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# Reflection Theory of Nichols Algebra

by

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# 南京大学研究生毕业论文中文摘要首页用纸

毕业论文题目：Nichols 代数的反射理论

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## 摘 要

本论文致力于建立具有双射对极的余拟 Hopf 代数上的 Nichols 代数的反射理论。虽然 Hopf 代数上的 Nichols 代数反射理论已较为成熟，但在余拟 Hopf 代数框架下，由于不具有结合律，相关理论尚未建立。为了克服这一困难，我们引入了余拟 Hopf 代数上的有理 Yetter-Drinfeld 模的概念，并证明了由对偶 Hopf 代数对联系的两个有理 Yetter-Drinfeld 模范畴之间存在辫子幺半范畴等价。

利用这一等价性，我们将反射理论推广到了余拟 Hopf 代数的情形。我们定义了反射算子，并证明了反射后的 Nichols 代数与原 Nichols 代数之间能够由上述辫子幺半范畴等价联系起来。更重要的是，我们证明了如果一组单的 Yetter-Drinfeld 模允许所有反射，则它们将生成一个半 Cartan 图。

作为反射理论的应用，本论文提供了判断余拟 Hopf 代数上的 Nichols 代数有限维性的新准则，并利用该理论重新证明了一类秩为 3 的非对角型 Nichols 代数是无限维的。此外，我们还展示了仿射 Nichols 代数可以在余拟 Hopf 代数上实现。

本文还研究了扭曲量子偶，它是一类特殊的拟 Hopf 代数。我们给出了有限交换群的扭曲量子偶与普通量子偶规范等价的充分条件，并确定了有限循环群的扭曲量子偶何时是真实的。作为一个应用，我们给出了上述所说的一类秩为 3 的非对角型 Nichols 代数是无限维的另一个证明。

最后，我们证明了交换群上的有限维余根基分次点余拟 Hopf 代数的余模范畴满足有限生成上同调的性质。

**关键词：** Nichols 代数；余拟 Hopf 代数；扭曲量子偶；上同调；张量范畴

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THESIS: Reflection Theory of Nichols Algebra

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## ABSTRACT

This thesis is dedicated to establishing the reflection theory of Nichols algebras over coquasi-Hopf algebras with bijective antipodes. Although the reflection theory for Nichols algebras over Hopf algebras is well-developed, the corresponding theory in the framework of coquasi-Hopf algebras has not yet been established due to the lack of associativity. To overcome this difficulty, we introduce the concept of rational Yetter-Drinfeld modules over coquasi-Hopf algebras and prove that there exists a braided monoidal equivalence between the categories of rational Yetter-Drinfeld modules related by a dual pair of Hopf algebras.

Using this equivalence, we extend the reflection theory to the setting of coquasi-Hopf algebras. We define reflection of Yetter-Drinfeld modules and prove that the reflected Nichols algebra and the original one can be related by the above braided monoidal equivalence. More importantly, we prove that if a tuple of simple Yetter-Drinfeld modules admits all reflections, they generate a semi-Cartan graph.

As an application of the reflection theory, this thesis provides a new criterion for determining the finite-dimensionality of Nichols algebras over coquasi-Hopf algebras and uses this theory to reprove that a class of rank 3 Nichols algebras of non-diagonal type is infinite-dimensional. Furthermore, we demonstrate that affine Nichols algebras can be realized over coquasi-Hopf algebras.

This thesis also investigates twisted quantum doubles, which are a special class of quasi-Hopf algebras. We provide sufficient conditions for the twisted quantum double of a finite abelian group to be gauge equivalent to an ordinary quantum double, and determine when the twisted quantum double of a finite cyclic group is genuine. As an application, we provide another proof for the infinite-dimensionality of the aforementioned class of rank 3 Nichols algebras of non-diagonal type.

Finally, we prove that the category of comodules of a finite-dimensional coradically graded pointed coquasi-Hopf algebra over an abelian group satisfies the property of finitely generated cohomology.

KEYWORDS: Nichols algebra; Coquasi-Hopf algebra; Twisted quantum double; Cohomology; Tensor category

# Contents

# Chapter 1

# Introduction

## 1.1 The classical setting: Nichols algebras over Hopf algebras

The classification of finite-dimensional pointed Hopf algebras has been one of the central problems in the theory of Hopf algebras over the past three decades. A crucial breakthrough in this direction was achieved through the systematic study of Nichols algebras. Heckenberger introduced the revolutionary concepts of semi-Cartan graphs and Weyl groupoids [46,47,48], which provide powerful combinatorial tools for understanding the structure and classification of Nichols algebras.

The reflection theory of Nichols algebras represents one of the most important developments in the field. Recall that a semi-Cartan graph is essentially a quadruple $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ consisting of a finite index set $\mathbb{I}$, a set of points $\mathcal{X}$, a set of reflection maps $r$, and a generalized Cartan matrix $A^X$ associated with each point $X \in \mathcal{X}$, satisfying specific compatibility axioms (see Definition 5.2.5 for details). The fundamental theorem states that a tuple of simple Yetter-Drinfeld modules over a Hopf algebra with bijective antipode will give rise to a semi-Cartan graph provided that it admits all reflections [8,45]. This result was proved again using a categorical approach in the setting of Hopf algebras in [51].

The interplay between Weyl groupoid theory and reflection theory has led to spectacular progress in the classification of finite-dimensional Nichols algebras. The classification program, initiated by Andruskiewitsch and Schneider in a series of papers [9,10,11,12], aims to determine all finite-dimensional pointed Hopf algebras over algebraically closed fields of characteristic zero whose group of group-like elements is abelian. A cornerstone of this classification is Heckenberger's complete classification of finite-dimensional Nichols algebras of diagonal type [47]. Building on this classification, Angiono made a breakthrough contribution by providing explicit presentations by generators and relations for all Nichols algebras of diagonal type [13,16]. The lifting problem of Nichols algebras of diagonal type has been addressed through cocycle deformation [3,15,2]. The classification of finite-dimensional Nichols algebras over non-abelian groups has made excellent progress as well [6,7,8,80,37,44,43,30,45,50,49,55,53,54].

## 1.2 From Hopf algebras to coquasi-Hopf algebras

In this work, we focus on the more general setting of coquasi-Hopf algebras. The classification of coradically graded pointed finite-dimensional coquasi-Hopf algebras over abelian groups has been completed in [61,57,62,59,58,60,68].

In the classification of pointed Hopf algebras over abelian groups, Nichols algebras of non-diagonal type do not appear. However, in the coquasi-Hopf algebra setting, Nichols algebras of non-diagonal type do occur and present greater challenges. The authors in [60] used extensive computations to establish that a certain class of such Nichols algebras is infinite-dimensional. We later provided an alternative proof of their main result via categorically Morita equivalence [68]. Nonetheless, these arguments are not entirely natural, and we hope that the theory of semi-Cartan graphs can be extended to coquasi-Hopf algebras. If such a generalization holds, it could be applied to the classification of finite-dimensional coquasi-Hopf algebras over non-abelian groups.

## 1.3 Braided monoidal equivalence induced by a dual pair and its application to reflection theory

While the reflection theory of Nichols algebras over Hopf algebras is mature, the parallel theory for coquasi-Hopf algebras remains less developed due to the lack of associativity. In the present work, we overcome several new difficulties that do not appear in the Hopf algebra settings. As a result, we extend the reflection theory to arbitrary coquasi-Hopf algebras $H$ with bijective antipode [69,71].

Inspired by methods in [51,52], we can define rational modules of $A$ in $C = {}^H_H\mathcal{YD}$. This allows us to define the category ${}^R_R\mathcal{YD}(C)_{\mathrm{rat}}$, which is proven to be a monoidal subcategory of ${}^R_R\mathcal{YD}(C)$. Now, consider two locally finite Hopf algebras $A$, $B$ in $C$ with bijective antipodes related by a non-degenerate Hopf pairing compatible with the grading, called a dual pair. We prove the following result:

**Theorem 1.3.1** (Theorem 3.4.6)**.** *There is a braided monoidal equivalence:*

$$\Omega : {}^B_B\mathcal{YD}(C)_{\mathrm{rat}} \longrightarrow {}^A_A\mathcal{YD}(C)_{\mathrm{rat}}.$$

Furthermore, we show that if $V$ is a finite-dimensional Yetter-Drinfeld module in ${}^H_H\mathcal{YD}$, there is a dual pair between $\mathcal{B}(V)$ and $\mathcal{B}(V^*)$. Thus the above theorem can be applied to Nichols algebras to get the following braided monoidal equivalence:

$$\Omega_V : {}^{\mathcal{B}(V)}_{\mathcal{B}(V)}\mathcal{YD}(C)_{\mathrm{rat}} \longrightarrow {}^{\mathcal{B}(V^*)}_{\mathcal{B}(V^*)}\mathcal{YD}(C)_{\mathrm{rat}}.$$

Now we are going to state our main result. Let $\theta \geq 1$ be a positive integer, $\mathbb{I} = \{1, 2, .., \theta\}$ and $M = (M_1, \ldots, M_\theta)$, where $M_1, \ldots, M_\theta \in {}^H_H\mathcal{YD}$ are finite-dimensional irreducible Yetter-Drinfeld modules. We say $M$ admits the $i$-th reflection for some $1 \leq i \leq \theta$ if for all $j \neq i$ there is a natural number $m_{ij}^M \geq 0$ such that $\mathrm{ad}(M_i)^{m_{ij}^M}(M_j)$ is a non-zero finite-dimensional subspace of $\mathcal{B}(M)$, and $\mathrm{ad}(M_i)^{m_{ij}^M+1}(M_j) = 0$. Assume $M$ admits the $i$-th reflection. Then we set $R_i(M) = (V_1, \ldots, V_\theta)$, where

$$V_j = \begin{cases} M_i^*, & \text{if } j = i, \\ (\mathrm{ad}\, M_i)^{m_{ij}^M}(M_j), & \text{if } j \neq i. \end{cases}$$

It is still a tuple of finite-dimensional irreducible Yetter-Drinfeld modules. By the above theorem, we have a braided monoidal equivalence:

$$\Omega_i : {}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(C)_{\mathrm{rat}} \longrightarrow {}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(C)_{\mathrm{rat}}.$$

Then we prove the following theorem.

**Theorem 1.3.2.** (Theorem 5.2.3) *With the above assumptions on $M$. Suppose $M$ admits the $i$-th reflection for $1 \le i \le \theta$. Then there is an isomorphism of Hopf algebras in ${}_H^H\mathcal{YD}$:*

$$\Theta_i : \mathcal{B}(R_i(M)) \cong \Omega_i\left(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}\right)\#\mathcal{B}(M_i^*). \tag{1.3.1}$$

Although our main conclusion appears consistent with the Hopf case, the specific details differ significantly. Some details require adjustment or reformulation in this more general setting. Some identities that are obviously valid in Hopf algebras require extensive computations to verify in our context.

As a corollary, suppose $M$ admits all reflections and let $\mathcal{X} = \{[X] \mid X \in \mathcal{F}_\theta(M)\}$, see Definition 5.2.4 for related notation. We denote the generalized Cartan matrix $A^{[X]} = (a_{ij}^X)_{i,j\in\mathbb{I}}$ for each $[X] \in \mathcal{X}$, where $(a_{ij}^X)_{i,j\in\mathbb{I}}$ is defined in Lemma 5.1.3. Let $r$ be the following map,

$$r : \mathbb{I}\times\mathcal{X} \to \mathcal{X},\ i\times[X] \mapsto [R_i(X)].$$

Then the quadruple

$$\mathcal{G}(M) = (\mathbb{I}, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$$

is a semi-Cartan graph.

## 1.4 An application of reflection theory

After establishing that $M$ gives rise to a semi-Cartan graph, several additional criteria for determining the finite-dimensionality of the Nichols algebra can also be derived. We show that if $\mathcal{B}(M)$ is finite-dimensional, then the semi-Cartan graph $\mathcal{G}(M)$ is a finite semi-Cartan graph. Furthermore, if $\mathcal{G}(M)$ is a standard finite semi-Cartan graph, that is $A^X = A^Y$, for all $X, Y \in \mathcal{X}$, then $A^M$ must be a finite Cartan matrix. At the same time, we prove that the dimension and Gelfand-Kirillov dimension of the Nichols algebras are invariant under reflections.

As an application of these results, we apply this machinery to a crucial test case: a rank-three Nichols algebra $\mathcal{B}(M) = \mathcal{B}(M_1 \oplus M_2 \oplus M_3)$ of non-diagonal type, originally identified in [60, Proposition 4.1]. We provide an alternative proof.

We compute the Weyl groupoid of $\mathcal{W}(\mathcal{G}(M))$, and show that the set of real roots of $\mathcal{G}(M)$ at each $X \in \mathcal{X}$ is just the set of real roots of the Kac-Moody Lie algebra of type $A_2^{(1)}$. Furthermore, we prove that $\mathcal{G}(M)$ is a Cartan graph. Finally, utilizing the correspondence between Cartan graphs and Tits cones [28,29], we demonstrate that this algebra is affine, meaning its Tits cone is a half-plane. Our observations show that affine Nichols algebras can be realized over coquasi-Hopf algebras.

## 1.5 About an important class of quasi-Hopf algebras——twisted quantum doubles

Given a finite group $G$ and a normalized 3-cocycle $\Phi \in Z^3(G, \mathbb{C}^*)$, Dijkgraaf-Pasquier-Roche defined a certain quasi-Hopf algebra(twisted quantum double) $D^\Phi(G)$ in [31]. The motivation for studying such a quasi-Hopf algebra comes from conformal field theory and vertex operator algebra, see [67,25,38]. Readers only need to understand that vertex operator algebras have a representation theory. In particular, Huang proved in [64] that if $\mathbb{V}$ is $C_2$-cofinite, rational, CFT type (i.e. $V(1)=\mathbb{C}1$), and self dual (i.e. $\mathbb{V}\cong\mathbb{V}'$), then $\mathrm{Rep}(\mathbb{V})$ is a modular tensor category. Then by reconstruction theory [75], there might be a weak quasi-Hopf algebra $H$ with the property that $\mathrm{Rep}(H)\cong\mathrm{Rep}(\mathbb{V})$ as a modular tensor category. In the context of [31], the authors conjectured that one can take $H$ to be a twisted quantum double $D^{\Phi}(G)$ of $G$ in the case when $\mathbb{V}$ is a so-called holomorphic orbifold model, that is there is a simple vertex operator algebra $\mathbb{W}$ and a finite group of automorphisms $G$ of $\mathbb{W}$ such that $\mathbb{V}=\mathbb{W}^G$, see also [76]. This conjecture was proven to be true in [32].

We consider the case when such a braided tensor equivalence holds:

$$\mathrm{Rep}(D^{\Phi_1}(G_1))\cong\mathrm{Rep}(D^{\Phi_2}(G_2)) \tag{1.5.1}$$

This is an interesting problem in its own right. The case in which the two twisted quantum doubles in question are commutative, i.e. the two groups are abelian and the 3-cocycles are abelian 3-cocycle was solved in [77]. In [42], the authors dealt with the case when $G_1$ to be an elementary abelian 2-group and $G_2$ turns out to be an extra-special 2-group. Here we are concerned with a particular case of (1.5.1) when taking $G_1$ as a finite abelian group, $\Phi_1$ is an arbitrary normalized 3-cocycle and $G_2$ is a finite group, $\Phi_2$ is trivial:

$$\mathrm{Rep}(D^{\Phi_1}(G_1))\cong\mathrm{Rep}(D(G_2)). \tag{1.5.2}$$

We give a sufficient condition when equivalence (1.5.2) holds.

**Theorem 1.5.1** (Theorem 6.3.4)**.** *Let $G$ be a finite abelian group and $\Phi$ a normalized 3-cocycle on $G$ as in (2.6.1). If the following condition holds:*

(i) $a_i=0$ *for all* $1\le i\le n$,

(ii) $A\cap B=\emptyset$.

*Then $D^{\Phi}(G)$ will be gauge equivalent to $D(G')$ for a finite group $G'$.*

For a quasi-Hopf algebra $H$, we say $H$ is genuine if it will never be gauge equivalent to a Hopf algebra. Studying the genuineness of a twisted quantum double is another question. [77, Example 9.5] gives us the first example of genuine twisted quantum double, say $D^{\Phi}(Z_2)$, where $\Phi$ is the nontrivial 3-cocycle on $Z_2$. In [72, Theorem 4.1], the authors showed that if $G$ is abelian, and $\Phi$ is an abelian cocycle, then $D^{\Phi}(G)$ is genuine if and only if there exists $V\in\mathrm{Rep}(D^{\Phi}(G))$ such that $\overline{\nu}(V)=0$, where $\overline{\nu}$ is the total Frobenius-Schur indicator of $\mathrm{Rep}(D^{\Phi}(G))$. Let $G$ be a finite cyclic group and $\Phi$ a nontrivial 3-cocycle on $G$. We provide a discriminant method for whether $D^{\Phi}(G)$ is genuine or not, also through using the explicit expression of 3-cocycles. Here is the result.

**Theorem 1.5.2** (Theorem 6.5.3)**.** *Let $G\cong Z_m$ be a finite cyclic group and $\Phi(g^i,g^j,g^k)=\zeta_m^{ai[\frac{j+k}{m}]}$ for $1\le a<m$. Then $D^{\Phi}(G)$ is genuine if and only if $(m,2a)\nmid(m,a)$.*

As an application, we gave another different proof of [60, Proposition 4.1] using Theorem 6.3.4. As a byproduct, we show that the Nichols algebras $\mathcal{B}(M_1\oplus M_2\oplus M_3)$ are infinite-dimensional where $M_1,M_2,M_3$ are three different simple Yetter-Drinfeld modules of $D_8$.

## 1.6 On cohomology of pointed finite tensor categories

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Given a finite tensor category $\mathcal{C}$ over $\Bbbk$, we may define the cohomology of $\mathcal{C}$ as follows:

$$\mathrm{H}^\bullet(\mathcal{C},V) = \mathrm{Ext}^\bullet_{\mathcal{C}}(\mathbf{1},V)$$

where $V$ is an object in $\mathcal{C}$ and $\mathbf{1}$ is the unit object in $\mathcal{C}$. We mainly focus on the cohomological finiteness property:

**Definition 1.6.1.** *A finite tensor category $\mathcal{C}$ over $\Bbbk$ is said to have finitely generated cohomology (**FGC** for brevity), if $\mathrm{H}^\bullet(\mathcal{C},\mathbf{1})$ is a finitely generated algebra and $\mathrm{H}^\bullet(\mathcal{C},V)$ is a finitely generated $\mathrm{H}^\bullet(\mathcal{C},\mathbf{1})$ -module for each $V$ in $\mathcal{C}$.*

In [36], Etingof and Ostrik proposed the well-known conjecture:

**Conjecture 1.6.2.** *Any finite tensor category over $\Bbbk$ satisfies **FGC**.*

For over twenty years, many beautiful results related to this conjecture have been obtained, especially from the perspective of Hopf algebras. For a finite-dimensional Hopf algebra $H$, the category $\mathrm{Rep}(H)$ of finite-dimensional representations is a finite tensor category. By abuse of terminology, we say $H$ satisfies **FGC** if $\mathrm{Rep}(H)$ does. Ginzburg and Kumar in [41] showed that all small quantum groups satisfy **FGC**. Moreover, they computed their cohomology rings explicitly. In [78], Mastnak, Pevtsova, Schauenberg and Witherspoon proved that any finite-dimensional pointed Hopf algebra over an abelian group (i.e., whose group-like elements form an abelian group) satisfies **FGC**, subject to mild restrictions on the order of the group. This result was later extended to all finite-dimensional pointed Hopf algebras over abelian groups in [4]. Related results for fields of positive characteristic can be found in [83] and [39].

On the other hand, some experts study this question from the perspective of category theory. In [81], Negron and Plavnik characterized when the dual category and the center of a finite tensor category $\mathcal{C}$ satisfy **FGC**, assuming $\mathcal{C}$ itself satisfies **FGC**. The condition of finitely generated cohomology is crucial in the theory of support varieties for finite tensor categories; further details may be found in [21] and [22].

The classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras over abelian groups was completed in [58] and [60]. Based on the type of the associated Nichols algebra, the classification divides into two cases: diagonal type and non-diagonal type. We address these cases separately to establish our main result on finitely generated cohomology [70].

**Theorem 1.6.3** (Theorem 8.4.5)**.** *Let $M$ be a finite-dimensional coradically graded pointed coquasi-Hopf algebra over abelian groups, and $\mathcal{C} := \mathrm{Comod}(M)$, then $\mathcal{C}$ satisfies **FGC**.*

## 1.7 Organization of this thesis

The paper is organized as follows. After reviewing preliminaries on coquasi-Hopf algebras, Yetter–Drinfeld modules,bosonization and Nichols algebras in Section 2, we introduce rational modules and establish equivalences of braided monoidal categories via dual pair in Section 3. In particular, we specialize to Nichols algebras and their pairings. Section 4 is devoted to the general reflection theory: we study the projection of a Nichols algebra. Then in Section 5 we study reflections of Nichols algebras and establish our main result. In section 6, we provide a sufficient condition for the twisted quantum double of a finite abelian group will be gauge equivalent to the ordinary quantum double of a finite group. Meanwhile, we give a criterion for a twisted quantum double of a finite cyclic group is genuine. Section 7 deals with a detailed example. We give a realization of an affine Nichols algebra over coquasi-Hopf algebra and show that it gives rise to a standard Cartan graph. Finally, in Section 8, we study the cohomology of some pointed finite tensor categories. As a result, we partially confirm the conjecture on finitely generated cohomology of finite tensor categories.

# Chapter 2

# Preliminaries

In this section, we briefly recall the fundamental definitions and fix the notation for coquasi-Hopf algebras and Yetter-Drinfeld modules. For a more detailed exposition and proofs of standard properties, we refer the reader to [69,58]. Throughout the paper, $\Bbbk$ denotes an algebraically closed field of characteristic zero.

## 2.1 Coquasi-Hopf algebras

By definition, coquasi-Hopf algebras are exactly the duals of Drinfeld's quasi-Hopf algebras [33]. Their formal definition can be given as follows.

**Definition 2.1.1.** *A coquasi-Hopf algebra is a coalgebra $(H,\Delta,\varepsilon)$ equipped with a compatible quasi-algebra structure and an antipode $(\mathcal{S},\alpha,\beta)$. Namely, there exist:*

- *Two coalgebra homomorphisms:*

$$m : H \otimes H \to H, \quad a \otimes b \mapsto ab,$$

$$\mu : \Bbbk \to H, \quad \lambda \mapsto \lambda 1_H,$$

- *A convolution-invertible map $\Phi : H^{\otimes 3} \to \Bbbk$ called an associator,*
- *A coalgebra antimorphism $\mathcal{S} : H \to H$,*
- *Two linear functions $\alpha, \beta : H \to \Bbbk$*

*such that for all $a,b,c,d \in H$ the following equalities hold:*

$$a_1(b_1c_1)\Phi(a_2,b_2,c_2) = \Phi(a_1,b_1,c_1)(a_2b_2)c_2, \tag{2.1.1}$$

$$1_H a = a = a 1_H, \tag{2.1.2}$$

$$\Phi(a_1,b_1,c_1d_1)\Phi(a_2b_2,c_2,d_2) = \Phi(b_1,c_1,d_1)\Phi(a_1,b_2c_2,d_2)\Phi(a_2,b_3,c_3), \tag{2.1.3}$$

$$\Phi(a,1_H,b) = \varepsilon(a)\varepsilon(b), \tag{2.1.4}$$

$$\mathcal{S}(a_1)\alpha(a_2)a_3 = \alpha(a)1_H, \quad a_1\beta(a_2)\mathcal{S}(a_3) = \beta(a)1_H, \tag{2.1.5}$$

$$\Phi(a_1,\mathcal{S}(a_3),a_5)\beta(a_2)\alpha(a_4) = \Phi^{-1}(\mathcal{S}(a_1),a_3,\mathcal{S}(a_5))\alpha(a_2)\beta(a_4) = \varepsilon(a). \tag{2.1.6}$$

Throughout this paper, we use the Sweedler sigma notation $\Delta(a) = a_1 \otimes a_2$ for the coproduct and $a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}$ for the result of the $n$-iterated application of $\Delta$ on $a$. We say $H$ has a bijective antipode if $\mathcal{S}$ is bijective.

A coquasi-Hopf algebra $H$ is pointed if the underlying coalgebra is so. Given a coquasi-Hopf algebra, let $\{H_n\}_{n\geq 0}$ be its coradical filtration, and

$$\mathrm{gr}(H) := H_0 \oplus H_1/H_0 \oplus H_2/H_1 \oplus \cdots$$

the corresponding coradically graded coalgebra. Then naturally $(\mathrm{gr}(H), \mathrm{gr}(\Phi))$ inherits from $H$ a graded coquasi-Hopf algebra structure. Furthermore, for homogeneous elements $a, b, c \in \mathrm{gr}(H)$, $\mathrm{gr}(\Phi)(a,b,c) = 0$ unless they all lie in $H_0$.

Following the definition in [17], we now recall the definition of a preantipode, which plays an important role in further calculations.

**Definition 2.1.2.** [17, Definition 3.6] *Let $(H, \Phi)$ be a coquasi-bialgebra, a preantipode for $H$ is a $\Bbbk$-linear map $\mathbb{S}: H \to H$ such that, for all $h \in H$,*

$$\begin{aligned} \mathbb{S}(h_1)_1 h_2 \otimes \mathbb{S}(h_1)_2 &= 1_H \otimes \mathbb{S}(h), \\ \mathbb{S}(h_2)_1 \otimes h_1 \mathbb{S}(h_2)_2 &= \mathbb{S}(h) \otimes 1_H, \\ \Phi(h_1, \mathbb{S}(h_2), h_3) &= \varepsilon(h). \end{aligned}$$

Furthermore, [17, Remark 3.7] provides us a useful property:

$$h_1 \mathbb{S}(h_2) = \varepsilon(\mathbb{S}(h)) 1_H = \mathbb{S}(h_1) h_2, \text{ for all } h \in H. \tag{2.1.7}$$

Meanwhile, when $H$ is equipped with the structure of a coquasi-Hopf algebra with antipode $(\mathcal{S}, \alpha, \beta)$, then a preantipode for $H$ always exists.

**Lemma 2.1.3.** [17, Theorem 3.10] *Let $H$ be a coquasi-Hopf algebra with antipode $(\mathcal{S}, \alpha, \beta)$, then*

$$\mathbb{S} := \beta * \mathcal{S} * \alpha \tag{2.1.8}$$

*is a preantipode for $H$, where $*$ denotes convolution product.*

**Example 2.1.4.** A baby example of coquasi-Hopf algebra is $(\Bbbk G, \Phi)$, where $G$ is a group and $\Phi$ a normalized 3-cocycle on $G$. It is well known that the group algebra $\Bbbk G$ is a Hopf algebra with:

$$\Delta(g) = g \otimes g, \quad \mathcal{S}(g) = g^{-1}, \quad \varepsilon(g) = 1 \quad \text{for any } g \in G.$$

By extending $\Phi$ linearly, $\Phi: (\Bbbk G)^{\otimes 3} \to \Bbbk$ becomes a convolution-invertible map. Define two linear functions $\alpha, \beta: \Bbbk G \to \Bbbk$ by:

$$\alpha(g) := \varepsilon(g), \quad \beta(g) := \frac{1}{\Phi(g, g^{-1}, g)}$$

for any $g \in G$. Then $\Bbbk G$ together with these $\Phi, \alpha$ and $\beta$ becomes a coquasi-Hopf algebra. We denote this resulting coquasi-Hopf algebra by $(\Bbbk G, \Phi)$. For all $g \in G$, we define

$$\mathbb{S}(g) = \frac{1}{\Phi(g, g^{-1}, g)} g^{-1}.$$

Then Lemma 2.1.3 shows $\mathbb{S}$ is a preantipode for $(\Bbbk G, \Phi)$.

## 2.2 Yetter-Drinfeld module categories over coquasi-Hopf algebras

We now turn our attention to the Yetter-Drinfeld module category structure. The definition of the Yetter-Drinfeld module category ${}^H_H\mathcal{YD}$ over an arbitrary coquasi-Hopf algebra $H$ was already given in [18]. Since it plays a crucial role in our paper, we recall this definition.

**Definition 2.2.1.** [18, Definition 3.1] *Let $H$ be a coquasi-Hopf algebra with associator $\Phi$. A left-left Yetter–Drinfeld module over $H$ is a triple $(V, \delta_V, \triangleright)$ such that:*

- *$(V, \delta_V)$ is a left comodule of $H$ and we denote $\delta_V(v)$ by $v_{-1} \otimes v_0$ as usual;*
- *$\triangleright : H \otimes V \to V$ is a $\Bbbk$-linear map satisfying for all $h, l \in H$ and $v \in V$:*

$$(hl) \triangleright v = \frac{\Phi(h_2, (l_2 \triangleright v_0)_{-1}, l_3)}{\Phi(h_1, l_1, v_{-1})\Phi((h_3 \triangleright (l_2 \triangleright v_0)_0)_{-1}, h_4, l_4)} (h_3 \triangleright (l_2 \triangleright v_0)_0)_0, \tag{2.2.1}$$

$$1_H \triangleright v = v, \tag{2.2.2}$$

$$(h_1 \triangleright v)_{-1} h_2 \otimes (h_1 \triangleright v)_0 = h_1 v_{-1} \otimes (h_2 \triangleright v_0). \tag{2.2.3}$$

*A morphism $f : (V, \delta_V, \triangleright) \to (V', \delta_{V'}, \triangleright')$ is a colinear map $f : (V, \delta_V) \to (V', \delta_{V'})$ such that $f(h \triangleright v) = h \triangleright' f(v)$ for all $h \in H$.*

The category ${}^H_H\mathcal{YD}$ is a $\Bbbk$-linear braided monoidal abelian category over the field $\Bbbk$. The unit object of ${}^H_H\mathcal{YD}$ is $\Bbbk$, which is regarded as an object in ${}^H_H\mathcal{YD}$ via trivial structures. For $V, W \in {}^H_H\mathcal{YD}$, the tensor product of Yetter-Drinfeld modules is defined by:

$$(V, \delta_V, \triangleright) \otimes (W, \delta_W, \triangleright) = (V \otimes W, \delta_{V \otimes W}, \triangleright),$$

where $\delta_{V \otimes W}$ is given by

$$\delta_{V \otimes W}(v \otimes w) = v_{-1} w_{-1} \otimes v_0 \otimes w_0,$$

and

$$h \triangleright (v \otimes w) = \frac{\Phi(h_1, v_{-1}, w_{-2})\Phi((h_2 \triangleright v_0)_{-1}, (h_4 \triangleright w_0)_{-1}, h_5)}{\Phi((h_2 \triangleright v_0)_{-2}, h_3, w_{-1})} (h_2 \triangleright v_0)_0 \otimes (h_4 \triangleright w_0)_0. \tag{2.2.4}$$

The braiding $c_{V,W} : V \otimes W \to W \otimes V$ is given by:

$$c_{V,W}(v \otimes w) = (v_{-1} \triangleright w) \otimes v_0. \tag{2.2.5}$$

In [19], the authors proved that the category $\mathcal{YD}^H_H$ of right-right Yetter-Drinfeld module is braided equivalent to $\mathcal{Z}_l(\mathcal{M}^H)$, which is the left center of the category of right comodules of $H$. The proof for ${}^H_H\mathcal{YD}$ is completely dual. We need to recall the definition of right center here for completeness.

**Definition 2.2.2.** *For a coquasi-Hopf algebra $H$ with bijective antipode, the right center of the category of left $H$-comodules $\mathcal{Z}_r({}^H\mathcal{M})$ has objects $(V, c_{-,V})$, where $V \in {}^H\mathcal{M}$ and $c_{W,V} : W \otimes V \longrightarrow V \otimes W$ is a natural transformation in ${}^H\mathcal{M}$ that satisfies the commutative diagram*

$$
\begin{array}{ccccc}
 & (X\otimes W)\otimes V & \xrightarrow{c_{X\otimes W,V}} & V\otimes(X\otimes W) & \\
 \swarrow{\scriptstyle a_{X,W,V}} & & & & \nwarrow{\scriptstyle a_{V,X,W}} \\
X\otimes(W\otimes V) & & & & (V\otimes X)\otimes W \\
 \searrow{\scriptstyle \mathrm{id}_X\otimes c_{W,V}} & & & & \nearrow{\scriptstyle c_{X,V}\otimes \mathrm{id}_W} \\
 & X\otimes(V\otimes W) & \xrightarrow{a^{-1}_{X,V,W}} & (X\otimes V)\otimes W &
\end{array}
$$

*for all $W, X \in^H \mathcal{M}$ and $c_{\Bbbk,V} = \mathrm{id}_V$. A morphism $f : (V, c_{-,V}) \longrightarrow (V', c_{-,V'})$ in the center is a right $H$-comodule map satisfying*

$$c_{W,V'} \circ (\mathrm{id}_W \otimes f) = (f \otimes \mathrm{id}_W) \circ c_{W,V}.$$

*for all $W \in {}^H\mathcal{M}$.*

The category $\mathcal{Z}_r({}^H\mathcal{M})$ is braided monoidal with tensor product $(V, c_{-,V}) \otimes (W, c_{-,W}) = (V \otimes W, c_{-,V\otimes W})$.

**Lemma 2.2.3.** *Let $H$ be a coquasi-Hopf algebra with bijective antipode. The category ${}^H_H\mathcal{YD}$ is braided monoidal isomorphic to the right center of the category of left $H$-comodules $\mathcal{Z}_r({}^H\mathcal{M})$.*

*Proof.* We construct mutually inverse braided monoidal functors between the two categories.

Let $(V, c_{V,-})$ be an object in $\mathcal{Z}({}^H\mathcal{M})$, we can define a left linear map on $V$ as follows:

$$\triangleright : H \otimes V \longrightarrow V, \quad h \triangleright v = (\mathrm{id}_V \otimes \varepsilon) c_{H,V}(h \otimes v) \tag{2.2.6}$$

Let $W \in {}^H\mathcal{M}$ and $w^* \in W^*$. Define the map $f_{w^*} : W \longrightarrow H, f_{w^*}(w) = w_{-1}w^*(w_0)$. Then $f_{w^*}$ is $H$-comodule map. By naturality of $c_{-,V}$, we have

$$\begin{aligned} c_{H,V} \circ (f_{w^*} \otimes \mathrm{id}_V)(w \otimes v) &= (\mathrm{id}_V \otimes f_{w^*}) \circ c_{W,V}(w \otimes v) \\ &= c_{H,V}(w_{-1}w^*(w_0) \otimes v). \end{aligned}$$

That is

$$c_{H,V}(w_{-1}w^*(w_0) \otimes v) = (\mathrm{id}_V \otimes f_{w^*}) \circ c_{W,V}(w \otimes v).$$

Applying $\mathrm{id}_V \otimes \varepsilon$ to both sides leads to

$$w^*(w_0)(w_{-1} \triangleright v) = (\mathrm{id}_V \otimes \varepsilon \circ f_{w^*}) \circ c_{W,V}(w \otimes v) = (\mathrm{id}_V \otimes w^*) \circ c_{W,V}(w \otimes v).$$

This implies

$$c_{W,V}(w \otimes v) = (w_{-1} \triangleright v) \otimes w_0.$$

for all objects $W \in {}^H\mathcal{M}$.

Since $c_{\Bbbk,V} = \mathrm{id}_V$ and the unit map $u : \Bbbk \longrightarrow H$ is a left comodule map, where $\Bbbk$ has trivial comodule structure and $H$ is a $H$-comodule via $\Delta$. We have (2.2.2) holds:

$$1_H \triangleright v = \varepsilon(1_H)v = v.$$

Next, for $h \in H$, $v \in V$, since $c_{H,V}$ is a left $H$-comodule map,

$$(h_1 \triangleright v)_{-1} h_2 \otimes (h_1 \triangleright v)_0 \otimes h_3 = \delta((h_1 \triangleright v) \otimes h_2) = h_1 v_{-1} \otimes (h_2 \triangleright v_0) \otimes h_3.$$

Applying $\varepsilon$ to last the factor we have (2.2.3). While relation (2.2.1) is a consequence of hexagon axiom applied to $W = X = H$. Note that $c_{H\otimes H,V}((h \otimes l) \otimes v) = (h_1 l_1 \triangleright v) \otimes (h_2 \otimes l_2)$. By hexagon axiom it equals to

$$\frac{\Phi(h_2,(l_2 \triangleright v_0)_{-1}, l_3)}{\Phi(h_1, l_1, v_{-1})\Phi((h_3 \triangleright (l_2 \triangleright v_0)_0)_{-1}, h_4, l_4)} (h_3 \triangleright (l_2 \triangleright v_0)_0)_0 \otimes (h_5 \otimes l_5).$$

By applying $\varepsilon$ to the second and third factors we get (2.2.1). Then $(V, \delta, \triangleright)$ becomes a left-left $H$ Yetter-Drinfeld module. It is easy to see for any morphism $f : (V, c_{-,V}) \longrightarrow (V', c_{-,V'})$, $f$ will be a morphism in ${}^H_H\mathcal{YD}$.

Conversely, for a Yetter-Drinfeld module $V$ and left $H$-comodules $W$, $X$, use relation (2.2.5) to define the natural transformation $c_{W,V}$. Then (2.2.1) implies that the hexagon axiom is fulfilled. The equality (2.2.2) implies $c_{\Bbbk,V} = \mathrm{id}_V$. The morphism $c_{W,V}$ being a left $H$-comodule map follows from (2.2.3). This together makes $(V, c_{-,V})$ an object in $\mathcal{Z}_r({}^H\mathcal{M})$. Similarly, a morphism in ${}^H_H\mathcal{YD}$ is a morphism in $\mathcal{Z}_r({}^H\mathcal{M})$.

The two constructions are clearly inverse to each other. Furthermore, the braiding in ${}^H_H\mathcal{YD}$ given by (2.2.5) coincides with the braiding of the right center under the above correspondence. Hence we obtain a braided monoidal isomorphism. □

In order to simplify notation, we recall the following linear maps $p_R$, $q_R$, $p_L$, $q_L \in \mathrm{Hom}(H \otimes H, \Bbbk)$, which is introduced in [19],

$$\begin{aligned}
p_R(h,g) &= \Phi^{-1}(h, g_1\beta(g_2), \mathcal{S}(g_3)),\\
q_R(h,g) &= \Phi(h, g_3, \alpha(\mathcal{S}^{-1}(g_2))\mathcal{S}^{-1}(g_1)),\\
p_L(h,g) &= \Phi(\mathcal{S}^{-1}(h_3)\beta(\mathcal{S}^{-1}(h_2)), h_1, g),\\
q_L(h,g) &= \Phi^{-1}(\mathcal{S}(h_1)\alpha(h_2), h_3, g),
\end{aligned}$$

for any $h, g \in H$. These maps satisfy the compatibility relations:

$$(h_1 g_1)\mathcal{S}(g_3) p_R(h_2, g_2) = h_2 p_R(h_1, g), \tag{2.2.7}$$
$$(h_2 g_3)\mathcal{S}^{-1}(g_1) q_R(h_1, g_2) = h_1 q_R(h_2, g), \tag{2.2.8}$$
$$\mathcal{S}(h_1)(h_3 g_2) q_L(h_2, g_1) = g_1 q_L(h, g_2), \tag{2.2.9}$$
$$q_R(h_1 g_1, \mathcal{S}(g_3)) p_R(h_2, g_2) = \varepsilon(h)\varepsilon(g), \tag{2.2.10}$$
$$p_L(\mathcal{S}(h_1), h_3 g_2) q_L(h_2, g_1) = \varepsilon(h)\varepsilon(g). \tag{2.2.11}$$

With the above notation established, we can now prove the following lemma.

**Lemma 2.2.4.** *Suppose $H$ has a bijective antipode.*

(1) *The equation (2.2.3) is equivalent to*

$$(h \triangleright v)_{-1} \otimes (h \triangleright v)_0 = p((h_3 \triangleright v_0)_{-1}, h_4) q(h_1 v_{-2}, \mathcal{S}(h_6))(h_2 v_{-1})\mathcal{S}(h_5) \otimes (h_3 \triangleright v_0)_0. \tag{2.2.12}$$

(2) *The equation (2.2.3) implies:*

$$v_{-1}\otimes(h\triangleright v_0)=t(h_3,v_{-1})s(\mathcal{S}(h_1),(h_4\triangleright v_0)_{-1}h_6)\mathcal{S}(h_2)((h_4\triangleright v_0)_{-2}h_5)\otimes(h_4\triangleright v_0)_0. \tag{2.2.13}$$

*Proof.* (1): Suppose $V\in{}^H_H\mathcal{YD}$ with a linear map $\triangleright: H\otimes V\to V$ satisfying (2.2.3). For any $v\in V$ and $h\in H$, we have

$$\begin{aligned}(h\triangleright v)_{-1}\otimes(h\triangleright v)_0&=\varepsilon(h_2)\varepsilon((h_1\triangleright v)_{-2})(h_1\triangleright v)_{-1}\otimes(h_1\triangleright v)_0\\&\overset{(2.2.8)}{=}q((h_1\triangleright v)_{-3}h_2,S(h_4))p((h_1\triangleright v)_{-2},h_3)(h_1\triangleright v)_{-1}\otimes(h_1\triangleright v)_0\\&\overset{(2.2.7)}{=}p((h_1\triangleright v)_{-1},h_4)q((h_1\triangleright v)_{-3}h_2,\mathcal{S}(h_6))((h_1\triangleright v)_{-2}h_3)\mathcal{S}(h_5)\otimes(h_1\triangleright v)_0\\&\overset{(2.2.3)}{=}p((h_3\triangleright v_0)_{-1},h_4)q(h_1v_{-2},\mathcal{S}(h_6))(h_2v_{-1})\mathcal{S}(h_5)\otimes(h_3\triangleright v_0)_0.\end{aligned}$$

Conversely, if (2.2.12) holds, we have

$$\varepsilon((h\triangleright v)_{-1})(h\triangleright v)_0=p((h_2\triangleright v_0)_{-1},h_3)q(h_1v_{-1},\mathcal{S}(h_4))(h_2\triangleright v_0)_0. \tag{2.2.14}$$

The origin equation comes from the following calculation,

$$\begin{aligned}(h_1\triangleright v)_{-1}h_2\otimes(h_1\triangleright v)_0&=p((h_3\triangleright v_0)_{-1},h_4)q(h_1v_{-2},\mathcal{S}(h_6))((h_2v_{-1})\mathcal{S}(h_5))h_7\otimes(h_3\triangleright v_0)_0\\&\overset{(2.2.8)}{=}p((h_3\triangleright v_0)_{-1},h_4)q(h_2v_{-1},\mathcal{S}(h_5))h_1v_{-2}\otimes(h_3\triangleright v_0)_0\\&\overset{(2.2.14)}{=}h_1v_{-1}\otimes(h_2\triangleright v_0).\end{aligned}$$

(2) We have

$$\begin{aligned}v_{-1}\otimes(h\triangleright v_0)&=\varepsilon(v_{-1})\varepsilon(h_1)v_{-2}\otimes(h_2\triangleright v_0)\\&\overset{(2.2.11)}{=}s(\mathcal{S}(h_1),h_3v_{-1})t(h_2,v_{-2})v_{-3}\otimes(h_4\triangleright v_0)\\&\overset{(2.2.9)}{=}s(\mathcal{S}(h_1),h_5v_{-1})t(h_3,v_{-3})\mathcal{S}(h_2)(h_4v_{-2})\otimes h_6\triangleright v_0\\&\overset{(2.2.3)}{=}s(\mathcal{S}(h_1),(h_4\triangleright v_0)_{-1}h_6)t(h_3,v_{-1})\mathcal{S}(h_2)((h_4\triangleright v_0)_{-2}h_5)\otimes(h_4\triangleright v_0)_0.\end{aligned}$$

□

**Remark 2.2.5.** Let $H$ be a coquasi-Hopf algebra with bijective antipode.

(1) [73, Theorem 2.8] The inverse braiding is given by:

$$\begin{aligned}c^{-1}_{V,W}(w\otimes v)=\Phi^{-1}(\mathcal{S}(v_{-2}),w_{-1},v_{-5})q_L(\mathcal{S}^{-1}(v_{-1}),w_{-2}v_{-6})p_R((\mathcal{S}^{-1}(v_{-3}\triangleright w_0)_{-1}),\mathcal{S}^{-1}(v_{-4}))\\v_0\otimes(\mathcal{S}^{-1}(v_{-3})\triangleright w_0)_0.\end{aligned} \tag{2.2.15}$$

(2) [73, Proposition 2.5] The coquasi-Hopf algebra $H$ itself can be viewed as an object in ${}^H_H\mathcal{YD}$ via the structures

$$\begin{aligned}&\delta_H(h)=h_1\mathcal{S}(h_3)\otimes h_2,\\&h\triangleright h'=\Phi(h_1,h'_1,\mathcal{S}(h'_7))\Phi(h_2h'_2,\mathcal{S}(h_4h'_4),h_7)g(h_5,h'_5)q_R(\mathcal{S}(h'_6),\mathcal{S}(h_6))h_3h'_3.\end{aligned} \tag{2.2.16}$$

Here

$$g(h,h') := \Phi^{-1}(\mathcal{S}(h_1h'_1), h_3h'_3, \mathcal{S}(h'_5)\mathcal{S}(h_5))\chi(h_4,h'_4)\alpha(h_2h'_2),$$

and

$$\chi(h,h') = \Phi(h_1h'_1, \mathcal{S}(h'_5), \mathcal{S}(h_4))\Phi^{-1}(h_2,h'_2,\mathcal{S}(h'_4))\beta(h_3)\beta(h'_3).$$

**Example 2.2.6.** For our purpose, we aim to describe Yetter-Drinfeld modules over coquasi-Hopf algebra of the form $(\Bbbk G, \Phi)$, where $G$ is a finite group and $\Phi$ a 3-cocycle on $G$. Assume that $V$ is a left $kG$-comodule with comodule structure map $\delta_L : V \to kG \otimes V$. Define

$${}^gV := \{v \in V \mid \delta_L(v) = g \otimes v\}.$$

Thus

$$V = \bigoplus_{g\in G} {}^gV.$$

Here we call $g$ the degree of the elements in ${}^gV$ and denote $\deg v = g$ for $v \in {}^gV$.

The left $\Bbbk G$-comodule $(V, \delta_L)$ is a left-left Yetter–Drinfeld module over the coquasi-Hopf algebra $H = (\Bbbk G, \Phi)$ if there is a linear map $\triangleright : G \otimes V \to V$ such that for all $e, f \in G$ and $v \in {}^gV$:

$$e \triangleright (f \triangleright v) = \frac{\Phi(e,f,g)\Phi(efgf^{-1}e^{-1},e,f)}{\Phi(e,fgf^{-1},f)}(ef) \triangleright v, \tag{2.2.17}$$

$$1_H \triangleright v = v, \tag{2.2.18}$$

$$e \triangleright v \in {}^{ege^{-1}}V. \tag{2.2.19}$$

The category of all left-left Yetter–Drinfeld modules over $(\Bbbk G, \Phi)$ is denoted by ${}^G_G\mathcal{YD}^\Phi$. Similarly, one can define left-right, right-left and right-right Yetter–Drinfeld modules over $(\Bbbk G, \Phi)$.

It is well-known that ${}^G_G\mathcal{YD}^\Phi$ is a braided tensor category. More precisely, for any $M, N \in {}^G_G\mathcal{YD}^\Phi$, the structure maps of $M \otimes N$ as a left-left Yetter–Drinfeld module are given by:

$$\delta_L(m_g \otimes n_h) := gh \otimes m_g \otimes n_h, \tag{2.2.20}$$

$$x \triangleright (m_g \otimes n_h) := \frac{\Phi(x,g,h)\Phi(xgx^{-1},xhx^{-1},x)}{\Phi(xgx^{-1},x,h)} x \triangleright m_g \otimes x \triangleright n_h, \tag{2.2.21}$$

for all $x, g, h \in G$ and $m_g \in {}^gM$, $n_h \in {}^hN$.

The associativity constraint $a$ and the braiding $c$ of ${}^G_G\mathcal{YD}^\Phi$ are given respectively by:

$$a((u_e \otimes v_f) \otimes w_g) = \Phi(e,f,g)^{-1} u_e \otimes (v_f \otimes w_g), \tag{2.2.22}$$

$$c(u_e \otimes v_f) = e \triangleright v_f \otimes u_e, \tag{2.2.23}$$

for all $e, f, g \in G$, $u_e \in {}^eU$, $v_f \in {}^fV$, $w_g \in {}^gW$ and $U, V, W \in {}^G_G\mathcal{YD}^\Phi$.

## 2.3 Bosonization for coquasi-Hopf algebras

Bosonization is a fundamental construction that builds a new Hopf algebra from a Hopf algebra in the Yetter–Drinfeld category. In [27], the authors introduced bosonization for quasi-Hopf algebras.

Dually, bosonization for coquasi-Hopf algebras is given in [18]. We recall this construction, as it provides the setting for our later study of reflections via projection and coinvariants.

Let $H$ be a coquasi-Hopf algebra with associator $\Phi$. Suppose $R$ is a Hopf algebra in ${}_H^H\mathcal{YD}$, we denote

$$r^1 \otimes r^2 := \Delta_R(r).$$

**Lemma 2.3.1.** [18, Theorem 5.2] *Let us consider on $M = R \otimes H$ the following structures:*

$$\begin{aligned}
(r\otimes h)(s\otimes k) &= \frac{\Phi(h_2, s_{-1}, k_2)\Phi(r_{-1}, (h_3 \triangleright s_0)_{-1}, h_5k_4)}{\Phi(r_{-2}, h_1, s_{-2}k_1)\Phi((h_3 \triangleright s_0)_{-2}, h_4, k_3)} r_0(h_3 \triangleright s_0)_0 \otimes h_6k_5,\\
u_M(k) &= k1_R \otimes 1_H,\\
\Delta_M(r\otimes h) &= \Phi^{-1}(r^1_{-1}, r^2_{-2}, h_1) r^1_0 \otimes r^2_{-1}h_2 \otimes r^2_0 \otimes h_3,\\
\varepsilon_M(r\otimes h) &= \varepsilon_R(r)\varepsilon_H(h),\\
\Phi_M((r\otimes h),(s\otimes k),(t\otimes l)) &= \varepsilon_R(r)\varepsilon_R(s)\varepsilon_R(t)\Phi(h,k,l),
\end{aligned}$$

*where $r,s,t \in R$, $h,k,l \in H$. With the above operations, $R\otimes H$ is a coquasi-bialgebra, which we denote by $R\#H$.*

**Remark 2.3.2.** The bosonization $R\#H$ can be further equipped with an antipode, making it a coquasi-Hopf algebra. The antipode $(\mathcal{S},\alpha,\beta)$ is given by:

$$\begin{aligned}
&\alpha(r\#h) = 0, \quad \alpha(1\#h) = \alpha(h),\\
&\beta(r\#h) = 0, \quad \beta(1\#h) = \beta(h),\\
&\mathcal{S}(r\#h) = m_{R\#H} \circ (\mathcal{S}_H \otimes \mathcal{S}_R) \circ c_{R,H}(r\otimes h) = \mathcal{S}_H(r_{-1} \triangleright h)\mathcal{S}_R(r_0).
\end{aligned}$$

In particular, if $r$ is a primitive element in $R$, then

$$\mathcal{S}(r\#1) = \mathcal{S}_H(r_{-2})\alpha(r_{-1})\mathcal{S}_R(y_0) = -\mathcal{S}_H(r_{-2})\alpha(r_{-1})y_0.$$

The antipode $\mathcal{S}$ is bijective if and only if $\mathcal{S}_H$ and $\mathcal{S}_R$ are bijective.

