 = \tau\varphi$.

(2) If we apply (2) in Proposition 3.1.3 twice, we get

$$\varphi\big(S^2(a)\big) = \varphi\big(S(a)\delta\big) = \varphi\big(S(\delta^{-1}a)\big) = \varphi\big((\delta^{-1}a)\delta\big).$$

So $\varphi\big((\delta^{-1}a)\delta\big) = \tau\varphi(a)$.

§3.2 Integral dual

In this section, we will construct the dual of an infinite dimensional Hopf quasigroup. This construction bases on the faithful integrals introduced in the last section. Here, we also start with defining the following subspace of the dual space H^*.

Definition 3.2.1 Let φ be a faithful left integral on a Hopf quasigroup H. We define $\widehat{H}$ as the space of linear functionals on H of the form $\varphi(\cdot h)$ where $h \in H$, i.e.,

$$\widehat{H} = \{\varphi(\cdot h) \,|\, h \in H\}.$$

Lemma 3.2.2 Let H be a Hopf quasigroup and φ (resp. ψ) be a left (resp. right) integral on H. If $a \in H$, then there is a $b \in H$ such that $\varphi(ax) = \psi(xb)$ for all $x \in H$. Similarly, given $q \in H$, we have $p \in H$ so that $\varphi(xp) = \psi(qx)$ for all $x \in H$.

Proof By the equations (3.1) in Lemma 3.1.1 and (3.2), we have for any $h, x \in H$,

$$\begin{aligned}
(\psi\otimes\varphi)\big(xq_{(1)}\otimes pS(q_{(2)})\big) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \varphi\big(p\underline{\psi(xq_{(1)})S(q_{(2)})}\big)\\
&\overset{(3.2)}{=} \varphi\big(p\psi(x_{(1)}q)x_{(2)}\big) = \psi(x_{(1)}q)\varphi\big(px_{(2)}\big)\\
&= \psi\big(\underline{x_{(1)}\varphi(px_{(2)})}q\big) \overset{(3.1)}{=} \psi\big(S(p_{(1)})\varphi(p_{(2)}x)q\big)\\
&= \varphi(p_{(2)}x)\psi\big(S(p_{(1)})q\big) = \varphi\Big(\big(\psi\big(S(p_{(1)})q\big)p_{(2)}\big)x\Big)\\
&= \varphi\Big((\psi\circ S\otimes id)\big((S^{-1}(q)\otimes 1)\Delta(p)\big)x\Big).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
(\psi\otimes\varphi)\big(xq_{(1)}\otimes pS(q_{(2)})\big) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \psi\Big(x\big(q_{(1)}\varphi\big(pS(q_{(2)})\big)\big)\Big)\\
&= \psi\Big(x(id\otimes\varphi\circ S)\big(\Delta(q)(1\otimes S^{-1}(p))\big)\Big).
\end{aligned}$$

By Theorem 4.5 in [27], Galois maps $T_1 : a\otimes b \mapsto \Delta(a)(1\otimes b)$ and $T_2 : a\otimes b \mapsto (a\otimes 1)\Delta(b)$

are bijective, then any element in H has the form $(\psi\circ S\otimes id)\big((S^{-1}(q)\otimes 1)\Delta(p)\big)$. Hence the above calculation will give us the formula $\varphi(ax)=\psi(xb)$ for all $x\in H$.

Similarly by computing $(\psi\otimes\varphi)\big(S(q_{(2)})x\otimes q_{(1)}S(p)\big)$, we get the second assertion. □

Remark (1) In the proof of second part, we need Galois maps $T_3: a\otimes b\mapsto \Delta(a)(b\otimes 1)$ and $T_4: a\otimes b\mapsto (1\otimes a)\Delta(b)$ are bijective, which follows the fact that the antipode S is bijective, and $T_3^{-1}: a\otimes b\mapsto b_{(2)}\otimes S^{-1}(b_{(1)})a$ and $T_4: a\otimes b\mapsto bS^{-1}(a_{(2)})\otimes a_{(1)}$.

(2) From Lemma 3.2.2, we get that

$$\widehat{H}=\{\varphi(\cdot h)\,|\,h\in H\}=\{\psi(h\cdot)\,|\,h\in H\}.$$
$$\text{and}\quad \{\varphi(h\cdot)\,|\,h\in H\}=\{\psi(\cdot h)\,|\,h\in H\}.$$

In the following, we need the following assumption to construct the dual.

Assumption 3.2.3

$$\varphi\big((\,\cdot h)h'\big),\varphi\big(h'(h\,\cdot\,)\big)\in\widehat{H},\quad \forall h,h'\in H. \tag{3.3}$$

Remark Following this assumption, Proposition 3.2.3 (2) and 3.2.5, we have

$$\{\varphi(\,\cdot h)\mid h\in H\}=\{\varphi(h\,\cdot\,)\mid h\in H\},\quad \forall h,h'\in H.$$

Therefore,

$$\widehat{H}=\{\varphi(\cdot h)\,|\,h\in H\}=\{\psi(h\cdot)\,|\,h\in H\}=\{\varphi(h\cdot)\,|\,h\in H\}=\{\psi(\cdot h)\,|\,h\in H\}$$

and $\psi\big((\cdot h)h'\big),\psi\big(h'(h\cdot)\big)\in\widehat{H}$.

We start by making $\widehat{H}$ into an algebra by dualizing the coproduct.

Proposition 3.2.4 For $w,w'\in\widehat{H}$, we can define a linear functional ww' on H by the formula

$$(ww')(h)=(w\otimes w')\Delta(h),\quad \forall h\in H. \tag{3.4}$$

Then $ww'\in\widehat{H}$. This product on $\widehat{H}$ ia associative and non-degenerate.

Proof Let $w, w' \in \widehat{H}$ and assume that $w' = \varphi(\cdot m)$ with $m \in H$. we have

$$\begin{aligned}
(ww')(h) &= (w \otimes \varphi(\cdot m))\Delta(h) = (w \otimes \varphi)\big(\Delta(h)(1 \otimes m)\big) \\
&= w\big(h_{(1)}\varphi(h_{(2)}m)\big) \overset{(3.1)}{=} w\Big(S^{-1}\big(m_{(1)}\varphi(hm_{(2)})\big)\Big) \\
&= \varphi\Big(h\big(wS^{-1}(m_{(1)})m_{(2)}\big)\Big)
\end{aligned}$$

We see that the product ww' is well-defined as alinear functional on H and it has the form $\varphi(\cdot g)$, where $g = wS^{-1}(m_{(1)})m_{(2)}$. So $ww' \in \widehat{H}$. Therefore, we have defined a product in $\widehat{H}$.

The associativity of this product in $\widehat{H}$ is an consequence of the coassociativity of Δ on H.

To prove that the product is non-degenerate, assume that $ww' = 0$ for all $w \in \widehat{H}$. From the above calculation, for any $h \in H$, $0 = (ww')(h) = \varphi\Big(h\big(wS^{-1}(m_{(1)})m_{(2)}\big)\Big)$, then $wS^{-1}(m_{(1)})m_{(2)} = 0$ because of the faithfulness of φ. This implies $wS^{-1}(m) = 0$ for all $w \in \widehat{H}$, i.e., $\varphi(S^{-1}(m)h) = 0$ for all $h \in H$. We conclude that $S^{-1}(m) = 0$ then $m = 0$, i.e., $w' = 0$. Similarly, $ww' = 0$ for all $w' \in \widehat{H}$ implies $w = 0$. $\square$

Remark Under the assumption, the elements of $\widehat{H}$ can be expressed in four different forms. When we use these different forms in the definition of product in $\widehat{H}$, we get the following useful expressions:

(1) $w\varphi(\cdot a) = \varphi(\cdot b)$ with $b = wS^{-1}(a_{(1)})a_{(2)}$; (2) $w\varphi(a\cdot) = \varphi(c\cdot)$ with $c = wS(a_{(1)})a_{(2)}$.

(3) $\psi(\cdot a)w = \psi(\cdot d)$ with $d = a_{(1)}wS(a_{(2)})$; (4) $\psi(a\cdot)w = \psi(e\cdot)$ with $e = a_{(1)}wS^{-1}(a_{(2)})$.

Moreover, the multiplier algebra $M(\widehat{H})$ of $\widehat{H}$ can be identified with the space H^*. Indeed, for $f \in H^*$ and $w \in \widehat{H}$, $fw, wf \in \widehat{H}$; he counit ε, as a linear functional on H, is in fact the unit in the multiplier algebra $M(\widehat{H})$; $fw = 0$ (resp. $wf = 0$) for all $w \in \widehat{H}$ implies $f = 0$.

Let us now define the comultiplication $\widehat{\Delta}$ on $\widehat{H}$. Roughly speaking, the coproduct is dual to the multiplication in H in the sense that

$$\langle \widehat{\Delta}(w), x \otimes y \rangle = \langle w, xy \rangle, \quad \forall x, y \in H.$$

Definition 3.2.5 Let $w_1, w_2 \in \widehat{H}$, then we put

$$\begin{aligned} \langle (w_1 \otimes 1)\widehat{\Delta}(w_2), x \otimes y \rangle &= \langle w_1 \otimes w_2, x_{(1)} \otimes x_{(2)}y \rangle & (3.5) \\ \langle \widehat{\Delta}(w_1)(1 \otimes w_2), x \otimes y \rangle &= \langle w_1 \otimes w_2, xy_{(1)} \otimes y_{(2)} \rangle & (3.6) \end{aligned}$$

for all $x, y \in H$.

We will first show that the functionals in Definition 3.2.7 are well-defined and again in $\widehat{H} \otimes \widehat{H}$.

Lemma 3.2.6 $(w_1 \otimes 1)\widehat{\Delta}(w_2), \widehat{\Delta}(w_1)(1 \otimes w_2) \in \widehat{H} \otimes \widehat{H}$. These above two formulas define $\widehat{\Delta}(w)$ as a multiplier in $M(\widehat{H} \otimes \widehat{H})$ for all $w \in \widehat{H}$.

Proof Let $w_1 = \psi(a\cdot)$ and $w_2 = \psi(b\cdot)$, where $a, b \in H$. For any $x, y \in H$, we have

$$\begin{aligned} \langle (w_1 \otimes 1)\widehat{\Delta}(w_2), x \otimes y \rangle &= \langle w_1 \otimes w_2, x_{(1)} \otimes x_{(2)}y \rangle \\ &= \psi(ax_{(1)})\psi\big(b(x_{(2)}y)\big) = \psi\big(b(\psi(ax_{(1)})x_{(2)}y)\big) \\ &\overset{(3.2)}{=} \psi\big(b(\psi(a_{(1)}x)S^{-1}(a_{(2)})y)\big) = \psi(a_{(1)}x)\psi\big(b(S^{-1}(a_{(2)})y)\big) \\ &= \Big(\psi(a_{(1)}\cdot) \otimes \psi\big(b(S^{-1}(a_{(2)})\cdot)\big)\Big)(x \otimes y). \end{aligned}$$

By the assumption, we obtain that $(w_1 \otimes 1)\widehat{\Delta}(w_2)$ is a well-defined element in $\widehat{H} \otimes \widehat{H}$. It is similar for the $\widehat{\Delta}(w_1)(1 \otimes w_2)$.

Using the fact that the product in $\widehat{H}$ is dual to the coproduct in H and Δ in H is coassociative, it easily followings that $\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big)(1 \otimes w_3) = (w_1 \otimes 1)\big(\widehat{\Delta}(w_2)(1 \otimes w_3)\big)$. Therefore, $\widehat{\Delta}(w)$ is defined as a two-side multiplier in $M(\widehat{H} \otimes \widehat{H})$. $\square$

Proposition 3.2.7 $\widehat{\Delta} : \widehat{H} \longrightarrow M(\widehat{H} \otimes \widehat{H})$ is an algebra homomorphism, and also $(1 \otimes w_1)\widehat{\Delta}(w_2), \widehat{\Delta}(w_1)(w_2 \otimes 1) \in \widehat{H} \otimes \widehat{H}$.

Proof It is straightforward that $\widehat{\Delta}$ is an algebra homomorphism, since for all $x, y \in H$

$$\begin{aligned} \langle \widehat{\Delta}(w_1w_2)(1 \otimes w_3), x \otimes y \rangle &= \langle w_1w_2 \otimes w_3, xy_{(1)} \otimes y_{(2)} \rangle \\ &= \langle w_1, x_{(1)}y_{(1)(1)} \rangle \langle w_2, x_{(2)}y_{(1)(2)} \rangle \langle w_3, y_{(2)} \rangle, \\ \langle \widehat{\Delta}(w_1)\widehat{\Delta}(w_2)(1 \otimes w_3), x \otimes y \rangle &= \langle \widehat{\Delta}(w_1)(f \otimes g), x \otimes y \rangle \qquad (\widehat{\Delta}(w_2)(1 \otimes w_3) := f \otimes g\ ) \end{aligned}$$

$$
\begin{aligned}
&= \langle \widehat{\Delta}(w_1)(1 \otimes g), x_{(1)} \otimes y \rangle \langle f, x_{(2)} \rangle \\
&= \langle w_1 \otimes g, x_{(1)} y_{(1)} \otimes y_{(2)} \rangle \langle f, x_{(2)} \rangle \\
&= \langle w_1, x_{(1)} y_{(1)} \rangle \langle f \otimes g, x_{(2)} \otimes y_{(2)} \rangle \\
&= \langle w_1, x_{(1)} y_{(1)} \rangle \langle \widehat{\Delta}(w_2)(1 \otimes w_3), x_{(2)} \otimes y_{(2)} \rangle \\
&= \langle w_1, x_{(1)} y_{(1)} \rangle \langle w_2 \otimes w_3, x_{(2)} y_{(2)(1)} \otimes y_{(2)(2)} \rangle \\
&= \langle w_1, x_{(1)} y_{(1)} \rangle \langle w_2, x_{(2)} y_{(2)(1)} \rangle \langle w_3, y_{(2)(2)} \rangle .
\end{aligned}
$$

By the coassociativity of Δ of H, we get $\widehat{\Delta}(w_1 w_2)(1 \otimes w_3) = \widehat{\Delta}(w_1)\widehat{\Delta}(w_2)(1 \otimes w_3)$ for all $w_3 \in \widehat{H}$. This implies $\widehat{\Delta}(w_1 w_2) = \widehat{\Delta}(w_1)\widehat{\Delta}(w_2)$.

With the bijective antipode, the proof of the second assertion is similar to the proof of Lemma 3.2.8. □

Let $w \in \widehat{H}$ and assume $w = \varphi(\cdot a)$ with $a \in H$ then. Define $\widehat{\varepsilon}(w) = \varphi(a) = w(1_H)$. Then $\widehat{\varepsilon}$ is a counit on $(\widehat{H}, \widehat{\Delta})$ as follows.

Proposition 3.2.8 $\widehat{\varepsilon} : \widehat{H} \longrightarrow k$ is an algebra homomorphism satisfying

$$(id \otimes \widehat{\varepsilon})\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big) = w_1 w_2 \tag{3.7}$$

$$(\widehat{\varepsilon} \otimes id)\big(\widehat{\Delta}(w_1)(1 \otimes w_2)\big) = w_1 w_2 \tag{3.8}$$

for all $w_1, w_2 \in \widehat{H}$.

Proof Firstly, let $w_1 = \varphi(a\cdot)$ and $w_2 = \varphi(b\cdot)$, then $w_1 w_2 = \varphi(c\cdot)$ with $c = \varphi\big(aS(b_{(1)})\big)b_{(2)}$. Therefore, if $\psi = \varphi \circ S$ we have

$$
\begin{aligned}
\widehat{\varepsilon}(w_1 w_2) &= \varphi(c) = \varphi\big(aS(b_{(1)})\big)\varphi(b_{(2)}) \\
&= \varphi\big(aS(b_{(1)}\varphi(b_{(2)}))\big) = \varphi(a)\varphi(b) \\
&= \widehat{\varepsilon}(w_1)\widehat{\varepsilon}(w_2).
\end{aligned}
$$

Secondly, let $w_1 = \psi(a\cdot)$ and $w_2 = \psi(b\cdot)$, then we have

$$
\begin{aligned}
(w_1 \otimes 1)\widehat{\Delta}(w_2) &= \psi(a_{(1)}\cdot) \otimes \psi\big(b(S^{-1}(a_{(2)})\cdot)\big), \\
\psi(a\cdot)\psi(b\cdot) &= \psi(a_{(1)}\cdot)\psi\big(bS^{-1}(a_{(2)})\big).
\end{aligned}
$$

Hence,

$$\begin{aligned}(id \otimes \widehat{\varepsilon})\big((w_1 \otimes 1)\widehat{\Delta}(w_2)\big) &= \psi(a_{(1)}\cdot)\psi\big(b(S^{-1}(a_{(2)})1)\big) \\ &= \psi(a_{(1)}\cdot)\psi\big(bS^{-1}(a_{(2)})\big) \\ &= w_1 w_2.\end{aligned}$$

Finally, the second formula is proven in a similar way, in this case letting $w_1 = \varphi(\cdot a)$ and $w_2 = \varphi(\cdot b)$. $\square$

Let $\widehat{S} : \widehat{H} \longrightarrow \widehat{H}$ be the dual to the antipode of H, i.e., $\widehat{S}(w) = w \circ S$. Then it is easy to see that $\widehat{S}(w) \in \widehat{H}$, and we have the following property.

Proposition 3.2.9 $\widehat{S}$ is antimultiplicative and coantimultiplicative such that

$$\begin{aligned} w' \otimes w &= (m \otimes id)(id \otimes \widehat{S} \otimes id)\Big((id \otimes \widehat{\Delta})\big((w' \otimes 1)\widehat{\Delta}(w)\big)\Big) \\ &= (m \otimes id)(\widehat{S} \otimes id \otimes id)\Big((id \otimes \widehat{\Delta})\big(\widehat{\Delta}(w)(\widehat{S}^{-1}(w') \otimes 1)\big)\Big) \\ &= (id \otimes m)(id \otimes \widehat{S} \otimes id)\Big((\widehat{\Delta} \otimes id)\big(\widehat{\Delta}(w')(1 \otimes w)\big)\Big) \\ &= (id \otimes m)(id \otimes id \otimes \widehat{S})\Big((\widehat{\Delta} \otimes id)\big((1 \otimes \widehat{S}^{-1}(w))\widehat{\Delta}(w')\big)\Big). \end{aligned}$$

Proof For $w_1, w_2 \in \widehat{H}$ and any $x \in H$,

$$\begin{aligned} \langle \widehat{S}(w_1 w_2), x \rangle &= \langle w_1 w_2, S(x) \rangle = \langle w_1, S(x_2) \rangle \langle w_2, S(x_1) \rangle \\ &= \langle \widehat{S}(w_1), x_2 \rangle \langle \widehat{S}(w_2), x_1 \rangle = \langle \widehat{S}(w_2)\widehat{S}(w_1), x \rangle \end{aligned}$$

This implies $\widehat{S}$ is antimultiplicative.

$$\begin{aligned} \langle \widehat{\Delta}\widehat{S}(w_1)(1 \otimes S(w_2)), x \otimes y \rangle &\overset{(3.6)}{=} \langle \widehat{S}(w_1) \otimes \widehat{S}(w_2), xy_{(1)} \otimes y_{(2)} \rangle \\ &= \langle w_1, S(xy_{(1)}) \rangle \langle w_2, S(y_{(2)}) \rangle \\ &= \langle w_1, S(y)_{(2)} S(x) \rangle \langle w_2, S(y)_{(1)} \rangle \\ &= \langle w_2 \otimes w_1, S(y)_{(1)} \otimes S(y)_{(2)} S(x) \rangle \\ &\overset{(3.5)}{=} \langle (w_2 \otimes 1)\widehat{\Delta}(w_1), S(y) \otimes S(x) \rangle \\ &= \langle (\widehat{S} \otimes \widehat{S})\widehat{\Delta}^{cop}(w_1)(1 \otimes S(w_2)), x \otimes y \rangle, \end{aligned}$$

We conclude $\widehat{S}$ is coantimultiplicative.

