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Post-doctoral Research Report

Structures and invariants of Hopf algebras with the dual Chevalley property

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Fundamental Mathematics

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研究报告题目: **具有对偶 Chevalley 性质的 Hopf 代数的结构与不变量**

基础数学 专业 **2019** 级博士后姓名: 李康桥

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摘　要

Pointed Hopf 代数提供了大量的 Hopf 代数基本例子，且其相关理论十分丰富。此报告致力于将 pointed Hopf 代数的部分经典结构以及不变量的结果，推广到具有对偶 Chevalley 性质（即余根是子 Hopf 代数）的 Hopf 代数上。除了 Hopf 代数理论中的传统工具外，此报告特别使用了所谓的可乘矩阵与本原矩阵作为主要研究工具。作为结果，我们考虑了具有对偶 Chevalley 性质 Hopf 代数的对极的零化多项式、指数、以及 link 不可分分支等。此报告分为如下五章。

第一章，我们介绍研究背景和主要结论。

第二章，我们回顾一些关于报告内容的基本概念和主要工具。

第三章，我们研究具有对偶 Chevalley 性质的有限维 Hopf 代数 H 的对极，并得到了它的一个零化多项式。这推广了 Taft 和 Wilson 在 1974 年的经典结论。此外我们得到了两个推论：1) H 的拟指数与其余根的指数相等，即 $\mathrm{qexp}(H)=\exp(H_0)$；2) $\mathrm{qexp}(H\rtimes\Bbbk\langle S^2\rangle)=\mathrm{qexp}(H)$。

第四章，我们讨论并比较了有限维 Hopf 代数指数的两种定义的性质，包括等价不变性和有限性。具体地，其中一种指数关于 twist 变换和 Drinfeld 偶运算都是不变的（这一点与另一个指数相同）。我们也发现对于具有对偶 Chevalley 性质的非余半单 Hopf 代数，两种指数在特征为 0 的基域上都是无限的，而在正特征的基域上都有限。

第五章，我们通过给出子余代数之间 link 关系的充分条件，证明了一个用于刻画具有对偶 Chevalley 性质的 Hopf 代数的 link 不可分分支乘积的公式。此外，我们证明了含单位元的分支是一个子 Hopf 代数，且每个分支都是它上面由一个单子余代数生成的忠实平坦模（事实上也是投射生成子）。这些结论推广了 Montgomery 在 1995 年对于 pointed Hopf 代数的相关结论。

关键词: Hopf 代数; 对偶 Chevalley 性质; 对极; 指数; Gauge 不变量; 不可分余代数; Link 关系; 忠实平坦性.

REPORT: Structures and invariants of Hopf algebras with the dual Chevalley property

SPECIALIZATION: Fundamental Mathematics

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Abstract

Pointed Hopf algebras include a good number of basic examples of Hopf algebras and they have a very rich theory. This report is devoted to generalizing some classical results on the structures and gauge invariants of pointed Hopf algebras to the case with the dual Chevalley property (which means that the coradical is a Hopf subalgebra). Besides a number of canonical theories for Hopf algebras, the main tools in this report are especially so-called multiplicative and primitive matrices over coalgebras. As conclusions, we study an annihilation polynomial for the antipode, the exponent, and the link-indecomposable components of Hopf algebras with the dual Chevalley property. This report is divided into five chapters.

In Chapter 1, we provide the research backgrounds, motivations and main results.

In Chapter 2, we recall some basic concepts and main tools related to this report.

In Chapter 3, we study the antipode of a finite-dimensional Hopf algebra H with the dual Chevalley property and obtain an annihilation polynomial for the antipode. This generalizes an old result given by Taft and Wilson in 1974. As consequences, we show that 1) the quasi-exponent of H is the same as the exponent of its coradical, that is, $\mathrm{qexp}(H) = \exp(H_0)$; 2) $\mathrm{qexp}(H \rtimes \Bbbk\langle S^2\rangle) = \mathrm{qexp}(H)$.

In Chapter 4, we discuss and compare properties of two notions of exponent of finite-dimensional Hopf algebras, including invariance and finiteness. Specifically, one notion is invariant under twisting and taking the Drinfeld double, exactly as the other one. We also find that if the non-cosemisimplicity and dual Chevalley property hold, both exponents are infinite in characteristic 0 but finite in positive characteristic.

In Chapter 5, we provide sufficient conditions for the link relation on simple subcoalgebras, and prove a formula on the products between link-indecomposable components of Hopf algebras with the dual Chevalley property. Furthermore, we show that each of its component is generated by a simple subcoalgebra, as a faithfully flat module (in fact, a projective generator) over a Hopf subalgebra which is the component containing the unit element. Our conclusions generalize some relevant results on pointed Hopf algebras, which were established by Montgomery in 1995.

Keywords: Hopf algebra; Dual Chevalley property; Antipode; Exponent; Gauge invariant; Indecomposable coalgebra; Link relation; Faithful flatness.

Chapter 1 Introduction

§1.1 Background

General theories for Hopf algebras over a field were begun in 1960s. As important cases of non-semisimple (or non-cosemisimple) ones among them, pointed Hopf algebras has particularly a large number of classical structural results in the literature. The main reason why pointed Hopf algebras were early considered is that the coradical is a group algebra, so that the theory of groups could be sufficiently applied. This assumption helps us to study pointed Hopf algebras with properties of their *grouplike and primitive elements*, which are immediately defined through the coradical.

Our motivation for this report is to generalize some classical results on the structures and invariants of pointed Hopf algebras, with the tools of so-called multiplicative and primitive matrices. Multiplicative and primitive matrices could be regarded as generalizations of grouplike and primitive elements, respectively. Indeed, pointed Hopf algebras must have the *dual Chevalley property*, which means that the coradical is a Hopf subalgebra and is in fact a quite weak condition for arbitrary Hopf algebras. Thus we aim to obtain some analogous results for the case with the dual Chevalley property. Now we introduce the original ones on pointed Hopf algebras for the remaining of this subsection.

Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$ with the antipode S, and denote the composition order of S^2 by $\mathrm{ord}(S^2)$. The order or annihilation polynomials of S^2 has been studied for more than 40 years. The first most general result was given by Radford [37] in 1976, which states that $\mathrm{ord}(S^2)$ is always finite. Then the people want to find an explicit bound for $\mathrm{ord}(S^2)$. Actually, the people made the progress at least in the following two different cases for H: semisimple case and pointed case.

As for the first case when H is semisimple, Kaplansky conjectured in [16] that semisimple Hopf algebras are all involutory (that is, $\mathrm{ord}(S^2) = 1$), which is well-known as the Kaplansky's fifth conjecture and still open in small positive characteristic. Moreover, when H is cosemisimple in addition, a positive answer was given by Etingof and Gelaki [11]. The other case when H is pointed was once studied by Taft and

Wilson [46] in 1974. They obtained the following annihilation polynomial:

$$(S^{2N} - \mathrm{id})^{L-1} = 0, \tag{1.1}$$

where N is the exponent of the coradical and L denotes the Loewy length. This formula implies directly that $\mathrm{ord}(S^2) \mid N$ in characteristic 0 ([13]) and $\mathrm{ord}(S^2) \mid Np^M$ in characteristic $p > 0$ for some positive integer M ([46]).

The notions of exponent of Hopf algebras have been introduced and studied since 1999. This process arised from a conjecture by Kashina [17,18] that nth Sweedler power $[n]$ is trivial on a semisimple and cosemisimple Hopf algebra of dimension n. She also verified this property on a number of known examples. One notion of exponents for a Hopf algebra H is considered to be the least positive integer n such that $[n]$ is trivial, which is denoted by $\exp_0(H)$ in this note. Actually it coincides with the exponent of a group G when H is the group algebra $\Bbbk G$.

However, the term "exponent" of Hopf algebras was first introduced by Etingof and Gelaki [12]. Their definition of the exponent, denoted by $\exp(H)$, is slightly different from $\exp_0(H)$ considered above. They provided at first various properties for $\exp(H)$ when H is finite-dimensional, especially the invariance properties. Specifically, this exponent is invariant under the duality, taking the opposite algebra, twisting and taking the Drinfeld double. One more important result in [12] is that $\exp(H)$ is finite and divides $\dim(H)^3$, as long as H is semisimple and cosemisimple (and thus involutory). This partially answers Kashina's conjecture, because of an immediate observation that $\exp(H) = \exp_0(H)$ holds when H is involutory (or pivotal). Another result about the finiteness is that $\exp(H) < \infty$ in positive characteristic when H is finite-dimensional.

On the other hand, $\exp_0(H)$ seems different from $\exp(H)$ for general Hopf algebras by definitions. One might ask whether $\exp_0(H)$ has similar properties with $\exp(H)$. Some properties are studied by Landers et al. [21]. For instance, they showed that $\exp_0(H)$ is also invariant under taking the opposite algebra. As for the finiteness, the author and Zhu [27] found that $\exp_0(H) = \infty$ if H is non-cosemisimple in characteristic 0 with the dual Chevalley property, and that $\exp_0(H) < \infty$ if H is finite-dimensional and pointed. Moreover, there are other researches involving exponents of Hopf algebras, such as [13], [20], [36] and [41], etc.

It is known in Kaplansky [16] that any coalgebra could be written uniquely as a direct sum of indecomposable subcoalgebras. The notion of the link relation (also known as the connected relation) on simple subcoalgebras is a theoretical way to determine the direct summands, which are referred as the link-indecomposable components. This was firstly shown by Shudo and Miyamoto [43]. Later in 1995, Montgomery [35] refined the related knowledge with the language of quivers, and studied properties of the link-indecomposable components of a pointed Hopf algebra. For any pointed Hopf algebra H, she established a formula on the products of link-indecomposable components. As consequences, H is free over a normal Hopf subalgebra $H_{(1)}$ which is exactly the link-indecomposable component containing the unit element, and H is furthermore a crossed product of a group over $H_{(1)}$.

§1.2 Main results

Our main result in Section 3 generalizes Formula (1.1) to the case when H has the dual Chevalley property:

Theorem 3.1.1 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property. Denote $N := \exp(H_0) < \infty$ and $L := \mathrm{Lw}(H)$. Then*

$$(S^{2N} - \mathrm{id})^{L-1} = 0$$

holds on H.

An immediate but meaningful conclusion on the order of the antipode is:

Corollary 3.1.3 *Let H, N and L be as in Theorem 3.1.1. Then*

(1) If $\mathrm{char}\, \Bbbk = 0$, then $\mathrm{ord}(S^2) \mid N$;

(2) If $\mathrm{char}\, \Bbbk = p > 0$, then $\mathrm{ord}(S^2) \mid \gcd(Np^M, \exp(H))$, where M is a natural number satisfying $p^M \geq L - 1$.

Section 4 is an attempt to complete comparisons of properties between two notions of the exponent $\exp_0(H)$ and $\exp(H)$ where H is a finite-dimensional Hopf algebra. Our first result is that $\exp_0(H)$ is also invariant under twisting and taking the Drinfeld double:

Proposition 4.1.5 *Let H be a finite-dimensional Hopf algebra. Suppose J be a (left or right) twist for H. Then $\exp_0(H^J) = \exp_0(H)$ and $\exp_0(D(H)) = \exp_0(H)$.*

Also, we show that if H is non-cosemisimple with the dual Chevalley property, $\exp(H)$ is infinite in characteristic 0 but finite in positive characteristic:

Theorem 4.3.6 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over $\Bbbk$. Then*

(1) If H is non-cosemisimple and $\operatorname{char}\Bbbk = 0$, then $\exp(H) = \infty$;

(2) If $\operatorname{char}\Bbbk = p > 0$, and denote $N := \operatorname{lcm}(\exp(H_0), \exp_0(H_0)) < \infty$ and $L := \operatorname{Lw}(H)$, then $\exp(H) \mid Np^M$, where M is a positive integer satisfying $p^M \geq L$.

Section 5 is devoted to generalize some of these main results in [35] to non-pointed Hopf algebras. Denote the link-indecomposable component of H containing the simple subcoalgebra E by $H_{(E)}$, which is a subcoalgebra of H, and note again that H is the direct sum of its different link-indecomposable components. Our final result states that:

Theorem 5.2.8 *Let H be a Hopf algebra over an arbitrary field $\Bbbk$ with the dual Chevalley property. Denote the set of all the simple subcoalgebras of H by $\mathcal{S}$. Then*

(1) For any $C \in \mathcal{S}$, $H_{(C)} = CH_{(1)} = H_{(1)}C$;

(2) For any $C, D \in \mathcal{S}$, $H_{(C)}H_{(D)} \subseteq \sum_{E\in\mathcal{S},\, E\subseteq CD} H_{(E)}$;

(3) $H_{(1)}$ is a Hopf subalgebra.

Moreover as a direct corollary, the faithful flatness of H over the Hopf subalgebra $H_{(1)}$ is followed:

Corollary 5.2.12(1) *Let H be a Hopf algebra with the dual Chevalley property. Then H is a projective generator of left as well as right $H_{(1)}$-modules.*

Chapter 2 Preliminaries

§2.1 Basic Structures and Invariants of Hopf algebras

We recall the most needed knowledge about coalgebras and Hopf algebras in this section. Through out this report, all vector spaces, coalgebras, bialgebras and Hopf algebras are assumed to be over a field $\Bbbk$. The tensor product over $\Bbbk$ is denoted simply by $\otimes$. For a coalgebra (H, Δ, ε) over a field $\Bbbk$, Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for $h \in H$ is always used.

§2.1.1 Coradical Filtration and Loewy Length

Recall the notion of the *wedge* on a coalgebra (H, Δ, ε) that

$$V \wedge W := \Delta^{-1}(V \otimes H + H \otimes W)$$

for any subspaces $V, W \subseteq H$. There are further notations as follows:

$$\begin{aligned} \wedge^0 V &:= V; \\ \wedge^n V &:= V \wedge \left(\wedge^{n-1} V\right) \quad (\forall n \geq 1). \end{aligned}$$

Denote the *coradical* of H by H_0. The *coradical filtration* of H is a sequence of subcoalgebras defined inductively as

$$H_{n+1} := H_0 \wedge H_n = \wedge^{n+1} H_0 \ (n \geq 0)$$

and always denoted by $\{H_n\}_{n\geq 0}$ in this report. These definitions could be found in [45, Chapter 9].

The *Loewy length* (cf. [14, Lemma 2.2]) of a coalgebra H is denoted as

$$\mathrm{Lw}(H) := \min\{l \geq 0 \mid H_{l-1} = H\}$$

with convention $H_{-1} = 0$ and $\min \varnothing = \infty$. It is apparent that $\mathrm{Lw}(H) < \infty$ if H is finite-dimensional.

§2.1.2 Dual Chevalley Property

A Hopf algebra H is said to have the *dual Chevalley property*, if its coradical H_0 is a Hopf subalgebra (or equivalently, H_0 is a subalgebra of H and $S(H_0) \subseteq H_0$). A well-known result about the dual Chevalley property is the following lemma (see e.g. [34, Lemma 5.2.8]).

Lemma 2.1.1 *Let H be a Hopf algebra H with the coradical filtration $\{H_n\}_{n\geq 0}$. Then the followings are equivalent:*

(1) H_0 is a Hopf subalgebra of H;

(2) $\{H_n\}_{n\geq 0}$ is a Hopf algebra filtration.

§2.1.3 Exponent

Let $(H, m, u, \Delta, \varepsilon)$ be a Hopf algebra with antipode S over a field $\Bbbk$. For convenience, we define following $\Bbbk$-linear maps for any positive integer n:

$$\begin{aligned} m_n: \quad & H^{\otimes n} \to H, \quad h_1 \otimes h_2 \otimes \cdots \otimes h_n \mapsto h_1 h_2 \cdots h_n; \\ \Delta_n: \quad & H \to H^{\otimes n}, \quad h \mapsto \sum h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n)}. \end{aligned}$$

When S is bijective, the notion of the *exponent* of H introduced in [12] by Etingof and Gelaki is defined as

$$\exp(H) := \min\{n \geq 1 \mid m_n \circ (\mathrm{id} \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n = u \circ \varepsilon\}$$

with convention $\min \varnothing = \infty$. One of their most crucial ways to study the exponent is the following identification [12, Theorem 2.5(2)], when H is finite-dimensional:

$$\exp(H) \text{ equals to the multiplication order of } u_{D(H)},$$

where $u_{D(H)}$ is the Drinfeld element of the Drinfeld double $D(H)$ ([9]). We remark that there is also another notion of "exponent" introduced by Kashina [17, 18] (see also Remark **??**).

It is known that if H is finite-dimensional, $D(H)$ is also a Hopf algebra, whose antipode is denoted by $S_{D(H)}$. Moreover, ${S_{D(H)}}^2$ is in fact an inner automorphism determined by $u_{D(H)}$ on $D(H)$. Thus ${S_{D(H)}}^{2\exp(H)}$ becomes the identity map on $D(H)$, as long as $\exp(H) < \infty$. Restricting this map onto the Hopf subalgebra $H \cong \varepsilon \bowtie H \subseteq D(H)$, we obtain the following fact immediately:

Corollary 2.1.2 *Let H be a finite-dimensional Hopf algebra with antipode S. If* $\exp(H) < \infty$*, then* $S^{2\exp(H)} = \mathrm{id}$ *(the identity map on H).*

There are two theorems [12, Theorem 4.3] and [12, Theorem 4.10] describing the finiteness of the exponent. We state them below:

Lemma 2.1.3 *Let H be a finite-dimensional Hopf algebra over $\Bbbk$.*

(1) If H is semisimple and cosemisimple, then $\exp(H)$ *is finite and divides* $\dim(H)^3$*;*

(2) If $\operatorname{char} \Bbbk > 0$*, then* $\exp(H) < \infty$.

As a conclusion of the theorems above, it is easy to check that the coradical of a finite-dimensional Hopf algebra with the dual Chevalley property always has finite exponent:

Corollary 2.1.4 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over $\Bbbk$. Then* $\exp(H_0) < \infty$.

Proof: If $\operatorname{char} \Bbbk > 0$, then this is a direct consequence of the (2) of the above lemma. If $\operatorname{char} \Bbbk = 0$, the cosemisimple Hopf algebra H_0 is also semisimple now by [23, Theorem 3.3]. Then (1) of the above lemma is applied.

§2.2 Coradical Orthonormal Idempotents

For any coalgebra H, its dual algebra with the convolution product is denoted by H^*. Now we refer a certain kind of family of idempotents in H^* introduced by Radford [39], which are called *coradical orthonormal idempotents* in this report. To introduce them, let $\mathcal{S}$ be the set of simple subcoalgebras of H and the classical Kronecker delta is denoted by δ.

Definition 2.2.1 *Let H be a coalgebra. A family of coradical orthonormal idempotents of H^* is a family of non-zero elements $\{e_C\}_{C\in\mathcal{S}}$ in H^* satisfying following conditions:*

(1) $e_C e_D = \delta_{C,D} e_C$ for $C, D \in \mathcal{S}$;

(2) $\sum_{C\in\mathcal{S}} e_C = \varepsilon$ *on* H *(distinguished condition);*

(3) $e_C|_D = \delta_{C,D}\varepsilon|_D$ *for* $C, D \in \mathcal{S}$*.*

The existence of (a family of) coradical orthonormal idempotents in H^* for any coalgebra H is affirmed in [39, Lemma 2] or [40, Corollary 3.5.15], using properties of injective comodules. It is always assumed that $\{e_C\}_{C\in\mathcal{S}}$ is a given family of coradical orthonormal idempotents in H^* for the remaining of this report.

We remark that when H is pointed, there is another way to construct coradical orthonormal idempotents from [34, Theorem 5.4.2] for example, and some convenient notations are used there. Similar notations for $\{e_C\}_{C\in\mathcal{S}}$ will be used in this report too:

$$^C h = h \leftharpoonup e_C, h^D = e_D \rightharpoonup h, {}^C h^D = e_D \rightharpoonup h \leftharpoonup e_C, \quad h \in H,\ C, D \in \mathcal{S},$$

where $\leftharpoonup$ and $\rightharpoonup$ are hit actions of H^* on H. Specially if $C = \Bbbk g$ is pointed, then we also denote ${}^g h := {}^C h$, $h^g := h^C$, and $V^C := e_C \rightharpoonup V$, etc.

Some direct properties, which can be found in [27, Proposition 2.2], are listed as follows:

Proposition 2.2.2 *Let* H *be a coalgebra. Then for all* $C, D \in \mathcal{S}$*, we have*

(1) ${}^C H_0{}^D = \delta_{C,D} C$*;*

(2) ${}^C H_1{}^D \subseteq \Delta^{-1}(C \otimes {}^C H_1{}^D + {}^C H_1{}^D \otimes D)$*;*

(3) ${}^C H^D \subseteq \mathrm{Ker}(\varepsilon)$ *if* $C \neq D$*;*

(4) Suppose $V \subseteq H$ *is a* $\Bbbk$*-subspace. We have following direct-sum decomposition:*

(i) $V = \bigoplus_{C\in\mathcal{S}} {}^C V$ *if* V *is a left coideal;*

(ii) $V = \bigoplus_{D\in\mathcal{S}} V^D$ *if* V *is a right coideal;*

(iii) $V = \bigoplus_{C,D\in\mathcal{S}} {}^C V^D$ *if* V *is a subcoalgebra.*

§2.3 Matrices over Coalgebras and Hopf algebras

As the main tools in this report are matrices over vector spaces, an evident lemma should be noted as first:

Lemma 2.3.1 *Let V be a vector space. For any matrix $\mathcal{A}$ over V, the followings are equivalent:*

(1) All the entries of $\mathcal{A}$ are linearly independent;

(2) All the entries of $P\mathcal{A}Q$ are linearly independent, for some invertible matrices P and Q over $\Bbbk$.

Moreover, we always say that two square matrices $\mathcal{A}$ and $\mathcal{B}$ over a vector space V are *similar*, if there exists an invertible matrix L over $\Bbbk$ such that $\mathcal{B} = L\mathcal{A}L^{-1}$. This is denoted by $\mathcal{A} \sim \mathcal{B}$ for simplicity.

§2.3.1 Multiplicative Matrices and Their Operations

The notion of the multiplicative matrices over coalgebras was once introduced in [30]. This helps us generalize some results of pointed coalgebras or Hopf algebras to the case of non-pointed ones. For our purposes, more properties of multiplicative matrices are considered in this subsection. Let us start by recalling notations and definitions.

Notation 2.3.2 *Let V and W be vector spaces.*

(1) For any matrix $\mathcal{A} := (v_{ij})_{m\times n}$ over V and matrix $\mathcal{B} := (w_{ij})_{n\times l}$ over W, denote the following matrix

$$\mathcal{A} \mathbin{\widetilde{\otimes}} \mathcal{B} := \left(\sum_{k=1}^{n} v_{ik} \otimes w_{kl}\right)_{m\times l};$$

(2) For any matrix $\mathcal{A} := (v_{ij})_{m\times n}$ over V, denote the following matrix

$$\mathcal{A}^{\mathrm{T}} := (v_{ji})_{n\times m};$$

(3) For any linear map $f : V \to W$ and a matrix $\mathcal{A} := (v_{ij})_{m\times n}$ over V, denote the following matrix

$$f(\mathcal{A}) := \left(f(v_{ij})\right)_{m\times n}.$$

Then multiplicative matrices could be defined simply as follows.

Definition 2.3.3 *Let (H, Δ, ε) be a coalgebra over $\Bbbk$.*

(1) A square matrix $\mathcal{G}$ over H is said to be multiplicative, if $\Delta(\mathcal{G}) = \mathcal{G} \widetilde{\otimes} \mathcal{G}$ and $\varepsilon(\mathcal{G}) = I$ (the identity matrix over $\Bbbk$) both hold;

(2) A multiplicative matrix $\mathcal{C}$ is said to be basic, if its entries are linearly independent.

Clearly, all the entries of a basic multiplicative matrix $\mathcal{C}$ span a simple subcoalgebra C of H. Conversely, when the base field $\Bbbk$ is *algebraically closed*, any simple coalgebra C has a basic multiplicative matrix $\mathcal{C}$ whose entries span C. Moreover, we could describe the uniqueness for $\mathcal{C}$ as follows:

Lemma 2.3.4 *Let C be a simple coalgebra over $\Bbbk$. Suppose that $\mathcal{C}$ is a basic multiplicative matrix of C. Then $\mathcal{D}$ is also a basic multiplicative matrix of C if and only if $\mathcal{D} \sim \mathcal{C}$.*

Proof: A particular case of Skolem-Noether theorem follows the fact that: Any two matric bases of a finite-dimensional matrix algebra are similar. Our desired lemma would be its dual version.