Now suppose $N$ and $H$ are both coquasi-Hopf algebras. Furthermore, assume there exist morphisms of coquasi-Hopf algebras

$$\pi : N \longrightarrow H \quad \text{and } \sigma : H \longrightarrow N$$

such that $\pi\sigma = \mathrm{id}_N$.

**Lemma 2.3.3.** [18, Theorem 5.8] *Under above assumptions, $L := N^{\mathrm{co}H}$ is a Hopf algebra in ${}_H^H\mathcal{YD}$. For all $a \in N$, we define the following linear map*

$$\tau(a) = \Phi_N(a_1, (\sigma\circ\mathbb{S}_H\circ\pi(a_3))_1, a_4)a_2(\sigma\circ\mathbb{S}_H\circ\pi(a_3))_2. \tag{2.3.1}$$

*Then $\tau : N \longrightarrow L$ is well-defined. For all $r,s \in L$, $h \in H$, $k \in \Bbbk$, the Yetter-Drinfeld module structure of $L$ is given by*

$$\begin{aligned}
\mathrm{ad}(h)(r) &:= \tau(\sigma(h)r) = \Phi_H(h_1r_{-1}, \mathbb{S}(h_3)_1, h_4)(h_2r_0)\mathbb{S}(h_3)_2,\\
r_{-1}\otimes r_0 &:= \rho_R(r) := \pi(r_1)\otimes r_2.
\end{aligned}$$

*Moreover, $L$ is a Hopf algebra in ${}^H_H\mathcal{YD}$ via*

$$m_L(r\otimes s) := rs, \qquad u_L(k) = k1_N,$$
$$\Delta_L(r) := \tau(r_1)\otimes\tau(r_2), \quad \varepsilon_L(r) := \varepsilon_N(r),$$
$$\mathcal{S}_R = \mathrm{ad}\circ(\mathrm{id}_H\otimes\tau\circ\mathcal{S}_N\circ i)\circ\rho_R,$$

*where $i : L \to N$ is the embedding map. Furthermore, there is an isomorphism of coquasi-Hopf algebra $\phi : L\#H \to N$ given by*

$$\phi(r\otimes h) = r\sigma(h), \quad \phi^{-1}(a) = \tau(a_1)\otimes\pi(a_2).$$

The following condition occurs frequently in this paper. We give an example here.

**Example 2.3.4.** Given $R, R'$ be two Hopf algebras in ${}^H_H\mathcal{YD}$ together with a surjective Hopf algebra map $\pi' : R' \to R$ and an injective map $\sigma' : R \to R'$ such that $\pi'\circ\sigma' = \mathrm{id}_R$. Let $M' = R'\#H$, and $M = R\#H$. Obviously, these maps naturally extend to yield a surjective coquasi-Hopf algebra map $\pi : M' \to M$ and an injective coquasi-Hopf algebra map $\sigma : M \to M'$, such that $\pi\circ\sigma = \mathrm{id}_M$.

Let $K := M'^{\mathrm{co}M}$, then $K$ is an object in ${}^M_M\mathcal{YD}$, with the following structures:

$$\mathrm{ad} : M\otimes K \to K, h\otimes k \mapsto \Phi_M(h_1k_{-1}, \mathbb{S}_M(h_3)_1, h_4)(h_2k_0)\mathbb{S}_M(h_3)_2,$$
$$\rho : K \to M\otimes K, k \mapsto \pi(k_1)\otimes k_2.$$

To analyze the linear map ad in concrete calculations, the following computation is useful. Let $y\in R$ be a primitive element. One can compute the preantipode $\mathbb{S}_M(y)$ in $M$.

$$\mathbb{S}_M(y) = \beta*\mathcal{S}_M*\alpha(y) = \beta(y_1)\mathcal{S}_M(y_2)\alpha(y_3) = \beta(y_{-1})\mathcal{S}_M(y_0) = -\beta(y_{-3})\mathcal{S}_H(y_{-2})\alpha(y_{-1})y_0.$$

Then we can proceed to calculate $\mathrm{ad}(y)(x)$, where $x\in K$,

$$\begin{aligned}\mathrm{ad}(y)(x) &= yx + (y_{-1}x)\mathbb{S}_M(y_0)\\ &= yx - (y_{-4}x)\beta(y_{-3})(\mathcal{S}_H(y_{-2})\alpha(y_{-1})y_0) \qquad (2.3.2)\\ &= yx - \Phi(y_{-7}x_{-1}, \mathcal{S}_H(y_{-3}), y_{-1})\beta(y_{-6})((y_{-5}x_0)\mathcal{S}_H(y_{-4}))\alpha(y_{-2})y_0.\end{aligned}$$

On the other hand, note that for all $h\in H$, $\Delta_H(\mathbb{S}(h)) = \beta(h_1)\mathcal{S}(h_3)\otimes\mathcal{S}(h_2)\alpha(h_4)$. Therefore, we have

$$\begin{aligned}yx - \mathrm{ad}(y_{-1})(x)y_0 &= yx - \Phi(y_{-4}x_{-1}, \mathbb{S}(y_{-2})_1, y_{-1}))((y_{-3}x_0)\mathbb{S}(y_{-2})_2)y_0\\ &= yx - \Phi(y_{-7}x_{-1}, \mathcal{S}(y_{-4}), y_{-1})\beta(y_{-6})((y_{-5}x_0)\mathcal{S}(y_{-3}))\alpha(y_{-2})y_0.\end{aligned}$$

This implies

$$\mathrm{ad}(y)(x) = yx - \mathrm{ad}(y_{-1})(x)y_0. \qquad (2.3.3)$$

## 2.4 Yetter-Drinfeld modules in an arbitrary braided monoidal category

In Section 3, we will investigate Yetter-Drinfeld modules over a Hopf algebra $A$ in the category ${}^H_H\mathcal{YD}$. Since ${}^H_H\mathcal{YD}$ is braided monoidal, we require the general framework of Yetter-Drinfeld modules in arbitrary braided monoidal categories. We recall the definition from [24], one may refer to [52, Section 3.4] for more details. Now we let $\mathcal{C}$ be a braided monoidal category and $R$ a Hopf algebra in $\mathcal{C}$. The category of Yetter-Drinfeld modules ${}^R_R\mathcal{YD}(\mathcal{C})$ is well-defined.

**Definition 2.4.1.** *Let $R$ be a Hopf algebra with bijective antipode in a braided monoidal category $\mathcal{C}$. Suppose that $X$ is a left module and left comodule over $R$ with structure map $\rho_X$ and $\delta_X$. The triple $(X,\rho_X,\delta_X)$ is a Yetter-Drinfeld module if in addition, in $\mathrm{Hom}_{\mathcal{C}}(R\otimes X, R\otimes X)$,*

$$\begin{aligned}(\mu_X\otimes\mathrm{id})\circ a^{-1}_{R,R,X}\circ(\mathrm{id}\otimes c_{X,R})\circ a_{R,X,R}\circ(\delta_X\circ\rho_X\otimes\mathrm{id})\circ a^{-1}_{R,X,R}\circ(\mathrm{id}\otimes c_{R,X})\circ a_{R,R,X}\circ(\Delta_R\otimes\mathrm{id})\\ =(\mu_R\otimes\rho_X)\circ a^{-1}_{R,R,R\otimes X}\circ(\mathrm{id}\otimes a_{R,R,X}\circ(\mathrm{id}\otimes c_{R,R})\otimes\mathrm{id})\circ(\mathrm{id}\otimes a^{-1}_{R,R,X})\circ a_{R,R,R\otimes X}\circ(\Delta_R\otimes\delta_X).\end{aligned}$$

It is known that the category ${}^R_R\mathcal{YD}(\mathcal{C})$ is a braided monoidal category. For $X,Y\in{}^R_R\mathcal{YD}(\mathcal{C})$ the braiding isomorphism in ${}^R_R\mathcal{YD}(\mathcal{C})$ is given by

$$\begin{aligned}&c^{\mathcal{YD}}_{X,Y}: X\otimes Y\to Y\otimes X,\\ &c^{\mathcal{YD}}_{X,Y}:=(\rho_Y\otimes\mathrm{id}_X)\circ a^{-1}_{R,Y,X}\circ(\mathrm{id}_R\otimes c_{X,Y})\circ a_{R,X,Y}\circ(\delta_X\otimes\mathrm{id}_Y).\end{aligned}$$

**Remark 2.4.2.** (1) According to the definition of $c^{\mathcal{YD}}_{X,Y}$, it is not hard to see, if $X\in{}^R\mathcal{C}$ and $Y\in{}_R\mathcal{C}$, then $c^{\mathcal{YD}}_{X,Y}$ is a morphism in $\mathcal{C}$.

(2) [52, Proposition 3.4.5] Let $V$ be an object in $\mathcal{C}$, $(V,\lambda)\in{}_R\mathcal{C}$, and $(V,\delta)\in{}^R\mathcal{C}$. Then the following are equivalent.

(i) $(V,\lambda,\delta)$ is a left Yetter-Drinfeld module over $R$.
(ii) For all $X\in{}_R\mathcal{C}$, $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_R\mathcal{C}$.
(iii) $c^{\mathcal{YD}}_{V,R}$ is a morphism in ${}_R\mathcal{C}$, where $R$ is a left $R$-module by the multiplication in $R$.
(iv) For all $X\in{}^R\mathcal{C}$, $c^{\mathcal{YD}}_{X,V}$ is a morphism in ${}^R\mathcal{C}$.

(3) In general, let $(\mathcal{C},\otimes,\mathbf{1},c)$ be a braided monoidal category, we can define the reverse monoidal category $\overline{\mathcal{C}}:=(\mathcal{C},\otimes,\mathbf{1},\overline{c})$, where for $X,Y\in\mathcal{C}$,

$$\overline{c}_{X,Y}:=c^{-1}_{Y,X}: X\otimes Y\to Y\otimes X.$$

For $X\in{}_R\mathcal{C}$ and $Y\in{}^R\mathcal{C}$, let

$$\overline{c}^{\mathcal{YD}}_{X,Y}=\overline{c}_{X,Y}\circ(\lambda_X\otimes\mathrm{id}_Y)\circ((\mathcal{S}^{-1}_R\otimes\mathrm{id}_X)\otimes\mathrm{id}_Y)\circ(\overline{c}_{X,R}\otimes\mathrm{id}_Y)\circ a^{-1}_{X,R,Y}\circ(\mathrm{id}_X\otimes\delta_Y).\qquad(2.4.1)$$

By [52, Proposition 3.4.8], $c^{\mathcal{YD}}_{Y,X}$ is an isomorphism in $\mathcal{C}$ with inverse $\overline{c}^{\mathcal{YD}}_{X,Y}$.

The definition of the bosonization naturally extends to the case where $K$ is a Hopf algebra in the braided monoidal category ${}^R_R\mathcal{YD}(\mathcal{C})$, where $\mathcal{C}$ is now an arbitrary braided monoidal category. We collect several key results from [24,23,1] that will be important for the construction.

**Lemma 2.4.3.** [24, Theorem 3.9.5] *There is a braided monoidal isomorphism from the Hopf bimodule category to the Yetter-Drinfeld module category:*

$$^R_R\mathcal{C}^R_R\cong{}^R_R\mathcal{YD}(\mathcal{C}).$$

**Lemma 2.4.4.** [23, Proposition 4.2.3] *Let $A$ be a Hopf algebra in $\mathcal{C}$ and $K$ a Hopf algebra in*

${}^{A}_{A}\mathcal{YD}(\mathcal{C})$. *There is an obvious isomorphism of braided monoidal categories*

$$ {}^{K\#A}_{K\#A}\mathcal{YD}(\mathcal{C}) \cong {}^{K}_{K}\mathcal{YD}({}^{A}_{A}\mathcal{YD}(\mathcal{C})). \tag{2.4.2} $$

**Lemma 2.4.5.** [1, Theorem 3.2] *Let $H$ and $A$ be Hopf algebras in a braided monoidal category $\mathcal{C}$. Let $\pi : H \to A$ and $\iota : A \to H$ be Hopf algebra morphisms such that $\pi \circ \iota = \mathrm{id}_A$. If $\mathcal{C}$ has equalizers and $A \otimes (-)$ preserves equalizers, there is a Hopf algebra $R$ in the braided monoidal category ${}^{A}_{A}\mathcal{YD}(\mathcal{C})$, such that*

$$ H \cong R\#A. $$

**Remark 2.4.6.** Let $H$ be a coquasi-Hopf algebra with bijective antipode.

(1) By [27, Proposition 3.9, Lemma 4.5] and Lemma 2.4.3, we have a braided monoidal equivalence

$$ {}^{R\#H}_{R\#H}\mathcal{YD} \cong {}^{R}_{R}\mathcal{YD}(\mathcal{C}), $$

where $\mathcal{C} = {}^{H}_{H}\mathcal{YD}$.

(2) Since ${}^{H}_{H}\mathcal{YD}$ is an abelian category, the conditions in Lemma 2.4.5 hold automatically. That is, we can talk about bosonization and taking coinvariants in ${}^{H}_{H}\mathcal{YD}$ without any restrictions.

## 2.5 Nichols algebras

Let $\mathcal{C}$ be an abelian braided monoidal category. We first recall the definition of Nichols algebras $\mathcal{B}(V)$ in $\mathcal{C}$, where $V \in \mathcal{C}$ is an arbitrary object. Then we are going to show that there is a non-degenerate Hopf pairing between $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$ when $V$ has a left dual $V^*$.

We denote by $T(V)$ the tensor algebra in $\mathcal{C}$ generated freely by $V$. Here we denote that

$$ V^{\otimes n} := (\cdots((V \otimes V) \otimes V) \cdots \otimes V). $$

Then $T(V)$ is isomorphic to $\bigoplus_{n\geq 0} V^{\otimes n}$ as an object. The tensor algebra $T(V)$ is naturally a graded Hopf algebra in $\mathcal{C}$.

Let $i_n : V^{\otimes n} \to T(V)$ be the canonical injection. We denote the total Woronowicz symmetriser by

$$ \mathrm{Wor}(c) = \bigoplus_{n\geq 1} i_n [n]!_c, $$

where $[n]!_c \in \mathrm{End}(V^{\otimes n})$ is the braided symmertriser. For explicit definition of $[n]!_c$, one may refer to [20, Section 5.6].

**Definition 2.5.1.** *Let $V \in \mathcal{C}$. The Nichols algebra of $V$ is defined to be the quotient Hopf algebra*

$$ \mathcal{B}(V) := T(V)/\ker \mathrm{Wor}(c). $$

An equivalent characterization is given as follows.

**Remark 2.5.2.** [52, Definition 1.6.17] Let $V \in \mathcal{C}$ and $I(V)$ be the largest coideal of $T(V)$ contained in $\bigoplus_{n\geq 2} T^n(V)$. The Nichols algebra of $V$ is defined by

$$ \mathcal{B}(V) := T(V)/I(V). $$

To facilitate further analysis, we assume $C$ is a $\Bbbk$-linear braided monoidal abelian category. In this setting, the following equivalent definition of the Nichols algebra is more convenient.

**Definition 2.5.3.** [79, Definition 2.4] *For a given object $V \in C$, the Nichols algebra $\mathcal{B}(V)$ is the unique Hopf algebra in $C$ that satisfies the following conditions:*

(1) *The Hopf algebra $\mathcal{B}(V)$ is graded by the non-negative integers.*

(2) *The zeroth component of the grading satisfies $\mathcal{B}(V)_0 = \Bbbk$.*

(3) *The first component of the grading satisfies $\mathcal{B}(V)_1 = V$ , and $\mathcal{B}(V)$ is generated by $V$ as an algebra in $C$.*

(4) *The subobject of primitive elements of $\mathcal{B}(V)$ is $V$.*

Suppose $C$ and $\mathcal{D}$ are braided monoidal abelian categories, and $F : C \to \mathcal{D}$ is a braided monoidal equivalence of abelian categories. The following lemma plays a crucial role in our theory.

**Lemma 2.5.4.** *Let $V \in C$ be an object and $\mathcal{B}(V)$ be its corresponding Nichols algebra in $C$. Then*

$$F(\mathcal{B}(V)) \cong \mathcal{B}(F(V))$$

*as Hopf algebras in $\mathcal{D}$.*

*Proof.* Let $V \in C$ and $J_{V,V} : F(V \otimes V) \cong F(V) \otimes F(V)$ be the monoidal structure of $F$. Using $J$ repeatedly, we have

$$F(V^{\otimes n}) \cong F(V)^{\otimes n}$$

for each $n \in \mathbb{N}$. Therefore we have $F(\bigoplus_{n\in\mathbb{N}} V^{\otimes n}) \cong \bigoplus_{n\in\mathbb{N}} F(V^{\otimes n}) \cong \bigoplus_{n\in\mathbb{N}} F(V)^{\otimes n}$. That is $F(T(V)) \cong T(F(V))$.

Recall that the Nichols algebras in $C$ are of the form $T(X)/I$ where $X \in C$ and $I$ is the unique maximal graded Hopf ideal in $T(V)$ generated by homogeneous elements of degree greater than or equal to 2. According to $F(T(V)) \cong T(F(V))$, we have

$$F(T(V)/I) \cong T(F(V))/F(I),$$

which is finite-dimensional as well. Here $F(I)$ is a homogeneous Hopf ideal of degree greater than or equal to 2 of $T(F(V))$. Note that Nichols algebra generated by $F(V)$ must be of the form $T(F(V))/J$, where $J$ is the unique maximal homogeneous graded Hopf ideal of $T(F(V)) \in \mathcal{D}$ with degree greater than or equal to 2. Hence $F(I) \subset J$ and $T(F(V))/J \subset T(F(V))/F(I)$. On the other hand, $F^{-1} : \mathcal{D} \longrightarrow C$ is inverse of $F$, which is an exact monoidal functor. Hence

$$F^{-1}(T(F(V))/J) \cong T(F^{-1}(F(V))/F^{-1}(J) \cong T(V)/F^{-1}(J) \supseteq T(V)/I.$$

So $F^{-1}(J) \subseteq I$, combining $F(I) \subseteq J$ implies $F(I) = J$, which leads to $F(T(V)/I) \cong T(F(V))/J$ and the proof is done. □

## 2.6 Classification results of finite-dimensional coquasi-Hopf algebras

We briefly summarize the classification of finite-dimensional coradically graded pointed coquasi-Hopf algebras over finite abelian groups, as established in [60].

By definition, coquasi-Hopf algebras are exactly the dual of Drinfeld' s quasi-Hopf algebras [33]. These are coassociative coalgebras that need not be associative. Analogous to the Hopf algebra case, the classification relies critically on: twisted Yetter-Drinfeld module categories ${}_G^G\mathcal{YD}^\Phi$ and twisted Nichols algebras, where $G$ is a finite abelian group and $\Phi$ is a normalized 3-cocycle over $G$. For formal definitions and examples, see [58, Section 2].

The classification of finite-dimensional coradically graded coquasi-Hopf algebra over abelian groups hinges on the structure of the coradical $(\Bbbk G, \Phi)$, where $\Phi$ is a 3-cocycle on $G$. A complete parametrization of 3-cohomology classes for finite abelian groups was achieved in [58] as follows. Let $G$ be a finite abelian group and so there is no harm to assume that $G = Z_{m_1} \times Z_{m_2} \cdots \times Z_{m_n}$ with $m_i \mid m_{i+1}$ for $1 \le i \le n-1$. Denote $\mathscr{A}$ the set of all $\mathbb{N}$-sequences:

$$\underline{a} = (a_1, a_2, ..., a_l, ..., a_n, a_{12}, a_{13}, ..., a_{st}, ..., a_{n-1,n}, a_{123}, ..., a_{rst}, ... a_{n-2,n-1,n})$$

such that $0 \le a_l < m_l$, $0 \le a_{st} < (m_s, m_t)$, $0 \le a_{rst} < (m_r, m_s, m_t)$, with indices ordered lexicographically. Let $g_i$ be the generator of $Z_{m_i}$, $1 \le i \le n$. For each $\underline{a} \in \mathscr{A}$, define

$$\Phi\left(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}\right) = \prod_{l=1}^{n} \zeta_{m_l}^{a_l i_l \left[\frac{j_l + k_l}{m_l}\right]} \prod_{1 \le s < t \le n} \zeta_{m_s}^{a_{st} k_s \left[\frac{i_t + j_t}{m_t}\right]} \prod_{1 \le r < s < t \le n} \zeta_{(m_r, m_s, m_t)}^{a_{rst} k_r j_s i_t} \tag{2.6.1}$$

Here $\zeta_m$ denotes a primitive $m$-th root of unity. We further define $\mathscr{A}'$ as the subset of $\mathscr{A}$ satisfying $a_{rst} = 0$ for all $1 \le r < s < t \le n$ and $\mathscr{A}''$ as the subset of $\mathscr{A}$ satisfying $a_i = 0$, $a_{st} = 0$ for all $1 \le i \le n$, $1 \le s < t \le n$.

**Lemma 2.6.1.** [58, Proposition 3.8] $\left\{\Phi_{\underline{a}} | \underline{a} \in \mathscr{A}\right\}$ *forms a complete set of representatives of the normalized* 3*-cocycles over* $G$ *up to* 3*-cohomology.*

Among all the 3-cocycles over abelian groups, the class of abelian cocycles is particularly significant. We may use $D^\Phi(G)$ to define abelian cocycles, which has been studied deeply in [77].

**Definition 2.6.2.** *A* 3*-cocycle* $\Phi$ *on an abelian group* $G$ *is called abelian if* $D^\Phi(G)$ *is a commutative algebra.*

There's a nice description when the 3-cocycle $\Phi_{\underline{a}}$ is abelian: which connected to the property that a Yetter-Drinfeld module $V$ over ${}_G^G\mathcal{YD}^\Phi$ is of diagonal type. For details, see [58, Section 3].

**Lemma 2.6.3.** [58, Corollary 3.13, Proposition 3.14] *Let* $G$ *be a finite abelian group and* $\Phi$ *a normalized* 3*-cocycle over* $G$ *as in (2.6.1), then*

$$\begin{aligned} \Phi \text{ is an abelian cocycle } &\Longleftrightarrow a_{rst} = 0 \text{ for all } 1 \le r < s < t \le n \\ &\Longleftrightarrow \text{Every Yetter-Drinfeld module over } (kG, \Phi) \text{ is of diagonal type.} \end{aligned}$$

Abelian cocycles possess an additional remarkable property: they can be resolved within an extended abelian group. Consider $\mathbb{G} \cong Z_{m_1^2} \times Z_{m_2^2} \times \cdots \times Z_{m_n^2} = \langle \mathbb{g}_1 \rangle \times \langle \mathbb{g}_2 \rangle \times \cdots \times \langle \mathbb{g}_n \rangle$. There is a group surjection $\pi: \mathbb{G} \longrightarrow G$, $\mathbb{g}_i \mapsto g_i$, for $1 \le i \le n$. We may pull back the 3-cocycles over $G$ and obtain a 3-cocycles over $\mathbb{G}$. That is, the map

$$\pi^*(\Phi): \mathbb{G} \times \mathbb{G} \times \mathbb{G} \longrightarrow \Bbbk^\times, \quad (g, h, k) \mapsto \Phi(\pi(g), \pi(h), \pi(k)).$$

is a 3-cocycle over $\mathbb{G}$. Actually, $\pi^*(\Phi)$ is a coboundary. If we define $J_{\underline{a}}$ via

$$J_{\underline{a}}: \mathbb{G}\times\mathbb{G} \longrightarrow \Bbbk^\times, \quad (g_1^{x_1}\cdots g_n^{x_n}, g_1^{y_1}\cdots g_n^{y_n}) \mapsto \prod_{l=1}^{n} \zeta_{m_l^2}^{a_l x_l (y_l - y_l')} \prod_{1\le s<t\le n} \zeta_{m_s m_t}^{a_{st} x_t (y_s - y_s')},$$

then $\pi^*(\Phi_{\underline{a}}) = \partial(J_{\underline{a}})$, This result corresponds precisely to [58].

The following theorem classifies finite-dimensional coradically graded pointed coquasi-Hopf algebras over finite abelian groups.

**Theorem 2.6.4.** [60, Theorem 5.2] *Let $M$ be a finite-dimensional coradically graded pointed coquasi-Hopf algebra over a finite abelian group with the coradical $M_0 = (\Bbbk G, \Phi)$. Then we have $M \cong \mathcal{B}(V)\#\Bbbk G$ for a Yetter-Drinfeld module of finite type $V \in {}^G_G\mathcal{YD}^\Phi$.*

**Remark 2.6.5.** We elaborate on Theorem 2.6.4 and present related results. Recall that if $V \in {}^G_G\mathcal{YD}^\Phi$, then we can define the support group $G_V := < g \in G \mid {}^g V \neq 0 >$, which is a subgroup of $G$.

(i) A Yetter-Drinfeld module ${}^G_G\mathcal{YD}^\Phi$ is said to be of finite type if

∘ $V$ has a standard basis. This is equivalent to the condition that $\omega\mid_{G_V}$ is an abelian cocycle.

∘ $\Delta_{\chi,E}$ is an arithmetic root system and $\mathrm{ht}(\alpha) < \infty$ for all $\alpha \in \Delta_{\chi,E}$, where $(\chi, E)$ is a pair associated to $V$.

(ii) By [60, Theorem 3.9], $\mathcal{B}(V)$ is finite-dimensional if and only if $V$ is finite type. Furthermore, in this case, $\mathcal{B}(V)$ is isomorphic to a Nichols algebra of diagonal type in ${}^{G_V}_{G_V}\mathcal{YD}^{\Phi|_{G_V}}$. Conversely, if $\Phi\mid_{G_V}$ is nonabelian, then $\mathcal{B}(V)$ is infinite-dimensional by [60, Theorem 3.13].

(iii) The classification divides into two cases:

∘ Finite-dimensional twisted Nichols algebras of diagonal type over $(G, \Phi)$ with $\Phi$ abelian;

∘ Finite-dimensional twisted Nichols algebras of non-diagonal type where $\Phi\mid_{G_V}$ is abelian.

# Chapter 3

# Rational Yetter-Drinfeld module over coquasi-Hopf algebras

In this section, $H$ always represents a coquasi-Hopf algebra with a bijective antipode, and $\mathcal{C} = {}^H_H\mathcal{YD}$ denotes the Yetter-Drinfeld module category of $H$ . We investigate rational Yetter-Drinfeld modules in ${}^H_H\mathcal{YD}$, a notion that will be important for our later study of Nichols algebras and their reflections.

## 3.1 Rational modules in ${}^H_H\mathcal{YD}$

Since $H$ is a non-associative algebra, we cannot talk about modules over $H$ in the usual sense. However, if $R$ is an algebra in $\mathcal{C} = {}^H_H\mathcal{YD}$, then the category ${}_R\mathcal{C}$ of left $R$-modules in $\mathcal{C}$ is well-defined. Motivated by [52, Definition 12.2.3], we now introduce the concept of rationality under the setting of coquasi-Hopf algebras.

**Definition 3.1.1.** *Let $R = \bigoplus_{n\geq 0} R(n)$ be an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$.*

(1) *A left module $(X, \lambda_X)$ over $R$ in $\mathcal{C}$ is called rational if for any $x \in X$, there exists a natural number $n_0$ such that $\lambda(R(n) \otimes x) = 0$ for all $n \geq n_0$. We denote the category of left rational module over $R$ in $\mathcal{C}$ by ${}_R\mathcal{C}_{\mathrm{rat}}$.*

(2) *A left Yetter-Drinfeld module $X$ over ${}^R_R\mathcal{YD}(\mathcal{C})$ is called rational if $X$ is rational as a module of $R$ in $\mathcal{C}$. We denote the corresponding category by ${}^R_R\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$.*

**Remark 3.1.2.** It is worth noting that for a locally finite $\mathbb{N}_0$-graded Hopf algebra $R$, this graded definition of rationality coincides with the classical one. Intuitively, the condition $\lambda(R(n) \otimes x) = 0$ for large $n$ implies that the action on any element $x$ is nontrivial on a finite-dimensional truncation of $R$, which recovers the standard notion of rationality in the non-graded setting.

A natural question arises: are rational modules closed under tensor products. The following lemma ensures this.

**Lemma 3.1.3.** *Let $R = \bigoplus_{n\geq 0} R(n)$ be an $\mathbb{N}_0$-graded Hopf algebra in $\mathcal{C}$.*

(1) *The category ${}_R\mathcal{C}_{\mathrm{rat}}$ is a monoidal subcategory of ${}_R\mathcal{C}$, which is closed under direct sums, subobjects and quotient objects.*

(2) *The category ${}^R_R\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$ is a monoidal subcategory of ${}^R_R\mathcal{YD}(\mathcal{C})$, which is closed under direct sums, subobjects and quotient objects.*

*Proof.* (1) Let $V, W \in {}_R\mathcal{C}_{\mathrm{rat}}$ with $R$-action $\lambda_V$, $\lambda_W$ respectively. It is standard that the tensor product $V \otimes W$ is an object in ${}_R\mathcal{C}$, with $R$-action

$$\begin{aligned}\lambda_{V\otimes W} = &(\lambda_V \otimes \lambda_W) \circ (a^{-1}_{R,V,R} \otimes \mathrm{id}_W) \circ a_{R\otimes V,R,W} \circ ((\mathrm{id}_R \otimes c_{R,V}) \otimes \mathrm{id}_W)\\ &\circ (a_{R,R,V} \otimes \mathrm{id}_W) \circ a^{-1}_{R\otimes R,V,W} \circ (\Delta_R \circ \mathrm{id}_{V\otimes W}).\end{aligned}$$

Now take $r \in R(m)$, we have

$$\lambda_{V\otimes W}(r \otimes (v \otimes w)) = \frac{\Phi(r^2_{-4}, v_{-1}, w_{-3})\Phi(r^1_{-1}, (r^2_{-3} \triangleright v_0)_{-1}, r^2_{-1} w_{-1})}{\Phi(r^1_{-2}, r^2_{-5}, v_{-2} w_{-4})\Phi((r^2_{-3} \triangleright v_0)_{-2}, r^2_{-2}, w_{-2})} \lambda_V(r^1_0 \otimes (r^2_{-3} \triangleright v_0)_0) \otimes \lambda_W(r^2_0 \otimes w_0).$$

Since $r^2_{-3} \in H$, and $V$ is an object in ${}^H_H\mathcal{YD}$, we have $(r^2_{-3} \triangleright v_0)_0 \in V$.

The expression $\lambda_{V\otimes W}(r \otimes (v \otimes w))$ is a finite sum. Thus, by the definition of the rational module, there exists a natural number $n_1$ such that for $n \geq n_1$,

$$\lambda_V(R(n) \otimes (r^2_{-3} \triangleright v_0)_0) = 0.$$

Similarly, there is a natural number $n_2$ such that

$$\lambda_W(R(n) \otimes w_0) = 0,$$

for all $n \geq n_2$.

Now we assume $m \geq 2\max\{n_1, n_2\}$, since $R$ is $\mathbb{N}_0$-graded, we have $\Delta_R(r) \in \bigoplus_{p+q=m} R(p) \otimes R(q)$. Therefore

$$\lambda_{V\otimes W}(R(m) \otimes (v \otimes w)) = 0.$$

This implies $V \otimes W \in {}_R\mathcal{C}_{\mathrm{rat}}$. Therefore, ${}_R\mathcal{C}_{\mathrm{rat}}$ is a monoidal subcategory of ${}^R_R\mathcal{YD}(\mathcal{C})$. The category ${}_R\mathcal{C}_{\mathrm{rat}}$ is obviously closed under arbitrary direct sums, subobjects and quotient objects.

(2) The forgetful functor ${}^R_R\mathcal{YD}(\mathcal{C}) \to {}_R\mathcal{C}$ is monoidal. Thus, for $V, W \in {}^R_R\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$, we have $V \otimes W \in {}_R\mathcal{C}_{\mathrm{rat}}$. By definition it implies $V \otimes W \in {}^R_R\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$.

□

## 3.2 Properties of pairings

We now turn to the study of pairings in the category $\mathcal{C} = {}^H_H\mathcal{YD}$, which will later facilitate the construction of dualities between comodules and rational modules.

Let $A$ and $B$ be objects in $\mathcal{C}$, and $\omega : A \otimes B \to \Bbbk$ is a morphism in $\mathcal{C}$, called a pairing. For subsets $X \subseteq A$ and $Y \subseteq B$, we define

$$\begin{aligned} X^\perp &:= \{b \in B \mid \omega(x, b) = 0, \text{ for all } x \in X\},\\ Y^\perp &:= \{a \in A \mid \omega(a, y) = 0, \text{ for all } y \in Y\}.\end{aligned}$$

We say $\omega$ is non-degenerate, if $A^\perp = 0$ and $B^\perp = 0$.

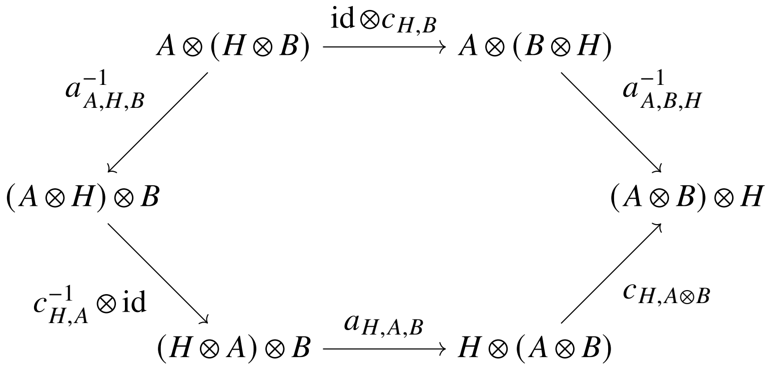
**Lemma 3.2.1.** *If $E$ is a subobject of $A$, then $E^\perp$ is a subobject of $B$. Similarly, if $F$ is a subobject of $B$, then $F^\perp$ is a subobject of $A$.*

*Proof.* Let $E$ be a subobject of $A$, we need to show that $E^\perp$ is an object in $\mathcal{C}$. The proof of $E^\perp$ being an $H$-comodule is the same as in the Hopf algebra case, see [51, Lemma 2.3].

To show $H \triangleright E^\perp \subseteq E^\perp$. One needs to verify that for any $b \in E^\perp$, $h \in H$ and $e \in E$,

$$\omega(e, h \triangleright b) = 0.$$

By hexagon axiom in $\mathcal{C}$, we have the following commutative diagram:




Since $E$ is an object in $\mathcal{C}$, for simplicity, we write $c^{-1}_{A,H}(e \otimes h) = \sum_{1 \le i \le n} h^i \otimes e^i$ for some natural number $n$. Here $h^i \in H$, $e^i \in E$. Now

$$\begin{aligned} c_{H,A\otimes B} \circ a_{H,A,B} \circ (c^{-1}_{A,H} \otimes \mathrm{id})((e \otimes h) \otimes b) &= \sum_{1 \le i \le n} c_{H,A\otimes B} \circ a_{H,A,B}((h^i \otimes e^i) \otimes b) \\ &= \sum_{1 \le i \le n} \Phi(h^i_1, e^i_{-1}, b_{-1}) c_{H,A\otimes B}(h^i_2 \otimes (e^i_0 \otimes b_0)) \\ &= \sum_{1 \le i \le n} \Phi(h^i_1, e^i_{-1}, b_{-1})(h^i_2 \triangleright (e^i_0 \otimes b_0)) \otimes h^i_3. \end{aligned} \tag{3.2.1}$$

Here $b_0 \in E^\perp$, because we have shown $E^\perp$ is a $H$-comodule. Also, $e^i_0 \in E$ for all $i$. Note that $\omega$ is a morphism in $\mathcal{C}$, and the braiding $c$ is a natural isomorphism, we have

$$(\omega \otimes \mathrm{id}) \circ c_{H,A\otimes B} \circ a_{H,A,B} \circ (c^{-1}_{A,H} \otimes \mathrm{id})((e \otimes h) \otimes b) = \sum_{1 \le i \le n} \Phi(h^i_1, e^i_{-1}, b_{-1}) \omega(e^i_0, b_0) \otimes h^i_2 = 0. \tag{3.2.2}$$

Note that $a^{-1}_{A,H,B}(e \otimes (h \otimes b)) \in (A \otimes H) \otimes B$, thus by the hexagon axiom, we obtain:

$$(\omega \otimes \varepsilon) \circ a^{-1}_{A,B,H} \circ (\mathrm{id} \otimes c_{H,B}) \circ a_{A,H,B}(a^{-1}_{A,H,B}(e \otimes (h \otimes b))) = 0.$$

That is

$$\Phi(e_{-1}, (h_1 \triangleright b)_{-1}, h_2)(\omega \otimes \varepsilon)((e_0 \otimes (h_1 \triangleright b)_0) \otimes h_3) = \Phi * (\omega \otimes \varepsilon)(e \otimes (h_1 \triangleright b) \otimes h_2) = 0. \tag{3.2.3}$$

Note that $\Phi$ is convolution invertible, thus

$$\omega(e, h \triangleright b) = 0 \tag{3.2.4}$$

for all $e \in E$, $b \in E^\perp$, $h \in H$. Consequently, $E^\perp$ is a subobject of $B$. The proof of the other direction is similar, and we omit it for simplicity. □

The following construction of the Yetter-Drinfeld submodule generated by a subcomodule is crucial for our subsequent analysis of rational modules.

Let $V \in {}^H_H\mathcal{YD}$ and let $V' \subseteq V$ be an $H$-subcomodule. We define the Yetter-Drinfeld submodule generated by $V'$, denoted by $\langle V' \rangle$, as the smallest Yetter-Drinfeld submodule of $V$ containing $V'$.

We provide an explicit filtration to construct $\langle V' \rangle$.

- Set $X_0 := V'$.
- Assume $X_n$ is defined. Let $Y_n := \text{span}\{h \triangleright x \mid h \in H, x \in X_n\}$.
- Define $X_{n+1}$ as the smallest $H$-subcomodule of $V$ containing $Y_n$:

$$X_{n+1} := \sum_{y \in Y_n} \text{span}\{y_0 \mid \delta_V(y) = y_{-1} \otimes y_0\}.$$

**Lemma 3.2.2.** *With the notation above, $\langle V' \rangle = \bigcup_{n \geq 0} X_n$.*

*Proof.* It is direct to verify that $\bigcup_{n \geq 0} X_n$ is closed under the $H$-action and $H$-coaction by construction. Since any Yetter-Drinfeld submodule containing $V'$ must contain $X_0$ and be closed under the action and coaction, it must contain each $X_n$. Thus, the union is minimal. □

Let $V, W \in C$, and $\omega$ a pairing of $A, B$ in $C$. We denote by $\mathrm{Hom}_{C,\text{rat}}(A \otimes V, W)$ the set of all $g$ in $\mathrm{Hom}_C(A \otimes V, W)$ such that for all $v \in V$ there is a finite-dimensional subobject $F \subseteq B$ in $C$ with $g(F^\perp \otimes v) = 0$.

**Proposition 3.2.3.** *Let $A, B \in C$, $\omega$ a non-degenerate pairing of $A, B$ in $C$, and $W \in C$. Assume that for every $b \in B$ there is a finite-dimensional subobject $F \subseteq B$ in $C$ containing $b$. Then for all $V \in C$, the map*

$$G_V : \mathrm{Hom}_C(V, B \otimes W) \longrightarrow \mathrm{Hom}_{C,\text{rat}}(A \otimes V, W),$$
$$f \mapsto (A \otimes V \xrightarrow{\mathrm{id}_A \otimes f} A \otimes (B \otimes W) \xrightarrow{a^{-1}_{A,B,W}} (A \otimes B) \otimes W \xrightarrow{\omega \otimes \mathrm{id}} W)$$

*is bijective.*

*Proof.* **Step1:**

We first show the map $G_V$ is well-defined and injective. Let $f \in \mathrm{Hom}_C(V, B \otimes W)$, and $g = G_V(f)$. For each $v \in V$, by assumption, there is a finite-dimensional subobject $F \subseteq B$ with $f(v) = \sum_{i \in I} b^i \otimes w^i \in F \otimes W$. Now for any $a \in F^\perp$, by definition of $G_V$ and Lemma 3.2.1, we have

$$g(a \otimes v) = \Phi(a_{-1}, b^i_{-1}, w^i_{-1})\omega(a_0, b^i_0) \otimes w^i_0 = 0.$$

This shows $G_v(f) \in \mathrm{Hom}_{C,\text{rat}}(A \otimes V, W)$. If $g = 0$, since $\omega$ is non-degenerate, we deduce that $f = 0$. Therefore $G_V$ is injective.

**Step2:**

Assume that $B$ is finite-dimensional. Then $B$ has a left dual $B^* = \mathrm{Hom}(B, \Bbbk)$. Since $\omega$ is non-degenerate, there is an isomorphism $A \cong B^*$ in $C$ via the following isomorphism:

$$\mathrm{Hom}_C(A \otimes B, \Bbbk) \cong \mathrm{Hom}_C(A, B^*).$$

Therefore $A$ is a left dual of $B$, which is finite-dimensional. We have such an isomorphism

$$\mathrm{Hom}_C(V, B\otimes W) \cong \mathrm{Hom}_C(A\otimes V, W).$$

Since $\mathrm{Hom}_{C,\mathrm{rat}}(A\otimes V, W) \subseteq \mathrm{Hom}_C(A\otimes V, W)$, and $G_V$ is injective, we deduce that $G_V$ is bijective when $B$ is finite-dimensional.

**Step3:**

We now extend the bijectivity to the general case when $B$ may be infinite-dimensional. Let $V' \subseteq V$ be a finite-dimensional $H$-comodule, we denote $\langle V'\rangle$ as the Yetter-Drinfeld submodule of $V$ generated by $V'$. Assume that $\langle V'\rangle = V$; we are going to prove that $G_V$ is surjective in this case. Let $g \in \mathrm{Hom}_{C,\mathrm{rat}}(A\otimes V, W)$. Since $V'$ is finite-dimensional, the rationality of $g$ implies the existence of a finite-dimensional subobject $F \subseteq B$ in $C$ such that $g(F^\perp \otimes V') = 0$. By Lemma 3.2.1, $F^\perp$ is an object in $C$. Since $g$ is a morphism in $C$, by similar method in proof of Lemma 3.2.1, for all $h \in H$, we have

$$g(F^\perp \otimes (h \triangleright V')) = 0.$$

By construction of $\langle V'\rangle$, if we denote $V'$ as $X_0$, this shows $g(F^\perp \otimes Y_0) = 0$.

Recall that $X_1 = \sum_{y\in Y_0} \mathrm{span}\{y_0 \mid \delta_V(y) = y_{-1}\otimes y_0\}$, We are going to show $g(F^\perp \otimes X_1) = 0$. Since $g$ is a morphism in $C$, we have

$$(\mathrm{id}_H \otimes g)\circ \delta(F^\perp \otimes Y_0) = \delta(g(F^\perp \otimes Y_0)) = 0.$$

By definition of $X_1$,the set $\mathrm{span}\{z_0 \mid \delta(z) = z_{-1}\otimes z_0,\ z \in F^\perp \otimes Y_0\} = F^\perp \otimes X_1$. This implies $g(F^\perp \otimes X_1) = 0$. By iterating this argument, we can prove that

$$g(F^\perp \otimes V) = 0.$$

Note that we have a non-degenerate pairing

$$A/F^\perp \otimes F \longrightarrow \Bbbk,\ \overline{a}\otimes b \mapsto \omega(a,b).$$

Since $F$ is finite-dimensional, by the situation considered before, we have a bijective map

$$\mathrm{Hom}_C(V, F\otimes W) \cong \mathrm{Hom}_{C,\mathrm{rat}}(A/F^\perp \otimes V, W).$$

Let $\overline{g} \in \mathrm{Hom}_{C,\mathrm{rat}}(A/F^\perp \otimes V, W)$ be the morphism induced by $g$ and $f \in \mathrm{Hom}(V, F\otimes W)$ be the preimage of $\overline{g}$. Then the preimage of $g$ is

$$i\circ f : V \longrightarrow B\otimes W,$$

where $i : F\otimes W \longrightarrow B\otimes W$ is an inclusion. Thus $G_V : \mathrm{Hom}_C(V, B\otimes W) \cong \mathrm{Hom}_{C,\mathrm{rat}}(A\otimes V, W)$ is bijective in this case.

**Step4:**

It remains to prove the bijectivity for an arbitrary object $V \in C$. Let

$$\mathcal{V} = \{\langle V'\rangle \mid V' \subseteq V \ \text{ finite-dimensional } \ H-\text{subcomodule}\}.$$

By the finiteness theorem for comodules, we have $V = \bigcup_{V' \text{ finite-dimensional}} V'$ as comodules. Note that for all $U_1, U_2 \in \mathcal{V}$, there is an object $U \in \mathcal{V}$ with $U_1 \cup U_2 \subseteq U$, since $U$ is closed under direct sums. Thus

$$V = \bigcup_{U \in \mathcal{V}} U$$

as an object in $\mathcal{C}$. Given $g \in \mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A \otimes V, W)$, let $g_U$ be the restriction of $g$ to $A \otimes U$, then there is a morphism $f_U : U \to B \otimes W$ with $G_V(f_U) = g_U$. For $U_1, U_2 \in \mathcal{V}$, let $U_1 \subseteq U_2$, we have $f_{U_2}|_{U_1} = f_{U_1}$, since $G_V$ is injective. Hence the maps $f_U$ define a linear map $f : V \to B \otimes W$ by $f(v) = f_U(v)$, where $U$ is an element in $\mathcal{V}$ containing $v$. This construction yields $G_V(f) = g$. Thus $G_V$ is bijective for arbitrary $V$. □

## 3.3 Bijection between comodules and rational modules

We begin by introducing the notion of Hopf pairings. For our purpose, we assume that $A$ and $B$ are locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C} = {}^H_H\mathcal{YD}$ in this section. Since this does not affect our study of Nichols algebras, we assume that $A$ and $B$ have bijective antipodes in $\mathcal{C}$.

**Definition 3.3.1.** *A morphism $\omega : A \otimes B \to \Bbbk$ in $\mathcal{C}$ is called a Hopf pairing, if the following identities hold:*

$$\omega \circ (\mu_A \otimes \mathrm{id}_B) = \omega \circ (\mathrm{id}_A \otimes \omega \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes a^{-1}_{A,B,B}) \circ a_{A,A,B\otimes B} \circ (\mathrm{id}_{A\otimes A} \otimes \Delta_B), \tag{3.3.1}$$

$$\omega \circ (\eta_A \otimes \mathrm{id}_B) = \varepsilon_B, \tag{3.3.2}$$

$$\omega \circ (\mathrm{id}_A \otimes \mu_B) = \omega \circ (\mathrm{id}_A \otimes \omega \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes a^{-1}_{A,B,B}) \circ a_{A,A,B\otimes B} \circ (\Delta_A \otimes \mathrm{id}_{B\otimes B}), \tag{3.3.3}$$

$$\omega \circ (\mathrm{id}_A \otimes \eta_B) = \varepsilon_A. \tag{3.3.4}$$

*Moreover, we call a Hopf pairing $\omega : A \otimes B \to \Bbbk$ a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C}$, if $\omega$ is non-degenerate and*

$$\omega(A(n), B(m)) = 0, \text{ if } n \neq m.$$

There are some useful properties of Hopf pairings, we list here for further use.

**Remark 3.3.2.** [52, Proposition 3.3.8] Under above assumptions, we have the following properties.

(1) Suppose $\omega : A \otimes B \to \mathbf{1}$ is a dual pair in $\mathcal{C}$, then

$$\omega \circ (\mathrm{id} \otimes \mathcal{S}_B) = \omega \circ (\mathcal{S}_A \otimes \mathrm{id}). \tag{3.3.5}$$

(2) If $\omega : A \otimes B \to \mathbf{1}$ is a dual pair, then

$$\omega^+ := \omega \circ c_{B,A} \circ (\mathcal{S}_B \otimes \mathcal{S}_A) : B \otimes A \to \Bbbk, \tag{3.3.6}$$

$$\omega^{+\mathrm{cop}} := \omega^+ \circ (\mathrm{id}_B \otimes \mathcal{S}_A^{-1}) : B^{\mathrm{cop}} \otimes A^{\mathrm{cop}} \to \Bbbk. \tag{3.3.7}$$

are dual pairs too.

The following lemma holds in general braided monoidal categories; here we cite this lemma for simplicity.

**Lemma 3.3.3.** [52, Proposition 3.3.9] *Let $\mathcal{C}$ be an arbitrary braided monoidal category. Suppose $A$, $B$ are Hopf algebras in $\mathcal{C}$. Assume that $p : A\otimes B \to \mathbf{1}$ is a Hopf pairing, then we have two strict monoidal functors:*

$$\begin{aligned} &D_1 : {}^{B}\mathcal{C} \longrightarrow {}_{A^{\mathrm{cop}}}\overline{\mathcal{C}}, \quad (V,\delta) \mapsto (V,\overline{\lambda}), \\ &\text{with } \overline{\lambda} = (A\otimes V \xrightarrow{\mathrm{id}_A\otimes\delta} A\otimes(B\otimes V) \xrightarrow{a^{-1}_{A,B,V}} (A\otimes B)\otimes V \xrightarrow{p\otimes \mathrm{id}_V} V) \end{aligned} \tag{3.3.8}$$

*and*

$$\begin{aligned} &D_2 : {}^{A^{\mathrm{cop}}}\overline{\mathcal{C}} \longrightarrow {}_{B}\mathcal{C}, \quad (V,\overline{\delta}) \mapsto (V,\lambda), \\ &\text{with } \lambda = (B\otimes V \xrightarrow{\mathrm{id}_B\otimes\overline{\delta}} B\otimes(A\otimes V) \xrightarrow{a^{-1}_{B,A,V}} (B\otimes A)\otimes V \xrightarrow{p^{+\mathrm{cop}}\otimes \mathrm{id}_V} V). \end{aligned} \tag{3.3.9}$$

The main result of this subsection is to give a bijection between $B$-comodules in $\mathcal{C}$ and rational $A^{\mathrm{cop}}$-modules in $\overline{\mathcal{C}}$.