Finally, we show $w'\otimes w=(m\otimes id)(id\otimes\widehat{S}\otimes id)\Big((id\otimes\widehat{\Delta})\big((w'\otimes 1)\widehat{\Delta}(w)\big)\Big)$, the other three formulas is similar.

$$\begin{aligned}
&\langle (m\otimes id)(id\otimes\widehat{S}\otimes id)\Big((id\otimes\widehat{\Delta})\big((w'\otimes 1)\widehat{\Delta}(w)\big)\Big), x\otimes y\rangle\\
\overset{(3.4)}{=}\ &\langle (id\otimes\widehat{\Delta})\big((w'\otimes 1)\widehat{\Delta}(w)\big), x_{(1)}\otimes S(x_{(2)})\otimes y\rangle\\
=\ &\langle \big((w'\otimes 1)\widehat{\Delta}(w)\big), x_{(1)}\otimes S(x_{(2)})y\rangle\\
\overset{(3.5)}{=}\ &\langle w'\otimes w, x_{(1)(1)}\otimes x_{(1)(2)}(S(x_{(2)})y)\rangle\\
=\ &\langle w'\otimes w, x\otimes y\rangle.
\end{aligned}$$

This completes the proof. $\square$

The equation in the Proposition 3.2.11 can be expressed by generalized Sweedler notation as follows.

$$\begin{aligned}
w'\otimes w &= w'\widehat{S}(w_{(1)})w_{(2)(1)}\otimes w_{(2)(2)}=w'w_{(1)}\widehat{S}(w_{(2)(1)})\otimes w_{(2)(2)}\\
&= w'_{(1)(1)}\otimes\widehat{S}(w'_{(1)(2)})w'_{(2)}w=w'_{(1)(1)}\otimes w'_{(1)(2)}\widehat{S}(w'_{(2)})w.
\end{aligned}$$

As a consequence, the antipode $\widehat{S}$ also satisfies

$$\begin{aligned}
m(id\otimes\widehat{S})\big((w_1\otimes 1)\widehat{\Delta}(w_2)\big)&=\widehat{\varepsilon}(w_2)w_1,\\
m(\widehat{S}\otimes id)\big(\widehat{\Delta}(w_1)(1\otimes w_2)\big)&=\widehat{\varepsilon}(w_1)w_2.
\end{aligned}$$

In fact, there is another way to prove.

$$\begin{aligned}
\langle m(id\otimes\widehat{S})\big((w_1\otimes 1)\widehat{\Delta}(w_2)\big), x\rangle &= \langle (id\otimes\widehat{S})\big((w_1\otimes 1)\widehat{\Delta}(w_2)\big), x_{(1)}\otimes x_{(2)}\rangle\\
&= \langle \big((w_1\otimes 1)\widehat{\Delta}(w_2)\big), x_{(1)}\otimes S(x_{(2)})\rangle\\
&\overset{(3.5)}{=} \langle w_1\otimes w_2), x_{(1)(1)}\otimes x_{(1)(2)}S(x_{(2)})\rangle\\
&= \langle w_1\otimes w_2, x\otimes 1\rangle\\
&= \widehat{\varepsilon}(w_2)w_1.
\end{aligned}$$

Let ψ be a right faithful integral on H. For $w=\psi(a\cdot)$ we set $\widehat{\varphi}(w)=\varepsilon(a)$. Then we have the following result.

Proposition 3.2.10 $\widehat{\varphi}$ defined above is a faithful left integral on $\widehat{H}$.

Proof It is clear that $\widehat{\varphi}$ is non-zero. Assume $w_1 = \psi(a\cdot)$ and $w_2 = \psi(b\cdot)$ with $a, b \in H$, then

$$(w_1 \otimes 1)\widehat{\Delta}(w_2) \quad = \quad \psi(a_{(1)}\cdot) \otimes \psi\big(b(S^{-1}(a_{(2)})\cdot)\big).$$

Therefore, we have

$$\begin{aligned}(id \otimes \widehat{\varphi})\Big((w_1 \otimes 1)\widehat{\Delta}(w_2)\Big) &= \psi(a_{(1)}\cdot) \otimes \widehat{\varphi}\psi\big(b(S^{-1}(a_{(2)})\cdot)\big) \\ &= \psi(a\cdot)\varepsilon(b) = \widehat{\varphi}(w_2)w_1.\end{aligned}$$

Next, we show that $\widehat{\varphi}$ is faithful. If $w_1, w_2 \in \widehat{H}$ and assume $w_1 = \psi(a\cdot)$ with $a \in H$, we have $w_1 w_2 = \psi\big(a_{(1)} w_2 S^{-1}(a_{(2)}) \cdot\big)$. Therefore, $\widehat{\varphi}(w_1 w_2) = w_2 S^{-1}(a)$. If this is 0 for all $a \in H$, then $w_2 = 0$, wile if this is 0 for all w_2 then $a = 0$. This proves the faithfulness of $\widehat{\varphi}$. $\square$

Now, we introduce an algebraic structure: multipler Hopf coquasigroup, generalizing the ordinary Hopf coquasigroup to a nonunital case. Let A be an (associative) algebra, may not has a unit, but the product, seen as a bilinear form, is non-degenerated.

Definition 3.2.11 A *multipler Hopf coquasigroup* is a nondegenerate associative algebra A equipped with algebra homomorphisms $\Delta : A \longrightarrow M(A \otimes A)$(coproduct), $\varepsilon : A \longrightarrow k$(counit) and a linear map $S : A \longrightarrow A$ (antipode) such that

(1) $T_1(a \otimes b) = \Delta(a)(1 \otimes b)$ and $T_2(a \otimes b) = (a \otimes 1)\Delta(b)$ belong to $A \otimes A$ for any $a, b \in A$.

(2) The counit satisfies $(\varepsilon \otimes id)T_1(a \otimes b) = ab = (id \otimes \varepsilon)T_2(a \otimes b)$.

(3) S is antimultiplicative and anticomultiplicative such that for any $a, b \in A$

$$S(a_{(1)})a_{(2)(1)} \otimes a_{(2)(2)} = 1 \otimes a = a_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)}, \tag{3.9}$$

$$a_{(1)(1)} \otimes S(a_{(1)(2)})a_{(2)} = a \otimes 1 = a_{(1)(1)} \otimes a_{(1)(2)}S(a_{(2)}). \tag{3.10}$$

If the antipode S is bijective, then multipler Hopf coquasigroup (A, Δ) is called *regular*.

Remark (1) In multipler Hopf coquasigroup (A, Δ), T_1 and T_2 are bijective. If (A, Δ) is regular, then T_3 and T_4 are also. In fact, from (3) in Definition 3.2.11 we can easily get

$$m(id \otimes S)\big((a \otimes 1)\Delta(b)\big) = \varepsilon(b)a,$$
$$m(S \otimes id)\big(\Delta(a)(1 \otimes b)\big) = \varepsilon(a)b.$$

(2) The equation (3.9) and (3.10) make sense. Take (3.9) for example, (3.10) is similar.

$$\begin{aligned} b \otimes ac &= ba_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)}c \\ &= (m \otimes id)(id \otimes S \otimes id)\Big((id \otimes \Delta)\big((b \otimes 1)\Delta(a)\big)(1 \otimes 1 \otimes c)\Big). \end{aligned}$$

$ba_{(1)} \otimes a_{(2)} = (b \otimes 1)\Delta(a) \in A \otimes A$, and then $a_{(2)(1)} \otimes a_{(2)(2)}c \in A \otimes A$. Therefore, $b \otimes ac = ba_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)}c$ holds for all $b, c \in A$. This implies $a_{(1)}S(a_{(2)(1)}) \otimes a_{(2)(2)} = 1 \otimes a$.

$$\begin{aligned} b \otimes ac &= S(a_{(1)})a_{(2)(1)}b \otimes a_{(2)(2)}c = S(a_{(1)})a_{(2)(1)}x_{(1)} \otimes a_{(2)(2)}x_{(2)}y \\ &= (m \otimes id)(S \otimes id \otimes id)\Big((id \otimes \Delta)\big(\Delta(a)(1 \otimes x)\big)(1 \otimes 1 \otimes y)\Big), \end{aligned}$$

where $b \otimes c = \Delta(x)(1 \otimes y)$. $b \otimes ac = S(a_{(1)})a_{(2)(1)}b \otimes a_{(2)(2)}c$ for all $b, c \in A$ implies $1 \otimes a = S(a_{(1)})a_{(2)(1)} \otimes a_{(2)(2)}$.

(3) The comultiplication may be not coassociative. Multipler Hopf coquasigroup weakens the coassociativity of coproduct in multiplier Hopf algebra, while algebraic quantum hypergroup in [7] weakens the homomorphism of coproduct. This is the main difference.

Following Definition 3.2.11, we get the main result of this section.

Theorem 3.2.12 Let (H, Δ) be an infinite dimensional Hopf quasigroup with a faithful integral φ and a bijective antipode S. Then under Assumption 3.2.3 the integral dual $(\widehat{H}, \widehat{\Delta})$ is a regular multipler Hopf coquasigroup with a faithful integral.

Because (infinite dimensional) Hopf quasigroup (H, Δ) has the unit 1_H, then there is a special element $\varphi = \varphi(\cdot 1_H) \in \widehat{H}$ such that for $w \in \widehat{H}$

$$(w\varphi)(h) = (w \otimes \varphi)\Delta(h) = \varphi(h)w(1) = \widehat{\varepsilon}(w)\varphi(h).$$

This implies $w\varphi = \widehat{\varepsilon}(w)\varphi$. We call φ a cointegral in $(\widehat{H}, \widehat{\Delta})$.

Analogous to multiplier Hopf algebra case in [24], we say that a regular multiplier Hopf coquasigroup with a faithful integral (A, Δ) is of discrete type, if there is a non-zero element $\xi \in A$ so that $a\xi = \varepsilon(a)\xi$ for all $a \in A$.

Then we have the integral dual $(\widehat{H}, \widehat{\Delta})$ of infinite dimensional Hopf quasigroup (H, Δ) is a multipler Hopf coquasigroup of discrete type.

§3.3 Multiplier Hopf coquasigroup: motivating example

In last section, we introduce the notion of multiplier Hopf coquasigroup, extending Hopf coquasigroup to a nonunital case, and provided a interesting construction: the integral dual of infinite dimensional Hopf quasigroups with integrals.

In the following, we firstly introduce the motivating example, where Assumption 4.3 naturally holds. And then we make some direct comments on multiplier Hopf coquasigroups.

Example 3.3.1 Let G be a infinite (IP) quasigroup with identity element e, by definition $u^{-1}(uv) = v = (vu)u^{-1}$ for all $u, v \in G$. We have that the quasigroup algebra kG is a Hopf quasigroup with the structure shown on the base element $\{u|u \in G\}$

$$\Delta(u) = u \otimes u, \quad \varepsilon(u) = 1, \quad S(u) = u^{-1}.$$

The function $\delta_u, u \in G$ on kG is given by $\delta_u(v) = \delta_{u,v}$, where $\delta_{u,v}$ is the kronecker delta. Then δ_e is the left and right integral on kG.

The integral dual $k(G) = \widehat{kG} = \{\delta_e(\cdot u)|u \in kG\} = \{\delta_{u^{-1}}|u \in kG\} = \{\delta_e(u\cdot)|u \in kG\}$. $\delta_e\big((\cdot u)v\big) = \delta_{v^{-1}u^{-1}} \in k(G)$ and $\delta_e\big(u(v\cdot)\big) = \delta_{v^{-1}u^{-1}} \in k(G)$. Assumption 3.2.3 naturally holds. Then $(k(G), \widehat{\Delta}, \widehat{\varepsilon}, \widehat{S})$ is a multipler Hopf coquasigroup with the structure as follows.

As an algebra, $k(G)$ is a nondegenerate algebra with the product

$$\delta_u \delta_v = \delta_{u,v} \delta_v,$$

and $1 = \sum_{u \in G} \delta_u$ is the unit in $M(k(G))$. The coproduct, counit and antipode are given by

$$\widehat{\Delta}(\delta_u) = \sum_{v \in G} \delta_v \otimes \delta_{v^{-1}u}, \quad \widehat{\varepsilon}(\delta_u) = \delta_{u,e}, \quad \widehat{S}(\delta_u) = \delta_{u^{-1}}.$$

By the definition of $\widehat{\varphi}$, we get the left integral on $k(G)$ is the function that maps every δ_u to 1.

As in the theory of multiplier Hopf algebra in [23], we also can define a multiplier Hopf $*$-coquasigroup (A, Δ) over $\mathbb{C}$, in which (A, Δ) is a regular multiplier Hopf coquasigroup with the coproduct, counit and antipode compatible with the involution $*$. i.e.,

(1) The comultiplication Δ is also a $*$-homomorphism (i.e., $\Delta(a^*) = \Delta(a)^*$);

(2) $\varepsilon(a^*) = \overline{\varepsilon(a)}$, where $\overline{(\cdot)}$ means the conjugation of complex numbers;

(3) $S(S(a)^*)^* = a$.

Example 3.3.2 In Example 3.3.1 if $k = \mathbb{C}$, then $\mathbb{C}(G)$ is a multiplier Hopf $*$-coquasigroup.

Proposition 3.3.3 Let (A, Δ) be a multipler Hopf $(*$-$)$coquasigroup. Then (A, Δ) is a multiplier Hopf $(*$-$)$algebra introduced, if and only if the comultiplication Δ is coassociative.

Proposition 3.3.4 If multipler Hopf coquasigroup (A, Δ) has the unit 1, then (A, Δ) is the usual Hopf coquasigroup.

Following these two results, multipler Hopf coquasigroup can be considered as the generalization of multiplier Hopf algebra and Hopf coquasigroup. Naturally, we can define flexible, alternative and Moufang multipler Hopf coquasigroup.

A multipler Hopf coquasigroup (A, Δ) is called *flexible* if

$$a_{(1)}a_{(2)(2)} \otimes a_{(2)(1)} = a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)}, \quad \forall a \in A, \tag{3.11}$$

and *alternative* if also

$$a_{(1)}a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)}a_{(1)(2)} \otimes a_{(2)}, \tag{3.12}$$

$$a_{(1)} \otimes a_{(2)(1)}a_{(2)(2)} = a_{(1)(1)} \otimes a_{(1)(2)}a_{(2)}, \quad \forall a \in A. \tag{3.13}$$

A is called *Moufang* if

$$a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}, \quad \forall a \in A, \tag{3.14}$$

Remark (1) By 'cover technique' introduced in [25], these four equations make sense.

(2) From the dual, we can get that the integral dual $(\widehat{H}, \widehat{\Delta})$ of infinite dimensional flexible (resp. alternative, Moufang) Hopf quasigroup (H, Δ) is a flexible (resp. alternative, Moufang) multipler Hopf coquasigroup.

Chapter 4 Multiplier Hopf coquasigroups

Let A be a multiplier Hopf coquasigroup introduced in last chapter. If the faithful integrals exist, then they are unique up to scalar. For a multiplier Hopf coquasigroup of discrete type A, its integral duality $\widehat{A}$ is a Hopf quasigroup, and the biduality $\widehat{\widehat{A}}$ is isomorphic to the original A as multiplier Hopf coquasigroups. This biduality theorem also holds for a class of Hopf quasigroups with faithful integrals.

§4.1 Integrals on a multiplier Hopf coquasigroup

Let (A, Δ) be a regular multiplier Hopf coquasigroup with a faithful integral φ. Just as in the case of algebraic quantum group (see Proposition 2.6 in [14]) or algebraic quantum hypergroups (see Proposition 1.6 in [7]), we show that the multiplier Hopf coquasigroup (A, Δ) must have local units in the sense of the following proposition.

Proposition 4.1.1 Let (A, Δ) be a regular multiplier Hopf coquasigroup with a non-zero integral φ. Given finite numbers of elements $\{a_1, a_2, \cdots, a_n\}$, there exists an element $e \in A$ such that $a_i e = a_i = e a_i$ for all i.

Proof It is similar to the proof of Proposition 1.6 in [7]. Set the linear space

$$V = \{(aa_1, aa_2, \cdots, aa_n, a_1a, a_2a, \cdots, a_na) \mid a \in A\} \subseteq A^{2n}.$$

Consider a linear functional on A^{2n} that is zero on V. This means that we have functionals w_i and ρ_i on A for $i = 1, 2, \cdots, n$, such that

$$\sum_{i=1}^{n} w_i(aa_i) + \sum_{i=1}^{n} \rho_i(a_ia) = 0 \quad \text{for all } a \in A.$$

Then for all $x, a \in A$ we have

$$\begin{aligned} & x\Big(\sum_{i=1}^{n}(w_i \otimes id)\big(\Delta(a)(a_i \otimes 1)\big) + \sum_{i=1}^{n}(\rho_i \otimes id)\big((a_i \otimes 1)\Delta(a)\big)\Big) \\ = \;& \sum_{i=1}^{n}(w_i \otimes id)\big((1 \otimes x)\Delta(a)(a_i \otimes 1)\big) + \sum_{i=1}^{n}(\rho_i \otimes id)\big((a_i \otimes 1)(1 \otimes x)\Delta(a)\big) \end{aligned}$$

$$= \quad 0,$$

since $(1\otimes x)\Delta(a)\in A\otimes A$. Because the product in A is nondegenerate, we have for all $a\in A$ that

$$\sum_{i=1}^{n}(w_i\otimes id)\big(\Delta(a)(a_i\otimes 1)\big)+\sum_{i=1}^{n}(\rho_i\otimes id)\big((a_i\otimes 1)\Delta(a)\big)=0.$$

Now applying φ on this expression, we get

$$\varphi(a)\Big(\sum_{i=1}^{n}w_i(a_i)+\sum_{i=1}^{n}\rho_i(a_i)\Big)=0\quad\text{for all } a\in A.$$

As integral φ is non-zero, we have

$$\sum_{i=1}^{n}w_i(a_i)+\sum_{i=1}^{n}\rho_i(a_i)=0.$$

So any lieaner functional on A^{2n} that is zero on the space V is also zero on the vector $(a_1,a_2,\cdots,a_n,a_1,a_2,\cdots,a_n)$. Therefore $(a_1,a_2,\cdots,a_n,a_1,a_2,\cdots,a_n)\in V$. This means that there exists an element $e\in A$ such that $a_ie=a_i=ea_i$ for all i. □

Recall from [7] that a linear functional f on A is called faithful if, for $a\in A$, we must have $a=0$ when either $f(ab)=0$ for all $b\in A$ or $f(ba)=0$ for all $b\in A$. Then under the faithfulness we can get the following result.