Remark 2.3.5 *One could easily verify that matrices similar to multiplicative ones are also multiplicative, even if they are not basic.*

This lemma states that for a simple coalgebra C, its basic multiplicative matrix would be unique up to the similarity relation (over $\Bbbk$). In fact as for an arbitrary multiplicative matrix, we claim in the followings that it could be "decomposed" into basic ones:

Proposition 2.3.6 *Suppose $\mathcal{G}$ is an $n \times n$ multiplicative matrix over a coalgebra H. Then*

(1) There exist basic multiplicative matrices $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ over H, such that

$$\mathcal{G} \sim \begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_{12} & \cdots & \mathcal{X}_{1t} \\ 0 & \mathcal{C}_2 & \cdots & \mathcal{X}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix},$$

where $\mathcal{X}_{ij}$'s are matrices over H for all $1 \leq i < j \leq t$;

(2) If all the entries of $\mathcal{G}$ belong to the coradical of H, then there exist basic multiplicative matrices $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ over H, such that

$$\mathcal{G} \sim \begin{pmatrix} \mathcal{C}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{C}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix}.$$

Proof: It is clear that all the entries of $\mathcal{G}$ span a subcoalgebra G of H. Define an n-dimensional $\Bbbk$-vector space $V := \Bbbk v_1 \oplus \Bbbk v_2 \oplus \cdots \oplus \Bbbk v_n$, which becomes a right G-comodule with structures

$$\rho(v_1, v_2, \cdots, v_n) := (v_1, v_2, \cdots, v_n) \,\widetilde{\otimes}\, \mathcal{G}.$$

(1) Evidently, V has at least one simple G-subcomodule, denoted by W. Suppose that W has a linear basis $\{w_1, w_2, \cdots, w_r\}$, and

$$\rho(w_1, w_2, \cdots, w_r) = (w_1, w_2, \cdots, w_r) \,\widetilde{\otimes} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rr} \end{pmatrix}$$

holds for some c_{ij}'s in G. Then according to [40, Theorem 3.2.11(d)] and its proof, $\{c_{ij} \mid 1 \leq i, j \leq r\}$ is linearly independent, and thus spans a simple subcoalgebra with a basic multiplicative matrix $\mathcal{C}_1 := (c_{ij})_{r\times r}$.

Now we suppose $\{w_1, w_2, \cdots, w_r, u_1, u_2 \cdots, u_{n-r}\}$ is another linear basis of V, which is extended from the basis of W mentioned above. Choose the $n \times n$ transition matrix L_1 over $\Bbbk$ such that

$$(v_1, v_2, \cdots, v_n) = (w_1, \cdots, w_r, u_1, \cdots, u_{n-r})L_1,$$

and consider the comodule structure ρ at this equation. We could compute to know that

$$\begin{aligned}
& (w_1, \cdots, w_r, u_1, \cdots, u_{n-r}) \widetilde{\otimes} L_1 \mathcal{G} \\
= \ & (w_1, \cdots, w_r, u_1, \cdots, u_{n-r}) L_1 \widetilde{\otimes} \mathcal{G} \\
= \ & (v_1, v_2, \cdots, v_n) \widetilde{\otimes} \mathcal{G} \ = \ \rho(v_1, v_2, \cdots, v_n) \\
= \ & \rho((w_1, \cdots, w_r, u_1, \cdots, u_{n-r}) L_1) \\
= \ & \rho(w_1, \cdots, w_r, u_1, \cdots, u_{n-r}) L_1 \\
= \ & (w_1, \cdots, w_r, u_1, \cdots, u_{n-r}) \widetilde{\otimes} \begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_1 \\ 0 & \mathcal{G}_1 \end{pmatrix} L_1,
\end{aligned}$$

where $\mathcal{G}_1$ is multiplicative (of size $n-r$) due to the axiom of comodules, and $\mathcal{X}_1$ is an $r \times (n-r)$ matrix over H. This follows that

$$L_1 \mathcal{G} L_1^{-1} = \begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_1 \\ 0 & \mathcal{G}_1 \end{pmatrix}.$$

If we repeat the process on $\mathcal{G}_1$ for several times, an invertible matrix L over $\mathbb{k}$ could be obtained, such that

$$L \mathcal{G} L^{-1} = \begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_{12} & \cdots & \mathcal{X}_{1t} \\ 0 & \mathcal{C}_2 & \cdots & \mathcal{X}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix}$$

holds for some basic multiplicative matrices $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ over G.

(2) The reason is similar to (1) but noting that G is cosemisimple, which follows that V is a completely irreducible G-comodule. In other words, there are simple G-comodules $W_1, W_2, \cdots, W_t$ of V, such that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_t$$

holds. If we choose linear bases for $W_1, W_2, \cdots, W_t$ respectively, then simple subcoalgebras with basic multiplicative matrices $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ are obtained as before.

The transition matrix L on V from $\{v_1, v_2, \cdots, v_n\}$ to the union of those bases chosen above for $W_1, W_2, \cdots, W_t$ would satisfy the property that

$$L\mathcal{G}L^{-1} = \begin{pmatrix} \mathcal{C}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{C}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix}.$$

Now we turn to mention a binary operations on multiplicative matrices:

Lemma 2.3.7 *Suppose $\mathcal{A} = (a_{ij})_{r\times r}$ and $\mathcal{B} = (b_{ij})_{s\times s}$ be multiplicative matrices over a coalgebra H. Then*

(1) The following $rs \times rs$ (block) matrix is multiplicative over the coalgebra $H \otimes H$:

$$\mathcal{G} := \begin{pmatrix} a_{11}\otimes\mathcal{B} & \cdots & a_{1r}\otimes\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{r1}\otimes\mathcal{B} & \cdots & a_{rr}\otimes\mathcal{B} \end{pmatrix}, \quad \textit{where } a_{ij}\otimes\mathcal{B} := \begin{pmatrix} a_{ij}\otimes b_{11} & \cdots & a_{ij}\otimes b_{1s} \\ \vdots & \ddots & \vdots \\ a_{ij}\otimes b_{s1} & \cdots & a_{ij}\otimes b_{ss} \end{pmatrix};$$

(2) If H is moreover a bialgebra, then the following $rs \times rs$ matrices are both multiplicative over H:

$$\mathcal{A}\odot\mathcal{B} := \begin{pmatrix} a_{11}\mathcal{B} & \cdots & a_{1r}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{r1}\mathcal{B} & \cdots & a_{rr}\mathcal{B} \end{pmatrix} \quad \textit{and } \mathcal{A}\odot'\mathcal{B} := \begin{pmatrix} \mathcal{A}b_{11} & \cdots & \mathcal{A}b_{1s} \\ \vdots & \ddots & \vdots \\ \mathcal{A}b_{s1} & \cdots & \mathcal{A}b_{ss} \end{pmatrix}.$$

Remark 2.3.8 *The matrix $\mathcal{A}\odot\mathcal{B}$ is supposed to be called the Kronecker product of $\mathcal{A}$ and $\mathcal{B}$. Clearly, the binary operation $\odot$ could be defined on arbitrary matrices over an algebra in the same ways.*

Proof:

(1) Consider the entry $a_{ij}\otimes b_{kl}$ in the block $a_{ij}\otimes\mathcal{B}$. It is direct that

$$\Delta(a_{ij}\otimes b_{kl}) = \sum_{r'=1}^{r}\sum_{s'=1}^{s}(a_{ir'}\otimes b_{ks'})\otimes(a_{r'j}\otimes b_{s'l}).$$

Then we compute the entry in $\mathcal{G} \mathbin{\widetilde{\otimes}} \mathcal{G}$ with the same position with $a_{ij} \otimes b_{kl}$ in $\mathcal{G}$. This entry is

$$\begin{aligned}
&\sum_{s'=1}^{s}(a_{i1} \otimes b_{ks'}) \otimes (a_{1j} \otimes b_{s'l}) + \sum_{s'=1}^{s}(a_{i2} \otimes b_{ks'}) \otimes (a_{2j} \otimes b_{s'l}) \\
&+ \cdots + \sum_{s'=1}^{s}(a_{ir} \otimes b_{ks'}) \otimes (a_{rj} \otimes b_{s'l}) \\
= \ &\sum_{r'=1}^{r}\sum_{s'=1}^{s}(a_{ir'} \otimes b_{ks'}) \otimes (a_{r'j} \otimes b_{s'l}).
\end{aligned}$$

In conclusion, $\Delta(\mathcal{G}) = \mathcal{G} \mathbin{\widetilde{\otimes}} \mathcal{G}$. Another requirement $\varepsilon(\mathcal{G}) = I_{rs}$ is evident, since $\varepsilon(a_{ij} \otimes b_{kl}) = \delta_{ij}\delta_{kl}$.

(2) Note that the multiplication $m : H \otimes H \to H$ is a coalgebra map. Thus $\mathcal{A} \odot \mathcal{B} = m(\mathcal{G})$ is multiplicative.

On the other hand, we consider the bialgebra H^{op}, whose multiplication is opposite to H. It could be seen that $\mathcal{A}$ and $\mathcal{B}$ are still multiplicative over H^{op}, since H and H^{op} share the same coalgebra structures. Therefore, $\mathcal{A} \odot' \mathcal{B}$ is the Kronecker product of $\mathcal{B}$ and $\mathcal{A}$ in H^{op} and thus multiplicative.

In the end of this subsection, some evident formulas on such Kronecker products should be noted for later computations. For a matrix $\mathcal{A} = (a_{ij})_{r\times s}$ over an algebra, we denote the transpose of $\mathcal{A}$ by $\mathcal{A}^{\mathrm{T}} := (a_{ji})_{s\times r}$.

Lemma 2.3.9 *Let H be an associative algebra with an algebra anti-endomorphism S. For any matrices $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ over H, we have*

$$S(\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n)^{\mathrm{T}} = S(\mathcal{A}_n)^{\mathrm{T}}S(\mathcal{A}_{n-1})^{\mathrm{T}}\cdots S(\mathcal{A}_1)^{\mathrm{T}}$$

as long as the product $\mathcal{A}_1\mathcal{A}_2\cdots\mathcal{A}_n$ is well-defined.

Proof: The equation holds due to direct calculations.

Lemma 2.3.10 *Let H be an algebra. Denote the identity matrix of size n by I_n.*

(1) Suppose that $\mathcal{A}_{m_1\times n_1}$ and $\mathcal{B}_{m_2\times n_2}$ are matrices over H. Then

$$(\mathcal{A}\odot\mathcal{B})^{\mathrm{T}}=\mathcal{A}^{\mathrm{T}}\odot\mathcal{B}^{\mathrm{T}};$$

(2) Suppose that $\mathcal{A}_{m_1\times n_1}$, $\mathcal{B}_{m_2\times n_2}$ and $\mathcal{B}'_{n_2\times l_2}$ are matrices over H. Then

$$(\mathcal{A}\odot\mathcal{B})(I_{n_1}\odot\mathcal{B}')=\mathcal{A}\odot\mathcal{B}\mathcal{B}'.$$

(3) If H is furthermore a Hopf algebra with bijective antipode S, then for any multiplicative matrix $\mathcal{G}_{n\times n}$ over H, we have

$$S(\mathcal{G})\mathcal{G}=\mathcal{G}S(\mathcal{G})=I_n \quad \textit{and} \quad S^{-1}(\mathcal{G})^{\mathrm{T}}\mathcal{G}^{\mathrm{T}}=\mathcal{G}^{\mathrm{T}}S^{-1}(\mathcal{G})^{\mathrm{T}}=I_n.$$

Proof: Equations in (1) and (2) could be verified directly. The former equation in (3) holds due to the definition of multiplicative matrices. As for the latter one, we compute according to Lemma 2.3.9 that

$$S^{-1}(\mathcal{G})^{\mathrm{T}}\mathcal{G}^{\mathrm{T}}=S^{-1}(\mathcal{G})^{\mathrm{T}}S^{-1}(S(\mathcal{G}))^{\mathrm{T}}=S^{-1}(S(\mathcal{G})\mathcal{G})^{\mathrm{T}}=S^{-1}(I_n)^{\mathrm{T}}=I_n,$$

since S^{-1} is an algebra anti-endomorphism on H. Of course, $\mathcal{G}^{\mathrm{T}}S^{-1}(\mathcal{G})^{\mathrm{T}}=I_n$ holds similarly.

§2.3.2 Non-Trivial Primitive Matrices

In this subsection, we turn to observe properties of primitive matrices. This notion is a non-pointed analogue of primitive elements (see [27, Definition 3.2]).

Definition 2.3.11 *Let (H,Δ,ε) be a coalgebra over $\Bbbk$. Suppose $\mathcal{C}_{r\times r}$ and $\mathcal{D}_{s\times s}$ are basic multiplicative matrices over H.*

(1) An $r\times s$ matrix $\mathcal{X}$ over H is said to be $(\mathcal{C},\mathcal{D})$-primitive, if $\Delta(\mathcal{X})=\mathcal{C}\,\widetilde{\otimes}\,\mathcal{X}+\mathcal{X}\,\widetilde{\otimes}\,\mathcal{D}$;

(2) A primitive matrix $\mathcal{X}$ is said to be non-trivial, if there exist some entries of $\mathcal{X}$ does not belong to the coradical H_0.

Remark 2.3.12 It is easy to show that $\varepsilon(\mathcal{X})=0$ for any primitive matrix $\mathcal{X}$.

Using the method of coradical orthonormal idempotents, we could express any element in H_1 as a sum of entries in multiplicative and primitive matrices. For any subspace $V \subseteq H$, we denote $V \cap \mathrm{Ker}(\varepsilon)$ by V^+. The following lemma is important for us (see [27, Theorem 3.1]).

Lemma 2.3.13 *Assume that $\Bbbk$ is algebraically closed. Let H be a coalgebra, C, D be simple subcoalgebras of H and $\mathcal{C} = (c_{i'i})_{r\times r}, \mathcal{D} = (d_{jj'})_{s\times s}$ be respectively basic multiplicative matrices for C and D. Then*

(1) If $C \neq D$, then for any $w \in {}^C H_1{}^D$, there exist rs-number of $(\mathcal{C}, \mathcal{D})$-primitive matrices

$$\mathcal{W}^{(i',j')} = \left(w_{ij}^{(i',j')}\right)_{r\times s} \quad (1 \leq i' \leq r, 1 \leq j' \leq s),$$

such that $w = \sum\limits_{i=1}^{r}\sum\limits_{j=1}^{s} w_{ij}^{(i,j)}$;

(2) If $C = D$ if we choose that $\mathcal{C} = \mathcal{D}$, then for any $w \in {}^C H_1{}^C$, there exist rs-number of $(\mathcal{C}, \mathcal{C})$-primitive matrices

$$\mathcal{W}^{(i',j')} = \left(w_{ij}^{(i',j')}\right)_{r\times s} \quad (1 \leq i' \leq r, 1 \leq j' \leq s)$$

such that $w - \sum\limits_{i=1}^{r}\sum\limits_{j=1}^{s} w_{ij}^{(i,j)} \in C$.

It is clear that entries of primitive matrices must belong to $H_1 := H_0 \wedge H_0$. Moreover, there are further properties for non-trivial primitive matrices:

Proposition 2.3.14 *Let $C, D \in \mathcal{S}$, and $\mathcal{C}_{r\times r}, \mathcal{D}_{s\times s}$ be their basic multiplicative matrices, respectively. Suppose $\mathcal{X} := (x_{ij})_{r\times s}$ is a $(\mathcal{C}, \mathcal{D})$-primitive matrix. Then the followings are equivalent:*

(1) $\mathcal{X}$ is non-trivial;

(2) $x_{ij} \notin H_0$ holds for all $1 \leq i \leq r$ and $1 \leq j \leq s$;

(3) $\{x_{ij} \mid 1 \leq j \leq s\}$ are linearly independent in H_1/H_0 (the quotient space) for each $1 \leq i \leq r$, and $\{x_{ij} \mid 1 \leq i \leq r\}$ are linearly independent H_1/H_0 for each $1 \leq j \leq s$.

Proof: Denote that

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rr} \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1s} \\ d_{21} & d_{22} & \cdots & d_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ d_{s1} & d_{s2} & \cdots & d_{ss} \end{pmatrix}.$$

(1)⇒(2): Assume (2) does not hold, and that is to say $x_{ij} \in H_0$ for some i, j. The condition that $\mathcal{X}$ is $(\mathcal{C}, \mathcal{D})$-primitive provides the equation

$$\Delta(x_{ij}) = \sum_{k=1}^{r} c_{ik} \otimes x_{kj} + \sum_{l=1}^{s} x_{il} \otimes d_{lj}.$$

Since $\{c_{ik} \mid 1 \le k \le r\}$ are linearly independent, we could find some linear functions $\{f_{k'} \mid 1 \le k' \le r\}$ on H, such that $\langle f_{k'}, c_{ik} \rangle = \delta_{k',k}$ holds for any $1 \le k', k \le r$. Then we obtain for each $1 \le k \le r$ that

$$(f_k \otimes \mathrm{id}) \circ \Delta(x_{ij}) = x_{ik} + \sum_{l=1}^{s} \langle f_k, x_{il} \rangle d_{lj},$$

which follows that

$$x_{ik} = (f_k \otimes \mathrm{id}) \circ \Delta(x_{ij}) - \sum_{l=1}^{s} \langle f_k, x_{il} \rangle d_{lj} \in H_0 + D \subseteq H_0,$$

due to our assumption that $x_{ij} \in H_0$.

Obviously there is a similar process on $\{d_{lj} \mid 1 \le l \le s\}$, and we conclude that the assumption $x_{ij} \in H_0$ would follow that $x_{ik} \in H_0$ and $x_{lj} \in H_0$ hold for all $1 \le k \le r$ and $1 \le l \le s$. Consequently it is found that all the entries of $\mathcal{X}$ belong to H_0, which contradicts (1).

(2)⇒(3): For any $1 \le i \le r$, suppose $\alpha_j \in \Bbbk$ $(1 \le j \le s)$ such that $\sum\limits_{j=1}^{s} \alpha_j x_{ij} \in H_0$. Then from the following computation

$$\begin{aligned} \Delta\left(\sum_{j=1}^{s} \alpha_j x_{ij}\right) &= \sum_{j=1}^{s} \alpha_j \Delta(x_{ij}) = \sum_{j=1}^{s} \alpha_j \left(\sum_{k=1}^{r} c_{ik} \otimes x_{kj} + \sum_{l=1}^{s} x_{il} \otimes d_{lj}\right) \\ &= \sum_{k=1}^{r} c_{ik} \otimes \left(\sum_{j=1}^{s} \alpha_j x_{ij}\right) + \sum_{j,l=1}^{s} \alpha_j x_{il} \otimes d_{lj}, \end{aligned}$$

we know that

$$\sum_{j,l=1}^{s} \alpha_j x_{il} \otimes d_{lj} = \Delta\left(\sum_{j=1}^{s} \alpha_j x_{ij}\right) - \sum_{k=1}^{r} c_{ik} \otimes \left(\sum_{j=1}^{s} \alpha_j x_{ij}\right) \in H_0 \otimes H_0.$$

As a consequence, (2) and the linear independence of $\{d_{lj} \mid 1 \leq l, j \leq s\}$ follow that $\alpha_j = 0$ for all $1 \leq j \leq s$. Thus we conclude that $\{x_{ij} \mid 1 \leq j \leq s\}$ are linearly independent in H_1/H_0.

The other desired linear independence in H_1/H_0 is obtained similarly.

(3)⇒(1): This is direct.

For the remaining of this report, each element $x \in H \setminus H_0$ is said to be *non-trivial* for convenience. Moreover, an arbitrary matrix $\mathcal{X}$ over H is also said to be *non-trivial*, if some of its entries does not belong to H_0. Of course, they would be called *trivial* otherwise.

Chapter 3 An Annihilation Polynomial for the Antipode

§3.1 An Annihilation Polynomial for the Antipode

Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property and H_0 be its coradical. We denote its Loewy length by $\mathrm{Lw}(H)$. In this section, our aim is to prove the following annihilation polynomial for $S^2 \in \mathrm{End}_{\Bbbk}(H)$:

Theorem 3.1.1 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property. Denote $N := \exp(H_0) < \infty$ and $L := \mathrm{Lw}(H)$. Then*

$$(S^{2N} - \mathrm{id})^{L-1} = 0$$

holds on H.

Remark 3.1.2 For a finite-dimensional pointed Hopf algebra, the same annihilation polynomial was established by Taft and Wilson 46 years ago [46, Theorem 5]. So above theorem can be regarded as a generalization since finite-dimensional pointed Hopf algebras clearly have the dual Chevalley property.

Before the proof, an immediate but meaningful conclusion on the order of the antipode should be noted as follows, which generalizes [13, Theorem 4.4] as well as [46, Corollary 6] for the pointed case. We denote the composition order of S^2 by $\mathrm{ord}(S^2)$.

Corollary 3.1.3 *Let H, N and L be as in Theorem 3.1.1. Then*

(1) If $\mathrm{char}\,\Bbbk = 0$*, then* $\mathrm{ord}(S^2) \mid N$*;*

(2) If $\mathrm{char}\,\Bbbk = p > 0$*, then* $\mathrm{ord}(S^2) \mid \gcd(Np^M, \exp(H))$*, where M is a natural number satisfying $p^M \geq L - 1$.*

Proof:

(1) The order of S is finite, for H is finite-dimensional [37, Theorem 1]. Then S^{2N} is semisimple in characteristic 0, but unipotent. It follows that $S^{2N} = \mathrm{id}$.

(2) It is evident that $S^{2Np^M} - \mathrm{id} = (S^{2N} - \mathrm{id})^{p^M} = 0$. On the other hand, Lemma 2.1.2 implies that $\mathrm{ord}\,(S^2) \mid \exp(H)$.

We divide the proof of Theorem 3.1.1 into several steps which occupy the following subsections.

§3.1.1 Antipode on Primitive Matrices

From now on, suppose that H is a finite-dimensional Hopf algebra with the antipode S. With the help of the lemma above, we could in fact calculate the image of a certain kind of primitive matrices under S^{2n} for each positive integer n.

Lemma 3.1.4 *Let $\mathcal{C} = (c_{ij})_{r\times r}$ be a multiplicative matrix, and let $\mathcal{X} = (x_1, x_2, \cdots, x_r)^{\mathrm{T}}$ be a $(\mathcal{C}, 1)$-primitive matrix over H.*

(1) $S(\mathcal{X}) = -S(\mathcal{C})\mathcal{X}$;

(2) $S^2(\mathcal{X}) = ((S(\mathcal{C})\mathcal{X})^{\mathrm{T}} S^2(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}}$. In other words, $S^2(x_i) = \sum\limits_{k_1,k_2=1}^{r} S(c_{k_2k_1})x_{k_1}S^2(c_{ik_2})$ for each $1 \leq i \leq r$;

(3) For any positive integer n,

$$S^{2n}(\mathcal{X}) = [[S^{2n-1}(\mathcal{C})\cdots((S(\mathcal{C})\mathcal{X})^{\mathrm{T}}S^2(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}}\cdots]^{\mathrm{T}}S^{2n}(\mathcal{C})^{\mathrm{T}}]^{\mathrm{T}}.$$

Specifically, the $(i,1)$-entry of $S^{2n}(\mathcal{X})$ is

$$\begin{aligned} S^{2n}(x_i) \;\;=\;\; & \sum_{k_1,k_2,\cdots,k_{2n}=1}^{r} S\left[c_{k_2k_1}S^2(c_{k_4k_3})\cdots S^{2n-4}(c_{k_{2n-2}k_{2n-3}})S^{2n-2}(c_{k_{2n}k_{2n-1}})\right] \\ & x_{k_1}S^2\left[c_{k_3k_2}S^2(c_{k_5k_4})\cdots S^{2n-4}(c_{k_{2n-1}k_{2n-2}})S^{2n-2}(c_{ik_{2n}})\right]. \end{aligned}$$

Proof:

(1) The definition of $(\mathcal{C}, 1)$-primitive matrices means that $\Delta(\mathcal{X}) = \mathcal{C} \,\widetilde{\otimes}\, \mathcal{X} + \mathcal{X} \,\widetilde{\otimes}\, 1$. We map $m \circ (S \otimes \mathrm{id})$ onto this equation and obtain

$$0 = \varepsilon(\mathcal{X}) = S(\mathcal{C})\mathcal{X} + S(\mathcal{X}),$$

which follows $S(\mathcal{X}) = -S(\mathcal{C})\mathcal{X}$ immediately.

(2) According to (1) and lemma 2.3.9, it is direct that

$$\begin{aligned} S^2(\mathcal{X}) &= S(-S(\mathcal{C})\mathcal{X}) = -(S(\mathcal{X})^{\mathrm{T}} S^2(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}} = -((-S(\mathcal{C})\mathcal{X})^{\mathrm{T}} S^2(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}} \\ &= ((S(\mathcal{C})\mathcal{X})^{\mathrm{T}} S^2(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}}. \end{aligned}$$

(3) We prove it by induction on n. Assume that the equation holds for any multiplicative matrix $\mathcal{C}$ and $(\mathcal{C},1)$-primitive matrix $\mathcal{X}$ in case $n-1$. Note that $S^2(\mathcal{C})$ is multiplicative and $S^2(\mathcal{X})$ is $(S^2(\mathcal{C}),1)$-primitive, because S^2 is a coalgebra endomorphism (as well as an isomorphism) on H. Then

$$\begin{aligned} & S^{2n}(x_i) = S^{2n-2}(S^2(x_i)) \\ = & \sum_{k_3,k_4,\cdots,k_{2n}=1}^{r} S\left[S^2(c_{k_4k_3})\cdots S^{2n-4}(c_{k_{2n-2}k_{2n-3}})S^{2n-2}(c_{k_{2n}k_{2n-1}})\right] \\ & S^2(x_{k_3})S^2\left[S^2(c_{k_5k_4})\cdots S^{2n-4}(c_{k_{2n-1}k_{2n-2}})S^{2n-2}(c_{ik_{2n}})\right] \\ = & \sum_{k_3,k_4,\cdots,k_{2n}=1}^{r} S\left[S^2(c_{k_4k_3})\cdots S^{2n-4}(c_{k_{2n-2}k_{2n-3}})S^{2n-2}(c_{k_{2n}k_{2n-1}})\right] \\ & \left(\sum_{k_1,k_2=1}^{r} S(c_{k_2k_1})x_{k_1}S^2(c_{k_3k_2})\right) S^2\left[S^2(c_{k_5k_4})\cdots S^{2n-4}(c_{k_{2n-1}k_{2n-2}})S^{2n-2}(c_{ik_{2n}})\right] \\ = & \sum_{k_1,k_2,k_3,k_4,\cdots,k_{2n}=1}^{r} S\left[S^2(c_{k_4k_3})\cdots S^{2n-4}(c_{k_{2n-2}k_{2n-3}})S^{2n-2}(c_{k_{2n}k_{2n-1}})\right] \\ & S(c_{k_2k_1})x_{k_1}S^2(c_{k_3k_2})S^2\left[S^2(c_{k_5k_4})\cdots S^{2n-4}(c_{k_{2n-1}k_{2n-2}})S^{2n-2}(c_{ik_{2n}})\right] \\ = & \sum_{k_1,k_2,k_3,k_4,\cdots,k_{2n}=1}^{r} S\left[c_{k_2k_1}S^2(c_{k_4k_3})\cdots S^{2n-4}(c_{k_{2n-2}k_{2n-3}})S^{2n-2}(c_{k_{2n}k_{2n-1}})\right] \\ & x_{k_1}S^2\left[c_{k_3k_2}S^2(c_{k_5k_4})\cdots S^{2n-4}(c_{k_{2n-1}k_{2n-2}})S^{2n-2}(c_{ik_{2n}})\right], \end{aligned}$$

which is exactly the required equation in case n.