**Proposition 3.3.4.** *Let $(A,B,\omega)$ be a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\mathcal{C} = {}^{H}_{H}\mathcal{YD}$. The functor*

$$\begin{aligned} &D_1 : {}^{B}\mathcal{C} \longrightarrow {}_{A^{\mathrm{cop}}}\overline{\mathcal{C}}_{\mathrm{rat}}, \quad (V,\delta) \mapsto (V,\lambda), \\ &\text{with } \lambda = (A\otimes V \xrightarrow{\mathrm{id}_A\otimes\delta} A\otimes(B\otimes V) \xrightarrow{a^{-1}_{A,B,V}} (A\otimes B)\otimes V \xrightarrow{\omega\otimes \mathrm{id}_V} V) \end{aligned}$$

*is a strict monoidal isomorphism of categories.*

*Proof.* Let $V$ be an object in ${}^{H}_{H}\mathcal{YD}$. By Proposition 3.2.3, we have a bijection:

$$\begin{aligned} &G_V : \mathrm{Hom}_{\mathcal{C}}(V, B\otimes V) \to \mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A\otimes V, V), \\ &f \mapsto \lambda = (A\otimes V \xrightarrow{\mathrm{id}_A\otimes f} A\otimes(B\otimes V) \xrightarrow{a^{-1}_{A,B,V}} (A\otimes B)\otimes V \xrightarrow{\omega\otimes \mathrm{id}} V). \end{aligned}$$

Note that $\mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A\otimes V,V)$ is the set of all $\lambda$ in $\mathrm{Hom}_{\mathcal{C}}(A\otimes V,V)$ such that for all $v\in V$, there is a finite-dimensional object $F\subseteq B$, such that $\lambda(F^{\perp}\otimes v)=0$. Since $B$ is locally finite, there is a natural number $n_0$ such that $F\subseteq \bigoplus_{i=0}^{n_0} B(i)$. Therefore $\lambda(A(n)\otimes v)=0$ for all $n>n_0$. Thus if $(V,\lambda)\in {}_{A}\mathcal{C}$, where $\lambda\in \mathrm{Hom}_{\mathcal{C},\mathrm{rat}}(A\otimes V,V)$, then $(V,\lambda)\in {}_{A}\mathcal{C}_{\mathrm{rat}}$ automatically.

Now we are going to prove $(V,\lambda)\in {}_{A}\mathcal{C}_{\mathrm{rat}}$ if and only if $(V,\delta)\in {}^{B}\mathcal{C}$. That the condition $(V,\delta)\in {}^{B}\mathcal{C}$ implies $(V,\lambda)\in {}_{A}\mathcal{C}_{\mathrm{rat}}$ follows from Lemma 3.3.3 and the above statement. On the other hand, $(V,\lambda)\in {}_{A}\mathcal{C}$ is equivalent to

$$\lambda\circ(\mathrm{id}_A\otimes\lambda) = \lambda\circ(\mu_A\otimes \mathrm{id}_V)\circ a^{-1}_{A,A,V}.$$

According to the definition of $\lambda$,

$$\lambda\circ(\mathrm{id}_A\otimes\lambda) = (\omega\otimes\mathrm{id}_V)\circ a^{-1}_{A,B,V}\circ(\mathrm{id}_A\otimes\delta)\circ(\mathrm{id}_A\otimes(\omega\otimes\mathrm{id}_V))\circ(\mathrm{id}_A\otimes a^{-1}_{A,B,V})\circ(\mathrm{id}_A\otimes(\mathrm{id}_A\otimes\delta)).$$

Since $\delta$, $\omega$ are morphisms in $C$, and the associative isomorphism is natural, we have

$$\begin{aligned}\lambda\circ(\mathrm{id}_A\otimes\lambda) &= (\omega\otimes\mathrm{id}_V)\circ a^{-1}_{A,B,V}\circ(\mathrm{id}_A\otimes(\omega\otimes\mathrm{id}_{B\otimes V}))\circ(\mathrm{id}_A\otimes a^{-1}_{A,B,B\otimes V})\circ(\mathrm{id}_A\otimes(\mathrm{id}_A\otimes(\mathrm{id}_B\otimes\delta)\circ\delta))\\ &= (\omega\otimes\mathrm{id}_V)\circ((\mathrm{id}_A\otimes(\omega\otimes\mathrm{id}_B))\otimes\mathrm{id}_V)\circ(a^{-1}_{A,A\otimes B,B}\otimes\mathrm{id}_V)\circ(\mathrm{id}_A\otimes a^{-1}_{A\otimes B,B,V})\\ &\circ(\mathrm{id}_A\circ a^{-1}_{A,B,B\otimes V})\circ(\mathrm{id}_A\otimes(\mathrm{id}_A\otimes(\mathrm{id}_B\otimes\delta)\circ\delta)).\end{aligned} \tag{3.3.10}$$

Now we write down explicit formula of $\lambda\circ(\mu_A\otimes\mathrm{id}_V)\circ a^{-1}_{A,A,V}$,

$$\lambda\circ(\mu_A\otimes\mathrm{id}_V)\circ a^{-1}_{A,A,V} = (\omega\otimes\mathrm{id}_V)\circ a^{-1}_{A,B,V}\circ(\mathrm{id}\otimes\delta)\circ(\mu_A\otimes\mathrm{id})\circ a^{-1}_{A,A,V}.$$

Since $\mu_A$ is a morphism in $C$, we have

$$\lambda\circ(\mu_A\otimes\mathrm{id}_V)\circ a^{-1}_{A,A,V} = (\omega\otimes\mathrm{id}_V)\circ((\mu_A\otimes\mathrm{id}_B)\otimes\mathrm{id}_V)\circ a^{-1}_{A\otimes A,B,V}\otimes(\mathrm{id}_{A\otimes A}\otimes\delta)\circ a^{-1}_{A,A,V}.$$

By equation (3.3.1), and by naturality of associative isomorphism,

$$\begin{aligned}&\lambda\circ(\mu_A\otimes\mathrm{id}_V)\circ a^{-1}_{A,A,V}\\ &= (\omega\otimes\mathrm{id}_V)\circ((\mathrm{id}_A\otimes(\omega\otimes\mathrm{id}_B))\otimes\mathrm{id}_V)\circ((\mathrm{id}_A\otimes a^{-1}_{A,B,B})\circ a_{A,A,B\otimes B}\circ(\mathrm{id}_{A\otimes A}\otimes\Delta_B)\otimes\mathrm{id}_V)\\ &\circ a^{-1}_{A\otimes A,B,V}\circ(\mathrm{id}_{A\otimes A}\otimes\delta)\circ a^{-1}_{A,A,V}\\ &= (\omega\otimes\mathrm{id}_V)\circ((\mathrm{id}_A\otimes(\omega\otimes\mathrm{id}_B))\otimes\mathrm{id}_V)\circ((\mathrm{id}_A\otimes a^{-1}_{A,B,B})\circ a_{A,A,B\otimes B}\otimes\mathrm{id}_V)\\ &\circ a^{-1}_{A\otimes A,B\otimes B,V}\circ a^{-1}_{A,A,(B\otimes B)\otimes V}\circ(\mathrm{id}_A\otimes(\mathrm{id}_A\otimes a^{-1}_{B,B,V}))\circ(\mathrm{id}_A\otimes(\mathrm{id}_A\otimes(a_{B,B,V}\circ((\Delta_B\otimes\mathrm{id}_V)\circ\delta)))).\end{aligned} \tag{3.3.11}$$

For all $a,a'\in A$, $v\in V$, we have $\lambda\circ(\mathrm{id}_A\otimes\lambda)(a\otimes(a'\otimes v)) = \lambda\circ(\mu_A\otimes\mathrm{id}_V)\circ a^{-1}_{A,A,V}(a\otimes(a'\otimes v))$. By non-degeneracy of $\omega$ and Mac-Lane's coherence theorem, this is equivalent to

$$(\mathrm{id}\otimes\delta)\circ\delta(v) = a_{B,B,V}\circ(\Delta_B\otimes\mathrm{id}_V)\circ\delta(v).$$

Furthermore, we have $v = (\varepsilon_B\otimes\mathrm{id}_V)\circ\delta(v)$ by equation (3.3.2). Therefore we deduce that $(V,\lambda)\in {}_A C_{\mathrm{rat}}$ if and only if $(V,\delta)\in {}^B C$.

Now let $(V,\delta),(V',\delta')\in {}^B C$, and $(V,\lambda)$, $(V',\lambda')$ be the corresponding modules in ${}_A C_{\mathrm{rat}}$, where $\lambda = G_V(\delta)$, $\lambda' = G'_V(\delta')$. It is direct that a map $f\in\mathrm{Hom}_C(V,V')$ is a $B$-comodule map if and only if $f$ is a $A$-module map in $C$.

Note that $(V,\lambda)\in {}_A C_{\mathrm{rat}}$ if and only if $(V,\lambda)\in {}_{A^{\mathrm{cop}}}\overline{C}_{\mathrm{rat}}$. Since the category ${}_{A^{\mathrm{cop}}}\overline{C}_{\mathrm{rat}}$ is a monoidal subcategory of ${}_{A^{\mathrm{cop}}}\overline{C}$ by Lemma 3.1.3 and $D_1$ is a strict monoidal functor by Lemma 3.3.3, we deduce that $D_1$ is a strict monoidal equivalence. Since $G_V$ is bijective, it is easy to see that the inverse functor is given by

$$D'_1 : {}_{A^{\mathrm{cop}}}\overline{C}_{\mathrm{rat}} \to {}^B C,\quad (V,\lambda')\mapsto (V,\delta'),$$

where $\delta'$ is determined by $G^{-1}_V(\lambda')$. Thus $D_1$ is a strict monoidal isomorphism. □

## 3.4 Monoidal equivalence between Yetter-Drinfeld module categories related by a dual pair

Let $(A,B,\omega)$ be a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras with bijective antipodes in $\mathcal{C}={}^H_H\mathcal{YD}$. The main result in this subsection is to show that there is a braided monoidal equivalence $(\Omega,\beta):{}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}\to{}^A_A\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}$.

By Remark 3.3.2, we have a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in $\overline{\mathcal{C}}$:

$$\begin{aligned}\omega^{+\mathrm{cop}}&:B^{\mathrm{cop}}\otimes A^{\mathrm{cop}}\longrightarrow\Bbbk,\\ \omega^{+\mathrm{cop}}&=\omega\circ c_{B,A}\circ(\mathcal{S}_B\otimes\mathcal{S}_A)\circ(\mathrm{id}_B\otimes\mathcal{S}_A^{-1})\\ &=\omega\circ c_{B,A}\circ(\mathcal{S}_B\otimes\mathrm{id}_A)\\ &=\omega\circ c_{B,A}\circ(\mathrm{id}_B\otimes\mathcal{S}_A).\end{aligned}$$

Recall that by Proposition 3.3.4, we now have two monoidal isomorphisms:

$$D_1:{}^B\mathcal{C}\to{}_{A^{\mathrm{cop}}}\overline{\mathcal{C}}_{\mathrm{rat}},\ D_2:{}^{A^{\mathrm{cop}}}\overline{\mathcal{C}}\to{}_B\mathcal{C}_{\mathrm{rat}},\tag{3.4.1}$$

$$D_1(V,\delta)=(V,\overline{\lambda}),\ \overline{\lambda}=(A\otimes V\xrightarrow{\mathrm{id}_A\otimes\delta}A\otimes(B\otimes V)\xrightarrow{a^{-1}_{A,B,V}}(A\otimes B)\otimes V\xrightarrow{\omega\otimes\mathrm{id}_V}V),\tag{3.4.2}$$

$$D_2(V,\overline{\delta})=(V,\lambda),\ \lambda=(B\otimes V\xrightarrow{\mathrm{id}_B\otimes\overline{\delta}}B\otimes(A\otimes V)\xrightarrow{a^{-1}_{B,A,V}}(A\otimes B)\otimes V\xrightarrow{\omega^{+\mathrm{cop}}\otimes\mathrm{id}_V}V).\tag{3.4.3}$$

Our first objective is to prove the following braided monoidal equivalence:

**Theorem 3.4.1.** *The functor*

$$\Gamma:\overline{{}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}}\longrightarrow{}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}},\ (V,\lambda,\delta)\mapsto(V,\overline{\lambda},\overline{\delta}),\tag{3.4.4}$$

*where $\overline{\lambda},\overline{\delta}$ are defined by (3.4.2), (3.4.3), and the morphism $f$ are mapped onto $f$, is an equivalence of braided monoidal categories.*

The following lemma, which holds in the general setting of a braided monoidal category $\mathcal{C}$ , is used to establish this theorem. Without loss of generality, we assume $\mathcal{C}$ is strict. Let $A$, $B$ be Hopf algebras in $\mathcal{C}$, and $p:A\otimes B\to\mathbf{1}$ is a Hopf pairing, now we have two braided monoidal categories $\overline{{}^B_B\mathcal{YD}(\mathcal{C})_{\mathrm{rat}}}$ and ${}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{\mathcal{C}})_{\mathrm{rat}}$, with braiding $\overline{c}^{\mathcal{YD}(\mathcal{C})}$ and $c^{\mathcal{YD}(\overline{\mathcal{C}})}$ respectively.

**Lemma 3.4.2.** *With above assumptions, let $(X,\overline{\delta}_X)\in{}^{A^{\mathrm{cop}}}\overline{\mathcal{C}}$, $(V,\delta)\in{}^B\mathcal{C}$, and define*

$$(X,\lambda_X)=D_2(X,\overline{\delta}_X),\ (V,\overline{\lambda})=D_1(V,\delta).$$

*where $D_1,D_2$ is given by Lemma 3.3.3, then we have*

$$\overline{c}^{\mathcal{YD}(\mathcal{C})}_{(X,\lambda_X),(V,\delta)}=c^{\mathcal{YD}(\overline{\mathcal{C}})}_{(X,\overline{\delta}_X),(V,\overline{\lambda})}.\tag{3.4.5}$$

*Proof.* By definition of $c^{\mathcal{YD}}$, properties of Hopf pairing and naturality of braiding, we have

$$\begin{aligned}
\overline{c}^{\mathcal{YD}(C)}_{(X,\lambda_X),(V,\delta)} &= \overline{c}_{X,V} \circ ((p^{+\mathrm{cop}} \otimes \mathrm{id}_X) \circ (\mathrm{id}_B \otimes \overline{\delta}_X) \otimes \mathrm{id}_V) \circ (\mathcal{S}_B^{-1} \otimes \mathrm{id}_X \otimes \mathrm{id}_V) \circ (\overline{c}_{X,B} \otimes \mathrm{id}_V) \circ (\mathrm{id}_X \otimes \delta) \\
&= \overline{c}_{X,V} \circ (p \circ c_{B,A} \otimes \mathrm{id}_X \otimes \mathrm{id}_V) \circ (\mathrm{id}_B \otimes \overline{\delta}_X \otimes \mathrm{id}_V) \circ (\overline{c}_{X,B} \otimes \mathrm{id}_V) \circ (\mathrm{id}_X \otimes \delta) \\
&= (p \otimes \overline{c}_{X,V}) \circ (c_{B,A} \otimes \mathrm{id}_X) \circ (\overline{c}_{A\otimes X,B} \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes \mathrm{id}_X \otimes \delta) \circ (\overline{\delta}_X \otimes \mathrm{id}_V) \\
&= (p \otimes \overline{c}_{X,V}) \circ (\mathrm{id}_A \otimes \overline{c}_{X,B} \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes \mathrm{id}_X \otimes \delta) \circ (\overline{\delta}_X \otimes \mathrm{id}_V) \\
&= (p \otimes \overline{c}_{X,B\otimes V}) \circ (\mathrm{id}_A \otimes \mathrm{id}_X \otimes \delta) \circ (\overline{\delta}_X \otimes \mathrm{id}_V) \\
&= ((p \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes \delta) \otimes \mathrm{id}_X) \circ (\mathrm{id}_A \otimes \overline{c}_{X,V}) \circ (\overline{\delta}_X \otimes \mathrm{id}_V) = c^{\mathcal{YD}(\overline{C})}_{(X,\overline{\delta}_X),(V,\overline{\lambda})}.
\end{aligned}$$

Here the second equality uses (3.3.5) and (3.3.7), the third and the sixth equalities use the naturality of the braiding, the fourth and the fifth equalities use the braiding axiom. □

Now we return to the setting of coquasi-Hopf algebras. We want to restrict the Yetter-Drinfeld module criterion to rational modules. Let $B$ be a locally finite $\mathbb{N}_0$-graded Hopf algebra in $C = {}^H_H\mathcal{YD}$ with bijective antipode. Let $(V,\lambda) \in {}_B C_{\mathrm{rat}}$, and $(V,\delta) \in {}^B C$.

**Lemma 3.4.3.** *Let $X \in {}_B C$, and assume that there is an index set $I$, a family $X_i, i \in I$, of objects in ${}_B C$, and morphisms $\pi_i : X \longrightarrow X_i$ in ${}_B C$ for all $i \in I$ with $\bigcap_{i\in I} \ker(\pi_i) = 0$. Then if $c^{\mathcal{YD}}_{V,X_i}$ is a morphism in ${}_B C$ for all $i \in I$, then $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_B C$.*

*Proof.* The proof of this lemma is parallel to [52, Lemma 12.2.5]. We omit here for simplicity. □

**Lemma 3.4.4.** *The following are equivalent.*

(1) *For all $(X,\lambda_X) \in {}_B C_{\mathrm{rat}}$, $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_B C_{\mathrm{rat}}$.*
(2) *The object $(V,\lambda,\delta)$ belongs to ${}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$.*

*Proof.* Suppose for all $(X,\lambda_X) \in {}_B C_{\mathrm{rat}}$, the map $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_B C_{\mathrm{rat}}$. Then by Remark 2.4.2, we only need to prove that

$$c^{\mathcal{YD}}_{V,B} : V \otimes B \longrightarrow B \otimes V$$

is a morphism in ${}_B C$. Now for all $n \geq 0$, let $X_n = B/\bigoplus_{i\geq n} B(i)$. The quotient map $\pi_n : B \longrightarrow X_n$ is a morphism in ${}_B C$. We have

$$\bigcap_{n\geq 0} \ker(\pi_n) = 0, \quad \mu_B(B(m) \otimes X_n) = 0$$

for all $m \geq n$. Since $X_n$ is a rational $B$-module in $C$, $c^{\mathcal{YD}}_{V,X_n}$ is a morphism in ${}_B C$. By Lemma 3.4.3, $c^{\mathcal{YD}}_{V,B}$ is a morphism in ${}_B C$. Hence $(V,\lambda,\delta) \in {}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$.

Suppose $(V,\lambda,\delta) \in {}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$, we deduce that $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_B C$ for any $(X,\lambda_X) \in {}_B C_{\mathrm{rat}}$ by Remark 2.4.2. Furthermore, ${}_B C_{\mathrm{rat}}$ is a monoidal full subcategory of ${}_B C$, and $V \otimes X, X \otimes V \in {}_B C_{\mathrm{rat}}$. Thus $c^{\mathcal{YD}}_{V,X}$ is a morphism in ${}_B C_{\mathrm{rat}}$. □

**Proposition 3.4.5.** *The following are equivalent.*

(1) *The object $(V,\lambda,\delta)$ belongs to ${}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$.*
(2) *The object $(V,\overline{\lambda},\overline{\delta})$ belongs to ${}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{C})_{\mathrm{rat}}$.*

(3) *For all* $(X,\lambda_X) \in {}_B\mathcal{C}_{\rm rat}$, *the following map is a morphism in* ${}_B\mathcal{C}_{\rm rat}$

$$\overline{c}^{\mathcal{YD}(\mathcal{C})}_{(X,\lambda_X),(V,\delta)} : (X,\lambda_X) \otimes (V,\lambda) \to (V,\lambda) \otimes (X,\lambda_X).$$

(4) *For all* $(X,\overline{\delta}_X) \in {}^{A^{\rm cop}}\overline{\mathcal{C}}$, *the following map is a morphism in* ${}^{A^{\rm cop}}\overline{\mathcal{C}}$.

$$c^{\mathcal{YD}(\overline{\mathcal{C}})}_{(X,\overline{\delta}_X),(V,\overline{\lambda})} : (X,\overline{\delta}_X) \otimes (V,\overline{\delta}) \to (V,\overline{\delta}) \otimes (X,\overline{\delta}_X).$$

*Proof.* (1) $\Leftrightarrow$ (3): The condition (1), by Lemma 3.3.2, is equivalent to the condition that for all $(X,\lambda_X) \in {}_B\mathcal{C}_{\rm rat}$, $c^{\mathcal{YD}}_{(V,\delta),(X,\lambda_X)}$ is a morphism in ${}_B\mathcal{C}_{\rm rat}$. Since we have assumed $B$ has a bijective antipode, $c^{\mathcal{YD}}_{(V,\delta),(X,\lambda_X)}$ is an isomorphism with inverse $\overline{c}^{\mathcal{YD}}_{(X,\lambda_X),(V,\delta)}$. Hence (1) is equivalent to (3).

(2) $\Leftrightarrow$ (4): This follows from Remark 2.4.2.

(3) $\Leftrightarrow$ (4): By Lemma 3.4.2,

$$\overline{c}^{\mathcal{YD}(\mathcal{C})}_{(X,\lambda_X),(V,\delta)} = c^{\mathcal{YD}(\overline{\mathcal{C}})}_{(X,\overline{\delta}_X),(V,\overline{\lambda})}$$

as a morphism in $\mathcal{C}$. Then the equivalence of (3) and (4) follows from $D_2 : {}^{A^{\rm cop}}\overline{\mathcal{C}} \to {}_B\mathcal{C}_{\rm rat}$ is a monoidal isomorphism. □

*Proof of Theorem 3.4.1.* Let $V \in \mathcal{C}$, and

$$S_1 = \{(\lambda,\delta) \mid (V,\lambda) \in {}_B\mathcal{C}_{\rm rat}, (V,\delta) \in {}^B\mathcal{C}\},$$
$$S_2 = \{(\overline{\lambda},\overline{\delta}) \mid (V,\overline{\lambda}) \in {}_{A^{\rm cop}}\overline{\mathcal{C}}_{\rm rat}, (V,\overline{\delta}) \in {}^{A^{\rm cop}}\overline{\mathcal{C}}\}.$$

We define a map $\phi : S_1 \to S_2$, by $(\lambda,\delta) \mapsto (\overline{\lambda},\overline{\delta})$, where $\overline{\lambda},\overline{\delta}$ are given by

$$D_1(V,\delta) = (V,\overline{\lambda}), \quad D_2(V,\overline{\delta}) = (V,\lambda).$$

Thus $\phi$ is bijective since $D_1$, $D_2$ are monoidal isomorphisms.

Now that given $(V,\lambda,\delta) \in {}^B_B\mathcal{YD}(\mathcal{C})_{\rm rat}$, the object $(V,\overline{\lambda},\overline{\delta}) \in {}^{A^{\rm cop}}_{A^{\rm cop}}\mathcal{YD}(\overline{\mathcal{C}})_{\rm rat}$ by Proposition 3.4.5. Since $\phi$ is bijective, we deduce that $\Gamma$ is a monoidal equivalence. It is strict since $D_1$ and $D_2$ is strict. To show $\Gamma$ is braided, let $X = (X,\lambda_X,\delta_X) \in {}^B_B\mathcal{YD}(\mathcal{C})_{\rm rat}$, and $\Gamma(X) = (X,\overline{\lambda}_X,\overline{\delta}_X) \in {}^{A^{\rm cop}}_{A^{\rm cop}}\mathcal{YD}(\overline{\mathcal{C}})_{\rm rat}$. We know that $\overline{c}^{\mathcal{YD}(\mathcal{C})}_{(X,\lambda_X),(V,\delta)}$ is the braiding in $\overline{{}^B_B\mathcal{YD}(\mathcal{C})_{\rm rat}}$, and $c^{\mathcal{YD}(\overline{\mathcal{C}})}_{(X,\overline{\delta}_X),(V,\overline{\lambda})}$ is the braiding of $\Gamma(X),\Gamma(V)$ in ${}^{A^{\rm cop}}_{A^{\rm cop}}\mathcal{YD}(\overline{\mathcal{C}})_{\rm rat}$. By Lemma 3.4.2, the functor $\Gamma$ is braided. Hence $\Gamma$ is a strict braided monoidal equivalence.

□

Our next goal is to build a monoidal equivalence between ${}^{A^{\rm cop}}_{A^{\rm cop}}\mathcal{YD}(\overline{\mathcal{C}})_{\rm rat}$ and $\overline{{}^A_A\mathcal{YD}(\mathcal{C})_{\rm rat}}$. It is standard that

$${}_A\mathcal{C} \to \mathcal{C}_A,\ (V,\lambda) \mapsto (V,\lambda_+),\ \text{where } \lambda_+ = \lambda \circ c_{V,A} \circ ({\rm id}_V \otimes \mathcal{S}_A),$$
$$\mathcal{C}_A \to {}_A\mathcal{C},\ (V,\lambda) \mapsto (V,\lambda_-),\ \text{where } \lambda_- = \lambda \circ \overline{c}_{A,V} \circ (\mathcal{S}_A^{-1} \otimes {\rm id}_V),$$

are inverse isomorphisms of monoidal categories. Furthermore, suppose $(V,\lambda) \in {}_A\mathcal{C}_{\rm rat}$, for $v \in V$, $c_{V,A}(v \otimes a)$ is a finite sum of elements in $A \otimes V$. By $\mathcal{S}_A$ is a $\mathbb{N}_0$-graded map, we have $(V,\lambda_+) \in \mathcal{C}_{A,{\rm rat}}$.

Similarly, suppose $(V,\lambda)\in C_{A,\mathrm{rat}}$, we can deduce that $(V,\lambda_-)\in {}_A C_{\mathrm{rat}}$.

Now we restrict the monoidal isomorphisms in [52, Theorem 3.4.15] to the rational Yetter-Drinfeld modules, we have a braided monoidal equivalence:

$$(F_{rl},\alpha): \mathcal{YD}(C)^A_{A,\mathrm{rat}} \to {}^A_A\mathcal{YD}(C)_{\mathrm{rat}},\ (V,\lambda,\delta)\mapsto (V,\lambda_-,(\mathcal{S}_A\otimes \mathrm{id}_V)\circ c_{V,A}\circ\delta), \tag{3.4.6}$$

$$\alpha_{X,Y} = c^{\mathcal{YD}(C)^A_A}_{Y,X}\circ \overline{c}_{X,Y}, \tag{3.4.7}$$

where morphisms $f$ are mapped onto $f$. From [52, Theorem 3.4.16], we have a braided strict monoidal equivalence:

$$F_{lr}: {}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{C})_{\mathrm{rat}} \to \overline{\mathcal{YD}(C)^A_{A,\mathrm{rat}}},\ (V,\lambda,\delta)\mapsto (V,\lambda_+,c_{A,V}\circ\delta), \tag{3.4.8}$$

where morphisms $f$ are mapped onto $f$.

Our main result in this section establishes the braided monoidal equivalence between the categories of rational modules over dual pairs. The construction of the functor $\Omega$ relies on the interplay between the equivalences established in the previous subsections, as illustrated in the following diagram:

$$\begin{array}{ccc}
\overline{{}^B_B\mathcal{YD}(C)_{\mathrm{rat}}} & \xrightarrow[\cong]{\Gamma} & {}^{A^{\mathrm{cop}}}_{A^{\mathrm{cop}}}\mathcal{YD}(\overline{C})_{\mathrm{rat}} \\
{\scriptstyle\Omega}\,\Big\downarrow & & \Big\downarrow\,{\scriptstyle F_{lr}} \\
\overline{{}^A_A\mathcal{YD}(C)_{\mathrm{rat}}} & \xleftarrow[\overline{F_{rl}}]{} & \overline{\mathcal{YD}(C)^A_{A,\mathrm{rat}}}
\end{array}$$

**Theorem 3.4.6.** *The following functor is a braided monoidal equivalence:*

$$(\Omega,\beta): {}^B_B\mathcal{YD}(C)_{\mathrm{rat}}\to {}^A_A\mathcal{YD}(C)_{\mathrm{rat}},\ \textit{where } (V,\lambda_B,\delta_B)\mapsto (V,\lambda_A,\delta_A), \textit{with}$$

$$\lambda_A = (A\otimes V\xrightarrow{\mathrm{id}\otimes\delta_B} A\otimes(B\otimes V)\xrightarrow{a^{-1}_{A,B,V}} (A\otimes B)\otimes V\xrightarrow{\omega\otimes\mathrm{id}} V),$$

$$\delta_A = (V\xrightarrow{\delta'} A\otimes V\xrightarrow{\mathcal{S}^2_A\otimes\mathrm{id}_V} A\otimes V\xrightarrow{c^2_{A,V}} A\otimes V),\quad \textit{where } \delta' \textit{ is defined by}$$

$$\lambda_B = (B\otimes V\xrightarrow{\mathrm{id}_V\otimes\delta'} B\otimes(A\otimes V)\xrightarrow{a^{-1}_{B,A,V}} (B\otimes A)\otimes V\xrightarrow{\omega^+\otimes\mathrm{id}_V} V),$$

*where morphisms $f$ are mapped onto $f$. The monoidal structure is given by*

$$(\Omega,\beta): {}^B_B\mathcal{YD}(C)_{\mathrm{rat}}\to {}^A_A\mathcal{YD}(C)_{\mathrm{rat}},\ \beta_{X,Y} = c^{{}^B_B\mathcal{YD}(C)}_{Y,X}\circ\overline{c}_{X,Y}.$$

*Proof.* As depicted in the diagram above, we define the functor $\Omega$ as the composition:

$$\Omega := F_{rl}\circ F_{lr}\circ\Gamma.$$

We now compute the explicit structure of this composition. Let $(V,\lambda_B,\delta_B)\in {}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$. Then

$$\begin{aligned}
F_{rl}F_{lr}\Gamma(V,\lambda_B,\delta_B) =& F_{rl}F_{lr}(V,\overline{\lambda_A},\overline{\delta_A})\\
&= F_{rl}(V,\overline{\lambda_A}_+,c_{A,V}\circ\overline{\delta_A})\\
&= (V,(\overline{\lambda_A}_+)_-,(\mathcal{S}_A\otimes\mathrm{id}_V)\circ c_{V,A}\circ c_{A,V}\circ\overline{\delta_A}).
\end{aligned}$$

Note that $(\overline{\lambda_{A^+}})_- = \overline{\lambda_A}$, while

$$\overline{\lambda_A} = (A\otimes V \xrightarrow{\mathrm{id}_A\otimes\delta_B} A\otimes(B\otimes V) \xrightarrow{a^{-1}_{A,B,V}} (A\otimes B)\otimes V \xrightarrow{\omega\otimes\mathrm{id}_V} V).$$

This coincides with our definition of $\lambda_A$.

Now we need to show $(\mathcal{S}_A\otimes\mathrm{id}_V)\circ c_{V,A}\circ c_{A,V}\circ\overline{\delta_A}$ equals $\delta_A$. Since $\mathcal{S}_A$ and $c_{A,V}$ are isomorphisms, we only need to prove:

$$\delta' = (\mathcal{S}_A^{-2}\otimes\mathrm{id}_V)\circ\overline{c}^2_{A,V}\circ(\mathcal{S}_A\otimes\mathrm{id}_V)\circ c^2_{A,V}\circ\overline{\delta_A} = (\mathcal{S}_A^{-1}\otimes\mathrm{id}_V)\circ\overline{\delta_A}.$$

It suffices to show

$$(\omega^+\otimes\mathrm{id}_V)\circ a^{-1}_{B,A,V}\circ(\mathrm{id}_B\otimes(\mathcal{S}_A^{-1}\otimes\mathrm{id}_V)\circ\overline{\delta_A}) = \lambda_B.$$

Since $a$ is a natural isomorphism, and $\omega^{+\mathrm{cop}} = \omega^+\circ(\mathrm{id}_B\otimes\mathcal{S}_A^{-1})$, we have

$$\begin{aligned}(\omega^+\otimes\mathrm{id}_V)\circ a^{-1}_{B,A,V}\circ(\mathrm{id}_B\otimes(\mathcal{S}_A^{-1}\otimes\mathrm{id}_V)\circ\overline{\delta_A}) &= (\omega^+\otimes\mathrm{id}_V)\circ((\mathrm{id}_B\otimes\mathcal{S}_A^{-1})\otimes\mathrm{id}_V)\circ a^{-1}_{B,A,V}\circ(\mathrm{id}_B\otimes\overline{\delta_A})\\ &= (\omega^{+\mathrm{cop}}\otimes\mathrm{id}_V)\circ a^{-1}_{B,A,V}\circ(\mathrm{id}_B\otimes\overline{\delta_A})\\ &= \lambda_B.\end{aligned}$$

Where the last equality follows from the relation between $\overline{\delta_A}$ and $\lambda_B$ via the functor $D_2$. Thus $\Omega$ induces a monoidal equivalence $\Omega : \overline{{}^B_B\mathcal{YD}(C)_{\mathrm{rat}}} \to \overline{{}^A_A\mathcal{YD}(C)_{\mathrm{rat}}}$.

For the monoidal structure of $\Omega$, note that only the functor $(F_{rl},\alpha)$ is non-strict. Since $F_{lr}$ and $\Gamma$ are braided, we have

$$\overline{\beta}_{X,Y} = \overline{c}^{\overline{\mathcal{YD}(C)^A_A}}_{F_{lr}\Gamma(Y),F_{lr}\Gamma(X)}\circ\overline{c}_{F_{lr}\Gamma(X),F_{lr}\Gamma(Y)} = \overline{c}^{\overline{{}^B_B\mathcal{YD}(C)}}_{Y,X}\circ\overline{c}_{X,Y}.$$

Thus $(\Omega,\overline{\beta}) : \overline{{}^B_B\mathcal{YD}(C)_{\mathrm{rat}}} \to \overline{{}^A_A\mathcal{YD}(C)_{\mathrm{rat}}}$ is a braided monoidal equivalence. Now we take the reverse braiding, and it is direct that

$$(\Omega,\beta) : {}^B_B\mathcal{YD}(C)_{\mathrm{rat}} \to {}^A_A\mathcal{YD}(C)_{\mathrm{rat}},\ \beta_{X,Y} = c^{{}^B_B\mathcal{YD}(C)}_{Y,X}\circ\overline{c}_{X,Y}$$

is a braided monoidal equivalence.

□

## 3.5 Applications to Nichols algebras over coquasi-Hopf algebras

We now specialize to the setting of Nichols algebras.

We are going to investigate the Hopf pairing between $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$ when $V$ has a left dual $V^*$. There is a unique Hopf pairing $\omega : T(V^*)\otimes T(V)\to\mathbf{1}$. It is shown in [74] that $\omega$ is given by

$$\omega\mid_{V^*\otimes V} = \mathrm{ev}_V.$$

and

$$\omega_n := \omega|_{V^{*\otimes n}\otimes V^{\otimes n}} = \mathrm{ev}_{V^{\otimes n}}\circ(\mathrm{id}_{V^{*\otimes n}}\otimes[n]!_c) = \mathrm{ev}_{V^{\otimes n}}\circ([n]!_{c^*}\otimes\mathrm{id}_{V^{\otimes n}}).$$

The Hopf pairing between $V^{\otimes n}$ and $V^{*\otimes m}$ is 0 unless $n=m$. Here $c^*$ denotes the braided symmetriser of $V^*$.

The Hopf pairing between $T(V)$ and $T(V^*)$ induces a Hopf pairing between $\mathcal{B}(V^*)$ and $\mathcal{B}(V)$. Note that the Hopf pairing $\omega$ vanishes on the kernels of the symmetrisers, and for all positive integer $n$, we have

$$\omega_n(\ker[n]!_c^*,\ V^{\otimes n}) = 0,\ \omega_n(V^{*\otimes n},\ \ker[n]!_c) = 0.$$

Now we return to the case of coquasi-Hopf algebra $H$ with bijective antipode. Let $V$ be a finite-dimensional object in ${}^H_H\mathcal{YD}$. Consequently, the Hopf pairing on the tensor algebras descends to a well-defined Hopf pairing

$$\omega : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to \Bbbk.$$

In particular,

$$\omega(V^{*\otimes m}, V^{\otimes n}) = 0,\ \text{for all } m \neq n,$$

and both $\mathcal{B}(V)$ and $\mathcal{B}(V^*)$ are locally finite $\mathbb{N}_0$-graded Hopf algebras in ${}^H_H\mathcal{YD}$.

We consider $T(V)^\perp$, by expression of $\omega$ it equals $\ker \mathrm{Wor}(c^*) = \bigoplus_{n\geq 1} i_n \ker([n]!_c^*)$. Similarly, $T(V^*)^\perp = \ker \mathrm{Wor}(c)$. Since $\mathcal{B}(V) = T(V)/\ker \mathrm{Wor}(c)$ and $\mathcal{B}(V^*) = T(V^*)/\ker \mathrm{Wor}(c^*)$, we deduce that $\omega : \mathcal{B}(V^*) \otimes \mathcal{B}(V^*) \to \Bbbk$ is a dual pair of locally finite $\mathbb{N}_0$-graded Hopf algebras in ${}^H_H\mathcal{YD}$.

**Corollary 3.5.1.** *For any finite-dimensional $V \in {}^H_H\mathcal{YD}$, there exists a braided monoidal equivalence:*

$$(\Omega_V, \beta_V) : {}^{\mathcal{B}(V)}_{\mathcal{B}(V)}\mathcal{YD}(C)_{\mathrm{rat}} \longrightarrow {}^{\mathcal{B}(V^*)}_{\mathcal{B}(V^*)}\mathcal{YD}(C)_{\mathrm{rat}}. \tag{3.5.1}$$

*Proof.* This follows immediately from the existence of a dual pair $\omega : \mathcal{B}(V^*) \otimes \mathcal{B}(V) \to \Bbbk$ and Theorem 3.4.6 □

Let $A$, $B$ be $\mathbb{N}_0$-graded locally finite Hopf algebra in $C$ with bijective antipodes and $\omega : A \otimes B \to \Bbbk$ a dual pair. Recall the braided monoidal equivalence in Theorem 3.4.6:

$$\Omega : {}^B_B\mathcal{YD}(C)_{\mathrm{rat}} \longrightarrow {}^A_A\mathcal{YD}(C)_{\mathrm{rat}}.$$

**Corollary 3.5.2.** *The Nichols algebra $\mathcal{B}(V)$ of any $V \in {}^B_B\mathcal{YD}(C)$ is rational if $V$ is.*

*Proof.* If $V$ is rational, then for any $n \in \mathbb{N}_{\geq 1}$, $V^{\otimes n}$ is rational in ${}^B_B\mathcal{YD}(C)$. Since ${}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$ is closed under direct sum and quotient by Lemma 3.1.3. Both objects $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ and $\mathcal{B}(V)$ are rational. □

Now for $Q \in {}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$, $\mathcal{B}(Q)$ is an $\mathbb{N}_0$-graded Hopf algebra in ${}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$ as well. The following result is straightforward.

**Corollary 3.5.3.** *Let $Q \in {}^B_B\mathcal{YD}(C)_{\mathrm{rat}}$, then*

$$\Omega(\mathcal{B}(Q)) \cong \mathcal{B}(\Omega(Q)) \tag{3.5.2}$$

*as $\mathbb{N}_0$-graded Hopf algebras in ${}^A_A\mathcal{YD}(C)_{\mathrm{rat}}$.*

*Proof.* It follows from Lemma 2.5.4 immediately. □

## 3.6 The functor $\Omega$ under $\mathbb{Z}$-gradings

Having established the duality for Nichols algebras, we now investigate how the functor $\Omega$ interacts with gradings. Let $\mathbb{N}_0$-Gr$\mathcal{M}_\Bbbk$(resp.$\mathbb{Z}$-Gr$\mathcal{M}_\Bbbk$) be the category of $\mathbb{N}_0$-graded vector spaces(resp.$\mathbb{Z}$-graded vector spaces). The functor $\mathbb{N}_0$-Gr$\mathcal{M}_k \to \mathbb{Z}$-Gr$\mathcal{M}_k$, which extends the $\mathbb{N}_0$-grading of an object $V$ in $\mathbb{N}_0$-Gr$\mathcal{M}_k$ to a $\mathbb{Z}$-grading by setting $V(n) = 0$ for all $n < 0$, is monoidal. This functor allows us to view $\mathbb{N}_0$-graded coalgebras and coquasi-Hopf algebras as $\mathbb{Z}$-graded coalgebras and $\mathbb{Z}$-graded coquasi-Hopf algebras, respectively.

Furthermore, suppose $H$ is a $\mathbb{N}_0$-graded coquasi-Hopf algebra. A $\mathbb{Z}$-graded Yetter-Drinfeld module $V$ over $H$ is by definition an object $V$ in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-Gr}\mathcal{M}_k)$. In other words, $V$ is a $\mathbb{Z}$-graded vector space such that the map $H \otimes V \to V$ and comodule structure maps $V \to H \otimes V$ are $\mathbb{Z}$-graded.

**Lemma 3.6.1.** *Suppose $H$ is a $\mathbb{Z}$-graded coquasi-Hopf algebra.*

*(1) Let $K$ be a Nichols algebra in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-Gr}\,\mathcal{M}_\Bbbk)$, and $K(1) = \oplus_{\gamma\in\mathbb{Z}} K(1)_\gamma$ a $\mathbb{Z}$-graded object in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-Gr}\,\mathcal{M}_\Bbbk)$. Then there is a unique $\mathbb{Z}$-grading on $K$ extending the grading on $K(1)$. Moreover, $K(n)$ is $\mathbb{Z}$-graded in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-Gr}\,\mathcal{M}_\Bbbk)$ for all $n \geq 0$.*

*(2) Let $K$ be a $\mathbb{Z}$-graded braided Hopf algebra in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-Gr}\mathcal{M}_k)$. Then the bosonization $K\#H$ is a $\mathbb{Z}$-graded Hopf algebra with $\deg K(\gamma)\#H(\lambda) = \gamma + \lambda$ for all $\gamma, \lambda \in \mathbb{Z}$.*

*(3) Let $H_0 \subseteq H$ be a coquasi-Hopf subalgebra of degree $0$, and $\pi : H \to H_0$ a coquasi-Hopf algebra map with $\pi|_{H_0} = \mathrm{id}$. Define $R = H^{\mathrm{co}\,H_0}$. Then $R$ is a $\mathbb{Z}$-graded braided Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}(\mathbb{Z}\text{-Gr}\mathcal{M}_k)$ with $R(\gamma) = R \cap H(\gamma)$ for all $\gamma \in \mathbb{Z}$.*

*Proof.* (1) The module and comodule maps of $K(1)$ are $\mathbb{Z}$-graded and hence the infinitesimal braiding $c \in \mathrm{Aut}(K(1) \otimes K(1))$, being determined by the module and comodule maps, is $\mathbb{Z}$-graded. Moreover, the structure map of $K(n)$ for $n \in \mathbb{N}$ are determined by $c$ and $K(1)$, since $K(1)$ generates $K$. Therefore $K$ is a $\mathbb{Z}$-graded objects in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-Gr}\,\mathcal{M}_\Bbbk)$.

(2) and (3) are the same as the $\mathbb{N}_0$-graded case. □

From now on, $H$ represents a coquasi-Hopf algebra with bijective antipode, let $A, B \in {}^H_H\mathcal{YD}$ be locally finite $\mathbb{N}_0$-graded Hopf algebras with dual pairing $\omega$. Moreover,

$$\omega(A(m), B(n)) = 0, \quad \text{for all } m \neq n.$$

To extend this construction to the $\mathbb{Z}$-graded setting, we extend the $\mathbb{N}_0$-grading to $\mathbb{Z}$-grading by

$$A(n) = 0, \; B(n) = 0 \;\; \text{for all } n < 0.$$

We endow $A$ with a new $\mathbb{Z}$-grading as follows: $\deg(A(n)) = -n$ for all $n \in \mathbb{Z}$, which ensures that $\omega : A \otimes B \to \Bbbk$ is $\mathbb{Z}$-graded. Here, the grading of $\Bbbk$ is given by $\Bbbk(n) = 0$, if $n \neq 0$, and $\Bbbk(0) = \Bbbk$.

Now we return to the purpose of this subsection. We first introduce the following notation for further use.

**Definition 3.6.2.** *Let $H$ be a coquasi-Hopf algebra with bijective antipode, and $C = {}^H_H\mathcal{YD}$. Suppose $R$ is an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Let $(X, \lambda_X)$ be a $R$-module in $C$ and $(Y, \delta_Y)$ be a $R$-*

*comodule in $\mathcal{C}$. For $n \geq 0$, we define:*

$$\mathcal{F}_n X = \{x \in X \mid \lambda_X(R(n) \otimes x) = 0 \text{ for all } i > n\},$$
$$\mathcal{F}^n Y = \{y \in Y \mid \delta_Y(y) \in \bigoplus_{i=0}^{n} R(i) \otimes Y\}.$$

With above notation, the functor $\Omega$ relates the operator $\mathcal{F}_n$ to $\mathcal{F}^n$.

**Lemma 3.6.3.** *Let $N$ be a finite-dimensional object in ${}^H_H\mathcal{YD}$. Recall we have the following equivalence of monoidal categories:*

$$\Omega : {}^{\mathcal{B}(N)}_{\mathcal{B}(N)}\mathcal{YD}(\mathcal{C})_{\text{rat}} \cong {}^{\mathcal{B}(N^*)}_{\mathcal{B}(N^*)}\mathcal{YD}(\mathcal{C})_{\text{rat}}.$$

*Let $V \in {}^{\mathcal{B}(N)}_{\mathcal{B}(N)}\mathcal{YD}(\mathcal{C})_{\text{rat}}$, then we have*

(1) *$\mathcal{F}_n\Omega(V) = \mathcal{F}^n V$ for all $n \geq 0$.*
(2) *$\mathcal{F}^n\Omega(V) = \mathcal{F}_n(V)$ for all $n \geq 0$.*
(3) *Suppose $V$ is a $\mathbb{Z}$-graded object in ${}^{\mathcal{B}(N)}_{\mathcal{B}(N)}\mathcal{YD}(\mathcal{C})$, then $\Omega(V)$ is a $\mathbb{Z}$-graded object in ${}^{\mathcal{B}(N^*)}_{\mathcal{B}(N^*)}\mathcal{YD}(\mathcal{C})$, where the grading of $\Omega(V)$ is given by $\Omega(V)(n) = V(-n)$ for all $n \in \mathbb{Z}$.*

*Proof.* Recall that the braided monoidal equivalence of abelian category

$$\Omega : {}^{\mathcal{B}(N)}_{\mathcal{B}(N)}\mathcal{YD}(\mathcal{C})_{\text{rat}} \longrightarrow {}^{\mathcal{B}(N^*)}_{\mathcal{B}(N^*)}\mathcal{YD}(\mathcal{C})_{\text{rat}},$$

sends

$$(V, \lambda_{\mathcal{B}(N)}, \delta_{\mathcal{B}(N)}) \mapsto (V, \lambda_{\mathcal{B}(N^*)}, \delta_{\mathcal{B}(N^*)}).$$

which is induced by the dual pair $\omega : \mathcal{B}(N^*) \otimes \mathcal{B}(N) \to \mathbb{k}$.

(1) Recall

$$\lambda_{\mathcal{B}(N^*)} = (\mathcal{B}(N^*) \otimes V \xrightarrow{\text{id}\otimes\delta_{\mathcal{B}(N)}} \mathcal{B}(N^*) \otimes (\mathcal{B}(N) \otimes V) \xrightarrow{a^{-1}_{\mathcal{B}(N^*),\mathcal{B}(N),V}} (\mathcal{B}(N^*) \otimes \mathcal{B}(N)) \otimes V \xrightarrow{\omega\otimes\text{id}_V} V),$$

Now for fixed $n \geq 0$, the kernel of the induced map:

$$\mathcal{B}(N^*) \otimes V \longrightarrow \text{Hom}(\mathcal{B}(N)(n), V), \quad a \otimes v \mapsto (b \mapsto \Phi(a_{-1}, b_{-1}, v_{-1})^{-1}\langle a_0, b_0\rangle v_0) \tag{3.6.1}$$

is $\bigoplus_{m\neq n} \mathcal{B}(N^*)(m) \otimes V$. This means, if $v \in \mathcal{F}^n V$, then for each $i > n$, $\lambda_{\mathcal{B}(N^*)}(\mathcal{B}(N^*)(i) \otimes v) = 0$ by (3.6.1). On the other hand, if $v \in \mathcal{F}_n(V, \lambda_{\mathcal{B}(N^*)})$, this will imply $\delta(v) \in \bigoplus_{i=0}^{n} \mathcal{B}(N)(i) \otimes V$. This completes the proof of (1): $\mathcal{F}_n\Omega(V) = \mathcal{F}_n(V, \lambda_{\mathcal{B}(N^*)}) = \mathcal{F}^n V$.

(2) For the second statement, recall $\delta_{\mathcal{B}(N^*)}$ is given by

$$\delta_{\mathcal{B}(N^*)} = (V \xrightarrow{\delta'} \mathcal{B}(N^*) \otimes V \xrightarrow{\mathcal{S}_{\mathcal{B}}(N^*)^2\otimes\text{id}_V} \mathcal{B}(N^*) \otimes V \xrightarrow{c^2_{\mathcal{B}(N^*),V}} \mathcal{B}(N^*) \otimes V), \quad \text{where } \delta' \text{ is defined by}$$

$$\lambda_{\mathcal{B}(N)} = (\mathcal{B}(N) \otimes V \xrightarrow{\text{id}_V\otimes\delta'} \mathcal{B}(N) \otimes (\mathcal{B}(N^*) \otimes V) \xrightarrow{a^{-1}_{\mathcal{B}(N),\mathcal{B}(N^*),V}} (\mathcal{B}(N) \otimes \mathcal{B}(N^*)) \otimes V \xrightarrow{\omega^+\otimes\text{id}_V} V).$$

By [52, Corollary 3.3.6, Lemma 12.2.11], $(V,\delta')$ is $\mathcal{B}(N^*)$-comodule in ${}^H_H\mathcal{YD}$, and we have:

$$\mathcal{F}^n(V,\delta') = \mathcal{F}^n\Omega(V).$$

A similar argument to that in (1) shows that $\mathcal{F}_n(V) = \mathcal{F}^n(V,\delta')$. Combining these together shows $\mathcal{F}_n(V) = \mathcal{F}^n(V,\delta') = \mathcal{F}^n\Omega(V)$.