Lemma 4.1.2 Let (A,Δ) be a multiplier Hopf coquasigroup. If f is a faithful linear functional on A, then for any $a\in A$ there is an element $e\in A$ such that

$$a=(id\otimes f)\big(\Delta(a)(1\otimes e)\big).$$

Proof Take $a\in A$ and set $V=\{(id\otimes f)\big(\Delta(a)(1\otimes b)\big)\mid b\in A\}$, we need to show $a\in V$. Suppose that $a\notin V$, then there is on A a functional $w\in A^*$ such that $w(a)\neq 0$ while $w|_V=0$, i.e.,

$$0=w\Big((id\otimes f)\big(\Delta(a)(1\otimes b)\big)\Big)=f\Big(\big((w\otimes id)\Delta(a)\big)b\Big)\quad\text{for all } b\in A.$$

Observe that $(w \otimes id)\Delta(a) \in M(A)$ and not necessarily belongs to A. However, we get $f\Big(\big((w \otimes id)\Delta(a)\big)b'b''\Big) = 0$ for all $b', b'' \in A$ and by the faithfulness of f, we must have $\big((w \otimes id)\Delta(a)\big)b' = 0$ for all $b' \in A$.

If we apply the counit, $w(a)\varepsilon(b') = 0$ for all $b' \in A$ and hence $w(a) = 0$. This is a contradiction. $\square$

Similarly we have for a regular multiplier Hopf coquasigroup (A, Δ),

$$
\begin{aligned}
a &\in \{(id \otimes f)\big((1 \otimes b)\Delta(a)\big) \mid b \in A\} \\
a &\in \{(f \otimes id)\big((b \otimes 1)\Delta(a)\big) \mid b \in A\} \\
a &\in \{(f \otimes id)\big(\Delta(a)(b \otimes 1)\big) \mid b \in A\}
\end{aligned}
$$

for any faithful $f \in A^*$. In particularly, when we assume that a left integral φ is faithful, then

$$
\begin{aligned}
A &= \mathrm{span}\{(id \otimes \varphi)\big(\Delta(a)(1 \otimes b)\big) \mid a, b \in A\}, \\
A &= \mathrm{span}\{(id \otimes \varphi)\big((1 \otimes a)\Delta(b)\big) \mid a, b \in A\},
\end{aligned}
$$

where 'span' means the linear span of a set of element.

Next, we give some equations on the left and right integral.

Proposition 4.1.3 Let φ (resp. ψ) be a left (resp. right) integral on A, then for $a, b \in A$

$$a_{(1)}\varphi\big(a_{(2)}S(b)\big) = \varphi\big(aS(b_{(1)})\big)b_{(2)}, \qquad a_{(1)}\varphi(ba_{(2)}) = S(b_{(1)})\varphi(b_{(2)}a). \tag{4.1}$$

$$\psi\big(S(a)b_{(1)}\big)b_{(2)} = \psi\big(S(a_{(2)})b\big)a_{(1)}, \qquad \psi(a_{(1)}b)a_{(2)} = \psi(ab_{(1)})S(b_{(2)}). \tag{4.2}$$

Proof We prove the first two equations on φ, and the others are similar.

$$
\begin{aligned}
a_{(1)}\varphi\big(a_{(2)}S(b)\big) &\overset{(2.5)}{=} a_{(1)}\big(S(b_{(1)(2)})b_{(2)}\big)\varphi\big(a_{(2)}S(b_{(1)(2)})\big) \\
&= \big(a_{(1)}S(b_{(1)(2)})\big)b_{(2)}\varphi\big(a_{(2)}S(b_{(1)(2)})\big) \\
&= \big(a_{(1)}S(b_{(1)})_{(1)}\big)b_{(2)}\varphi\big(a_{(2)}S(b_{(1)})_{(2)}\big) \\
&= \big(aS(b_{(1)})\big)_{(1)}b_{(2)}\varphi\big((aS(b_{(1)}))_{(2)}\big)
\end{aligned}
$$

$$= \quad \varphi\big(aS(b_{(1)})\big)b_{(2)},$$

and

$$\begin{aligned}
a_{(1)}\varphi(ba_{(2)}) &\overset{(2.4)}{=} \big(S(b_{(1)})b_{(2)(1)}\big)a_{(1)}\varphi\big(b_{(2)(2)}a_{(2)}\big) \\
&= S(b_{(1)})\big(b_{(2)(1)}a_{(1)}\big)\varphi\big(b_{(2)(2)}a_{(2)}\big) \\
&= S(b_{(1)})\big(b_{(2)}a\big)_{(1)}\varphi\big((b_{(2)}a)_{(2)}\big) \\
&= S(b_{(1)})\varphi(b_{(2)}a).
\end{aligned}$$

This completes the proof. $\square$

Remark (1) Following Lemma 4.1.2, we can easily check that (3.1) and (3.2) make sense.

(2) These formulas are useful in the following part. We take the first one for example. When the antipode of (A, Δ) is bijective, $a_{(1)}\varphi\big(a_{(2)}S(b)\big) = \varphi\big(aS(b_{(1)})\big)b_{(2)}$ is equivalent to

$$S(id \otimes \varphi)\big(\Delta(a)(1 \otimes b)\big) = (id \otimes \varphi)\big((1 \otimes a)\Delta(b)\big),$$

which is used to define the antipode in algebraic quantum hypergroup (see Definition 1.9 in [7]).

Set $x = (id \otimes \varphi)\big(\Delta(a)(1 \otimes b)\big)$ and apply ε on the above equation, we have $\varepsilon\big(S(x)\big) = \varphi(ab) = \varepsilon(x)$. By Lemma 4.1.2 we get $\varepsilon \circ S = \varepsilon$.

In the following, we will show the uniqueness of faithful left integrals.

Theorem 4.1.4 Let φ' be another faithful left integral on (A, Δ), then $\varphi' = \lambda\varphi$ for some scalar $\lambda \in k$, i.e., the faithful left integral on A is unique up to scalar.

Proof From Proposition 4.1.3, we have $a_{(1)}\varphi(ba_{(2)}) = S(b_{(1)})\varphi(b_{(2)}a)$ for all $a, b \in A$. Apply φ' to both expressions in this equation. Because $\varphi' \circ S$ is a right integral, the right hand side will give

$$\varphi' S(b_{(1)})\varphi(b_{(2)}a) = \varphi(\varphi' S(b_{(1)})b_{(2)}a) = \varphi(\varphi' S(b)1_{M(A)} \cdot a) = \varphi' S(b)\varphi(a).$$

For the left hand side,

$$\varphi'(a_{(1)}\varphi(ba_{(2)})) = \varphi'(a_{(1)})\varphi(ba_{(2)}) = \varphi(b\varphi'(a_{(1)})a_{(2)}) = \varphi(b\delta_a),$$

where $\delta_a = \varphi'(a_{(1)})a_{(2)}$. Therefore, $\varphi' S(b)\varphi(a) = \varphi(b\delta_a)$ for all $a, b \in H$.

We claim that there is an element $\delta \in M(A)$ such that $\delta_a = \varphi(a)\delta$ for all $a \in A$. Indeed, for any $a' \in A$

$$\begin{aligned}
\varphi(b\varphi(a')\delta_a) &= \varphi(a')\varphi(b\delta_a) = \varphi(a')\varphi' S(b)\varphi(a) \\
&= \varphi(a)\varphi' S(b)\varphi(a') = \varphi(a)\varphi(b\delta_{a'}) \\
&= \varphi(b\varphi(a)\delta_{a'}),
\end{aligned}$$

then $\varphi(a')\delta_a = \varphi(a)\delta_{a'}$ for all $a, a' \in A$, since φ is faithful. Choose an $a' \in A$ such that $\varphi(a') = 1$ and denote $\delta = \delta_{a'}$, then $\delta_a = \varphi(a)\delta$.

If we apply ε, we get

$$\begin{aligned}
\varphi(a)\varepsilon(\delta) &= \varepsilon(\delta_a) = \varepsilon(\varphi'(a_{(1)})a_{(2)}) \\
&= \varphi'(a_{(1)})\varepsilon(a_{(2)}) = \varphi'(a_{(1)}\varepsilon(a_{(2)})) \\
&= \varphi'(a)
\end{aligned}$$

for all $a \in A$ and with $\lambda = \varepsilon(\delta)$, we find the desired result. □

Remark (1) Similarly, the right faithful integral on A is unique up to scalar. However, as in the special infinite dimensional Hopf algebra case, the non-zero faithful integrals do not always exist in infinite dimensional case.

(2) The uniqueness of the faithful integral also provides the uniqueness of the antipode as in [7].

Proposition 4.1.5 There is a unique invertible element $\delta \in M(A)$ such that for all $a \in A$

(1) $(\varphi \otimes id)\Delta(a) = \varphi(a)\delta$ and $(id \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$.

(2) $\varphi S(a) = \varphi(a\delta)$.

Proof In the proof of Theorem 4.1.4, $\varphi(a)\delta = \delta_a = \varphi'(a_{(1)})a_{(2)}$ and $\varphi' S(b)\varphi(a) = \varphi(b\delta_a)$, we take $\varphi' = \varphi$ and get a element $\delta \in M(A)$ such that $(\varphi \otimes id)\Delta(a) = \varphi(a)\delta$ and $\varphi S(a) = \varphi(a\delta)$. This gives the first part of (1) and (2).

If we apply ε on the first equation, we find $\varepsilon(\delta) = 1$. Because S flips the coproduct and if we let $\psi = \varphi \circ S$, we get

$$\begin{aligned}(id \otimes \psi)\Delta(a) &= S^{-1}(S \otimes \psi)\Delta(a) = S^{-1}(S \otimes \varphi \circ S)\Delta(a) \\ &= S^{-1}(id \otimes \varphi)(S \otimes S)\Delta(a) = S^{-1}(id \otimes \varphi)\Delta^{cop}(S(a)) \\ &= S^{-1}(\varphi \otimes id)\Delta(S(a)) \overset{(1)}{=} S^{-1}(\varphi(S(a))\delta) \\ &= \psi(a)S^{-1}(\delta).\end{aligned}$$

It remains to show $S^{-1}(\delta) = \delta^{-1}$.

If we apply φ to the formula (3.2) $\psi(a_{(1)}b)a_{(2)} = \psi(ab_{(1)})S(b_{(2)})$, we get

$$\begin{aligned}\psi(b)\varphi(a) &= \psi(ab_{(1)})\varphi S(b_{(2)}) = \psi(ab_{(1)}\psi(b_{(2)})) = \psi(a\psi(b)S^{-1}(\delta)) \\ &= \psi(b)\psi(aS^{-1}(\delta))\end{aligned}$$

for all $a, b \in A$. Then $\varphi(a) = \psi(aS^{-1}(\delta))$ for all $a \in A$. Therefore, $\varphi(a) = \varphi S(aS^{-1}(\delta)) = \varphi\big(aS^{-1}(\delta)\delta\big)$ and so $S^{-1}(\delta)\delta = 1_{M(A)}$. On the other hand, $\psi(a) = \varphi S(a) = \varphi(a\delta) = \psi(a\delta S^{-1}(\delta))$ and so $\delta S^{-1}(\delta) = 1_{M(A)}$. Hence, δ is invertible and $S^{-1}(\delta) = \delta^{-1}$, equivalently $S(\delta) = \delta^{-1}$. $\square$

Remark (1) The square S^2 leaves the coproduct invariant, it follows that the composition $\varphi \circ S^2$ of the faithful left integral φ with S^2 will again a faithful left integral. By the uniqueness of faithful left integrals, there is a number $\tau \in k$ such that $\varphi \circ S^2 = \tau\varphi$.

(2) If we apply (2) in Proposition 4.1.3 twice, we get

$$\varphi\big(S^2(a)\big) = \varphi\big(S(a)\delta\big) = \varphi\big(S(\delta^{-1}a)\big) = \varphi\big((\delta^{-1}a)\delta\big) = \varphi\big(\delta^{-1}a\delta\big).$$

So $\varphi\big(\delta^{-1}a\delta\big) = \tau\varphi(a)$.

(3) We call δ the modular element as in algebraic quantum group. Here we cannot conclude that $\Delta(\delta) = \delta \otimes \delta$ due to lack of the coassociativity of Δ.

Finally, just as in the algebraic quantum and algebraic quantum hypergroup case, we will show the existence of the modular automorphism.

Proposition 4.1.6 (1) There is a unique automorphism σ of A such that $\varphi(ab) = \varphi\big(b\sigma(a)\big)$ for all $a, b \in A$. We also have $\varphi\big(\sigma(a)\big) = \varphi(a)$ for all $a \in A$.

(2) Similarly, there is a unique automorphism σ' of A satisfying $\psi(ab) = \psi\big(b\sigma'(a)\big)$ for all $a, b \in A$. And also $\psi\big(\sigma'(a)\big) = \psi(a)$ for all $a \in A$.

Proof (1) For any $p, q, x \in A$,

$$\begin{aligned}
(\psi \otimes \varphi)\big(xq_{(1)} \otimes pS(q_{(2)})\big) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \varphi\big(p\underline{\psi(xq_{(1)})S(q_{(2)})}\big) \\
&\overset{(4.2)}{=} \varphi\big(p\psi(x_{(1)}q)x_{(2)}\big) = \psi(x_{(1)}q)\varphi\big(px_{(2)}\big) \\
&= \psi\big(\underline{x_{(1)}\varphi(px_{(2)})}q\big) \overset{(4.3)}{=} \psi\big(S(p_{(1)})\varphi(p_{(2)}x)q\big) \\
&= \varphi(p_{(2)}x)\psi\big(S(p_{(1)})q\big) = \varphi\Big(\big(\psi\big(S(p_{(1)})q\big)p_{(2)}\big)x\Big) \\
&= \varphi\Big((\psi \circ S \otimes id)\big((S^{-1}(q) \otimes 1)\Delta(p)\big)x\Big).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
(\psi \otimes \varphi)\big(xq_{(1)} \otimes pS(q_{(2)})\big) &= \psi(xq_{(1)})\varphi(pS(q_{(2)})) = \psi\Big(x(q_{(1)}\varphi\big(pS(q_{(2)})\big))\Big) \\
&= \psi\Big(x(id \otimes \varphi \circ S)\big(\Delta(q)(1 \otimes S^{-1}(p))\big)\Big).
\end{aligned}$$

Now assume that $\psi = \varphi \circ S$. then we have $\psi \circ S = \tau\varphi$ and $\psi(y) = \varphi(y\delta)$ by Proposition 4.1.5 (2). Then the above calculation will give us

$$\begin{aligned}
\varphi(ax) &= \psi(xb) = \frac{1}{\tau}\varphi \circ S(xb) \\
&= \frac{1}{\tau}\varphi(xb\delta) = \varphi\big(x(\frac{1}{\tau}b\delta)\big) \\
&= \varphi\big(xb\big)
\end{aligned}$$

for all $x \in A$, where $a = (\psi \circ S \otimes id)\big((S^{-1}(q) \otimes 1)\Delta(p)$, $b' = (id \otimes \varphi \circ S)\big(\Delta(q)(1 \otimes S^{-1}(p))\big)$ and $b = b'\delta$.

Because φ is faithful, the element b is uniquely determined by the element a. So we can define $\sigma(a) = b$. Moreover, by Lemma 4.1.2 and its remark, all element in A are of the form a above, the map σ is defined on all of A. The map σ is injective by the faithfulness of φ, it is also surjective because all element in A are also of the form b above.

Take $a, b, c \in A$, then

$$\begin{aligned}\varphi\big(c\sigma(ab)\big) &= \varphi\big((ab)c\big) = \varphi(abc) = \varphi\big(a(bc)\big) \\ &= \varphi\Big((bc)\sigma(a)\Big) = \varphi\Big(b\big(c\sigma(a)\big)\Big) = \varphi\Big(\big(c\sigma(a)\big)\sigma(b)\Big) \\ &= \varphi\Big(c\big(\sigma(a)\sigma(b)\big)\Big).\end{aligned}$$

It follows from the faithfulness of φ that $\sigma(ab) = \sigma(a)\sigma(b)$. So $\sigma : A \longrightarrow A$ is an algebraic homomorphism. Apply this result twice, we have

$$\varphi(ab) = \varphi\big(b\sigma(a)\big) = \varphi\big(\sigma(a)\sigma(b)\big) = \varphi\big(\sigma(ab)\big) \quad \text{for all } a, b \in A.$$

By Proposition 4.1.1 A has local units, then $A^2 = A$, so φ is σ-invariant.

(2) Using that $\psi = \varphi \circ S$ we can easily get the statement for ψ.

$$\begin{aligned}\psi(ab) &= \varphi S(ab) = \varphi\big(S(b)S(a)\big) \\ &= \varphi\Big(\sigma^{-1}S(a)S(b)\Big) = \varphi S\Big(bS^{-1}\sigma^{-1}S(a)\Big) \\ &= \psi\Big(bS^{-1}\sigma^{-1}S(a)\Big).\end{aligned}$$

Therefore, $\sigma' = S^{-1}\sigma^{-1}S$. $\square$

Remark In the proof of Proposition 4.1.6, we have

$$\varphi\Big((\psi \circ S \otimes id)\big((S^{-1}(q) \otimes 1)\Delta(p)\big)x\Big) = \psi\Big(x(id \otimes \varphi \circ S)\big(\Delta(q)(1 \otimes S^{-1}(p))\big)\Big).$$

According to Lemma 4.1.2, we have that if $a \in A$ then there is a $b \in A$ such that $\varphi(ax) = \psi(xb)$ for all $x \in A$. This result will be used in the next section.

As in the algebraic quantum group and hypergroup cases, the automorphism σ and σ' are called the modular automorphisms of A associated with φ and ψ respectively. There are some extra properties derived from the above proposition.

Proposition 4.1.7 With the notation of above, we have

(1) $\sigma' = S^{-1}\sigma^{-1}S$ and $\sigma'(a) = \delta\sigma(a)\delta^{-1}$.

(2) $\sigma(\delta) = \frac{1}{\tau}\delta$ and $\sigma'(\delta) = \frac{1}{\tau}\delta$

(3) The modular automorphisms σ and σ' commute with each other.

(4) The modular automorphisms σ and σ' commute with S^2.

(5) For all $a \in A$, $\Delta\big(\sigma(a)\big) = (S^2 \otimes \sigma)\Delta(a)$ and $\Delta\big(\sigma'(a)\big) = (\sigma' \otimes S^{-2})\Delta(a)$.