It is suggested in Lemma 3.1.4 that the mapping S^2 on $(\mathcal{C},1)$-primitive matrices is somehow similar to a "conjugate action by by $\mathcal{C}$". We show next that such an action has finite order when H has finite exponent.

Lemma 3.1.5 *Let $\mathcal{C}=(c_{ij})_{r\times r}$ be a basic multiplicative matrix of a simple subcoalgebra*

$C \in \mathcal{S}$ of H. Assume that $N := \exp(H) < \infty$. Then for $1 \le i, j \le r$,

$$\begin{aligned}&\sum_{k_2,k_3,\cdots,k_{2N}=1}^{r} c_{k_2j} S^2(c_{k_4k_3}) \cdots S^{2N-4}(c_{k_{2N-2}k_{2N-3}}) S^{2N-2}(c_{k_{2N}k_{2N-1}}) \\ &\qquad \otimes c_{k_3k_2} S^2(c_{k_5k_4}) \cdots S^{2N-4}(c_{k_{2N-1}k_{2N-2}}) S^{2N-2}(c_{ik_{2N}}) \\ &= \delta_{ij}(1 \otimes 1),\end{aligned}$$

which is the (i,j)-entry of the identity matrix $\mathcal{I}_r \mathbin{\widetilde{\otimes}} \mathcal{I}_r \in \mathcal{M}_r(H \otimes H)$.

Proof: Firstly it is known that $\exp(H^{*\mathrm{op}}) = \exp(H) = N$ according to [12, Proposition 2.2(4) and Corollary 2.6]. Here the antipode of dual Hopf algebra H^* is also denoted as S, and then the antipode of $H^{*\mathrm{op}}$ is actually S^{-1}.

Let us prove the equation by taking values of the function $f \otimes g \in H^* \otimes H^*$ for arbitrary $f, g \in H^*$. Note that

$$\Delta_{2N}(c_{ij}) = \sum_{k_2,k_3,\cdots,k_{2N}=1}^{r} c_{ik_{2N}} \otimes c_{k_{2N}k_{2N-1}} \otimes \cdots \otimes c_{k_3k_2} \otimes c_{k_2j}.$$

The value of $f \otimes g$ on the left side of the required equation is then

$$\begin{aligned}&\sum_{k_2,k_3,\cdots,k_{2N}=1}^{r} \langle f, c_{k_2j} S^2(c_{k_4k_3}) \cdots S^{2N-4}(c_{k_{2N-2}k_{2N-3}}) S^{2N-2}(c_{k_{2N}k_{2N-1}}) \rangle \\ &\qquad \langle g, c_{k_3k_2} S^2(c_{k_5k_4}) \cdots S^{2N-4}(c_{k_{2N-1}k_{2N-2}}) S^{2N-2}(c_{ik_{2N}}) \rangle \\ &= \sum_{k_2,k_3,\cdots,k_{2N}=1}^{r} \langle g_{(N)}, S^{2N-2}(c_{ik_{2N}}) \rangle \langle f_{(N)}, S^{2N-2}(c_{k_{2N}k_{2N-1}}) \rangle \\ &\qquad \langle g_{(N-1)}, S^{2N-4}(c_{k_{2N-1}k_{2N-2}}) \rangle \langle f_{(N-1)}, S^{2N-4}(c_{k_{2N-2}k_{2N-3}}) \rangle \\ &\qquad \cdots \langle g_{(2)}, S^2(c_{k_5k_4}) \rangle \langle f_{(2)}, S^2(c_{k_4k_3}) \rangle \langle g_{(1)}, c_{k_3k_2} \rangle \langle f_{(1)}, c_{k_2j} \rangle \\ &= \langle \sum S^{2N-2}(g_{(N)}) S^{2N-2}(f_{(N)}) S^{2N-4}(g_{(N-1)}) S^{2N-4}(f_{(N-1)}) \\ &\qquad \cdots S^2(g_{(2)}) S^2(f_{(2)}) g_{(1)} f_{(1)}, c_{ij} \rangle \\ &= \Big\langle \sum S^{2N-2}(g_{(N)} f_{(N)}) S^{2N-4}(g_{(N-1)} f_{(N-1)}) \cdots S^2(g_{(2)} f_{(2)}) g_{(1)} f_{(1)}, c_{ij} \Big\rangle \\ &= \big\langle m_N^{*\mathrm{op}} \circ \big(\mathrm{id} \otimes (S^{-1})^{-2} \otimes \cdots \otimes (S^{-1})^{-2N+2}\big) \circ \Delta_N^*(gf), c_{ij} \big\rangle \\ &= \langle gf, 1 \rangle \langle \varepsilon, c_{ij} \rangle \ = \ \delta_{ij} \langle f, 1 \rangle \langle g, 1 \rangle \ = \ \langle f \otimes g, \delta_{ij}(1 \otimes 1) \rangle.\end{aligned}$$

The proof is now complete since f and g are arbitrary linear functions.

The following proposition is a conclusion of two lemmas above.

Proposition 3.1.6 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property. Denote $N := \exp(H_0) < \infty$. Then for any basic multiplicative matrix $\mathcal{C}$ and any $(\mathcal{C}, 1)$-primitive matrix $\mathcal{X}$, we have $S^{2N}(\mathcal{X}) = \mathcal{X}$.*

Proof: Denote $\mathcal{C} = (c_{ij})_{r\times r}$ and $\mathcal{X} = (x_1, x_2, \cdots, x_r)^{\mathrm{T}}$. Since $\mathcal{C}$ is a multiplicative matrix over H_0 as well, Lemma 3.1.4(3) and Lemma 3.1.5 imply that for any $1 \leq i \leq r$,

$$\begin{aligned} S^{2N}(x_i) &= \sum_{k_1,k_2,\cdots,k_{2N}=1}^{r} S\left[c_{k_2k_1}S^2(c_{k_4k_3})\cdots S^{2N-4}(c_{k_{2N-2}k_{2N-3}})S^{2N-2}(c_{k_{2N}k_{2N-1}})\right] \\ &\quad x_{k_1}S^2\left[c_{k_3k_2}S^2(c_{k_5k_4})\cdots S^{2N-4}(c_{k_{2N-1}k_{2N-2}})S^{2N-2}(c_{ik_{2N}})\right] \\ &= \sum_{k_1=1}^{r} \delta_{ik_1}S(1)x_{k_1}1 \;=\; x_i. \end{aligned}$$

That is to say, $S^{2N}(\mathcal{X}) = \mathcal{X}$.

In Subsection §2.2, we have made the convention that a family of coradical orthonormal idempotents $\{e_C\}_{C\in\mathcal{S}}$ is always given. Now recall that according to Proposition 2.2.2(4), the left coideal ${H_1}^1$ could be decomposed as a direct sum:

$${H_1}^1 = \bigoplus_{C\in\mathcal{S}} {}^C{H_1}^1.$$

On the other hand, if we assume that $\Bbbk$ is algebraically closed, then every simple subcoalgebra $C \in \mathcal{S}$ has a basic multiplicative matrix $\mathcal{C}$, and thus each element in ${}^C{H_1}^1$ is a sum of some entries in $(\mathcal{C}, 1)$-primitive matrices and some elements in H_0. This is followed from Lemma 2.3.13. As a consequence, we could obtain the following corollary that the transformation S^{2N} on ${H_1}^1$ equals to identity in this case.

Corollary 3.1.7 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over an algebraically closed field $\Bbbk$. Denote $N := \exp(H_0) < \infty$. Then $S^{2N}\mid_{{H_1}^1} = \mathrm{id}_{{H_1}^1}$.*

Proof: This is because S^{2N} keeps any basic multiplicative matrix $\mathcal{C}$ as well as any $(\mathcal{C}, 1)$-primitive matrix.

§3.1.2 Antipode on H_1

Our first goal in this subsection is to give the following proposition.

Proposition 3.1.8 *Let H be a Hopf algebra with the dual Chevalley property. Then $H_1 = {H_1}^1 \cdot H_0$.*

Proof: It seems that this is known and we give an approach to prove it for safety. At first, using comultiplication and multiplication one can show that there is a right H_0-Hopf module structure on H_1/H_0. Secondly, it is straightforward to show that the space of coinvariants of this right Hopf module is exactly $({H_1}^1 + H_0)/H_0$. At last, we can apply the fundamental theorem of Hopf modules ([25, Propostion 1]) to get the result.

Proposition 3.1.8 provides that ${H_1}^1$ and H_0 generate H_1 by multiplication. Then we can continue to investigate whether S^{2N} could be identified with the identity map on H_1 or not.

Proposition 3.1.9 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over an algebraically closed field $\Bbbk$. Denote $N := \exp(H_0) < \infty$. Then $S^{2N}\mid_{H_1} = \mathrm{id}_{H_1}$.*

Proof: We already know that the algebra morphism S^{2N} restricted to subspaces H_0 or ${H_1}^1$ is supposed to be the identity (Corollaries 2.1.2 and 3.1.7). Now that Proposition 3.1.8 gives $H_1 = {H_1}^1 \cdot H_0$, consequently $S^{2N}\mid_{H_1} = \mathrm{id}_{H_1}$ holds in this case.

§3.1.3 Proof of Theorem 3.1.1

We remark firstly a classical result on coradical filtrations in [46, Proposition 4], which holds for non-pointed coalgebras as well.

Lemma 3.1.10 (*[46, Proposition 4]*) *Let H be an arbitrary coalgebra. Let i be a positive integer and $\varphi : H \to H$ be a coalgebra endomorphism, such that $(\varphi - \mathrm{id})(H_j) \subseteq H_{j-1}$ for all $0 \leq j \leq i$. Then $(\varphi - \mathrm{id})(H_{i+1}) \subseteq H_i$.*

Proof of Theorem 3.1.1 when $\Bbbk$ is algebraically closed. Combining Lemma 3.1.10 and Proposition 3.1.9, it is clear that the statement of Theorem 3.1.1 holds when the base field $\Bbbk$ is algebraically closed. □

Next we want to prove Theorem 3.1.1 using the method of field extensions. For the purpose, it is necessary to show that $\exp(H_0)$ and $\mathrm{Lw}(H)$ are invariant under field extensions due to the dual Chevalley property. The following lemma seems also known.

Lemma 3.1.11 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over $\mathbb{k}$. Suppose K is a field extension of $\mathbb{k}$, and $H \otimes K$ denotes the extended finite-dimensional K-Hopf algebra. Then*

(1) $H \otimes K$ has the dual Chevalley property with coradical $H_0 \otimes K$;

(2) The coradical filtration of $H \otimes K$ is $\{H_n \otimes K\}_{n\geq 0}$;

(3) Moreover $\exp(H_0 \otimes K) = \exp(H_0)$ and $\mathrm{Lw}(H \otimes K) = \mathrm{Lw}(H)$.

Proof: The definition of $H \otimes K$ could be found in [40, Exercise 7.1.8] for example.

(1) Regard $H_0 \otimes K \hookrightarrow H \otimes K$ as a subspace canonically, which is in fact a Hopf subalgebra over K. Meanwhile, $H_0 \otimes K$ is cosemisimple because H_0 is ([22, Lemma 1.3]). Thus $H_0 \otimes K$ is contained in the coradical $(H \otimes K)_0$.

On the other hand, the coalgebra structure of $H \otimes K$ follows naturally that $\{\wedge^n(H_0 \otimes K)\}_{n\geq 0}$ is a coalgebra filtration of $H \otimes K$, which implies that $H_0 \otimes K \supseteq (H \otimes K)_0$ by [45, Proposition 11.1.1].

As a conclusion, the coradical $(H \otimes K)_0 = H_0 \otimes K$, and the dual Chevalley property for $H \otimes K$ could be obtained since $H_0 \otimes K$ is closed under the multiplication evidently.

(2) This could be inferred with $(H \otimes K)_0 = H_0 \otimes K$ as well as

$$(H_0 \otimes K) \wedge (H_n \otimes K) = (H_0 \wedge H_n) \otimes K$$

for each $n \geq 0$.

(3) The equation $\mathrm{Lw}(H \otimes K) = \mathrm{Lw}(H)$ follows immediately from (2). The exponent is invariant under field extensions is stated in [12, Proposition 2.2(8)].

Proof of Theorem 3.1.1 for general $\mathbb{k}$. Let $K := \overline{\mathbb{k}}$ be the algebraic closure of $\mathbb{k}$. Then $H \otimes K$ is a finite-dimensional Hopf algebra over the algebraically closed field K. We know from Lemma 3.1.11 that $H \otimes K$ has the dual Chevalley property, $\exp(H_0 \otimes K) = N$ and $\mathrm{Lw}(H \otimes K) = L$ according to our notations.

Therefore, if we denote the antipode and identity map of $H \otimes K$ by $S_{H\otimes K}$ and $\mathrm{id}_{H\otimes K}$ respectively, then we already know that

$$({S_{H\otimes K}}^{2N} - \mathrm{id}_{H\otimes K})^{L-1} = 0 \tag{3.1}$$

holds in $\mathrm{End}_K(H \otimes K)$. Now we apply Equation (3.1) on element $h \otimes 1_K \in H \otimes K$ for $h \in H$, and obtain

$$(S^{2N} - \mathrm{id})^{L-1}(h) \otimes 1_K = 0.$$

This implies that $(S^{2N} - \mathrm{id})^{L-1} = 0$ holds in $\mathrm{End}_{\Bbbk}(H)$ too. □

§3.2 Two Applications

In this section, we want to give two applications of our main result. Both of them concern with an important gauge invariant which is called the quasi-exponent.

§3.2.1 A Generalization

In [13], they introduced a gauge invariant called the quasi-exponent for finite-dimensional Hopf algebras, which has similar properties with the exponent but always finite. Precisely, the *quasi-exponent* of a finite-dimensional Hopf algebra H, denoted by $\mathrm{qexp}(H)$, is defined to be the least positive integer n such that the nth power of the Drinfeld element in $D(H)$ is unipotent ([13, Definition 2.1]).

When H is moreover pointed over $\mathbb{C}$, there is a description of $\mathrm{qexp}(H)$ in the following Etingof-Gelaki's theorem (see [13, Theorem 4.6]).

Theorem 3.2.1 *Let H be a finite-dimensional pointed Hopf algebra over $\mathbb{C}$. Then* $\mathrm{qexp}(H) = \exp(G(H))$.

Its proof is based on following lemma which is a combination of [13, Lemma 4.2] and [13, Proposition 4.3].

Lemma 3.2.2 *Let H be a finite-dimensional Hopf algebra over $\mathbb{C}$.*

(1) If H is filtered and let $\mathrm{gr}H$ be its associated graded Hopf algebra. Then $\mathrm{qexp}(H) = \mathrm{qexp}(\mathrm{gr}H)$.

(2) Assume that H is graded with zero part $H_{(0)}$. Then

$$\mathrm{qexp}(H) = \mathrm{lcm}(\mathrm{qexp}(H_{(0)}), \mathrm{ord}(S^2)),$$

where lcm *denotes the least common multiple.*

With the help of the lemma above and Corollary 3.1.3 (1), we generalize Theorem 3.2.1 to finite-dimensional Hopf algebras with the dual Chevalley property.

Theorem 3.2.3 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over $\mathbb{C}$. Then* $\mathrm{qexp}(H) = \exp(H_0)$.

Proof: As mentioned in Lemma 2.1.1, the dual Chevalley property implies that H is a filtered Hopf algebra with the filtration $\{H_n\}_{n\geq 0}$. Thus

$$\mathrm{qexp}(H) = \mathrm{qexp}(\mathrm{gr}H)$$

holds by Lemma 3.2.2 (1). Meanwhile, Lemma 3.2.2 (2) provides the equation

$$\mathrm{qexp}(\mathrm{gr}H) = \mathrm{lcm}(\mathrm{qexp}((\mathrm{gr}H)_{(0)}), \mathrm{ord}({S_{\mathrm{gr}H}}^2)).$$

In fact, it could be shown that $\mathrm{ord}({S_{\mathrm{gr}H}}^2) = \mathrm{ord}(S^2)$ in our situation. Precisely, according to the definition of the associated graded Hopf algebra (e.g. [45, Chapter 11]), $\mathrm{ord}({S_{\mathrm{gr}H}}^2) \leq \mathrm{ord}(S^2)$ holds evidently. On the other hand, if we assume $\mathrm{ord}({S_{\mathrm{gr}H}}^2) = M < \infty$, the definition of $\mathrm{gr}H$ follows that $(S^{2M} - \mathrm{id})^{\mathrm{Lw}(H)} = 0$ on H. Thus $S^{2M} = \mathrm{id}$ holds as well, since S is semisimple in characteristic 0.

As a conclusion, we have

$$\begin{aligned} \mathrm{qexp}(H) &= \mathrm{qexp}(\mathrm{gr}H) = \mathrm{lcm}(\mathrm{qexp}((\mathrm{gr}H)_{(0)}), \mathrm{ord}({S_{\mathrm{gr}H}}^2)) \\ &= \mathrm{lcm}(\mathrm{qexp}(H_0), \mathrm{ord}(S^2)) = \mathrm{qexp}(H_0), \end{aligned}$$

as long as we note that the zero part $(\mathrm{gr}H)_{(0)} = H_0$. Besides, the last equation holds because of Corollary 3.1.3 (1) by noting that $\mathrm{qexp}(H_0) = \exp(H_0)$.

Since the quasi-exponent is a gauge invariant, we have the following corollary which was known for pointed Hopf algebras (see [13, Corollary 4.8]):

Corollary 3.2.4 *Let H and H' be two finite-dimensional Hopf algebras with the dual Chevalley property over $\mathbb{C}$. If they are twist equivalent, then* $\exp(H_0) = \exp({H'}_0)$.

Note that the quasi-exponent is invariant under taking duals of finite-dimensional Hopf algebras. Thus a dual version of Theorem 3.2.3 could be given, which holds for Hopf algebras with the Chevalley property. Recall that a finite-dimensional Hopf algebra H is said to have the *Chevalley property*, if the tensor product of any two simple H-modules is semisimple, or, equivalently, if the radical of H is a Hopf ideal. And it is clear that H has the Chevalley property if and only if H^* has the dual Chevalley property. The dual version of Theorem 3.2.3 could be regarded as a generalization of [13, Propostion 4.13]:

Corollary 3.2.5 *Let H be a finite-dimensional Hopf algebra with the Chevalley property over $\mathbb{C}$, and let $H/\mathrm{Rad}(H)$ be its semisimple quotient. Then* $\mathrm{qexp}(H) = \exp(H/\mathrm{Rad}(H))$.

§3.2.2 Quasi-Exponent of a Pivotal Hopf Algebra

Chapter 4 Invariance and Finiteness for the Exponent

§4.1 Exponents and Their Invariance

We start by recalling the definitions and basic properties of exponents. Let $(H, m, u, \Delta, \varepsilon)$ be a Hopf algebra with bijective antipode S over a field $\Bbbk$. Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for $h \in H$ is always used. We also denote following $\Bbbk$-linear maps for convenience:

$$\begin{aligned} m_n : \quad & H^{\otimes n} \to H, \quad h_1 \otimes h_2 \otimes \cdots \otimes h_n \mapsto h_1 h_2 \cdots h_n, \\ \Delta_n : \quad & H \to H^{\otimes n}, \quad h \mapsto \sum h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n)} \end{aligned}$$

and the *Sweedler power* $[n] := m_n \circ \Delta_n$ for each positive integer n. The two notions of *exponent* ([17, 18] and [12]) of H are defined respectively as:

$$\begin{aligned} \exp(H) &:= \min\{n \geq 1 \mid m_n \circ (\mathrm{id} \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n = u \circ \varepsilon\}, \\ \exp_0(H) &:= \min\{n \geq 1 \mid [n] := m_n \circ \Delta_n = u \circ \varepsilon\}. \end{aligned}$$

Moreover in this report, we always make following conventions to cover infinite cases:

- $\min \varnothing = \infty$;
- Each positive integer divides ∞, and ∞ divides ∞;
- Any positive integer (or ∞) divided by ∞ must be ∞ itself.

It is immediate that whenever finite and infinite, $\exp_0(H) = \exp(H)$ when H is involutory.

Now we list invariance properties for $\exp(H)$ below, which are introduced in [12]. Recall that an element $J \in H \otimes H$ is called a *left twist* for H ([9]), if J is invertible satisfying

$$(J \otimes 1) \cdot (\Delta \otimes \mathrm{id})(J) = (1 \otimes J) \cdot (\mathrm{id} \otimes \Delta)(J),$$

A Hopf algebra $(H^J, m, u, \Delta^J, \varepsilon)$ with antipode S^J could be constructed afterwards. Besides, the Drinfeld double of H is denoted by $D(H)$.

Lemma 4.1.1 *([12]) Let H be a finite-dimensional Hopf algebra. Then*

(1) If $\exp(H) < \infty$, then $m_n \circ (\mathrm{id} \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n = u \circ \varepsilon$ if and only if $\exp(H) \mid n$;

(2) $\exp(H^*) = \exp(H)$;

(3) $\exp(H^{\mathrm{op}}) = \exp(H^{\mathrm{cop}}) = \exp(H)$;

(4) If H' is a Hopf subalgebra or quotient of H, then $\exp(H') \mid \exp(H)$;

(5) Let H' be another Hopf algebra. Then $\exp(H \otimes H') = \mathrm{lcm}(\exp(H), \exp(H'))$;

(6) Let J be a (left or right) twist for H. Then $\exp(H^J) = \exp(H)$;

(7) $\exp(D(H)) = \exp(H)$.

Remark 4.1.2 *Items (1), (2), (4) and (5) are contained in [12, Proposition 2.2]. Item (3) is [12, Corollaray 2.6], while (6) and (7) are respectively [12, Theorem 3.3 and Corollary 3.4].*

Lemma 4.1.3 *([17] and [21]) Let H be a finite-dimensional Hopf algebra. Then*

(1) If $\exp_0(H) < \infty$, then $[n] := m_n \circ \Delta_n = u \circ \varepsilon$ if and only if $\exp_0(H) \mid n$;

(2) $\exp_0(H^*) = \exp_0(H)$;

(3) $\exp_0(H^{\mathrm{op}}) = \exp_0(H^{\mathrm{cop}}) = \exp_0(H)$;

(4) If H' is a Hopf subalgebra or quotient of H, then $\exp_0(H') \mid \exp_0(H)$;

(5) Let H' be another Hopf algebra. Then $\exp_0(H \otimes H') = \mathrm{lcm}(\exp_0(H), \exp_0(H'))$.

Remark 4.1.4 *Items (1), (2), (4) and (5) are direct and found in [17], and (3) is [21, Proposition 2.2(2)].*

We end this section by verifying that $\exp_0(H)$ is invariant under twisting and taking the Drinfeld double as well.

Proposition 4.1.5 *Let H be a finite-dimensional Hopf algebra. Then*

(6) Let J be a (left or right) twist for H. Then $\exp_0(H^J) = \exp_0(H)$;

(7) $\exp_0(D(H)) = \exp_0(H)$.

Proof:

(6) Let $J = \sum_i J_i \otimes J^i \in H \otimes H$ be a left twist for H, and its inverse is denoted by $J^{-1} = \sum_i (J^{-1})_i \otimes (J^{-1})^i$. We remark that the definition of S^J provides

$$S^J(h) = \sum_{k,l} J_k S(J^k) S(h) S((J^{-1})_l)(J^{-1})^l \quad (\forall h \in H).$$

In order to compute the Sweedler power on H^J, we use the equation in [19, Lemma 2.5] that

$$h^{[n+1]} = m_{n+2} \circ (\mathrm{id} \otimes \Delta_n^J \otimes \mathrm{id}) \left[(1 \otimes J)(1 \otimes \Delta(h))(J^{-1} \otimes 1)\right] \quad (\forall h \in H)$$

for each positive integer n. Meanwhile, the equation also holds when $n = 0$ with conventions $m_0 = u$ and $\Delta_0^J = \varepsilon$.