(3) We regard $\mathcal{B}(N)$ and $\mathcal{B}(N^*)$ as $\mathbb{Z}$-graded Hopf algebras in ${}^H_H\mathcal{YD}$. Let $V = \bigoplus_{n\in\mathbb{Z}} V(n)$ be its gradation, and set $\Omega(V)(n) = V(-n)$ for all $n \in \mathbb{Z}$. Then for $m \geq 0$, by definition of $\rho_{\mathcal{B}(N^*)}$, we have

$$\lambda_{\mathcal{B}(N^*)}(\mathcal{B}(N^*)(m)\otimes\Omega(V)(n)) = \lambda_{\mathcal{B}(N^*)}(\mathcal{B}(N^*)(m)\otimes V(-n)) \subseteq V(-m-n) = \Omega(V(m+n)).$$

On the other hand, by definition of $\delta_{\mathcal{B}(N^*)}$

$$\delta_{\mathcal{B}(N^*)}(\Omega(V)(n)) = \delta_{\mathcal{B}(N^*)}(V(-n)) \subseteq \bigoplus_{i\in\mathbb{Z}} \mathcal{B}(N^*)(i)\otimes V(i-n) = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}(N^*)(i)\otimes\Omega(V(n-i)).$$

The above two equations imply that $\Omega(V)$ is a well-defined $\mathbb{Z}$-graded object in ${}^{\mathcal{B}(N^*)}_{\mathcal{B}(N^*)}\mathcal{YD}(\mathcal{C})$. $\square$

# Chapter 4

# Projections of Nichols algebras

Let $H$ be a coquasi-Hopf algebra with bijective antipode. In this section, we study the projection of Nichols algebras in ${}^H_H\mathcal{YD}$, which will play an important role when considering reflection of simple Yetter-Drinfeld modules.

## 4.1 The structure of the space of coinvariants $K$

We first consider the projection of Nichols algebras for further study of reflection of Nichols algebras.

For any $M, N \in {}^H_H\mathcal{YD}$, there is a canonical surjection of braided Hopf algebra in ${}^H_H\mathcal{YD}$

$$\pi_{\mathcal{B}(N)} : \mathcal{B}(M \oplus N) \to \mathcal{B}(N), \quad \pi_{\mathcal{B}(N)}\,|_N = \mathrm{id}, \quad \pi_{\mathcal{B}(N)}\,|_M = 0.$$

This surjection naturally induces a canonical projection of coquasi-Hopf algebras:

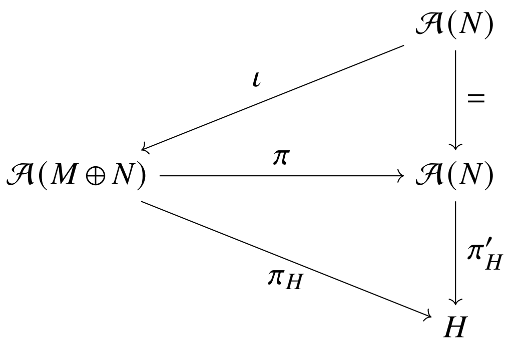
$$\pi = \pi_{\mathcal{B}(N)} \# \mathrm{id}_H : \mathcal{B}(M \oplus N)\#H \to \mathcal{B}(N)\#H.$$

To simplify our notation, we introduce the following conventions:

$$\mathcal{A}(M \oplus N) := \mathcal{B}(M \oplus N)\#H, \ \mathcal{A}(N) := \mathcal{B}(N)\#H.$$

This projection allows us to construct the associated space of coinvariants. There is a natural injection $\iota : \mathcal{A}(N) \to \mathcal{A}(M \oplus N)$ such that $\pi \circ \iota = \mathrm{id}_{\mathcal{A}(N)}$.

Let $\pi_H : \mathcal{A}(M \oplus N) \to H$, and $\pi'_H : \mathcal{A}(N) \to H$ be the natural projection. These maps satisfy a compatibility condition, namely, there exists a commutative diagram relating these projections:



Let

$$K = \mathcal{A}(M \oplus N)^{\mathrm{co}\,\mathcal{A}(N)}$$

be the space of right $\mathcal{A}(N)$-coinvariant elements with respect to the projection $\pi$. By virtue of Lemma 2.3.3, $K$ inherits the structure of a braided Hopf algebra in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ with $\mathcal{A}(N)$-coaction:

$$\delta : K \to \mathcal{A}(N) \otimes K, \quad x_{-1} \otimes x_0 =: \delta(x) := \pi(x_1) \otimes x_2,$$

and a linear map:

$$\mathrm{ad} : \mathcal{A}(N) \otimes K \to K, \quad a \otimes x \mapsto \mathrm{ad}(a)(x) = \Phi(a_1 x_{-1}, \mathbb{S}(a_3)_1, a_4)(a_2 x_0)\mathbb{S}(a_3)_2. \tag{4.1.1}$$

Here, $\mathbb{S}$ denotes the preantipode of $\mathcal{A}(N)$. We have the following two observations.

**Lemma 4.1.1.** *The space $K$ coincides with $\mathcal{B}(M \oplus N)^{co\mathcal{B}(N)}$, which is precisely the space of right $\mathcal{B}(N)$-coinvariant elements with respect to $\pi_{\mathcal{B}(N)}$.*

*Proof.* Let $X \in K$, then $X_1 \otimes \pi_H(X_2) = X_1 \otimes \pi'_H \circ \pi(X_2) = X \otimes 1$. Hence $X \in \mathcal{A}(M \oplus N)^{\mathrm{co}\,H} = \mathcal{B}(M \oplus N)$. Now, we assume $X$ is homogeneous. Under this assumption, we have

$$X \otimes 1 = (\mathrm{id} \otimes \pi)\Delta_{\mathcal{A}(M\oplus N)}(X) = X^1 \# x^2 \otimes \pi(X^2) = X^1 \# x^2 \otimes \pi_{\mathcal{B}(N)}(X^2).$$

On the other hand, consider the projection $\gamma = \mathrm{id}\#\varepsilon : \mathcal{A}(M \oplus N) \to \mathcal{B}(M \oplus N)$. Applying this projection yields

$$X \otimes 1 = (\gamma \otimes \mathrm{id})(X \otimes 1) = X^1 \otimes \pi_{\mathcal{B}(N)}(X^2) = (\mathrm{id} \otimes \pi_{\mathcal{B}(N)})(\Delta_{\mathcal{B}(M\oplus N)}(X)).$$

This demonstrates that $X \in \mathcal{B}(M \oplus N)^{\mathrm{co}\,\mathcal{B}(N)}$, completing one direction of the proof.

Conversely, let $X \in \mathcal{B}(M \oplus N)$ be a homogeneous element with

$$(\mathrm{id} \otimes \pi_{\mathcal{B}(N)})\Delta_{\mathcal{B}(M\oplus N)}(X) = X^1 \otimes \pi_{\mathcal{B}(N)}(X^2) = X \otimes 1.$$

Since $\pi_{\mathcal{B}(N)}$ is $\Bbbk G$-colinear, we can deduce that

$$X \otimes 1 \otimes 1 = X^1 \otimes x^2 \otimes \pi_{\mathcal{B}(N)}(X^2).$$

which implies $X \otimes 1 = X^1 \# x^2 \otimes \pi_{\mathcal{B}(N)}(X^2)$. Furthermore, observe that

$$(\mathrm{id} \otimes \pi)\Delta_{\mathcal{A}(M\oplus N)}(X\#1) = X^1 \# x^2 \otimes \pi(X^2) = X^1 \# x^2 \otimes \pi_{\mathcal{B}(N)}(X^2) = X \otimes 1.$$

Therefore, we conclude that $X \in K$, establishing the reverse inclusion.

□

Having characterized the coinvariant space $K$, we now turn our attention to its primitive elements. Indeed, the space of primitive elements of $K$ remains an object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$.

**Lemma 4.1.2.** *The space of primitive elements $P(K) = \{x \in K \mid \Delta_K(x) = x \otimes 1 + 1 \otimes x\}$ is a subobject of $K$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$.*

*Proof.* Let $\Delta_K : K \to K \otimes K$ be its comultiplication map and $\widetilde{\Delta} : K \to K \otimes K$, $x \mapsto 1 \otimes x + x \otimes 1$. We consider the following map

$$\widehat{\Delta}: K \to K \otimes K, \quad x \mapsto \Delta_K(x) - \widetilde{\Delta}(x).$$

$\widehat{\Delta}$ is a comodule map since both $\Delta_K$ and $\widetilde{\Delta}$ are colinear. Furthermore, for all $A \in {}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ and $X \in K$,

$$\begin{aligned}\widehat{\Delta}(\mathrm{ad}(A)(X)) &= \Delta_K(\mathrm{ad}(A)(X)) - \mathrm{ad}(A)(X) \otimes 1 - 1 \otimes \mathrm{ad}(A)(X) \\ &= \mathrm{ad}(A)(\Delta_K(X)) - \mathrm{ad}(A)(1 \otimes X + X \otimes 1) \\ &= \mathrm{ad}(A)(\Delta_K(X) - \widetilde{\Delta}(X)).\end{aligned}$$

Therefore $\widehat{\Delta}$ is a morphism in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. We conclude that $P(K)$ is a subobject of $K$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, because $P(K)$ is precisely the kernel of the map $\widehat{\Delta}$. □

One may expect that $K$ is a Nichols algebra generated by $\mathrm{ad}(\mathcal{B}(N))(M)$. The following lemma establishes that $\mathrm{ad}(\mathcal{B}(N))(M)$ is a subobject of $K$.

**Lemma 4.1.3.** *Let $M, N \in {}^H_H\mathcal{YD}$, $K = (\mathcal{A}(M \oplus N))^{\mathrm{co}\,\mathcal{A}(N)}$. The standard $\mathbb{N}_0$-grading of $\mathcal{B}(M \oplus N)$ induces an $\mathbb{N}_0$-grading on*

$$L := \mathrm{ad}(\mathcal{B}(N))(M) = \bigoplus_{n \in \mathbb{N}} \mathrm{ad}(N)^n(M)$$

*with degree $\deg(\mathrm{ad}(N)^n(M)) = n+1$. Then $L$ is a $\mathbb{N}_0$-graded object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. Moreover, $L \subseteq K$ is a subobject of $K$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$.*

*Proof.* Let $a \in N$ and $x \in \mathcal{B}(M \oplus N)$ be homogeneous elements. By equation (2.3.2), we have

$$\mathrm{ad}(a)(x) = ax - \mathrm{ad}(a_{-1})(x)a_0.$$

The element $\mathrm{ad}(a)(x)$ is of degree $\deg(X)+1$ in the Nichols algebra $\mathcal{B}(M \oplus N)$. Moreover, we have $\mathrm{ad}(N)^n(M) \cap \mathrm{ad}(N)^m(M) = 0$ for $n \neq m$. This implies the direct sum decomposition of $L$.

Since $M \subseteq K$ and $K \in {}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, we conclude that $L = \mathrm{ad}(\mathcal{B}(N))(M) \subseteq K$. Note that $\mathrm{ad}(\mathcal{B}(N))(M) \subseteq \mathrm{ad}(\mathcal{A}(N))(M)$. Conversely, since $M \in {}^H_H\mathcal{YD}$, we have

$$\mathrm{ad}(\mathcal{A}(N))(M) \subseteq \mathrm{ad}(\mathcal{B}(N))(\mathrm{ad}(H)(M)) \subseteq \mathrm{ad}(\mathcal{B}(N))(M).$$

Hence $L = \mathrm{ad}(\mathcal{A}(N))(M)$.

It is clear that the map

$$\mathrm{ad}: \mathcal{A}(N) \otimes L \to L$$

is well-defined since ad satisfies the first axiom of Yetter-Drinfeld modules. Furthermore, the map $\mathrm{ad}: \mathcal{A}(N) \otimes L \to L$ is $\mathbb{N}_0$-graded. Next, we want to show $L$ is a $\mathcal{A}(N)$-comodule. The comodule structure is given by $\delta_K$. For $a \in N$ and $x \in L$. Since $a$ is primitive in $\mathcal{B}(M \oplus N)$, we have

$$\Delta_{\mathcal{A}(M \oplus N)}(a) = a_{-1} \otimes a_0 + a \otimes 1.$$

By Lemma 2.2.4(1), a direct computation shows:

$$\begin{aligned}\delta_K(\mathrm{ad}(a)(x)) &= p_R((\mathrm{ad}(a_3)(x_0))_{-1}, a_4) q_R(a_1 x_{-2}, \mathcal{S}(a_6))(a_2 x_{-1})\mathcal{S}(a_5) \otimes (\mathrm{ad}(a_3)(x_0))_0 \\ &= p_R((\mathrm{ad}(a_0)(x_0))_{-1}, 1) q_R(a_{-2} x_{-2}, 1) a_{-1} x_{-1} \otimes (\mathrm{ad}(a_0)(x_0))_{-1} \\ &+ p_R((\mathrm{ad}(a_{-2})(x_0))_{-1}, a_{-1}) q_R(a_{-4} x_{-2}, 1)(a_{-3} x_{-1})\mathcal{S}(a_0) \otimes (\mathrm{ad}(a_{-2})(x_0))_0 \\ &= a_{-1} x_{-1} \otimes \mathrm{ad}(a_0)(x_0) + p_R((\mathrm{ad}(a_{-2})(x_0))_{-1}, a_{-1})(a_{-3} x_{-1})\mathcal{S}(a_0) \otimes (\mathrm{ad}(a_{-2})(x_0))_0.\end{aligned}$$

Since $x \in L = \bigoplus_{n\geq 0} \mathrm{ad}(N)^n(M)$. We proceed by induction on $n$. For $n=0$, note that $\mathrm{ad}(a_0)(x_0) \in L$, hence the first term lies in $\mathcal{A}(N)\otimes L$. For the second term, note that $\mathrm{ad}(a_{-2})(x_0) \in M$, since $a_{-2} \in H$ has degree zero. Therefore $(\mathrm{ad}(a_{-2})(x_0))_0 \in M$. Consequently, $\delta_K(\mathrm{ad}(a)(x)) \in \mathcal{A}(N)\otimes L$.

Now we assume that for some fixed integer $n_0$, we have $\delta_K(\mathrm{ad}(a)(x)) \in \mathcal{A}(N)\otimes L$ for all $x \in \mathrm{ad}(N)^{n'}(M)$, where $n' \leq n_0$. Then for $a \in N$ and $x \in \mathrm{ad}(N)^{n_0+1}(M)$, since by definition $\mathrm{ad}(N)^{n_0+1}(M) = \mathrm{ad}(N)(\mathrm{ad}(N)^{n_0}(M))$, we have $\delta_K(x) \in \mathcal{A}(N)\otimes L$ by the inductive hypothesis. Thus $\mathrm{ad}(a_0)(x_0) \in \mathrm{ad}(N)(L) \subseteq L$. On the other hand, $\mathrm{ad}(a_{-2})(x_0) \in L$ as well. Since $\mathrm{ad}(a_{-2})(x_0) \in \bigoplus_{i=0}^{n_0+1} \mathrm{ad}(N)^i(M)$, we have $(\mathrm{ad}(a_{-2})(x_0))_0 \in L$ by inductive hypothesis. This completes the induction, and we show $\delta_K : L \longrightarrow \mathcal{A}(N)\otimes L$ is well-defined. The third axiom holds automatically since $L \subseteq K$. Thus $L$ is a $\mathbb{N}_0$-graded object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ and is a subobject of $K$.

□

The following lemma is used to prove the semisimplicity of the object $L$ in the case that $M$ is semisimple. It guarantees $L$ to be a $\mathcal{B}(N)$-comodule in ${}^H_H\mathcal{YD}$ via restricting the following composition to $L$,

$$\delta : \mathcal{B}(M\oplus N) \xrightarrow{\Delta_{\mathcal{B}(M\oplus N)}} \mathcal{B}(M\oplus N)\otimes\mathcal{B}(M\oplus N) \xrightarrow{\pi_{\mathcal{B}(N)}\otimes \mathrm{id}} \mathcal{B}(N)\otimes\mathcal{B}(M\oplus N).$$

**Lemma 4.1.4.** *For all $x \in L$, we have $\Delta_{\mathcal{B}(M\oplus N)}(x) - x\otimes 1 \in \mathcal{B}(N)\otimes L$.*

*Proof.* Recall $L = \bigoplus_{n\in\mathbb{N}_0} \mathrm{ad}(N)^n(M)$. We proceed by induction on $n$. Now let $a \in N$ and $x \in M$ be homogeneous elements. A direct computation yields:

$$\begin{aligned}&\Delta_{\mathcal{B}(M\oplus N)}(\mathrm{ad}(a)(x)) = \Delta_{\mathcal{B}(M\oplus N)}(ax - (\mathrm{ad}(a_{-1})(x)a_0)) \\ &= ax\otimes 1 + 1\otimes ax + a\otimes x + \mathrm{ad}(a_{-1})(x)\otimes a_0 - \mathrm{ad}(a_{-1})(x)a_0\otimes 1 - 1\otimes \mathrm{ad}(a_{-1})(x)a_0 - \mathrm{ad}(a_{-1})(x)\otimes a_0 \\ &- \mathrm{ad}((\mathrm{ad}(a_{-1})(x))_{-1})(a_0)\otimes(\mathrm{ad}(a_{-1})(x))_0 \\ &= \mathrm{ad}(a)(x)\otimes 1 + 1\otimes \mathrm{ad}(a)(x) + a\otimes x - \mathrm{ad}((\mathrm{ad}(a_{-1})(x))_{-1})(a_0)\otimes(\mathrm{ad}(a_{-1})(x))_0.\end{aligned}$$

Note that $x = \mathrm{ad}(1)(x) \in L$. Since $\mathrm{ad}(a_{-1})(x) \in L$, we have $(\mathrm{ad}(a_{-1})(x))_0 \in L$ by Lemma 4.1.3. Therefore $\Delta_{\mathcal{B}(M\oplus N)}(\mathrm{ad}(a)(x)) - \mathrm{ad}(a)(x)\otimes 1 \in \mathcal{B}(N)\otimes L$.

For a fixed $m \in \mathbb{N}_0$. We assume that for all $y \in \mathrm{ad}(N)^{m'}(M)$, $1 \leq m' \leq m$, we have

$$\Delta_{\mathcal{B}(M\oplus N)}(y) - y\otimes 1 \in \mathcal{B}(N)\otimes L.$$

Now let $a \in N$ and $x \in \mathrm{ad}(N)^m(M)$ be homogeneous, we have

$$\begin{aligned}\Delta_{\mathcal{B}(M\oplus N)}(\mathrm{ad}(a)(x)) &= \Delta_{\mathcal{B}(M\oplus N)}(ax-(\mathrm{ad}(a_{-1})(x))a_0)\\ &=(a\otimes 1+1\otimes a)(x^1\otimes x^2)\\ &-\frac{\Phi(a_{-5},x^1_{-1},x^2_{-2})\Phi((\mathrm{ad}(a_{-4})(x^1_0))_{-1},(\mathrm{ad}(a_{-2})(x^2_0))_{-1},a_{-1})}{\Phi((\mathrm{ad}(a_{-4})(x^1_0))_{-2},a_{-3},x^2_{-1})}\times\\ &((\mathrm{ad}(a_{-4})(x^1_0))_0\otimes((\mathrm{ad}(a_{-2})(x^2_0))_0)(1\otimes a_0+a_0\otimes 1).\end{aligned}$$

Since the term $x\otimes 1$ lies in $\Delta_{\mathcal{B}(M\oplus N)}(x)$, let us consider this expression separately. The term $\mathrm{ad}(a)(x)\otimes 1$ will occur in $\Delta_{\mathcal{B}(M\oplus N)}(\mathrm{ad}(a)(x))$. Now the remaining terms are

$$\begin{aligned}&\sum_{x^1\otimes x^2\neq x\otimes 1}\Phi(a_{-1},x^1_{-1},x^2_{-1})a_0x^1_0\otimes x^2_0+\frac{\Phi(a_{-3},x^1_{-1},x^2_{-2})}{\Phi((\mathrm{ad}(a_{-2})(x^1_0))_{-1},a_{-1},x^2_{-1})}(\mathrm{ad}(a_{-2})(x^1_0))_0\otimes a_0x^2_0\\ &-\frac{\Phi(a_{-6},x^1_{-1},x^2_{-2})\Phi((\mathrm{ad}(a_{-5})(x^1_0))_{-2},(\mathrm{ad}(a_{-3})(x^2_0))_{-2},a_{-2})}{\Phi((\mathrm{ad}(a_{-5})(x^1_0))_{-3},a_{-4},x^2_{-1})\Phi((\mathrm{ad}(a_{-5})(x^1_0))_{-1},(\mathrm{ad}(a_{-3})(x^2_0))_{-1},a_{-1})}\times\\ &(\mathrm{ad}(a_{-5})(x^1_0))_0\otimes(\mathrm{ad}(a_{-3})(x^2_0))_0a_0\\ &-\frac{\Phi(a_{-5},x^1_{-1},x^2_{-2})\Phi((\mathrm{ad}(a_{-4})(x^1_0))_{-2},(\mathrm{ad}(a_{-2})(x^2_0))_{-3},a_{-1})}{\Phi((\mathrm{ad}(a_{-4})(x^1_0))_{-3},a_{-3},x^2_{-1})}\times\\ &\Phi((\mathrm{ad}(a_{-4})(x^1_0))_{-1},(\mathrm{ad}((\mathrm{ad}(a_{-2})(x^2_0))_{-2})(a_0))_{-1},(\mathrm{ad}(a_{-2})(x^2_0))_{-1})\times\\ &(\mathrm{ad}(a_{-4})(x^1_0))_0(\mathrm{ad}((\mathrm{ad}(a_{-2})(x^2_0))_{-2})(a_0))_0\otimes(\mathrm{ad}(a_{-2})(x^2_0))_0.\end{aligned}$$

By assumptions,

$$\Delta_{\mathcal{B}(M\oplus N)}(x)-x\otimes 1\in\mathcal{B}(N)\otimes L.$$

Therefore the first term lies in $\mathcal{B}(N)\otimes L$. For the third term, note that $\Phi$ is convolution invertible, thus the third term becomes

$$\sum_{x^1\otimes x^2\neq x\otimes 1}\frac{\Phi(a_{-3},x^1_{-1},x^2_{-2})}{\Phi((\mathrm{ad}(a_{-2})(x^1_0))_{-1},a_{-1},x^2_{-1})}(\mathrm{ad}(a_{-2})(x^1_0))_0\otimes\mathrm{ad}(a_{-1})(x^2_0)a_0.$$

Combining the second and third terms together yields

$$\sum_{x^1\otimes x^2\neq x\otimes 1}\frac{\Phi(a_{-3},x^1_{-1},x^2_{-2})}{\Phi((\mathrm{ad}(a_{-2})(x^1_0))_{-1},a_{-1},x^2_{-1})}(\mathrm{ad}(a_{-2})(x^1_0))_0\otimes\mathrm{ad}(a_0)(x^2_0).$$

By assumption, $\sum_{x^1\otimes x^2\neq x\otimes 1}x^1\otimes x^2\in\mathcal{B}(N)\otimes L$, therefore this term lies in $\mathcal{B}(N)\otimes L$. The last term belongs to $\mathcal{B}(N)\otimes L$ by similar reason, which completes the induction and proves the lemma.

□

## 4.2 The semisimplicity of the object $L$

We now turn to the structural analysis of the object $L$, whose properties are useful to our subsequent arguments regarding Nichols algebras.

Recall that a graded coalgebra is a coalgebra $C$ provided with a grading $C = \bigoplus_{m\in\mathbb{N}_0} C^m$ such that $\Delta(C^m) \subseteq \bigoplus_{i+j=m} C^i \otimes C^j$. Let $\Delta_{i,j} : C^m \to C^i \otimes C^j$ denote the map $\mathrm{pr}_{i,j}\Delta$, where $m = i + j$ and $\mathrm{pr}_{i,j} : C \otimes C \to C^i \otimes C^j$. More generally, if $i_1, \ldots, i_n \in \mathbb{N}_0$ and $i_1 + \cdots + i_n = m$, then $\Delta_{i_1,\ldots,i_n}$ is defined by the commutative diagram

$$\begin{array}{ccc} C^m & \xrightarrow{\Delta^{n-1}} & \bigoplus_{j_1+j_2+\cdots+j_n=m} C^{j_1} \otimes C^{j_2} \otimes \cdots \otimes C^{j_n} \\ \Big\downarrow \Delta_{i_1,..,i_n} & \swarrow \mathrm{pr}_{i_1,..,i_n} & \\ C^{i_1} \otimes C^{i_2} \otimes \cdots \otimes C^{i_n} & & \end{array}$$

This framework naturally leads us to consider the maximal coideal in a graded coalgebra. For notational convenience, we write $\Delta_{1,1,\ldots,1}$ as $\Delta_{1^n}$. Let $I_C(n) = \ker(\Delta_{1^n})$ for all $n \geq 1$, and

$$I_C = \bigoplus_{n\geq 1} I_C(n) = \bigoplus_{n\geq 2} I_C(n).$$

Note that $I_C(1) = 0$ since $\Delta_1 = \mathrm{id}$.

**Lemma 4.2.1.** [52, Lemma 1.3.13, Proposition 1.3.14]
(1) *Assume that $C$ is connected, that is, $C^0$ is one dimensional. Then $I_C \subseteq C$ is a coideal of $C$.*
(2) *Assume that $C$ is connected. Then the following are equivalent.*
  $(a)$ *$C$ is strictly graded, that is, $P(C) = C^1$.*
  $(b)$ *For all $n \geq 2$, $\Delta_{1^n} : C^n \to (C^1)^{\otimes n}$ is injective.*
  $(c)$ *For all $n \geq 2$, $\Delta_{1,n-1} : C^n \to C^1 \otimes C^{n-1}$ is injective.*
  $(d)$ *$I_C = 0$.*

The following technical result provides the necessary machinery to analyze the semisimplicity properties of $L$.

**Lemma 4.2.2.** [52,Proposition 13.2.3] *Let $C$ be an $\mathbb{N}_0$-graded coalgebra, $X$ an $\mathbb{N}_0$-graded left $C$-comodule, and $Y$ a $C$-subcomodule of $X$. Let $k > 0$ be an integer. Assume that the map $\delta^X_{n-k,k} : X(n) \to C(n-k) \otimes X(k)$ is injective for all $n \geq k$, and $Y$ is not contained in $\bigoplus_{i=0}^{k-1} X(i)$. Then $Y \cap \bigoplus_{i=0}^{k} X(i) \neq 0$.*

**Lemma 4.2.3.** *If $L' \subseteq L$ is a non-zero $\mathcal{A}(N)$-subcomodule, then $L' \cap M \neq 0$.*

*Proof.* We begin by observing that $\mathcal{B}(M \oplus N)$ carries a natural left $\mathcal{B}(N)$-comodule structure in the category ${}^H_H\mathcal{YD}$, defined via the composition:

$$\delta : \mathcal{B}(M \oplus N) \xrightarrow{\Delta_{\mathcal{B}(M\oplus N)}} \mathcal{B}(M \oplus N) \otimes \mathcal{B}(M \oplus N) \xrightarrow{\pi_{\mathcal{B}(N)}\otimes \mathrm{id}} \mathcal{B}(N) \otimes \mathcal{B}(M \oplus N).$$

Now let $X \in L$, then $\delta(X) \in \mathcal{B}(N) \otimes L$ by Lemma 4.1.4, which implies that L inherits an $\mathbb{N}_0$-graded $\mathcal{B}(N)$-comodule structure via the restriction of $\delta : L \to \mathcal{B}(N) \otimes L$.

Suppose $X \in L(n)$ for $n \geq 1$, consider the component map $\delta_{n-1,1} : L(n) \to \mathcal{B}(N)(n-1) \otimes L(1) = \mathcal{B}(N)(n-1) \otimes M$. By Lemma 4.1.4, $\Delta_{\mathcal{B}(M\oplus N)}(X) = X \otimes 1 + \mathcal{B}(N) \otimes L$, it follows that

$$\delta_{n-1,1}(X) = (\Delta_{\mathcal{B}(M\oplus N)})_{n-1,1}(X).$$

As Nichols algebras are strictly graded connected coalgebras, the map $(\Delta_{\mathcal{B}(M\oplus N)})_{n-1,1}$ is injective, and hence so is $\delta_{n-1,1}$. We now apply Lemma 4.2.2 with $C = {}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, $X = L$, $Y = L'$ and $k = 1$. Since $L(0) = 0$ by its gradation, we conclude that $L' \cap L(1) = L' \cap M \neq 0$, which completes the proof. ◻

Now we are ready to show the semisimplicity of $L$.

**Lemma 4.2.4.** (1) *Assume $M = \bigoplus_{i\in I} M_i$ is a direct sum in ${}^H_H\mathcal{YD}$. Let $L_i = \operatorname{ad}\mathcal{B}(N)(M_i)$ for all $i \in I$. Then we have such a decomposition in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$*

$$L = \bigoplus_{i\in I} L_i.$$

(2) *If $M$ is irreducible in ${}^H_H\mathcal{YD}$, then $L = \operatorname{ad}(\mathcal{B}(N))(M)$ is irreducible in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$.*

*Proof.* (1) It suffices to show that the sum is direct. Suppose, for contradiction, that the sum is not a direct sum. Then there exists an index $k \in I$ such that $L_k \cap \sum_{i\neq k} L_i \neq 0$. By Lemma 4.2.3, this implies

$$(L_k \cap \sum_{i\neq k} L_i) \cap M_k \neq 0,$$

hence $\sum_{i\neq k} L_i \cap M_k \neq 0$. But since $M_k$ lies in degree one in $L_k$, we obtain

$$\sum_{i\neq k} M_i \cap M_k \neq 0,$$

contradicting the directness of the sum $M = \bigoplus_{i\in I} M_i$.

(2) Now let $0 \neq L' \subseteq L$ be a subobject in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. Then $L' \cap M \neq 0$ by Lemma 4.2.3. Furthermore, $L' \cap M$ is a subobject of $M$ in ${}^H_H\mathcal{YD}$. But $M$ is irreducible in ${}^H_H\mathcal{YD}$, then $L' \cap M = M$. Hence

$$L = \operatorname{ad}(\mathcal{A}(N))(M) = \operatorname{ad}(\mathcal{A}(N))(L' \cap M) \subseteq L',$$

which implies $L' = L$. We conclude that $L$ is irreducible. ◻

## 4.3 $K$ is a Nichols algebra

We now proceed to prove that $K$ possesses the structure of a Nichols algebra in an appropriate Yetter-Drinfeld module category. To complete our analysis, we require an additional result concerning coalgebra filtrations. Let $C$ be a coalgebra. Recall that an $\mathbb{N}_0$-filtration of $C$ is a family of subspaces $C_n$, $n \geq 0$, of $C$ satisfying

- $C_n$ is a subspace of $C_m$ for all $m, n \in \mathbb{N}_0$ with $n \leq m$
- $C = \bigcup_{n\in\mathbb{N}_0} C_n$,
- $\Delta_C(x) \in \sum_{i=0}^n C_i \otimes C_{n-i}$ for all $x \in C_n$, $n \in \mathbb{N}_0$.

**Lemma 4.3.1.** *Let $C$ be a coalgebra having an $\mathbb{N}_0$-filtration $\{C_n\}_{n\geq 0}$. Let $U$ be a non-zero comodule of $C$. Then there exists $u \in U \setminus \{0\}$ such that $\delta(u) \in C_0 \otimes U$.*

*Proof.* Let $x \in U \setminus \{0\}$. Then there exists $n \in \mathbb{N}_0$ with $\delta(x) \in C_n \otimes U$. If $n = 0$, we are done. Assume now that $n \geq 1$ and let $\pi_0 : C \to C/C_0$ be the canonical linear map. Since $C = \bigcup_{n\in\mathbb{N}_0} C_n$ is a coalgebra filtration, there is a maximal $m \in \mathbb{N}_0$ such that

$$\pi_0(x_{-m}) \otimes \cdots \otimes \pi_0(x_{-1}) \otimes x_0 \neq 0.$$

Let $f_1, \ldots, f_m \in C^*$ with $f_i|_{C_0} = 0$ for all $i \in \{1, \ldots, m\}$ such that

$$y := f_1(x_{-m}) \cdots f_m(x_{-1}) x_0 \neq 0.$$

Then $\delta(y) = f_1(x_{-m-1}) \cdots f_m(x_{-2}) x_{-1} \otimes x_0 \in C_0 \otimes U$ by the maximality of $m$. □

**Theorem 4.3.2.** *There is an isomorphism*

$$K \cong \mathcal{B}(L) \tag{4.3.1}$$

*of Hopf algebras in the category ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. In particular, $P(K) = L$.*

*Proof.* We endow a non-standard grading on $\mathcal{B}(M \oplus N)$ via setting $\deg(M) = 1$ and $\deg(N) = 0$. Under this grading, $\mathcal{B}(M \oplus N)$ remains a $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Furthermore, $\mathcal{B}(M \oplus N)\#H$ becomes a $\mathbb{N}_0$-graded coquasi-Hopf algebra with $\deg(H) = 0$. Now we are going to show $K$ is a Nichols algebra by verifying the axiom in Definition 2.5.3.

(1) It follows directly that $K$ inherits an $\mathbb{N}_0$-graded Hopf algebra in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ with $K(n) = K \cap (\mathcal{B}(M \oplus N)\#H)(n)$.

(2) We have $K(0) = K \cap (\mathcal{B}(M \oplus N)\#H)(0) = K \cap (\mathcal{B}(M \oplus N)(0) = \Bbbk$.

(3) Note that $K(1) = K \cap (\mathcal{B}(M \oplus N)\#H)(1) = K \cap (\mathcal{B}(M \oplus N)(1)) = \mathrm{ad}(\mathcal{B}(N))(M) = L$. We are going to show $K$ is generated by its degree one component $L$. Let $K'$ be the subalgebra of $K$ generated by $L$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, which is an object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ with structures induced by tensor product of $L$. Now we let $W$ be the image of $K'\#\mathcal{B}(N)$ under the isomorphism $K\#\mathcal{B}(N) \cong \mathcal{B}(M \oplus N)$. It suffices to show that $W = \mathcal{B}(M \oplus N)$. Note that $M \oplus N \subseteq W$ since $M, N \subseteq W$. Thus, we need only verify that $W$ forms a subalgebra of $\mathcal{B}(M \oplus N)$.

The key observation is the stability of $K'$ under the map of $\mathcal{B}(N)$. Indeed, $L$ itself is stable under the map of $\mathcal{B}(N)$. Now consider arbitrary homogeneous elements $y, z \in L$ and $a \in \mathcal{A}(N)$, by (2.2.4),

$$\begin{aligned} &\mathrm{ad}(a)(y \otimes z) \\ &= \frac{\Phi(a_1, y_{-1}, z_{-2})\Phi((\mathrm{ad}(a_2)(y_0))_{-1}, (\mathrm{ad}(a_4)(z_0))_{-1}, a_5))}{\Phi((\mathrm{ad}(a_2)(y_0))_{-2}, a_3, z_{-1})} (\mathrm{ad}(a_2)(y_0))_0 \otimes (\mathrm{ad}(a_4)(z_0))_0. \end{aligned}$$

Since $L$ is an object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, it follows that $\mathrm{ad}(a)(y \otimes z) \in L \otimes L$. Thus $K'$ remains stable under the map induced by $\mathcal{B}(N)$ by induction.

As both $K'$ and $\mathcal{B}(N)$ are subalgebras of $\mathcal{B}(M \oplus N)$, we only need to examine the case when $a \in N$ and $x \in K'$ . In this situation:

$$\mathrm{ad}(a)(x) = ax - \mathrm{ad}(a_{-1})(x)a_0.$$

Since $\mathrm{ad}(a)(x) \in K'$ and $\mathrm{ad}(a_{-1})(x)a_0 \in K'\#\mathcal{B}(N)$, we conclude that $ax \in K'\#\mathcal{B}(N)$. Hence its image lies in $W$. Thus $W$ is a subalgebra of $\mathcal{B}(M \oplus N)$, which establishes that $K$ is generated by $L$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$.

(4) To show $P(K) = K(1)$, that is, there is no primitive element in $K(n)$, $n \geq 2$. Suppose, for contradiction, that there exists a nonzero subspace $U \subseteq P(K(n))$ for some positive integer $n$. By Lemma 4.1.2, $U$ is an object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, since $\deg(N) = \deg(H) = 0$. Now, consider the coalgebra filtration on $\mathcal{B}(N)\#H$ given by $(\mathcal{B}(N)\#H)_0 = H$, $(\mathcal{B}(N)\#H)_1 = H + N$. Applying Lemma 4.3.1, we find a nonzero element $u \in U$ such that $\delta_K(u) = u_{-1} \otimes u_0 \in H \otimes U$. Note that

$$\delta_K(u) = (\pi \otimes \mathrm{id})\Delta_{\mathcal{A}(M\oplus N)}(u).$$

Therefore

$$\Delta_{\mathcal{A}(M\oplus N)}(u) = u \otimes 1 + u_{-1} \otimes u_0.$$

Applying the map $\mathrm{id}\#\varepsilon \otimes \mathrm{id}$, we obtain

$$\Delta_{\mathcal{B}(M\oplus N)}(u) = (\mathrm{id}\#\varepsilon \otimes \mathrm{id})\Delta(u) = 1 \otimes u + u \otimes 1,$$

which implies that $u$ is a primitive element in $\mathcal{B}(M \oplus N)$. However, since $K(n)$ is generated by $L = \mathrm{ad}(\mathcal{B}(N))(M)$, the element $u$ must have degree at least $n$ in the standard grading of $\mathcal{B}(M \oplus N)$. This leads to a contradiction, as primitive elements in a Nichols algebra lie in degree one.

□

## 4.4 From the Nichols algebra back to the space of coinvariants

We now turn to a converse of the preceding result under an additional restriction.

Let $C$ be a coalgebra and $D \subseteq C$ be a subcoalgebra. Let $V$ be a left comodule of $C$ with comodule structure $\delta : V \to C \otimes V$. We denote the largest $D$-subcomodule of $V$ by

$$V(D) = \{v \in V \mid \delta(v) \in D \otimes V\}.$$

**Lemma 4.4.1.** *Let $N \in {}^H_H\mathcal{YD}$ and $W \in {}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. Assume that $W = \bigoplus_{i\in I} W_i$ is a decomposition into irreducible objects in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}(\mathbb{Z}\text{-}\mathrm{Gr}\,\mathcal{M}_{\Bbbk})$. Let $M = W(\Bbbk G)$, and $M_i = M \cap W_i$ for all $i \in I$.*

*(1) $M = \bigoplus_{i\in I} M_i$ is a decomposition into irreducible objects in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-}\mathrm{Gr}\,\mathcal{M}_{\Bbbk})$.*

*(2) For all $i \in I$, $M_i$ is the $\mathbb{Z}$-homogeneous component of $W_i$ of minimal degree, and $W_i = \mathcal{B}(N) \triangleright M_i = \bigoplus_{n\geq 0} N^{\otimes n} \triangleright M_i$.*

*Proof.* Let $W = \bigoplus_{n\in\mathbb{Z}} W(n)$ be the $\mathbb{Z}$-grading of $W$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. Then $M$ is a $\mathbb{Z}$-graded object in ${}^H_H\mathcal{YD}$ with homogeneous components $M(n) = M \cap W(n)$ for all $n \in \mathbb{Z}$. It is clear that $M = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{i\in I} M_i(n) = \bigoplus_{i\in I} M_i$, where $M_i = M \cap W_i = W_i(H)$.

We now show that each $M_i$ is an irreducible object in ${}^H_H\mathcal{YD}(\mathbb{Z}\text{-}\mathrm{Gr}\,\mathcal{M}_{\Bbbk})$ for each $i \in I$. First, by Lemma 4.3.1, we have $M_i \neq 0$. Suppose $0 \neq M_i' \subseteq M_i$ is a homogeneous subobject in ${}^H_H\mathcal{YD}$, and let $m_i$ be the degree of $M_i'$. Define

$$W_i' := \mathcal{B}(N) \triangleright M_i'.$$

Using a method similar to that in Lemma 4.1.3, one verifies that $W_i'$ is an $\mathbb{Z}$-graded $\mathcal{A}(N)$-subcomodule. Moreover, $\mathcal{A}(N) \triangleright (\mathcal{B}(N) \triangleright M_i') \subseteq \mathcal{A}(N) \triangleright M_i' = \mathcal{B}(N) \triangleright M_i'$. Therefore $W_i'$ is a $\mathbb{Z}$-graded subobject of $W_i$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. The object $W_i'$ is concentrated in degrees $\geq m_i$, and its degree $m_i$'s component is exactly $M_i'$. Since $W_i$ is irreducible, we must have $W_i' = W_i$. Therefore,

$M_i' = W_i' \cap M = W_i \cap M = M_i$, which establishes the irreducibility of $M_i$. Moreover, $M_i$ is indeed the homogeneous component of $W_i$ of minimal degree. Finally, for each $i \in I$ and $n \in \mathbb{N}_0$,

$$\deg(N^{\otimes n} \triangleright M_i) = n + \deg(M_i),$$

since the map $\mathcal{B}(N)\#\Bbbk G \otimes W_i \longrightarrow W_i$ is $\mathbb{Z}$-graded. It follows that $\mathcal{B}(N) \triangleright M_i = \bigoplus_{n\geq 0} N^{\otimes n} \triangleright M_i$ as required. ◻

**Theorem 4.4.2.** *Let $N \in {}^H_H\mathcal{YD}$ and $(W, \triangleright, \delta)$ be a semisimple object in the category ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}(\mathbb{Z}\text{-}\mathrm{Gr}\,\mathcal{M}_\Bbbk)$ with linear map $\triangleright : \mathcal{A}(N) \otimes W \to W$. Here $\mathcal{A}(N)$ is equipped with the standard $\mathbb{N}_0$-grading. Let $\mathcal{B}(W)$ be the Nichols algebra of $W$ in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ and define $M = W(H)$. Then there is a unique isomorphism*

$$\mathcal{B}(W)\#\mathcal{B}(N) \cong \mathcal{B}(M \oplus N) \tag{4.4.1}$$

*of braided Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on $M \oplus N$.*

*Proof.* Let $W = \bigoplus_{i\in I} W_i$ be the decomposition of $W$ into irreducible objects in the category ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}(\mathbb{Z}\text{-}\mathrm{Gr}\,\mathcal{M}_\Bbbk)$. Then $M$ is an object in ${}^H_H\mathcal{YD}$ with decomposition into irreducible objects $M = \bigoplus_{i\in I} M_i := \bigoplus_{i\in I} M \cap W_i$. Moreover, we have $W_i = \mathcal{B}(N) \triangleright M_i = \bigoplus_{n\in\mathbb{N}_0} N^{\otimes n} \triangleright M_i$ for each $i \in I$.

Let $\deg(M_i) = m_i$ for each $i \in I$. We endow $W_i$ a new grading by $\widetilde{W_i} = W_i$ with

$$\widetilde{W_i}(n) = W(n + m_i - 1) = N^{n-1} \triangleright M_i, \text{ for all } n \in \mathbb{N}_0.$$

With the new grading, $\widetilde{W} = \bigoplus_{i\in I} \widetilde{W_i}$ remains an object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ since degree shifting preserves graded modules and graded comodules. Moreover, $\widetilde{W}(n) = 0$ for all $n \leq 0$. So the Nichols algebra $\mathcal{B}(\widetilde{W})$ is an $\mathbb{N}_0$-graded Hopf algebra in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. Note that $\mathcal{B}(\widetilde{W}) \cong \mathcal{B}(W)$, as both are determined by the same module and comodule maps. Under this grading, $\widetilde{W}(0) = 0$ and $\widetilde{W}(1) = M$, and the degree zero and degree one part of the Nichols algebra $\mathcal{B}(\widetilde{W})$ are

$$\mathcal{B}(\widetilde{W})(0) = \Bbbk, \quad \mathcal{B}(\widetilde{W})(1) = M.$$

Moreover, $\mathcal{B}(W)\#(\mathcal{B}(N)\#H)$ is an $\mathbb{N}_0$-graded coquasi-Hopf algebra, and

$$R := \mathcal{B}(W)\#\mathcal{B}(N) = (\mathcal{B}(W)\#(\mathcal{B}(N)\#H))^{\mathrm{co}H}$$

is a $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. Its degree $0$ part is $\Bbbk$, and its degree $1$ part is $M \oplus N$.

We are going to show $R$ is a pre-Nichols algebra in ${}^H_H\mathcal{YD}$. It remains to show $R$ is generated by $M \oplus N$ as an algebra in ${}^H_H\mathcal{YD}$. It is direct to see that $R$ is generated by $K(1) = \mathcal{B}(N) \triangleright M$ and $N$. It therefore suffices to prove that $\mathcal{B}(N) \triangleright M = \bigoplus_{n\geq 0} N^{\otimes n} \triangleright M$ is contained in the subalgebra generated by $\mathcal{B}(N)$ and $M$. To see this, we proceed by induction on $n$. For the base case, take elements $x \in M$ and $y \in N$. Inside the coquasi-Hopf algebra $\mathcal{B}(W)\#(\mathcal{B}(N)\#H)$, we have

$$yx = (y_1 \triangleright x)y_2 = (y_{-1} \triangleright x)y_0 + (y \triangleright x).$$

Since $M \in {}^H_H\mathcal{YD}$ and $y_{-1} \in H$, the term $y \triangleright x$ is contained in the subalgebra generated by $\mathcal{B}(N)$ and $M$. Since $\mathcal{B}(N) \triangleright M$ is an object in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, $\delta(y \triangleright x) \in \mathcal{A}(N) \otimes (N \triangleright M)$. This implies terms like $(y \triangleright x)_0$ are contained in the subalgebra generated by $\mathcal{B}(N)$ and $M$. Let us fix $k \geq 0$, for all $z \in \mathcal{B}(N)(k')$ and $x \in M$, where $k' \geq k$. We assume the element $z \triangleright x$ as well as $(z \triangleright x)_0$ lie in the subalgebra generated by $\mathcal{B}(N)$ and $M$, then for homogeneous $y \in N$,

$$\begin{aligned}(yz) \triangleright x &= \frac{\Phi(y_{-1},(z_2 \triangleright x_0)_{-1},z_3)}{\Phi(y_{-2},z_1,x_{-1})} y_0 \triangleright (z_2 \triangleright x_0)_0 \\ &= \frac{\Phi(y_{-2},(z_2 \triangleright x_0)_{-1},z_3))}{\Phi(y_{-3},z_1,x_{-1})}(y_{-1} \triangleright (z_2 \triangleright x_0)_0)y_0 - \frac{\Phi(y_{-1},(z_2 \triangleright x_0)_{-1},z_3)}{\Phi(y_{-2},z_1,x_{-1})} y_0(z_2 \triangleright x_0)_0.\end{aligned}$$

By the induction hypothesis, $(yz) \triangleright x$ lies in the subalgebra generated by $\mathcal{B}(N)$ and $M$. Therefore, $R$ is generated by $M \oplus N$, and hence a pre-Nichols algebra of $M \oplus N$.

By the universal property of $\mathcal{B}(M \oplus N)$, there exists a surjective morphism of $\mathbb{N}_0$-graded Hopf algebras in ${}^H_H\mathcal{YD}$:

$$\rho : R \to \mathcal{B}(M \oplus N), \rho\,|_{M\oplus N} = \mathrm{id}.$$

This induces a surjective coquasi-Hopf algebra map:

$$\rho\#\mathrm{id} : R\#H \to \mathcal{B}(M \oplus N)\#H.$$

Let $K = (\mathcal{B}(M \oplus N)\#H)^{\mathrm{co}\,\mathcal{B}(N)\#H}$. Then we obtain two bijective coquasi-Hopf algebra maps:

$$R\#H \to \mathcal{B}(W)\#(\mathcal{B}(N)\#H), \quad K\#(\mathcal{B}(N)\#H) \to \mathcal{B}(M \oplus N)\#H.$$

Then the map $\rho\#\mathrm{id}$ thus induces a surjective map of coquasi-Hopf algebras

$$\rho' : \mathcal{B}(W)\#(\mathcal{B}(N)\#H) \to K\#(\mathcal{B}(N)\#H), \quad \rho'\,|_{(M\oplus N)} = \mathrm{id}.$$

Moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}(W)\#(\mathcal{B}(N)\#H) & \xrightarrow{\quad\rho'\quad} & K\#(\mathcal{B}(N)\#H) \\ \Big\downarrow{\scriptstyle \varepsilon_{\mathcal{B}(W)}\#\mathrm{id}_{\mathcal{B}(N)\#H}} & & \Big\downarrow{\scriptstyle \varepsilon_K\#\mathrm{id}_{\mathcal{B}(N)\#H}} \\ \mathcal{B}(N)\#H & \xrightarrow{\quad\mathrm{id}\quad} & \mathcal{B}(N)\#H \end{array}$$

since $\rho\,|_{M\oplus N} = \mathrm{id}$. As $\mathcal{B}(N)\#H$ acts on $K$ via adjoint map. Theorem 4.3.2 implies that $K \cong \mathcal{B}(\mathrm{ad}(\mathcal{B}(N))(M))$. Therefore $\rho'$ induces a surjective map

$$\phi : \mathcal{B}(W) \to \mathcal{B}(\mathrm{ad}(\mathcal{B}(N))(M))$$

in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$ between the right coinvariant subspaces of $\varepsilon_{\mathcal{B}(W)}\#\mathrm{id}_{\mathcal{B}(N)\#H}$ and $\varepsilon_K\#\mathrm{id}_{\mathcal{B}(N)\#H}$, satisfying $\phi\,|_M = \mathrm{id}$. Furthermore, there is a surjective map in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$:

$$\phi_1 : \mathcal{B}(N) \triangleright M \to \mathrm{ad}(\mathcal{B}(N))(M), \quad \phi_1\,|_M = \mathrm{id}.$$

Since $M = \bigoplus_{i\in I} M_i$ is a decomposition into irreducible objects in ${}^H_H\mathcal{YD}$, each $\mathrm{ad}(\mathcal{B}(N))(M_i)$ is irreducible in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. On the other hand, each $\mathcal{B}(N) \triangleright M_i$ is irreducible as well. Thus $\phi_1$ is an isomorphism in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$, and it follows that $\phi$ is an isomorphism of Hopf algebra in ${}^{\mathcal{A}(N)}_{\mathcal{A}(N)}\mathcal{YD}$. Consequently, $\rho\#\mathrm{id}_H = \phi\#\mathrm{id}_{\mathcal{A}(N)}$ is an isomorphism of coquasi-Hopf algebras. Therefore,

$$\rho = (\rho\#\mathrm{id}_H)^{\mathrm{co}\,H} : \mathcal{B}(W)\#\mathcal{B}(N) \to \mathcal{B}(M \oplus N)$$

is an isomorphism of Hopf algebras in ${}^{H}_{H}\mathcal{YD}$.