Compared with algebraic quantum groups, Δ on multiplier Hopf coquasigroup is not necessarily coassociatve. This is a significant difference between the two objects. Other than that, the proof is similar.

§4.2 Duality of discrete multiplier Hopf coquasigroups

In this section, we will construct the dual of (infinite dimensional) multiplier Hopf coquasigroup of discrete type. The construction bases on the faithful integrals introduced in the last section. Here, we also start with defining the following subspace of the dual space A^*.

Definition 4.2.1 Let φ be a faithful left integral on a regular multiplier Hopf coquasigroup (A, Δ). We define $\widehat{A}$ as the space of linear functionals on A of the form $\varphi(\cdot a)$ where $a \in A$, i.e.,

$$\widehat{A} = \{\varphi(\cdot a) \mid a \in A\}.$$

Because of Proposition 4.1.3 and the following remark, we have

$$\widehat{A} = \{\varphi(\cdot a) \mid a \in A\} = \{\psi(a\cdot) \mid a \in A\} = \{\varphi(a\cdot) \mid a \in A\} = \{\psi(\cdot a) \mid a \in A\}.$$

Recall from [28] a regular multiplier Hopf coquasigroup (A, Δ) with a faithful integral φ is called of discrete type, if there is a non-zero element $\xi \in A$ so that $a\xi = \varepsilon(a)\xi$ for all $a \in A$.

The element ξ ia called a *left cointegral*. Similarly a *right cointegral* is a non-zero element $\eta \in A$ so that $\eta a = \varepsilon(a)\eta$. The antipode will turn a left cointegral into a right one and a right one into a left one.

Following this definition, we conclude $\varphi(\xi) \neq 0$. (If not, $0 = \varepsilon(a)\varphi(\xi) = \varphi(a\xi)$ for all $a \in A$, then $\xi = 0$ by the faithfulness of φ. this is contradiction.)

We start by making a discrete multiplier Hopf coquasigroup (A, Δ) into an unital algebra by dualizing the coproduct.

Proposition 4.2.2 For $w, w' \in \widehat{A}$, we can define a linear functional ww' on A by the formula

$$(ww')(x) = (w \otimes w')\Delta(x), \quad \forall x \in A. \tag{4.3}$$

Then $ww' \in \widehat{A}$. This product on $\widehat{A}$ is not necessarily associative, but has a unit.

Proof Let $w, w' \in \widehat{A}$ and assume that $w' = \varphi(\cdot a)$ with $a \in A$. we have

$$\begin{aligned}
(ww')(x) &= (w \otimes \varphi(\cdot a))\Delta(x) = (w \otimes \varphi)\big(\Delta(x)(1 \otimes a)\big) \\
&= w\big(x_{(1)}\varphi(x_{(2)}a)\big) \overset{(3.1)}{=} w\Big(S^{-1}\big(a_{(1)}\varphi(xa_{(2)})\big)\Big) \\
&= \varphi\Big(x\big((wS^{-1} \otimes id)\Delta(a)\big)\Big)
\end{aligned}$$

We see that the product ww' is well-defined as a linear functional on A and it has the form $\varphi(\cdot b)$, where $b = (wS^{-1} \otimes id)\Delta(a)$. So $ww' \in \widehat{A}$. Therefore, we have defined a product in $\widehat{A}$.

The associativity of this product in $\widehat{A}$ is a consequence of the coassociativity of Δ on A, and A is not necessarily coassociative.

To prove that $\widehat{A}$ has a unit, assume that there is a cointegral $\xi \in A$ so that $a\xi = \varepsilon(a)\xi$ for all $a \in A$.

$$\begin{aligned}
\varphi\big(\cdot \frac{1}{\varphi(\xi)}\xi\big)(a) &= \frac{1}{\varphi(\xi)}\varphi\big(a\xi\big) = \frac{1}{\varphi(\xi)}\varphi\big(\varepsilon(a)\xi\big) \\
&= \varepsilon(a),
\end{aligned}$$

so $\varepsilon = \varphi\big(\cdot \frac{1}{\varphi(\xi)}\xi\big) \in \widehat{A}$. $\square$

Remark (1) Under the assumption, the elements of $\widehat{A}$ can be expressed in four different forms. When we use these different forms in the definition of product in $\widehat{A}$, we get the following useful expressions:

(1) $w\varphi(\cdot a) = \varphi(\cdot b)$ with $b = wS^{-1}(a_{(1)})a_{(2)}$; (2) $w\varphi(a\cdot) = \varphi(c\cdot)$ with $c = wS(a_{(1)})a_{(2)}$.

(3) $\psi(\cdot a)w = \psi(\cdot d)$ with $d = a_{(1)}wS(a_{(2)})$; (4) $\psi(a\cdot)w = \psi(e\cdot)$ with $e = a_{(1)}wS^{-1}(a_{(2)})$.

(2) The reason for being restricted to the discrete case is that there is no definition of multiplier algebra $M(A)$ for a non-associative algebra A.

Let us now define the comultiplication $\widehat{\Delta}$ on the unital algebra $\widehat{A}$. Roughly speaking, the coproduct is dual to the multiplication in A in the sense that

$$\langle \widehat{\Delta}(w), x \otimes y \rangle = \langle w, xy \rangle, \quad \forall x, y \in A.$$

We will first show that the above functional is well-defined and again in $\widehat{A} \otimes \widehat{A}$.

Proposition 4.2.3 Let $w \in \widehat{A}$, then we have $\widehat{\Delta}(w) \in \widehat{A} \otimes \widehat{A}$ and $\widehat{\Delta}$ is coassociative.

Proof The unit $1_{\widehat{A}} = \varepsilon = \psi\Big(\frac{1}{\psi S(\xi)} S(\xi)\cdot\Big) \in \widehat{A}$, and let $w = \psi(b\cdot)$. Then

$$\begin{aligned}
\langle \widehat{\Delta}(w), x \otimes y \rangle &= \langle (\varepsilon \otimes 1_{\widehat{A}})\widehat{\Delta}(w), x \otimes y \rangle = \langle \varepsilon \otimes w, x_{(1)} \otimes x_{(2)} y \rangle \\
&= \langle \frac{1}{\psi S(\xi)} \psi\Big(S(\xi)\cdot\Big) \otimes \psi(b\cdot), x_{(1)} \otimes x_{(2)} y \rangle \\
&= \frac{1}{\psi S(\xi)} \psi\Big(S(\xi)x_{(1)}\Big)\psi(b x_{(2)} y) = \frac{1}{\psi S(\xi)} \psi\Big(b\underline{\psi\big(S(\xi)x_{(1)}\big)x_{(2)}}y\Big) \\
&\overset{(4.2)}{=} \frac{1}{\psi S(\xi)} \psi\Big(b\psi\big(S(\xi_{(2)})x\big)\xi_{(1)} y\Big) = \psi\big(\frac{1}{\psi S(\xi)} S(\xi_{(2)})x\big)\psi\Big(b\xi_{(1)} y\Big) \\
&= \langle \psi\big(\frac{1}{\psi S(\xi)} S(\xi_{(2)})\cdot\big) \otimes \psi\Big(b\xi_{(1)}\cdot\Big), x \otimes y \rangle
\end{aligned}$$

Hence $\widehat{\Delta}(w) = \psi\big(\frac{1}{\psi S(\xi)} S(\xi_{(2)})\cdot\big) \otimes \psi\Big(b\xi_{(1)}\cdot\Big) \in \widehat{A} \otimes \widehat{A}$.

The coassociativity is a direct consequence of the product associativity in A. $\square$

Proposition 4.2.4 $\widehat{\Delta} : \widehat{A} \longrightarrow \widehat{A} \otimes \widehat{A}$ is an algebra homomorphism.

Proof It is straightforward that $\widehat{\Delta}$ is an algebra homomorphism, since for all $x, y \in H$

$$\begin{aligned}
\langle \widehat{\Delta}(w_1 w_2), x \otimes y \rangle &= \langle w_1 w_2, xy \rangle = \langle w_1 \otimes w_2, \Delta(xy) \rangle \\
&= \langle w_1, x_{(1)} y_{(1)} \rangle \langle w_2, x_{(2)} y_{(2)} \rangle \\
\langle \widehat{\Delta}(w_1)\widehat{\Delta}(w_2), x \otimes y \rangle &= \langle \widehat{\Delta}(w_1) \otimes \widehat{\Delta}(w_2), (x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)}) \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \widehat{\Delta}(w_1), x_{(1)} \otimes y_{(1)} \rangle \langle \otimes \widehat{\Delta}(w_2), x_{(2)} \otimes y_{(2)} \rangle \\
&= \langle w_1, x_{(1)} y_{(1)} \rangle \langle w_2, x_{(2)} y_{(2)} \rangle
\end{aligned}
$$

This completes the proof. □

Let $w \in \widehat{A}$ and assume $w = \varphi(\cdot a)$ with $a \in A$. Define $\widehat{\varepsilon}(w) = \varphi(a) = w(1_{M(A)})$. Then $\widehat{\varepsilon}$ is a counit on $(\widehat{A}, \widehat{\Delta})$ as follows.

Proposition 4.2.5 $\widehat{\varepsilon} : \widehat{A} \longrightarrow k$ is an algebra homomorphism satisfying

$$
(id \otimes \widehat{\varepsilon})\widehat{\Delta}\big(w\big) = w = (\widehat{\varepsilon} \otimes id)\widehat{\Delta}\big(w\big) \tag{4.4}
$$

for all $w \in \widehat{A}$.

Proof Firstly, let $w_1 = \varphi(a\cdot)$ and $w_2 = \varphi(b\cdot)$, then $w_1 w_2 = \varphi(c\cdot)$ with $c = \varphi\big(aS(b_{(1)})\big)b_{(2)}$. Therefore, if $\psi = \varphi \circ S$ we have

$$
\begin{aligned}
\widehat{\varepsilon}(w_1 w_2) &= \varphi(c) = \varphi\big(aS(b_{(1)})\big)\varphi(b_{(2)}) \\
&= \varphi\big(aS(b_{(1)}\varphi(b_{(2)}))\big) = \varphi(a)\varphi(b) \\
&= \widehat{\varepsilon}(w_1)\widehat{\varepsilon}(w_2).
\end{aligned}
$$

Secondly, let $w = \varphi(\cdot a)$, then we have

$$
\begin{aligned}
\langle \widehat{\Delta}(w), x \otimes y \rangle &= \langle \widehat{\Delta}(w)(1_{\widehat{A}} \otimes \varepsilon), x \otimes y \rangle = \langle w \otimes \varepsilon, x y_{(1)} \otimes y_{(2)} \rangle \\
&= \langle \varphi(\cdot a) \otimes \varphi\big(\cdot \frac{1}{\varphi(\xi)}\xi\big), x y_{(1)} \otimes y_{(2)} \rangle \\
&= \frac{1}{\varphi(\xi)}\varphi(x y_{(1)} a)\varphi\big(y_{(2)}\xi\big) = \frac{1}{\varphi(\xi)}\varphi\Big(x\underline{y_{(1)}\varphi\big(y_{(2)}\xi\big)}a\Big) \\
&\overset{(3.1)}{=} \varphi\Big(xS^{-1}(\xi_{(1)})\varphi\big(y\xi_{(2)}\big)a\Big) \\
&= \langle \varphi\Big(\cdot \frac{1}{\varphi(\xi)}S^{-1}(\xi_{(1)})a\Big) \otimes \varphi\big(\cdot \xi_{(2)}\big), x \otimes y \rangle.
\end{aligned}
$$

Hence $\widehat{\Delta}(w) = \varphi\Big(\cdot \frac{1}{\varphi(\xi)}S^{-1}(\xi_{(1)})a\Big) \otimes \varphi\big(\cdot \xi_{(2)}\big)$. Therefore,

$$
\begin{aligned}
(id \otimes \widehat{\varepsilon})\widehat{\Delta}(w) &= \varphi\Big(\cdot \frac{1}{\varphi(\xi)}S^{-1}(\xi_{(1)})a\Big)\varphi\big(\xi_{(2)}\big) \\
&= \varphi\Big(\cdot \frac{1}{\varphi(\xi)}S^{-1}\big(\xi_{(1)}\varphi(\xi_{(2)})\big)a\Big) = \varphi\Big(\cdot a\Big)
\end{aligned}
$$

$$= \quad w.$$

Finally, from Proposition 4.2.3 $\widehat{\Delta}(\psi(b\cdot)) = \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\cdot\big) \otimes \psi\Big(b\xi_{(1)}\cdot\Big)$. Then $(\widehat{\varepsilon}\otimes id)\widehat{\Delta}(w) = \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\big)\psi\Big(b\xi_{(1)}\cdot\Big) = \frac{1}{\psi S(\xi)}\psi\Big(b\xi_{(1)}\psi\big(S(\xi_{(2)})\big)\cdot\Big) = \psi(b\cdot)$. This completes the proof. $\square$

Let $\widehat{S}:\widehat{A}\longrightarrow\widehat{A}$ be the dual to the antipode of A, i.e., $\widehat{S}(w) = w\circ S$. Then it is easy to see that $\widehat{S}(w)\in\widehat{A}$, and we have the following property.

Proposition 4.2.6 $\widehat{S}$ is antimultiplicative and coantimultiplicative such that

$$m(id\otimes m)(\widehat{S}\otimes id\otimes id)(\widehat{\Delta}\otimes id) = \widehat{\varepsilon}\otimes id = m(id\otimes m)(id\otimes\widehat{S}\otimes id)(\widehat{\Delta}\otimes id),$$
$$m(m\otimes id)(id\otimes\widehat{S}\otimes id)(id\otimes\widehat{\Delta}) = id\otimes\widehat{\varepsilon} = m(m\otimes id)(id\otimes id\otimes\widehat{S})(id\otimes\widehat{\Delta}).$$

Proof For $w_1, w_2\in\widehat{H}$ and any $x\in H$,

$$\begin{aligned}\langle\widehat{S}(w_1w_2), x\rangle &= \langle w_1w_2, S(x)\rangle = \langle w_1, S(x_{(2)})\rangle\langle w_2, S(x_{(1)})\rangle\\ &= \langle\widehat{S}(w_1), x_{(2)}\rangle\langle\widehat{S}(w_2), x_{(1)}\rangle = \langle\widehat{S}(w_2)\widehat{S}(w_1), x\rangle\end{aligned}$$

This implies $\widehat{S}$ is antimultiplicative.

$$\begin{aligned}\langle\widehat{\Delta}\widehat{S}(w), x\otimes y\rangle &= \langle\widehat{S}(w), xy\rangle = \langle w, S(xy)\rangle = \langle w, S(y)S(x)\rangle\\ &= \langle\widehat{\Delta}(w), S(y)\otimes S(x)\rangle = \langle\widehat{\Delta}^{cop}(w), S(x)\otimes S(y)\rangle\\ &= \langle(\widehat{S}\otimes\widehat{S})\widehat{\Delta}^{cop}(w), x\otimes y\rangle,\end{aligned}$$

We conclude $\widehat{S}$ is coantimultiplicative.

Finally, we show $m(id\otimes m)(\widehat{S}\otimes id\otimes id)(\widehat{\Delta}\otimes id) = \widehat{\varepsilon}\otimes id$, the other three formulas is similar.

$$\begin{aligned}&\langle m(id\otimes m)(\widehat{S}\otimes id\otimes id)(\widehat{\Delta}\otimes id)\big(w\otimes w'\big), x\rangle\\ =\ &\langle(\widehat{S}\otimes id\otimes id)(\widehat{\Delta}\otimes id)\big(w\otimes w'\big), (id\otimes\Delta)\Delta(x)\rangle\\ =\ &\langle(\widehat{\Delta}\otimes id)\big(w\otimes w'\big), (S\otimes id\otimes id)(id\otimes\Delta)\Delta(x)\rangle\\ =\ &\langle w\otimes w', (m\otimes id)(S\otimes id\otimes id)(id\otimes\Delta)\Delta(x)\rangle\\ =\ &\langle w\otimes w', 1_{M(A)}\otimes x\rangle\end{aligned}$$

$$= \widehat{\varepsilon}(w)w'(x).$$

This completes the proof. $\square$

From now, we get the one main result of this section.

Theorem 4.2.7 Let (A,Δ) be a regular multiplier Hopf coquasigroup of discrete type with a faithful left integral φ. Then $(\widehat{A},\widehat{\Delta})$ is a Hopf quasigroup introduced in [19].

Let ψ be a right faithful integral on A. For $w=\psi(a\cdot)$ we set $\widehat{\varphi}(w)=\varepsilon(a)$. Then we have the following result.

Proposition 4.2.8 The functional $\widehat{\varphi}$ defined above is a faithful left integral on Hopf quasigroup $(\widehat{A},\widehat{\Delta})$.

Proof It is clear that $\widehat{\varphi}$ is non-zero. Assume $w=\psi(b\cdot)$, then by Proposition 4.2.3

$$\widehat{\Delta}(w)=\psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\cdot\big)\otimes\psi\Big(b\xi_{(1)}\cdot\Big).$$

Therefore, we have

$$\begin{aligned}(id\otimes\widehat{\varphi})\widehat{\Delta}(w) &= \psi\big(\frac{1}{\psi S(\xi)}S(\xi_{(2)})\cdot\big)\varepsilon\Big(b\xi_{(1)}\Big)\\ &= \psi\big(\frac{1}{\psi S(\xi)}S(\xi)\cdot\big)\varepsilon(b)=\widehat{\varphi}(w)1_{\widehat{A}}.\end{aligned}$$

Next, we show that $\widehat{\varphi}$ is faithful. If $w_1,w_2\in\widehat{A}$ and assume $w_1=\psi(a\cdot)$ with $a\in A$, we have $w_1w_2=\psi\big(a_{(1)}w_2S^{-1}(a_{(2)})\cdot\big)$. Therefore, $\widehat{\varphi}(w_1w_2)=w_2S^{-1}(a)$. If this is 0 for all $a\in H$, then $w_2=0$, while if this is 0 for all w_2 then $a=0$. This proves the faithfulness of $\widehat{\varphi}$. $\square$

If we set $\widehat{\psi}=\widehat{\varphi}\circ\widehat{S}$ as we do for the multiplier Hopf coquasigroup (A,Δ), we find that when $w=\varphi(\cdot a)$

$$\widehat{\psi}(w)=\widehat{\varphi}\circ\widehat{S}(w)=\widehat{\varphi}(w\circ S)=\widehat{\varphi}\big(\varphi S(S^{-1}(a)\cdot)\big)=\varepsilon\big(S^{-1}(a)\big)=\varepsilon(a).$$

§4.3 Biduality

Recall from [28] that the integral dual $\widehat{H}$ of an infinite dimensional Hopf quasigroup H is a regular multiplier Hopf coquasigroup of discrete type. Specifically, let H be an infinite dimensional Hopf quasigroup with a faithful left integral φ and $\widehat{H} = \varphi(\cdot H)$, if $\varphi(\cdot H) = \varphi(H \cdot)$ and $\varphi\big((\cdot h)h'\big), \varphi\big(h'(h \cdot)\big) \in \widehat{H}$ for all $h, h' \in H$, then $\widehat{H}$ is a regular multiplier Hopf coquasigroup of discrete type.