Assume $\exp_0(H^J) := N_0 < \infty$. It follows that $m_{N_0-1} \circ \Delta_{N_0-1}^J = S^J$ on H^J, because they are both the convolution inverses of $\mathrm{id} \in \mathrm{Hom}_{\Bbbk}(H^J, H)$. Then for all $h \in H$, we make computations:

$$\begin{aligned}
h^{[N_0]} &= m_{N_0+1} \circ (\mathrm{id} \otimes \Delta_{N_0-1}^J \otimes \mathrm{id}) \left[(1 \otimes J)(1 \otimes \Delta(h))(J^{-1} \otimes 1)\right] \\
&= m_3 \circ (\mathrm{id} \otimes S^J \otimes \mathrm{id}) \left[(1 \otimes J)(1 \otimes \Delta(h))(J^{-1} \otimes 1)\right] \\
&= \sum m_3 \circ (\mathrm{id} \otimes S^J \otimes \mathrm{id}) \left[\sum_{i,j} (J^{-1})_j \otimes J_i h_{(1)} (J^{-1})^j \otimes J^i h_{(2)}\right] \\
&= \sum_{i,j} (J^{-1})_j S^J \left(J_i h_{(1)} (J^{-1})^j\right) J^i h_{(2)} \\
&= \sum_{i,j,k,l} (J^{-1})_j J_k S(J^k) S\left(J_i h_{(1)} (J^{-1})^j\right) S((J^{-1})_l)(J^{-1})^l J^i h_{(2)} \\
&= \sum_{i,j,k,l} (J^{-1})_j J_k S\left((J^{-1})_l J_i h_{(1)} (J^{-1})^j J^k\right) (J^{-1})^l J^i h_{(2)}.
\end{aligned}$$

Note that

$$\sum_{j,k}(J^{-1})_j J_k \otimes (J^{-1})^j J^k = \sum_{i,l}(J^{-1})_l J_i \otimes (J^{-1})^l J^i = J^{-1}J = 1 \otimes 1.$$

Thus

$$\begin{aligned} h^{[N_0]} &= \sum_{i,j,k,l}(J^{-1})_j J_k S\left((J^{-1})_l J_i h_{(1)} (J^{-1})^j J^k\right)(J^{-1})^l J^i h_{(2)} \\ &= \sum S(h_1)h_{(2)} \;=\; \varepsilon(h)1. \end{aligned}$$

That is to say $[N_0] = u \circ \varepsilon$, and thus $\exp_0(H) \mid \exp_0(H^J)$ by Lemma 4.1.3(1). However, it is known that J^{-1} is a left twist for H^J and $H = (H^J)^{J^{-1}}$. Therefore, $\exp_0(H^J) \mid \exp_0(H)$ holds similarly. As a consequence, we have $\exp_0(H^J) = \exp_0(H)$. Of course, the process above also shows that $\exp_0(H^J) = \infty$ if and only if $\exp_0(H) = \infty$.

The property holds when J is a right twist as well, since J^{-1} is a left twist for H at that time.

(7) As mentioned in [21, Section 2], this could be inferred by the following isomorphism between Hopf algebras introduced in [8]:

$$D(H) = (H^{*\mathrm{cop}} \otimes H)_\sigma,$$

where $\sigma : (f \otimes h, f' \otimes h') \mapsto \langle f, 1\rangle\langle f', h\rangle\langle \varepsilon, h'\rangle$ is a left 2-cocycle for $H^{*\mathrm{cop}} \otimes H$, and $(H^{*\mathrm{cop}} \otimes H)_\sigma$ denotes the corresponding 2-cocycle deformation.

Specifically, according to Lemma 4.1.3 and the duality between (left) 2-cocycles and twists, we could know that

$$\begin{aligned} \exp_0(D(H)) &= \exp_0((H^{*\mathrm{cop}} \otimes H)_\sigma) \;=\; \exp_0(((H^{*\mathrm{cop}} \otimes H)_\sigma)^*) \\ &= \exp_0(((H^{*\mathrm{cop}} \otimes H)^*)^{\sigma^*}) \;=\; \exp_0((H^{*\mathrm{cop}} \otimes H)^*) \\ &= \exp_0(H^{*\mathrm{cop}} \otimes H) \;=\; \mathrm{lcm}(\exp_0(H^{*\mathrm{cop}}), \exp_0(H)) \\ &= \mathrm{lcm}(\exp_0(H^*), \exp_0(H)) \;=\; \mathrm{lcm}(\exp_0(H), \exp_0(H)) \\ &= \exp_0(H), \end{aligned}$$

where σ^* denotes the left twist for $(H^{*\mathrm{cop}} \otimes H)^*$ dual to σ.

§4.2 Exponent and Quasi-Exponent of the Pivotal Semidirect Product $H \rtimes \Bbbk\langle S^2 \rangle$

In this subsection, we concentrate on a kind of semidirect product of a Hopf algebra H, which is denoted by $H \rtimes \Bbbk\langle S^2 \rangle$. It is a pivotal Hopf algebra containing H and appears in some researches such as [42]. Thus we think that it is interesting to investigate the exponent and quasi-exponent of $H \rtimes \Bbbk\langle S^2 \rangle$. Let us begin by recalling the corresponding concepts.

In fact, $\exp(H) = \exp_0(H)$ holds as long as H is pivotal. Recall that a Hopf algebra H is said to be *pivotal*, if there exists a grouplike element $g \in H$ such that

$$\forall h \in H,\ S^2(h) = ghg^{-1}.$$

Such a grouplike element g is called a *pivotal element* of H. The claim above could be implied by the following lemma:

Lemma 4.2.1 *([42, Lemma 4.2]) Let H be a Hopf algebra with a grouplike element $g \in H$. Define φ to be the inner automorphism on H determined by g. Then*

$$(hg)^{[n]} = \sum h_{(1)}\varphi(h_{(2)}) \cdots \varphi^{n-1}(h_{(n)})g^n$$

holds for each $n \geq 1$ and all $h \in H$.

Corollary 4.2.2 *Let H be a pivotal Hopf algebra. Then* $\exp_0(H) = \exp(H)$.

Proof: Suppose that the grouplike element g satisfies $S^2(h) = ghg^{-1}$ for all $h \in H$, and then the automorphism S^{-2} is inner and determined by g^{-1}. It is provided by Lemma 4.2.1 that

$$(hg^{-1})^{[n]} = \sum h_{(1)}S^{-2}(h_{(2)}) \cdots S^{-2n+2}(h_{(n)})g^{-n} \quad (\forall h \in H,\ \forall n \geq 1).$$

Note that clearly the (multiplication) order of g divides $\operatorname{lcm} \exp_0(H), \exp(H)$. Therefore

$$[n] = u \circ \varepsilon \quad \text{if and only if} \quad m_n \circ (\mathrm{id} \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n = u \circ \epsilon,$$

which means that $\exp_0(H) = \exp(H)$.

A known result is that any finite-dimensional H could be embedded into a pivotal Hopf algebra, namely a semidirect product $H \rtimes \Bbbk\langle S^2 \rangle$. (See Step 2 in the proof of [42, Theorem 4.1] for example, but the details would be recalled in the following definition.) Thus, $\exp(H)$ and $\exp_0(H)$ are bounded by the equal exponents of $H \rtimes \Bbbk\langle S^2 \rangle$, especially when the latter is finite.

Let us recall the definition. First it is clear from [37, Theorem 1] that the subgroup generated by $S^2 \in \mathrm{End}_{\Bbbk}(H)$ is finite, which is denoted by $\langle S^2 \rangle$ in this report.

Definition 4.2.3 *Let H be a finite-dimensional Hopf algebra. The semidirect product (or, smash product) $H \rtimes \Bbbk\langle S^2 \rangle$ of H is defined through*

- $H \rtimes \Bbbk\langle S^2 \rangle = H \otimes \Bbbk\langle S^2 \rangle$ *as a coalgebra;*
- *The multiplication is that* $(h \rtimes S^{2i})(k \rtimes S^{2j}) := hS^{2i}(k) \rtimes S^{2(i+j)}$ *for all* $h, k \in H$ *and* $i, j \in \mathbb{Z}$*;*
- *The unit element is* $1 \rtimes \mathrm{id}$*;*
- *The antipode is then* $S_{H \rtimes \Bbbk\langle S^2 \rangle} : h \rtimes S^{2i} \mapsto S^{-2i+1}(h) \rtimes S^{-2i}$.

Note that this is indeed a pivotal Hopf algebra (e.g. [33, Theorem 2.13] and [44, Proposition 2.3(1)]) with a pivotal element $1 \rtimes S^2$, and $H \cong H \rtimes \mathrm{id} \hookrightarrow H \rtimes \Bbbk\langle S^2 \rangle$ is an inclusion of Hopf algebras.

The remaining of this section is devoted to establish a formula for $\exp(H \rtimes \Bbbk\langle S^2 \rangle)$. For this purpose, following notation should be given, which could be regarded as special case of *twisted exponents* introduced in [41, Definition 3.1] and [36, Definition 3.1].

Notation 4.2.4 *Let H be a finite-dimensional Hopf algebra. For any $i \in \mathbb{Z}$, we denote*

$$\exp_{2i}(H) := \min\{n \geq 1 \mid m_n \circ (\mathrm{id} \otimes S^{2i} \otimes \cdots \otimes S^{2(n-1)i}) \circ \Delta_n = u \circ \varepsilon\}.$$

Of course $\exp(H)$ is exactly $\exp_{-2}(H)$ with the notation.

The formula for $\exp(H \rtimes \Bbbk\langle S^2 \rangle)$ would be established by steps.

Lemma 4.2.5 *Let H be a finite-dimensional Hopf algebra. Then*

(1) We have

$$\exp(H \rtimes \Bbbk\langle S^2\rangle) = \operatorname{lcm}(\exp_{2i}(H) \mid i \in \mathbb{Z});$$

(2) If H is quasitriangular, then for all $i \in \mathcal{Z}$,

$$\exp_{4i}(H) = \exp_0(H) \quad \textit{and} \quad \exp_{4i-2}(H) = \exp(H)$$

hold, and thus

$$\exp(H \rtimes \Bbbk\langle S^2\rangle) = \operatorname{lcm}(\exp_0(H), \exp(H)).$$

Proof:

(1) Note that we have $\exp(H \rtimes \Bbbk\langle S^2\rangle) = \exp_0(H \rtimes \Bbbk\langle S^2\rangle)$. Now for any positive integer n and each $h \in H$, $i \in \mathbb{Z}$, we calculate that

$$(h \rtimes S^{2i})^{[n]} = \sum h_{(1)} S^{2i}(h_{(2)}) \cdots S^{2(n-1)i}(h_{(n)}) \rtimes S^{2ni}.$$

Therefore, the nth Sweedler power $[n]_{H \rtimes \Bbbk\langle S^2\rangle}$ on $H \rtimes \Bbbk\langle S^2\rangle$ is trivial if and only if $S^{2ni} = \mathrm{id}$ and

$$m_n \circ (\mathrm{id} \otimes S^{2i} \otimes \cdots \otimes S^{2(n-1)i}) \circ \Delta_n = u \circ \varepsilon$$

both hold for all $i \in \mathbb{Z}$. In other words,

$$[n]_{H \rtimes \Bbbk\langle S^2\rangle} \text{ is trivial } \iff \operatorname{lcm}\big(\operatorname{ord}(S^{2i}), \exp_{2i}(H) \mid i \in \mathbb{Z}\big) \mid n.$$

However, we know that $\operatorname{ord}(S^2) \mid \exp_{-2}(H)$ by Corollary 2.1.2. As a conclusion, $\exp(H \rtimes \Bbbk\langle S^2\rangle) = \operatorname{lcm}(\exp_{2i}(H) \mid i \in \mathbb{Z})$ is obtained.

(2) Suppose that H is quasitriangular. According to [10, Section 3], there exists a grouplike element $g \in H$ determining the inner automorphism S^4 on H. Thus Lemma 4.2.1 provides that

$$(hg^i)^{[n]} = \sum h_{(1)} S^{4i}(h_{(2)}) \cdots S^{4(n-1)i}(h_{(n)}) g^{ni} \quad (\forall i \in \mathcal{Z})$$

holds for all $h \in H$ and each $n \geq 1$. It is clear that the (multiplication) order of g divides $\gcd(\exp_0(H), \exp_{4i}(H))$. Thus the choice of n being either $\exp_0(H)$ or $\exp_{4i}(H)$ in the equation must imply that n is divisible by the other one. We conclude that $\exp_{4i}(H) = \exp_0(H)$ for each $i \in \mathcal{Z}$.

Similarly, the equation $S^{4i-2}(h) = g^i S^{-2}(h) g^{-i}$ implies that

$$\begin{aligned} & m_n \circ (\mathrm{id} \otimes S^{-2} \otimes \cdots \otimes S^{-2n+2}) \circ \Delta_n(hg^i) \\ = \; & \sum (h_{(1)}g^i) S^{-2}(h_{(2)}g^i) \cdots S^{-2n+2}(h_{(n)}g^i) \\ = \; & \sum h_{(1)} \left(g^i S^{-2}(h_{(2)}) g^{-i}\right) \cdots \left(g^{(n-1)i} S^{-2n+2}(h_{(n)}) g^{-(n-1)i}\right) \cdot g^{ni} \\ = \; & \sum h_{(1)} S^{4i-2}(h_{(2)}) \cdots S^{(n-1)(4i-2)}(h_{(n)}) g^{ni} \quad (\forall i \in \mathcal{Z}) \end{aligned}$$

holds for all $h \in H$ and each $n \geq 1$. The (multiplication) order of g also divides $\gcd(\exp(H), \exp_{4i-2}(H))$. Thus $\exp_{4i-2}(H) = \exp(H)$ for each $i \in \mathcal{Z}$ due to the same reason.

Corollary 4.2.6 *Let H be a finite-dimensional Hopf algebra. Then*

$$\exp(D(H) \rtimes \Bbbk\langle {S_{D(H)}}^2 \rangle) = \exp(H \rtimes \Bbbk\langle S^2 \rangle).$$

Proof: It is clear that $\mathrm{ord}({S_{D(H)}}^2) = \mathrm{ord}(S^2)$, and as a consequence $\exp(H \rtimes \Bbbk\langle S^2 \rangle) \mid \exp(D(H) \rtimes \Bbbk\langle {S_{D(H)}}^2 \rangle)$ holds because of the inclusion of Hopf algebras:

$$\begin{aligned} H \rtimes \Bbbk\langle S^2 \rangle & \hookrightarrow D(H) \rtimes \Bbbk\langle {S_{D(H)}}^2 \rangle \\ h \rtimes S^{2i} & \mapsto (\varepsilon \bowtie h) \rtimes {S_{D(H)}}^{2i}. \end{aligned}$$

On the other hand, it is well-known that $D(H)$ is quasitriangular. Then according to the invariance under taking the Drinfeld double (Proposition 4.1.5(7) and Lemma 4.1.1(7)), as well as Lemma 4.2.5(2),

$$\begin{aligned} \exp(D(H) \rtimes \Bbbk\langle {S_{D(H)}}^2 \rangle) & = \mathrm{lcm}\,(\exp_0(D(H)), \exp(D(H))) \\ & = \mathrm{lcm}\,(\exp_0(H), \exp(H)) \;\mid\; \exp(H \rtimes \Bbbk\langle S^2 \rangle). \end{aligned}$$

The proof is then completed.

Corollary 4.2.7 *Let H be a finite-dimensional Hopf algebra. Then*

$$\exp(H \rtimes \Bbbk\langle S^2\rangle) = \mathrm{lcm}(\exp_0(H), \exp(H)).$$

Proof: This is a consequence of Lemma 4.2.5, Corollary 4.2.6, and the invariance under taking the Drinfeld double (Lemma 4.1.1(7) and Proposition 4.1.5):

$$\begin{aligned}\exp(H \rtimes \Bbbk\langle S^2\rangle) &= \exp(D(H) \rtimes \Bbbk\langle {S_{D(H)}}^2\rangle) \\ &= \mathrm{lcm}\left(\exp_0(D(H)), \exp(D(H))\right) \\ &= \mathrm{lcm}\left(\exp_0(H), \exp(H)\right).\end{aligned}$$

Remark 4.2.8 *A similar result is [36, Theorem 3.4]. They describe the exponent of smash coproduct $H \natural \Bbbk^G$ with $\exp(G)$ and twisted exponents of H, where G acts as Hopf algebra automorphisms on the involutory Hopf algebra H.*

We finish this subsection by describing the quasi-exponent of $H \rtimes \mathbb{C}\langle S^2\rangle$ when H has the dual Chevalley property over $\mathbb{C}$. In this case $H \rtimes \mathbb{C}\langle S^2\rangle$ also has the dual Chevalley property with the coradical $H_0 \rtimes \mathbb{C}\langle S^2\rangle$.

Proposition 4.2.9 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over $\mathbb{C}$. Then*

$$\mathrm{qexp}(H \rtimes \mathbb{C}\langle S^2\rangle) = \exp(H_0) = \mathrm{qexp}(H).$$

Proof: By Theorem 3.2.3, we know that $\exp(H_0) = \mathrm{qexp}(H)$, and $\mathrm{qexp}(H \rtimes \mathbb{C}\langle S^2\rangle) = \exp(H_0 \rtimes \mathbb{C}\langle S^2\rangle)$ which equals to $\mathrm{lcm}(\exp_{2i}(H_0) \mid i \in \mathbb{Z})$ by Lemma 4.2.5(1). Since H_0 is semisimple over $\mathbb{C}$, $(S\mid_{H_0})^2 = \mathrm{id}_{H_0}$. This implies that $\mathrm{lcm}(\exp_{2i}(H_0) \mid i \in \mathbb{Z}) = \exp(H_0)$.

§4.3 Finiteness of Exponents

In the final section, we study the finiteness (and upper bounds) of $\exp(H)$ and $\exp_0(H)$ for a finite-dimensional Hopf algebra H. First recall in [12, Theorem 4.3] that $\exp(H) < \infty$ when H is semisimple and cosemisimple, and H must be involutory in this case ([11, Theorem 3.1]) which follows that $\exp_0(H) = \exp(H) < \infty$ holds. Therefore, we are supposed to focus on the case when H is non-cosemisimple or non-semisimple.

Our results are mainly divided into two situations whether the characteristic of the base field $\Bbbk$ is 0 or not. Note that when $\operatorname{char} \Bbbk = 0$, the semisimplicity and cosemisimplicity for H are equivalent ([23, Theorem 3.3]). Hence it is enough for us to consider the following two cases:

- H is non-cosemisimple in characteristic 0;
- H is finite-dimensional in positive characteristic.

§4.3.1 Finiteness of $\exp_0(H)$

Recall in [12, Theorem 4.10] that $\exp(H) < \infty$ as long as H is finite-dimensional in positive characteristic. With the help of the semidirect product $H \rtimes \Bbbk\langle S^2\rangle$, we could directly infer that $\exp_0(H)$ is also finite in this case:

Proposition 4.3.1 *Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$ of positive characteristic. Then $\exp_0(H) < \infty$.*

Proof: Since $H \rtimes \Bbbk\langle S^2\rangle$ is finite-dimensional and pivotal over $\Bbbk$, we know that

$$\exp_0(H \rtimes \Bbbk\langle S^2\rangle) = \exp(H \rtimes \Bbbk\langle S^2\rangle) < \infty.$$

However, $H \hookrightarrow H \rtimes \Bbbk\langle S^2\rangle$ is a Hopf subalgebra. Thus $\exp_0(H) < \infty$.

When H has the dual Chevalley property, we could discuss the finiteness of $\exp_0(H)$ more specifically.

Proposition 4.3.2 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over $\Bbbk$. Then*

(1) If H is non-cosemisimple and $\operatorname{char} \Bbbk = 0$, *then* $\exp_0(H) = \infty$;

(2) *If* char $\Bbbk = p > 0$*, and denote* $N := \exp_0(H_0) < \infty$ *and* $L := \mathrm{Lw}(H)$*, then* $\exp_0(H) \mid Np^M$*, where* M *is a positive integer satisfying* $p^M \geq L$*.*

Proof:

(1) This is [27, Theorem 4.1].

(2) We would show the divisibility by the same method as the proof of [42, Lemma 4.11].

Note that $N < \infty$ holds according to Proposition 4.3.1. Thus,

$$\mathrm{id}^{*N}(h) = h^{[N]} = u \circ \varepsilon(h)$$

for all $h \in H_0$, where $*$ denotes the convolution in $\mathrm{End}_{\Bbbk}(H)$. One could also write $(\mathrm{id}^{*N} - u \circ \varepsilon) \mid_{H_0} = 0$. According to [42, Lemma 2.3], we know that

$$(\mathrm{id}^{*N} - u \circ \varepsilon)^{*L} = 0$$

holds on H.

As a consequence, $\mathrm{id}^{*Np^M} - u \circ \varepsilon = (\mathrm{id}^{*N} - u \circ \varepsilon)^{*p^M} = 0$ if $p^M \geq L$, whenever the characteristic p is 2 or an odd prime number. The desired divisibility is then obtained, since $\exp_0(H)$ is exactly the convolution order of $\mathrm{id} \in \mathrm{End}_{\Bbbk}(H)$

Remark 4.3.3 *Item (2) generalizes [27, Theorem 5.1], which holds when* H *is pointed in positive characteristic.*

The author apologizes that in our previous work [27, Theorem 5.1], the upper bound $Np(\lfloor \frac{L-1}{p} \rfloor + 1)$ for $\exp_0(H)$ (with the notations in Proposition 4.3.2(2)) is written incorrect. This is because the incorrect upper bound is obtained by the wrong equation $\mathcal{Z}^{\left[dp(\lfloor \frac{n}{p} \rfloor + 1)\right]} = \mathcal{Z}^{dp(\lfloor \frac{n}{p} \rfloor + 1)} = I$ appearing in [27, Proposition 5.1], where $\lfloor - \rfloor$ stands for the floor function.

The equation should be corrected as follows:

$$\mathcal{Z}^{\left[dp^{\lfloor \log_p n \rfloor + 1}\right]} = \mathcal{Z}^{dp^{\lfloor \log_p n \rfloor + 1}} = I,$$

which is deduced by a computation correcting the proof of [27, Proposition 5.1]:

$$\mathcal{Z}^{dp^{\lfloor \log_p n \rfloor+1}} = (I+\mathcal{W})^{p^{\lfloor \log_p n \rfloor+1}} = I+\mathcal{W}^{p^{\lfloor \log_p n \rfloor+1}} = I$$

in characteristic $p>0$, for $\mathcal{Z}^d = I+\mathcal{W}$, $\mathcal{W}^{n+1}=0$ and $p^{\lfloor \log_p n \rfloor+1} > n$. As a result, the correct upper bound for $\exp_0(H)$ in [27, Theorem 5.1] should be written as $p^{\lfloor \log_p (L-1) \rfloor+1}$, or Np^M for a positive integer satisfying $p^M \geq L$. The latter form is the same as the one in Proposition 4.3.2(2) in this section.

§4.3.2 Primitive Matrices over Hopf Algebras

In this subsection, we introduce further properties of multiplicative and primitive matrices over Hopf algebras.

As for later uses, we focus on existence and operation properties for certain non-trivial primitive matrices over H, especially when H is non-cosemisimple with the dual Chevalley property. The set of all the simple subcoalgebras of H is denoted by $\mathcal{S}$ for convenience.

Lemma 4.3.4 *Let H be a finite-dimensional non-cosemisimple Hopf algebra. Then:*

(1) There exists a non-trivial $(\mathcal{C},1)$-primitive matrix for some $C\in\mathcal{S}$ with a basic multiplicative matrix $\mathcal{C}$;

(2) If H has the dual Chevalley property, and suppose $\Lambda_0 \in 1+\sum\limits_{D\in\S\setminus\{\Bbbk 1\}} D$, then for any basic multiplicative matrix $\mathcal{C}$ and any non-trivial $(\mathcal{C},1)$-primitive matrix $\mathcal{X}$, we have

$$\Lambda_0\mathcal{X}\neq 0 \quad \textit{and} \quad \mathcal{X}\Lambda_0 \neq 0.$$

Proof:

(1) This is a conclusion of [27, Proposition 4.3] and [27, Theorem 3.1].

Specifically, [27, Proposition 4.3] provides that there is a non-zero subspace denoted as follows:

$$({}^CH_1{}^1)^+ := {}^CH_1{}^1 \cap \mathrm{Ker}(\varepsilon) \neq 0$$

for some $C\in\mathcal{S}$. Choose some $0\neq w\in({}^CH_1{}^1)^+$, and actually $w\notin H_0$ since $({}^CH_1{}^1)^+\cap H_0 = 0$ (due to the fact that ${}^CH_0{}^1 = \delta_{C,1}\Bbbk 1$).

When $C = \Bbbk 1$, then evidently w is a non-trivial primitive element, which is also regarded as a non-trivial $(1,1)$-primitive matrix; On the other hand when $C \neq \Bbbk 1$, it is followed from [27, Theorem 3.1(1)] that $w \in ({}^C H_1{}^1)^+$ is a sum of some entries of $(\mathcal{C}, 1)$-primitive matrices. Hence the non-triviality of w implies that one of these matrices must be non-trivial.