□

# Chapter 5

# Reflection of Nichols algebras over coquasi-Hopf algebra with bijective antipode

In this chapter, we develop reflections of Nichols algebras over arbitrary coquasi-Hopf algebras with bijective antipode. The reflection allows us to systematically relate different Nichols algebra realizations through a well-defined transformation procedure.

## 5.1 Definition of reflection and basic properties

Fix a positive integer $\theta$ and denote the index set $\mathbb{I} = \{1, 2, ..., \theta\}$.

**Definition 5.1.1.** *Let $\mathcal{F}_\theta$ denote the class of all $\theta$-tuples $M = (M_1, \ldots, M_\theta)$, where $M_1, \ldots, M_\theta \in {}^H_H\mathcal{YD}$ are finite-dimensional Yetter-Drinfeld modules. If $M \in \mathcal{F}_\theta$, we define*

$$\mathcal{B}(M) := \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta).$$

*Two tuples $M, M' \in \mathcal{F}_\theta$ are called isomorphic, denoted $M \cong M'$, if $M_j \cong M'_j$ in ${}^H_H\mathcal{YD}$ for all $j$. The isomorphism class of $M \in \mathcal{F}_\theta$ is denoted by $[M]$.*

*For $1 \le i \le \theta$ and $M \in \mathcal{F}_\theta$, we say the tuple $M$ admits the $i$-th reflection $R_i(M)$ if for all $j \ne i$ there is a natural number $m_{ij}^M \ge 0$ such that $(\mathrm{ad}\, M_i)^{m_{ij}^M}(M_j)$ is a non-zero finite-dimensional subspace of $\mathcal{B}(M)$, and $(\mathrm{ad}\, M_i)^{m_{ij}^M+1}(M_j) = 0$. Assume $M$ admits the $i$-th reflection. Then we set $R_i(M) = (V_1, \ldots, V_\theta)$, where*

$$V_j = \begin{cases} M_i^*, & \text{if } j = i, \\ (\mathrm{ad}\, M_i)^{m_{ij}^M}(M_j), & \text{if } j \ne i. \end{cases}$$

Such a question arises naturally: how do the irreducibility properties behave under reflections? The following result provides a reassuring answer.

**Lemma 5.1.2.** *Suppose $M \in \mathcal{F}_\theta$ admits the $i$-th reflection for some $i \in \mathbb{I}$, and $M_j$ is irreducible for each $j \in \mathbb{I}$. Then each $R_i(M)_j$ is irreducible in ${}^H_H\mathcal{YD}$ for $1 \le j \le \theta$.*

*Proof.* By definition, $R_i(M)$ is defined, thus $R_i(M)_i \cong M_i^*$ is irreducible since $M_i$ is. For $j \ne i$, we observe that $\mathrm{ad}(\mathcal{B}(M_i))(M_j) = \bigoplus_{n=0}^{m_{ij}} \mathrm{ad}(M_i)^n(M_j)$. Consider the embedding $H \to \mathcal{A}(M_i)$ and the projection $\mathcal{A}(M_i) \to H$. The object $\mathrm{ad}(M_i)^{m_{ij}}(M_j)$ belongs to ${}^H_H\mathcal{YD}$. Moreover, it is obvious that $\mathrm{ad}(\mathcal{B}(M_i))(M_j)$ is generated by $\mathrm{ad}(M_i)^{m_{ij}}(M_j)$ as a $\mathcal{A}(M_i)$-comodule. Now suppose $0 \neq P \subset \mathrm{ad}(M_i)^{m_{ij}}(M_j)$ is a Yetter-Drinfeld submodule in ${}^H_H\mathcal{YD}$. Let $\langle P \rangle$ be the $\mathcal{A}(M_i)$-subcomodule of $\mathrm{ad}\,\mathcal{B}(M_i)(M_j)$ generated by $P$, defined explicitly as

$$\langle P \rangle := \{\langle f, x_{-1} \rangle x_0 \mid x \in P, f \in \mathrm{Hom}(\mathcal{A}(M_i), \Bbbk)\}.$$

We want to prove $\langle P \rangle$ is a Yetter-Drinfeld submodule of $\mathrm{ad}\,\mathcal{B}(M_i)(M_j)$ in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$. To see this, for all $a \in \mathcal{A}(M_i)$ and $x \in P$, by Lemma 2.2.4,

$$\begin{aligned}
\mathrm{ad}(a)(\langle f, x_{-1} \rangle x_0) &= \langle f, x_{-1} \rangle \, \mathrm{ad}(a)(x_0) \\
&= \langle f, p_L(\mathcal{S}(a_1), (\mathrm{ad}(a_4)(x_0))_{-1} a_6) q_L(a_3, x_{-1}) \mathcal{S}(a_2)((\mathrm{ad}(a_4)(x_0))_{-2} a_5) \rangle (\mathrm{ad}(a_4)(x_0))_0.
\end{aligned}$$

One may view $s := p_L(\mathcal{S}(a_1), \underline{\quad} \cdot a_6)$ and $t := \langle f, q_L(a_3, x_{-1}) \mathcal{S}(a_2)(\underline{\quad} \cdot a_5) \rangle$ as two linear functions, thus

$$\mathrm{ad}(a)(\langle f, x_{-1} \rangle x_0) = t * s\,((\mathrm{ad}(a_4)(x_0))_{-1})\,(\mathrm{ad}(a_4)(x_0))_0.$$

Hence $\mathrm{ad}(a)(\langle f, x_{-1} \rangle x_0) \in \langle P \rangle$. This ensures the first axiom of a Yetter-Drinfeld module, and the third axiom holds automatically. Therefore $\langle P \rangle$ is a Yetter-Drinfeld submodule of $\mathrm{ad}\,\mathcal{B}(M_i)(M_j)$ in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$.

Since $\mathrm{ad}(\mathcal{B}(M_i))(M_j)$ is irreducible in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$ by Lemma 4.2.4 (2), and $P \neq 0$, we have $\langle P \rangle = \mathrm{ad}(\mathcal{B}(M_i))(M_j)$. It is direct to see that $\langle P \rangle = \bigoplus_{i=0}^{m_{ij}} \langle P \rangle \cap \mathcal{B}^i(M_i \oplus M_j)$. Then

$$P = \langle P \rangle \cap \mathcal{B}^{m_{ij}}(M_i \oplus M_j) = \mathrm{ad}\,\mathcal{B}(M_i)(M_j) \cap \mathcal{B}^{m_{ij}}(M_i \oplus M_j) = \mathrm{ad}(M_i)^{m_{ij}}(M_j).$$

This establishes the irreducibility. □

**Lemma 5.1.3.** *Suppose $M \in \mathcal{F}_\theta$ and $M$ admits $i$-th reflection for each $i \in \mathbb{I}$. We define $a_{ii}^M = 2$ for all $1 \le i \le \theta$ and define $a_{ij}^M = -m_{ij}^M$, then $(a_{ij}^M)_{i,j \in \mathbb{I}}$ is a generalized Cartan matrix.*

*Proof.* To establish this result, we need only verify: if $1 \le i < j \le \theta$ such that $a_{ij}^M = 0$, then $a_{ji}^M = 0$.

Let $X \in M_i, Y \in M_j$ be homogeneous. A direct computation yields:

$$\begin{aligned}
\Delta_{\mathcal{B}(M)}(\mathrm{ad}(X)(Y)) &= \Delta_{\mathcal{B}(M)}(XY - (\mathrm{ad}(x)(Y))X) \\
&= (1 \otimes X + X \otimes 1)(1 \otimes Y + Y \otimes 1) - (\mathrm{ad}(x)(Y) \otimes 1 + 1 \otimes \mathrm{ad}(x)(Y))(1 \otimes X + X \otimes 1) \\
&= XY \otimes 1 + X \otimes Y + 1 \otimes XY + \mathrm{ad}(x)(Y) \otimes X \\
&\quad - \mathrm{ad}(x)(Y) \otimes X - \mathrm{ad}(x)(Y)X \otimes 1 - 1 \otimes \mathrm{ad}(x)(Y)X - \mathrm{ad}(xyx^{-1})(X) \otimes \mathrm{ad}(x)(Y) \\
&= \mathrm{ad}(X)(Y) \otimes 1 + 1 \otimes \mathrm{ad}(X)(Y) + X \otimes Y - c^2(X \otimes Y).
\end{aligned}$$

Note that $a_{ij}^M = 0$ implies $\mathrm{ad}(X)(Y) = 0$, from which we deduce $X \otimes Y - c^2(X \otimes Y) = 0$. This establishes $(\mathrm{id} - c^2)(M_i \otimes M_j) = 0$. Furthermore, we have $(\mathrm{id} - c^2)c(M_j \otimes M_i) = 0$. Since the braiding $c$ is invertible, it follows that $(\mathrm{id} - c^2)(M_j \otimes M_i) = 0$. Consequently, $\Delta_{\mathcal{B}(M)}(\mathrm{ad}(Y)(X)) = 0$ in $\mathcal{B}(M_i \oplus M_j)$, for all $X \in M_i, Y \in M_j$, which implies $\mathrm{ad}(Y)(X) = 0$. We conclude that $a_{ji}^M = 0$, as required. □

**Lemma 5.1.4.** *Suppose $M \cong N$ in $\mathcal{F}_\theta$, if $M$ admits the $i$-th reflection for some $i \in \mathbb{I}$, so does $N$. Furthermore, $R_i(M) \cong R_i(N)$ and $a_{ij}^M = a_{ij}^N$ for each $j \in \mathbb{I}$.*

*Proof.* Let $\phi : M \to N$ be an isomorphism of tuples with each component $\phi_j : M_j \to N_j$ an isomorphism of Yetter-Drinfeld modules. We first demonstrate that $\phi_i$ induces a unique morphism $\mathcal{B}(\phi_i) : \mathcal{B}(M_i) \to \mathcal{B}(N_i)$.

The tensor algebra map $T(\phi_i) : T(M_i) \to T(N_i)$ is a morphism of bialgebras in ${}^H_H\mathcal{YD}$. Since $T(\phi_i)$ is a $\mathbb{N}_0$-graded coalgebra map, it sends the coideal $I(M_i) = \bigoplus_{n\geq 2} \ker(\delta_{1^n})$ to $I(N_i)$. Hence $\phi_i$ descends to a well-defined morphism $\mathcal{B}(\phi_i) : \mathcal{B}(M_i) \to \mathcal{B}(N_i)$. This morphism is $\mathbb{N}_0$-graded, as its restriction to each homogeneous component $\mathcal{B}(V)(n)$ is induced by $\phi_i^{\otimes n}$. Explicitly, for any $n \in \mathbb{N}_0$ and $\gamma, \gamma_1, \dots, \gamma_n \in \mathbb{N}_0$ with $\gamma = \gamma_1 + \cdots + \gamma_n$, we have

$$\mathcal{B}(\phi_i)(V(\gamma_1)\cdots V(\gamma_n)) = \phi_i(V(\gamma_1))\cdots \phi_i(V(\gamma_n)) \subseteq \mathcal{B}(W)(\gamma).$$

The claim on the surjectivity of $\mathcal{B}(\phi_i)$ is immediate, while injectivity follows from the equation

$$\Delta_{1^n}\phi_i^{\otimes n} = \phi_i^{\otimes n}\Delta_{1^n}$$

for all $n \in \mathbb{N}_0$. We thus obtain an equivalence of categories: ${}^{A(M_i)}_{A(M_i)}\mathcal{YD} \cong {}^{A(N_i)}_{A(N_i)}\mathcal{YD}$. For each $j \neq i \in \mathbb{I}$,

$$\operatorname{ad}(\mathcal{B}(\phi_i))(\phi_j) : \operatorname{ad}(\mathcal{B}(M_i))(M_j) \cong \operatorname{ad}(\mathcal{B}(N_i))(N_j)$$

is an isomorphism of $\mathbb{N}_0$-graded objects in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$. In particular, $\operatorname{ad}(M_i)^{m_{ij}^M}(M_j) \cong \operatorname{ad}(N_i)^{m_{ij}^M}(N_j) \in {}^H_H\mathcal{YD}$ and $\operatorname{ad}(N_i)^{m_{ij}^M+1}(N_j) = 0$. This establishes both $a_{ij}^M = a_{ij}^N$ and $R_i(M) \cong R_i(N)$. □

## 5.2 Main result

Recall from Remark 2.4.2, let $A$ be a Hopf algebra in ${}^H_H\mathcal{YD}$ with bijective antipode, we have such a braided monoidal isomorphism

$$F : {}^A_A\mathcal{YD}(C) \cong {}^{A\#H}_{A\#H}\mathcal{YD}.$$

Now restricting to rational Yetter-Drinfeld modules, we denote the image of $F$ by ${}^{A\#H}_{A\#H}\mathcal{YD}_{\text{rat}}$, which is a monoidal full subcategory of ${}^{A\#H}_{A\#H}\mathcal{YD}$.

From now on, we always fix a tuple $M = (M_1, \dots, M_\theta) \in \mathcal{F}_\theta$, with each component $M_i$ being irreducible for $i \in \mathbb{I}$. By Corollary 3.5.1, we have such a braided tensor equivalence for each $i \in \mathbb{I}$.

$$\Omega_i, : {}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(C)_{\text{rat}} \longrightarrow {}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(C)_{\text{rat}}. \tag{5.2.1}$$

By abusing notation, we again denote the following monoidal equivalence by $\Omega_i$.

$${}^{A(M_i)}_{A(M_i)}\mathcal{YD}_{\text{rat}} \cong {}^{\mathcal{B}(M_i)}_{\mathcal{B}(M_i)}\mathcal{YD}(C)_{\text{rat}} \to {}^{\mathcal{B}(M_i^*)}_{\mathcal{B}(M_i^*)}\mathcal{YD}(C)_{\text{rat}} \cong {}^{A(M_i^*)}_{A(M_i^*)}\mathcal{YD}_{\text{rat}}$$

**Lemma 5.2.1.** *The following are equivalent:*

(1) *$M$ admits $i$-th reflection for some $i \in \mathbb{I}$.*

(2) *$\mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)}$ belongs to ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}_{\text{rat}}$.*

*Proof.* (1) ⇒ (2): Let $W = \bigoplus_{j\neq i} M_j$, and define $Q = \operatorname{ad}\mathcal{B}(M_i)(W)$. By Lemma 4.2.4, we have the decomposition $Q = \bigoplus_{j\neq i} Q_j$, where each $Q_j = \operatorname{ad}\mathcal{B}(M_i)(M_j)$ is irreducible in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$. Applying

Theorem 4.3.2 yields an isomorphism of Hopf algebras:

$$\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)} \cong \mathcal{B}(Q).$$

Now by assumption, there is an integer $m$ such that $\mathrm{ad}(M_i)^m(W) = 0$. This implies that $\mathrm{ad}(M_i)^m(Q) = 0$. Thus $Q$ is rational, thus $\mathcal{B}(Q)$ as well as $\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}$ are rational by Lemma 3.1.3(3).

(2) ⇒ (1): The condition (2) implies that $Q$ is rational in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$. Furthermore, $M$ admits the $i$-th reflection by definition. □

**Lemma 5.2.2.** *With above assumptions on $M$, and suppose $M$ admits the $i$-th reflection with $R_i(M) = (V_1, \dots, M_i^*, \dots, V_\theta)$, then the following equalities hold.*

$$\mathrm{ad}(\mathcal{B}(M_i)(M_j)) = \bigoplus_{n=0}^{m_{ij}} \mathrm{ad}(M_i)^n(M_j), \qquad V_j = \mathcal{F}_0(\mathrm{ad}\,\mathcal{B}(M_i)(M_j)),$$
$$\Omega_i(\mathrm{ad}(\mathcal{B}(M_i)(M_j))) = \bigoplus_{n=0}^{m_{ij}} \mathrm{ad}(M_i^*)^n(V_j), \qquad M_j \cong \mathrm{ad}(M_i^*)^{m_{ij}}(V_j).$$

*Proof.* The first equality follows directly from the decomposition:

$$\mathrm{ad}\left(\mathcal{B}(M_i)(M_j)\right) = \bigoplus_{n\geq 0} \mathrm{ad}(M_i)^n(M_j).$$

For the second statement, the inclusion $V_j \subseteq \mathcal{F}_0(\mathrm{ad}\,\mathcal{B}(M_i)(M_j))$ is immediate. Suppose, for contradiction, that the containment is proper. Then there exists a nonzero $X \in \mathrm{ad}(M_i)^l(M_j)$ for $0 \leq l \leq m_{ij} - 1$ such that $X \in \mathcal{F}_0(\mathrm{ad}\,\mathcal{B}(M_i)(M_j))$. The Yetter-Drinfeld submodule generated by $X$ is contained in $\bigoplus_{k=0}^{l} \mathrm{ad}(M_i)^k(M_j)$. However, since $V_j \neq 0$, and $\mathrm{ad}(\mathcal{B}(M_i)(M_j))$ is irreducible in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$ by Lemma 4.2.4(ii), this leads to a contradiction. Hence, $V_j = \mathcal{F}_0(\mathrm{ad}\,\mathcal{B}(M_i)(M_j))$.

For the third statement, we consider the category ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}(\mathbb{Z}\text{-Gr}\,\mathcal{M}_{\Bbbk})$ by setting $\deg(M_k) = 1$ for $1 \leq k \leq \theta$. Then $\mathrm{ad}(M_i)^l(M_j)$ has degree $l+1$ for $1 \leq l \leq m_{ij}$. By Lemma 3.6.3, $\Omega_i(\mathrm{ad}(M_i)^l(M_j)) = \mathrm{ad}(M_i)^l(M_j)$ is $\mathbb{Z}$-graded with $\deg(\Omega_i(\mathrm{ad}(M_i)^l(M_j))) = -1 - l$. Moreover,

$$\mathcal{F}^0\Omega_i(\mathrm{ad}\,\mathcal{B}(M_i)(M_j)) = \mathcal{F}_0(\mathrm{ad}\,\mathcal{B}(M_i)(M_j)) = \mathrm{ad}(M_i)^{m_{ij}}(M_j) = V_j.$$

Since $\Omega_i(\mathrm{ad}(\mathcal{B}(M_i)(M_j)))$ is irreducible in ${}^{A(M_i^*)}_{A(M_i^*)}\mathcal{YD}$, with the minimal degree part $V_j$. By Lemma 4.4.1,

$$\Omega_i\left(\mathrm{ad}\left(\mathcal{B}(M_i)(M_j)\right)\right) = \mathrm{ad}\left(\mathcal{B}(M_i^*)(V_j)\right) = \bigoplus_{n=0}^{m_{ij}} \mathrm{ad}(M_i^*)^n(V_j).$$

In particular, we have

$$M_j = \mathcal{F}^0\,\mathrm{ad}\,\mathcal{B}(M_i)(M_j) = \mathcal{F}_0\,\mathrm{ad}\,\mathcal{B}(M_i^*)(V_j) = \mathrm{ad}(M_i^*)^{m_{ij}}(V_j).$$

□

The next theorem gives a natural explanation of reflections of tuples of Yetter-Drinfeld modules. Although the proof strategy parallels that of Hopf algebra cases, we include the details here for the sake of completeness and the reader's convenience.

**Theorem 5.2.3.** *With above assumptions on $M$, and suppose $M$ admits the $i$-th reflection, then there is an isomorphism of Hopf algebras in ${}^H_H\mathcal{YD}$:*

$$\Theta_i : \mathcal{B}(R_i(M)) \cong \Omega_i\left(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}\right)\#\mathcal{B}(M_i^*). \tag{5.2.2}$$

*Proof.* Let $N = \bigoplus_{j\neq i} M_j$, and define $Q = \mathrm{ad}\,\mathcal{B}(M_i)(N)$. By Lemma 4.2.4(1), we have the decomposition $Q = \bigoplus_{j\neq i} Q_j$, where each $Q_j = \mathrm{ad}\,\mathcal{B}(M_i)(M_j)$ is irreducible in ${}^{A(M_i)}_{A(M_i)}\mathcal{YD}$. Applying Theorem 4.3.2 yields an isomorphism of Hopf algebras:

$$\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)} \cong \mathcal{B}(Q).$$

Now observe that the functor $\Omega_i$ sends Nichols algebras to Nichols algebras by Corollary 3.5.3. Therefore,

$$\Omega_i\left(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}\right) \cong \Omega_i(\mathcal{B}(Q)) \cong \mathcal{B}(\Omega_i(Q)).$$

Hence, in the category ${}^H_H\mathcal{YD}$,

$$\Omega_i\left(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}\right)\#\mathcal{B}(M_i^*) \cong \mathcal{B}(\Omega_i(Q))\#\mathcal{B}(M_i^*).$$

Since $Q$ is a direct sum of irreducible objects, it is semisimple. The equivalence $\Omega_i$ preserves semisimplicity, so $\Omega_i(Q)$ is also semisimple in ${}^{A(M_i^*)}_{A(M_i^*)}\mathcal{YD}(\mathbb{Z}\text{-Gr}\,\mathcal{M}_{\Bbbk})$ by Lemma 3.6.3(3). Then we apply Theorem 4.4.2 to obtain

$$\mathcal{B}\left(\Omega_i\left(Q\right)\right)\#\mathcal{B}(M_i^*) \cong \mathcal{B}\left(\mathcal{F}^0(\Omega_i(Q)) \oplus M_i^*\right).$$

According to Lemma 5.2.2 and Lemma 3.6.3(2), $\mathcal{F}^0(\Omega_i(Q)) = \mathcal{F}_0(Q) = \bigoplus_{j\neq i} V_j$. Thus

$$\mathcal{B}(\Omega_i(Q))\#\mathcal{B}(M_i^*) \cong \mathcal{B}\left(\bigoplus_{j\neq i} V_j \oplus M_i^*\right) \cong \mathcal{B}(R_i(M)).$$

Combining the above isomorphisms, we conclude that in ${}^H_H\mathcal{YD}$, there is an isomorphism of Hopf algebras:

$$\Theta_i : \mathcal{B}(R_i(M)) \cong \Omega_i\left(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}\right)\#\mathcal{B}(M_i^*).$$

□

Having defined individual reflections, we now extend this notion to sequences of reflections, which will be important for our study of repeated reflections of tuples.

**Definition 5.2.4.** *Let $M \in \mathcal{F}_\theta$, with each $M_i$ is irreducible for $i \in \mathbb{I}$. For $l \in \mathbb{N}_0$ and $i_1, i_2, ..., i_l \in \mathbb{I}$.*

(1) *We say $M$ admits the reflection sequence $(i_1, i_2, ..., i_l)$ if $l = 0$ or $M$ admits the $i_1$-th reflection and $R_{i_1}(M)$ satisfies the reflection sequence $(i_2, i_3, ..., i_l)$.*

(2) *We say $M$ admits all reflection sequences if $M$ admits reflection sequence $(i_1, i_2, ..., i_l)$ for all $l \in \mathbb{N}_0$ and $i_1, i_2, ..., i_l \in \mathbb{I}$.*

For $M \in \mathcal{F}_\theta$ admitting all reflections, we denote

$$\mathcal{F}_\theta(M) = \{R_{i_1}(...(R_{i_l}(M))...) \mid l \in \mathbb{N}_0,\ i_1,..,i_l \in \mathbb{I}\}.$$

**Definition 5.2.5.** *Let $\mathbb{I}$ be a non-empty finite set, $\mathcal{X}$ a non-empty set, $r : \mathbb{I}\times\mathcal{X} \to \mathcal{X}$, $A : \mathbb{I}\times\mathbb{I}\times\mathcal{X} \to \mathbb{Z}$ maps. For all $i, j \in \mathbb{I}$ and $X \in \mathcal{X}$ we write $r_i(X) = r(i, X)$, $a^X_{ij} = A(i, j, X)$ and $A^X = (a^X_{ij})_{i,j\in\mathbb{I}} \in \mathbb{Z}^{\mathbb{I}\times\mathbb{I}}$. The quadruple $\mathcal{G} = \mathcal{G}(\mathbb{I},\mathcal{X},(r_i)_{i\in I},(A^X)_{X\in\mathcal{X}})$ is called a semi-Cartan graph if for all $X \in \mathcal{X}$, the matrix $A^X$ is a generalized Cartan matrix, and the following axioms hold.*
*$(CG1)$ For all $i \in \mathbb{I}$, the map $r_i$ satisfies $r_i^2 = \mathrm{id}_\mathcal{X}$.*
*$(CG2)$ For all $i \in \mathbb{I}$, $X \in \mathcal{X}$, $A^X$ and $A^{r_i(X)}$ have the same $i$-th row.*

As a consequence of Theorem 5.2.3, we obtain the following corollary.

**Corollary 5.2.6.** *Let $M = (M_1,\dots,M_\theta) \in \mathcal{F}_\theta$ with each component $M_i$ is irreducible.*

(1) *For each $i \in \mathbb{I}$, suppose $M$ admits $i$-th reflection, then $R_i(M)$ admits $i$-th reflection. Furthermore, we have:*

$$R_i^2(M) \cong M, \tag{5.2.3}$$

*and*

$$a^M_{ij} = a^{R_i(M)}_{ij}, \tag{5.2.4}$$

*for all $1 \le j \le \theta$.*

(2) *Suppose $M$ admits all reflections. We define the set*

$$\mathcal{X} = \{[P] \mid P \in \mathcal{F}_\theta(M)\},$$

*and the map*

$$r : \mathbb{I}\times\mathcal{X} \to \mathcal{X},\ i\times[X] \mapsto [R_i(X)].$$

*Furthermore, we assume each $\mathcal{B}(P_i)$ is finite-dimensional for $P \in \mathcal{X}$ and $i \in \mathbb{I}$, then*

$$\mathcal{G}(M) = (\mathbb{I},\mathcal{X},r,(A^X)_{X\in\mathcal{X}}),$$

*where $A^{[X]} = (a^X_{ij})_{i,j\in\mathbb{I}}$ for all $[X] \in \mathcal{X}$, is a semi-Cartan graph.*

*Proof.* (1) By Lemma 5.2.2, $R_i(M) = (V_1,...,M_i^*,...,V_\theta)$. Note that

$$\mathrm{ad}\,\mathcal{B}(M_i^*)(V_j) = \Omega_i\left(\mathrm{ad}\,\mathcal{B}(M_i)(M_j)\right).$$

It follows that

$$\mathrm{ad}\,\mathcal{B}(M_i^*)^{1-a^M_{ij}}(V_j) = 0,$$

which implies that $R_i(M)$ admits $i$-th reflection and $a^M_{ij} = a^{R_i(M)}_{ij}$ for all $j \ne i$. It is clear that the $i$-th position of $R_i^2(M)$ is isomorphic to $M_i$. For $j \ne i$,

$$\mathcal{F}_0\left(\mathrm{ad}\,\mathcal{B}(M_i^*)(V_j)\right) = \mathcal{F}_0\left(\Omega_i\left(\mathrm{ad}\left(\mathcal{B}\left(M_i\right)\left(M_j\right)\right)\right)\right) = \mathcal{F}^0(\mathrm{ad}(\mathcal{B}(M_i))(M_j)) = M_j.$$

Therefore $R_i^2(M)_j \cong M_j$ for all $j \ne i$, and hence $R_i^2(M) \cong M$.

(2) It follows by (1) and the definition of semi-Cartan graph. □

## 5.3 A criterion for finite-dimensional Nichols algebras

Building upon the framework developed in the preceding sections, we now present some applications of the semi-Cartan graph theory for Nichols algebras over certain coquasi-Hopf algebras. We establish several criteria that connect the finite-dimensionality of a Nichols algebra to the finiteness and structure of its associated semi-Cartan graph and Weyl groupoid. Then we apply this theory to provide a new proof for the key result in [60].

We still work on the setting of ${}_H^H\mathcal{YD}$, where $H$ is coquasi-Hopf algebra with bijective antipode. Now fix a positive integer $\theta$ and denote the index set $\mathbb{I} = \{1,2,...,\theta\}$. Let $M = (M_1,\dots,M_\theta) \in \mathcal{F}_\theta$. We endow a $\mathbb{Z}^\theta$-grading on $\mathcal{B}(M)$ by setting

$$\deg(M_i) = \alpha_i,\ 1 \le i \le \theta,$$

where $\alpha_i$, $1 \le i \le \theta$ are the standard basis of $\mathbb{Z}^\theta$. For each $\beta \in \mathbb{Z}^\theta$, define the homogeneous component $\mathcal{B}(M)_\beta = \{x \in \mathcal{B}(M) \mid \deg(x) = \beta\}$, and the support of $\mathcal{B}(M)$ as

$$\operatorname{Supp}\mathcal{B}(M) := \{\beta \in \mathbb{Z}^\theta \mid \mathcal{B}(M)_\beta \ne 0\}.$$

**Lemma 5.3.1.** *For each $i \in \mathbb{I}$, suppose $M$ admits $i$-th reflection,*

$$\operatorname{Supp}\mathcal{B}(M) \cup \operatorname{Supp}\mathcal{B}(M^*) = \operatorname{Supp}\mathcal{B}(R_i(M)) \cup \operatorname{Supp}\mathcal{B}(R_i(M)^*), \tag{5.3.1}$$
$$s_i^M(\operatorname{Supp}\mathcal{B}(M) \cup \operatorname{Supp}\mathcal{B}(M^*)) = \operatorname{Supp}\mathcal{B}(R_i(M)) \cup \operatorname{Supp}\mathcal{B}(R_i(M)^*). \tag{5.3.2}$$

*Proof.* Regarding $\mathcal{B}(M)$ as an $\mathbb{N}_0$-graded object in ${}_H^H\mathcal{YD}$, by [79, Lemma 2.6] the graded dual of $\mathcal{B}(M)$ is isomorphic to $\mathcal{B}(M^*)$. We thus obtain an isomorphism of $\mathbb{N}_0$-graded objects in ${}_H^H\mathcal{YD}$:

$$\mathcal{B}(M) \otimes \mathcal{B}(M^*) \cong K_i \otimes \mathcal{B}(M_i) \otimes K_i^{\text{gr-dual}} \otimes \mathcal{B}(M_i^*).$$

Similarly, for the reflected tuple, we have

$$\mathcal{B}(R_i(M)) \otimes \mathcal{B}(R_i(M)^*) \cong K_i \otimes \mathcal{B}(M_i^*) \otimes K_i^{\text{gr-dual}} \otimes \mathcal{B}(M_i).$$

Since $A \otimes B \cong B \otimes A$ as $\mathbb{Z}^\theta$-graded vector spaces. the supports on both sides coincide, which establishes the first claim.

The second claim follows directly from the definition of $s_i^M$. □

Next, we show that reflections preserve the dimension of Nichols algebras.

**Lemma 5.3.2.** *Suppose $\dim\mathcal{B}(M) < \infty$, then $M$ gives rise to a semi-Cartan graph, and $\dim\mathcal{B}(R_i(M)) = \dim\mathcal{B}(M)$ for each $i \in \mathbb{I}$.*

*Proof.* For each $i \in \mathbb{I}$, as vector spaces, we have the isomorphism $\mathcal{B}(M) \cong K_i \otimes \mathcal{B}(M_i)$. Therefore $K_i$ is finite-dimensional, and $\operatorname{ad}(\mathcal{B}(M_i))(M_j)$ is finite-dimensional for each $j \in \mathbb{I}$. It follows by definition that $M$ admits the $i$-th reflection. Applying the equivalence $\Omega_i$, we obtain $\Omega_i(K_i) = K_i \in {}_{A(M_i^*)}^{A(M_i^*)}\mathcal{YD}$ and $\mathcal{B}(R_i(M)) \cong K_i \otimes \mathcal{B}(M_i^*)$ as vector space. Since $\mathcal{B}(M_i)$ is finite-dimensional, we have $\dim\mathcal{B}(M_i) = \dim\mathcal{B}(M_i^*)$. Therefore $\dim\mathcal{B}(R_i(M)) = \dim\mathcal{B}(M)$ for each $i \in \mathbb{I}$. By iterated reflections, $M$ admits all reflections and gives rise to a semi-Cartan graph. □

Before we state the next lemma, we recall the definition of a finite semi-Cartan graph.

**Definition 5.3.3.** *We denote by* $\mathcal{D}(\mathcal{X}, \mathrm{End}(\mathbb{Z}^{\mathbb{I}}))$ *to be the category with objects* $\mathrm{Ob}\,\mathcal{D}(\mathcal{X}, \mathrm{End}(\mathbb{Z}^{\mathbb{I}})) = \mathcal{X}$*, and morphisms*

$$\mathrm{Hom}(X,Y) = \{(Y, f, X) \mid f \in \mathrm{End}(\mathbb{Z}^{\mathbb{I}})\},$$

*where the composition of morphisms is defined by*

$$(Z,g,Y)\circ(Y,f,X) = (Z, gf, X), \text{ for all } X,Y,Z \in \mathcal{X},\ f,g \in \mathrm{End}(\mathbb{Z}^{\mathbb{I}}).$$

*Let* $\alpha_i$*,* $1 \le i \le \theta$ *be the standard basis of* $\mathbb{Z}^{\mathbb{I}}$*, and*

$$s_i^X \in \mathrm{Aut}(\mathbb{Z}^{\mathbb{I}}),\ s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i, \text{for all } j.$$

*We call the smallest subcategory of* $\mathcal{D}(\mathcal{X}, \mathrm{End}(\mathbb{Z}^{\mathbb{I}}))$ *which contains all morphisms* $(r_i(X), s_i^X, X)$ *with* $i \in \mathbb{I}$*,* $X \in \mathcal{X}$ *the Weyl groupoid of* $\mathcal{G}$*, denoted by* $\mathcal{W}(\mathcal{G})$*.*

**Definition 5.3.4.** *Let* $\mathcal{G} = (\mathbb{I}, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ *be a semi-Cartan graph. For all* $X \in \mathcal{X}$*, the set*

$$\Delta^{X,\mathrm{re}} = \{\omega(\alpha_i) \in \mathbb{Z}^{\mathbb{I}} \mid \omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X), i \in \mathbb{I}\}$$

*is called the set of real roots of* $\mathcal{G}$ *at* $X$*. We call the semi-Cartan graph* $\mathcal{G}$ *is finite, if* $\Delta^{X,\mathrm{re}}$ *is finite for all* $X \in \mathcal{X}$*.*

We now relate the finiteness of the Nichols algebra to the structure of the associated semi-Cartan graph.

**Lemma 5.3.5.** *Suppose* $\mathcal{B}(M)$ *is finite-dimensional, then the semi-Cartan graph* $\mathcal{G}(M)$ *is finite.*

*Proof.* Let $W = M_1 \oplus \cdots \oplus M_\theta$. Clearly, the homogeneous degrees of elements in $W$ lie in the union $W \subseteq \mathrm{Supp}\,\mathcal{B}(M) \cup \mathrm{Supp}\,\mathcal{B}(M^*)$. For $1 \le j \le \theta$, we apply equations (5.3.1),(5.3.2), together with the fact that $R_i^2(M) \cong M$ to obtain

$$\begin{aligned} s_i^{R_i(M)}(\alpha_j) &\subseteq s_i^{R_i(M)}(\mathrm{Supp}\,\mathcal{B}(M) \cup \mathrm{Supp}\,\mathcal{B}(M^*)) \\ &= s_i^{R_i(M)}(\mathrm{Supp}\,\mathcal{B}(R_i(M)) \cup \mathrm{Supp}\,\mathcal{B}(R_i(M^*))) \\ &= \mathrm{Supp}\,\mathcal{B}(M) \cup \mathrm{Supp}\,\mathcal{B}(M^*). \end{aligned}$$

We recall that every element $\omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), M)$ can be expressed as an iteration of simple reflections. It follows that $\Delta^{M,\mathrm{re}} \subseteq \mathrm{Supp}\,\mathcal{B}(M) \cup \mathrm{Supp}\,\mathcal{B}(M^*)$. Now, since $\mathcal{B}(M)$ is finite-dimensional, the set $\mathrm{Supp}\,\mathcal{B}(M)$ is finite. Moreover, as vector spaces, $\mathcal{B}(M) \cong \mathcal{B}(M^*)^{\text{gr-dual}}$, so $\mathrm{Supp}\,\mathcal{B}(M^*)$ is also finite. Combining these observations, we conclude that $\Delta^{M,\mathrm{re}}$ is finite. The same reasoning applies to any $N \in X$: we have

$$\Delta^{N,\mathrm{re}} \subseteq \mathrm{Supp}\,\mathcal{B}(N) \cup \mathrm{Supp}\,\mathcal{B}(N^*).$$

Then $\Delta^{N,\mathrm{re}}$ is again finite by Lemma 5.3.2. Therefore, $\mathcal{G}(M)$ is a finite semi-Cartan graph. □

**Lemma 5.3.6.** [45, Lemma 5.1] *Suppose* $\mathcal{G} = (\mathbb{I}, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ *is a semi-Cartan graph. Then* $\mathcal{G}$ *is finite semi-Cartan graph if and only if the Weyl groupoid* $\mathcal{W}(\mathcal{G})$ *is finite.*

**Definition 5.3.7.** *Suppose $\mathcal{G} = (\mathbb{I}, \mathcal{X}, r, (A^X)_{X\in\mathcal{X}})$ is a semi-Cartan graph. If $A^X = A^Y$ for all $X, Y \in \mathcal{X}$, then $\mathcal{G}$ is called a standard semi-Cartan graph.*

The proof of the next theorem is similar to that in the Hopf algebra situations; we write it down here for completeness.

**Theorem 5.3.8.** *Let $M = (M_1, \ldots, M_\theta) \in \mathcal{F}_\theta$, where each $M_i$ is irreducible for all $i \in \mathbb{I}$. If $\dim \mathcal{B}(M) < \infty$ and $\mathcal{G}(M)$ is standard, then $A^M$ must be a finite Cartan matrix.*

*Proof.* Since $\dim \mathcal{B}(M) < \infty$, Lemma 5.3.5 implies that $\mathcal{G}(M)$ is a finite Semi-Cartan graph. By Lemma 5.3.6, $\mathcal{W}(\mathcal{G}(M))$ is finite, in particular, $\mathrm{Hom}(\mathcal{W}(\mathcal{G}(M)))$ is finite.

Let $W(A^M)$ be the Weyl group of the Kac-Moody Lie algebra of the Cartan matrix $A^M$. Note that

$$W(A^M) = \langle s_i^M \in \mathrm{Aut}(\mathbb{Z}^\theta) \mid 1 \le i \le \theta \rangle.$$

Since $\mathcal{G}(M)$ is standard, we know that

$$\mathrm{Hom}(\mathcal{W}(\mathcal{G}(M))) \longrightarrow W(A^M), \quad (Y, s, X) \mapsto s$$

is well-defined and surjective. Therefore $W(A^M)$ is finite, which implies that $A^M$ is of finite type. □

**Proposition 5.3.9.** *Let $M = (M_1, \ldots, M_\theta) \in \mathcal{F}_\theta$ with each component $M_i$ is irreducible. Suppose $M$ admits $i$-th reflection, then*

$$\mathrm{GKdim}\, \mathcal{B}(M) = \mathrm{GKdim}\, \mathcal{B}(R_i(M)).$$

*Proof.* Since $M_i$ is finite-dimensional, the Nichols algebra $\mathcal{B}(M_i)$ is finitely generated in degree 1. Consequently, the Gelfand-Kirillov dimension is well-defined. Now that $M$ admits $i$-th reflection, we have

$$\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1) = \bigoplus_{j\neq i} (\mathrm{ad}\, \mathcal{B}(M_i))(M_j)$$

is finite-dimensional. We have $\mathcal{B}(M) \simeq \mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)} \# \mathcal{B}(M_i)$ by Theorem 5.2.3. Further, $\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}$ is generated by $\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1)$ by Theorem 4.3.2. Then we have $\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1) + M_i$ generates $\mathcal{B}(M)$ and $\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1) + M_i^*$ generates $\Omega_i(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}) \# \mathcal{B}(M_i^*)$, and

$$(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1) + M_i)^n = \bigoplus_{k=0}^{n} \mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1)^{n-k} \# M_i^k \quad \text{in } \mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)} \# \mathcal{B}(M_i),$$
$$(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1) + M_i^*)^n = \bigoplus_{k=0}^{n} \mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}(1)^{n-k} \# (M_i^*)^k \quad \text{in } \Omega_i(\mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}) \# \mathcal{B}(M_i^*)$$

for all $n$, where $M_i^k$ means $k$-fold product of $M_i$ in $\mathcal{B}(M_i)$. Recall that both $\mathcal{B}(M_i)$ and $\mathcal{B}(M_i^*)$ are finitely generated $\mathbb{N}_0$-graded algebras, and there is a non-degenerate dual pairing between them which is compatible with the grading. Thus $\dim M_i^l = \dim (M_i^*)^l < \infty$ for all $l \in \mathbb{N}_0$. Therefore the definition of GKdim implies that

$$\begin{aligned}\operatorname{GKdim}\mathcal{B}(M) &= \limsup_{n\to\infty}\frac{\log\dim(\mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)}(1)+M_i)^n}{\log n}\\ &= \limsup_{n\to\infty}\frac{\log\dim(\mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)}(1)+M_i^*)^n}{\log n} = \operatorname{GKdim}\Omega_i(\mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)})\#\mathcal{B}(M_i^*).\end{aligned}$$

Hence the theorem holds since $\mathcal{B}(R_i(M)) \simeq \Omega_i(\mathcal{B}(M)^{\operatorname{co}\mathcal{B}(M_i)})\#\mathcal{B}(M_i^*)$ as Hopf algebras in ${}^H_H\mathcal{YD}$. $\square$

# Chapter 6

# On gauge equivalence of twisted quantum double

We study the quantum double of a finite abelian group $G$ twisted by a 3-cocycle and give a sufficient condition for when such a twisted quantum double will be gauge equivalent to an ordinary quantum double of a finite group. Moreover, we will determine when a twisted quantum double of a cyclic group is genuine.

## 6.1 Twisted quantum doubles

We recall the definition of the twisted quantum double for the sake of completeness of the thesis.

**Definition 6.1.1.** *The twisted quantum double $D^{\Phi}(G)$ of a finite group $G$ with respect to the 3-cocycle $\Phi$ on $G$ is the semisimple quasi-Hopf algebra with underlying vector space $(\Bbbk G)^* \otimes \Bbbk G$ in which multiplication, comultiplication $\Delta$, associator $\phi$, counit $\varepsilon$, antipode $\mathcal{S}$, $\alpha$ and $\beta$ are given by*

$$\begin{aligned}
&(e(g)\otimes x)(e(h)\otimes y) = \theta_g(x,y)\delta_{g^x,h}e(g)\otimes xy,\\
&\Delta(e(g)\otimes x) = \sum_{hk=g}\gamma_x(h,k)e(h)\otimes x\otimes e(k)\otimes x,\\
&\phi = \sum_{g,h,k\in G}\Phi(g,h,k)^{-1}e(g)\otimes 1\otimes e(h)\otimes 1\otimes e(k)\otimes 1,\\
&\mathcal{S}(e(g)\otimes x) = \theta_{g^{-1}}(x,x^{-1})^{-1}\gamma_x(g,g^{-1})^{-1}e(x^{-1}g^{-1}x)\otimes x^{-1},\\
&\varepsilon(e(g)\otimes x) = \delta_{g,1}, \quad \alpha = 1, \quad \beta = \sum_{g\in G}\Phi(g,g^{-1},g)e(g)\otimes 1,
\end{aligned}$$

*where $\{e(g)|g\in G\}$ is the dual basis of $\{g\in G\}$, and $\delta_{g,1}$ is the Kronecker delta, $g^x = x^{-1}gx$, and*

$$\begin{aligned}
\theta_g(x,y) &= \frac{\Phi(g,x,y)\Phi(x,y,(xy)^{-1}gxy)}{\Phi(x,x^{-1}gx,y)},\\
\gamma_g(x,y) &= \frac{\Phi(x,y,g)\Phi(g,g^{-1}xg,g^{-1}yg)}{\Phi(x,g,g^{-1}yg)}
\end{aligned}$$

*for any $x,y,g\in G$.*

## 6.2 Module category and categorically Morita equivalence

Module categories are important tools in the theory of tensor categories. It is parallel to module theory over a ring. Its definition is similar to that of a tensor category, see [35, Definition 7.1.1] for explicit definitions. The theory of categorically Morita equivalence is a categorical analogue of Morita equivalence in ring theory, which plays an important role in the theory of module category.

**Definition 6.2.1.** *Let $\mathcal{C}$ be a tensor category with enough projective objects. A module category $\mathscr{M}$ over $\mathcal{C}$ is called exact if for any projective object $P \in \mathcal{C}$ and any object $M \in \mathscr{M}$ the object $P \otimes M$ is projective in $\mathscr{M}$.*

For an exact indecomposable right module category, one can form the dual category $\mathcal{C}^*_{\mathscr{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathscr{M}, \mathscr{M})$, that is, the category of module functors from $\mathscr{M}$ to itself. It is known that $\mathcal{C}^*_{\mathscr{M}}$ is also a tensor category.

**Definition 6.2.2.** *Let $\mathcal{C}$, $\mathcal{D}$ be tensor categories. We will say that $\mathcal{C}$ and $\mathcal{D}$ are categorically Morita equivalent if there is an exact indecomposable $\mathcal{C}$-module category $\mathscr{M}$ and a tensor equivalence $\mathcal{D}^{op} \cong \mathcal{C}^*_{\mathscr{M}}$.*

Here is a basic example of categorically Morita equivalence.

**Example 6.2.3.** Let $G$ be a finite group and let $\mathcal{C} = \mathrm{Vec}_G$. The category Vec is an exact $\mathrm{Vec}_G$-module category via the forgetful tensor functor $\mathrm{Vec}_G \to \mathrm{Vec}$. Consider the dual category $(\mathrm{Vec}_G)^*_{\mathrm{Vec}}$. By definition, a $\mathrm{Vec}_G$-module endofunctor $F$ of Vec consists of a vector space $V := F(\Bbbk)$ and a collection of isomorphisms

$$\gamma_g \in \mathrm{Hom}(F(\delta_g \otimes \Bbbk), \delta_g \otimes F(\Bbbk)) = \mathrm{End}_{\Bbbk}(V).$$

By axioms of module functors, the map $g \mapsto \gamma_g : G \longrightarrow \mathrm{GL}(V)$ is a representation of $G$ on $V$. Conversely, any such representation determines a $\mathrm{Vec}_G$-module endofunctor of Vec. The homomorphisms of representations are precisely morphisms between the corresponding module functors. Thus, $(\mathrm{Vec}_G)^*_{\mathrm{Vec}} \cong \mathrm{Rep}(G)^{\mathrm{op}}$, i.e., the categories $\mathrm{Vec}_G$ and $\mathrm{Rep}(G)$ are categorically Morita equivalent.

## 6.3 On gauge equivalence between $D^{\Phi}(G)$ and $D(G')$

Throughout this section, let $G = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_n}$ with $m_i \mid m_{i+1}$ for $1 \le i \le n-1$ and $\Phi$ be a normalized 3-cocycle with the following form:

$$\Phi\left(g_1^{i_1}\cdots g_n^{i_n}, g_1^{j_1}\cdots g_n^{j_n}, g_1^{k_1}\cdots g_n^{k_n}\right) = \prod_{l=1}^{n} \zeta_{m_l}^{a_l i_l \left[\frac{j_l+k_l}{m_l}\right]} \prod_{1\le s<t\le n} \zeta_{m_s}^{a_{st} k_s \left[\frac{i_t+j_t}{m_t}\right]} \prod_{1\le r<s<t\le n} \zeta_{(m_r,m_s,m_t)}^{a_{rst} k_r j_s i_t}. \quad (6.3.1)$$

We first recall the result of categorically Morita equivalence in [85]:

**Lemma 6.3.1.** [85, Theorem 3.9] *Let $G$ and $G'$ be finite groups, $\eta \in Z^3(G, \mathbb{C}^*)$ and $\hat{\eta} \in Z^3(G', \mathbb{C}^*)$ be normalized 3-cocycles. Then the tensor categories $Vec_G^{\eta}$ and $Vec_{G'}^{\hat{\eta}}$ are categorically Morita equivalent if and only if the following conditions are satisfied:*

(1) *There exist isomorphism of groups:*

$$\phi : H \underset{F}{\bowtie} K \xrightarrow{\cong} G, \quad \phi' : \widehat{H} \underset{\hat{F}}{\bowtie} K \xrightarrow{\cong} G' \quad (6.3.2)$$

*for some finite group $K$ acting on the abelian normal group $H$, with $F \in Z^2(K,H)$ and $\hat{F} \in Z^2(K,\widehat{H})$ where $\widehat{H} := \mathrm{Hom}(H,\mathbb{C}^*)$.*

(2) *There exists $\varepsilon : K^3 \longrightarrow \mathbb{C}^*$ such that*

$$\hat{F} \wedge F = \delta_K \varepsilon. \tag{6.3.3}$$

*Here $\hat{F} \wedge F(k_1,k_2,k_3,k_4) := \hat{F}(k_1,k_2)(F(k_3,k_4))$.*

(3) *The cohomology classes satisfy the equations $[\phi^*\eta] = [\Phi]$ and $[\phi'^*\hat{\eta}] = [\Phi']$ with*

$$\begin{aligned} \Phi((h_1,k_1),(h_2,k_2),(h_3,k_3)) &:= \hat{F}(k_1,k_2)(h_3)\varepsilon(k_1,k_2,k_3), \\ \Phi'((\rho_1,k_1),(\rho_2,k_2),(\rho_3,k_3)) &:= \varepsilon(k_1,k_2,k_3)\rho_1(F(k_2,k_3)). \end{aligned} \tag{6.3.4}$$

For simplicity, we will regard $\Phi$ (resp. $\Phi'$) as a normalized 3-cocycle on $G$ and $H \underset{\hat{F}}{\rtimes} K$ (resp. $G'$ and $\widehat{H} \underset{\hat{F}}{\rtimes} K$ ) simultaneously in what follows. A simple but useful application of this lemma is the following:

**Corollary 6.3.2.** *Let $G$ be a finite abelian group. If $\mathrm{Vec}_G^\Phi$ is categorically Morita equivalent to $\mathrm{Vec}_{G'}$ for a finite group $G'$, then*

(i) *The choice of $\varepsilon$ in Lemma 6.3.1 must be $\varepsilon(k_1,k_2,k_3) = 1$ for all $k_1$, $k_2$, $k_3 \in K$.*

(ii) *The crossed product $G = H \underset{F}{\rtimes} K$ in Lemma 6.3.1 is actually a direct product. That is, the decomposition of $G$ must be of the form $G = H \times K$ for an abelian normal subgroup $H$.*

*Proof.* Suppose $\mathrm{Vec}_G^\Phi$ is categorically Morita equivalence to $\mathrm{Vec}_{G'}$. By Lemma 6.3.1, There exists isomorphism of groups: $H \underset{F}{\rtimes} K \xrightarrow{\cong} G$, $\widehat{H} \underset{\hat{F}}{\rtimes} K \xrightarrow{\cong} G'$ for abelian normal subgroup $H$ of $G$. By assumption, the normalized 3-cocycle $\Phi'$ of $G'$ is trivial. That is

$$\Phi'((\rho_1,k_1),(\rho_2,k_2),(\rho_3,k_3)) = \varepsilon(k_1,k_2,k_3)\rho_1(F(k_2,k_3)) \equiv 1$$

for all $k_1,k_2,k_3 \in K$ and $\rho_1,\rho_2,\rho_3 \in \widehat{H}$.