And by Theorem 4.7 the integral dual $\widehat{\widehat{H}}$ of the regular multiplier Hopf coquasigroup of discrete type $\widehat{H}$ is Hopf quasigroup. Then, how is the relation between H and $\widehat{\widehat{H}}$? Similarly, for a discrete multiplier Hopf coquasigroup A, $\widehat{A}$ is a Hopf quasigroup, the relation of A and $\widehat{\widehat{A}}$ is what we care about. This is the content of the following theorem (biduality theorem).

Theorem 4.3.1 Let (H, Δ) be a Hopf quasigroup, and $(\widehat{H}, \widehat{\Delta})$ be the dual mutiplier Hopf coquasigroup of discrete type. For $h \in H$ and $f \in \widehat{H}$, we set $\Gamma(h)(f) = f(h)$. Then $\Gamma(h) \in \widehat{\widehat{H}}$ for all $h \in H$. Moreover, Γ is an isomorphism between the Hopf quasigroups (H, Δ) and $(\widehat{\widehat{H}}, \widehat{\widehat{\Delta}})$.

Proof For $h \in H$, first we show that $\Gamma(h)$, as a linear functional on $\widehat{H}$, is in $\widehat{\widehat{H}}$. Indeed, let $f = \varphi\big(\cdot S(h)\big)$ and take any $f' \in \widehat{H}$. By Proposition 4.4 in [28], $f'f = \varphi(\cdot h')$ where $h' = f'(h_{(2)})S(h_{(1)})$. Therefore,

$$\widehat{\psi}(f'f) = \varepsilon(h') = f'(h) = \Gamma(h)(f').$$

So $\Gamma(h) = \widehat{\psi}(\cdot f)$ and $\Gamma(h) \in \widehat{\widehat{H}}$.

It is clear that Γ is bijective between the linear space H and $\widehat{\widehat{H}}$ because of the bijection of the antipode. Γ respects the multiplication and comultiplication is straightforward because in both case the product is dual to the coproduct and vice versa. For details,

$$\begin{aligned}
\langle \Gamma(hh'), f \rangle &= \langle f, hh' \rangle = \langle \widehat{\Delta}(f), h \otimes h' \rangle \\
&= \langle (\Gamma \otimes \Gamma)(h \otimes h'), \widehat{\Delta}(f) \rangle = \langle \Gamma(h)\Gamma(h'), \widehat{\Delta}(f) \rangle, \\
\langle \widehat{\widehat{\Delta}}\Gamma(h), f \otimes f' \rangle &= \langle \Gamma(h), ff' \rangle = \langle ff', h \rangle \\
&= \langle f \otimes f', \Delta(h) \rangle = \langle (\Gamma \otimes \Gamma)\Delta(h), f \otimes f' \rangle.
\end{aligned}$$

Hence, Γ is an isomorphism between H and $\widehat{\widehat{H}}$. $\square$

Similarly, we can get another isomorphism.

Theorem 4.3.2 Let (A, Δ) be a discrete multiplier Hopf coquasigroup, and $(\widehat{A}, \widehat{\Delta})$ be the dual Hopf quasigroup. For $a \in A$ and $w \in \widehat{A}$, we set $\Gamma(a)(w) = w(a)$. Then $\Gamma(a) \in \widehat{\widehat{A}}$ for all $a \in A$. Moreover, Γ is an isomorphism between the multiplier Hopf coquasigroup (A, Δ) and $(\widehat{\widehat{A}}, \widehat{\widehat{\Delta}})$.

As in the cases of algebraic quantum group and algebraic quantum group hypergroup, all the results also hold for (flexible (resp. alternative, Moufang)) multiplier Hopf ($*$-) coquasigroups. At the end of this section, we return to our motivating example of multipler Hopf coquasigroups.

Example 4.3.3 Let G be a infinite (IP) quasigroup with identity element e, by definition $u^{-1}(uv) = v = (vu)u^{-1}$ for all $u, v \in G$. The quasigroup algebra kG has a natrual Hopf quasigroup structure. δ_e is the left and right integral on kG. The integral dual $k(G)$ introduced in [28] is a multipler Hopf coquasigroup of discrete type with the structure as follows.

As an algebra, $k(G)$ is a nondegenerate algebra with the product

$$\delta_u \delta_v = \delta_{u,v} \delta_v,$$

and $1 = \sum_{u \in G} \delta_u$ is the unit in $M(k(G))$. The coproduct, counit and antipode are given by

$$\widehat{\Delta}(\delta_u) = \sum_{v \in G} \delta_v \otimes \delta_{v^{-1}u}, \quad \widehat{\varepsilon}(\delta_u) = \delta_{u,e}, \quad \widehat{S}(\delta_u) = \delta_{u^{-1}}.$$

The left integral $\widehat{\varphi}$ and right integral $\widehat{\psi}$ on $k(G)$ is the function that maps every δ_u to 1. δ_e is the left and right cointegral in $k(G)$.

Now, we construct the dual of $k(G)$ as introduced in Section 4. Then

$$\widehat{k(G)} = \{\widehat{\varphi}(\cdot \delta_u) \mid u \in G\}.$$

The element $\widehat{\varphi}(\cdot\delta_u) = \widehat{\psi}(\cdot\delta_u)$ maps δ_u to 1 and maps $\delta_v (v \neq u)$ to 0.

By Theorem 4.3.2, $kG \cong \widehat{k(G)}$ as Hopf quasigroups. The isomorphism $\Gamma : kG \longrightarrow \widehat{k(G)}$ is given by

$$\begin{aligned} \Gamma(u) &= \widehat{\psi}\Big(\cdot \varphi\big(\cdot S(u)\big)\Big) = \widehat{\psi}\Big(\cdot \delta_e\big(\cdot u^{-1}\big)\Big) \\ &= \widehat{\psi}\Big(\cdot \delta_u\Big). \end{aligned}$$

So if we identify $\widehat{\psi}(\cdot\delta_u)$ with u, then $\widehat{k(G)} = kG$.

By Theorem 4.3.3, $k(G) \cong \widehat{kG}$ as multiplier Hopf coquasigroups. The isomorphism $\Gamma : k(G) \longrightarrow \widehat{kG}$ is given by

$$\begin{aligned} \Gamma(\delta_u) &= \widehat{\widehat{\psi}}\Big(\cdot \widehat{\varphi}\big(\cdot S(\delta_u)\big)\Big) = \delta_e\Big(\cdot \widehat{\varphi}\big(\cdot \delta_{u^{-1}}\big)\Big) \\ &= \delta_e\Big(\cdot u^{-1}\Big) = u. \end{aligned}$$

So $\widehat{kG} = k(G)$.

Chapter 5 Some special cases

In this chapter, we apply the techniques introduced in multiplier Hopf algebra and multiplier Hopf coquasigroup theories to some special cases and get some interesting results.

§5.1 Diagonal crossed products

For an infinite dimensional coFrobenius Hopf algebra H with $\alpha, \beta \in Aut(H)$, the diagonal crossed product $\bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha, \beta)$ is a quasitriangular multiplier Hopf T-coalgebra. And its unital representation category is isomorphic to the generalized Yetter-Drinfeld category $\mathcal{YD}(H)$ introduced in [20] as braided T-categories.

§5.1.1 (α, β)-quantum double

Let H be a Hopf algebra, and $\alpha, \beta \in Aut_{Hopf}(H)$. Denote $G = Aut_{Hopf}(H) \times Aut_{Hopf}(H)$, a group with multiplication

$$(\alpha, \beta) * (\gamma, \delta) = (\alpha\gamma, \delta\gamma^{-1}\beta\gamma), \tag{5.1.1}$$

The unit is (ι, ι) and $(\alpha, \beta)^{-1} = (\alpha^{-1}, \alpha\beta^{-1}\alpha^{-1})$.

Recall from [20], an (α, β)-Yetter-Drinfel'd module over H is a vector space M, such that M is a left H-module (with notation $h \otimes m \mapsto h \cdot m$) and a right H-comodule (with notation $M \to M \otimes H, m \mapsto m_{(0)} \otimes m_{(1)}$) with the following compatibility condition:

$$(h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = h_{(2)} \cdot m_{(0)} \otimes \beta(h_{(3)}) m_{(1)} \alpha S^{-1}(h_{(1)}). \tag{5.1.2}$$

We denote by ${}_H\mathcal{YD}^H(\alpha, \beta)$ the category of (α, β)-Yetter-Drinfel'd modules, morphism being the H-linear H-colinear maps.

If H is "finite-dimensional", then

$${}_H\mathcal{YD}^H(\alpha, \beta) \cong {}_{H^* \bowtie H(\alpha,\beta)}\mathcal{M},$$

where $H^* \bowtie H(\alpha, \beta)$ is the diagonal crossed product, whose product is given by: for all $p, q \in H^*$ and $h, l \in H$,

$$(p \bowtie h)(q \bowtie l) = p\big(\alpha(h_{(1)}) \blacktriangleright q \blacktriangleleft S^{-1}\beta(h_{(3)})\big) \bowtie h_{(2)}l. \tag{5.1.3}$$

One main question naturally arises: Does this isomorphism also hold for some "infinite-dimensional" Hopf algebras?

For this question, we need to construct the diagonal crossed products of infinite-dimensional Hopf algebras, which is a multiplier Hopf algebra rather than the usual Hopf algebra. Let H be a coFrobenius Hopf algebra with a left integral φ, then by [34] $\widehat{H} = \varphi(\cdot H)$ is a regular multiplier Hopf algebra with integrals.

In the following proposition, we will show that the diagonal crossed product (5.1.3) also holds for infinite-dimensional coFrobenius Hopf algebra.

Proposition 5.1.1 Let H be a coFrobenius Hopf algebra with its dual multiplier Hopf algebra $\widehat{H}$. Then $\mathcal{A} = \bigoplus_{(\alpha,\beta)\in G} \mathcal{A}_{(\alpha,\beta)} = \bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha, \beta)$ is a quasitriangular G-cograded multiplier Hopf algebra with the following strucrures:

- For any $(\alpha, \beta) \in G$, $\mathcal{A}_{(\alpha,\beta)}$ has the multiplication given by

$$(p \bowtie h)(q \bowtie l) = p\big(\alpha(h_{(1)}) \blacktriangleright q \blacktriangleleft S^{-1}\beta(h_{(3)})\big) \bowtie h_{(2)}l$$

  for $p, q \in \widehat{H}$ and $h, l \in H$.

- The comultiplication on $\mathcal{A}$ is given by:

$$\Delta_{(\alpha,\beta),(\gamma,\delta)} : \mathcal{A}_{(\alpha,\beta)*(\gamma,\delta)} \longrightarrow M(\mathcal{A}_{(\alpha,\beta)} \otimes \mathcal{A}_{(\gamma,\delta)}),$$
$$\Delta_{(\alpha,\beta),(\gamma,\delta)}(p \bowtie h) = \Delta^{cop}(p)(\gamma \otimes \gamma^{-1}\beta\gamma)\Delta(h).$$

- The counit $\varepsilon_{\mathcal{A}}$ on $\mathcal{A}_{(\iota,\iota)} = D(H)$ is the counit on the Drinfel'd double of H.

- For any $(\alpha, \beta) \in G$, the antipode is given by

$$S : \mathcal{A}_{(\alpha,\beta)} \longrightarrow \mathcal{A}_{(\alpha,\beta)^{-1}},$$
$$S_{(\alpha,\beta)}(p \bowtie h) = T(\alpha\beta S(h) \otimes S^{-1}(p)) \text{ in } \mathcal{A}_{(\alpha,\beta)^{-1}} = \mathcal{A}_{(\alpha^{-1},\alpha\beta^{-1}\alpha^{-1})}.$$

- A crossing action $\xi : G \longrightarrow Aut(\mathcal{A})$ is given by

$$\xi^{(\gamma,\delta)}_{(\alpha,\beta)} : \mathcal{A}_{(\gamma,\delta)} \longrightarrow \mathcal{A}_{(\alpha,\beta)*(\gamma,\delta)*(\alpha,\beta)^{-1}} = \mathcal{A}_{(\alpha\gamma\alpha^{-1},\alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})},$$
$$\xi^{(\gamma,\delta)}_{(\alpha,\beta)}(p \bowtie h) = p \circ \beta\alpha^{-1} \bowtie \alpha\gamma^{-1}\beta^{-1}\gamma(h).$$

- A generalized R-matrix is given by

$$R = \sum_{(\alpha,\beta),(\gamma,\delta)\in G} R_{(\alpha,\beta),(\gamma,\delta)} = \sum_{(\alpha,\beta),(\gamma,\delta)\in G} \varepsilon \bowtie \beta^{-1}(u) \otimes v \bowtie 1.$$

Example 5.1.2 Let C be an infinite cyclic group with generator c and let m be a positive integer. Let $i \in N$, the set of natural integers and $\lambda \in \mathbb{C}$ such that λ^i is a primitive mth root of 1. Let B be the algebra with generators c and X satisfying relations: $cX = \lambda Xc$ and $Xm = 0$. Then B is a Hopf algebra with the structure given by $\Delta(c) = c \otimes c$, $\Delta(X) = c^i \otimes X + X \otimes 1$, $\varepsilon(c) = 1$, $\varepsilon(X) = 0$, $S(c) = c^{-1}$ and $S(X) = -c^{-i}X$.

In [[8], 2.2.1], the authors constructed the dual of infinite dimensional Hopf algebra B, which is a multiplier Hopf algebra $A = \widehat{B}$ with the linear basis $\{\omega_{p,0}Y^l \mid p \in \mathbb{Z}, l \in \mathbb{N}, l < m\}$. For the details, the multiplier Hopf structure on A is given by the formula

$$\omega_{p,q}\omega_{k,l} = \delta_{p-k,il}\tbinom{l+q}{q}_{-i}\omega_{k,l+q},$$
$$\Delta(\omega_{p,0}) = \sum_{k\in\mathbb{Z}} \omega_{k,0} \otimes \omega_{p-k,0}, \qquad \Delta(Y) = D \otimes Y + Y \otimes 1,$$
$$\varepsilon(\omega_{p,0}) = \delta_{p,0}, \qquad \varepsilon(Y) = 0,$$
$$S(\omega_{p,0}) = \omega_{-p,0}, \qquad S(Y) = -D^{-1}Y,$$

where $D = \sum_{j\in\mathbb{Z}} {}^j\omega_{j,0}$ and $Y = \sum_{s\in\mathbb{Z}} {}^s\omega_{s,1}$. Notice that $DY = YD$, $Y^m = 0$ and $D\omega_{k,0} = {}^k\omega_{k,0} = \omega_{k,0}D$.

Then by Proposition 5.1.1 we can construct a G-cograded multiplier Hopf algebra $DT(H)$ with the multiplier Hopf structure as follows: The product in $\mathcal{A}_{(\alpha,\beta)}$ is given by

$$(1 \otimes c)(\omega_{k,0} \otimes 1) = \omega_{k,0} \otimes c, \quad (1 \otimes X)(\omega_{k,0} \otimes 1) = \omega_{k-i,0} \otimes X,$$
$$(1 \otimes c)(Y \otimes 1) = {}^{-1}Y \otimes c, \quad (1 \otimes X)(Y \otimes 1) = -\alpha^{-1}1 \otimes c^i + Y \otimes X + \beta^{-1}D \otimes 1.$$

The comultiplication, counit in $\mathcal{A}_{(\iota,\iota)}$, antipode are given by

$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(\omega_{p,0}\otimes c)=\sum_{k\in\mathbb{Z}}(\omega_{p-k,0}\otimes c)\otimes(\omega_{k,0}\otimes c),$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(\omega_{p,0}\otimes X)=\sum_{k\in\mathbb{Z}}(\omega_{p-k,0}\otimes c^{i})\otimes(\omega_{k,0}\otimes\alpha^{-1}X)$$
$$+\sum_{k\in\mathbb{Z}}(\omega_{p-k,0}\otimes\gamma^{-1}X)\otimes(\omega_{k,0}\otimes 1),$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(Y\otimes c)=(Y\otimes c)\otimes(D\otimes c)+(1\otimes c)\otimes(Y\otimes c),$$
$$\Delta_{(\alpha,\beta),(\lambda,\gamma)}(Y\otimes X)=(Y\otimes c^{i})\otimes(D\otimes\alpha^{-1}X)+(Y\otimes\gamma^{-1}X)\otimes(D\otimes 1)$$
$$+(1\otimes c^{i})\otimes(Y\otimes\alpha^{-1}X)+(1\otimes\gamma^{-1}X)\otimes(Y\otimes 1),$$
$$\varepsilon(\omega_{p,0}Y^{l}\otimes c^{n}X^{s})=\delta_{p,0}\,\delta_{l,0}\,\delta_{s,0},\quad S_{(\alpha,\beta)}(\omega_{k,0}\otimes c)=\omega_{-k,0}\otimes c^{-1},$$
$$S_{(\alpha,\beta)}(\omega_{k,0}\otimes X)=-\beta^{-1}\alpha^{-1}\omega_{-i-k,0}\otimes c^{-i}X,$$
$$S_{(\alpha,\beta)}(Y\otimes c)=-\sum_{p\in\mathbb{Z}}{}^{1-p}\omega_{p+i,0}Y\otimes c^{-1},\ \text{and}$$
$$S_{(\alpha,\beta)}(Y\otimes X)=\alpha^{-1}1\otimes c^{-i}+\beta{-}1\alpha{-}1\sum_{p\in\mathbb{Z}}{}^{-p}\omega_{p+i,0}Y\otimes c^{-i}X-\beta^{-1}\sum_{p\in\mathbb{Z}}{}^{-p}\omega_{p,0}\otimes 1.$$

We can check that $u\otimes v=\sum_{k,l}c^{k}X^{l}\otimes\omega_{k,l}$. Then by Theorem 5.1.4, the R-matrix is given by

$$R=\sum_{(\alpha,\beta),(\lambda,\gamma)\in G(\mathbb{K}^{\times})}\sum_{j\in\mathbb{Z},l<m}\alpha^{l}(1\otimes c^{j}X^{l})\otimes(\omega_{j,l}\otimes 1).$$

§5.1.2 Representation category of the diagonal crossed product

In the following, we will consider the properties of the representation category ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$ of unital $\widehat{H}\bowtie H(\alpha,\beta)$-modules.