(2) The dual Chevalley property of H implies that $H_0H_1 + H_1H_0 \subseteq H_1$ (e.g. [34, Lemma 5.2.8]). Let x be an entry of $\mathcal{X}$ satisfying $x \notin H_0$. Since $\Delta(x) \in C \otimes H_1 + H_1 \otimes 1$, it is evident that for any $D \in \mathcal{S} \setminus \{\Bbbk 1\}$,

$$\begin{aligned} \Delta(Dx) &\subseteq DC \otimes DH_1 + DH_1 \otimes D \subseteq H_0 \otimes H_1 + H_1 \otimes D, \\ \Delta(xD) &\subseteq CD \otimes H_1D + H_1D \otimes D \subseteq H_0 \otimes H_1 + H_1 \otimes D, \end{aligned}$$

both hold.

Now we choose a linear function $e : H \to \Bbbk$ such that

$$e(1) = 1 \quad \text{while} \quad e(D) = 0 \;\; (\forall D \in \S \setminus \{\Bbbk 1\}).$$

We find immediately that $(\mathrm{id} \otimes e) \circ \Delta(x) \in Ce(H_1) + xe(1) \subseteq H_0 + x$, because $\mathcal{X}$ is $(\mathcal{C}, 1)$-primitive. Further computations show that

$$\begin{aligned} (\mathrm{id} \otimes e) \circ \Delta(\Lambda_0 x) &\in (\mathrm{id} \otimes e) \circ \Delta(x + \sum_{D \in \mathcal{S} \setminus \{\Bbbk 1\}} Dx) \\ &= (\mathrm{id} \otimes e) \circ \Delta(x) + \sum_{D \in \mathcal{S} \setminus \{\Bbbk 1\}} (\mathrm{id} \otimes e) \circ \Delta(Dx) \\ &\subseteq (H_0 + x) + H_0 = x + H_0, \end{aligned}$$

and similarly $(\mathrm{id} \otimes e) \circ \Delta(x\Lambda_0) \in x + H_0$. However $x \notin H_0$, which follows that $\Lambda_0 x$ and $x\Lambda_0$ are non-zero, then so are $\Lambda_0 \mathcal{X}$ and $\mathcal{X}\Lambda_0$.

The lemma above is actually enough for us to estimate $\exp_0(H)$ for a non-cosemisimple Hopf algebra H. However if we try to compute $\exp(H)$, we need to know how S^2 acts on primitive matrices by Proposition 3.1.6.

Proposition 4.3.5 *Let $\Bbbk$ be an algebraically closed field of characteristic 0. Suppose H is a finite-dimensional non-cosemisimple Hopf algebra with the dual Chevalley property* *over* $\Bbbk$. *Denote* $N := \exp(H_0)$. *Then there exists a non-trivial* $(\mathcal{C}, 1)$*-primitive matrix* $\mathcal{X}$ *for some basic multiplicative matrix* $\mathcal{C}$*, such that*

(1) $S^2(\mathcal{C}) = \mathcal{C}$*,*

(2) $S^2(\mathcal{X}) = q\mathcal{X}$*, and*

(3) $S(\mathcal{C})\mathcal{X} = q(X^{\mathrm{T}} S(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}}$*,*

where $q \in \Bbbk$ *is an* N*th root of unity.*

Proof:

(1) Consider the finite-dimensional cosemisimple Hopf subalgebra H_0 in characteristic 0, which is involutory by [24, Theorem 4]. In other words, $S^2 \mid_{H_0}$ is the identity. However, since all the entries of the multiplicative matrix $\mathcal{C}$ lie in H_0, we could know that $S^2(\mathcal{C}) = \mathcal{C}$ holds.

(2) We know by Lemma 4.3.4(1) that there do exist non-trivial $(\mathcal{C}, 1)$-primitive matrices for some basic multiplicative matrix $\mathcal{C}$. Let $P \neq 0$ be the finite-dimensional space of all the $(\mathcal{C}, 1)$-primitive matrices over H.

Note that the dual Chevalley property ensures that H_0 is involutory in characteristic 0. Thus P is stable under S^2. In fact, if we denote $\mathcal{C} = (c_{ij})_{r\times r}$, then

$$\begin{aligned}
\Delta(S^2(w_i)) &= (S^2 \otimes S^2)\left(\sum_{k=1}^{r} c_{ik} \otimes w_k + w_i \otimes 1\right) \\
&= \sum_{k=1}^{r} S^2(c_{ik}) \otimes S^2(w_k) + S^2(w_i) \otimes S^2(1) \\
&= \sum_{k=1}^{r} c_{ik} \otimes S^2(w_k) + S^2(w_i) \otimes 1 \quad (\forall 1 \leq i \leq r)
\end{aligned}$$

for each $(\mathcal{C}, 1)$-primitive matrix $\mathcal{W} = (w_1, w_2, \cdots, w_r)^{\mathrm{T}}$, and the equations show that $S^2(\mathcal{W})$ is also $(\mathcal{C}, 1)$-primitive.

Now we consider the representation of the cyclic group $\mathcal{Z}_N$ on P defined by

$$n : \mathcal{W} \mapsto S^{2n}(\mathcal{W}) \quad (n \in \mathcal{Z}_N,\ \mathcal{W} \in P),$$

which is well defined due to Proposition 3.1.6.

Clearly this is a direct sum of 1-dimensional representations. We claim that one of these subrepresentations must have non-trivial basis $\mathcal{X}$, otherwise there would be no non-trivial $(\mathcal{C}, 1)$-primitive matrices in P, a contradiction. Of course the subrepresentation $\Bbbk\mathcal{X}$ provides that $S^2(\mathcal{X}) = q\mathcal{X}$, where q is an Nth root of unity.

(3) It follows directly from the definition of (basic) multiplicative matrices that

$$S(\mathcal{C})\mathcal{C} = \mathcal{C}S(\mathcal{C}) = I \quad \text{(the identity matrix over } H).$$

Now we show that because H_0 is involutory, the equations $S(\mathcal{C})^{\mathrm{T}}\mathcal{C}^{\mathrm{T}} = \mathcal{C}^{\mathrm{T}}S(\mathcal{C})^{\mathrm{T}} = I$ hold as well. In fact, we could compute that

$$S(\mathcal{C})^{\mathrm{T}}\mathcal{C}^{\mathrm{T}} = S(\mathcal{C})^{\mathrm{T}}S^2(\mathcal{C})^{\mathrm{T}} = S(S(\mathcal{C})\mathcal{C})^{\mathrm{T}} = S(I)^{\mathrm{T}} = I,$$

etc, where the second equality is due to the following computations if we denote $\mathcal{C} := (c_{ij})_{r\times r}$:

$$\begin{aligned}
S(\mathcal{C})^{\mathrm{T}}S^2(\mathcal{C})^{\mathrm{T}} &= (S(c_{ij}))_{r\times r}^{\mathrm{T}} \left(S^2(c_{ij})\right)_{r\times r}^{\mathrm{T}} = (S(c_{ji}))_{r\times r} \left(S^2(c_{ji})\right)_{r\times r} \\
&= \Big(\sum_{k=1}^{r} S(c_{ki})S^2(c_{jk})\Big)_{r\times r} = \Big(\sum_{k=1}^{r} S\left(S(c_{jk})c_{ki}\right)\Big)_{r\times r} \\
&= S\Big(\sum_{k=1}^{r} S(c_{jk})c_{ki}\Big)_{r\times r} = S\Big(\sum_{k=1}^{r} S(c_{ik})c_{kj}\Big)_{r\times r}^{\mathrm{T}} \\
&= S\left((S(c_{ij}))_{r\times r}(c_{ij})_{r\times r}\right)^{\mathrm{T}} = S(S(\mathcal{C})\mathcal{C})^{\mathrm{T}}.
\end{aligned}$$

Another equation concerned is according to Lemma 3.1.4(2) that

$$S^2(\mathcal{X}) = ((S(\mathcal{C})\mathcal{X})^{\mathrm{T}}S^2(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}} = ((S(\mathcal{C})\mathcal{X})^{\mathrm{T}}\mathcal{C}^{\mathrm{T}})^{\mathrm{T}},$$

which could be transformed as

$$S(\mathcal{C})\mathcal{X} = (S^2(\mathcal{X})^{\mathrm{T}}S(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}} = q(\mathcal{X}^{\mathrm{T}}S(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}}.$$

The latter equality is due to item (2).

§4.3.3 Finiteness with the Dual Chevalley Property

The finiteness of $\exp(H)$ is discussed at the end of this section. When H is non-semisimple in characteristic 0, it is estimated that $\exp(H)$ is usually (and probably always) infinite in [13, Section 1]. We show that this property is true as long as H has the dual Chevalley property.

Theorem 4.3.6 *Let H be a finite-dimensional Hopf algebra with the dual Chevalley property over $\Bbbk$. Then*

(1) If H is non-cosemisimple and $\operatorname{char}\Bbbk = 0$, *then* $\exp(H) = \infty$;

(2) If $\operatorname{char}\Bbbk = p > 0$, *and denote* $N := \operatorname{lcm}(\exp(H_0), \exp_0(H_0)) < \infty$ *and* $L := \operatorname{Lw}(H)$, *then* $\exp(H) \mid Np^M$, *where M is a positive integer satisfying $p^M \geq L$.*

Proof:

(1) Without the loss of generality, $\Bbbk$ is assumed to be algebraically closed ([12, Proposition 2.2(8)]). We begin with the non-cosemisimplicity of H. According to Proposition 4.3.5, there must be a non-trivial $(\mathcal{C}, 1)$-primitive matrix $\mathcal{X}$ for some basic multiplicative matrix $\mathcal{C}$, satisfying

$$S^2(\mathcal{X}) = q\mathcal{X} \quad \text{and} \quad S(\mathcal{C})\mathcal{X} = q(X^{\mathrm{T}}S(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}} \quad (0 \neq q \in \Bbbk).$$

Now we focus on showing that

$$\begin{aligned} & m_n \circ (\mathrm{id} \otimes S^2 \otimes \cdots \otimes S^{2n-2}) \circ \Delta_n(\mathcal{X}) \qquad (4.1) \\ = \; & \mathcal{X} + \mathcal{C}S^2(\mathcal{X}) + \cdots + \mathcal{C}^{n-1}S^{2n-2}(\mathcal{X}) \\ = \; & \mathcal{X} + q\mathcal{C}\mathcal{X} + \cdots q^{n-1}\mathcal{C}^{n-1}\mathcal{X} \\ = \; & \sum_{i=0}^{n-1} q^i\mathcal{C}^i\mathcal{X} \end{aligned}$$

is non-zero for any $n \geq 1$. This claim would imply that $\exp_2(H) = \infty$, since $\varepsilon(\mathcal{X}) = 0$ by Remark 2.3.12.

Denote the integral of H_0 by Λ_0 which belongs in $1 + \sum_{D \in \S\setminus\{\Bbbk 1\}} D$. Clearly it satisfies that $S(\mathcal{C})^{\mathrm{T}}\Lambda_0 = I\Lambda_0$. We compute first that for each $i \geq 0$,

$$\begin{aligned}
& q^{-i}S(\mathcal{C})^i\mathcal{X}\Lambda_0 \\
= \ & q^{-i}S(\mathcal{C})^{i-1}(S(\mathcal{C})\mathcal{X})\Lambda_0 \ = \ q^{-i}S(\mathcal{C})^{i-1}(q^{-1}(\mathcal{X}^{\mathrm{T}}S(\mathcal{C})^{\mathrm{T}})^{\mathrm{T}})\Lambda_0 \\
= \ & q^{-i+1}S(\mathcal{C})^{i-1}(\mathcal{X}^{\mathrm{T}}S(\mathcal{C})^{\mathrm{T}}\Lambda_0)^{\mathrm{T}} \ = \ q^{-i+1}S(\mathcal{C})^{i-1}(\mathcal{X}^{\mathrm{T}}\Lambda_0)^{\mathrm{T}} \\
= \ & q^{-i+1}S(\mathcal{C})^{i-1}\mathcal{X}\Lambda_0 \ = \ \cdots \\
= \ & q^{-1}S(\mathcal{C})\mathcal{X}\Lambda_0 \ = \ \mathcal{X}\Lambda_0
\end{aligned}$$

holds, where the second equality is due to Proposition 4.3.5. As a consequence, for any $n \geq 1$,

$$\begin{aligned}
q^{-n+1}S(\mathcal{C})^{n-1}\left(\sum_{i=0}^{n-1} q^i\mathcal{C}^i\mathcal{X}\right)\Lambda_0 \ &= \ \sum_{i=0}^{n-1} q^{-(n-1-i)}S(\mathcal{C})^{n-1-i}\mathcal{X}\Lambda_0 \\
&= \ \sum_{i=0}^{n-1} q^{-i}S(\mathcal{C})^i\mathcal{X}\Lambda_0 \ = \ n\mathcal{X}\Lambda_0 \neq 0
\end{aligned}$$

according to Lemma 4.3.4(2) and char $\Bbbk = 0$. It clearly follows that (4.1) must be non-zero for any positive integer n, and thus $\exp_2(H) = \infty$.

Finally, it follows from Lemma 4.2.5(2) that

$$\exp(H) = \exp(D(H)) = \exp_2(D(H)) \geq \exp_2(H) = \infty.$$

Note that the inequality holds since H is a Hopf subalgebra of its Drinfeld double $D(H)$.

(2) We claim that $H \rtimes \Bbbk\langle S^2\rangle$ has the dual Chevalley property and the Lowey length L when H does so. In fact, recall that its coalgebra structure is $H \otimes \Bbbk\langle S^2\rangle$. Then according to [40, Corollary 4.1.8], the coradical is exactly

$$(H \rtimes \Bbbk\langle S^2\rangle)_0 = H_0 \rtimes \Bbbk\langle S^2\rangle,$$

which is a Hopf subalgebra by the definition. It also implies that

$$(H \rtimes \Bbbk\langle S^2\rangle)_n = \sum_{i=0}^{n} H_{n-i} \rtimes (\Bbbk\langle S^2\rangle)_i = H_n \rtimes \Bbbk\langle S^2\rangle$$

by [40, Corollary 4.2.2(c)], and we know that $\mathrm{Lw}(H \rtimes \Bbbk\langle S^2\rangle) = \mathrm{Lw}(H)$ as a consequence.

On the other hand, the coradical $H_0 \rtimes \Bbbk\langle S^2\rangle$ is pivotal and has exponent

$$\begin{aligned}\exp(H_0 \rtimes \Bbbk\langle S^2\rangle) &= \exp_0(H_0 \rtimes \Bbbk\langle S^2\rangle) \\ &= \mathrm{lcm}(\exp(H_0), \exp_0(H_0)) = N.\end{aligned}$$

According to Proposition 4.3.2(2), we find that $\exp_0(H \rtimes \Bbbk\langle S^2\rangle)$ divides Np^M where M is a positive integer satisfying $p^M \geq L$. Consequently

$$\exp(H) \mid Np^M$$

holds as well, since $\exp(H)$ divides $\exp(H \rtimes \Bbbk\langle S^2\rangle)$ which equals to $\exp_0(H \rtimes \Bbbk\langle S^2\rangle)$.

Chapter 5 The Link-Indecomposable Components

§5.1 Link-Indecomposable Coalgebras and Decompositions

§5.1.1 Link Relations and Matric Condition

The definitions involving *link-indecomposable components* were introduced in [35]. They were later presented by [40, Section 4.8] in a slightly different way, which will be listed as follows in this report. Let H be a coalgebra over $\Bbbk$, and denote the set of all its simple subcoalgebras by $\mathcal{S}$. Besides, the wedge product operation on H is denoted by $\wedge$.

Definition 5.1.1 *Suppose that $C, D \in \mathcal{S}$.*

(1) C and D are said to be directly linked in H, if $C + D \subsetneq C \wedge D + D \wedge C$;

(2) C and D are said to be linked in H, if there is an $n \in \mathbb{N}$ and $E_0, E_1, \cdots, E_n \in \mathcal{S}$, such that $C = E_0$, $D = E_n$, and E_i and E_{i+1} are directly linked in H for $0 \leq i < n$.

Note that the link relation in H is an equivalence relation on $\mathcal{S}$. It could be remarked that this relation is the same as which in [43]. Some relevant concepts and results in the literature are recalled as follows.

Definition 5.1.2 *(1) A link-indecomposable subcoalgebra of H is a subcoalgebra $H' \subseteq H$, such that any two simple subcoalgebras of H' are linked in H';*

(2) A link-indecomposable component of H is a maximal link-indecomposable subcoalgebra of H.

It is known that the link-indecomposable components are closely related to the decomposition of coalgebras. This could be seen by following lemmas.

Lemma 5.1.3 *([40, Lemma 4.8.3]) Suppose $H = H' \oplus H''$ is the direct sum of subcoalgebras H' and H''. Let $C, D \in \mathcal{S}$ be simple subcoalgebras of H. Then:*

(1) If $C \subseteq H'$ and $D \subseteq H''$, then C and D are not directly linked in H;

(2) If C and D are linked in H, then $C, D \subseteq H'$ or $C, D \subseteq H''$.

Lemma 5.1.4 *([35, Theorem 2.1] and [40, Theorem 4.8.6])*

(1) H is the direct sum of its link-indecomposable components;

(2) Suppose that $H = \bigoplus_i H_{(i)}$ is the direct sum of non-zero link-indecomposable subcoalgebras of H. Then $H_{(i)}$'s are the link-indecomposable components of H.

Now we provide some sufficient conditions for simple subcoalgebras to be linked, with the help of non-trivial matrices over H. For the purpose, we introduce a family of so-called *coradical orthonormal idempotents* $\{e_C\}_{C\in\mathcal{S}}$ in H^*, whose existence is affirmed in [39, Lemma 2] or [40, Corollary 3.5.15] for any coalgebra H:

Definition 5.1.5 *Let H be a coalgebra. $\{e_C\}_{C\in\mathcal{S}} \subseteq H^*$ is called a family of coradical orthonormal idempotents in H^*, if*

$$e_C|_D = \delta_{C,D}\varepsilon|_D, \quad e_Ce_D = \delta_{C,D}e_C \quad (\textit{for any } C, D \in \mathcal{S}), \quad \sum_{C\in\mathcal{S}} e_C = \varepsilon.$$

It should be remarked that the sum $\sum_{C\in\mathcal{S}} e_C$ is well-defined in H^*, as long as $\{e_C\}_{C\in\mathcal{S}}$ is orthogonal (hence linearly independent). This is because each $h \in H$ belongs to a finite-dimensional subcoalgebra vanishing through all but finitely many e_C's, as explained in the second paragraph of [39, Section 1].

Also, we would use following notations for convenience:

$${}^Ch = h \leftharpoonup e_C, \quad h^D = e_D \rightharpoonup h, \quad {}^Ch^D = e_D \rightharpoonup h \leftharpoonup e_C \quad (\text{for any } h \in H \text{ and } C, D \in \mathcal{S}),$$

where $\leftharpoonup$ and $\rightharpoonup$ are hit actions of H^* on H. Notations such as $V^C := e_C \rightharpoonup V$ for a subspace V of H are used as well.

It is shown in the next lemma how the coradical orthonormal idempotents are applied to connect non-trivial wedges with non-trivial primitive matrices:

Lemma 5.1.6 *Let $C, D \in \mathcal{S}$.*

(1) Suppose $\{e_E\}_{E\in\mathcal{S}}$ is a family of coradical orthonormal idempotents in H^*. If $C \wedge D \supsetneq C + D$, then there exists some $x \in C \wedge D$ such that

$$x = {}^Cx^D \notin H_0.$$

(2) Let $\mathcal{C}, \mathcal{D}$ be basic multiplicative matrices of C and D, respectively. Then $C \wedge D \supsetneq C + D$ if and only if there is a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix over H.

Proof:

(1) Choose $y \in (C \wedge D) \setminus (C + D)$ and consider the sum

$$y = \varepsilon \rightharpoonup y \leftharpoonup \varepsilon = \sum_{E,F \in \mathcal{S}} (e_F \rightharpoonup y \leftharpoonup e_E) = \sum_{E,F \in \mathcal{S}} {}^{E}y^{F}.$$

We claim that:

- ${}^{E}y^{F} \in D$ holds when $E \neq C$, and
- ${}^{E}y^{F} \in C$ holds when $F \neq D$.

In fact, since $\Delta(y) \in C \otimes H + H \otimes D$, when $E \neq C$ we find that

$${}^{E}y^{F} = ({}^{E}y)^{F} \in (\langle e_E, C \rangle H + \langle e_E, H \rangle D)^{F} \subseteq D^{F} \subseteq D.$$

The second claim holds similarly.

As a conclusion, we know that the summand ${}^{C}y^{D} \notin C + D$, because of our choice of y. Now we choose $x := {}^{C}y^{D}$. Clearly, $x = {}^{C}x^{D}$ holds by the fact that e_C and e_D are idempotents. Meanwhile, one could verify that the condition $x \notin C + D$ implies $x \notin H_0$ with a proof by contradiction, according to the hit actions by $\{e_E\}_{E \in \mathcal{S}}$.

(2) Suppose that $C \wedge D \supsetneq C + D$ holds, and then there exists some element

$$x = {}^{C}x^{D} \in (C \wedge D) \setminus H_0$$

according to (1). In fact we could know by direct computations that $x \in {}^{C}{H_1}^{D} \setminus H_0$ holds, where $H_1 = H_0 \wedge H_0$. In order to show that C and D are linked, we might assume $C \neq D$. Therefore, due to [27, Theorem 3.1](1), we could obtain a finite number of $(\mathcal{C}, \mathcal{D})$-primitive matrices, such that x is exactly the sum of some of their entries. Now since $x \notin H_0$ is non-trivial, there must be a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix $\mathcal{X}$ within, and this could be our desired one.

On the other hand, suppose that $\mathcal{X}$ is a $(\mathcal{C},\mathcal{D})$-primitive matrix. It is not hard to know all the entries of a $\mathcal{X}$ must lie in $C\wedge D$. Thus, non-trivial ones would belong to $(C\wedge D)\setminus H_0$, which follows that $C\wedge D\supsetneq C+D$ holds as well.

Lemma 5.1.6(2) could be regarded as a non-pointed generalization of [40, Lemma 15.2.2], which provides a condition for simple subcoalgebras to be directly linked. Furthermore, a sufficient condition for the link relation could also be verified. Before that, we need a lemma on triviality properties of block upper-triangular multiplicative matrices (with basic diagonal) over a coalgebra H:

Lemma 5.1.7 *Let $\{e_E\}_{E\in\mathcal{S}}$ be a family of coradical orthonormal idempotents in H^*. Suppose that*

$$\begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_{12} & \cdots & \mathcal{X}_{1t} \\ 0 & \mathcal{C}_2 & \cdots & \mathcal{X}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix} \tag{5.1}$$

is a (block) multiplicative matrix over H, where $\mathcal{C}_1,\mathcal{C}_2,\cdots,\mathcal{C}_t$ are basic multiplicative matrices for $C_1,C_2,\cdots,C_t\in\mathcal{S}$ respectively. Then

(1) $\mathcal{X}_{1t}-{}^{C_1}\mathcal{X}_{1t}{}^{C_t}$ is trivial;

(2) ${}^{D}\mathcal{X}_{1t}$ is trivial for any $D\in\mathcal{S}\setminus\{C_1\}$, and $\mathcal{X}_{1t}{}^{D'}$ is trivial for any $D'\in\mathcal{S}\setminus\{C_t\}$.

Proof: At first we claim that (2) is a direct consequence of (1). In fact e_D and e_{C_1} are orthogonal in H^* when $D\neq C_1$, and then

$${}^{D}\mathcal{X}_{1t}={}^{D}\mathcal{X}_{1t}-{}^{D}({}^{C_1}\mathcal{X}_{1t}{}^{C_t})={}^{D}(\mathcal{X}_{1t}-{}^{C_1}\mathcal{X}_{1t}{}^{C_t})$$

holds. Thus, ${}^{D}\mathcal{X}_{1t}$ is trivial because $\mathcal{X}_{1t}-{}^{C_1}\mathcal{X}_{1t}{}^{C_t}$ is so. Similarly, $\mathcal{X}_{1t}{}^{D'}$ is also trivial when $D'\in\mathcal{S}\setminus\{C_t\}$.

Now we try to prove (1) by inductions on $t\geq 2$. The case $t=2$ is not hard to verify: Since $\Delta(\mathcal{X}_{12})=\mathcal{C}_1\,\widetilde{\otimes}\,\mathcal{X}_{12}+\mathcal{X}_{12}\,\widetilde{\otimes}\,\mathcal{C}_2$, we could obtain ${}^{C_1}\mathcal{X}_{12}=\mathcal{X}_{12}+\langle e_{C_1},\mathcal{X}_{12}\rangle\mathcal{C}_2$ and then

$${}^{C_1}\mathcal{X}_{12}{}^{C_2}=\mathcal{X}_{12}+\mathcal{C}_1\langle e_{C_2},\mathcal{X}_{12}\rangle+\langle e_{C_1},\mathcal{X}_{12}\rangle\mathcal{C}_2. \tag{5.2}$$

It follows that $\mathcal{X}_{12}-{}^{C_1}\mathcal{X}_{12}{}^{C_2}$ is trivial as desired.

Assume that (1) holds for $2, 3, \cdots, t-1$, and then (2) holds as well. Note that we could actually obtain by the inductive assumption that

$$\mathcal{X}_{ij} - {}^{C_i}\mathcal{X}_{ij}{}^{C_j}, \ \ {}^{D}\mathcal{X}_{ij} \ \text{ and } \ \mathcal{X}_{ij}{}^{D'} \ \text{ are all trivial,}$$

for each $1 \leq i < j \leq t$ satisfying $j - i \leq t-2$ and any $D \neq C_i$, $D' \neq C_j$. This is due to the fact

$$\begin{pmatrix} \mathcal{C}_i & \mathcal{X}_{i\,i+1} & \cdots & \mathcal{X}_{ij} \\ 0 & \mathcal{C}_{i+1} & \cdots & \mathcal{X}_{i+1\,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_j \end{pmatrix}$$

is a multiplicative submatrix.