We first assume $\varepsilon$ is nontrivial, then there exist $k',k'',k''' \in K$ such that $\varepsilon(k',k'',k''') \neq 1$, then

$$\Phi'\left((1_{\widehat{H}},k'),(\rho_2,k''),(\rho_3,k''',)\right) = \varepsilon(k',k'',k''')1_{\widehat{H}}(F(k_1,k_2)) = \varepsilon(k',k'',k''') \neq 1.$$

This implies $\Phi'$ will never be identically equal to $1$, which is a contradiction.

Suppose the crossed product is not a direct product. Then $F \in Z^2(K,H)$ is nontrivial, and there exist $k'$, $k'' \in K$ such that $F(k',k'') \neq 1_H$. So we can choose a character $\rho \in \hat{H}$ such that $\rho(F(k',k'')) \neq 1$ and consider the ratio

$$\frac{\Phi'\left((\rho,k_1),(1_{\hat{H}},k_2),(1_{\hat{H}},k_3)\right)}{\Phi'\left((1_{\hat{H}},k_1),(1_{\hat{H}},k_2),(1_{\hat{H}},k_3)\right)} = \rho(F(k_1,k_2)) \neq 1.$$

Thus one of the values of $\Phi'$ cannot be one. This leads to a contradiction as well. □

Keeping the notation above, we will give a sufficient condition for categorically Morita equivalence between $\mathrm{Vec}_G^\Phi$ and $\mathrm{Vec}_{G'}$ in this subsection. Now let $G = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_n}$ with $m_i \mid m_{i+1}$ for $1 \leq i \leq n-1$. The 3-cocycle $\Phi$ as in (6.3.1).

$$\underline{a} = (a_1,a_2,\ldots,a_l,\ldots,a_n,a_{12},a_{13},\ldots,a_{st},\ldots,a_{n-1,n},a_{123},\ldots,a_{rst},\ldots a_{n-2,n-1,n}) \in \mathscr{A}$$

where $0 \le a_l < m_l$, $0 \le a_{st} < (m_s, m_t)$, $0 \le a_{rst} < (m_r, m_s, m_t)$. For a fixed $\underline{a} \in \mathscr{A}$, define the following sets:

$$A_1 := \left\{i | a_{ij} \ne 0, 1 \le i < j \le n\right\}, \qquad A_2 := \left\{i | a_{ijk} \ne 0, 1 \le i < j < k \le n\right\},$$
$$B_1 := \left\{j | a_{ij} \ne 0, 1 \le i < j \le n\right\}, \qquad B_2 := \left\{j, k | a_{ijk} \ne 0, 1 \le i < j < k \le n\right\}.$$

Let $A = A_1 \cup A_2$, $B = B_1 \cup B_2$.

**Theorem 6.3.3.** *Let $G$ be a finite abelian group and $\Phi$ is a normalized 3-cocycle on $G$ as in (2.6.1). If*

(i) *$a_i = 0$ for all $1 \le i \le n$,*
(ii) *$A \cap B = \emptyset$.*

*Then $\mathrm{Vec}_G^{\Phi}$ is categorically Morita equivalent to $\mathrm{Vec}_{G'}$ for a finite group $G'$.*

*Proof.* Let $a_i = 0$ for all $1 \le i \le n$ and $A \cap B = \emptyset$. Denote $I = \{1, 2, \cdots, n\}$. Clearly $A, B \subset I$. Now take $H = \prod_{i \in A} Z_{m_i} = \prod_{i \in A} \langle g_i \rangle$, and $K = \prod_{j \in I \setminus A} Z_{m_j} = \prod_{j \in I \setminus A} \langle g_j \rangle$, then $G \cong H \times K$. Define

$$\hat{F}\left(\prod_{m \in I \setminus A} g_m^{i_m}, \prod_{m \in I \setminus A} g_m^{j_m}\right) = \prod_{\substack{p<q \\ p \in A_1, q \in B_1}} (\chi_p)^{a_{pq}[\frac{i_q + j_q}{m_q}]} \prod_{\substack{r<s<t \\ r \in A_2, s, t \in B_2}} (\chi_r)^{\frac{a_{rst} m_r}{(m_r, m_s, m_t)} j_s i_t}$$

where $\chi_p \in \widehat{Z_{m_p}}$ is primitive. $\hat{F}$ lies in $Z^2(K, \hat{H})$ by direct computation. We are now going to show $\mathrm{Vec}_G^{\Phi}$ is categorically Morita equivalent to $\mathrm{Vec}_{H \rtimes_{\hat{F}} K}$ by Lemma 6.3.1:

Equation (6.3.2) is satisfied. If we set $\varepsilon : K^3 \to k^{\times}$ to be identically 1, then (6.3.3) is satisfied since $\hat{F} \wedge F(k_1, k_2, k_3, k_4) = \hat{F}(k_1, k_2)(F(k_3, k_4)) = 1 = \delta_K \varepsilon$. Note

$$\begin{aligned}\Phi\left(g_1^{i_1} \cdots g_n^{i_n}, g_1^{j_1} \cdots g_n^{j_n}, g_1^{k_1} \cdots g_n^{k_n}\right) &= \prod_{1 \le p < q \le n} \zeta_{m_p}^{a_{pq} k_p \left[\frac{i_q + j_q}{m_q}\right]} \prod_{1 \le r < s < t \le n} \zeta_{(m_r, m_s, m_t)}^{a_{rst} k_r j_s i_t} \\ &= \prod_{\substack{p<q \\ p \in A_1, q \in B_1}} \zeta_{m_p}^{a_{pq} k_p [\frac{i_q + j_q}{m_q}]} \prod_{\substack{r<s<t \\ r \in A_2, s, t \in B_2}} \zeta_{(m_r, m_s, m_t)}^{a_{rst} k_r j_s i_t}\end{aligned}$$

since $a_{pq} = 0$ if $p \notin A_1$ or $q \notin B_1$ and $a_{rst} = 0$ if $r \notin A_2$ or $s \notin B_2$ or $t \notin B_2$. On the other hand

$$\begin{aligned}\hat{F}\left(\prod_{n \in I \setminus A} g_n^{i_n}, \prod_{n \in I \setminus A} g_n^{j_n}\right)\left(\prod_{m \in A} g_m^{k_m}\right) &= \prod_{\substack{p<q \\ p \in A_1, q \in B_1}} \zeta_{m_p}^{a_{pq} k_p [\frac{i_q + j_q}{m_q}]} \prod_{\substack{r<s<t \\ r \in A_2, s, t \in B_2}} \zeta_{m_r}^{a_{rst} \frac{m_r k_r j_s i_t}{(m_r, m_s, m_t)}} \\ &= \prod_{\substack{p<q \\ p \in A_1, q \in B_1}} \zeta_{m_p}^{a_{pq} k_p [\frac{i_q + j_q}{m_q}]} \prod_{\substack{r<s<t \\ r \in A_2, s, t \in B_2}} \zeta_{(m_r, m_s, m_t)}^{a_{rst} k_r j_s i_t}.\end{aligned}$$

Hence

$$\begin{aligned}\Phi\left(g_1^{i_1}\cdots g_n^{i_n},g_1^{j_1}\cdots g_n^{j_n},g_1^{k_1}\cdots g_n^{k_n}\right)&=\Phi((\prod_{m\in A}g_m^{i_m},\prod_{n\in I\setminus A}g_n^{i_n}),(\prod_{m\in A}g_m^{j_m},\prod_{n\in I\setminus A}g_n^{j_n}),(\prod_{m\in A}g_m^{k_m},\prod_{n\in I\setminus A}g_n^{k_n}))\\&=\prod_{\substack{p<q\\p\in A_1,q\in B_1}}\zeta_{m_p}^{a_{pq}k_p[\frac{i_q+j_q}{m_q}]}\prod_{\substack{r<s<t\\r\in A_2,s,t\in B_2}}\zeta_{(m_r,m_s,m_t)}^{a_{rst}k_rj_si_t}\\&=\hat{F}(\prod_{n\in I\setminus A}g_n^{i_n},\prod_{n\in I\setminus A}g_n^{j_n})(\prod_{m\in A}g_m^{k_m}).\end{aligned}$$

Thus the first equation of (6.3.4) has been verified.

Since $G\cong H\times K=H\underset{F}{\rtimes}K$ where $F(k_1,k_2)=1_H$ for all $k_1,k_2\in K$. Then $\rho(F(k_1,k_2))=1$ for all $\rho\in\hat{H}$ and $k_1,k_2\in K$. Thus

$$\Phi'((\rho_1,k_1),(\rho_2,k_2),(\rho_3,k_3))=1=\rho_1(F(k_2,k_3)).$$

We have verified all conditions in Lemma 6.3.1. Hence $\mathrm{Vec}_G^\Phi$ is categorically Morita equivalent to $\mathrm{Vec}_{\hat{H}\underset{\hat{F}}{\rtimes}K}$ if $a_i=0$ for all $1\le i\le n$ and $A\cap B=\emptyset$. □

This theorem implies Theorem 6.3.4 directly.

**Theorem 6.3.4.** *Let $G$ be a finite abelian group and $\Phi$ a normalized 3-cocycle on $G$ as in (2.6.1). If the following condition holds:*

(i) *$a_i=0$ for all $1\le i\le n$.*

(ii) *$A\cap B=\emptyset$.*

*Then $D^\Phi(G)$ will be gauge equivalent to $D(G')$ for a finite group $G'$.*

*Proof.* According to Theorem 6.3.3, if $a_i=0$ for all $1\le i\le n$ and $A\cap B=\emptyset$, then $\mathrm{Vec}_G^\Phi$ is categorically Morita equivalent to $\mathrm{Vec}_{G'}$ for some finite group $G'$. By [35, Theorem 3.1], the centers of these two fusion categories are braided equivalent. It is known that the center is equivalent to the representation category of the corresponding Drinfeld double [84]. That is, $\mathrm{Rep}(D^\Phi(G))$ is braided tensor equivalent to $\mathrm{Rep}(D(G'))$. Hence $D^\Phi(G)$ will be gauge equivalent to $D(G')$ by Theorem 2.2 in [82]. □

A natural question is under what conditions $G'$ can be a finite abelian group. The following corollary provides the answer.

**Corollary 6.3.5.** *Let $G$ be a finite abelian group and $\Phi$ a normalized 3-cocycle on $G$ as above. Then $\mathrm{Vec}_G^\Phi$ is categorically Morita equivalent to $\mathrm{Vec}_{G'}$ for a finite abelian group $G'$ if*

(1) *$a_i=0$ for all $1\le i\le n$ and $a_{rst}=0$ for all $1\le r<s<t\le n$,*

(2) *$A_1\cap B_1=\emptyset$.*

*Proof.* We first assume the condition (i) and (ii) in Theorem 6.3.3 hold. Thus there's a finite group $G'$ such that $\mathrm{Vec}_G^\Phi$ is categorically Morita equivalence to $\mathrm{Vec}_{G'}$. By construction, $G'\cong$

$(\prod_{i\in A} Z_{m_i}) \rtimes_{\hat{F}} (\prod_{j\in I\setminus A} Z_{m_j})$ where $\hat{F}$ is defined to be

$$\hat{F}\left(\prod_{m\in I\setminus A} g_m^{i_m}, \prod_{m\in I\setminus A} g_m^{j_m}\right) = \prod_{\substack{p<q\\ p\in A_1, q\in B_1}} (\chi_p)^{a_{pq}[\frac{i_q+j_q}{m_q}]} \prod_{\substack{r<s<t\\ r\in A_2, s,t\in B_2}} (\chi_r)^{\frac{a_{rst}m_r}{(m_r,m_s,m_t)} j_s i_t}.$$

Then $G'$ is abelian $\Leftrightarrow (\prod_{m\in A} g_m^{i_m}, \prod_{n\in I\setminus A} g_n^{i_n})\cdot(\prod_{m\in A} g_m^{j_m}, \prod_{n\in I\setminus A} g_n^{j_n}) = (\prod_{m\in A} g_m^{j_m}, \prod_{n\in I\setminus A} g_n^{j_n})\cdot(\prod_{m\in A} g_m^{i_m}, \prod_{n\in I\setminus A} g_n^{i_n})$

$$\Leftrightarrow \hat{F}\left(\prod_{n\in I\setminus A} g_n^{i_n}, \prod_{n\in I\setminus A} g_n^{j_n}\right) = \hat{F}\left(\prod_{n\in I\setminus A} g_n^{j_n}, \prod_{n\in I\setminus A} g_n^{i_n}\right)$$

$$\Leftrightarrow \prod_{\substack{p<q\\ p\in A_1, q\in B_1}} (\chi_p)^{a_{pq}[\frac{i_q+j_q}{m_q}]} \prod_{\substack{r<s<t\\ r\in A_2, s,t\in B_2}} (\chi_r)^{\frac{a_{rst}m_r j_s i_t}{(m_r,m_s,m_t)}} = \prod_{\substack{p<q\\ p\in A_1, q\in B_1}} (\chi_p)^{a_{pq}[\frac{j_q+i_q}{m_q}]} \prod_{\substack{r<s<t\\ r\in A_2, s,t\in B_2}} (\chi_r)^{\frac{a_{rst}m_r i_s j_t}{(m_r,m_s,m_t)}}$$

$$\Leftrightarrow A_2, B_2 = \emptyset.$$

This is equivalent to $a_{rst} = 0$ for all $1 \le r < s < t \le n$. Thus if (1) and (2) hold, then $G'$ is abelian. □

If $G$ is a cyclic group, then conditions (i),(ii) in Theorem 6.3.3 are also necessary. In fact, for any cyclic group $G \cong Z_m =< g | g^m = 1 >$ with a normalized 3-cocycle $\Phi_a$ given by $\Phi_a(g^i, g^j, g^k) = \zeta_m^{ai[\frac{j+k}{m}]}$, where $0 \le a, i, j, k < m$, we have the following proposition.

**Proposition 6.3.6.** *The fusion category* $\mathrm{Vec}_G^{\Phi_a}$ *is categorically Morita equivalent to* $\mathrm{Vec}_{G'}$ *for a finite group* $G'$ *if and only if* $a = 0$.

*Proof.* The sufficiency follows from Theorem 6.3.3. Now suppose that $\mathrm{Vec}_G^{\Phi_a}$ is categorically Morita equivalent to $\mathrm{Vec}_{G'}$ for a finite group $G'$. By Corollary 6.3.2, $G$ must be direct product of two subgroups, like $G \cong H \times K$ and the function $\varepsilon$ should be 1. Since $G$ is cyclic, then $H$ and $K$ must be cyclic subgroups. Moreover, $|H|$ should be coprime to $|K|$, hence $\mathrm{H}^2(K, \hat{H}) = \{1\}$. Thus $\Phi_a$ should be 1 by formula (6.3.4). That is, $a = 0$. □

But in general the conditions (i) and (ii) in Theorem 6.3.3 both are not necessary as the following example shows.

**Example 6.3.7.** Let $G \cong Z_2 \times Z_2 \times Z_2 = \langle g_1\rangle \times \langle g_2\rangle \times \langle g_3\rangle$ and

$$\Phi\left(g_1^{i_1}g_2^{i_2}g_3^{i_3}, g_1^{j_1}g_2^{j_2}g_3^{j_3}, g_1^{k_1}g_2^{k_2}g_3^{k_3}\right) = (-1)^{i_2[\frac{j_2+k_2}{2}]}(-1)^{k_1[\frac{i_2+j_2}{2}]}(-1)^{k_1[\frac{i_3+j_3}{2}]}(-1)^{k_2[\frac{i_3+j_3}{2}]}.$$

In this case, $\underline{a} = (0,1,0,1,1,1,0) \in \mathscr{A}$, $a_2 \ne 0$ and $A_1 \cap B_1 \ne 0$.

Take $H \cong Z_2 = \langle g_1\rangle$ and $K \cong Z_2 \times Z_2 = \langle g_1 g_2\rangle \times \langle g_3\rangle$. Obviously, $G \cong H \times K$. Define

$$\hat{F}: K \times K \to \hat{H}, \quad \hat{F}((g_1g_2)^{i_2}, g_3^{i_3}, (g_1g_2)^{j_2}, g_3^{j_3}) = \chi_1^{[\frac{i_2+j_2}{2}]}\chi_1^{[\frac{i_3+j_3}{2}]},$$

where $\chi_1$ generates $\hat{H}$. Let $G' = \hat{H} \rtimes_{\hat{F}} K$, we are going to show $\mathrm{Vec}_G^{\Phi}$ is categorically Morita equivalent to $\mathrm{Vec}_{G'}$.

Define $\varepsilon : K^3 \longrightarrow \mathbb{C}^*$ as $\varepsilon \equiv 1$, then equation (6.3.3) holds. Note that

$$\Phi(g_1^{(i_1-i_2)'}((g_1g_2)^{i_2}, g_3^{i_3}), g_1^{(j_1-j_2)'}((g_1g_2)^{j_2}g_3^{j_3}), g_1^{(k_1-k_2)'}((g_1g_2)^{k_2}g_3^{k_3}))$$
$$= \Phi\left(g_1^{i_1}g_2^{i_2}g_3^{i_3}, g_1^{j_1}g_2^{j_2}g_3^{j_3}, g_1^{k_1}g_2^{k_2}g_3^{k_3}\right) = (-1)^{i_2[\frac{j_2+k_2}{2}]}(-1)^{k_1[\frac{i_2+j_2}{2}]}(-1)^{k_1[\frac{i_3+j_3}{2}]}(-1)^{k_2[\frac{i_3+j_3}{2}]},$$

and

$$\widehat{F}\left(((g_1g_2)^{i_2}, g_3^{i_3}), ((g_1g_2)^{j_2}, g_3^{j_3})\right)(g_1^{(k_1-k_2)'}) = (-1)^{(k_1-k_2)'[\frac{i_2+j_2}{2}]}(-1)^{(k_1-k_2)'[\frac{i_3+j_3}{2}]}$$

for $0 \le i_1, i_2, j_1, j_2, k_1, k_2 \le 1$. Actually,

$$\begin{aligned}
&\frac{(-1)^{i_2[\frac{j_2+k_2}{2}]}(-1)^{k_1[\frac{i_2+j_2}{2}]}(-1)^{k_1[\frac{i_3+j_3}{2}]}(-1)^{k_2[\frac{i_3+j_3}{2}]}}{(-1)^{(k_1-k_2)'[\frac{i_2+j_2}{2}]}(-1)^{(k_1-k_2)'[\frac{i_3+j_3}{2}]}} \\
&= \frac{(-1)^{i_2[\frac{j_2+k_2}{2}]}(-1)^{k_1[\frac{i_2+j_2}{2}]}(-1)^{k_1[\frac{i_3+j_3}{2}]}(-1)^{k_2[\frac{i_3+j_3}{2}]}}{(-1)^{k_1[\frac{i_2+j_2}{2}]}(-1)^{-k_2[\frac{i_2+j_2}{2}]}(-1)^{k_1[\frac{i_3+j_3}{2}]}(-1)^{-k_2[\frac{i_3+j_3}{2}]}} \\
&= (-1)^{k_2[\frac{i_2+j_2}{2}]}\cdot(-1)^{i_2[\frac{j_2+k_2}{2}]} = 1.
\end{aligned}$$

Thus $\Phi\left(g_1^{i_1}g_2^{i_2}g_3^{i_3}, g_1^{j_1}g_2^{j_2}g_3^{j_3}, g_1^{k_1}g_2^{k_2}g_3^{k_3}\right) = \widehat{F}\left(((g_1g_2)^{i_2}, g_3^{i_3}), ((g_1g_2)^{j_2}, g_3^{j_3})\right)(g_1^{(k_1-k_2)'})$ and the first equation in (6.3.4) holds. Obviously, if we define the 3-cocycle $\Phi'$ on $G'$ as

$$\Phi' \equiv 1.$$

Then the second equation in (6.3.4) holds. Hence we have proved $\mathrm{Vec}_G^{\Phi}$ is categorically Morita equivalent to $\mathrm{Vec}_{G'}$.

## 6.4 The structure of $D^{\Phi}(G)$

In the article [31], the authors asked whether or not $D^{\Phi}(G)$ can be obtained by twisting a Hopf algebra. In [77, Example 9.5], the authors have shown that $D^{\Phi}(Z_2)$ is genuine for $\Phi$ being the normalized 3-cocycle on $G$ whose cohomology class is nontrivial. This gives a negative answer to the above question. In this section, we will investigate when a twisted quantum double of a cyclic group is genuine, that is, it cannot be obtained by twisting a Hopf algebra. Let $G$ be an abelian group and $\Phi$ an abelian 3-cocycle on $G$. Let $\Gamma^{\Phi}$ be the group of all group-like elements in $D^{\Phi}(G)$, and denote $\Phi_g(x,y) = \frac{\Phi(g,x,y)\Phi(x,y,g)}{\Phi(x,g,y)}$ for $g,x,y \in G$.

**Lemma 6.4.1.** [77, Corollary 3.6] *With the notation above, $D^{\Phi}(G)$ is spanned by the set of group-like elements $\Gamma^{\Phi}$ and it is a commutative algebra. In particular, $\Phi_g$ is a 2-coboundary for any $g \in G$.*

Moreover, $\Gamma^{\Phi}$ can be seen as an abelian extension, which may help us to figure out the explicit structure of $D^{\Phi}(G)$.

**Lemma 6.4.2.** [77 Proposition 3.8] *Let $\hat{G}$ be the character group of $G$, then $\Gamma^{\Phi}$ is an extension*

$$1 \longrightarrow \hat{G} \longrightarrow \Gamma^{\Phi} \longrightarrow G \longrightarrow 1. \tag{6.4.1}$$

*For each $g \in G$, let $\Phi_g = \delta\tau_g$ for a 1-cochain $\tau_g : G \to \mathbb{C}^\times$. The 2-cocycle $\beta$ associated to this central extension is given by*

$$\beta(x,y)(g) = \frac{\tau_x(g)\tau_y(g)}{\tau_{xy}(g)}\Phi_g(x,y). \tag{6.4.2}$$

From now on, let $G = Z_m = \langle g \rangle$ be a finite cyclic group and $\Phi(g^i, g^j, g^k) = \zeta_m^{ai[\frac{j+k}{m}]}$ be a nontrivial normalized 3-cocycle. In this case, $\widehat{G} = \widehat{Z_m} = \langle \chi \rangle$, where $\chi(g) = \zeta_m$. We will determine when $D^\Phi(G)$ is genuine. The first task is to figure out the group structure of $\Gamma^\Phi$. Since $\Gamma^\Phi$ is totally determined by $D^\Phi(G)$, it is independent of the choice of $\tau_x$ for each $x \in G$.

**Lemma 6.4.3.** *Let $\tau_{g^i}(g^j) = \zeta_{m^2}^{aij}$ for all $0 \le i,j \le m$. then $\delta\tau_{g^i} = \Phi_{g^i}$. Further, $\beta(g^i,g^j) = \chi^{2a[\frac{i+j}{m}]}$ in this case.*

*Proof.* Direct computation shows that

$$\delta\tau_{g^i}(g^j,g^k) = \frac{\tau_{g^i}(g^j)\tau_{g^i}(g^k)}{\tau_{g^i}(g^{(j+k)'})} = \frac{\zeta_{m^2}^{aij}\zeta_{m^2}^{aik}}{\zeta_{m^2}^{ai(j+k)'}} = \zeta_m^{ai[\frac{j+k}{m}]} = \Phi_{g^i}(g^j,g^k).$$

Here $(j+k)'$ denotes the reminder of $j+k$ modulo $m$. Now

$$\beta(g^j,g^k)(g^i) = \frac{\tau_{g^j}(g^i)\tau_{g^k}(g^i)}{\tau_{g^{(j+k)}}(g^i)}\Phi_{g^i}(g^j,g^k) = \zeta_m^{2ai[\frac{j+k}{m}]}.$$

Hence $\beta(g^i,g^j) = \chi^{2a[\frac{i+j}{m}]}$. □

Note that $\Gamma^\Phi$ consists of all group-like elements, hence it is beneficial to explicit formulas of all group-like elements. By [77], a nonzero element $u$ in $D^\Phi(G)$ is a group-like element if and only if

$$u = \sigma_\tau(\alpha,x) = \sum_{g\in G}\alpha(g)\tau_x(g)e(g)\otimes x. \tag{6.4.3}$$

for $\alpha \in \widehat{G}$ and $x \in G$. Here we have assumed $G$ is a cyclic group, we can simplify the expression of $\sigma_\tau(\alpha,x)$.

**Lemma 6.4.4.** (i) *We have $\sigma_\tau(\chi^j,1) = \chi^j \otimes 1$ and $\sigma_\tau(\chi^j,g) = \sum_{i=0}^{m-1}\zeta_{m^2}^{ai}\zeta_m^{ij}e(g^i)\otimes g$, where $0 \le j \le m-1$.*

(ii) *Let $s = \sigma_\tau(\chi,1), t = \sigma_\tau(1,g)$, then $\Gamma^\Phi$ has the following presentation:*

$$\left\langle s,t \middle| t^{\frac{m^2}{(m,2a)}} = s^m = 1,\ s^{2a} = t^m,\ st = ts\right\rangle. \tag{6.4.4}$$

*Proof.* First, by direct computation

$$e(g^i) = 1_i = \frac{1}{m}\sum_{l=0}^{m-1}\zeta_m^{-li}\chi^l, \tag{6.4.5}$$

$$\chi^j = \sum_{i=0}^{m-1}\zeta_m^{ij}e(g^i). \tag{6.4.6}$$

Then

$$\sigma_\tau(\chi^j,1) = \sum_{i=0}^{m-1}\chi^j(g^i)\tau_1(g^i)e(g^i)\otimes 1 = \sum_{i=0}^{m-1}\zeta_m^{ij}e(g^i)\otimes 1 = \chi^j\otimes 1,$$

and

$$\sigma_\tau(\chi^j,g) = \sum_{i=0}^{m-1}\chi^j(g^i)\tau_g(g^i)e(g^i)\otimes g = \sum_{i=0}^{m-1}\zeta_{m^2}^{ai}\zeta_m^{ij}e(g^i)\otimes g.$$

By multiplication rule of twisted quantum double,

$$\sigma_\tau(1,g)\cdot\sigma_\tau(\chi,1) = \sigma_\tau(\chi,g) = \sigma_\tau(\chi,1)\cdot\sigma_\tau(1,g).$$

Suppose $0 \le l \le m-1$, we have

$$\sigma_\tau(1,g)^l = \sum_{i=0}^{m-1}\zeta_{m^2}^{ail}e(g^i)\otimes g^l = \sigma_\tau(1,g^l).$$

Moreover,

$$\sigma_\tau(1,g)^m = \sum_{i=0}^{m-1}\zeta_{m^2}^{mai}e(g^i)\theta_{g^i}(g^{m-1},g)\otimes 1 = \sum_{i=0}^{m-1}\zeta_m^{2ai}e(g^i)\otimes 1 = \chi^{2a}\otimes 1 = \sigma_\tau(\chi,1)^{2a}.$$

It is easy to verify that $\sigma_\tau(\chi,1)^m = 1$ and thus $\sigma_\tau(\chi,1)^{2a}$ has order $\frac{m}{(m,2a)}$. This implies that $\sigma_\tau(1,g)$ has order $\frac{m^2}{(m,2a)}$. Obviously, each $\sigma_\tau(\chi^j,g^k)$ can be expressed as a production of some powers of $\sigma_\tau(1,g)$ and $\sigma_\tau(\chi,1)$. Thus we get the desired presentation of $\Gamma^\Phi$. □

$\Gamma^\Phi$ is actually a metacyclic group, for details, see [56]. In general, it is not easy to determine the group structure of $\Gamma^\Phi$, while in our case $\Gamma^\Phi$ can be obtained without much difficulty.

**Proposition 6.4.5.** *We have* $\Gamma^\Phi \cong Z_{(2a,m)}\times Z_{\frac{m^2}{(2a,m)}}$.

*Proof.* It is obvious that $\Gamma^\Phi$ is an abelian group and has order $m^2$. By the presentation of $\Gamma^\Phi$, the number of generators of $\Gamma^\Phi$, must be equal to or less than 2. Thus we may write $\Gamma^\Phi \cong Z_{m_1}\times Z_{m_2}$, where $m_1 \mid m_2$. Consider the element $\sigma_\tau(1,g)$ and we know that its order is $\frac{m^2}{(2a,m)}$. Hence $\Gamma^\Phi$ has a cyclic subgroup $\langle\sigma_\tau(1,g)\rangle$ of order $\frac{m^2}{(2a,m)}$. If $(2a,m)=1$, then $\sigma_\tau(1,g)$ has order $m^2$. So $\Gamma^\Phi \cong Z_{m^2} = \langle\sigma_\tau(1,g)\rangle$. Actually, we may regard it as $Z_1\times Z_{m^2}$ for consistency.

If $(2a,m)\neq 1$, then $\frac{m^2}{(2a,m)}$ is strictly less than $m^2$. We claim that for an arbitrary element $h = \sigma_\tau(\chi^i,g^j)$, $0\le i,j<m$, the order of $h$ will be less than or equal to $\frac{m^2}{(2a,m)}$. The case $i=j=0$ is trivial and for the case $i\neq 0$ but $j=0$, we haveord$(h) = \frac{m}{(m,i)} \le m \le \frac{m^2}{(m,2a)}$. The remaining case is when $j\neq 0$, by direct computation.

$$\begin{aligned} h^{\frac{m^2}{(m,2aj)}} &= (\sigma_\tau(\chi,1)^{im}\cdot\sigma_\tau(1,g)^{jm})^{\frac{m}{(m,2aj)}} \\ &= (\sigma_\tau(\chi,1)^{2aj})^{\frac{m}{(m,2aj)}} \\ &= 1. \end{aligned}$$

So $\mathrm{ord}(h) \le \frac{m^2}{(m,2aj)}$. Note that $(m,2aj)\ge(m,2a)$, hence $\frac{m^2}{(m,2aj)} \le \frac{m^2}{(m,2a)}$. So $\langle\sigma_\tau(1,g)\rangle$ is a maximal subgroup of $\Gamma^\Phi$. Since $\Gamma^\Phi \cong Z_{m_1} \times Z_{m_2}$ with $m_1 \mid m_2$, $\langle \sigma_\tau(1,g) \rangle$ must be isomorphic to $Z_{m_2}$. Hence $Z_{m_1}$ has order $(2a,m)$. So $\Gamma^\Phi \cong Z_{(2a,m)} \times Z_{\frac{m^2}{(2a,m)}}$. □

Let us recall an approach to determine whether a 3-cohomology on a finite abelian group is nontrivial or not.

Let $H \cong Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_n}$ be a finite abelian group and $(B_\bullet, \partial_\bullet)$ be the bar resolution of $H$. By applying $\mathrm{Hom}_{\mathbb{Z}H}(-, \Bbbk^\times)$ we get a complex $(B_\bullet^*, \partial_\bullet^*)$, where $\Bbbk^\times$ is a trivial $H$-module. In [58], the authors defined another free resolution $(K_\bullet, d_\bullet)$ for arbitrary abelian groups and constructed a chain map $F_\bullet$ from $(K_\bullet, d_\bullet)$ to $(B_\bullet, \partial_\bullet)$. For our purpose, we only need the morphism $F_3$, see [58 Lemma 3.9] :

$$
\begin{aligned}
&F_3 : K_3 \to B_3, \\
&\Psi_{r,s,t} \mapsto [g_r, g_s, g_t] - [g_s, g_r, g_t] - [g_r, g_t, g_s] + [g_t, g_r, g_s] + [g_s, g_t, g_r] - [g_t, g_s, g_r], \\
&\Psi_{r,r,s} \mapsto \sum_{l=0}^{m_r-1} \left( \left[g_r^l, g_r, g_s\right] - \left[g_r^l, g_s, g_r\right] + \left[g_s, g_r^l, g_r\right] \right), \\
&\Psi_{r,s,s} \mapsto \sum_{l=0}^{m_s-1} \left( \left[g_r, g_s^l, g_s\right] - \left[g_s^l, g_r, g_s\right] + \left[g_s^l, g_s, g_r\right] \right), \\
&\Psi_{r,r,r} \mapsto \sum_{l=0}^{m_r-1} \left[g_r, g_r^l, g_r\right],
\end{aligned}
$$

for $1 \le r < s < t \le n$, where the symbols like $\Psi_{r,r,r}$ are terms in the resolutions $(K_\bullet, d_\bullet)$. Moreover, we have the following observation since $F_3^*$ induces an isomorphism between 3-cohomology groups.

**Lemma 6.4.6.** *Let $\phi$ in $(B_\bullet^*, \partial_\bullet^*)$ be a 3-cocycle. Then $\phi$ is a 3-coboundary if and only if $F_3^*(\phi)$ is a 3-coboundary.*

The following lemma provides a criterion for whether a 3-cochain $f \in \mathrm{Hom}_{\mathbb{Z}H}(K_3, \Bbbk^\times)$ is 3-coboundary.

**Lemma 6.4.7.** [58 Lemma 3.3] *The 3-cochain $f \in \mathrm{Hom}_{\mathbb{Z}H}(K_3, \Bbbk^\times)$ is 3-coboundary if and only if for all $1 \le i < j \le n$, there are $g_{i,j} \in \Bbbk^\times$ such that*

$$
f(\Psi_{i,,i,j}) = g_{i,j}^{m_i}, \quad f(\Psi_{i,j,j}) = g_{i,j}^{-m_j}, \quad \textit{and} \quad f(\Psi_{l,l,l}) = 1, \quad f(\Psi_{r,s,t}) = 1. \tag{6.4.7}
$$

*for $1 \le l \le n$ and $1 \le r < s < t \le n$.*

## 6.5 On genuineness of twisted quantum double

In [72] , the authors gave a criterion for when a twisted quantum double with an abelian cocycle to be genuine.

**Lemma 6.5.1.** [72, Theorem 4.1, Lemma 4.5] *Let $G$ be a finite abelian group, and $\Phi$ a normalized abelian 3-cocycle of $G$. Then $D^\Phi(G)$ is a genuine quasi-Hopf algebra if, and only if $\Phi' \in Z^3(\Gamma^\Phi, \mathbb{C}^\times)$ is a nontrivial 3-cocycle of $\Gamma^\Phi$, where $\Phi' \in Z^3(\Gamma^\Phi, \mathbb{C}^\times)$ is the inflation of $\Phi^{-1}$ along the above map $\Gamma^\Phi \longrightarrow G$.*

Now it suffices to determine whether $\Phi'$ is nontrivial on $\Gamma^{\Phi} \cong Z_{(2a,m)} \times Z_{\frac{m^2}{(2a,m)}}$ or not. Obviously, if $\Phi'$ is nontrivial on $Z_{\frac{m^2}{(2a,m)}}$, then $\Phi'$ will be nontrivial on $\Gamma^{\Phi}$. Hence, we first consider this condition.

**Proposition 6.5.2.** *Let $G \cong Z_m$ be a finite cyclic group and $\Phi(g^i, g^j, g^k) = \zeta_m^{ai[\frac{j+k}{m}]}$ for $1 \leq a < m$. If $(m,2a) \nmid (m,a)$, then $\Phi'$ is nontrivial on $\Gamma^{\Phi}$.*

*Proof.* Since $\Gamma^{\Phi}$ is the extension of $G$ by $\widehat{G}$. there is a obvious group surjection

$$\pi : \Gamma^{\Phi} \longrightarrow Z_m : \sigma_\tau(\chi,1)^j \mapsto 1, \\ \sigma_\tau(1,g)^i \mapsto g^i.$$

Hence $\pi^*(\Phi^{-1})$ will actually be the restriction of $\Phi'$ to $Z_{\frac{m^2}{(2a,m)}}$.

To show $\pi^*(\Phi^{-1})$ is nontrivial, it suffices to show $F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,1,1})$ does not equal 1 by Lemmas 6.4.6 and 6.4.7. By definition of $F_3$,

$$\begin{aligned} F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,1,1}) &= \pi^*(\Phi^{-1})(\sum_{l=0}^{\frac{m^2}{(2a,m)}-1} [\sigma_\tau(1,g), \sigma_\tau(1,g)^l, \sigma_\tau(1,g)]) \\ &= \Phi^{-1}(\sum_{l=0}^{\frac{m^2}{(2a,m)}-1} [g, g^l, g]) \\ &= (\zeta_m^{-a})^{\frac{m}{(2a,m)}}. \end{aligned}$$

Note that $(\zeta_m^{-a})^{\frac{m}{(2a,m)}} = 1$ if and only if $\frac{m}{(m,a)} \mid \frac{m}{(m,2a)}$, that is, $(m,2a)$ should divide $(m,a)$. Hence if $(m,2a) \nmid (m,a)$, then $\Phi'$ is nontrivial on $Z_{\frac{m^2}{(2a,m)}}$, hence on $\Gamma^{\Phi}$. ◻

**Theorem 6.5.3.** *Let $G \cong Z_m$ be a finite cyclic group and $\Phi(g^i, g^j, g^k) = \zeta_m^{ai[\frac{j+k}{m}]}$ for $1 \leq a < m$. Then $D^{\Phi}(G)$ is genuine if and only if $(m,2a) \nmid (m,a)$.*

The necessity of Theorem 6.5.3 is obvious by Proposition 6.5.2, since $\Phi'$ being a 3-coboundary on $\Gamma^{\Phi}$ will imply $(m,2a) \mid (m,a)$.

Now we need to deal with the case $(m,2a) \mid (m,a)$. Unfortunately, it is difficult to write down the explicit generator of $Z_{(2a,m)}$. We avoid this difficulty via the following result. By [59, Lemma 2.16], it is harmless to assume $Z_{(2a,m)} = \langle \sigma_\tau(\chi,1) \cdot \sigma_\tau(1,g)^b \rangle = \langle \sigma_\tau(\chi, g^b) \rangle$ for $0 \leq b \leq (2a,m)$. Note that this assumption requires

$$m \mid b(2a,m), \quad \text{and} \quad m \mid (2a,m) + 2a[\frac{b(2a,m)}{m}].$$

since $\sigma_\tau(\chi, g^b)^{(2a,m)} = 1$. With all preparations complete, we now prove Theorem 6.5.3.

*Proof of Theorem 6.5.3.* We only need to show $\Phi'$ is a 3-coboundary on $\Gamma^{\Phi}$ if $(m,2a) \mid (m,a)$. For consistency, we regard $\sigma_\tau(1,g)$ as the first generator and $\sigma_\tau(\chi, g^b)$ as the second generator. We have already shown $F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,1,1}) = 1$. It remains to verify the conditions in Lemma 6.4.7.

By direct computations, we have

$$\begin{aligned}F_3^*(\pi^*(\Phi^{-1}))(\Psi_{2,2,2}) &= \pi^*(\Phi^{-1})(\sum_{l=0}^{(2a,m)-1}[\sigma_\tau(\chi,g^b),\sigma_\tau(\chi,g^b)^l,\sigma_\tau(\chi,g^b)]\\ &= \Phi^{-1}(\sum_{l=0}^{(2a,m)-1}[g^b,(g^b)^l,g^b])\\ &= \prod_{l=0}^{(2a,m)-1}(\zeta_m^{-a})^{b[\frac{(bl)'+b}{m}]}.\end{aligned}$$

We have $m \mid ab$ since $(2a,m) \mid (m,a),(m,a) \mid a$ and $m \mid (2a,m)b$ by assumption, thus

$$F_3^*(\pi^*(\Phi^{-1}))(\Psi_{2,2,2}) = 1.$$

Next we are going to compute $F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,2,2})$ and $F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,1,2})$. We have

$$\begin{aligned}&F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,1,2})\\ &= \pi^*(\Phi^{-1})(\sum_{l=0}^{\frac{m^2}{(2a,m)}-1}[\sigma_\tau(1,g)^l,\sigma_\tau(1,g),\sigma_\tau(\chi,g^b)]-[\sigma_\tau(1,g)^l,\sigma_\tau(\chi,g^b),\sigma_\tau(1,g)]\\ &+[\sigma_\tau(\chi,g^b),\sigma_\tau(1,g)^l,\sigma_\tau(1,g)])\\ &= \prod_{l=0}^{\frac{m^2}{(2a,m)}-1}\frac{\Phi^{-1}(g^{l'},g,g^b)\Phi^{-1}(g^b,g^{l'},g)}{\Phi^{-1}(g^{l'},g^b,g)}\\ &= \prod_{l=0}^{\frac{m^2}{(2a,m)}-1}(\zeta_m^{-a})^{b[\frac{l'+1}{m}]} = 1.\end{aligned}$$

since $m \mid ab$ by the analysis above. On the other hand,

$$\begin{aligned}&F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,2,2})\\ &= \pi^*(\Phi^{-1})(\sum_{l=0}^{(2a,m)-1}[\sigma_\tau(1,g),\sigma_\tau(\chi,g^b)^l,\sigma_\tau(\chi,g^b)]-[\sigma_\tau(\chi,g^b)^l,\sigma_\tau(1,g),\sigma_\tau(\chi,g^b)]\\ &+[(\sigma_\tau(\chi,g^b)^l,\sigma_\tau(\chi,g^b),\sigma_\tau(1,g)])\\ &= \prod_{l=0}^{(2a,m)-1}\frac{\Phi^{-1}(g,g^{(bl)'},g^b)\Phi^{-1}(g^{(bl)'},g^b,g)}{\Phi^{-1}(g^{(bl)'},g,g^b)}\\ &= \prod_{l=0}^{(2a,m)-1}(\zeta_m^{-a})^{[\frac{(bl)'+b}{m}]}.\end{aligned}$$

If $b = 0$, then $F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,2,2}) = 1$. If we take $g_{1,2} = 1$, then equation (6.4.7) holds, thus $\Phi'$ is a 3-coboundary. If $b \neq 0$, then $(b((2a,m)-1))'+b$ equals $m$ since $m \mid b(2a,m)$ by assumption. Thus

$$F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,2,2}) = (\zeta_m^{-a})^{\frac{b(2a,m)}{m}}.$$

In this case, take $g_{1,2} = \zeta_m^{\frac{ab}{m}}$. Since $(m,2a) \mid m$ and $m \mid ab$, we have

$$g_{1,2}^{-(2a,m)} = (\zeta_m^{-a})^{\frac{b(2a,m)}{m}} = F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,2,2}),$$

and

$$g_{1,2}^{\frac{m^2}{(2a,m)}} = \zeta_m^{\frac{ab}{m}\frac{m}{(2a,m)}m} = 1 = F_3^*(\pi^*(\Phi^{-1}))(\Psi_{1,1,2}).$$

As a result, if $(m,2a) \mid (m,a)$, then $\Phi'$ is a 3-coboundary on $\Gamma^{\Phi}$. Hence $D^{\Phi}(G)$ is genuine if and only if $(m,2a) \nmid (m,a)$. □

Next we will investigate when $(m,2a) \nmid (m,a)$ holds. This can provide a more intuitive criterion.

**Theorem 6.5.4.** *Let $G \cong Z_m$ be a finite cyclic group and $\Phi(g^i, g^j, g^k) = \zeta_m^{ai[\frac{j+k}{m}]}$ for $1 \leq a < m$. Let $m = 2^n \prod_i p_i^{a_i}$ and $a = 2^{n'} \prod_j p_j^{b_j}$ be their prime decompositions, where $n, n' \geq 0$. Then $D^{\Phi}(G)$ is genuine if and only if $n' < n$.*

*Proof.* Suppose $m = 2^n \prod_i p_i^{a_i}$ and $a = 2^{n'} \prod_j p_j^{b_j}$ be their prime decompositions. Then

$$(m,2a) = (2^n \prod_i p_i^{a_i}, 2^{n'+1} \prod_j p_j^{b_j}) = (2^n, 2^{n'+1}) \cdot (\prod_i p_i^{a_i}, \prod_j p_j^{b_j}).$$

and

$$(m,a) = (2^n \prod_i p_i^{a_i}, 2^{n'} \prod_j p_j^{b_j}) = (2^n, 2^{n'}) \cdot (\prod_i p_i^{a_i}, \prod_j p_j^{b_j}).$$

Thus $(m,2a) \nmid (m,a)$ if and only if $(2^n, 2^{n'+1}) \nmid (2^n, 2^{n'})$. This is equivalent to $n' < n$. □

**Remark 6.5.5.** (i) Note that if $m$ is odd, then $D^{\Phi}(G)$ will never be genuine for arbitrary $0 \leq a < m$. This conclusion is consistent with [77, Theorem 9.4].

(ii) According to Proposition 6.3.6, if $G$ is cyclic, $D^{\Phi}(G)$ will never be gauge equivalent to $D(G')$ for arbitrary finite group $G'$ by the theory of categorically Morita equivalence, but $D^{\Phi}(G)$ may be gauge equivalent to a Hopf algebra by Theorem 6.5.3.

# Chapter 7

# On infinite-dimensionality of a class of Nichols algebras

In this section, we consider a rank three Nichols algebra $\mathcal{B}(M)$ of non-diagonal type in [69], which plays a central role in classification of finite-dimensional Nichols algebra in ${}_{G}^{G}\mathcal{YD}^{\Phi}$, where $G$ is a finite abelian group and $\Phi$ is a 3-cocycle on $G$. We prove it to be infinite-dimensional in different ways. Moreover, we prove $\mathcal{G}(M)$ gives rise to a standard Cartan graph and find a root system over $\mathcal{G}(M)$. At the same time, we show that $\mathcal{B}(M)$ is affine in the sense of [29].

## 7.1 The first proof of Theorem 7.1.2

The purpose of this section is to give a new proof of Proposition 4.1 in [60] through applying our previous observations.

Let $\mathbb{G} = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_3} = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle$ with $m_1 \mid m_2$, $m_2 \mid m_3$, and

$$\Psi\left(g_1^{i_1} g_2^{i_2} g_3^{i_3}, g_1^{j_1} g_2^{j_2} g_3^{j_3}, g_1^{k_1} g_2^{k_2} g_3^{k_3}\right) = (-1)^{k_1 j_2 i_3} \tag{7.1.1}$$

be a 3-cocycle on $G$, where $0 \le i_l < m_1$, $0 \le j_l < m_2$, $0 \le k_l < m_3$, $1 \le l \le 3$.

Let $V_1, V_2, V_3 \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\Psi}$ be simple modules such that $\deg(V_1) = g_1$, $\deg(V_2) = g_2$, $\deg(V_3) = g_3$ and $\dim(V_1) = \dim(V_2) = \dim(V_3) = 2$. Moreover, $V_1 = \{X_1', X_2'\}, V_2 = \{Y_1', Y_2'\}, V_3 = \{Z_1', Z_2'\}$ can be assumed satisfying the following equations:

$$\begin{cases} g_1 \triangleright X_i' = -X_i', \ i = 1,2, \\ g_2 \triangleright X_1' = X_1', g_2 \triangleright X_2' = -X_2', \\ g_3 \triangleright X_1' = X_2', g_3 \triangleright X_2' = X_1'. \end{cases} \quad \begin{cases} g_2 \triangleright Y_i' = -Y_i', \ i = 1,2, \\ g_3 \triangleright Y_1' = Y_1', g_3 \triangleright Y_2' = -Y_2', \\ g_1 \triangleright Y_1' = Y_2', g_1 \triangleright Y_2' = Y_1', \end{cases} \quad \begin{cases} g_3 \triangleright Z_i' = -Z_i', \ i = 1,2, \\ g_2 \triangleright Z_1' = Z_1', g_2 \triangleright Z_2' = -Z_2', \\ g_1 \triangleright Z_1' = Z_2', g_1 \triangleright Z_2' = Z_1'. \end{cases}$$

**Remark 7.1.1.** (i) Note that $m_1, m_2, m_3$ must be even. Indeed, from the relation $X_i = e^{m_1} \triangleright X_i = (-1)^{m_1} X_i$, it follows that $m_1$ is even. Similarly $m_2, m_3$ are even.

(ii) $\mathcal{B}(V_1)$, $\mathcal{B}(V_2)$, $\mathcal{B}(V_3)$ are finite-dimensional Nichols algebras by [60, Proposition 2.12]. Moreover, $\mathcal{B}(V_1 \oplus V_2)$, $\mathcal{B}(V_1 \oplus V_3)$, $\mathcal{B}(V_2 \oplus V_3)$ are finite-dimensional by [63, Proposition 5.1].

(iii) The requirement $\dim(V_1) = \dim(V_2) = \dim(V_3) = 2$ is in fact necessary, otherwise the corresponding Nichols algebra must be infinite-dimensional.

**Theorem 7.1.2.** [60, Proposition 4.1] *With above notation, $\mathcal{B}(V_1 \oplus V_2 \oplus V_3)$ is infinite-dimensional.*

It should be emphasized that the above theorem plays the key role in that paper and the original proof relays on heavy computations. The purpose of the following context is to give a new proof of this theorem through applying reflection theory of Nichols algebras.