Proposition 5.1.3 If $M\in{}_{H}\mathcal{YD}^{H}(\alpha,\beta)$, then $M\in{}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$ with the structure

$$(p\bowtie h)\cdot m=p((h\cdot m)_{(1)})(h\cdot m)_{(0)}.$$

Proof It is straightforward to check that $(p\bowtie h)\cdot\big((q\bowtie l)\cdot m\big)=\big((p\bowtie h)(q\bowtie$

$l)\big) \cdot m$. In fact,

$$
\begin{aligned}
& (p \bowtie h) \cdot \big((q \bowtie l) \cdot m\big) \\
= \; & (p \bowtie h) \cdot \big(q((l \cdot m)_{(1)})(q \cdot l)_{(0)}\big) \\
= \; & (p \bowtie h) \cdot \big(q(\beta(l_{(3)})m_{(1)}\alpha S^{-1}(l_{(1)}))l_{(2)} \cdot m_{(0)}\big) \\
= \; & q\big(\beta(l_{(3)})m_{(1)}\alpha S^{-1}(l_{(1)})\big)p\big((hl_{(2)} \cdot m_{(0)})_{(1)}\big)(hl_{(2)} \cdot m_{(0)})_{(0)} \\
= \; & q\big(\beta(l_{(5)})m_{(2)}\alpha S^{-1}(l_{(1)})\big)p\big(\beta(h_{(3)}l_{(4)})m_{(1)}\alpha S^{-1}(h_{(1)}l_{(2)})\big)(h_{(2)}l_{(3)}) \cdot m_{(0)},
\end{aligned}
$$

and

$$
\begin{aligned}
& \big((p \bowtie h)(q \bowtie l)\big) \cdot m \\
= \; & \big(\langle q_{(1)}, S^{-1}\beta(h_{(3)})\rangle\langle q_{(3)}, \alpha(h_{(1)})\rangle(pq_{(2)} \bowtie h_{(2)}l)\big) \cdot m \\
= \; & \langle q_{(1)}, S^{-1}\beta(h_{(3)})\rangle\langle q_{(3)}, \alpha(h_{(1)})\rangle(pq_{(2)})((h_{(2)}l \cdot m)_{(1)}) \otimes (h_{(2)}l \cdot m)_{(0)} \\
= \; & \langle q_{(1)}, S^{-1}\beta(h_{(5)})\rangle\langle q_{(3)}, \alpha(h_{(1)})\rangle(pq_{(2)})\big(\beta(h_{(4)}l_{(3)})m_{(1)}\alpha S^{-1}(h_{(2)}l_{(1)})\big)(h_{(3)}l_{(2)}) \cdot m_{(0)} \\
= \; & \underline{\langle q_{(1)}, S^{-1}\beta(h_{(7)})\rangle\langle q_{(3)}, \alpha(h_{(1)})\rangle}\langle p, \beta(h_{(5)}l_{(4)})m_{(1)}\alpha S^{-1}(h_{(3)}l_{(2)})\rangle \\
& \underline{\langle q_{(2)}, \beta(h_{(6)}l_{(5)})m_{(2)}\alpha S^{-1}(h_{(2)}l_{(1)})\rangle}(h_{(4)}l_{(3)}) \cdot m_{(0)} \\
= \; & \langle q, \beta(l_{(5)})m_{(2)}\alpha S^{-1}(l_{(1)})\rangle\langle p, \beta(h_{(3)}l_{(4)})m_{(1)}\alpha S^{-1}(h_{(1)}l_{(2)})\rangle(h_{(2)}l_{(3)}) \cdot m_{(0)}.
\end{aligned}
$$

■

Recall from Lemma 11 in [34], If M is a left unital $\widehat{H}$-module, then

$$
\begin{aligned}
\rho: \quad & M \longrightarrow M \otimes H, \\
& m \mapsto \sum S^{-1}(\varphi_{(1)}) \cdot m \otimes t(\cdot\varphi_{(2)}) = v \cdot m \otimes u
\end{aligned}
$$

gives the H-comodule structure on M. Following this lemma, we get the following proposition.

Proposition 5.1.4 If $M \in {}_{\widehat{H} \bowtie H(\alpha,\beta)}\mathcal{M}$, then $M \in {}_{H}\mathcal{YD}^{H}(\alpha, \beta)$ with structures

$$
\begin{aligned}
& h \cdot m = (\varepsilon \bowtie h) \cdot m, \\
& m \mapsto m_{(0)} \otimes m_{(1)} = (S^{-1}(\varphi_{(1)}) \bowtie 1) \cdot m \otimes t(\cdot\varphi_{(2)}).
\end{aligned}
$$

Proof Here we treat $\widehat{H}$ and H as subalgebras of $\widehat{H} \bowtie H(\alpha, \beta)$ in the usual way, then it is easy to get M is an H-module and H-comodule.

To show $M \in {}_H\mathcal{YD}^H(\alpha, \beta)$, i.e., $\rho(h \cdot m) = h_{(2)} \cdot m_{(0)} \otimes \beta(h_{(3)})m_{(1)}\alpha S^{-1}(h_{(1)})$, it is enough to verify that

$$(S^{-1}(\varphi_{(1)}) \bowtie h) \otimes t(\cdot\varphi_{(2)}) = (\varepsilon \bowtie h_{(2)})(S^{-1}(\varphi_{(1)}) \bowtie 1) \otimes \beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)}).$$

Viewing $\widehat{H} \bowtie H(\alpha, \beta) \otimes H$ as a subspace of $Hom(\widehat{H}, (\widehat{H} \bowtie H)(\alpha, \beta))$ in a natural way, we only need to check that

$$\begin{aligned} p \bowtie h &\overset{(3.5)}{=} (S^{-1}(\varphi_{(1)}) \bowtie h)t(p\varphi_{(2)}) \\ &= (\varepsilon \bowtie h_{(2)})(S^{-1}(\varphi_{(1)}) \bowtie 1)\langle p, \beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)})\rangle \end{aligned}$$

holds for any $p \in \widehat{H}$. Indeed, for any $p' \in \widehat{H}$,

$$\begin{aligned} & (p' \bowtie h_{(2)})(S^{-1}(\varphi_{(1)}) \bowtie 1)\langle p, \beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)})\rangle \\ = \ & (p' \bowtie h_{(2)})(S^{-1}(\varphi_{(1)}) \bowtie 1)\langle p, \beta(h_{(3)})t(\cdot\varphi_{(2)})\alpha S^{-1}(h_{(1)})\rangle \\ = \ & (p' \bowtie h_{(2)})(p_{(2)} \bowtie 1)\langle p_{(1)}, \beta(h_{(3)})\rangle\langle p_{(3)}, \alpha S^{-1}(h_{(1)})\rangle \\ = \ & \langle p_{(1)}, \beta(h_{(5)})\rangle\langle p_{(3)}, \alpha S^{-1}(h_{(1)})\rangle\langle p_{(2)}, S^{-1}\beta(h_{(4)})\rangle\langle p_{(4)}, \alpha(h_{(1)})\rangle(p'p_{(3)} \bowtie h_{(3)}) \\ = \ & p'p \bowtie h. \end{aligned}$$

This completes the proof. ■

Next, we get the main result of this section, generalizing the conclusion in [20] and giving an answer to the question introduced in Section 1.

Theorem 5.1.5 For a coFrobenius Hopf algebra H,

$$_H\mathcal{YD}^H(\alpha, \beta) \cong {}_{\widehat{H} \bowtie H(\alpha,\beta)}\mathcal{M}. \tag{5.1.1}$$

Proof The correspondence easily follows from Proposition 5.1.1 and 5.1.2. Let $f : M \to N$ be a morphism in ${}_H\mathcal{YD}^H(\alpha, \beta)$, i.e., f is a module and comodule map.

Then in ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$,

$$\begin{aligned}(f\otimes\iota)\rho(m) &= f\big((S^{-1}(\varphi_{(1)})\bowtie 1)\cdot m\big)\otimes t(\cdot\varphi_{(2)})\\ \rho(f(m)) &= (S^{-1}(\varphi_{(1)})\bowtie 1)\cdot f(m)\otimes t(\cdot\varphi_{(2)}).\end{aligned}$$

$(f\otimes\iota)\rho(m)=\rho(f(m))$ implies f is a $\widehat{H}\bowtie 1$-module map, and so a ${}_{H}\mathcal{YD}^{H}(\alpha,\beta)$-module map. We define a functor $F_{(\alpha,\beta)}: {}_{H}\mathcal{YD}^{H}(\alpha,\beta)\longrightarrow {}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$ as follows,

$$F_{(\alpha,\beta)}(M)=M,\qquad \text{and}\qquad F_{(\alpha,\beta)}(f)=f.$$

Conversely, if $f: M\to N$ be a morphism in ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$, then

$$\begin{aligned}(f\otimes\iota)\rho(m) &= f\big((S^{-1}(\varphi_{(1)})\bowtie 1)\cdot m\big)\otimes t(\cdot\varphi_{(2)})\\ &= (S^{-1}(\varphi_{(1)})\bowtie 1)\cdot f(m)\otimes t(\cdot\varphi_{(2)})\\ &= \rho(f(m)).\end{aligned}$$

This shows that f is a H-comodule map. Then we similarly define a functor $G_{(\alpha,\beta)}$: ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}\longrightarrow {}_{H}\mathcal{YD}^{H}(\alpha,\beta)$ by

$$G_{(\alpha,\beta)}(M)=M,\qquad \text{and}\qquad G_{(\alpha,\beta)}(f)=f.$$

From above, F and G preserve the morphisms from each other. Also $F_{(\alpha,\beta)}G_{(\alpha,\beta)}=1_{(\alpha,\beta)}$ and $G_{(\alpha,\beta)}F_{(\alpha,\beta)}=1_{(\alpha,\beta)}$. We have established the equivalence between ${}_{H}\mathcal{YD}^{H}(\alpha,\beta)$ and ${}_{\widehat{H}\bowtie H(\alpha,\beta)}\mathcal{M}$. ■

When $\alpha=\beta=\iota$, then $\widehat{H}\bowtie H(\iota,\iota)=D(H)$ the quantum double of a coFrobenius Hopf algebra. Then we have the following corollary, which is the main result in [34].

Corollary 5.1.6 For a coFrobenius Hopf algebra H, ${}_{H}\mathcal{YD}^{H}\cong {}_{\widehat{H}\bowtie H}\mathcal{M}$.

Let $\mathcal{YD}(H)$ be the disjoint union of ${}_{H}\mathcal{YD}^{H}(\alpha,\beta)$ for every $(\alpha,\beta)\in G$. Then following Section 3 in [20] or [31, 32] (H is a special multiplier Hopf algebra), we have ${}_{H}\mathcal{YD}^{H}$ is a braided T-category with the structures as follows

- Tensor product: if $V\in {}_{H}\mathcal{YD}^{H}(\alpha,\beta)$ and $W\in {}_{H}\mathcal{YD}^{H}(\gamma,\delta)$ with $\alpha,\beta,\gamma,\delta\in$

$Aut(H)$, then $V \otimes W \in {}_H\mathcal{YD}^H(\alpha\gamma, \delta\gamma^{-1}\beta\gamma)$, with the structures as follows:

$$h \cdot (v \otimes w) = \gamma(h_{(1)}) \cdot v \otimes \gamma^{-1}\beta\gamma(h_{(2)}) \cdot w,$$
$$v \otimes w \mapsto (v \otimes w)_{(0)} \otimes (v \otimes w)_{(1)} = (v_{(0)} \otimes w_{(0)}) \otimes w_{(1)}v_{(1)}$$

for all $v \in V, w \in W$.

- Crossed functor: Let $W \in {}_H\mathcal{YD}^H(\gamma, \delta)$, define $\xi_{(\alpha,\beta)}(W) = {}^{(\alpha,\beta)}W = W$ as vector space, with structures: for all $a, a' \in A$ and $w \in W$

$$a \rightharpoonup w = \gamma^{-1}\beta\gamma\alpha^{-1}(a) \cdot w,$$
$$w \mapsto w_{<0>} \otimes w_{<1>} = w_{(0)} \otimes \alpha\beta^{-1}(w_{(1)}).$$

  Then ${}^{(\alpha,\beta)}W \in {}_H\mathcal{YD}^H((\alpha,\beta)\#(\gamma,\delta)\#(\alpha,\beta)^{-1}) = {}_H\mathcal{YD}^H(\alpha\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\gamma\alpha^{-1})$. The functor $\xi_{(\alpha,\beta)}$ acts as identity on morphisms.

- Braiding: If $V \in {}_H\mathcal{YD}^H(\alpha, \beta)$, and $W \in {}_H\mathcal{YD}^H(\gamma, \delta)$. Take ${}^VW = {}^{(\alpha,\beta)}W$, define a map $C_{V,W} : V \otimes W \longrightarrow {}^VW \otimes V$ by

$$C_{(\alpha,\beta),(\gamma,\delta)}(v \otimes w) = w_{(0)} \otimes \beta^{-1}(w_{(1)}) \cdot v$$

  for all $v \in V$ and $w \in W$.

Following from Theorem 5.1.3, we obtain the following result, generalizing Theorem 3.10 in [20].

Theorem 5.1.7 For a coFrobenius Hopf algebra H and its G-cograded multiplier Hopf algebra $\mathcal{A} = \bigoplus_{(\alpha,\beta)\in G} \widehat{H} \bowtie H(\alpha, \beta)$, $Rep(\mathcal{A})$ and $\mathcal{YD}(H)$ are isomorphic as braided T-categories over G.

§5.2 Notes on Drinfeld twists of multiplier Hopf algebras

This section determines how the integral changes under a Drinfeld twist in multiplier Hopf algebras case. For a multiplier Hopf algebra A with a Drinfeld twist J, we construct a new multiplier Hopf algebra A^J. Furthermore, if A is quasitriangular, then

A^J is also. Finally, for a counimodular algebraic quantum group A, A^J is an algebraic quantum group, and as an application we give the integral formula of H^J, where H is an infinite dimensional counimodular coFrobenius Hopf algebra.

§5.2.1 Drinfeld twists for multiplier Hopf algebras

In [2], the properties of Drinfeld twist for finite-dimensional Hopf algebras were studied. For a given Hopf algebra or quasitriangular Hopf algebra, one gets another such structure by twisting it with a Drinfeld twist, and the authors also determine how the integral of the dual to finite-dimensional unimodular Hopf algebra changes under a twist (see Theorem 3.4 in [2]).

However, the following question remains open:

(Q1) *Does [[2], Theorem 3.4] remains valid for any infinite-dimensional Hopf algebras or any multiplier Hopf algebras?*

To answer the question (Q1), we first define a twist of a multiplier Hopf algebra as follows. Let (A, Δ) be a regular multiplier Hopf algebra. We first generalize the Drinfeld twist to the multiplier Hopf algebra case, and then construct some new multiplier Hopf algebras by this twist.

Definition 5.2.1 A twist of a regular multiplier Hopf algebra A is an invertible element $J \in M(A \otimes A)$, which satisfies

$$(\Delta \otimes \iota)(J)(J \otimes 1) = (\iota \otimes \Delta)(J)(1 \otimes J). \tag{5.2.2}$$

Remark (1) Taking the inverse of (5.2.2), we can get the equivalent equation:

$$(J^{-1} \otimes 1)(\Delta \otimes \iota)(J^{-1}) = (1 \otimes J^{-1})(\iota \otimes \Delta)(J^{-1}).$$

Let $\mathcal{R}$ be a quasitriangular structure for a regular multiplier Hopf algebra (A, Δ) (the definition will be recalled in Section 3), then following Proposition 3 in [34], we can get that $\mathcal{R}^{-1}$ is a Drinfeld twist.

(2) Applying $(\iota \otimes \varepsilon \otimes \iota)$ to the equation (2.1), one sees that as in the Hopf case $c = (\varepsilon \otimes \iota)(J) = (\iota \otimes \varepsilon)(J)$ is a non-zero scalar for the twist J. One can always replace

J by $c^{-1}J$ to normalize the twist in such a way that

$$(\varepsilon \otimes \iota)(J) = (\iota \otimes \varepsilon)(J) = 1. \tag{5.2.3}$$

In the following, we will always assume that J is normalized in this way.

(3) Let $x \in M(A)$ be an invertible element such that $\varepsilon(x) = 1$. If J is a twist of A, then so is $J^x := \Delta(x)J(x^{-1} \otimes x^{-1})$. Indeed, it is similar to the one in Hopf algebra case except that we should take the (unique) extension for the homomorphism $\iota \otimes \Delta$ and $\Delta \otimes \iota$ from $A \otimes A$ to $M(A \otimes A)$. The twists J and J^x are said to be *gauge equivalent.*

Example 5.2.2 (1) Let A and B be regular multiplier Hopf algebras and $\langle A, B\rangle$ be a multiplier Hopf algebra pairing. Let $W \in M(A \otimes B)$ be the canonical element defined in Definintion 4.1 in [8]. Then it is straightforward to check that

$$J = W_{14} = (\iota \otimes \iota \otimes \tau)(\iota \otimes \tau \otimes \iota)(W \otimes 1 \otimes 1)$$

is a twist for the multiplier Hopf algebra $A^{op} \otimes B$. Indeed, $(\Delta \otimes \iota)(J)(J \otimes 1) = W_{14}W_{16}W_{36} = (\iota \otimes \Delta)(J)(1 \otimes J)$. Moreover, if A is an algebraic quantum group, $\widehat{A}$ is its dual, and $W \in M(\widehat{A} \otimes A)$ be the canonical element, then $J = W_{14}$ is a twist for multiplier Hopf algebra $\widehat{A}^{op} \otimes A$.

(2) Let G be an infinite group. Denote by $B = kG$ the group algebra and by $A = k(G)$ the classic multiplier Hopf algebra. Then $\langle k(G), kG\rangle$ is a multiplier Hopf algebra pairing, and

$$J = \sum_{g \in G} (\delta_g \otimes e) \otimes (1 \otimes g)$$

is a twist for multiplier Hopf algebra $k(G)^{op} \otimes kG$, where e is the unit of group and $1 = \sum_{g \in G} \delta_g \in M(A)$.

(3) Let H be a coFrobenius Hopf algebra with a left integral φ and $A = \varphi(\cdot H)$ be the dual multiplier Hopf algebra. a left cointegral $t \in H$ satisfying $\varphi(t) = 1$. Then following from Lemma 9 in [34] the element $J = \sum(\varphi(\cdot t_{(2)}) \otimes 1) \otimes (\varepsilon \otimes t_{(1)})$ is a twist for multiplier Hopf algebra $A^{op} \otimes H$.

(4) Let H be a Hopf algebra with a twist J, and A be a multiplier Hopf algebra.

Then $H \otimes A$ is a multiplier Hopf algebra with the product, coproduct, counit and antipode as follows.

$$
\begin{aligned}
&(h \otimes a)(h' \otimes a') = hh' \otimes aa', \quad \Delta(h \otimes a) = (\iota \otimes \tau \otimes \iota)(\Delta(h) \otimes \Delta(a)),\\
&\varepsilon(h \otimes a) = \varepsilon(h)\varepsilon(a), \qquad S(h \otimes a) = S(h) \otimes S(a).
\end{aligned}
$$

In this case, there is a twist J_{13} on $H \otimes A$, where $J_{13} = (\iota \otimes \tau \otimes \iota)(J \otimes 1 \otimes 1)$.

Furthermore, suppose that $J(1 \otimes a), (1 \otimes a)J \in M(A) \otimes A$ and $J(a \otimes 1), (a \otimes 1)J \in A \otimes M(A)$ for all $a \in A$. At this time, so is J^{-1} and we call J the Drinfeld twist for A. We denote $(a \otimes 1)J = aJ^{(1)} \otimes J^{(2)}$ and $J(a \otimes 1) = J^{(1)}a \otimes J^{(2)}$ in $A \otimes M(A)$.