Consider the equation induced by the multiplicative matrix (5.1) that

$$\Delta(\mathcal{X}_{1t}) = \mathcal{C}_1 \mathbin{\widetilde{\otimes}} \mathcal{X}_{1t} + \mathcal{X}_{12} \mathbin{\widetilde{\otimes}} \mathcal{X}_{2t} + \cdots + \mathcal{X}_{1t} \mathbin{\widetilde{\otimes}} \mathcal{C}_t, \tag{5.3}$$

which also follows

$$\begin{aligned} \Delta({}^{C_1}\mathcal{X}_{1t}{}^{C_t}) &= {}^{C_1}\mathcal{C}_1 \mathbin{\widetilde{\otimes}} \mathcal{X}_{1t}{}^{C_t} + {}^{C_1}\mathcal{X}_{12} \mathbin{\widetilde{\otimes}} \mathcal{X}_{2t}{}^{C_t} + \cdots + {}^{C_1}\mathcal{X}_{1t} \mathbin{\widetilde{\otimes}} \mathcal{C}_t{}^{C_t} \\ &= \mathcal{C}_1 \mathbin{\widetilde{\otimes}} {}^{C_1}\mathcal{X}_{1t}{}^{C_t} + {}^{C_1}\mathcal{X}_{12} \mathbin{\widetilde{\otimes}} \mathcal{X}_{2t}{}^{C_t} + \cdots + {}^{C_1}\mathcal{X}_{1t}{}^{C_t} \mathbin{\widetilde{\otimes}} \mathcal{C}_t. \end{aligned} \tag{5.4}$$

However, we know by computations that

$$\mathcal{X}_{1k} \mathbin{\widetilde{\otimes}} \mathcal{X}_{kt} = \sum_{E \in \mathcal{S}} \mathcal{X}_{1k}{}^{E} \mathbin{\widetilde{\otimes}} {}^{E}\mathcal{X}_{kt} = \mathcal{X}_{1k}{}^{C_k} \mathbin{\widetilde{\otimes}} {}^{C_k}\mathcal{X}_{kt} + \sum_{D \in \mathcal{S}\setminus\{C_k\}} \mathcal{X}_{1k}{}^{D} \mathbin{\widetilde{\otimes}} {}^{D}\mathcal{X}_{kt}$$

holds for each $2 \leq k \leq t-1$. Hence the inductive assumption implies that all the entries of matrices

$$\mathcal{X}_{1k} \mathbin{\widetilde{\otimes}} \mathcal{X}_{kt} - \mathcal{X}_{1k}{}^{C_k} \mathbin{\widetilde{\otimes}} {}^{C_k}\mathcal{X}_{kt} \quad (2 \leq k \leq t-1)$$

belong to $H_0 \otimes H_0$. Comparing Equations (5.3) with (5.4), we find that all entries of the matrix

$$\Delta(\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}) - \mathcal{C}_1 \mathbin{\widetilde{\otimes}} (\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}) - (\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}) \mathbin{\widetilde{\otimes}} \mathcal{C}_t$$

would still belong to $H_0 \otimes H_0$ as a result. Consequently, a similar process with Equation (5.2) provides that

$$^{C_1}(\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t})^{C_t} - (\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}) - \mathcal{C}_1\langle e_{C_1}, \mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}\rangle - \langle e_{C_t}, \mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}\rangle\mathcal{C}_t$$

is trivial, but in fact

$$^{C_1}(\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t})^{C_t} = {}^{C_1}\mathcal{X}_{1t}{}^{C_t} - {}^{C_1}({}^{C_1}\mathcal{X}_{1t}{}^{C_t})^{C_t} = 0.$$

We conclude in the end that $\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}$ must be trivial.

Finally, we would show that the existence of non-trivial multiplicative matrices of form (5.1) are sufficient for the desired link relation.

Proposition 5.1.8 *Suppose $C, D \in \mathcal{S}$.*

(1) Let $\{e_E\}_{E\in\mathcal{S}}$ be a family of coradical orthonormal idempotents in H^*. If ${}^CH^D \backslash H_0 \neq \varnothing$, then C and D are linked;

(2) Suppose that

$$\begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_{12} & \cdots & \mathcal{X}_{1t} \\ 0 & \mathcal{C}_2 & \cdots & \mathcal{X}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix} \tag{5.5}$$

is a (block) multiplicative matrix over H, where $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ are basic multiplicative matrices for $C_1, C_2, \cdots, C_t \in \mathcal{S}$ respectively. If $\mathcal{X}_{1t}$ is non-trivial, then C_1 and C_t are linked.

Proof:

(1) Denote the coradical filtration of H by $\{H_n\}_{n\geq 0}$ in this proof. It is evident that

$$^CH^D = {}^C(\bigcup_{n\geq 0} H_n)^D = \bigcup_{n\geq 0} {}^CH_n{}^D,$$

and we would show by induction on $n \geq 1$ that for each $C, D \in \mathcal{S}$, C and D are linked if ${}^CH_n{}^D \setminus H_0 \neq \varnothing$.

Consider the case when $n = 1$. At first we could find that ${}^C H_1{}^D \subseteq C \wedge D$. In fact, this is due to following computations:

$$\begin{aligned} \Delta({}^C H_1{}^D) &\subseteq {}^C H_0 \otimes H_1{}^D + {}^C H_1 \otimes H_0{}^D \\ &\subseteq C \otimes H + H \otimes D. \end{aligned}$$

Therefore, $(C \wedge D) \setminus (C + D) \supseteq {}^C H_1{}^D \setminus H_0 \neq \varnothing$ holds, which follows that C and D are directly linked.

Now we assume that the above claim holds for $1, 2, \cdots, n-1$, and suppose that ${}^C H_n{}^D \setminus H_0 \neq \varnothing$. Without the loss of generality, one might moreover assume that ${}^C H_n{}^D \setminus H_1 \neq \varnothing$, otherwise ${}^C H_n{}^D = {}^C H_1{}^D$ and this case is solved in the previous paragraph. However, we could compute directly to know that

$$\Delta({}^C H_n{}^D) \subseteq \sum_{i=0}^{n} {}^C H_i \otimes H_{n-i}{}^D \subseteq \sum_{E \in \mathcal{S}} \sum_{i=0}^{n} {}^C H_i{}^E \otimes {}^E H_{n-i}{}^D.$$

Discuss the following classified situations:

a) There exist some $E \in \mathcal{S}$ and some $1 \leq i \leq n-1$ such that

$$ {}^C H_i{}^E \setminus H_0 \neq \varnothing \quad \text{and} \quad {}^E H_{n-i}{}^D \setminus H_0 \neq \varnothing $$

both hold. Then by our inductive assumption, C and E are linked, and meanwhile E and D are linked.

b) For every $E \in \mathcal{S}$ and $1 \leq i \leq n-1$, we always have

$$ {}^C H_i{}^E \subseteq H_0 \quad \text{or} \quad {}^E H_{n-i}{}^D \subseteq H_0. $$

This implies that

$$\begin{aligned} {}^C H_i{}^E \otimes {}^E H_{n-i}{}^D &\subseteq H_0 \otimes {}^E H_{n-i}{}^D + {}^C H_i{}^E \otimes H_0 \\ &\subseteq H_0 \otimes H_n + H_n \otimes H_0 \end{aligned}$$

holds for each $E \in \mathcal{S}$ and $1 \leq i \leq n-1$. In this situation, we find that

$$\begin{aligned}\Delta({}^C H_n{}^D) &\subseteq H_0 \otimes H_n + \sum_{E\in\mathcal{S}} \sum_{i=1}^{n-1} \left({}^C H_i{}^E \otimes {}^E H_{n-i}{}^D\right) + H_n \otimes H_0 \\ &\subseteq H_0 \otimes H_n + H_n \otimes H_0,\end{aligned}$$

which follows that ${}^C H_n{}^D \subseteq H_0 \wedge H_0 = H_1$, a contradiction to our additional assumption ${}^C H_n{}^D \setminus H_1 \neq \varnothing$.

As a conclusion, C and D must be linked.

(2) We know by Lemma 5.1.7(1) that $\mathcal{X}_{1t} - {}^{C_1}\mathcal{X}_{1t}{}^{C_t}$ must be trivial. Therefore, ${}^{C_1}\mathcal{X}_{1t}{}^{C_t}$ would also be non-trivial according to our requirement on $\mathcal{X}_{1t}$. On the other hand, evidently all the entries of ${}^{C_1}\mathcal{X}_{1t}{}^{C_t}$ lie in the subspace ${}^{C_1}H^{C_t}$, and thus non-trivial ones would belong to ${}^{C_1}H^{C_t} \setminus H_0$. It is concluded that C_1 and C_t are linked according to (1).

§5.1.2 Products of Link-Indecomposable Components

This subsection is devoted to study link-indecomposable components of a (non-pointed) Hopf algebra. For the purpose and convenience in this chapter, we should probably extend the definition of link relations onto arbitrary pairs of subcoalgebras at first. Of course, it coincides with Definition 5.1.1 on simple subcoalgebras.

Definition 5.1.9 *Let H be a coalgebra, and let H', H'' be its subcoalgebras. We say that H' and H'' are linked, if both of following conditions hold:*

- *For each $C \in \mathcal{S}$ contained in H', there exists an $D \in \mathcal{S}$ contained in H'', such that C and D are linked in H (in the sense of Definition 5.1.1);*
- *For each $D \in \mathcal{S}$ contained in H'', there exists an $C \in \mathcal{S}$ contained in H', such that C and D are linked in H.*

Remark 5.1.10 *Suppose that subcoalgebras H' and H'' are linked (in the sense of Definition 5.1.9). A direct discussion follows that for any $E \in \mathcal{S}$, $H' \cap H_{(E)} \neq 0$ if and only if $H'' \cap H_{(E)} \neq 0$. In particular, H' is linked with some $E \in \mathcal{S}$, if and only if $H' \subseteq H_{(E)}$.*

We turn to consider link relations for a Hopf algebra H. We need to mention at first that when the antipode S of is bijective, it is a bijection on $\mathcal{S}$ and $S(H_0) \subseteq H_0$. Now for each $C \in \mathcal{S}$, denote the link-indecomposable component containing C by $H_{(C)}$. The following result is not hard:

Corollary 5.1.11 *Let H be a Hopf algebra over a field $\Bbbk$ with the bijective antipode S. Then for any $C \in \mathcal{S}$, $S(H_{(C)}) = H_{S(C)}$.*

Proof: It is known by [40, Lemma 15.2.1] that $C_1, C_2 \in \mathcal{S}$ are linked, if and only if simple subcoalgebras $S(C_1)$ and $S(C_2)$ are linked. This fact implies that $S(H_{(C)})$ is link-indecomposable and thus contained in $H_{S(C)}$.

On the other hand, the same reason concerning the coalgebra anti-isomorphism S^{-1} follows that $S^{-1}(H_{S(C)}) \subseteq H_{(S^{-1}\circ S(C))} = H_{(C)}$, which means that $H_{S(C)} \subseteq S(H_{(C)})$. As a conclusion, $S(H_{(C)}) = H_{S(C)}$ holds.

Now the products of link-indecomposable components of a Hopf algebra could be considered. With the language of Definition 5.1.9, we start our process by describing how products of simple subcoalgebras preserve their link relations:

Lemma 5.1.12 *Let H be a Hopf algebra over an algebraically closed field $\Bbbk$ with the bijective antipode S.*

(1) Suppose $C_1, C_2, D \in \mathcal{S}$, and that C_1 and C_2 are directly linked. If

$$((C_1D)_0 + (C_2D)_0)(S(D) + S^{-1}(D)) \subseteq H_0 \tag{5.6}$$

holds, then C_1D and C_2D are linked;

(2) Suppose $C, D_1, D_2 \in \mathcal{S}$, and that D_1 and D_2 are directly linked. If

$$(S(C) + S^{-1}(C))((CD_1)_0 + (CD_2)_0) \subseteq H_0 \tag{5.7}$$

holds, then CD_1 and CD_2 are linked.

Here $(C_1D)_0$ denotes the coradical of the subcoalgebra C_1D, and so on in conditions (5.6) and (5.7).

Proof:

(1) Assume that $C_1 \wedge C_2 \supsetneqq C_1 + C_2$ without the loss of generality. Suppose $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}$ are basic multiplicative matrices of C_1, C_2, D with sizes r_1, r_2, s, respectively. Then by Lemma 5.1.6(2), there exists a non-trivial $(\mathcal{C}_1, \mathcal{C}_2)$-primitive matrix $\mathcal{X}$. Define a square matrix

$$\mathcal{G} := \begin{pmatrix} \mathcal{C}_1 & \mathcal{X} \\ 0 & \mathcal{C}_2 \end{pmatrix} \odot \mathcal{D} = \begin{pmatrix} \mathcal{C}_1 \odot \mathcal{D} & \mathcal{X} \odot \mathcal{D} \\ 0 & \mathcal{C}_2 \odot \mathcal{D} \end{pmatrix}.$$

Note that according to Lemma 2.3.7(2), matrices $\mathcal{C}_1 \odot \mathcal{D}$, $\mathcal{C}_2 \odot \mathcal{D}$ and $\mathcal{G}$ are all multiplicative.

Now we try to observe properties of the $r_1 s \times r_2 s$ matrix $\mathcal{X} \odot \mathcal{D}$ in details. Of course, each row of $\mathcal{X} \odot \mathcal{D}$ is a vector in $H^{r_1 s \times 1}$, which denotes the space of all row vectors with $r_1 s$ entries from H. Moreover, we might regard these rows as vectors in $(H/H_0)^{r_1 s \times 1}$ with entries from the quotient space H/H_0. Similar conventions are made for column vectors and spaces $H^{1 \times r_2 s}$ and $(H/H_0)^{1 \times r_2 s}$. We aim to show that the following properties (i) and (ii) for the matrix $\mathcal{X} \odot \mathcal{D}$ both hold:

(i) The set of all its row vectors is linearly independent over H/H_0;

(ii) The set of all its column vectors is linearly independent over H/H_0.

At first we try to show that $\mathcal{X} \odot \mathcal{D}$ has property (i). Clearly, all the entries of $\mathcal{X} \odot \mathcal{D}$ must belong to $C_1 D \wedge C_2 D$, and thus trivial ones among them would belong to $(C_1 D)_0 + (C_2 D)_0$.

Assume on the contrary that (i) does not hold for $\mathcal{X} \odot \mathcal{D}$, or equivalently, there is an non-zero $1 \times r_1 s$ matrix P over $\mathbb{k}$ such that $P(\mathcal{X} \odot \mathcal{D})$ is trivial as a row vector in $H^{1 \times r_2 s}$. Moreover, clearly $\mathcal{X} \odot \mathcal{D}$ is actually a matrix over the subcoalgebra $C_1 D \wedge C_2 D$. Thus all entries of the trivial vector $P(\mathcal{X} \odot \mathcal{D})$ would belong to $(C_1 D)_0 + (C_2 D)_0$.

However, we could compute by Lemma 2.3.10(2) that

$$P(\mathcal{X} \odot \mathcal{D})(I_{r_2} \odot S(\mathcal{D})) = P(\mathcal{X} \odot \mathcal{D} S(\mathcal{D})) = P(\mathcal{X} \odot I_s),$$

whose entries all lie in $((C_1D)_0 + (C_2D)_0)S(D) \subseteq H_0$ due to our condition (5.6). This is a contradiction to the fact that $P(\mathcal{X} \odot I_s)$ is a non-trivial row vector, because $\mathcal{X} \odot I_s$ must have property (i) by the definition of our Kronecker product $\odot$ as well as Proposition 2.3.14(3).

On the other hand, a similar argument would follow that the matrix $(\mathcal{X} \odot \mathcal{D})^{\mathrm{T}}$ has property (i) as well. Specifically, for any non-zero $1 \times r_2 s$ matrix Q over $\Bbbk$, we could compute by Lemma 2.3.10 again to know that:

$$\begin{aligned} Q(\mathcal{X} \odot \mathcal{D})^{\mathrm{T}}(I_{r_1} \odot S^{-1}(\mathcal{D}))^{\mathrm{T}} &= Q(\mathcal{X}^{\mathrm{T}} \odot \mathcal{D}^{\mathrm{T}})(I_{r_1} \odot S^{-1}(\mathcal{D})^{\mathrm{T}}) \\ &= Q(\mathcal{X}^{\mathrm{T}} \odot \mathcal{D}^{\mathrm{T}} S^{-1}(\mathcal{D})^{\mathrm{T}}) \\ &= Q(\mathcal{X}^{\mathrm{T}} \odot I_s), \end{aligned}$$

whose entries would all lie in $((C_1D)_0 + (C_2D)_0)S^{-1}(D) \subseteq H_0$ by the condition (5.6). Of course, this is equivalent to say $\mathcal{X} \odot \mathcal{D}$ has property (ii).

Next we turn to deal with $\mathcal{G}$. It is followed by Proposition 2.3.6(1) that there exist invertible matrices L_1 and L_2 over $\Bbbk$, such that

$$L_1(\mathcal{C}_1 \odot \mathcal{D})L_1^{-1} = \begin{pmatrix} \mathcal{E}_1 & \mathcal{Y}_{12} & \cdots & \mathcal{Y}_{1t} \\ 0 & \mathcal{E}_2 & \cdots & \mathcal{Y}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{E}_t \end{pmatrix} \quad \text{and}$$

$$L_2(\mathcal{C}_2 \odot \mathcal{D})L_2^{-1} = \begin{pmatrix} \mathcal{F}_1 & \mathcal{Z}_{12} & \cdots & \mathcal{Z}_{1u} \\ 0 & \mathcal{F}_2 & \cdots & \mathcal{Z}_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{F}_u \end{pmatrix}$$

both hold, where $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t$ and $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_u$ are basic multiplicative matrices over H. Meanwhile we denote

$$L_1(\mathcal{X} \odot \mathcal{D})L_2^{-1} = \begin{pmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} & \cdots & \mathcal{X}_{1u} \\ \mathcal{X}_{12} & \mathcal{X}_{22} & \cdots & \mathcal{X}_{2u} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}_{t1} & \mathcal{X}_{t2} & \cdots & \mathcal{X}_{tu} \end{pmatrix},$$

where for each $1 \leq i \leq t$ and $1 \leq j \leq u$, the matrix $\mathcal{X}_{ij}$ has the same number of rows with $\mathcal{E}_i$, and has the same number of columns with $\mathcal{F}_j$. These notations are concluded as follows:

$$\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \mathcal{G} \begin{pmatrix} L_1^{-1} & 0 \\ 0 & L_2^{-1} \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1 & \cdots & \mathcal{Y}_{1t} & \mathcal{X}_{11} & \cdots & \mathcal{X}_{1u} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{E}_t & \mathcal{X}_{t1} & \cdots & \mathcal{X}_{tu} \\ & & & \mathcal{F}_1 & \cdots & \mathcal{Z}_{1u} \\ & 0 & & \vdots & \ddots & \vdots \\ & & & 0 & \cdots & \mathcal{F}_u \end{pmatrix}$$

as multiplicative matrices.

Recall that we has shown that $\mathcal{X} \odot \mathcal{D}$ has properties (i) and (ii). It is then not hard to know that (i) and (ii) both hold for $L_1(\mathcal{X} \odot \mathcal{D})L_2^{-1}$ as well. Therefore, we could obtain following two facts:

(I) For each $1 \leq i \leq t$, there is some $1 \leq j \leq u$ such that $\mathcal{X}_{ij}$ is non-trivial. Meanwhile,

(II) For each $1 \leq j \leq u$, there is some $1 \leq i \leq t$ such that $\mathcal{X}_{ij}$ is non-trivial.

Finally according to Proposition 5.1.8(2), the non-trivially of $\mathcal{X}_{ij}$ implies that the simple subcoalgebras corresponding to $\mathcal{E}_i$ and $\mathcal{F}_j$ are linked. As a conclusion, C_1D and C_2D are linked in the sense of Definition 5.1.9.

(2) Consider the opposite Hopf algebra H^{op} with multiplication $\cdot^{\text{op}}$ and antipode S^{-1}, where our condition (5.7) becomes

$$((D_1 \cdot^{\text{op}} C)_0 + (D_2 \cdot^{\text{op}} C)_0) \cdot^{\text{op}} (S^{-1}(C) + S(C)) \subseteq H_0.$$

Of course D_1 and D_2 are also directly linked in H^{op}, and thus subcoalgebras $D_1 \cdot^{\text{op}} C$ and $D_2 \cdot^{\text{op}} C$ are linked according to (1). This is exactly our desired result.

With Lemma 5.1.12, a sufficient condition for $H_{(1)}$ to be a Hopf subalgebra could be given as follows.

Proposition 5.1.13 *Let H be a Hopf algebra over an algebraically closed field $\Bbbk$ with the bijective antipode S. If*

$$({H_{(1)})_0}^3 \subseteq H_0 \tag{5.8}$$

holds, then $H_{(1)}$ is a Hopf subalgebra.

Proof: Clearly, the unit element 1 belongs to the subcoalgebra $H_{(1)}$, and we know by Corollary 5.1.11 that $S(H_{(1)}) \subseteq H_{(1)}$ holds as well. It remains to prove that $H_{(1)}$ is closed under the multiplication, which is written as ${H_{(1)}}^2 \subseteq H_{(1)}$. However by Remark 5.1.10, we only need to show that each simple subcoalgebra E of ${H_{(1)}}^2$ is linked with $\Bbbk 1$.

In fact, it is known that $(H_{(1)} \otimes H_{(1)})_0 = (H_{(1)})_0 \otimes (H_{(1)})_0$, since $\Bbbk$ is algebraically closed ([40, Corollary 4.1.8] for example). Consider the multiplication on H as an epimorphism $H_{(1)} \otimes H_{(1)} \to {H_{(1)}}^2$ of coalgebras, and it is followed by [34, Corollary 5.3.5] that

$$({H_{(1)}}^2)_0 \subseteq {(H_{(1)})_0}^2 = \sum_{\substack{C\in\mathcal{S}\\ C\subseteq H_{(1)}}} \sum_{\substack{D\in\mathcal{S}\\ D\subseteq H_{(1)}}} CD.$$

Therefore, each simple subcoalgebra E of ${H_{(1)}}^2$ must be contained in some subcoalgebra CD, where C and D are both linked with $\Bbbk 1$.

Note that condition (5.8) and the fact $S(H_{(1)}) \subseteq H_{(1)}$ would imply that any triples of simple subcoalgebras of $H_{(1)}$ would satisfy conditions (5.6) as well as (5.7). As a consequence, for any $C, D \in \mathcal{S}$ linked with $\Bbbk 1$, we find that CD is linked with $(\Bbbk 1)^2 = \Bbbk 1$ in final. It is concluded that ${H_{(1)}}^2$ and $\Bbbk 1$ are linked, and the desired result is obtained.

In order to study the products for arbitrary link-indecomposable components $H_{(C)}$ and $H_{(D)}$, we might require a stronger condition for H. Recall in the literature that a finite-dimensional Hopf algebra H is said to have the dual Chevalley property, if its coradical H_0 is a Hopf subalgebra. In this chapter, we also use the term *dual Chevalley property* to indicate a Hopf algebra H with its coradical H_0 as a Hopf subalgebra, even if H is infinite-dimensional.

Evidently, when the antipode S is bijective, the dual Chevalley property is equivalent to the requirement that ${H_0}^2 \subseteq H_0$. On the other hand, the bijectivity of S is in fact a consequence of the dual Chevalley property:

Lemma 5.1.14 *([38, Corollary 3.6]) Let H be a Hopf algebra. Suppose that H has the dual Chevalley property, which means that its coradical H_0 is a Hopf subalgebra. Then the antipode S is bijective.*

The following direct corollary is due to a similar argument as the end of the proof of Proposition 5.1.13. Namely, the dual Chevalley property follows a fact that any triples in $\mathcal{S}$ would satisfy conditions (5.6) and (5.7), since the antipode is bijective in this case.

Corollary 5.1.15 *Let H be a Hopf algebra over an algebraically closed field $\Bbbk$ with the dual Chevalley property. Suppose $C_1, C_2, D_1, D_2 \in \mathcal{S}$. If C_1, C_2 are linked, and D_1, D_2 are also linked, then C_1D_1 and C_2D_2 are linked.*

Our main result could be a generalized version of [35, Theorem 3.2(1)]:

Proposition 5.1.16 *Let H be a Hopf algebra over an algebraically closed field $\Bbbk$ with the dual Chevalley property. Then*

(1) For any $C, D \in \mathcal{S}$, $H_{(C)}H_{(D)} \subseteq \sum_{E\in\mathcal{S},\ E\subseteq CD} H_{(E)}$;

(2) $H_{(1)}$ is a Hopf subalgebra.

Proof:

(1) The proof is basically similar to which of Proposition 5.1.13. By Remark 5.1.10 as well, it is sufficient to show that each simple subcoalgebra E' of $H_{(C)}H_{(D)}$ is linked with some $E \in \mathcal{S}$ contained in CD.