To simplify the proof, we proceed with the following reduction. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle$, and

$$\Phi\left(g_1^{i_1} g_2^{i_2} g_3^{i_3}, g_1^{j_1} g_2^{j_2} g_3^{j_3}, g_1^{k_1} g_2^{k_2} g_3^{k_3}\right) = (-1)^{k_1 j_2 i_3},$$

be a 3-cocycle on $H$, where $0 \le i_l, j_l, k_l \le 1,\ 1 \le l \le 3$. By [60 Lemma 3.5], we may give a complete list of irreducible Yetter-Drinfeld modules over ${}_G^G\mathcal{YD}^\Phi$. In particular, when restricting to those irreducible Yetter-Drinfeld modules that generate finite-dimensional Nichols algebras, we see that the isomorphism classes are given by the set

$$S = \{[{}^hM] \mid {}^hM \text{ is irreducible, } h \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 - \{1, h_1h_2h_3\}\}. \tag{7.1.2}$$

There are exactly six isomorphism classes. Here ${}^hM$ represents the simple Yetter–Drinfeld module has comodule structure

$${}^hM := \{w \in M \mid \delta_M(w) = h \otimes w\}.$$

We now choose explicit representatives of $S$. For $1 \le i \le 6$, let $M_i \in S$ be pairwise non-isomorphic simple modules such that $\dim(M_i) = 2$, and $\deg(M_1) = g_1$, $\deg(M_2) = g_2$, $\deg(M_3) = g_3$, $\deg(M_4) = g_1g_2$, $\deg(M_5) = g_1g_3$, $\deg(M_6) = g_2g_3$. Moreover, we may take

$$M_1 = \{X_1, X_2\}, \quad M_2 = \{Y_1, Y_2\}, \quad M_3 = \{Z_1, Z_2\},$$
$$M_4 = \{R_1, R_2\}, \quad M_5 = \{S_1, S_2\}, \quad M_6 = \{T_1, T_2\},$$

satisfying the following relations:

$$\begin{cases} g_1 \triangleright X_i = -X_i,\ i = 1,2, \\ g_2 \triangleright X_1 = X_1, g_2 \triangleright X_2 = -X_2, \\ g_3 \triangleright X_1 = X_2, g_3 \triangleright X_2 = X_1. \end{cases} \quad \begin{cases} g_2 \triangleright Y_i = -Y_i,\ i = 1,2, \\ g_3 \triangleright Y_1 = Y_1, g_3 \triangleright Y_2 = -Y_2, \\ g_1 \triangleright Y_1 = Y_2, g_1 \triangleright Y_2 = Y_1. \end{cases} \quad \begin{cases} g_3 \triangleright Z_i = -Z_i,\ i = 1,2, \\ g_2 \triangleright Z_1 = Z_1, g_2 \triangleright Z_2 = -Z_2, \\ g_1 \triangleright Z_1 = Z_2, g_1 \triangleright Z_2 = Z_1. \end{cases}$$

$$\begin{cases} (g_1g_2) \triangleright R_i = -R_i,\ i = 1,2, \\ g_1 \triangleright R_1 = R_1, g_1 \triangleright R_2 = -R_2, \\ g_3 \triangleright R_1 = R_2, g_3 \triangleright R_2 = R_1. \end{cases} \quad \begin{cases} (g_1g_3) \triangleright T_i = -T_i,\ i = 1,2, \\ g_1 \triangleright T_1 = T_1, g_1 \triangleright T_2 = -T_2, \\ g_2 \triangleright T_1 = T_2, g_1 \triangleright T_2 = T_1. \end{cases} \quad \begin{cases} (g_2g_3) \triangleright S_i = -S_i,\ i = 1,2, \\ g_2 \triangleright S_1 = S_1, g_2 \triangleright S_2 = -S_2, \\ g_1 \triangleright S_1 = S_2, g_1 \triangleright S_2 = S_1. \end{cases}$$

**Remark 7.1.3.** The category ${}_G^G\mathcal{YD}^\Phi{}_{\mathrm{fd}}$ is rigid. It is straightforward to verify that for each $1 \le i \le 6$, $M_i^*$ does exist with $M_i^* \cong M_i$. This implies $M_i^* \in S$ as well. Moreover, $\mathcal{B}(M_i) \cong \mathcal{B}(M_i^*)$ as Hopf algebras in ${}_G^G\mathcal{YD}^\Phi$.

**Theorem 7.1.4.** *Let $M = (M_1, M_2, M_3)$ be the 3-tuple, then $M$ admits all reflections and $\mathcal{G}(M)$ is a standard semi-Cartan graph. In particular, the Cartan matrix is*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

To establish this result, we first prove the following key lemma:

**Lemma 7.1.5.** *For all* $1 \le i \ne j \le 6$*, if* $\deg(M_i)\cdot\deg(M_j) \ne 1, h_1h_2h_3$*, then* $\mathrm{ad}(M_i)^2(M_j) = 0$*. Moreover,* $\mathrm{ad}(M_i)(M_j) \in S$*, and as Yetter-Drinfeld module in* ${}^G_G\mathcal{YD}^\Phi$*, we have*

$$\mathrm{ad}(M_i)(M_j) \cong \mathrm{ad}(M_j)(M_i). \tag{7.1.3}$$

*We list the following isomorphism for further use*

$$\begin{aligned}
&\mathrm{ad}(M_1)(M_2) \cong M_4, \mathrm{ad}(M_1)(M_3) \cong M_5, \mathrm{ad}(M_1)(M_4) \cong M_2,\\
&\mathrm{ad}(M_1)(M_5) \cong M_3, \mathrm{ad}(M_2)(M_3) \cong M_6, \mathrm{ad}(M_2)(M_4) \cong M_1,\\
&\mathrm{ad}(M_2)(M_6) \cong M_3, \mathrm{ad}(M_3)(M_5) \cong M_2, \mathrm{ad}(M_3)(M_6) \cong M_2\\
&\mathrm{ad}(M_4)(M_5) \cong M_6, \mathrm{ad}(M_4)(M_6) \cong M_5, \mathrm{ad}(M_5)(M_6) \cong M_4.
\end{aligned} \tag{7.1.4}$$

*Proof.* This proof involves extensive but analogous computations. We begin by establishing that $\mathrm{ad}(M_1)^2(M_2) = 0$. To this end, we compute the coproducts:

$$\begin{aligned}
\Delta(\mathrm{ad}(X_1)(Y_1)) &= 1 \otimes \mathrm{ad}(X_1)(Y_1) + \mathrm{ad}(X_1)(Y_1) \otimes 1 + X_1 \otimes (Y_1 + Y_2),\\
\Delta(\mathrm{ad}(X_1)(Y_2)) &= 1 \otimes \mathrm{ad}(X_1)(Y_2) + \mathrm{ad}(X_1)(Y_2) \otimes 1 + X_1 \otimes (Y_1 + Y_2),\\
\Delta(\mathrm{ad}(X_2)(Y_1)) &= 1 \otimes \mathrm{ad}(X_2)(Y_1) + \mathrm{ad}(X_2)(Y_1) \otimes 1 + X_2 \otimes (Y_1 - Y_2),\\
\Delta(\mathrm{ad}(X_2)(Y_2)) &= 1 \otimes \mathrm{ad}(X_2)(Y_2) + \mathrm{ad}(X_2)(Y_2) \otimes 1 + X_2 \otimes (Y_2 - Y_1).
\end{aligned}$$

From these we deduce the relations:

$$\begin{aligned}
&\mathrm{ad}(X_1)(Y_1) - \mathrm{ad}(X_1)(Y_2) = 0,\\
&\mathrm{ad}(X_2)(Y_1) + \mathrm{ad}(X_2)(Y_2) = 0.
\end{aligned} \tag{7.1.5}$$

Therefore $\mathrm{ad}(M_1)(M_2) = \mathrm{span}\{\mathrm{ad}(X_1)(Y_1), \mathrm{ad}(X_2)(Y_1)\}$. Now since $X_1^2 = 0$, we have

$$\mathrm{ad}(X_1)^2(Y_1) = X_1(X_1Y_1 - Y_2X_1) - h_1 \triangleright ((X_1Y_1 - Y_2X_1))X_1 = 0$$

which implies $\mathrm{ad}(X_1)^2(Y_1) = \mathrm{ad}(X_1)^2(Y_2) = 0$. Meanwhile,

$$\mathrm{ad}(X_2)(\mathrm{ad}(X_1)(Y_1)) = \mathrm{ad}(X_2)(\mathrm{ad}(X_1)(Y_2)) = 0.$$

Since

$$\begin{aligned}
\Delta(\mathrm{ad}(X_2)(\mathrm{ad}(X_1)(Y_1))) &= (1 \otimes X_2 + X_2 \otimes 1)(1 \otimes \mathrm{ad}(X_1)(Y_1) + \mathrm{ad}(X_1)(Y_1) \otimes 1 + X_1 \otimes (Y_1 + Y_2)\\
&\quad - (1 \otimes \mathrm{ad}(X_1)(Y_1) + \mathrm{ad}(X_1)(Y_1) \otimes 1 + X_1 \otimes (Y_1 + Y_2))(1 \otimes X_2 + X_2 \otimes 1),\\
&= 1 \otimes \mathrm{ad}(X_2)(\mathrm{ad}(X_1)(Y_1)) + \mathrm{ad}(X_2)(\mathrm{ad}(X_1)(Y_1)) \otimes 1
\end{aligned}$$

where we have used equation (7.1.5). A similar argument shows $\mathrm{ad}(X_2)^2(Y_1) = 0$ and $\mathrm{ad}(X_1)(\mathrm{ad}(X_2)(Y_1)) = 0$. We conclude that $\mathrm{ad}(M_1)^2(M_2) = 0$.

By Lemma 5.1.2, the object $\mathrm{ad}(M_1)(M_2) \in {}^G_G\mathcal{YD}^\Phi$ is simple. Since $\deg(\mathrm{ad}(M_1)(M_2)) = h_1h_2$, $\mathrm{ad}(M_1)(M_2) \cong M_4$. Furthermore, by proof of Lemma 5.1.4, we have an isomorphism of $\mathbb{N}_0$-graded objects $\mathrm{ad}\,\mathcal{B}(M_1)(M_4) \cong \mathrm{ad}\,\mathcal{B}(M_1)(\mathrm{ad}(M_1)(M_2))$. Since $\mathrm{ad}(M_1)^2(M_2) = 0$, we have $\mathrm{ad}(M_1)^2(M_4) = 0$ since it has same degree as $\mathrm{ad}(M_1)^2(M_2)$ in the standard $\mathbb{N}_0$-grading. Therefore, $\mathrm{ad}(M_1)(M_4)$ is irreducible and one readily verifies that $\mathrm{ad}(M_1)(M_4) \cong M_2$ in ${}^G_G\mathcal{YD}^\Phi$.

Analogous computations yield:

$$\mathrm{ad}(M_2)^2(M_1) = \mathrm{ad}(M_4)^2(M_1) = 0.$$

These identities further imply:

$$\mathrm{ad}(M_2)(M_1) \cong M_4, \ \mathrm{ad}(M_4)(M_1) \cong M_2,$$

and

$$\begin{aligned}&\mathrm{ad}(M_2)^2(M_4) = \mathrm{ad}(M_4)^2(M_2) = 0,\\ &\mathrm{ad}(M_2)(M_4) \cong M_1, \ \mathrm{ad}(M_4)(M_2) \cong M_1.\end{aligned}$$

We now proceed to prove $\mathrm{ad}(M_1)^2(M_3) = 0$:

$$\begin{aligned}\Delta(\mathrm{ad}(X_1)(Z_1)) &= 1 \otimes \mathrm{ad}(X_1)(Z_1) + \mathrm{ad}(X_1)(Z_1) \otimes 1 + X_1 \otimes Z_1 + X_2 \otimes Z_2,\\ \Delta(\mathrm{ad}(X_1)(Z_2)) &= 1 \otimes \mathrm{ad}(X_1)(Z_2) + \mathrm{ad}(X_1)(Z_2) \otimes 1 + X_1 \otimes Z_2 + X_2 \otimes Z_1,\\ \Delta(\mathrm{ad}(X_2)(Z_1)) &= 1 \otimes \mathrm{ad}(X_2)(Z_1) + \mathrm{ad}(X_2)(Z_1) \otimes 1 + X_2 \otimes Z_1 + X_1 \otimes Z_2,\\ \Delta(\mathrm{ad}(X_2)(Z_2)) &= 1 \otimes \mathrm{ad}(X_2)(Z_2) + \mathrm{ad}(X_2)(Z_2) \otimes 1 + X_2 \otimes Z_2 + X_1 \otimes Z_1.\end{aligned}$$

It is straightforward to observe that:

$$\begin{aligned}&\mathrm{ad}(X_1)(Z_1) - \mathrm{ad}(X_2)(Z_2) = 0,\\ &\mathrm{ad}(X_2)(Z_1) + \mathrm{ad}(X_1)(Z_2) = 0.\end{aligned} \tag{7.1.6}$$

Hence $\mathrm{ad}(M_1)(M_3) = \mathrm{span}\{\mathrm{ad}(X_1)(Z_1), \mathrm{ad}(X_2)(Z_1)\}$. By $X_1^2 = X_2^2 = 0$,

$$\begin{aligned}\mathrm{ad}(X_1)^2(Z_1) &= X_1(X_1Z_1 - Z_2X_1) - h_1 \triangleright (X_1Z_1 - Z_2X_1)X_1 = 0,\\ \mathrm{ad}(X_1)^2(Z_2) &= X_1(X_1Z_2 - Z_1X_1) - h_1 \triangleright (X_1Z_2 - Z_1X_1)X_1 = 0,\\ \mathrm{ad}(X_2)^2(Z_1) &= X_2(X_2Z_1 - Z_2X_2) - h_1 \triangleright (X_2Z_1 - Z_2X_2)X_2 = 0,\\ \mathrm{ad}(X_2)^2(Z_2) &= X_2(X_2Z_2 - Z_1X_2) - h_1 \triangleright (X_2Z_2 - Z_1X_2)X_2 = 0.\end{aligned}$$

Combining this with (7.1.6) implies $\mathrm{ad}(M_1)^2(M_3) = 0$. As in the first case, we can derive the following identities:

$$\begin{aligned}\mathrm{ad}(M_3)^2(M_1) = \mathrm{ad}(M_1)^2(M_5) = \mathrm{ad}(M_3)^2(M_5) = \mathrm{ad}(M_5)^2(M_1) = \mathrm{ad}(M_5)^2(M_3) = 0.\\ \mathrm{ad}(M_1)(M_3) \cong \mathrm{ad}(M_3)(M_1) \cong M_5,\\ \mathrm{ad}(M_1)(M_5) \cong \mathrm{ad}(M_5)(M_1) \cong M_3,\\ \mathrm{ad}(M_3)(M_5) \cong \mathrm{ad}(M_5)(M_3) \cong M_1.\end{aligned}$$

Now we turn to proving $\mathrm{ad}(M_2)^2(M_3) = 0$:

$$\begin{aligned}\Delta(\mathrm{ad}(Y_1)(Z_1)) &= 1 \otimes \mathrm{ad}(Y_1)(Z_1) + \mathrm{ad}(Y_1)(Z_1) \otimes 1,\\ \Delta(\mathrm{ad}(Y_1)(Z_2)) &= 1 \otimes \mathrm{ad}(Y_1)(Z_2) + \mathrm{ad}(Y_1)(Z_2) \otimes 1 + 2Y_1 \otimes Z_2,\end{aligned}$$

$$\Delta\left(\mathrm{ad}(Y_2)\left(Z_1\right)\right) = 1\otimes \mathrm{ad}(Y_2)\left(Z_1\right) + \mathrm{ad}(Y_2)\left(Z_1\right)\otimes 1 + 2Y_2\otimes Z_1,$$
$$\Delta\left(\mathrm{ad}(Y_2)\left(Z_2\right)\right) = 1\otimes \mathrm{ad}(Y_2)\left(Z_2\right) + \mathrm{ad}(Y_2)\left(Z_2\right)\otimes 1.$$

Then $\mathrm{ad}(Y_1)(Z_1) = \mathrm{ad}(Y_2)(Z_2) = 0$ and $\mathrm{ad}(M_2)(M_3) = \{\mathrm{ad}(Y_1)(Z_2), \mathrm{ad}(Y_2)(Z_1)\}$. By $Y_1^2 = Y_2^2 = 0$,

$$\mathrm{ad}(Y_1)^2(Z_2) = Y_1(Y_1Z_2 + Z_2Y_1) - h_2 \triangleright (Y_1Z_2 + Z_2Y_1)Y_1 = 0,$$
$$\mathrm{ad}(Y_2)^2\left(Z_1\right) = Y_2(Y_2Z_1 - Z_1Y_2) - h_2 \triangleright (Y_2Z_1 - Z_1Y_2)Y_2 = 0.$$

On the other hand, since $\mathrm{ad}Y_2(Z_2) = 0$, and $Y_1Y_2 + Y_2Y_1 = 0$. The following identity holds.

$$\begin{aligned}\Delta(\mathrm{ad}(Y_2)\left(\mathrm{ad}(Y_1)\left(Z_2\right)\right)) &= (1\otimes Y_2 + Y_2\otimes 1)(1\otimes \mathrm{ad}(Y_1)\left(Z_2\right) + \mathrm{ad}(Y_1)\left(Z_2\right)\otimes 1 + 2Y_1\otimes Z_2)\\ &\quad - (1\otimes \mathrm{ad}(Y_1)\left(Z_2\right) + \mathrm{ad}(Y_1)\left(Z_2\right)\otimes 1 + 2Y_1\otimes Z_2)(1\otimes Y_2 + Y_2\otimes 1)\\ &= 1\otimes \mathrm{ad}(X_2)\left(\mathrm{ad}(X_1)\left(Y_1\right)\right) + \mathrm{ad}(X_2)\left(\mathrm{ad}(X_1)\left(Y_1\right)\right)\otimes 1.\end{aligned}$$

Similarly, $\mathrm{ad}(Y_1)(\mathrm{ad}(Y_2)(Z_1)) = 0$. Hence $\mathrm{ad}(M_2)^2(M_3) = 0$. We list all identities which can be obtained in this case.

$$\mathrm{ad}(M_3)^2(M_2) = \mathrm{ad}(M_2)^2(M_6) = \mathrm{ad}(M_6)^2(M_2) = \mathrm{ad}(M_3)^2(M_6) = \mathrm{ad}(M_6)^2(M_3) = 0.$$
$$\mathrm{ad}(M_2)(M_3) \cong \mathrm{ad}(M_3)(M_2) \cong M_6,$$
$$\mathrm{ad}(M_2)(M_6) \cong \mathrm{ad}(M_6)(M_2) \cong M_3,$$
$$\mathrm{ad}(M_3)(M_6) \cong \mathrm{ad}(M_6)(M_3) \cong M_2.$$

The final case to confirm is $\mathrm{ad}(M_4)^2(M_5) = 0$:

$$\Delta\left(\mathrm{ad}(R_1)\left(T_1\right)\right) = 1\otimes \mathrm{ad}(R_1)\left(T_1\right) + \mathrm{ad}(R_1)\left(T_1\right)\otimes 1 + R_1\otimes T_1 - R_2\otimes T_2,$$
$$\Delta\left(\mathrm{ad}(R_1)\left(T_2\right)\right) = 1\otimes \mathrm{ad}(R_1)\left(T_2\right) + \mathrm{ad}(R_1)\left(T_2\right)\otimes 1 + R_1\otimes T_2 + R_2\otimes T_1,$$
$$\Delta\left(\mathrm{ad}(R_2)\left(T_1\right)\right) = 1\otimes \mathrm{ad}(R_2)\left(T_1\right) + \mathrm{ad}(R_2)\left(T_1\right)\otimes 1 + R_2\otimes T_1 + R_1\otimes T_2,$$
$$\Delta\left(\mathrm{ad}(R_2)\left(T_2\right)\right) = 1\otimes \mathrm{ad}(R_2)\left(T_2\right) + \mathrm{ad}(R_2)\left(T_2\right)\otimes 1 + R_2\otimes T_2 - R_1\otimes T_1.$$

Then it is direct to see that

$$\begin{aligned}&\mathrm{ad}(R_1)\left(T_1\right) + \mathrm{ad}(T_2)\left(R_2\right) = 0,\\ &\mathrm{ad}(R_2)\left(T_1\right) - \mathrm{ad}(R_1)\left(T_2\right) = 0.\end{aligned} \tag{7.1.7}$$

Hence $\mathrm{ad}(M_4)(M_5) = \mathrm{span}\{\mathrm{ad}(R_1)(T_1), \mathrm{ad}(R_2)(T_1)\}$. By $R_1^2 = R_2^2 = 0$,

$$\mathrm{ad}(R_1)^2\left(T_1\right) = R_1(R_1T_1 + T_2R_1) - (h_1h_2) \triangleright (R_1T_1 + T_2R_1)R_1 = 0,$$
$$\mathrm{ad}(R_1)^2(T_2) = R_1(R_1T_2 - T_1R_1) - (h_1h_2) \triangleright (R_1T_2 - T_1R_1)R_1 = 0,$$
$$\mathrm{ad}(R_2)^2\left(T_1\right) = R_2(R_2T_1 + T_2R_2) - (h_1h_2) \triangleright (R_2T_1 + T_2R_2)R_2 = 0,$$
$$\mathrm{ad}(R_2)^2(T_2) = R_2(R_2T_2 - T_1R_2) - (h_1h_2) \triangleright (R_2T_2 - T_1R_2)R_2 = 0.$$

Combining this with (7.1.7) implies $\mathrm{ad}(M_4)^2(M_5) = 0$.

The following identities can be obtained in this case.

$$\mathrm{ad}(M_4)^2(M_6) = \mathrm{ad}(M_6)^2(M_4) = \mathrm{ad}(M_6)^2(M_5) = \mathrm{ad}(M_5)^2(M_6) = \mathrm{ad}(M_5)^2(M_4) = 0.$$
$$\mathrm{ad}(M_4)(M_5) \cong \mathrm{ad}(M_5)(M_4) \cong M_6,$$
$$\mathrm{ad}(M_4)(M_6) \cong \mathrm{ad}(M_6)(M_4) \cong M_5,$$
$$\mathrm{ad}(M_5)(M_6) \cong \mathrm{ad}(M_6)(M_5) \cong M_4.$$

We have listed all cases, which proves this lemma. □

To establish that the tuple $M$ admits all reflections, the following structural observation is essential.

**Lemma 7.1.6.** *Suppose $M$ admits the reflection sequence $(i_1, i_2, .., i_l)$, where $i_1, .., i_l \in \{1,2,3\}$. Then for $j \in \{1,2,3\}$,*

$$\deg(R_{i_l}(...(R_{i_1}(M))...)_j) \neq 1, h_1h_2h_3. \tag{7.1.8}$$

*and*

$$\prod_{j=1}^{3} \deg(R_{i_l}(...(R_{i_1}(M))...)_j) = h_1h_2h_3.$$

*Proof.* We prove it by induction. For the base case $l = 1$, Lemma 5.1.4 gives,

$$R_1(M) \cong (M_1, M_4, M_5),\ R_2(M) \cong (M_4, M_2, M_6),\ R_3(M) \cong (M_5, M_6, M_3).$$

It is straightforward to verify that for each $1 \leq i, j \leq 3$, $\prod_{j=1}^{3} R_i(M) = h_1h_2h_3$.

Now assume, for some $l \geq 2$, that $M$ admits the reflection sequence $(i_1, i_2, .., i_{l-1})$, and

$$\prod_{j=1}^{3} \deg(R_{i_{l-1}}(...(R_{i_1}(M))...)_j) = h_1h_2h_3.$$

Let us denote $\deg(R_{i_{l-1}}(...(R_{i_1}(M))...)_j) = h'_j$ for each $1 \leq j \leq 3$, where $h'_j \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then for any $1 \leq i_l \leq 3$, we compute

$$\begin{aligned}\prod_{j=1}^{3} \deg(R_{i_l}(...(R_{i_1}(M))...)_j) &= (\prod_{k \neq i_l} h'_k h_{i_l}) h_{i_l}^{-1} \\ &= h_1h_2h_3,\end{aligned}$$

which establishes the second claim.

Without loss of generality, suppose for the sake of contradiction that $\deg(R_{i_l}(...(R_{i_1}(M))...)_1) = h_1h_2h_3$. We must have $\deg(R_{i_l}(...(R_{i_1}(M))...)_2) = \deg(R_{i_l}(...(R_{i_1}(M))...)_3) = h'$ by the second claim. If $i_l = 1$, then

$$R_{i_{l-1}}(...(R_{i_1}(M))...) = (h_1h_2h_3, h_1h_2h_3h', h_1h_2h_3h'),$$

which contradicts the assumption. If $i_1 = 2$,the degrees become

$$R_{i_{l-1}}(...(R_{i_1}(M))...) = (h_1h_2h_3h', h', 1),$$

and if $i_1 = 3$

$$R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots) = (h_1h_2h_3h', 1, h').$$

Both of which again lead to contradictions.

A similar contradiction arises if we assume $\deg(R_{i_l}(\ldots(R_{i_1}(M))\ldots)_1) = 1$, $\deg(R_{i_l}(\ldots(R_{i_1}(M))\ldots)_2) = k'$, $\deg(R_{i_l}(\ldots(R_{i_1}(M))\ldots)_3) = k''$, where $k' \cdot k'' = h_1h_2h_3$. Now if $i_l = 1$, we have

$$R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots) = (1, k', k'').$$

If $i_l = 2$, we obtain

$$R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots) = (k', k', k'k''),$$

and if $i_l = 3$

$$R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots) = (k'', k'k'', k'').$$

All cases lead to contradictions. This completes the induction. □

*Proof of Theorem 7.1.4.* We proceed by induction to show that $M$ admits all reflection sequences. Without loss of generality, we first consider the 1st-reflection. By proof of Lemma 7.1.5, for all $1 \le j \le 3$, we have $R_1(M)_j \in S$. Recall that two tuples $M$ and $M'$ are isomorphic if and only if their corresponding components are isomorphic. $R_1(M) \cong (M_1, M_4, M_5)$. By Lemma 5.1.4, $R_i(R_1(M)) \cong R_i(M_1, M_4, M_5)$ for $1 \le i \le 3$. Applying Lemma 7.1.5 again, we conclude that $(M_1, M_4, M_5)$ admits the $i$-th reflection by analogous reasoning.

Now, suppose $M$ admits the reflection sequence $(i_1, i_2, \ldots, i_{l-1})$, where $1 \le i_1, \ldots, i_{l-1} \le 3$. Then we have $\deg(R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots)_j) \ne 1, h_1h_2h_3$ for $1 \le j \le 3$ and $\prod_{j=1}^{3} \deg(R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots)_j) = h_1h_2h_3$. Consequently, for any $1 \le m < n \le 3$, the product

$$\deg(R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots)_m) \cdot \deg(R_{i_{l-1}}(\ldots(R_{i_1}(M))\ldots)_n)$$

is neither 1 nor $h_1h_2h_3$. Therefore $M$ admits the $(i_1, i_2, \ldots, i_{l-1}, i_l)$-th reflection by Lemma 7.1.5. This completes the induction and shows that $M$ admits all reflection sequences.

For any $N \in \mathcal{F}_3(M)$, we have $\deg(N_i) \ne 1, h_1h_2h_3$ for $1 \le i \le 3$ by Lemma 7.1.6. Therefore, $\mathcal{B}(N_i)$ is finite-dimensional since $N_i$ belongs to $S$. Thus $M$ gives rise to a semi-Cartan graph $\mathcal{G}(M)$.

Now for all $N \in \mathcal{F}_3(M)$, Lemma 7.1.5 implies that the generalized Cartan matrix $A^N$ is given by

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Thus $\mathcal{G}(M)$ is a standard semi-Cartan graph. □

**Corollary 7.1.7.** *The Nichols algebra $\mathcal{B}(M)$ is infinite-dimensional.*

*Proof.* $\mathcal{G}(M)$ is a standard semi-Cartan graph by Theorem 7.1.4. Suppose, for contradiction, that $\mathcal{B}(M)$ is finite-dimensional, then $A^M$ is a finite Cartan matrix by Theorem 5.3.8. However

$$A^M = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

is not a finite Cartan matrix. This contradiction implies that $\mathcal{B}(M)$ must be infinite-dimensional. □

*Proof of Theorem 7.1.2.* By Remark 7.1.1(1), there is a group surjection satisfying

$$\pi : \mathbb{G} \longrightarrow G, \ \mathrm{g}_i \mapsto g_i, \ 1 \le i \le 3.$$

Furthermore, $\pi^*(\Phi) = \Psi$.

We construct such a linear map for $1 \le j \le 2$,

$$F : V_1 \oplus V_2 \oplus V_3 \to M_1 \oplus M_2 \oplus M_3, \ X'_j \mapsto X_j, Y'_j \mapsto Y_j, Z'_j \mapsto Z_j$$

A direct verification shows that for each $1 \le i \le 3$, $v_i \in V_i$ and $\mathrm{g} \in \mathbb{G}$, the following compatibilities hold:

$$(\pi \otimes F)\delta_{V_i}(v_i) = \delta_{M_i}(F(v_i)),$$
$$F(g \triangleright v_i) = \pi(g) \triangleright F(v_i).$$

By [58, Lemma 4.4], we deduce that

$$\mathcal{B}(V) \cong \mathcal{B}(M)$$

in the sense of [58, Definition 4.3]. It follows that $\mathcal{B}(V)$ is infinite-dimensional. □

## 7.2 Nichols algebras over $D_8$ of rank 3

We first recall the basic notation of irreducible Yetter-Drinfeld modules over groups. Let $G$ be a finite group, $\mathcal{O}$ a conjugacy class of $G$ , $s \in \mathcal{O}$ fixed, $(\rho, V)$ an irreducible representation of $G^s$, where $G^s$ is the centralizer of $s$ in $G$. Let $t_1 = s, \ldots, t_M$ be a numeration of $\mathcal{O}$ and let $g_i \in G$ such that $g_i s g_i^{-1} = t_i$ for all $1 \le i \le M$. Then the corresponding irreducible Yetter-Drinfeld module $M(\mathcal{O}, \rho)$ is defined as follows: As a vector space, it is simply $\bigoplus_{1 \le i \le M} g_i \otimes V$. Let $g_i v := g_i \otimes v \in M(\mathcal{O}, \rho)$, $1 \le i \le M$, $v \in V$. If $v \in V$ and $1 \le i \le M$, then the coaction and the action of $g \in G$ are given by

$$\delta(g_i v) = t_i \otimes g_i v, \qquad g \triangleright (g_i v) = g_j(\gamma \circ v),$$

where $g g_i = g_j \gamma$ and $\gamma \circ v = \rho(\gamma)(v)$ for some $1 \le j \le M, \gamma \in G^s$. The Yetter-Drinfeld module $M(\mathcal{O}, \rho)$ is a braided vector space with braiding given by

$$c(g_i v \otimes g_j w) = t_i \triangleright (g_j w) \otimes g_i v = g_h(\gamma \circ v) \otimes g_i v$$

for any $1 \le i, j \le M$, $v, w \in V$, where $t_i g_j = g_h \gamma$ for unique $h$, $1 \le h \le M$ and $\gamma \in G^s$.

Next, we describe the well-known classification result of finite-dimensional Nichols algebras generated by irreducible Yetter-Drinfeld modules over $D_8$. Recall that the dihedral group $D_8$ is generated by $x$ and $y$ with the following presentation

$$\langle x, y \mid y^2 = 1 = x^4, yxy = x^{-1} \rangle$$

and let $\chi$ be a character of $\langle x \rangle$ such that $\chi(x) = \omega$ is a primitive 4-th root of unity.

**Lemma 7.2.1.** [5 Theorem 3.1] *Let $M(\mathcal{O}, \rho)$ be the irreducible Yetter-Drinfeld module over $D_8$*

*corresponding to a pair $(\mathcal{O},\rho)$. Assume that its Nichols algebra is finite-dimensional, then $(\mathcal{O},\rho)$ is one of the following:*

(i) $(\mathcal{O}_{x^2},\rho)$, *where* $\rho \in \widehat{D_8}$ *satisfies* $\rho(x^2)=1$.

(ii) $(\mathcal{O}_{x^h},\chi^j)$, *where* $h=1$ *or* $3$, *and* $\omega^{hj}=-1$.

(iii) $(\mathcal{O}_y,\mathrm{sgn}\otimes\mathrm{sgn})$ *or* $(\mathcal{O}_y,\mathrm{sgn}\otimes\varepsilon)$, *where* $\mathrm{sgn}\otimes\mathrm{sgn}$, $\mathrm{sgn}\otimes\mathrm{sgn} \in \widehat{D_8^y}$, $D_8^y=\langle y\rangle\oplus\langle x^2\rangle \cong Z_2\times Z_2$.

(iv) $(\mathcal{O}_{xy},\mathrm{sgn}\otimes\mathrm{sgn})$ *or* $(\mathcal{O}_{xy},\mathrm{sgn}\otimes\varepsilon)$, *where* $\mathrm{sgn}\otimes\mathrm{sgn}$, $\mathrm{sgn}\otimes\mathrm{sgn} \in \widehat{D_8^{xy}}$, $D_8^{xy}=\langle xy\rangle\oplus\langle x^2\rangle \cong Z_2\times Z_2$.

**Remark 7.2.2.** (i) In all above cases, $\dim M(\mathcal{O},\rho)=2$ and $\dim\mathcal{B}(\mathcal{O},\rho)=4$.

(ii) It is obvious that

$$M(\mathcal{O}_x,\chi)\cong M(\mathcal{O}_{x^3},\chi^3)$$

as irreducible Yetter-Drinfeld modules. Meanwhile, there are isomorphisms of braided vector spaces

$$M(\mathcal{O}_y,\mathrm{sgn}\otimes\mathrm{sgn})\cong M(\mathcal{O}_{xy},\mathrm{sgn}\otimes\mathrm{sgn}),$$

$$M(\mathcal{O}_y,\mathrm{sgn}\otimes\varepsilon)\cong M(\mathcal{O}_{xy},\mathrm{sgn}\otimes\varepsilon).$$

We will prove all Nichols algebras generated by three pairwise nonisomorphic Yetter-Drinfeld modules over $D_8$ are infinite-dimensional. Our main ingredients are generalized Cartan matrix and Heckenberger's classification of finite-dimensional Nichols algebra of rank $\geq 3$. We first recall the definition of the Cartan matrix. We assume $I$ is a finite non-abelian group in this subsection.

**Definition 7.2.3.** *Let ${}^I_I\mathcal{YD}$ be the Yetter-Drinfeld module category over $I$ and $\theta\in\mathbb{N}$ and $I=\{1,\ldots,\theta\}$. For $N=(N_1,N_2,\ldots,N_\theta)$ where $N_i$ are simple Yetter-Drinfeld module for all $i$, let*

$$a_{ij}^N=\begin{cases}-\infty & \text{if } (\mathrm{ad}\,N_i)^m\left(N_j\right)\neq 0 \text{ for all } m\geq 0,\\ -\sup\left\{m\in\mathbb{N}_0:(\mathrm{ad}\,N_i)^m\left(N_j\right)\neq 0\right\} & \text{otherwise}\end{cases}$$

*for all $i\in I$ and $j\in I\setminus\{i\}$. Moreover, let $a_{ii}^N=2$ for all $i\in I$. Then $A^N=(a_{ij}^N)_{i,j\in I}$ is called the generalized Cartan matrix of N.*

SThus far, the classification of finite-dimensional Nichols algebras in the usual Yetter-Drinfeld module category has made significant progress. In [54], Heckenberger has classified all finite-dimensional Nichols algebra over a non-abelian group of rank $\geq 3$. We briefly recall the relevant results.

**Definition 7.2.4.** [54, Definition 2.1] *Let $\theta\in\mathbb{N}$, $M=(M_1,M_2,\ldots,M_\theta)\in{}^I_I\mathcal{YD}$ with each $M_i$ simple is called braid-indecomposable if there exists no decomposition $M'\oplus M''$ of $\bigoplus_{i=1}^{\theta}M_i$ with $M',M''\neq 0$ such that $(\mathrm{id}-\mathrm{c}^2)(M'\otimes M'')=0$.*

**Definition 7.2.5.** [54, Definition 2.2] *Let $\theta\in\mathbb{N}$, $M=(M_1,M_2,\ldots,M_\theta)\in{}^I_I\mathcal{YD}$ with each $M_i$ simple. Let $A=(a_{ij})$ be the generalized Cartan matrix of $M$, we say $M$ has a skeleton if:*

(1) *for all $1\leq i\leq\theta$, there exists $s_i\in\mathrm{supp}\,M_i$, and $\sigma_i\in\widehat{G^{s_i}}$ such that $M_i\cong M(\mathcal{O}_{s_i},\sigma_i)$, and*

(2) *for all $1\leq i<j\leq\theta$ with $a_{ij}\neq 0$, at least one of $a_{ij},a_{ji}$ is $-1$.*

In this case the skeleton of $M$ is a partially oriented partially labeled loopless graph with $\theta$ vertices with the following properties:

• For all $1 \le i \le \theta$, the $i$-th vertex is symbolized by $|\operatorname{supp} M_i| = \dim M_i$ points. If $\dim M_i = 1$, then the vertex is labeled by $\sigma_i(s_i)$. If $\dim M_i = 2$ and there is an additional restriction on $p = \sigma_i\left(s_i' s_i^{-1}\right)$, where $\operatorname{supp} M_i = \left\{s_i, s_i'\right\}$, then the $i$-th vertex is labeled by $(p)$. Otherwise there is no label.

• For all $i, j \in \{1, \ldots, \theta\}$ with $i \neq j$ there are $a_{ij} a_{ji}$ edges between the $i$-th and $j$-th vertex. The edge is oriented towards $j$ if and only if $a_{ij} = -1, a_{ji} < -1$.

• Let $1 \le i < j \le \theta$ with $a_{ij} < 0$. If $\operatorname{supp} M_i$ and $\operatorname{supp} M_j$ commute, then the connection between the $i$-th and $j$-th vertex consists of continuous lines. Otherwise the connection consists of dashed lines. The connection is labeled with $\sigma_i(s_j)\sigma_j(s_i)$ if $\dim M_i = 1$ or $\dim M_j = 1$, and otherwise it is not labeled.

The next Theorem gives a criterion to determine when $\mathcal{B}(M) \in {}^I_I\mathcal{YD}$ is finite-dimensional.

**Theorem 7.2.6.** [54,Theorem 2.5] *Let $\theta \in \mathbb{N}_{\ge 3}$. Let $I$ be a non-abelian group and $M = (M_1, M_2, ..., M_\theta)$ with each $M_i$ simple and $\operatorname{supp} M$ generates $I$. Assume that $M$ is braid-indecomposable. Then the following are equivalent:*

(1) *$M$ has a skeleton of finite type.*

(2) *$\mathcal{B}(M)$ is finite-dimensional.*

(3) *$M$ admits all reflections and the Weyl groupoid $\mathcal{W}(M)$ of $M$ is finite.*

A complete classification result of skeletons of finite type with at least three vertices over arbitrary field is given simultaneously, see [54].

Let us return to the dihedral group $D_8$ case. There are six nonisomorphic irreducible Yetter-Drinfeld modules over $D_8$. For simplicity, denote $M_1 = M(O_{x^2}, \rho) = \operatorname{span}\{1u_1, 1u_2\}$, $M_2 = M(O_x, \chi) = \operatorname{span}\{1v, yv\}$, $M_3 = M(O_y, \operatorname{sgn} \otimes \operatorname{sgn}) = \operatorname{span}\{1w_1, xw_1\}$, $M_4 = M(O_y, \operatorname{sgn} \otimes \varepsilon) = \operatorname{span}\{1w_2, xw_2\}$, $M_5 = M(O_{xy}, \operatorname{sgn} \otimes \operatorname{sgn}) = \operatorname{span}\{1w_3, xw_3\}$, $M_6 = M(O_{xy}, \operatorname{sgn} \otimes \varepsilon) = \operatorname{span}\{1w_4, xw_4\}$. For simplicity, we denote $S := \left\{M = (M_i, M_j, M_k) \mid 1 \le i < j < k \le 6\right\}$. Now we are going to state the main result of this subsection. Actually, it follows directly from computations.

**Theorem 7.2.7.** *The Nichols algebra $\mathcal{B}(M) = \mathcal{B}(M_i \oplus M_j \oplus M_k)$ is infinite-dimensional for all $1 \le i < j < k \le 6$.*

This theorem relies on the following lemmas. We deal with the cases where $\operatorname{supp}(M)$ is an abelian group at first. In these cases, $M$ can be reduced to a diagonal type Yetter-Drinfeld module over $\operatorname{supp}(M)$.

**Lemma 7.2.8.** *Let $M = (M_i, M_j, M_k)$, where $1 \le i < j < k \le 6$. Suppose*

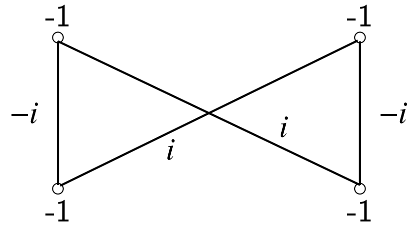
$$M \in S_1 := \{(M_1, M_2, M_k) \mid 3 \le k \le 6\}.$$

*Then $\mathcal{B}(M)$ is infinite-dimensional.*

*Proof.* Note $\operatorname{supp}(M_1 \oplus M_2) = \langle x \rangle \cong Z_4$. By restriction, $\mathcal{B}(M_1 \oplus M_2) \in {}^{Z_4}_{Z_4}\mathcal{YD}$ is of diagonal type. Hence we choose a new basis of $M_1$ by setting $t_1 = 1u_1 + i(1u_2)$ and $t_2 = 1u_1 - i(1u_2)$. Then $M_1 \oplus M_2 = \operatorname{span}\{t_1, t_2, 1v, yv\}$. Direct computation gives the braiding matrix:

$$\begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -i & i & -1 & -1 \\ i & -i & -1 & -1 \end{pmatrix}.$$

The corresponding generalized Dynkin diagram is of the form




which does not appear in the classification of arithmetic root system [47]. So $\mathcal{B}(M_1 \oplus M_2)$ is infinite-dimensional, hence $\mathcal{B}(M)$ is infinite-dimensional.

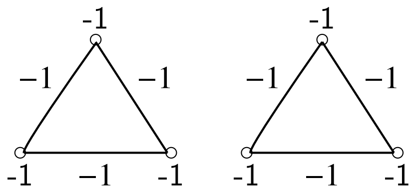
□

**Lemma 7.2.9.** *If $M \in S_2 := \{(M_1, M_3, M_4),\ (M_1, M_5, M_6)\}$, then $\mathcal{B}(M)$ is infinite-dimensional.*

*Proof.* We will prove $\mathcal{B}(M) = \mathcal{B}(M_1 \oplus M_3 \oplus M_4)$ is infinite-dimensional, the other case is similar. Note that $\operatorname{supp}(M_1 \oplus M_4 \oplus M_5) \cong Z_2 \times Z_2 = \langle y \rangle \times \langle x^2 \rangle$ By restriction, $\mathcal{B}(M) \in {}^{Z_2\times Z_2}_{Z_2\times Z_2}\mathcal{YD}$ is of diagonal type, We choose a new basis of $M_1$ via $t_1 = 1u_1 + 1u_2$, $t_2 = 1u_1 - 1u_2$. Then $M = \operatorname{span}\{t_1, t_2, 1w_1, xw_1, 1w_2, xw_2\}$ by direct computation, the corresponding braiding matrix is

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}.$$

The corresponding generalized Dynkin diagram is of the form




which does not appear in the classification of arithmetic root system [47]. So $\mathcal{B}(M)$ is infinite-dimensional. □

We are going to deal with the cases which not appear in Lemmas 7.2.8 and 7.2.9. It is obvious that $\operatorname{supp}(M) = D_8$ in these cases. We are going to use Theorem 7.2.6 to show $\mathcal{B}(M)$ are all infinite-dimensional.

**Lemma 7.2.10.** *Suppose $M = (M_i, M_j, M_k) \in S \setminus S_1 \cup S_2$, then $M$ is braid-indecomposable.*

*Proof.* It is not difficult to observe that as long as for any $i, j$ $1 \le i < j \le 6$, $(\mathrm{id} - c^2)(M_i \otimes M_j) \neq 0$, then $M$ is braid-indecomposable for all $M \in S \setminus S_1 \cup S_2$. In particular, we will not consider braid-indecomposability of $M_1 \oplus M_2$ since $\mathcal{B}(M_1 \oplus M_2)$ is infinite-dimensional by Lemma 7.2.8. We will compute one case as an example and and list the complete results in the following table.

We are going to show $M_1 \otimes M_3$ is braid-indecomposable. Choose $1u_1 \otimes 1w_1 \in M_1 \otimes M_3$. Note $\delta(1u_1) = x^2 \otimes 1u_1$ and $\delta(1w_1) = y \otimes 1w_1$. Then $c(1u_1 \otimes 1w_1) = x^2 \triangleright (1w_1) \otimes 1u_1 = -1w_1 \otimes 1u_1$, and $c(-1w_1 \otimes 1u_1) = -y \triangleright (1u_1) \otimes 1w_1 = 1u_2 \otimes 1w_1$. Hence

$$(\mathrm{id} - c^2)(1u_1 \otimes 1w_1) = 1u_1 \otimes 1w_1 - 1u_2 \otimes 1w_1 \neq 0.$$

**Table 7.1:** Braiding-Indecomposability of $M$

| $M_i \otimes M_j$ | $x \otimes y \in M_i \otimes M_j$, s.t. $(\mathrm{id} - c^2)(x \otimes y) \neq 0$. | $M_i \otimes M_j$ | $x \otimes y \in M_i \otimes M_j$, s.t. $(\mathrm{id} - c^2)(x \otimes y) \neq 0$. |
|---|---|---|---|
| $M_1 \otimes M_3$ | $1u_1 \otimes 1w_1$ | $M_1 \otimes M_4$ | $1u_1 \otimes xw_2$ |
| $M_1 \otimes M_5$ | $1u_1 \otimes 1w_3$ | $M_1 \otimes M_6$ | $1u_1 \otimes xw_4$ |
| $M_2 \otimes M_3$ | $1v \otimes xw_1$ | $M_2 \otimes M_4$ | $1v \otimes xw_2$ |
| $M_2 \otimes M_5$ | $1v \otimes 1w_3$ | $M_2 \otimes M_6$ | $1v \otimes 1w_4$ |
| $M_3 \otimes M_4$ | $1w_1 \otimes xw_2$ | $M_3 \otimes M_5$ | $1w_1 \otimes xw_3$ |
| $M_3 \otimes M_6$ | $1w_1 \otimes xw_4$ | $M_4 \otimes M_5$ | $1w_2 \otimes xw_3$ |
| $M_4 \otimes M_6$ | $1w_2 \otimes xw_4$ | $M_5 \otimes M_6$ | $1w_3 \otimes xw_4$ |

□

**Proposition 7.2.11.** *Suppose $M = (M_i, M_j, M_k) \in S \setminus S_1 \cup S_2$, then $\mathcal{B}(M) = \mathcal{B}(M_i \oplus M_j \oplus M_k)$ is infinite-dimensional.*

*Proof.* We choose a few cases to compute, as the others are similar. The key is to determine the generalized Cartan matrix for each $M$, then draw the corresponding skeleton and finally apply the Theorem 7.2.6 finally.

Take $M = (M_1, M_3, M_5)$, we first calculate the number $a_{13}^M$. Note that $\mathrm{ad}_{1u_1}(1w_1) = 1u_1 \cdot 1w_1 + 1w_1 \cdot 1u_1$, $\mathrm{ad}_{1u_1}(xw_2) = 1u_1 \cdot xw_1 + xw_1 \cdot 1u_1$, $\mathrm{ad}_{1u_2}(1w_1) = 1u_2 \cdot 1w_1 + 1w_1 \cdot 1u_2$, and $\mathrm{ad}_{1u_2}(xw_1) = 1u_2 \cdot xw_1 + xw_1 \cdot 1u_2$. Then we take coproduct of these elements.

$$\begin{aligned}
\Delta(\mathrm{ad}_{1u_1}(1w_1)) &= 1 \otimes \mathrm{ad}_{1u_1}(1w_1) + \mathrm{ad}_{1u_1}(1w_1) \otimes 1 + 1u_1 \otimes 1w_1 - 1u_2 \otimes 1w_1, \\
\Delta(\mathrm{ad}_{1u_1}(xw_1)) &= 1 \otimes \mathrm{ad}_{1u_1}(xw_1) + \mathrm{ad}_{1u_1}(xw_1) \otimes 1 + 1u_1 \otimes xw_1 + 1u_2 \otimes xw_1, \\
\Delta(\mathrm{ad}_{1u_2}(1w_1)) &= 1 \otimes \mathrm{ad}_{1u_2}(1w_1) + \mathrm{ad}_{1u_2}(1w_1) \otimes 1 + 1u_2 \otimes 1w_1 - 1u_1 \otimes 1w_1, \\
\Delta(\mathrm{ad}_{1u_2}(xw_1)) &= 1 \otimes \mathrm{ad}_{1u_2}(xw_1) + \mathrm{ad}_{1u_2}(xw_1) \otimes 1 + 1u_2 \otimes xw_1 - 1u_1 \otimes xw_1.
\end{aligned}$$

Obviously, $\mathrm{ad}_{1u_1}(1w_1) + \mathrm{ad}_{1u_2}(1w_1) = 0$ and $\mathrm{ad}_{1u_1}(xw_1) + \mathrm{ad}_{1u_2}(xw_1) = 0$ by their coproduct. Hence $\mathrm{ad}_{M_1}(M_3) = \mathrm{span}\{\mathrm{ad}_{1u_1}(1w_1), \mathrm{ad}_{1u_2}(xw_1)\}$.

Next, since $1u_1^2 = 1u_2^2 = 1w_1^2 = xw_1^2 = 0$, we have $\mathrm{ad}_{1u_1}(\mathrm{ad}_{1u_1}(1w_1)) = 1u_1 \cdot (1u_1 \cdot 1w_1 + 1w_1 \cdot 1u_1) - x^2 \triangleright (1u_1 \cdot 1w_1 + 1w_1 \cdot 1u_1) 1u_1 = 0$ as well as $\mathrm{ad}_{1u_2}(\mathrm{ad}_{1u_2}(xw_1)) = 0$. Moreover

$$\begin{aligned}
\mathrm{ad}_{1u_2}(\mathrm{ad}_{1u_1}(1w_1)) &= 1u_2 \cdot (1u_1 \cdot 1w_1 + 1w_1 \cdot 1u_1) - x^2 \triangleright (1u_1 \cdot 1w_1 + 1w_1 \cdot 1u_1) \cdot 1u_2 \\
&= 1u_2 \cdot 1u_1 \cdot 1w_1 + 1u_2 \cdot 1w_1 \cdot 1u_1 - 1u_1 \cdot 1w_1 \cdot 1u_2 - 1w_1 \cdot 1u_1 \cdot 1u_2 \\
&= 1u_2 \cdot (-1u_2 \cdot 1w_1 - 1w_1 \cdot 1u_2) - (-1u_2 \cdot 1w_1 - 1w_1 \cdot 1u_2) \cdot 1u_2 = 0.
\end{aligned}$$

where in the last equation we use the fact that $\mathrm{ad}_{1u_1}(1w_1) + \mathrm{ad}_{1u_2}(1w_1) = 0$.We can prove $\mathrm{ad}_{1u_1}(\mathrm{ad}_{1u_2}(xw_1)) = 0$ similarly. Thus $\mathrm{ad}_{M_1}^2(M_3) = 0$.