Remark here the above assumptions are reasonable. Take the nontrival Example 5.2.2 (1) for example, the canonical elements W in many specific examples (e.g. Example 5.2.2 (2) and Example 4.9 in [8]) satisfy the condition $W(1 \otimes b), (1 \otimes b)W \in M(A) \otimes B$ and $W(a \otimes 1), (a \otimes 1)W \in A \otimes M(B)$ for all $a \in A$ and $b \in B$, Hence the above assumption holds.

Under the above assumption, we can define a multiplier $Q_J = S(J^{(1)})J^{(2)} \in M(A)$ by

$$
\begin{aligned}
Q_J a &= m(S \otimes \iota)(J(1 \otimes a)) \in A,\\
aQ_J &= m(S \otimes \iota)(J(S^{-1}(a) \otimes 1)) \in A
\end{aligned}
$$

for all $a \in A$. This multiplier Q_J is invertible with the inverse $Q_J^{-1} = J^{-(1)}S(J^{-(2)})$. Indeed, for any $a, b \in A$

$$
\begin{aligned}
aQ_J^{-1}Q_J b &= aJ^{-(1)}S(J^{-(2)})S(J^{(1)})J^{(2)}b\\
&= aJ^{-(1)}S(J^{(1)}J^{-(2)})J^{(2)}b\\
&= aJ'^{(1)}J^{-(1)}S(J^{(1)}J^{-(2)})\varepsilon(J'^{(2)})J^{(2)}b\\
&= aJ'^{(1)}J^{-(1)}S(J'^{(2)}_{(1)}J^{(1)}J^{-(2)})J'^{(2)}_{(2)}J^{(2)}b\\
&= aJ^{(1)}_{(1)}J'^{(1)}J^{-(1)}S(J^{(1)}_{(2)}J'^{(2)}J^{-(2)})J^{(2)}b\\
&= aJ^{(1)}_{(1)}S(J^{(1)}_{(2)})J^{(2)}b
\end{aligned}
$$

$$= \quad ab$$

and similarly $aQ_J^{-1}Q_Jb = ab$.

The element Q_J satisfies

$$\Delta(Q_J) = (S \otimes S)(J_{21}^{-1})(Q_J \otimes Q_J)J^{-1}, \tag{5.2.4}$$

where $J_{21}^{-1} = \tau J^{-1}$. These equations make sense because of the extension of (anti-) homomorphism (see Proposition A.5 in [23]).

By the Drinfeld twist J, we can get a new multiplier Hopf algebra as follows, which generalizes the result in [2], and gives a positive answer to the question in the beginning of this section.

Proposition 5.2.3 Let (A, Δ) be a regular multiplier Hopf algebra. Then (A^J, Δ^J) is also a regular multiplier Hopf algebra with the same algebra structure and counit as (A, Δ), and the comultiplication and antipode are given by

$$\Delta^J(a) = J^{-1}\Delta(a)J,$$
$$S^J(a) = Q_J^{-1}S(a)Q_J$$

for all $a \in A$.

Proof It is sufficient to check the equivalent definition, i.e., equations (2.2) and (2.3). Firstly, it is easy to check that Δ^J is a homomorphism and S^J is a bijective anti-homomorphism. In the following, we only check (2.2) and (2.3) is similar.

$$\begin{aligned}(\varepsilon \otimes \iota)(\Delta^J(a)(1 \otimes b)) &= (\varepsilon \otimes \iota)(J^{-1}\Delta(a)J(1 \otimes b)) \\ &= (\varepsilon \otimes \iota)(J^{-1})(\varepsilon \otimes \iota)(\Delta(a)(\varepsilon \otimes \iota)(J)b \\ &= ab,\end{aligned}$$

where the second equation holds because ε ia a homomorphism.

$$\begin{aligned}m(S^J \otimes \iota)(\Delta^J(a)(1 \otimes b)) &= m(S^J \otimes \iota)(J^{-1}\Delta(a)J(1 \otimes b)) \\ &= m(S^J \otimes \iota)(J^{-(1)}a_{(1)}J'^{(1)} \otimes J^{-(2)}a_{(2)}J'^{(2)}b)\end{aligned}$$

$$
\begin{aligned}
&= Q_J^{-1}S(J^{-(1)}a_{(1)}J'^{(1)})Q_J J^{-(2)}a_{(2)}J'^{(2)}b \\
&= Q_J^{-1}S(J'^{(1)})S(a_{(1)})S(J^{-(1)})Q_J J^{-(2)}a_{(2)}J'^{(2)}b \\
&= \varepsilon(a)b.
\end{aligned}
$$

The equation $J^{-1}\Delta(a)J(1\otimes b) = J^{-(1)}a_{(1)}J'^{(1)} \otimes J^{-(2)}a_{(2)}J'^{(2)}b$ makes sense, since $J(1\otimes b) \in A\otimes A$ and denote it as $J'^{(1)} \otimes J'^{(2)}b$, and then $\Delta(a)(J'^{(1)} \otimes J'^{(2)}b) \in A\otimes A$ by the "cover" technique shown in [25]. $\square$

Remark The regular multiplier Hopf algebra A^J admits the Drinfeld twist J^{-1}, because $(\Delta^J \otimes \iota)(J^{-1})(J^{-1}\otimes 1) = (\iota\otimes\Delta^J)(J^{-1})(1\otimes J^{-1})$ is equivalent to (5.2.2). It follows from Proposition 5.2.4 that the regular multiplier Hopf algebra $(A^J)^{J^{-1}}$ is canonically isomorphic to A.

Proposition 5.2.4 Let J be a Drinfeld twist for multiplier Hopf algebra A. Then for any $a, b\in A$

$$
\begin{aligned}
&aS(J^{(1)})J^{(2)}_{(1)} \otimes bJ^{(2)}_{(2)} = (a\otimes b)(Q_J\otimes 1)J^{-1}, \\
&aJ^{-(1)}S(J^{-(2)}_{(1)}) \otimes bS(J^{-(2)}_{(2)}) = (aQ_J^{-1}\otimes b)(S\otimes S)(J).
\end{aligned}
$$

Proof We only check the first equation, and the second one is similar.

$$
\begin{aligned}
\big(aS(J^{(1)})J^{(2)}_{(1)} \otimes bJ^{(2)}_{(2)}\big)J &= aS(J^{(1)})J^{(2)}_{(1)}\bar{J}^{(1)} \otimes bJ^{(2)}_{(2)}\bar{J}^{(2)} \\
&\overset{(5.2.2)}{=} aS(J^{(1)}_{(1)}\bar{J}^{(1)})J^{(1)}_{(2)}\bar{J}^{(2)} \otimes bJ^{(2)} \\
&= aS(\bar{J}^{(1)})S(J^{(1)}_{(1)})J^{(1)}_{(2)}\bar{J}^{(2)} \otimes bJ^{(2)} \\
&= aS(\bar{J}^{(1)})\bar{J}^{(2)} \otimes b \\
&= aQ_J \otimes b,
\end{aligned}
$$

so $\big(aS(J^{(1)})J^{(2)}_{(1)} \otimes bJ^{(2)}_{(2)}\big)J = aQ_J\otimes b$ for any $a, b\in A$. From the nondegenerate property of the product, we have $\big(S(J^{(1)})J^{(2)}_{(1)} \otimes J^{(2)}_{(2)}\big)J = Q_J\otimes 1$, by multiplier J^{-1} from the right side, we get the assertion. $\square$

Recall from [11], let A be a regular multiplier Hopf algebra, an algebra X is called an A-module algebra if X is a unital A-module (we denote the A-action by $\triangleright$, then $A \triangleright X = X$) and $a \triangleright (xx') = (a_{(1)} \triangleright x)(a_{(2)} \triangleright x')$ for any $a \in A$ and $x, x' \in X$. In the following, we consider two kind of A^J-module algebras and their relations.

Proposition 5.2.5 Let A be a regular multiplier Hopf algebra with a Drinfeld twist J, and X an A-module algebra (not necessarily with unit). Then there exists an A^J-module algebra $X_\star$, which has the same K-module structure as A and the action of A^J on $X_\star$ is that of A on X. The product of $X_\star$ is defined by

$$x \star y = m(J \triangleright (x \otimes y)) \tag{5.2.5}$$

for any $x, y \in X$.

Proof First it is easy to show the product is well-defined. Then we check the associativity of this new product as follows: for all $x, y, z \in X$,

$$\begin{aligned}
(x \star y) \star z &= m(J \triangleright (x \otimes y)) \star z = m(J \triangleright (m(J \triangleright (x \otimes y)) \otimes z)) \\
&= m(m \otimes \iota)\big((\Delta \otimes \iota)(J)(J \otimes 1) \triangleright (x \otimes y \otimes z)\big) \\
&= m(\iota \otimes m)\big((\iota \otimes \Delta)(J)(1 \otimes J) \triangleright (x \otimes y \otimes z)\big) \\
&= m(J \triangleright (x \otimes m(J \triangleright (y \otimes z))) \\
&= x \star (y \star z).
\end{aligned}$$

Finally, we need to prove that the product in $A_\star$ is compatible with the multiplier Hopf algebra structure of A^J. For all $a \in A^J$ and $x, y \in X$,

$$\begin{aligned}
a \triangleright (x \star y) &= m(\Delta(a) J \triangleright (x \otimes y)) = m(J \Delta^J(a) \triangleright (x \otimes y)) \\
&= m(J \triangleright (a^J_{(1)} \triangleright x \otimes a^J_{(2)} \triangleright y)) = (a^J_{(1)} \triangleright x) \star (a^J_{(2)} \triangleright y),
\end{aligned}$$

where $\Delta^J(a)(1 \otimes b) = a^J_{(1)} \otimes a^J_{(2)} b$. □

We recall from [3] that an (X, Y)-bimodule is a left X-module and a right Y-module V satisfying the compatibility condition: for $x \in X, y \in Y$ and $v \in V$, $(x \cdot v) \cdot y = x \cdot (v \cdot y)$. Now, we consider a unital (X, Y)-bimodule V (i.e., $X \cdot V = V$ and $V \cdot Y = V$), where X and Y are A-module algebras and V is also a left unital A-module.

Compatibility between the structure of A and the (X, Y)-bimodule structure lead to the following covariance requirement.

Definition 5.2.6 Let A be a regular multiplier Hopf algebra, and X, Y be A-module algebras. A left A-module (X, Y)-bimodule (or ${}_{A,X}\mathcal{M}_Y$-module) is an (X, Y)-bimodule V, which is also a unital left A-module such that for any $a \in A$, $x \in X$, $y \in Y$ and $v \in V$,

$$a \triangleright (x \cdot v) = (a_{(1)} \triangleright x) \cdot (a_{(2)} \triangleright v), \tag{5.2.6}$$

$$a \triangleright (v \cdot y) = (a_{(1)} \triangleright v) \cdot (a_{(2)} \triangleright y). \tag{5.2.7}$$

An algebra E is a left A- module (X, Y)-bimodule algebra (or ${}_{A,X}\mathcal{M}_Y$-algebra), if E is an ${}_{A,X}\mathcal{M}_Y$-module and also an A-module algebra.

Proposition 5.2.7 Let A be a regular multiplier Hopf algebra with a Drinfeld twist J, X and Y be A-module algebras. Given a ${}_{A,X}\mathcal{M}_Y$-module V, there exists an ${}_{A^J,X_\star}\mathcal{M}_{Y_\star}$-module $V_\star$. The module $V_\star = V$ as vector spaces and the left action of A^J on $V_\star$ is that of A on V. The $X_\star$ and $Y_\star$ action on $V_\star$ are respectively given by

$$x \star v = \cdot \circ J \triangleright (x \otimes v), \tag{5.2.8}$$

$$v \star y = \cdot \circ J \triangleright (v \otimes y). \tag{5.2.9}$$

If $V = E$ is further an ${}_{A,X}\mathcal{M}_Y$-algebra, then $E_\star$ is an ${}_{A^J,X_\star}\mathcal{M}_{Y_\star}$-algebra with the product given in Proposition 5.2.5.

Proof First, we need to check that the two actions on $V_\star$ are well-defined, i.e., $(xx') \star v = x \star (x' \star v)$ and $v \star (yy') = (v \star y) \star y'$. Here we only check the first equation, the second one is similar.

$$\begin{aligned}
(xx') \star v &= (m(J \triangleright (x \otimes y))) \star v \\
&= \cdot \circ J \triangleright (m(J \triangleright (x \otimes y)) \otimes v), \\
&= \cdot \circ (m \otimes \iota)\Big((\Delta \otimes \iota)(J)(J \otimes 1) \triangleright (x \otimes x' \otimes v)\Big) \\
&= \cdot \circ (\iota \otimes \cdot)\Big((\iota \otimes \Delta)(J)(1 \otimes J) \triangleright (x \otimes x' \otimes v)\Big) \\
&= \cdot \circ J \triangleright \Big(x \otimes \cdot \circ J \triangleright (x' \otimes v)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \cdot \circ J \triangleright \Big(x \otimes x' \star v \Big) \\
&= x \star (x' \star v).
\end{aligned}
$$

Then we check the compatibility between the left A^J-action and the left $X_\star$-action. For any $a \in A^J$, $x \in X$ and $v \in V$,

$$
\begin{aligned}
a \triangleright (x \star v) &= a \triangleright \Big(\cdot \circ J \triangleright (x \otimes v) \Big) = \cdot \circ (\Delta(a)J) \triangleright (x \otimes v) \\
&= \cdot \circ (J\Delta^J(a)) \triangleright (x \otimes v) = \cdot \circ J \triangleright \circ \Delta^J(a) \triangleright (x \otimes v) \\
&= (a^J_{(1)} \triangleright x) \star (a^J_{(2)} \triangleright v).
\end{aligned}
$$

Compatibility between the left A^J-action and the right $Y_\star$-action is similar.

Finally, if $V = E$ is an A-module algebra, then by Proposition 5.2.5, $E_\star$ is A^J-module algebra. Combining the above result, we can easily get $E_\star$ is an ${}_{A^J,X_\star}\mathcal{M}_{Y_\star}$-algebra. □

Remark The equations (5.2.8) and (5.2.9) are well-defined. Indeed, for e.g. (5.2.8), because X and V are unital A-modules, there exists $a_i, b_j \in A$, $x_i \in X$ and $v_i \in V$ such that $x = \sum_i a_i \triangleright x_i$ and $v = \sum_j b_j \triangleright v_j$. Then

$$
\cdot \circ J \triangleright (x \otimes v) = \sum_{i,j} \cdot \circ J(a_i \otimes b_j) \triangleright (x_i \otimes v_j) = \sum_{i,j} (J^{(1)}a_i \triangleright x_i) \cdot (J^{(2)}b_j \triangleright v_j),
$$

where $\sum_{i,j} J^{(1)}a_i \otimes J^{(2)}b_j \in A \otimes A$. Therefore the equation (5.2.8) is reasonable, and (5.2.9) is similar.

Given a unital A-bimodule V, we can consider the adjoint action: for $a \in A$ and $v \in V$, $a \blacktriangleright v = a_{(1)} \cdot v \cdot S(a_{(2)}) := a_{(1)} v S(a_{(2)})$. In the following, we will consider the algebra isomorphism between algebra X and $X_\star$.

Proposition 5.2.8 Consider a regular multiplier Hopf algebra A and an A-bimodule X that is also an algebra (not necessarily with unit). If for all $a \in A$ and $x, x' \in X$, the "generalized associativity" conditions

$$
(xx')a = x(x'a), (xa)x' = x(ax'), a(xx') = (ax)x',
$$

hold, then the adjoint action makes X an A-module algebra. Given a twist J of the regular multiplier Hopf algebra A, the twist deformed algebra $X_\star$ is isomorphic to X via the map

$$\begin{aligned} D_J: \quad & X_\star \longrightarrow X, \\ & x \mapsto D_J(x) := (J^{(1)} \blacktriangleright x) J^{(2)}. \end{aligned}$$

Proof Firstly, we check that X is an A-module algebra, i.e., $a \blacktriangleright (xx') = (a_{(1)} \blacktriangleright x)(a_{(2)} \blacktriangleright x')$. Indeed,

$$\begin{aligned} (a_{(1)} \blacktriangleright x)(a_{(2)} \blacktriangleright x') &= \big(a_{(1)} x S(a_{(2)})\big)\big(a_{(3)} x' S(a_{(4)})\big) \\ &= a_{(1)}\Big(\big(xS(a_{(2)})\big)\big(a_{(3)} x' S(a_{(4)})\big)\Big) \\ &= a_{(1)}\Big(x\Big[S(a_{(2)})\big(a_{(3)} x' S(a_{(4)})\big)\Big]\Big) \\ &= a_{(1)}\Big(x\Big[S(a_{(2)}) a_{(3)}\big(x' S(a_{(4)})\big)\Big]\Big) \\ &= a_{(1)}\Big(x\big(x' S(a_{(2)})\big)\Big) = a_{(1)}\big(xx'\big)S(a_{(2)}) \\ &= a \blacktriangleright (xx'). \end{aligned}$$

Then, we need to show that D_J is an isomorphism. Obviously D_J is a K-linear map. It is also an algebra homomorphism, since for $x, x' \in X$,

$$\begin{aligned} & D_J(x \star x') \\ = \;& D_J\Big((J^{(1)} \blacktriangleright x)(J^{(2)} \blacktriangleright x')\Big) = \bar{J}^{(1)} \blacktriangleright \Big((J^{(1)} \blacktriangleright x)(J^{(2)} \blacktriangleright x')\Big)\bar{J}^{(2)} \\ = \;& \Big((\bar{J}^{(1)}_{(1)} J^{(1)} \blacktriangleright x)(\bar{J}^{(1)}_{(2)} J^{(2)} \blacktriangleright x')\Big)\bar{J}^{(2)} \overset{(5.2.2)}{=} (\bar{J}^{(1)} \blacktriangleright x)(\bar{J}^{(2)}_{(1)} J^{(1)} \blacktriangleright x')\bar{J}^{(2)}_{(2)} J^{(2)} \\ = \;& (\bar{J}^{(1)} \blacktriangleright x)\Big(\bar{J}^{(2)}_{(1)} \blacktriangleright (J^{(1)} \blacktriangleright x')\Big)\bar{J}^{(2)}_{(2)} J^{(2)} = (\bar{J}^{(1)} \blacktriangleright x)\Big(\bar{J}^{(2)}_{(1)}(J^{(1)} \blacktriangleright x')S(\bar{J}^{(2)}_{(2)})\Big)\bar{J}^{(2)}_{(3)} J^{(2)} \\ = \;& (\bar{J}^{(1)} \blacktriangleright x)\bar{J}^{(2)}(J^{(1)} \blacktriangleright x')J^{(2)} = D_J(x)D_J(x'). \end{aligned}$$

Finally, we need to check that if D_J is invertible. In fact, for $x \in X$, the inverse is given by $D_J^{-1}(x) = J^{(1)} x J^{-(1)} S(J^{(2)} J^{-(2)}) = J^{(1)} x Q_J^{-1} S(J^{(2)})$. □

Remark Using (5.2.2) we can easily get that

$$D_J(x) \;=\; (J^{(1)} \blacktriangleright x)J^{(2)} = J^{(1)}_{(1)} x S(J^{(1)}_{(2)}) J^{(2)}$$

$$
\begin{aligned}
&= J^{(1)}J^{(-1)}xS(J^{(2)}_{(1)}J'^{(1)}J^{(-2)})J^{(2)}_{(2)}J'^{(2)} \\
&= J^{(-1)}xS(J^{(-2)})Q_J,
\end{aligned}
$$

and $D_{J^{-1}}(x) = D_J^{-1}(x)$.