The same argument follows $(H_{(C)}H_{(D)})_0 \subseteq (H_{(C)})_0(H_{(D)})_0$ at first, though in fact the dual Chevalley property implies

$$(H_{(C)}H_{(D)})_0 = (H_{(C)})_0(H_{(D)})_0 = \sum_{\substack{C'\in\mathcal{S} \\ C'\subseteq H_{(C)}}} \sum_{\substack{D'\in\mathcal{S} \\ D'\subseteq H_{(D)}}} C'D'.$$

Therefore, each simple subcoalgebra E' of $H_{(C)}H_{(D)}$ must be contained in some $C'D'$, where C', C are linked, and D', D are linked.

Note that CD is linked with this $C'D'$, according to Corollary 5.1.15. As a consequence, we could know each simple subcoalgebra E' of $H_{(C)}H_{(D)}$ is linked with some $E \in \mathcal{S}$ contained in CD, and the desired result is obtained.

(2) This is a particular case of Proposition 5.1.13, as the condition (5.8) would be followed by the dual Chevalley property of H.

Remark 5.1.17 *In fact, the assumption that $\Bbbk$ is algebraically closed is not necessary. This would be shown as Theorem 5.2.8 in the next section.*

§5.2 Further Presentations of Link-Indecomposable Components

In this section, we try to find much more explicit presentations of the link-indecomposable components for a Hopf algebra H with the dual Chevalley property. Our main idea is to identify H with the smash coproduct $H_0 \blacktriangleright\!\!< H/{H_0}^+H$ as left H_0-module coalgebras. However, the link-indecomposable decomposition of $H_0 \blacktriangleright\!\!< H/{H_0}^+H$ is in fact determined by a semisimple decomposition of H_0 as a sense of relative $(H_0, (H_{(1)})_0)$-Hopf bi(co)modules. This process would be stated in the following subsections.

§5.2.1 Relative Hopf Bi(co)modules over Cosemisimple Hopf Algebras

We begin with a cosemisimple Hopf algebra J over an arbitrary field $\Bbbk$, whose antipode is then bijective according to [22, Theorem 3.3]. Suppose that K is a Hopf subalgebra of J. Clearly K is also cosemisimple, and it is known that J is faithfully flat (or a projective generator) over K by [4, Theorem 2.1].

Let ${}^J\mathcal{M}_K$ (resp. $\mathcal{M}_K^J$) denote the category consisting of *relative* (J, K)*-Hopf modules* which are right K-modules as well as left (resp. right) J-comodules. One could see [32, Section 1] for details of the definition. Now consider the $\Bbbk$-linear abelian category ${}^J\mathcal{M}_K^J$ consisting of (J, J)-bicomodules M equipped with right K-module structure which are Hopf modules both in ${}^J\mathcal{M}_K$ and $\mathcal{M}_K^J$.

Proposition 5.2.1 *With notations above, ${}^J\mathcal{M}_K^J$ is semisimple.*

Proof: We aim to show that every short exact sequence

$$0 \to L \xrightarrow{f} M \to N \to 0$$

in ${}^J\mathcal{M}^J_K$ splits. This, regarded as a short exact sequence in ${}^J\mathcal{M}_K$, splits K-linearly since N is K-projective by [32, Corollary 2.9]. Consequently, there is a right K-module map $g : M \to L$ such that $gf = \mathrm{id}_L$.

Note that due to the cosemisimplicity (or rather, the coseparability) of J, we have a (J, J)-bicomodule map $\phi : J \to \Bbbk$ such that $\phi(1) = 1$. Moreover, if we denote the (J, J)-bicomodule structures of L and M by

$$\rho_M : M \to J \otimes M \otimes J,\ m \mapsto \sum m_{(-1)} \otimes m_{(0)} \otimes m_{(1)},$$
$$\rho_L : L \to J \otimes L \otimes J,\ l \mapsto \sum l_{(-1)} \otimes l_{(0)} \otimes l_{(1)},$$

then a retraction of ρ_L (in ${}^J\mathcal{M}^J_K$) could be obtained as in [7, Theorem 1], which is given by

$$r_L : J \otimes L \otimes J \to L,\ a \otimes l \otimes b \mapsto \sum \phi(l_{(-1)}S^{-1}(a))l_{(0)}\phi(l_{(1)}S(b)).$$

Specifically, consider the right K-module and (J, J)-bicomodule structures of $J \otimes L \otimes J \in {}^J\mathcal{M}^J_K$ defined by

$$(a \otimes l \otimes b) \cdot c := \sum ac_{(1)} \otimes lc_{(2)} \otimes bc_{(3)} \quad \text{and} \quad a \otimes l \otimes b \mapsto \sum a_{(1)} \otimes a_{(2)} \otimes l \otimes b_{(1)} \otimes b_{(2)}$$

respectively for $a, b \in J$, $c \in K$ and $l \in L$, which clearly make $J \otimes L \otimes J$ an object in ${}^J\mathcal{M}^J_K$. One could find that r_L is indeed a morphism in ${}^J\mathcal{M}^J_K$ satisfying $r_L\rho_L = \mathrm{id}_L$ by direct computations. In fact, for any $a, b \in J$, $c \in K$ and $l \in L$,

$$\begin{aligned}
r_L((a \otimes l \otimes b) \cdot c) &= \sum r_L(ac_{(1)} \otimes lc_{(2)} \otimes bc_{(3)}) \\
&= \sum \phi\left((lc_{(2)})_{(-1)}S^{-1}(ac_{(1)})\right)(lc_{(2)})_{(0)}\phi\left((lc_{(2)})_{(1)}S(bc_{(3)})\right) \\
&= \sum \phi\left(l_{(-1)}c_{(2)}S^{-1}(c_{(1)})S^{-1}(a)\right)l_{(0)}c_{(3)}\phi\left(l_{(1)}c_{(4)}S(c_{(5)})S(b)\right) \\
&= \sum \phi\left(l_{(-1)}S^{-1}(a)\right)l_{(0)}c\phi\left(l_{(1)}S(b)\right) \\
&= \sum \phi\left(l_{(-1)}S^{-1}(a)\right)l_{(0)}\phi\left(l_{(1)}S(b)\right)c \\
&= r_L(a \otimes l \otimes b)c.
\end{aligned}$$

We also compute that

$$
\begin{aligned}
& \rho_L \circ r_L(a\otimes l\otimes b) \\
= & \sum \rho_L\left(\phi(l_{(-1)}S^{-1}(a))l_{(0)}\phi(l_{(1)}S(b))\right) \\
= & \sum \phi(l_{(-1)}S^{-1}(a))\rho_L(l_{(0)})\phi(l_{(1)}S(b)) \\
= & \sum \phi(l_{(-2)}S^{-1}(a))l_{(-1)}\otimes l_{(0)}\otimes l_{(1)}\phi(l_{(2)}S(b)) \\
= & \sum \phi(l_{(-2)}S^{-1}(a_{(3)}))l_{(-1)}S^{-1}(a_{(2)})a_{(1)}\otimes l_{(0)}\otimes l_{(1)}S(b_{(2)})b_{(3)}\phi(l_{(2)}S(b_{(1)})) \\
= & \sum \phi(l_{(-1)}S^{-1}(a_{(2)}))a_{(1)}\otimes l_{(0)}\otimes b_{(2)}\phi(l_{(1)}S(b_{(1)})) \\
= & \sum a_{(1)}\otimes\phi(l_{(-1)}S^{-1}(a_{(2)}))l_{(0)}\otimes\phi(l_{(1)}S(b_{(1)}))b_{(2)} \\
= & \sum a_{(1)}\otimes r_L\big(a_{(2)}\otimes l\otimes b_{(1)}\big)\otimes b_{(2)}
\end{aligned}
$$

due to the (J,J)-bicomodule map $\phi : J \to \Bbbk$. Above computations are essentially the same as the proof of [7, Lemma (3)(ii) on Page 101]

On the other hand, it is not hard to know that ρ_M and $\mathrm{id}\otimes g\otimes\mathrm{id}$ are both morphisms in ${}^J\mathcal{M}^J_K$, since g is a right K-module map. Consequently, a morphism $r_L(\mathrm{id}\otimes g\otimes\mathrm{id})\rho_M$ from M to L is found in ${}^J\mathcal{M}^J_K$. However, we could know by $gf=\mathrm{id}_L$ that

$$r_L(\mathrm{id}\otimes g\otimes\mathrm{id})\rho_M\circ f = r_L(\mathrm{id}\otimes g\otimes\mathrm{id})(\mathrm{id}\otimes f\otimes\mathrm{id})\rho_L = r_L\rho_L = \mathrm{id}_L,$$

and thus the desired splitting is obtained.

It is evident that J is an object in ${}^J\mathcal{M}^J_K$. A simple subobject of J in ${}^J\mathcal{M}^J_K$ is precisely a non-zero minimal right K-module subcoalgebra of J, which would be called a *simple right K-module subcoalgebra* of J. In fact, we could give more explicit presentations of these simple right K-module subcoalgebras in our situation:

Corollary 5.2.2 *Let J and K be as above. Then:*

(1) J is the direct sum of all simple right K-module subcoalgebras of J;

(2) For every simple subcoalgebra C of J, CK is a simple right K-module subcoalgebra of J. In particular, K is a simple right K-module subcoalgebra of J;

(3) Each simple right K-module subcoalgebra K_i is of the form CK, where C could be chosen as any simple subcoalgebra contained in K_i;

(4) Given simple subcoalgebras C and D of J, we have that $CK = DK$ if and only if $C \subseteq DK$ if and only if $CK \supseteq D$.

Proof:

(1) This is followed by Proposition (5.2.1) directly.

(2) According to (1), every simple subcoalgebra C of J would be contained in some simple right K-module subcoalgebra K_i of J. However, CK is clearly a non-zero subobject of K_i in ${}^{J}\mathcal{M}_{K}^{J}$. Thus $CK = K_i$ holds due to the minimality of K_i.

(3) The reason is similar to the proof of (2), since K_i must contain some simple subcoalgebra C of J.

(4) It follows by (2) that CK and DK are both simple right K-module subcoalgebras of J. The desired claim is then obtained according to (3).

§5.2.2 A Smash Coproduct and its Link-Indecomposable Components

We still let J be a cosemisimple Hopf algebra in this subsection, but let Q be a irreducible coalgebra whose coradical Q_0 is spanned by a grouplike element g. Moreover, suppose Q is a right J-comodule coalgebra such that J coacts trivially on g, or namely, $g \mapsto g \otimes 1$ under this coaction $Q \to Q \otimes J$. Then the *smash coproduct* ([6, Section 4.2]) of Q with J is constructed as

$$H := J \blacktriangleright\!\!< Q,$$

whose coalgebra structure is defined by

$$\Delta(a \blacktriangleright\!\!< q) := \sum (a_{(1)} \blacktriangleright\!\!< q_{(1)(0)}) \otimes (a_{(2)} q_{(1)(1)} \blacktriangleright\!\!< q_{(2)}) \quad \text{and} \quad \varepsilon(a \blacktriangleright\!\!< q) = \varepsilon(a)\varepsilon(q)$$

for any $a \in J$ and $q \in Q$. Furthermore, this is evident a left J-module coalgebra, via the J-module structure

$$J \otimes H \to H,\ b \otimes (a \blacktriangleright\!\!< q) \mapsto ba \blacktriangleright\!\!< q.$$

Also, we have following straightforward descriptions of the coradical H_0 of H:

Lemma 5.2.3 *Suppose J, Q are defined as above, and $H := J \blacktriangleright\!\!< Q$. Then:*

(1) The coradical H_0 is $J \otimes \Bbbk g \cong J$ as coalgebras;

(2) If we let

$$\pi : H = J \blacktriangleright\!\!< Q \to J \otimes (Q/Q^+) \cong J,\ a \blacktriangleright\!\!< q \mapsto a\varepsilon(q) \tag{5.9}$$

denote the natural left J-module coalgebra projection, then for any subcoalgebra H' of H, $\pi(H')$ is the coradical of H'.

Proof:

(1) Since the right J-coaction on the unique grouplike $g \in Q$ is trivial, we find that the cosemisimple subcoalgebra $J \otimes \Bbbk g$ cogenerates H (via wedge products). In fact, one could verify that for any $n \geq 1$,

$$\Delta(J \blacktriangleright\!\!< Q_n) \subseteq (J \blacktriangleright\!\!< \Bbbk g) \otimes H + H \otimes (J \blacktriangleright\!\!< Q_{n-1})$$

holds according to $\Delta(Q_n) \subseteq \Bbbk g \otimes Q + Q \otimes Q_{n-1}$, where $\{Q_n\}_{n\geq 0}$ denotes the coradical filtration.

(2) Evidently π is a left J-module coalgebra map. On the other hand, as an irreducible coalgebra, clearly $Q = \Bbbk g \oplus Q^+$ holds. Note that Q^+ is a coideal of Q. Therefore, $J \blacktriangleright\!\!< Q^+$ is also a coideal of H, and $H = H_0 \oplus (J \blacktriangleright\!\!< Q^+)$ holds. It is not hard to see that the coalgebra map π is exactly the projection from H to H_0, with respect to this direct sum. Then the desired claim $\pi(H') = H'_0$ could be shown, as a direct application of [40, Proposition 4.1.7(a)] to $\pi_{H'}$ for example.

Another notion required is the coefficient space for comodules. In general, given a right comodule $V = (V, \rho)$ over a coalgebra Z, the *coefficient space* $C_Z(V)$ is the smallest subspace (indeed, subcoalgebra) Z' of Z such that $\rho(V) \subseteq V \otimes Z'$. See [40, Theorem 3.2.11] for example.

Now with the construction of the smash coproduct $H = J \blacktriangleright\!\!< Q$, its indecomposable decomposition of H could be determined under specific assumptions:

Lemma 5.2.4 *Let K denote the smallest Hopf subalgebra of the cosemisimple Hopf algebra J that includes the coefficient space $C_J(Q)$ of the right J-comodule Q. With notations above, we have:*

(1) The coalgebra H decomposes as a direct sum of subcoalgebras:

$$H = \bigoplus_i K_i \blacktriangleright\!\!< Q, \tag{5.10}$$

where K_i are the simple right K-module subcoalgebras of J such that $J = \bigoplus_i K_i$;

(2) If the coradical $(H_{(1)})_0$ of $H_{(1)}$ is a Hopf subalgebra of J, and the coradical of each link-indecomposable component of H is stable under the right multiplication by $(H_{(1)})_0$, then (5.10) gives the link-indecomposable decomposition of the coalgebra H.

Proof:

(1) Since K_i is a right K-module coalgebra, it then follows from our assumption $K \supseteq C_J(Q)$ that

$$\Delta(K_i \blacktriangleright\!\!< Q) \subseteq (K_i \blacktriangleright\!\!< Q) \otimes (K_i C_J(Q) \blacktriangleright\!\!< Q) \subseteq (K_i \blacktriangleright\!\!< Q) \otimes (K_i \blacktriangleright\!\!< Q)$$

by definition. As a result, $K_i \blacktriangleright\!\!< Q$ is a subcoalgebra of H. This together with Corollary 5.2.2(1) proves the desired claim.

(2) Recall that we have regarded $H_0 = J \otimes \Bbbk g \cong J$, and hence the coradical of each link-indecomposable component of H is identified as a subcoalgebra of the Hopf algebra J.

Suppose that H' is a link-indecomposable component of H, and E is a subcoalgebra of the coradical H'_0 of H'. We claim firstly that the coefficient space $C_J(E \blacktriangleright\!\!< Q)$ of the right coideal $E \blacktriangleright\!\!< Q$ of H, regarded as a right J-comodule through π (5.9), is included in H'_0.

In fact, as a right coideal, $E \blacktriangleright\!\!< Q$ is a direct summand of H and hence an injective comodule. On the other hand, its socle $(E \blacktriangleright\!\!< Q)_0$, namely the direct sum of all the simple right H-subcomodules, is of course included by H_0. Therefore

$$(E \blacktriangleright\!\!< Q)_0 = \pi((E \blacktriangleright\!\!< Q)_0) \subseteq \pi(E \blacktriangleright\!\!< Q) = E.$$

As a conclusion, $E \blacktriangleright\!\!< Q$ is contained in the injective hull of E as a right coideal by

[40, Proposition 3.5.6(1)]. Hence we could know that $E \blacktriangleright\!\!< Q$ is also contained in the component H' according to [40, Lemma 3.7.1]. This implies that $H' \supseteq C_H(E \blacktriangleright\!\!< Q)$, and consequently $H'_0 \supseteq C_J(E \blacktriangleright\!\!< Q)$ holds as well.

Now consider the link-indecomposable component $H_{(1)}$, and choose the subcoalgebra $E = \Bbbk 1$. The claim above provides $(H_{(1)})_0 \supseteq C_J(Q)$. It follows that if $(H_{(1)})_0$ is a Hopf subalgebra of J, then it includes $K = (K \blacktriangleright\!\!< Q)_0$. We conclude that $H_{(1)} \supseteq K \blacktriangleright\!\!< Q$ as subcoalgebras. Moreover, since $H_{(1)}$ is a indecomposable direct summand of the coalgebra H, we then see from (5.10) that

$$H_{(1)} = K \blacktriangleright\!\!< Q \tag{5.11}$$

holds, and also $(H_{(1)})_0 = K$ as their coradicals.

Finally for any $C \in \mathcal{S}$, assume that the coradical $(H_{(C)})_0$ of the component $H_{(C)}$ is right K-stable. We could know according to Corollary 5.2.2 that

$$(H_{(C)})_0 \supseteq (H_{(C)})_0 K \supseteq CK = K_i$$

for some simple right K-module subcoalgebra K_i of J. Then $H_{(C)} = K_i \blacktriangleright\!\!< Q$ holds, as is seen from (5.10) by the same argument again. In a word, each link-indecomposable component of H is of form $K_i \blacktriangleright\!\!< Q$, which completes the proof.

Remark 5.2.5 *It is not clear whether $K_i \blacktriangleright\!\!< Q$ in (5.10) is indecomposable without the assumptions of (2), so that the decomposition above may not be indecomposable.*

§5.2.3 Presentations of Components with the Dual Chevalley Property

In this subsection, H is assumed to be a Hopf algebra with the dual Chevalley property over an arbitrary field $\Bbbk$. For convenience, let $J = H_0$ be its coradical, and let

$$Q := H/J^+H,$$

which is clearly a quotient right H-module coalgebra of H. In addition, Q is a irreducible coalgebra with the one-dimensional coradical $Q_0 = \Bbbk\overline{1}$ spanned by the natural image $\overline{1}$ of $1 \in H$, because Q is cogenerated by

$$\overline{H_0} = H_0/(H_0^+ H \cap H_0) = H_0/H_0^+ = \Bbbk\overline{1}.$$

Furthermore, it could be known by [31, Theorem 3.1] that H is isomorphic to the smash coproduct $J \blacktriangleright\!\!< Q$ constructed. We recall this result with a bit more details, all of which could be found in [31, Section 4].

Lemma 5.2.6 *([31, Theorem 3.1]) Let H be a Hopf algebra with the dual Chevalley property. Denote $J := H_0$ and $Q := H/H_0^+ H$ respectively as above. Then:*

(1) There exists a left J-module coalgebra retraction $\gamma : H \to J$ of the inclusion $J \hookrightarrow H$;

(2) Q turns into a right J-comodule coalgebra through the structure $\rho : Q \to Q \otimes J$ induced from

$$H \to Q \otimes J,\ h \mapsto \sum \overline{h_{(2)}} \otimes S(\gamma(h_{(1)}))\gamma(h_{(3)}),$$

where $h \mapsto \overline{h}$ represents the projection $H \twoheadrightarrow Q$;

(3) The resulting left J-module coalgebra of smash coproduct of Q with J is isomorphic to H through

$$H \xrightarrow{\cong} J \blacktriangleright\!\!< Q,\ h \mapsto \sum \gamma(h_{(1)}) \otimes \overline{h_{(2)}}. \tag{5.12}$$

We remark that by taking the associated grading gr with respect to the coradical filtration, the isomorphism (5.12) turns into the canonical isomorphism

$$\mathsf{gr}\ H \xrightarrow{\cong} J \ltimes R$$

of the graded Hopf algebra $\mathsf{gr}\ H$ onto the associated bosonization, where $R = \mathsf{gr}\ Q$, a Nichols algebra in ${}^J_J\mathcal{YD}$. Since R is generated by $(\mathsf{gr}\ Q)(1) = P(Q)$, the space of the primitives in Q, it follows that the smallest Hopf subalgebras of J that includes the following three coefficient spaces

$$C_J(Q),\ \ C_J(R),\ \ C_J(P(H))$$

coincide. We let K denote the coinciding Hopf subalgebra.

The following proposition could be proved with the usage of Proposition 5.1.16 over the algebraic closure $\overline{\Bbbk}$ of the base field $\Bbbk$.

Proposition 5.2.7 *Let H be a Hopf algebra with the dual Chevalley property over $\Bbbk$. Denote $J := H_0$ and $Q := H/H_0^+H$ respectively. Then (5.10) gives the link-indecomposable decomposition of H.*

Proof: We aim to verify the assumption of Lemma 5.2.4(2) for H. Firstly, consider the Hopf algebra $H \otimes \overline{\Bbbk}$ over the algebraically closed field $\overline{\Bbbk}$. Note that $H \otimes \overline{\Bbbk}$ also has the dual Chevalley property, since it is cogenerated by its cosemisimple Hopf subalgebra $J \otimes \overline{\Bbbk}$ ([22, Lemma 1.3]).

Now it follows from Proposition 5.1.16(2) that the coradical of $(H \otimes \overline{\Bbbk})_{(1)}$ is a Hopf $\overline{\Bbbk}$-subalgebra of $(H \otimes \overline{\Bbbk})_0 = J \otimes \overline{\Bbbk}$. Besides, since

$$(H \otimes \overline{\Bbbk})/(J \otimes \overline{\Bbbk})^+(H \otimes \overline{\Bbbk}) = (H/J^+H) \otimes \overline{\Bbbk} = Q \otimes \overline{\Bbbk},$$

the smallest Hopf subalgebra of $H \otimes \overline{\Bbbk}$ including $C_{J\otimes\overline{\Bbbk}}(Q \otimes \overline{\Bbbk})$ is exactly $K \otimes \overline{\Bbbk}$. Thus we see from Equation (5.11) on $H \otimes \overline{\Bbbk}$ in the proof of Lemma 5.2.4(2) that

$$(H \otimes \overline{\Bbbk})_{(1)} = (K \otimes \overline{\Bbbk}) \blacktriangleright\!\!<_{\overline{\Bbbk}} (Q \otimes \overline{\Bbbk}) = (K \blacktriangleright\!\!< Q) \otimes \overline{\Bbbk}, \tag{5.13}$$

which is indecomposable as a subcoalgebra of $H \otimes \overline{\Bbbk}$. Therefore, $K \blacktriangleright\!\!< Q$ in H must be indecomposable, and this is exactly $H_{(1)}$ because we know that $(H \otimes \overline{\Bbbk})_{(1)} \subseteq H_{(1)} \otimes \overline{\Bbbk}$ holds. As a result, we have $(H_{(1)})_0 = K$, a Hopf subalgebra of J.

On the other hand, applying the projection (5.9) on the subcoalgebra $(H \otimes \overline{\Bbbk})_{(1)}$, we know by Equation (5.13) that its coradical is

$$((H \otimes \overline{\Bbbk})_{(1)})_0 = K \otimes \overline{\Bbbk}.$$

It then follows from Proposition 5.1.16(1) that each link-indecomposable component of $H \otimes \overline{\Bbbk}$ is right $K \otimes \overline{\Bbbk}$-stable. Consequently, a component of H turns, after the base extension $\otimes\overline{\Bbbk}$, into a direct sum of some components in $H \otimes \overline{\Bbbk}$, which are right $K \otimes \overline{\Bbbk}$-stable. Therefore, the original component of H and its coradical as well, are right K-stable.

Now we could improve the presentation for the link-indecomposable components of H as follows, where $\mathcal{S}$ still denotes the set of all the simple subcoalgebras:

Theorem 5.2.8 *Let H be a Hopf algebra over an arbitrary field $\Bbbk$ with the dual Chevalley property. Then*

(1) For any $C \in \mathcal{S}$, $H_{(C)} = CH_{(1)} = H_{(1)}C$;

(2) For any $C, D \in \mathcal{S}$, $H_{(C)}H_{(D)} \subseteq \sum\limits_{E \in \mathcal{S},\, E \subseteq CD} H_{(E)}$;

(3) $H_{(1)}$ is a Hopf subalgebra.

Proof:

(1) Combining Proposition 5.2.7 and Corollary 5.2.2 with notations in this subsection, we know that $H_{(C)} = CK \blacktriangleright\!\!< Q$ holds for any $C \in \mathcal{S}$. Moreover, it follows from the identification (5.12) $H = J \blacktriangleright\!\!< Q$ as left J-module coalgebras that

$$H_{(C)} = CK \blacktriangleright\!\!< Q = C(K \blacktriangleright\!\!< Q) = CH_{(1)}.$$

The other equation $H_{(C)} = H_{(1)}C$ is obtained by the opposite-sided results above.

(2) This is followed by (1), namely,

$$H_{(C)}H_{(D)} = CH_{(1)}DH_{(1)} = CDH_{(1)} = \left(\sum_{E \in \mathcal{S},\, E \subseteq CD} E \right) H_{(1)} \subseteq \sum_{E \in \mathcal{S},\, E \subseteq CD} H_{(E)}$$

for any $C, D \in \mathcal{S}$.