Example 5.2.9 Given a unital left A-module algebra X (not necessarily with unit), we consider the smash product $A\#X$. By definition as a vector space $A\#X = A \otimes X$, and the product is given by $(a\#x)(a'\#x') = a(a'_{(1)} \cdot x)\#a'_{(2)}x'$, which we simply rewrite as

$$
axa'x' = a(a'_{(1)} \cdot x)a'_{(2)}x'.
$$

The algebra $A\#X$ is an A-module algebra with the action $a \blacktriangleright (xa') = (a_{(1)} \cdot x)(a_{(2)} \blacktriangleright a')$. The right A-module structure is given by $(xa)a' = x(aa')$. Then by the above proposition we can get that deformed algebra $(A\#X)_\star$ is isomorphic to $A\#X$.

In the hypotheses of Proposition 5.2.8, the algebra X has an A^J-module algebra structure given by the A^J-adjoint action: for $a \in A^J$ and $x \in X$,

$$
a \blacktriangleright_J x := a^J_{(1)} x S^J(a^J_{(2)}). \tag{5.2.10}
$$

We denote this A^J-module algebra by $(X, \blacktriangleright_J)$, then we have the following result, which generalizes Theorem 3.10 in [3].

Theorem 5.2.10 The algebra isomorphism $D_J : X_\star \longrightarrow X$ is also an isomorphism between the A^J-module algebras $(X_\star, \blacktriangleright)$ and $(X, \blacktriangleright_J)$, i.e., D_J intertwines A^J-action $\blacktriangleright$ and $\blacktriangleright_J$: for any $a \in A^J$ and $x \in X$,

$$
D_J(a \blacktriangleright x) = a \blacktriangleright_J D_J(x). \tag{5.2.11}
$$

Proof For any $a \in A^J$ and $x \in X$,

$$
\begin{aligned}
D_J(a \blacktriangleright x) &= J^{(-1)}(a \blacktriangleright x)S(J^{(-2)})Q_J \\
&= J^{(-1)}a_{(1)}xS(a_{(2)})S(J^{(-2)})Q_J \\
&= J^{(-1)}a_{(1)}J^{(1)}J'^{-(1)}xS(J^{(-2)}a_{(2)}J^{(2)}J'^{-(2)})Q_J
\end{aligned}
$$

$$
\begin{aligned}
&= a^J_{(1)}J'^{-(1)}xS(a^J_{(2)}J'^{-(2)})Q_J = a^J_{(1)}J'^{-(1)}xS(J'^{-(2)})S(a^J_{(2)})Q_J \\
&= a^J_{(1)}J'^{-(1)}xS(J'^{-(2)})Q_JQ_J^{-1}S(a^J_{(2)})Q_J \\
&= a^J_{(1)}D_J(x)S^J(a^J_{(2)}) \\
&= a \blacktriangleright_J D_J(x).
\end{aligned}
$$

This completes the proof. □

§5.2.2 Quasitriangular structure and integral under a twist

Recall from [34], a regular multiplier Hopf algebra (A, Δ) is called *quasitriangular* if there exists an invertible multiplier $\mathcal{R}$ in $M(A \otimes A)$ which is subject to

(1) $(\Delta \otimes \iota)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{23}$, $(\iota \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{13}\mathcal{R}^{12}$,

(2) $\mathcal{R}\Delta(a) = \Delta^{cop}(a)\mathcal{R}$ for all $a \in A$,

(3) $(\iota \otimes \varepsilon)(\mathcal{R}) = 1 = (\varepsilon \otimes \iota)(\mathcal{R})$.

Let (A, Δ) be a multiplier Hopf algebra. By Proposition 5.2.1.4, we get (A^J, Δ^J) is also a multiplier Hopf algebra. When (A, Δ) is quasitriangular, how about (A^J, Δ^J)?

Theorem 5.2.11 Let (A, Δ) be a quasitriangular multiplier Hopf algebra with generalized R-matrix $\mathcal{R}$. Then (A^J, Δ^J) is also quasitriangular, and the quasitriangular structure given by

$$\mathcal{R}^J = J_{21}^{-1}\mathcal{R}J.$$

Proof It is sufficient to check the equation (1), (2) and (3) in the definition of quasitriangular. Firstly, we check $\mathcal{R}^J\Delta^J(a) = (\Delta^J)^{cop}(a)\mathcal{R}^J$. In fact,

$$
\begin{aligned}
\mathcal{R}^J\Delta^J(a) &= J_{21}^{-1}\mathcal{R}JJ^{-1}\Delta(a)J = J_{21}^{-1}\mathcal{R}\Delta(a)J, \\
(\Delta^J)^{cop}(a)\mathcal{R}^J &= \tau(J^{-1}\Delta(a)J)J_{21}^{-1}\mathcal{R}J = J_{21}^{-1}\Delta^{cop}(a)J_{21}J_{21}^{-1}\mathcal{R}J \\
&= J_{21}^{-1}\Delta^{cop}(a)\mathcal{R}J.
\end{aligned}
$$

Because (A, Δ) is quasitriangular, $\mathcal{R}\Delta(a) = \Delta^{cop}(a)\mathcal{R}$, then the equation holds.

Secondly, by the extension of homomorphism ε, it is easy to check that $(\iota \otimes \varepsilon)(\mathcal{R}^J) = 1 = (\varepsilon \otimes \iota)(\mathcal{R}^J)$.

Finally, we need to check $(\Delta^J \otimes \iota)\mathcal{R}^J = \mathcal{R}^{J^{13}}\mathcal{R}^{J^{23}}$ and $(\iota \otimes \Delta^J)\mathcal{R}^J = \mathcal{R}^{J^{13}}\mathcal{R}^{J^{12}}$. Here, we only show the proof of the first equation, and the second one is similar to verify.

For any $x \otimes y \in A \otimes A$, $x \otimes y = \sum_i \Delta^J(a_i)(1 \otimes b_i)$ because of Proposition 5.2.3. Thus

$$
\begin{aligned}
& (\Delta^J \otimes \iota)(\mathcal{R}^J)(x \otimes y \otimes z) \\
= \;& (\Delta^J \otimes \iota)(\mathcal{R}^J)(\sum_i \Delta^J(a_i)(1 \otimes b_i) \otimes z) = \sum_i (\Delta^J \otimes \iota)(\mathcal{R}^J(a_i \otimes z))(1 \otimes b_i \otimes 1) \\
= \;& \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1}\mathcal{R}J(a_i \otimes z))(J \otimes 1)(1 \otimes b_i \otimes 1) \\
= \;& \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})(\Delta \otimes \iota)(\mathcal{R})(\Delta \otimes \iota)(J(a_i \otimes z))(J \otimes 1)(1 \otimes b_i \otimes 1) \\
= \;& \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})\mathcal{R}^{13}\mathcal{R}^{23}(\Delta \otimes \iota)(J)(\Delta \otimes \iota)(a_i \otimes z)(J \otimes 1)(1 \otimes b_i \otimes 1) \\
= \;& \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})\mathcal{R}^{13}\mathcal{R}^{23}(\Delta \otimes \iota)(J)(J \otimes 1) \\
& (J^{-1} \otimes 1)(\Delta \otimes \iota)(a_i \otimes z)(J \otimes 1)(1 \otimes b_i \otimes 1) \\
= \;& \sum_i (J^{-1} \otimes 1)(\Delta \otimes \iota)(J_{21}^{-1})\mathcal{R}^{13}\mathcal{R}^{23}(\iota \otimes \Delta)(J)(1 \otimes J)(J^{-1}\Delta(a_i)J \otimes 1)(1 \otimes b_i \otimes z) \\
= \;& (\dot{J}^{-(1)}J^{-(2)}_{(1)} \otimes \dot{J}^{-(2)}J^{-(2)}_{(2)} \otimes J^{-(1)})\mathcal{R}^{13}\mathcal{R}^{23}(\iota \otimes \Delta)(J)(1 \otimes J)(x \otimes y \otimes z) \\
= \;& \underline{\dot{J}^{-(1)}J^{-(2)}_{(1)}}\mathcal{R}^{(1)}J'^{(1)}x \otimes \underline{\dot{J}^{-(2)}J^{-(2)}_{(2)}}\mathcal{R}'^{(1)}J'^{(2)}_{(1)}\dot{J}'^{(1)}y \otimes \underline{J^{-(1)}}\mathcal{R}^{(2)}\mathcal{R}'^{(2)}J'^{(2)}_{(2)}\dot{J}'^{(2)}z \\
= \;& J^{-(2)}\dot{J}^{-(1)}_{(2)}\mathcal{R}^{(1)}J'^{(1)}x \otimes \dot{J}^{-(2)}\underline{\mathcal{R}'^{(1)}J'^{(2)}_{(1)}}\dot{J}'^{(1)}y \otimes J^{-(1)}\dot{J}^{-(1)}_{(1)}\mathcal{R}^{(2)}\underline{\mathcal{R}'^{(2)}J'^{(2)}_{(2)}}\dot{J}'^{(2)}z \\
= \;& J^{-(2)}\underline{\dot{J}^{-(1)}_{(2)}\mathcal{R}^{(1)}}J'^{(1)}x \otimes \dot{J}^{-(2)}J'^{(2)}_{(2)}\mathcal{R}'^{(1)}\dot{J}'^{(1)}y \otimes J^{-(1)}\underline{\dot{J}^{-(1)}_{(1)}\mathcal{R}^{(2)}}J'^{(2)}_{(1)}\mathcal{R}'^{(2)}\dot{J}'^{(2)}z \\
= \;& J^{-(2)}\mathcal{R}^{(1)}\underline{\dot{J}^{-(1)}_{(1)}J'^{(1)}}x \otimes \underline{\dot{J}^{-(2)}J'^{(2)}_{(2)}}\mathcal{R}'^{(1)}\dot{J}'^{(1)}y \otimes J^{-(1)}\mathcal{R}^{(2)}\underline{\dot{J}^{-(1)}_{(2)}J'^{(2)}_{(1)}}\mathcal{R}'^{(2)}\dot{J}'^{(2)}z \\
= \;& J^{-(2)}\mathcal{R}^{(1)}J^{(1)}x \otimes \dot{J}^{-(2)}\mathcal{R}'^{(1)}\dot{J}'^{(1)}y \otimes J^{-(1)}\mathcal{R}^{(2)}J^{(2)}J'^{-(1)}\mathcal{R}'^{(2)}\dot{J}'^{(2)}z \\
= \;& \mathcal{R}^{J^{13}}\mathcal{R}^{J^{23}}(x \otimes y \otimes z),
\end{aligned}
$$

where the penultimate equation holds because of $(\Delta\otimes\iota)(J^{-1})(\iota\otimes\Delta)(J) = (J\otimes1)(1\otimes J)$.
□

From Theorem 5.2.6 and Proposition 2.6 in [10], we can easily get the following result.

Proposition 5.2.12 Let (A, Δ) be a quasitriangular multiplier Hopf algebra and J a twist. Then for all $a \in A$

$$(S^J)^4(a) = gag^{-1},$$

where $g = \mu S(\mu)^{-1}$ and $\mu = S^J(\mathcal{R}^{J(2)})\mathcal{R}^{J(1)}$.

Because the integral play an important role in the Pontryagin duality, then the following queation naturally arise:

(Q2) *For a regular multiplier Hopf algebra (A, Δ) with an integral, does (A^J, Δ^J) also admits a integral?*

Before answering this question, we first consider a multiplier $u_J = Q_J^{-1}S(Q_J)$ in $M(A)$. By equation (5.2.4), we have

$$\begin{aligned}\Delta(u_J) &= \Delta(Q_J^{-1}S(Q_J)) = J(Q_J^{-1}S(Q_J) \otimes Q_J^{-1}S(Q_J))(S^2 \otimes S^2)(J^{-1}) \\ &= J(u_J \otimes u_J)(S^2 \otimes S^2)(J^{-1}).\end{aligned}$$

Theorem 5.2.13 Let (A, Δ) be a counimodular algebraic quantum group with a non-zero left (resp. right) integral φ (resp. ψ) and J be a Drinfeld twist. Then the elements $\varphi^J = u_J \rightharpoonup \varphi$ and $\psi^J = \psi \leftharpoonup u_J^{-1}$ are non-zero left and right integrals on (A^J, Δ^J) respectively.

Proof We need to check that $(\iota \otimes \varphi^J)\Delta^J(a) = \varphi^J(a)1$, equivalently $S(\iota \otimes \varphi^J)\Delta^J(a) = \varphi^J(a)1$. In deed, for any $x \in A$, there exists $b \in A$ such that $x = S(b)$. Thus

$$\begin{aligned}
& S(\iota \otimes \varphi^J)(\Delta^J(a))x = S(\iota \otimes u_J \rightharpoonup \varphi)(J^{-1}\Delta(a)J)x \\
= \ & S(\iota \otimes \varphi)(J^{-1}\Delta(a)J(1 \otimes u_J))S(b) = S(\iota \otimes \varphi)((b \otimes 1)J^{-1}\Delta(a)J(1 \otimes u_J)) \\
= \ & S(\iota \otimes \varphi)(bJ^{-(1)}a_{(1)}\dot{J}^{(1)} \otimes J^{-(2)}a_{(2)}\dot{J}^{(2)}u_J) = S(bJ^{-(1)}a_{(1)}\dot{J}^{(1)})\varphi(J^{-(2)}a_{(2)}\dot{J}^{(2)}u_J) \\
= \ & S(\dot{J}^{(1)})\underline{S(a_{(1)})}S(J^{-(1)})S(b)\underline{\varphi(a_{(2)}\dot{J}^{(2)}u_J S^2(J^{-(2)}))} \\
\overset{(1)}{=} \ & S(\dot{J}^{(1)})\dot{J}^{(2)}_{(1)}u_{J(1)}S^2(J^{-(2)}_{(1)})S(J^{-(1)})S(b)\varphi(a\dot{J}^{(2)}_{(2)}u_{J(2)}S^2(J^{-(2)}_{(2)})) \\
= \ & (\iota \otimes \varphi)\Big((1 \otimes a)\big(\underline{S(\dot{J}^{(1)})\dot{J}^{(2)}_{(1)} \otimes \dot{J}^{(2)}_{(2)}}\big)\Delta(u_J)\big((S \otimes S)(b\underline{\underline{J^{-(1)}S(J^{-(2)}_{(1)}) \otimes S(J^{-(2)}_{(2)})}})\big)\Big) \\
\overset{(2)}{=} \ & (\iota \otimes \varphi)\Big((1 \otimes a)\big((Q_J \otimes 1)J^{-1}\big)\Delta(u_J)\big((S \otimes S)((bQ_{J^{-1}} \otimes 1)(S \otimes S)(J))\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= (\iota\otimes\varphi)\Big((1\otimes a)\big((Q_J\otimes 1)J^{-1}\big)(J(u_J\otimes u_J)(S^2\otimes S^2)(J^{-1}))\\
&\quad \big((S^2\otimes S^2)(J)(S(Q_{J^{-1}})\otimes 1)(S(b)\otimes 1)\big)\Big)\\
&= (\iota\otimes\varphi)\Big((1\otimes a)(Q_J\otimes 1)(u_J\otimes u_J)(S(Q_{J^{-1}})\otimes 1)(S(b)\otimes 1)\big)\Big)\\
&= (\iota\otimes\varphi)\Big((1\otimes a)(1\otimes u_J)(S(b)\otimes 1)\big)\Big)\\
&= \varphi^J(a)x,
\end{aligned}
$$

where equation (1) holds because $S(\iota\otimes\varphi)(\Delta(a)(1\otimes b)) = (\iota\otimes\varphi)((1\otimes a)\Delta(b))$ and (2) holds because of Proposition 2.4. □

Note that Theorem 5.2.13 contains an answer to the question (Q) in Introduction. Following Theorem 5.2.13 we can easily get the follwing result in the Hopf algebra case.

Corollary 5.2.14 Let (H,Δ) be an infinite-dimensional counimodular coFrobenius Hopf algebra. φ (resp. ψ) is the non-zero left (resp. right) integral on H and $J\in H\otimes H$ is a twist. Then the elements $\varphi^J = u_J \rightharpoonup \varphi$ and $\psi^J = \psi \leftharpoonup u_J^{-1}$ are non-zero left and right integrals on (H^J,Δ^J) respectively.

Example 5.2.15 Let G be an (infinite) abelian group, let kG be its group algebra with coefficients in a field k, and let $k(G)$ be the dual multiplier Hopf algebra. Then $D(G) = k(G)^{cop}\bowtie kG$ is the Drinfeld double with the quasitriangular multiplier Hopf algebra structure as follows. For any $g,h,p,q\in G$,

$$
\begin{aligned}
&(\delta_g\bowtie p)(\delta_h\bowtie q) = \delta_g\delta_{php^{-1}}\bowtie pq, \qquad \varepsilon(\delta_g\bowtie p) = \delta_{g,e},\\
&\Delta(\delta_g\bowtie p) = \sum_{s\in G}(\delta_{s^{-1}g}\bowtie p)\otimes(\delta_s\bowtie p), \quad S(\delta_g\bowtie p) = \delta_{p^{-1}g^{-1}p}\bowtie p^{-1},\\
&\mathcal{R} = \sum_{g\in G}(1\bowtie g)\otimes(\delta_g\bowtie e).
\end{aligned}
$$

In this case, the Drinfeld twist $J = \mathcal{R}^{-1} = \sum_{g\in G}(1\bowtie g)\otimes(\delta_{g^{-1}}\bowtie e)$, where $1 = \sum_{g\in G}\delta_g$. By Theorem 5.2.2.1 the quasitriangular structure in $D(G)^J$ is given by

$$\mathcal{R}^J = J_{21}^{-1}\mathcal{R}J = \sum_{g\in G}(\delta_g\bowtie e)\otimes(1\bowtie g).$$

Since $Q_J = S(J^{(1)})J^{(2)} = \sum_{g\in G} \delta_g \bowtie g^{-1}$, $u_J = Q_J^{-1}S(Q_J) = 1 \bowtie e$, by Theorem 5.2.3 the left and right integrals on $(D(G)^J, \Delta^J)$ are given by $\varphi^J = \psi^J = f \otimes \delta_e$, where f maps every $\delta_g, g \in G$ to 1.

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