(3) This is directly followed by (2) and Corollary 5.1.11.

Remark 5.2.9 *Lemma 5.1.12 as well as Theorem 5.2.8 might fail for a Hopf algebra H without the dual Chevalley property. A counter-example is presented as Example 5.3.2 in the next section.*

In addition, we introduce an equivalence relation on $\mathcal{S}$, defining that C and D are related if $CK = DK$ (or equivalently $KC = KD$), where $K = (H_{(1)})_0$ as before. Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be a full set of chosen non-related representatives with respect to this equivalence relation. Then a presentation of the indecomposable decomposition of H is stated as follows:

Corollary 5.2.10 *Let H be a Hopf algebra over an arbitrary field $\Bbbk$ with the dual Chevalley property. Then the link-indecomposable decomposition of H could be presented as*

$$H = \bigoplus_{C\in\mathcal{S}_0} CH_{(1)}.$$

Proof: It suffices to show that for any $C, D \in \mathcal{S}$, $CK = DK$ if and only if C and D are linked. This could be known from the fact $CK \blacktriangleright\!\!< Q = H_{(C)}$ appearing in the proof of Theorem 5.2.8(1) which holds for each $C \in \mathcal{S}$.

This corollary partially generalizes Montgomery's result [35, Theorem 3.2(4)] (and its proof) for a pointed Hopf algebra H:

$$H = \bigoplus_{g\in G(H)/G(H_{(1)})} gH_{(1)},$$

where $G(H)$ and $G(H_{(1)})$ denote the set of all the grouplikes of H and $H_{(1)}$, respectively. Indeed when H is pointed, simple subcoalgebras spanned by $g, g' \in G(H)$ are related (in the sense preceding Corollary 5.2.10) if and only if $gG(H_{(1)}) = g'G(H_{(1)})$. We also remark that $G(H_{(1)})$ is a normal subgroup of $G(H)$ by [35, Theorem 3.2(3)].

One more corollary to mention is that the construction of $H_{(1)}$ is compatible with base extension:

Corollary 5.2.11 *Let H be a Hopf algebra with the dual Chevalley property over $\Bbbk$. Then for any field extension $\mathbb{F}/\Bbbk$,*

$$(H\otimes\mathbb{F})_{(1)} = H_{(1)}\otimes\mathbb{F}.$$

Proof: Let $\mathbb{F}$ replace $\overline{\Bbbk}$ in the proof of Proposition 5.2.7. Then by Theorem 5.2.8, the coradical of $(H\otimes\mathbb{F})_{(1)}$ is also a Hopf subalgebra. Consequently, equations as (5.13)

$$(H\otimes\mathbb{F})_{(1)} = (K \blacktriangleright\!\!< Q)\otimes\mathbb{F}$$

and $H_{(1)} = K \blacktriangleright\!\!< Q$ hold in this situation as well.

§5.2.4 Properties for H over the Hopf Subalgebra $H_{(1)}$

At the final of this section, the faithfully flatness and freeness of H as an $H_{(1)}$-module are discussed when H has the dual Chevalley property. In fact, the faithful flatness could be replaced by a stronger property that H is a projective generator as a $H_{(1)}$-module:

Corollary 5.2.12 *Let H be a Hopf algebra with the dual Chevalley property. Then:*

(1) H is a projective generator of left as well as right $H_{(1)}$-modules;

(2) For any $C \in \mathcal{S}$, $H_{(C)}$ is a finitely generated projective generator of left as well as right $H_{(1)}$-modules.

Proof:

(1) By Lemma 5.1.4, there is a direct sum $H = H_{(1)} \oplus M$, where M denotes the direct sum of all link-indecomposable components of H excluding $H_{(1)}$. It follows by Theorem 5.2.8(2) that $H_{(1)}M$ and $MH_{(1)}$ are both contained in M. The faithful flatness of H over $H_{(1)}$ is then obtained according to a combination of [4, Propositions 1.4 and 1.6].

Also, recall that Hopf algebras H and $H_{(1)}$ must have bijective antipodes, since they both have the dual Chevalley property. As a result, we could find by [32, Corollary 2.9] that H is even a projective generator as a left as well as right $H_{(1)}$-module.

(2) Clearly, it follows by Theorem 5.2.8(1) that each $H_{(C)}$ is finitely generated over $H_{(1)}$ (on both sides). On the other hand, since each $H_{(C)}$ is a non-zero object in ${}_{H_{(1)}}\mathcal{M}^H$ as well as ${}^H\mathcal{M}_{H_{(1)}}$, the desired result is obtained from (1) according to [32, Corollary 2.9] as well.

However, we could show that H is not always free over $H_{(1)}$ by an example in the following.

We begin with an example of commutative cosemisimple Hopf algebras J which is not free over some of its Hopf subalgebra K (see [47, Section 5] e.g.). One could choose also a right K-comodule V, such that the coefficient space $C_K(V)$ and its image $S(C_K(V))$ under the antipode S generate K. Let R be the symmetric algebra on V. It is routine to verify that via diagonal K-coactions, R forms a right K-comodule Hopf algebra containing the elements of V as primitives. According to [1, Section 4], the construction of smash coproduct of R with K gives rise to a commutative Hopf algebra

$$K \blacktriangleright\!\!< R,$$

which is the tensor product $K \otimes R$ as an algebra.

Besides, since R is naturally a right J-comodule Hopf algebra, we have the analogous Hopf algebra

$$H := J \blacktriangleright\!\!< R,$$

which includes $K \blacktriangleright\!\!< R$ as a Hopf subalgebra. Evidently H has coradical $J \otimes 1$ which is also a Hopf subalgebra. Moreover, note that R is irreducible as a coalgebra with grouplike element 1 on which J coacts trivially, and K is in fact the smallest Hopf subalgebra of J that includes $C_J(R)$ by our choice of V. Thus we know that $H_{(1)} = K \blacktriangleright\!\!< R$, as is seen from Proposition 5.2.7 as well as Lemma 5.2.4. We conclude that this H is not free over $H_{(1)}$. Otherwise, if it were free, the tensor product $\otimes_R R/R^+$ would imply that J were free over K, a contradiction to our chosen example.

Finally, recall in [34, Theorem 3.2] that $H_{(1)}$ must be a normal Hopf subalgebra for any pointed Hopf algebra H. When H is non-pointed, we pose the following question on the normality of $H_{(1)}$, although there are several positive examples such as $T_\infty(2,1,-1)^\circ$ presented in Subsection §5.3.2,

Question 5.2.13 *Let H be a Hopf algebra with the dual Chevalley property. Is the Hopf subalgebra $H_{(1)}$ normal in H?*

It is not clear to us whether the answer to this question is positive or not, even for $\mathsf{gr}\ H$, for which $(\mathsf{gr}\ H)_{(1)}$ is normal in $\mathsf{gr}\ H$ if and only if K is normal in J. This last condition should be related with possible structures on $P(R)$ as a Yetter-Drinfeld module, or namely, as an object in $\mathcal{YD}_J^J$.

§5.3 Examples

For the remaining of this section, $\Bbbk$ is always assumed to be an algebraically closed field of characteristic 0. Before specific examples, we provide an evident lemma which helps us determine link-indecomposable components:

Lemma 5.3.1 *Let H be a coalgebra, and $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ be basic multiplicative matrices of $C_1, C_2, \cdots, C_t \in \mathcal{S}$, respectively. Suppose that there is a multiplicative matrix of form*

$$\mathcal{G} := \begin{pmatrix} \mathcal{C}_1 & \mathcal{X}_{12} & \cdots & \mathcal{X}_{1t} \\ 0 & \mathcal{C}_2 & \cdots & \mathcal{X}_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{C}_t \end{pmatrix}. \tag{5.14}$$

If $C_1, C_2, \cdots, C_t$ are linked, then all the entries of $\mathcal{G}$ belong to this link-indecomposable component $H_{(C_1)}$.

Proof: Since $\mathcal{G}$ is multiplicative, all its entries would span a subcoalgebra H'. Also, $C_1, C_2, \cdots, C_t$ are exactly all the simple subcoalgebras of H'. Thus if $C_1, C_2, \cdots, C_t$ are linked, then H' is link-indecomposable and thus contained in the link-indecomposable component.

§5.3.1 Without the Dual Chevalley Property

As mentioned in the end of Section §5.1, the dual Chevalley property might be necessary for Lemma 5.1.12 or Corollary 5.1.15 in a way. We would show that the following Hopf algebra, denoted by $D(2, 2, \sqrt{-1})$, does not satisfy the property in Lemma 5.1.12. The structure is in fact a particular example of a certain classification $D(m, d, \xi)$ introduced in [50, Section 4.1], where m and d are both chosen to be 2.

Example 5.3.2 *Let $\sqrt{-1}$ be a fixed square root of -1. As an algebra, $D(2, 2, \sqrt{-1})$ is generated by $x^{\pm 1}, g^{\pm 1}, y, u_0, u_1$ with relations:*

$$xx^{-1} = x^{-1}x = 1, \;\; gg^{-1} = g^{-1}g = 1,$$
$$xy = yx, \;\; gx = xg, \;\; yg = -gy, \;\; y^2 = 1 - x^4 = 1 - g^2,$$
$$u_i x = x^{-1}u_i, \;\; u_i g = (-1)^i g^{-1} u_i, \;\; yu_i = (1 + (-1)^i x^2)u_{1-i} = \sqrt{-1}x^2 u_i y$$

for $i = 0, 1$, *and*

$$u_0^2 = \frac{1}{2}x(1+x^2)g^{-1}, \;\; u_0u_1 = \frac{\sqrt{-1}}{2}xg^{-1}y, \;\; u_1u_0 = -\frac{1}{2}xg^{-1}y, \;\; u_1^2 = -\frac{\sqrt{-1}}{2}x(1-x^2)g^{-1}.$$

The coalgebra structure and antipode are given by:

$$\begin{aligned}
&\Delta(x) = x \otimes x, \;\; \Delta(g) = g \otimes g, \;\; \Delta(y) = 1 \otimes y + y \otimes g, \\
&\Delta(u_0) = u_0 \otimes u_0 - u_1 \otimes x^{-2}gu_1, \;\; \Delta(u_1) = u_0 \otimes u_1 + u_1 \otimes x^{-2}gu_0, \\
&\varepsilon(x) = \varepsilon(g) = \varepsilon(u_0) = 1, \;\; \varepsilon(y) = \varepsilon(u_1) = 0, \\
&S(x) = x^{-1}, \;\; S(g) = g^{-1}, \;\; S(y) = g^{-1}y, \;\; S(u_0) = x^{-3}gu_0, \;\; S(u_1) = -\sqrt{-1}x^{-1}u_1.
\end{aligned}$$

With the application of the Diamond Lemma [2], we could know that $D(2, 2, \sqrt{-1})$ has a linear basis

$$\{x^ig^jy^l \mid 0 \leq i \leq 3, \; j \in \mathbb{Z}, \; 0 \leq l \leq 1\} \cup \{x^ig^ju_l \mid i \in \mathbb{Z}, \; 0 \leq j, l \leq 1\}. \tag{5.15}$$

An equivalent but more general version is [49, Lemma 3.3], but we write the basis in this form (5.15) for our purposes. Furthermore, all the simple subcoalgebras and their basic multiplicative matrices are also needed:

Proposition 5.3.3 *The set of all the simple subcoalgebras of* $D(2, 2, \sqrt{-1})$ *is*

$$\mathcal{S} = \{\Bbbk x^ig^j \mid 0 \leq i \leq 3, \; j \in \mathbb{Z}\} \cup \{x^iC \mid i \in \mathbb{Z}\},$$

where $C := \Bbbk\{x^{-2j}g^ju_l \mid 0 \leq j, l \leq 1\}$ *with a basic multiplicative matrix*

$$\mathcal{C} := \begin{pmatrix} u_0 & u_1 \\ -x^{-2}gu_1 & x^{-2}gu_0 \end{pmatrix},$$

and $x^iC \neq x^{i'}C$ *as long as* $i \neq i'$.

Proof: Verified by the structure of $D(2, 2, \sqrt{-1})$ and direct computations. One could see [49, Proposition 3.2] for more general cases.

Now we know that $D(2, 2, \sqrt{-1})$ does not have the dual Chevalley property, since for $u_0, u_1 \in C$, their products u_0u_1 and u_1u_0 do not belong to the coradical.

Proposition 5.3.4 *The link-indecomposable decomposition of $H := D(2,2,\sqrt{-1})$ is*

$$H = \left(\bigoplus_{0\le i\le 3} H_{(x^i)}\right) \oplus \left(\bigoplus_{i\in\mathbb{Z}} H_{(x^iC)}\right),$$

where $H_{(x^i)} = \Bbbk\{x^ig^jy^l \mid 0 \le j,l \le 1\} = x^iH_{(1)}$ and $H_{(x^iC)} = x^iC$.

Proof: On the one hand, note that $\Delta(x^ig^jy) = x^ig^j \otimes x^ig^jy + x^ig^jy \otimes x^ig^{j+1}$ always holds. Thus, for each fixed $0 \le i \le 3$, the simple subcoalgebras (or grouplike elements)

$$\cdots,\ x^ig^{-2},\ x^ig^{-1},\ x^i,\ x^ig,\ x^ig^2,\ x^ig^3,\ \cdots,$$

or equivalently

$$\cdots,\ x^{i-4},\ x^{i-4}g,\ x^i,\ x^ig,\ x^{i+4},\ x^{i+4}g,\ \cdots,$$

are linked, and x^ig^jy belongs to this link-indecomposable component $H_{(x^i)}$. We conclude that

$$\Bbbk\{x^ig^jy^l \mid j\in\mathbb{Z},\ 0\le l\le 1\} \subseteq H_{(x^i)} \quad (0\le i\le 3). \tag{5.16}$$

On the other hand, the remaining non-pointed simple subcoalgebras clearly satisfy

$$\Bbbk\{x^{i-2j}g^ju_l \mid 0\le j,l\le 1\} = x^iC \subseteq H_{(x^iC)} \quad (i\in\mathbb{Z}). \tag{5.17}$$

However, the direct sum of the left-hand sides of (5.16) and (5.17) become exactly $D(2,2,\sqrt{-1})$, according to the form of the basis (5.15). The desired link-indecomposable decomposition is then obtained as the direct sum of the right-hand sides.

Remark 5.3.5 *Note that as a pointed subcoalgebra, $H_{(1)}$ would satisfy condition (5.8). Thus it is a Hopf subalgebra, even though $H = D(2,2,\sqrt{-1})$ does not have the dual Chevalley property.*

Finally we could verify that $D(2,2,\sqrt{-1})$ does not have the property in Lemma 5.1.12(1). Consider simple subcoalgebras $\Bbbk 1$, $\Bbbk g$ and C, and note that

$$C = \Bbbk\{x^{-2j}g^ju_l \mid 0\le j,l\le 1\} = \Bbbk\{x^{2j}g^{-j}u_l \mid 0\le j,l\le 1\}$$

holds since $x^{-2}g = x^2g^{-1}$. Clearly, $\Bbbk 1$ and $\Bbbk g$ are linked, but we could compute that

$$\begin{aligned} gC &= g\cdot \Bbbk\{x^{2j}g^{-j}u_l \mid 0\leq j,l\leq 1\} \;=\; \Bbbk\{x^{2j}g^{1-j}u_l \mid 0\leq j,l\leq 1\} \\ &= \Bbbk\{x^{2j}g^{1-j}u_l \mid 0\leq 1-j,l\leq 1\} \;=\; \Bbbk\{x^{2(1-j)}g^{j}u_l \mid 0\leq j,l\leq 1\} \\ &= \Bbbk\{x^{2(1-j)}g^{2j}g^{-j}u_l \mid 0\leq j,l\leq 1\} \;=\; \Bbbk\{x^{2(1-j)}x^{4j}g^{-j}u_l \mid 0\leq j,l\leq 1\} \\ &= \Bbbk\{x^{2+2j}g^{-j}u_l \mid 0\leq j,l\leq 1\} \;=\; x^2C, \end{aligned}$$

which is not linked with C. That is to say, $(\Bbbk 1)C$ and $(\Bbbk g)C$ are *not* linked, and hence the property in Lemma 5.1.12(1) does not hold.

Moreover, one could find by direct computations that

$$H_{(1)}H_{(C)} = H_{(C)} \oplus H_{(x^2C)} \nsubseteq H_{(C)}$$

for example. Thus $D(2,2,\sqrt{-1})$ dissatisfies the properties in Items (1) and (2) of Theorem 5.2.8.

§5.3.2 Non-Degenerate Hopf Pairings

When H is infinite-dimensional, sometimes $H_{(1)}$ could be an idea for constructing non-degenerate Hopf pairings. The notion of pairings of bialgebras or Hopf algebras are due to [29]. This is also regarded as a sense of a quantum group in [48].

Definition 5.3.6 *Let H and $H^\bullet$ be Hopf algebras. A linear map $\langle,\rangle : H^\bullet\otimes H\to \Bbbk$ is called a Hopf pairing (on H), if*

(i) $\langle ff',h\rangle = \sum\langle f,h_{(1)}\rangle\langle f',h_{(2)}\rangle$, (ii) $\langle f,hh'\rangle = \sum\langle f_{(1)},h\rangle\langle f_{(2)},h'\rangle$,
(iii) $\langle 1,h\rangle = \varepsilon(h)$, (iv) $\langle f,1\rangle = \varepsilon(f)$,
(v) $\langle f,S(h)\rangle = \langle S(f),h\rangle$

hold for all $f,f'\in H^\bullet$ and $h,h'\in H$. Moreover, it is said to be non-degenerate, if for any $f\in H^\bullet$ and any $h\in H$,

$$\langle f,H\rangle = 0 \text{ implies } f=0, \text{ and } \langle H^\bullet,h\rangle = 0 \text{ implies } h=0.$$

Consider one of the infinite-dimensional Taft algebras ([28, Example 2.7]), denoted by $T_\infty(2,1,-1)$. Suppose $T_\infty(2,1,-1)^\bullet$ is chosen as the link-indecomposable component of the finite dual $T_\infty(2,1,-1)^\circ$ containing the unit element. We would show that the evaluation $\langle , \rangle : T_\infty(2,1,-1)^\bullet \otimes T_\infty(2,1,-1) \to \Bbbk$ is a non-degenerate Hopf pairing.

Let us recall the structure of $T_\infty(2,1,-1)$ and $T_\infty(2,1,-1)^\circ$. We remark that the finite dual of infinite-dimensional Taft algebra $T_\infty(n,v,\xi)$ are once determined in [15, Lemma 6.9] and [5, Corollary 4.4.6(III)] (see also [3, Proposition 7.2]). Here we introduce the structure of $T_\infty(2,1,-1)^\circ$ stated in [26, Section 3] with generators and relations:

Example 5.3.7 *(1) As an algebra, $T_\infty(2,1,-1)$ is generated by g and x with relations:*

$$g^2 = 1, \quad xg = -gx.$$

Then $T_\infty(2,1,-1)$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\begin{aligned}
&\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0, \\
&S(g) = g, \quad S(x) = gx.
\end{aligned}$$

Moreover, $T_\infty(2,1,-1)$ has a linear basis $\{g^j x^l \mid 0 \le j \le 1,\ l \in \mathbb{N}\}$.

(2) As an algebra, $T_\infty(2,1,-1)^\circ$ is generated by ψ_λ $(\lambda \in \Bbbk)$, ω, E_2, E_1 with relations

$$\begin{aligned}
&\psi_{\lambda_1}\psi_{\lambda_2} = \psi_{\lambda_1+\lambda_2}, \quad \psi_0 = 1, \quad \omega^2 = 1, \quad E_1^2 = 0, \\
&\omega\psi_\lambda = \psi_\lambda\omega, \quad E_2\omega = \omega E_2, \quad E_1\omega = -\omega E_1, \\
&E_2\psi_\lambda = \psi_\lambda E_2, \quad E_1\psi_\lambda = \psi_\lambda E_1, \quad E_1E_2 = E_2E_1
\end{aligned}$$

for all $\lambda, \lambda_1, \lambda_2 \in \Bbbk$. The coalgebra structure and antipode are given by:

$$\begin{aligned}
&\Delta(\omega) = \omega \otimes \omega, \quad \Delta(E_1) = 1 \otimes E_1 + E_1 \otimes \omega, \\
&\Delta(E_2) = 1 \otimes E_2 + E_1 \otimes \omega E_1 + E_2 \otimes 1, \\
&\Delta(\psi_\lambda) = (\psi_\lambda \otimes \psi_\lambda)(1 \otimes 1 + \lambda E_1 \otimes \omega E_1), \\
&\varepsilon(\omega) = \varepsilon(\psi_\lambda) = 1, \quad \varepsilon(E_1) = \varepsilon(E_2) = 0, \\
&S(\omega) = \omega, \quad S(E_1) = \omega E_1, \quad S(E_2) = -E_2, \quad S(\psi_\lambda) = \psi_{-\lambda},
\end{aligned}$$

for $\lambda \in \Bbbk$. *Note that* $\{\psi_\lambda \omega^j E_2^s E_1^l \mid \lambda \in \Bbbk,\ 0 \le j, l \le 1,\ s \in \mathbb{N}\}$ *is a linear basis.*

Lemma 5.3.8 *([26, Section 6])* $T_\infty(2,1,-1)^\circ$ *has a Hopf subalgebra*

$$T_\infty(2,1,-1)^\bullet := \Bbbk\{\omega^j E_2^s E_1^l \mid 0 \le j, l \le 1,\ s \in \mathbb{N}\},$$

such that the evaluation $\langle , \rangle : T_\infty(n,v,\xi)^\bullet \otimes T_\infty(n,v,\xi) \to \Bbbk$ *is a non-degenerate Hopf pairing.*

Finally, a similar process as Subsection §5.3.1 follows the link-decomposition of $T_\infty(2,1,-1)^\circ$, by which we could identify the Hopf subalgebra $T_\infty(2,1,-1)^\bullet$ with a link-indecomposable component:

Proposition 5.3.9 *The Hopf subalgebra* $T_\infty(2,1,-1)^\bullet$ *is exactly the link-indecomposable component of* $T_\infty(2,1,-1)^\circ$ *containing the unit element* 1.

Proof: Denote the Hopf algebra $T_\infty(2,1,-1)^\circ$ simply by H. We claim that

$$H = H_{(1)} \oplus \left(\bigoplus_{\lambda \in \Bbbk^*} H_{(C_\lambda)} \right), \tag{5.18}$$

where

$$\begin{aligned} H_{(1)} &= \Bbbk\{\omega^j E_2^s E_1^l \mid 0 \le j, l \le 1,\ s \in \mathbb{N}\} = T_\infty(2,1,-1)^\bullet, \\ H_{(C_\lambda)} &= \Bbbk\{\psi_\lambda \omega^j E_2^s E_1^l \mid 0 \le j, l \le 1,\ s \in \mathbb{N}\}. \end{aligned}$$

In details, evidently the set of simple subcoalgebras $\mathcal{S}$ contains

$$\{\Bbbk 1,\ \Bbbk\omega,\ C_\lambda \mid \lambda \in \Bbbk^*\},$$

where C_λ has a basic multiplicative matrix $\mathcal{C}_\lambda := \begin{pmatrix} \psi_\lambda & \lambda\psi_\lambda E_1 \\ \psi_\lambda \omega E_1 & \psi_\lambda \omega \end{pmatrix}$ for each $\lambda \in \Bbbk^*$, and hence $\omega C = C\omega = C$.

One could find that

$$\mathcal{E} := \begin{pmatrix} 1 & E_1 & E_2 \\ 0 & \omega & \omega E_1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a multiplicative matrix. Clearly $\Bbbk 1$ and $\Bbbk\omega$ are linked. For any $0 \leq j, l \leq 1,\ s \in \mathbb{N}$, the element $\omega^j E_2^s E_1^l$ is an entry (with some non-zero scalar) of the multiplicative matrix $\mathcal{E}^{\odot s}$. Thus

$$\Bbbk\{\omega^j E_2^s E_1^l \mid 0 \leq j, l \leq 1,\ s \in \mathbb{N}\} \subseteq H_{(1)}.$$

On the other hand, for any $\lambda \in \Bbbk^*$ and $0 \leq j, l \leq 1,\ s \in \mathbb{N}$, the element $\psi_\lambda \omega^j E_2^s E_1^l$ is an entry (with some scalar) of the multiplicative matrix $\mathcal{E}^{\odot s} \odot \mathcal{C}_\lambda$, whose diagonal is made up with basic multiplicative matrices of C_λ. Thus

$$\Bbbk\{\psi_\lambda \omega^j E_2^s E_1^l \mid 0 \leq j, l \leq 1,\ s \in \mathbb{N}\} \subseteq H_{(C_\lambda)}.$$

It could be concluded that $\mathcal{S} = \{\Bbbk 1,\ \Bbbk\omega,\ C_\lambda \mid \lambda \in \Bbbk^*\}$ and (5.18) holds.

Remark 5.3.10 *When H is pointed, it is stated in [34, Theorem 3.2] that $H_{(1)}$ is always a normal Hopf subalgebra. As for the example $H = T_\infty(2, 1, -1)^\circ$ in this subsection, one could verify that $H_{(1)}$ is also normal as a Hopf subalgebra, according the equations such as*

$$\psi_\lambda \omega^j E_2^s E_1^l \psi_{-\lambda} = \psi_\lambda \psi_{-\lambda} \omega^j E_2^s E_1^l = \omega^j E_2^s E_1^l.$$

Some other examples might also be verified.

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