Using the same method, we can prove $\mathrm{ad}_{M_1}(M_5) \neq 0$ and $\mathrm{ad}_{M_3}(M_5) \neq 0$. But $\mathrm{ad}^2_{M_1}(M_5) = \mathrm{ad}^2_{M_3}(M_5) = 0$. Hence $M = (M_1, M_3, M_5)$ has Cartan matrix $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.

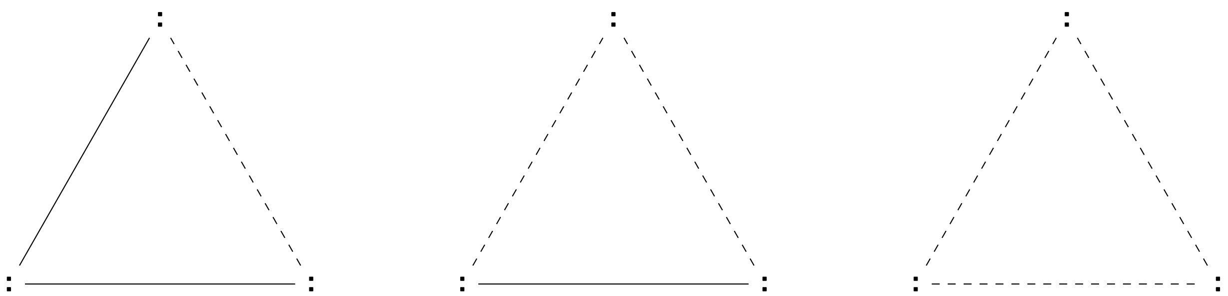
It is not surprising that for all $M \in S \setminus S_1 \cup S_2$, the Cartan matrix of $M$ are all $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$, because their Yetter-Drinfeld module structures are similar. We omit the proof for simplicity.

Although they have the same Cartan matrix, the corresponding skeletons may be different.

If $(i,j,k) \in \{(1,3,5),\ (1,3,6),\ (1,4,5),\ (1,4,6)\}$, the corresponding skeleton will be the first picture.

If $(i,j,k) \in \{(2,3,4),\ (3,4,5),\ (3,4,6),\ (3,5,6),\ (4,5,6),\ (2,5,6)\}$, the corresponding skeleton will be the second picture.

If $(i,j,k) \in \{(2,3,5),\ (2,3,6),\ (2,4,5),\ (2,4,6)\}$, the corresponding skeleton will be the third picture.



None of the skeletons above appear in [54 Figure2.1], hence by Theorem 7.2.6, $\dim(\mathcal{B}(M)) = \infty$ for all $M \in S \setminus S_1 \cup S_2$. □

*Proof of Theorem 7.2.7.* It is direct from Lemma 7.2.8, 7.2.9 and Proposition 7.2.11. □

## 7.3 A second proof of Theorem 7.1.2

Now let $G$ be an abelian group with a nontrivial 3-cocycle $\Phi$ and $H$ is a finite group. Let ${}^G_G\mathcal{YD}^\Phi_{\mathrm{fd}}$ denote the full subcategory of ${}^G_G\mathcal{YD}^\Phi$consisting of all finite-dimensional twisted Yetter-Drinfeld modules. Suppose $F : {}^G_G\mathcal{YD}^\Phi_{\mathrm{fd}} \longrightarrow {}^I_I\mathcal{YD}_{\mathrm{fd}}$ is an equivalence of fusion categories, we have the following lemma.

**Lemma 7.3.1.** *For each $X \in {}^G_G\mathcal{YD}^\Phi_{\mathrm{fd}}$, we have the following equations with respect to dimensions:*

$$\dim_{\Bbbk}(X) = \mathrm{FPdim}(X) = \mathrm{FPdim}(F(X)) = \dim_{\Bbbk}(F(X)).$$

*Proof.* There's an equivalence of fusion categories: ${}^G_G\mathcal{YD}^\Phi_{\mathrm{fd}} \cong \mathcal{Z}(\mathrm{Vec}^\Phi_G) \cong \mathrm{Rep}(D^\Phi(G))$. By [35, Example 5.13.8], $\dim_{\Bbbk}(X) = \mathrm{FPdim}(X)$ for all $X \in {}^G_G\mathcal{YD}^\Phi_{\mathrm{fd}}$.

Note ${}^G_G\mathcal{YD}^\Phi_{\mathrm{fd}}$ and ${}^I_I\mathcal{YD}_{\mathrm{fd}}$ are all fusion categories and $F$ is a tensor functor, then by [35, Proposition 4.5.7]

$$\mathrm{FPdim}_{{}^G_G\mathcal{YD}^\Phi_{\mathrm{fd}}}(X) = \mathrm{FPdim}_{{}^I_I\mathcal{YD}_{\mathrm{fd}}}(F(X)).$$

The last equation $\mathrm{FPdim}(F(X)) = \dim_{\Bbbk}(F(X))$ can be obtained via using the same result as the first equation. □

We return to the situation considered in ${}_G^G\mathcal{YD}^\Phi$, recall here $M_1, M_2, M_3 \in {}_G^G\mathcal{YD}^\Phi$ be three 2-dimensional pairwise non-isomorphic simple objects with $\deg(M_1) = g_1$, $\deg(M_2) = g_2$, $\deg(M_3) = g_3$.

**Proposition 7.3.2.** *The category ${}_G^G\mathcal{YD}_{\mathrm{fd}}^\Phi$ is braided fusion equivalent to ${}_{D_8}^{D_8}\mathcal{YD}_{\mathrm{fd}}$.*

*Proof.* The existence of the finite group $H$ follows immediately from Theorem 6.3.3. Explicitly, take $A = \langle e\rangle$, $K = \langle f\rangle \times \langle g\rangle$, $F = 1$ in Lemma 6.3.1. Let

$$\hat{F}(f^{i_2}g^{i_3}, f^{j_2}g^{j_3}) = \chi^{j_2 i_3},$$

where $\chi \in \widehat{A}$ is primitive such that $\chi(g_1) = -1$. and let $\varepsilon \equiv 1$, By Theorem 6.3.3, $\mathrm{Vec}_G^\omega$ and $\mathrm{Vec}_{\widehat{Z_2}\rtimes_{\hat{F}}(Z_2\times Z_2)}$ are categorically Morita equivalent. Let $H = \widehat{Z_2} \rtimes_{\hat{F}} Z_2 \times Z_2$. Actually, $H$ is isomorphic to $D_8$ since $H$ has the following presentation

$$\begin{aligned} < (1,(f,g)),\ (1,(f,1)) \mid (1,(f,g))^4 = (1,(1,1)) = (1,(f,1))^2, \\ (1,(f,1))\cdot(1,(f,g))\cdot(1,(f,1)) = (1,(f,g))^{-1} > . \end{aligned}$$

Hence

$${}_G^G\mathcal{YD}_{\mathrm{fd}}^\Phi \simeq \mathcal{Z}(\mathrm{Vec}_G^\omega) \simeq \mathcal{Z}(\mathrm{Vec}_{D_8}) \simeq {}_{D_8}^{D_8}\mathcal{YD}_{\mathrm{fd}}$$

as braided fusion category. □

Now we are going to prove Theorem 7.1.4.

*Proof of Theorem 7.1.4.* Let $M = (M_1, M_2, M_3)$ be the 3-tuple. Since $M_1, M_2, M_3$ are pairwise non-isomorphic, simple and $F : {}_G^G\mathcal{YD}_{\mathrm{fd}}^\Phi \longrightarrow {}_{D_8}^{D_8}\mathcal{YD}_{\mathrm{fd}}$ is a braided fusion equivalence then $F(M_1)$, $F(M_2)$ and $F(M_3)$ are pairwise nonisomorphic and simple.

Suppose $\mathcal{B}(M) \in {}_G^G\mathcal{YD}^\Phi{}_{\mathrm{fd}}$ is finite-dimensional, then $F(\mathcal{B}(M))$ must be a finite-dimensional Nichols algebra of rank 3 in ${}_{D_8}^{D_8}\mathcal{YD}_{\mathrm{fd}}$ since $F$ maps Nichols algebra in ${}_G^G\mathcal{YD}^\Phi{}_{\mathrm{fd}}$ to Nichols algebra in ${}_{D_8}^{D_8}\mathcal{YD}_{\mathrm{fd}}$ by Lemma 2.5.4. But all Nichols algebra generated by three pairwise nonisomorphic simple Yetter-Drinfeld module over $D_8$ are infinite-dimensional by Theorem 7.2.7. This is a contradiction, so $\mathcal{B}(M)$ is infinite-dimensional. □

## 7.4 The Cartan graph of $\mathcal{B}(M)$

We denote the Weyl groupoid structure on $\mathcal{G}(M)$ by $\mathcal{W}(\mathcal{G}(M))$. Furthermore, the set of real roots of $\mathcal{G}(M)$ at $X$

$$\Delta^{X,\mathrm{re}} = \{\omega(\alpha_i) \in \mathbb{Z}^{\mathbb{I}} \mid \omega \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}(M)), X), i \in \mathbb{I}\}$$

can be formulated explicitly. Since $\mathcal{G}(M)$ is standard, it is obvious that $\Delta^{X,\mathrm{re}} = \Delta^{Y,\mathrm{re}}$ for all $X, Y \in F_3(M)$ and

$$\mathrm{Hom}(\mathcal{W}(\mathcal{G}(M)), X) \to W(A_2^{(1)}), \quad (Y, s, X) \mapsto s$$

is bijective, where $W(A_2^{(1)})$ is the Weyl group of the affine Lie algebra of type $A_2^{(1)}$.

For an affine Lie algebra of type $A_2^{(1)}$ (and more generally for any symmetrizable Kac–Moody algebra), the set of real roots is closed under the action of the Weyl group, and every real root can be obtained from a simple root by applying a sequence of fundamental reflections, see [66, Proposition 5.1]. If we denote $\delta = \alpha_1 + \alpha_2 + \alpha_3$. Then

$$\Delta^{M,\mathrm{re}} = \{\alpha + k\delta \mid \alpha \in \Phi_{A_2},\ k \in \mathbb{Z}\}. \tag{7.4.1}$$

Here $\Phi_{A_2} = \pm\{\alpha_2, \alpha_3, \alpha_2 + \alpha_3\}$, which corresponds to the root of Lie algebra of type$A_2$. This agreement is expected because the semi-Cartan graph is standard, and thus $\Delta^{M,\mathrm{re}}$ coincides with the set of real roots for the affine Lie algebra of type $A_2^{(1)}$. Unfortunately, $\Delta^{M,\mathrm{re}}$ is an infinite set, by definition $\mathcal{G}(M)$ is not a finite semi-Cartan graph.

**Definition 7.4.1.** (1) *A semi-Cartan graph $\mathcal{G}$ is called connected if the groupoid $\mathcal{W}(\mathcal{G})$ is connected, that is, if for any two objects $X, Y$ of $\mathcal{G}$ there is a morphism from $X$ to $Y$ in $\mathcal{W}(\mathcal{G})$.*

(2) *A semi-Cartan graph $\mathcal{G}$ is called simply connected if for any two points $X, Y$ of $\mathcal{G}$ there is at most one morphism from $X$ to $Y$ in $\mathcal{W}(\mathcal{G})$.*

(3) *For any $X \in \mathcal{X}$ and $i, j \in \mathbb{I}$ let*

$$m_{ij}^X = |\Delta^{X\,re} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|.$$

*We say that $\mathcal{G}$ is a Cartan graph if the following hold.*

(CG3) *For all $X \in \mathcal{X}$, the set $\Delta^{X\,re}$ consists of positive and negative roots.*

(CG4) *Let $X \in \mathcal{X}$, and $i, j \in \mathbb{I}$. If $m_{ij}^X < \infty$, then $(r_i r_j)^{m_{ij}^X}(X) = X$.*

**Proposition 7.4.2.** *The quadruple $\mathcal{G}(M)$ is a connected and simply connected Cartan graph.*

*Proof.* Since $M$ admits all reflections, $\mathcal{G}(M)$ is a connected semi-Cartan graph. Moreover, the fact that $\mathrm{Hom}([M],[M]) = \mathrm{id}_{[M]}$ implies that $\mathcal{G}(M)$ is simply connected.

The condition (CG3) is obvious, since $\Delta^{X,\mathrm{re}} = \Delta^{Y,\mathrm{re}}$ for all $X, Y \in F_3(M)$ and $\Delta^{M,\mathrm{re}}$ consists of positive and negative roots.

For (CG4), by definition,

$$m_{12}^M = |\ \Delta^{M,\mathrm{re}} \cap (\mathbb{N}_0\alpha_1 + \mathbb{N}_0\alpha_2)\ | = 3.$$

Now by direct computation of reflection, see [69, Lemma 6.12],

$$\begin{aligned}(r_1r_2)^3(M) &\cong r_1r_2r_1r_2r_1(M_4, M_2, M_6) \cong r_1r_2r_1r_2(M_4, M_1, M_5) \cong r_1r_2r_1(M_2, M_1, M_3)\\ &\cong r_1r_2(M_2, M_4, M_6) \cong r_1(M_1, M_4, M_5) \cong (M_1, M_2, M_3).\end{aligned}$$

We deduce that $(r_1r_2)^3([M]) = [M]$. Similarly, for all $i, j \in \{1,2,3\}$ and $X \in \mathcal{X}$, one can prove $m_{ij}^X = 3$, and

$$(r_ir_j)^3([X]) = [X].$$

Hence $\mathcal{G}(M)$ is a Cartan graph. □

**Definition 7.4.3.** *Let $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be a Cartan graph. For all $X \in \mathcal{X}$ let $R^X$ be a subset of $\mathbb{Z}^{\mathbb{I}}$ with the following properties.*

(1) *$0 \notin R^X$ and $\alpha_i \in R^X$ for all $X \in \mathcal{X}$ and $i \in \mathbb{I}$.*

(2) $R^X \subseteq \mathbb{N}_0^{\mathbb{I}} \cup -\mathbb{N}_0^{\mathbb{I}}$ *for all* $X \in \mathcal{X}$.
(3) *For any* $X \in \mathcal{X}$ *and* $i \in \mathbb{I}$, $s_i^X(R^X) = R^{r_i(X)}$.
(4) *If* $i, j \in \mathbb{I}$ *and* $X \in \mathcal{X}$ *such that* $i \neq j$ *and* $m_{ij}^X$ *in Definition 7.4.1 is finite, then* $(r_i r_j)^{m_{ij}^X}(X) = X$.
*Then we say that the pair* $(\mathcal{G}, (R^X)_{X\in\mathcal{X}})$ *is a root system over* $\mathcal{G}$. *A root system over* $\mathcal{G}$ *is said to be reduced if for all* $X \in \mathcal{X}$ *and* $\alpha \in R^X$ *the roots* $\alpha$ *and* $-\alpha$ *are the only rational multiples of* $\alpha$ *in* $R^X$. *A root system is said to be finite if for all* $X \in \mathcal{X}$, $R^X$ *is finite.*

If $\mathcal{G}$ is a Cartan graph, the pair $(\mathcal{G}, (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}})$ is a reduced root system over $\mathcal{G}$ by [52, Example 10.4.4]. The following corollary follows immediately.

**Corollary 7.4.4.** *Let* $M$ *be the* 3*-tuple as above, then* $(\mathcal{G}(M), (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}})$ *is a reduced root system over* $\mathcal{G}(M)$. *Furthermore, there is no finite root system over* $\mathcal{G}$.

*Proof.* The second claim follows from the fact that $\mathcal{G}$ is not a finite Cartan graph and [52, Theorem 10.4.7]. □

## 7.5 The Tits cone induced by $(\mathcal{G}(M), (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}})$ is a half plane

In [28,29], the authors showed that most Nichols algebras over Hopf algebras can define a Tits cone, via their root systems. In particular, they call a Nichols algebra affine if the corresponding Tits cone is a half plane. The following is their main observation:

**Theorem 7.5.1.** [29, Theorem 1.1] *There exists a one-to-one correspondence between connected, simply connected Cartan graphs permitting a root system and crystallographic Tits arrangements with reduced root system.*

*Under this correspondence, equivalent Cartan graphs correspond to combinatorially equivalent Tits arrangements and vice versa, giving rise to a one-to-one correspondence between the respective equivalence classes.*

A crystallographic Tits arrangement consists of a pair $(\mathcal{A}, T)$, where $\mathcal{A}$ is a set of linear hyperplanes in $\mathbb{R}^r$, and $T$ is a Tits cone in $\mathbb{R}^r$, satisfying some additional conditions. For explicit definition of crystallographic Tits arrangements, one may refer to [29, Definition 3.1].

Now return to the 3-tuple $M$ in ${}_G^G\mathcal{YD}^\Phi$, we have proved that $(\mathcal{G}(M), (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}})$ is a connected and simply connected Cartan graph with a reduced root system, thus it should produce a Tits cone in $\mathbb{R}^3$. Our goal in this subsection is to calculate the corresponding Tits cone.

We briefly recall the calculation of the Tits cone when given a connected simply connected Cartan graph $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, (r_i)_{i\in I}, (A^X)_{X\in\mathcal{X}})$ with a root system $R = (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}}$. Fix an object $A \in \mathcal{X}$. Let $V = \mathbb{R}^r$ (where $r = |I|$) and fix a linear isomorphism $\psi : \mathbb{Z}^I \to V^*$ sending the standard basis to a basis $\{\phi_1, \ldots, \phi_r\}$ of $V^*$. Define the set of roots $\mathcal{R} = \psi(\Delta^{A,\mathrm{re}}) \subset V^*$.

For any object $B \in \mathcal{X}$, since $\mathcal{G}$ is connected and simply connected, there exists a unique morphism $w_B \in \mathrm{Hom}(A, B)$ in the Weyl groupoid. Define the linear map $\psi_B = \psi \circ w_B^{-1} : \mathbb{Z}^I \to V^*$. Then the set $\mathcal{R}^B := \psi_B(\{e_i\}_{i\in I})$ forms a basis of $V^*$, called the root basis at $B$. The corresponding chamber is defined as the open simplicial cone

$$K^B := \bigcap_{\beta\in\mathcal{R}^B} \beta^+ = \{x \in V \mid \beta(x) > 0 \text{ for all } \beta \in \mathcal{R}^B\}.$$

Let $\mathcal{K} = \{K^B \mid B \in \mathcal{X}\}$. The set of hyperplanes is

$$\mathcal{A} := \{\phi^\perp \mid \phi \in \mathcal{R}\}.$$

Finally, the Tits cone is defined as the convex hull of all chambers:

$$T := \operatorname{conv}\left(\bigcup_{X\in\mathcal{X}} K^X\right).$$

Theorem 7.5.1 guarantees that $(\mathcal{A}, T)$ is a crystallographic Tits arrangement. The next proposition seems to be new, although those who are familiar with affine Nichols algebras may regard it as a known fact.

**Proposition 7.5.2.** *Let $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be a connected, simply connected standard Cartan graph, with $(\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}}$ a reduced root system over $\mathcal{G}$. If the Cartan matrix $A$ is affine, then the corresponding Tits cone $T$ in the sense of Theorem 7.5.1 is a half plane.*

*Proof.* We denote $r = |\mathbb{I}|$. Since $A$ is affine, it is automatically symmetrizable, then there exists a positive vector $v = (v_1, v_2, .., v_r) \in \mathbb{Z}_{\geq 0}$ such that $v^T A = 0$, that is

$$\sum_{i=1}^{r} v_i c_{ij} = 0 \text{ for all } j.$$

For simplicity, we use $X$ to represent its isomorphism class $[X] \in \mathcal{X}$. We fix a base object $M$. Let $V = \mathbb{R}^r$ and fix a linear isomorphism $\psi : \mathbb{Z}^r \to V^*$ sending the standard basis vectors $\{\alpha_1, \alpha_2, ..., \alpha_r\}$ to a basis $\{\phi_1, \phi_2, ..., \phi_r\}$ of $V^*$. Define the set of roots $\mathcal{R} = \psi(\Delta^{M,\mathrm{re}}) \subset V^*$.

For any object $X \in \mathcal{X}$, there is a unique morphism $w_X \in \mathrm{Hom}(M, X)$ in the Weyl groupoid. Define $\psi_X = \psi \circ w_X^{-1} : \mathbb{Z}^r \to V^*$. The set of root basis at $X$ is

$$B^X = \psi_X(\{\alpha_1, \alpha_2, ..., \alpha_r\}),$$

and the associated chamber is

$$K^X = \bigcap_{\beta\in B^X} \beta^+ = \{x \in V \mid \beta(x) > 0 \text{ for all } \beta \in B^X\}.$$

We now prove that

$$T = \operatorname{conv}\left(\bigcup_{X\in\mathcal{X}} K^X\right)$$

coincides with the half-space $\delta^+$ where $\delta = v_1\phi_1 + v_2\phi_2 + \ldots + v_r\phi_r$.

Now let $X, Y \in \mathcal{X}$, such that $r_i(X) = Y$ for some $i \in I$. By [29, Proposition 3.5], if $B^X = \{\beta_1, \beta_2, ...\beta_r\}$ is indexed compatibly with $B^Y = \{\beta'_1, \beta'_2, \beta'_r\}$, then

$$\beta'_i = -\beta_i, \qquad \beta'_j = \beta_j - c_{ji}\beta_i \quad (j \neq i).$$

Consequently,

$$\sum_{j=1}^{r} v_j\beta_j' = \sum_{j\neq i} v_j(\beta_j - c_{ji}\beta_i) - v_i\beta_i$$
$$= \sum_{j\neq i} v_j\beta_j - \sum_{j\neq i}(v_jc_{ji} - v_i)\beta_i$$
$$= \sum_{j\neq i} v_j\beta_j + v_i\beta_i = \sum_{j=1}^{r} v_j\beta_j.$$

Note that in this case $K^X$ and $K^Y$ are adjacent chambers, hence the sum $\delta_X := \sum_{\beta\in B^X}\beta$ is invariant under adjacent chambers. Since the Cartan graph is connected, this sum is independent of the chamber $X$; we denote this common value by $\delta$.

For any $X \in \mathcal{X}$ and any $x \in K^X$, we have $\beta(x) > 0$ for all $\beta \in B^X$. Hence

$$\delta(x) = \sum_{\beta\in B^X} \beta(x) > 0.$$

So $K^X \subset \delta^+$. Since $\delta^+$ is convex, the convex hull $T$ of the union of all chambers also satisfies $T \subseteq \delta^+$.

Consider the affine plane $H_1 = \{x \in V \mid \delta(x) = 1\}$. The intersection of each chamber $K^X$ with $H_1$ is non-empty because for any $x \in K^X$ we have $\delta(x) > 0$, so $x/\delta(x) \in K^X \cap H_1$. Moreover, each chamber $K^X$ is a r-dimensional open simplicial cone. The intersection of $K^X$ with $H_1$ is a non-empty $(r-1)$-dimensional simplex. In the theory of affine reflection groups, $\Delta := K^X \cap H_1$ is known as a fundamental alcove. A classical result [65, Chapter 4] states that the affine Weyl group $W = W(A)$ acts simply transitively on the set of alcoves and that the closures of the alcoves tile the Euclidean space $H_1$, i.e.

$$H_1 = \bigcup_{w\in W} w(\overline{\Delta}).$$

and the interiors of distinct alcoves are disjoint. Hence the collection $\{K^X \cap H_1\}_{X\in\mathcal{X}}$ forms a tessellation of $H_1$. It covers $H_1$ completely and the interiors of different members do not overlap.

The point $x/\delta(x)$ must lie in the closure of some simplex. Consequently, we can find finitely many vertices $p_1, \ldots, p_n$ of these simplices, where each vertex belongs to some chamber $K^{X_i} \cap H_1$ and non-negative coefficients $t_1, \ldots, t_n$ with $\sum_{i=1}^n t_i = 1$ such that

$$x = \sum_{i=1}^{n} \delta(x)t_ip_i = \sum_{i=1}^{n} t_i(\delta(x)p_i).$$

Each $\delta(x)p_i$ belongs to $K^{X_i}$ because $K^{X_i}$ is a cone. Hence $x$ is a convex combination of points from $\bigcup_X K^X$, so $x \in T$. This proves $\delta^+ \subseteq T$. Now we obtain $T = \delta^+$. Thus, by the philosophy of Theorem 7.5.1, the Tits cone corresponding to $(\mathcal{G}, (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}})$ is a half plane. □

Now return to the 3-tuple $M$ in ${}^G_G\mathcal{YD}^\Phi$, we have proved that $(\mathcal{G}(M), (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}})$ is a connected and simply connected standard Cartan graph with a reduced root system, thus it should produce a Tits cone in $\mathbb{R}^3$.

**Corollary 7.5.3.** *Under the equivalence of Theorem 7.5.1, the Tits cone corresponding to $(\mathcal{G}(M), (\Delta^{X,\mathrm{re}})_{X\in\mathcal{X}})$ is a half plane. That is, the Nichols algebra $\mathcal{B}(M)$ is an affine Nichols algebra.*

**Remark 7.5.4.** (1) This means we realize an affine Nichols algebra over a coquasi-Hopf algebra, which broadens the approaches to realizing affine Nichols algebras.

(2) We hope to prove that a semi-Cartan graph induced by a Nichols algebra over ${}^H_H\mathcal{YD}$ is actually a Cartan graph, which would endow it with more beautiful properties.

# Chapter 8

# Cohomology of Pointed Finite Tensor Categories

We consider the finite generation property for cohomology algebra of pointed finite tensor categories via de-equivariantization and exact sequence of finite tensor categories. As a result, we prove that for all finite-dimensional coradically graded coquasi-Hopf algebras over abelian group, their corepresentation categories have finitely generated cohomology.

## 8.1 De-equivariantization of finite tensor categories

In this subsection, we review the definition of de-equivariantization and present key results related to this construction.

**Definition 8.1.1.** (i) *Let $\mathcal{C}$ be a finite braided tensor category. Suppose there is a braided tensor functor $\widetilde{i}: \mathcal{C} \to Z(\mathcal{D})$, such that the functor $Q: \mathcal{C} \xrightarrow{\widetilde{i}} Z(\mathcal{D}) \xrightarrow{F} \mathcal{D}$ is a fully faithful tensor functor, where $F$ is the forgetful functor. Then we say $i$ is a central embedding.*

(ii) *For any central embedding $\mathrm{Rep}(G) \to \mathcal{D}$, we can define the de-equivariantization $\mathcal{D}_G$ , which is the tensor category of $\mathcal{O}(G)$-modules in $\mathcal{D}$, where $\mathcal{O}(G)$ is the linear dual of the group algebra.*

Let $G$ be a finite group. There exists a profound relationship between braided central Hopf subalgebras of a Hopf algebra $H$ and central embeddings $\mathrm{Rep}(G) \to {}^H_H\mathcal{YD}$.

**Definition 8.1.2.** *Let $H$ be a finite-dimensional Hopf algebra. A braided central Hopf subalgebra of $H$ is a pair $(K, r)$, where $K \subset H$ is a Hopf subalgebra, and $r: H \otimes K \to k$ is a bilinear form such that:*

$$r(hh', k) = r(h', k_1)\, r(h, k_2), \tag{8.1.1}$$

$$r(h, kk') = r(h_1, k)\, r(h_2, k'), \tag{8.1.2}$$

$$r(h,1) = \varepsilon(h), \quad r(1,k) = \varepsilon(k), \tag{8.1.3}$$

$$r(h_1, k_1)\, k_2 h_2 = h_1 k_1 r(h_2, k_2), \tag{8.1.4}$$

$$r(k, k') = \varepsilon(kk'), \tag{8.1.5}$$

*for all $k, k' \in K$, $h, h' \in H$.*

**Lemma 8.1.3.** [14, Theorem 3.4] *Let $H$ be a Hopf algebra and $K \subset H$ a commutative Hopf subalgebra. Then the following set of data are equivalent:*

(1) *A map $r : H \otimes K \to k$ such that $(K, r)$ is a braided central Hopf subalgebra of $H$.*

(2) *A braided tensor functor $F : {}^K\mathcal{M} \to \mathcal{Z}({}^H\mathcal{M}) = {}^H_H\mathcal{YD}$ such that the composition with the forgetful functor $\mathcal{Z}({}^H\mathcal{M}) = {}^H_H\mathcal{YD} \to {}^H\mathcal{M}$ is fully faithful.*

## 8.2 Cohomology of finite tensor categories

In this section, $\mathcal{C}$ and $\mathcal{D}$ are finite tensor categories. We present essential lemmas regarding their cohomology for subsequent applications.

**Lemma 8.2.1.** [40, Theorem 1.2.1] *Let $H$ be a finite-dimensional pointed Hopf algebra whose group of group-like elements is abelian. Then $\mathrm{Comod}(H)$ as well as $\mathrm{Rep}(H)$ satisfies* **FGC**.

**Lemma 8.2.2.** [81, Proposition 3.3] *If $F : \mathcal{D} \to \mathcal{C}$ is a surjective tensor functor, and $\mathcal{D}$ satisfies* **FGC**, then $\mathcal{C}$ satisfies **FGC**.

**Lemma 8.2.3.** [81, Theorem 4.4] *Suppose $F : \mathcal{D} \to \mathcal{C}$ is a de-equivariantization of $\mathcal{D}$ with respect to a central embedding $\mathrm{Rep}(G) \to \mathcal{D}$, or equivalently an equivariantization of $\mathcal{C}$ with respect to a $G$-action. Then $\mathcal{D}$ satisfies* **FGC** if and only if $\mathcal{C}$ satisfies **FGC**.

The authors of [34] generalized Bruguieres and Natale's definition of exact sequences of tensor categories, introducing a new notion of exact sequences with respect to module categories. For brevity, we omit the explicit definition here.

**Lemma 8.2.4.** [81, Corollary 8.13] *Let*

$$\mathcal{B} \longrightarrow \mathcal{C} \longrightarrow \mathcal{D} \boxtimes \mathrm{End}(\mathcal{M})$$

*be an exact sequence of finite tensor categories. In particular, if we assume that $\mathcal{D}$ is a fusion category, then there is a natural identification of $\Bbbk$-algebras*

$$\mathrm{H}^\bullet(\mathcal{B}, 1) = \mathrm{H}^\bullet(\mathcal{C}, 1).$$

*Furthermore, for any object $V$ in $\mathcal{C}$ , there is an identification of $\mathrm{H}^\bullet(\mathcal{C}, 1)$-modules, $\mathrm{H}^\bullet(\mathcal{B}, \mathscr{H}^0_{\mathcal{C}}(\mathcal{D}, V)) = \mathrm{H}^\bullet(\mathcal{C}, V)$, where the definition of $\mathscr{H}^0_{\mathcal{C}}$ is given in [81, Definition 8.10].*

## 8.3 Proof of diagonal type case

In this section, given a finite-dimensional coradically graded coquasi-Hopf algebra of diagonal type, we prove that $\mathrm{Comod}(M)$ satisfies **FGC**. We begin by recalling key classification results.

In this section, let $G \cong Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_n} = \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_n \rangle$ with an abelian 3-cocycle $\omega_{\underline{a}}$ and $\mathbb{G} \cong Z_{m_1^2} \times Z_{m_2^2} \times \cdots \times Z_{m_n^2} = \langle \mathrm{g}_1 \rangle \times \langle \mathrm{g}_2 \rangle \times \cdots \times \langle \mathrm{g}_n \rangle$. We assume $\mathcal{B}(V) \in {}^G_G\mathcal{YD}^{\Phi_{\underline{a}}}$ is a finite-dimensional Nichols algebra of diagonal type. Without loss of generality, we can assume $G = G_V$ (the support group of $V$). In [58 Section 4], the authors provided a method to classify such twisted Nichols algebras. Here, we adopt an equivalent approach to describe the 'return trip' transformation between twisted and ordinary Nichols algebras for later use.

$$\begin{array}{c} \text{Given } \mathcal{B}(V) \in {}_G^G\mathcal{YD}^{\Phi_{\underline{a}}} \\ \downarrow \\ \mathcal{B}(V) \cong \mathcal{B}(V_{\mathcal{D}}) \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})} \\ \downarrow \\ \mathcal{B}(V_{\mathcal{D}}) \text{ is twisted equivalent to } \mathcal{B}(V_{\mathcal{D}})^{J_{\underline{a}}^{-1}} \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD} \end{array}$$

Figure I

$$\begin{array}{c} \mathcal{B}(V')^{J_{\underline{a}}} \cong \mathcal{B}(V) \in {}_G^G\mathcal{YD}^{\Phi_{\underline{a}}} \\ \uparrow \text{If [58, (4.8)] holds} \\ \mathcal{B}(V')^{J_{\underline{a}}} \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\partial(J_{\underline{a}})} \\ \uparrow \\ \text{Given Nichols algebra } \mathcal{B}(V') \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD} \end{array}$$

Figure II

Let us illustrate the above figures explicitly via the language of root data. Let $\mathcal{D} = (\mathcal{D}_{\chi,E}, S, X)$ be a root data, where:

- $\mathcal{D}_{\chi,E}$ is a Dynkin diagram of an arithmetic root system $\Delta_{\chi,E}$;
- $S = (s_{ij})_{m\times n}$, $X = (x_{ij})_{n\times m}$ are two matrices satisfying certain conditions (see [58, Definition 4.14]).

By [58, Corollary 4.13], for each root data $\mathcal{D}$, there exists unique $\underline{a} \in \mathcal{A}'$ such that the following equation holds for all $1 \le i \le n$ and $1 \le j \le m$.

$$\zeta_{m_i^2}^{m_i x_{ij}} = \zeta_{m_i^2}^{a_i s_{ji} m_i} \prod_{i<k\le n} \zeta_{m_i m_k}^{a_{ik} s_{jk} m_i} \tag{8.3.1}$$

For each root data $\mathcal{D}$, we can define a Nichols algebra $\mathcal{B}(V_{\mathcal{D}}) \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ as follows. Let $V_{\mathcal{D}}$ be the Yetter–Drinfeld module in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ with a canonical basis $\{X_i \mid 1 \le i \le m\}$ such that

$$\delta_L(X_i) = \prod_{k=1}^{n} \mathrm{g}_k^{s_{ik}} \otimes X_i, \quad \mathrm{g}_i \triangleright X_j = \zeta_{m_i^2}^{x_{ij}} \frac{J_{\underline{a}}\left(g_i, \prod_{k=1}^n \mathrm{g}_k^{s_{ik}}\right)}{J_{\underline{a}}\left(\prod_{k=1}^n \mathrm{g}_k^{s_{ik}}, g_i\right)} X_j.$$

The second statement of [58, Theorem 4.15] states each twisted Nichols algebra in ${}_G^G\mathcal{YD}^{\Phi_{\underline{a}}}$ must be isomorphic to $\mathcal{B}(V_{\mathcal{D}}) \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ for some root data $\mathcal{D}$. This is exactly what Figure I says.

The first part of that theorem shows that each Nichols algebra in ${}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ satisfying equation (8.3.1) must be isomorphic to $\mathcal{B}(V_{\mathcal{D}}) \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ for some root data $\mathcal{D}$. Moreover, $\mathcal{B}(V_{\mathcal{D}})$ will be uniquely isomorphic to a twisted Nichols algebra in ${}_G^G\mathcal{YD}^{\Phi_{\underline{a}}}$. This corresponds to Figure II.

This completes the classification of finite-dimensional twisted Nichols algebras of diagonal type, and consequently, all finite-dimensional coquasi-Hopf algebras of diagonal type.

For each $V' \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$, one may define a finite-dimensional Nichols algebra $\mathcal{B}(V') \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*(\Phi_{\underline{a}})}$ and thus a finite-dimensional coquasi-Hopf algebra $H^{J_{\underline{a}}} = \mathcal{B}(V')\#\Bbbk\mathbb{G}$. Through the processes of twisting and bosonization, we obtain a finite-dimensional coradically graded pointed Hopf algebra:

$$H = \mathcal{B}(V')^{J_{\underline{a}}^{-1}}\#\Bbbk\mathbb{G}.$$

On the other hand, if $\underline{a}$ satisfies equation (8.3.1), we can define a finite-dimensional coquasi-Hopf algebra

$$M \cong \mathcal{B}(V)\#\Bbbk G,$$

where $\mathcal{B}(V)$ is isomorphic to $\mathcal{B}(V')$. Note that $\mathbb{G}$ contains a subgroup $\widetilde{\mathbb{G}} = \langle \mathrm{g}_1^{m_1} \rangle \times \langle \mathrm{g}_2^{m_2} \rangle \times \cdots \times$

$\langle g_n^{m_n} \rangle \cong G$. We now state our main result:

**Proposition 8.3.1.** *With the above notions, if equation (8.3.1) holds for all $1 \le i \le n$ and $1 \le j \le m$, then we have such a tensor equivalence*

$$\mathrm{Comod}(H)_{\widetilde{\mathbb{G}}} \cong \mathrm{Comod}(M). \tag{8.3.2}$$

*Proof.* Let $\chi_i$ be the generator of $\widehat{Z_{m_i^2}}$ such that $\chi_i(g_i) = \zeta_{m_i^2}$ for all $1 \le i \le n$. Define $\phi : \widetilde{\mathbb{G}} \to \widehat{\mathbb{G}}$ by:

$$\phi((g_1^{m_1})^{y_1} \cdots (g_n^{m_n})^{y_n}) = \prod_{l=1}^{n} \chi_l^{a_l y_l m_l} \prod_{1 \le j < k \le n} \chi_k^{a_{jk} y_j m_k}. \tag{8.3.3}$$

Let $k = (g_1^{m_1})^{y_1} \cdots (g_n^{m_n})^{y_n} \in \widetilde{\mathbb{G}}$. Then for any $k' \in \widetilde{\mathbb{G}}$ and each generator $X_j$, $1 \le j \le m$, we have

$$< k', \phi(k) >= 1,$$

$$k \triangleright X_j = \prod_{i=1}^{n} \zeta_{m_i^2}^{m_i y_i x_{ij}} X_j \overset{(8.3.1)}{=} \prod_{i=1}^{n} \zeta_{m_i^2}^{a_i y_i s_{ji} m_i} \prod_{1 \le i < l \le n} \zeta_{m_i m_l}^{a_{il} s_{jl} y_i m_i} X_j =< \prod_{l=1}^{n} g_l^{s_{jl}}, \phi(k) > X_j.$$

Hence, the pair $(\widetilde{\mathbb{G}}, \phi)$ satisfies the condition (b) in [14, Proposition 4.3], yielding a braided central Hopf subalgebra $(\Bbbk\widetilde{\mathbb{G}}, r)$ of $H$. As $\widetilde{\mathbb{G}}$ is abelian, we have $\mathrm{Rep}(\widetilde{\mathbb{G}}) \cong \mathrm{Comod}(\Bbbk\widetilde{\mathbb{G}})$ as fusion categories. By Lemma 8.1.3, there exists a braided tensor functor $F : \mathrm{Rep}(\widetilde{\mathbb{G}}) \to \mathcal{Z}({}^H\mathcal{M}) = {}^H_H\mathcal{YD}$ whose composition with the forgetful functor $\mathcal{Z}({}^H\mathcal{M}) = {}^H_H\mathcal{YD} \to {}^H\mathcal{M}$ is fully faithful. This guarantees the existence of the de-equivariantization $\mathrm{Comod}(H)$, which is defined as the tensor category of left $\Bbbk\widetilde{\mathbb{G}}$-modules in $\mathrm{Comod}(H)$.

On the other hand, there is an epimorphism of coquasi-Hopf algebras:

$$\pi := (f \otimes p) : \ H^{J_{\underline{a}}} = \mathcal{B}(V_{\mathcal{D}}) \# \Bbbk\mathbb{G} \longrightarrow M = \mathcal{B}(V) \# \Bbbk G.$$

where $p : \Bbbk\mathbb{G} \to \Bbbk G$ is given by $g_i \mapsto g_i$ is surjective and $f : \mathcal{B}(V_{\mathcal{D}}) \to \mathcal{B}(V)$ is an isomorphism. Direct computation shows:

$$\Bbbk\widetilde{G} = (H^{J_{\underline{a}}})^{\mathrm{co}\pi} := \left\{x \in H^{J_{\underline{a}}} \mid (\mathrm{id} \otimes \pi)(\Delta(b)) = b \otimes 1\right\}.$$

It follows by [26, Example 6.4] that $\Bbbk\widetilde{\mathbb{G}}$ admits a structure of commutative algebra in $\mathcal{Z}(\mathrm{Comod}(H^{J_{\underline{a}}}))$, making the tensor category of left $\Bbbk\widetilde{\mathbb{G}}$-modules in $\mathrm{Comod}(H^{J_{\underline{a}}})$ tensor equivalent to $\mathrm{Comod}(M)$. Note that, $\mathrm{Comod}(H) \cong \mathrm{Comod}(H^{J_{\underline{a}}})$ as tensor categories since $H^{J_{\underline{a}}}$ and $H$ differ by a cocycle-deformation. Hence the tensor category of left $\Bbbk\widetilde{\mathbb{G}}$-modules in $\mathrm{Comod}(H)$ is tensor equivalent to $\mathrm{Comod}(M)$. We conclude: $\mathrm{Comod}(H)_{\widetilde{\mathbb{G}}} \cong \mathrm{Comod}(M)$. □

**Theorem 8.3.2.** *For each finite-dimensional coradically graded coquasi-Hopf algebra $M$ of diagonal type,* $\mathrm{Comod}(M)$ *satisfies* **FGC**.

*Proof.* Given a finite-dimensional coradically graded pointed coquasi-Hopf algebra $M \cong \mathcal{B}(V) \# \Bbbk G$ of diagonal type, Figure I associates it to a unique Hopf algebra $H \cong \mathcal{B}(V_{\mathcal{D}})^{J_{\underline{a}}^{-1}} \# \Bbbk\mathbb{G}$. The equation (8.3.1) holds automatically for $B(V_{\mathcal{D}})$ by [58, Theorem 4.15]. Then by Proposition 8.3.1, there is a subgroup $\widetilde{\mathbb{G}} \subseteq \mathbb{G}$ such that $\mathrm{Comod}(H)_{\widetilde{\mathbb{G}}} \cong \mathrm{Comod}(M)$. Since $\mathrm{Comod}(H)$ satisfies **FGC** by Lemma 8.2.2, so does $\mathrm{Comod}(M)$ by Lemma 8.2.3. □

**Remark 8.3.3.** The referee pointed out that [14, Proposition 4.3] can simplify the proof of Proposition 8.3.1. We acknowledge the referee for providing this better approach.

## 8.4 Non-diagonal type case and proof of Theorem 8.4.5

The classification result of finite-dimensional twisted Nichols algebras of non-diagonal type is much simpler, but the procedure is more difficult, see [60]. We first characterize all finite-dimensional simple twisted Nichols algebras of non-diagonal type over abelian groups.

**Lemma 8.4.1.** [60, Proposition 2.12] *Suppose $V \in {}_G^G\mathcal{YD}^\Phi$ is a simple twisted Yetter-Drinfeld module of non-diagonal type, $\deg(V) = g$. Then $\mathcal{B}(V)$ is finite-dimensional if and only if V is one of the following two cases:*

(C1) $g \triangleright v = -v$ for all $v \in V$,

(C2) $\dim(V) = 2$ and $g \triangleright v = \zeta_3 v$ for all $v \in V$.

**Remark 8.4.2.** In [60], the authors showed the following results.

(1) For a rank-2 Yetter-Drinfeld module $V = V_1 \oplus V_2$ with $V_1, V_2$ are simple but nonisomorphic, $\mathcal{B}(V)$ is finite-dimensional if and only if $\dim(V_1) = \dim(V_2) = 2$ and $V_1, V_2$ are of type (C1) with some additional restriction.

(2) Any $V$ of rank 3 with pairwise nonisomorphic simple twisted Yetter-Drinfeld modules yields an infinite-dimensional Nichols algebra $\mathcal{B}(V)$.

For brevity, we focus on simple twisted Yetter-Drinfeld modules of non-diagonal type, as our methods extend directly to rank 2. Consider the group

$$G \cong \langle g_1, g_2, \ldots, g_n \rangle$$

for $n \geq 3$, since $\omega$ is a non-abelian 3-cocycle. Let $V \in {}_G^G\mathcal{YD}^\Phi$ be of either type (C1) or (C2) with $\deg(V) = g_1$ and $\dim(V) = m \geq 2$. By [60, Proposition 3.10, Theorem 3.9], there is a finite abelian group

$$\mathbb{G} \cong \langle \mathrm{g}_1 \rangle \times \langle \mathrm{g}_2 \rangle \times \cdots \times \langle \mathrm{g}_n \rangle$$

with generators $\mathrm{g}_i$, such that $\mathrm{ord}(\mathrm{g}_i) = \mathrm{ord}(g_i)^2$. Moreover, there exists a group epimorphism $\pi : \mathbb{G} \longrightarrow G$ satisfying $\pi(\mathrm{g}_i) = g_i$ and a section $\iota$ to $\pi$ such that $\iota(g_i) = \mathrm{g}_i$ for all $1 \leq i \leq n$. Furthermore, there exists $\widetilde{V} \in {}_{\mathbb{G}}^{\mathbb{G}}\mathcal{YD}^{\pi^*\Phi}$ such that $\mathcal{B}(\widetilde{V}) \cong \mathcal{B}(V)$, $\pi^*(\omega) \in \mathscr{A}''$ and the support group $\mathbb{G}_{\widetilde{V}}$ of $\widetilde{V}$ is $\langle \mathrm{g}_1 \rangle$. The following lemma is immediate.

**Lemma 8.4.3.** *The bosonization $H := \mathcal{B}(\widetilde{V})\#\Bbbk\mathbb{G}_{\widetilde{V}}$ is a Hopf algebra and satisfies* **FGC**.

*Proof.* Since $\pi^*\omega \in \mathscr{A}''$, the restriction $\pi^*\omega\,|_{\mathbb{G}_{\widetilde{V}} \times \mathbb{G}_{\widetilde{V}} \times \mathbb{G}_{\widetilde{V}}} \equiv 1$. By [60, Corollary 3.16], $\mathcal{B}(\widetilde{V})$ is isomorphic to a Nichols algebra of diagonal type in ${}_{\mathbb{G}_{\widetilde{V}}}^{\mathbb{G}_{\widetilde{V}}}\mathcal{YD}$(retaining the notation $\mathcal{B}(\widetilde{V})$ for simplicity). Thus $H := \mathcal{B}(\widetilde{V})\#\Bbbk\mathbb{G}_{\widetilde{V}}$ is a Hopf algebra, with an abelian group of group-like elements. Hence $H$ satisfies **FGC** by Lemma 8.2.1. □

Let $M := \mathcal{B}(V)\#\Bbbk G$ and $\widetilde{M} := \mathcal{B}(\widetilde{V})\#\Bbbk\mathbb{G}$. To prove $\mathrm{Comod}(M)$ satisfies **FGC**, we are going to show $\mathrm{Comod}(\widetilde{M})$ does, leveraging the surjective coquasi-Hopf algebra map $\widetilde{M} \to M$.

**Theorem 8.4.4.** *Both* $\mathrm{Comod}(\widetilde{M})$ *and* $\mathrm{Comod}(M)$ *satisfy* **FGC**.

*Proof.* Clearly, $\widetilde{M}$ and $H$ are generated by group-like and skew primitive elements. Since $\dim(V) = m$ and $\mathcal{B}(V) \cong \mathcal{B}(\widetilde{V})$. We may assume $\widetilde{V} = \operatorname{span}\{x_1, x_2, ..., x_m\}$. Thus, $\widetilde{M}$ is generated by

$$\left\{x_1, x_2, ..., x_m, g_j^i \mid 0 \le i < m_j^2,\ 1 \le j \le n\right\},$$

while $H$ is generated by

$$\{x_1, x_2, ..., x_m, g_1^i \mid 0 \le i < m_1^2\}.$$

The injective coquasi-Hopf algebra map $i : H \to \widetilde{M}$, $x_j \# g_1^i \mapsto x_j \# g_1^i$ for $0 \le i \le n-1$, $1 \le j \le m$, induces a fully faithful tensor functor

$$\iota : \mathrm{Comod}(H) \longrightarrow \mathrm{Comod}(\widetilde{M}). \tag{8.4.1}$$

Note that $p : \mathbb{G} \to \mathbb{G}/\mathbb{G}_{\widetilde{V}}$ is a group epimorphism given by $p(g_1) = 1$ and $p(g_i) = g_i$ for all $2 \le i \le n$ and $\pi : \mathcal{B}(V) \to \mathbb{k}, 1_{\mathcal{B}(\mathcal{V})} \mapsto 1_{\mathcal{B}(V)},\ \ x_i \mapsto 0$ is an algebra map and a coalgebra map. This yields a coquasi-Hopf algebra surjection:

$$f = (\pi \otimes p) : \mathcal{B}(\widetilde{V}) \# \mathbb{k}\mathbb{G} \longrightarrow \mathbb{k}\mathbb{G}/\mathbb{G}_{\widetilde{V}},\ x \# g \mapsto \pi(x) \# p(g).$$

which induces a surjective tensor functor:

$$F : \mathrm{Comod}(\widetilde{M}) \to \mathrm{Vec}_{\mathbb{G}/\mathbb{G}_{\widetilde{V}}}^{\omega_{\mathbb{G}/\mathbb{G}_{\widetilde{V}}}}. \tag{8.4.2}$$

We denote the fusion category $\mathrm{Vec}_{\mathbb{G}/\mathbb{G}_{\widetilde{V}}}^{\omega_{\mathbb{G}/\mathbb{G}_{\widetilde{V}}}}$ as $\mathcal{D}$ for simplicity. Since $f \circ i(x \# g_1^i) = \pi(x) \# 1 \in \mathbb{k}$, we see that $F \circ \iota(X) \in \mathrm{Vec}$ for all $X \in \mathrm{Comod}(H)$ by the definitions of $\iota$ and $F$. That is $\mathrm{Comod}(H) \subseteq \mathrm{Ker}(F)$.

All categories involved are comodule categories of finite-dimensional coquasi-Hopf algebras. Hence $\mathrm{FPdim}(\mathrm{Comod}(H)) = \dim_{\mathbb{k}}(H)$, $\mathrm{FPdim}(\mathrm{Comod}(\widetilde{M})) = \dim_{\mathbb{k}}(\widetilde{M})$ and $\mathrm{FPdim}(\mathcal{D}) = \dim_{\mathbb{k}} \mathbb{k}(\mathbb{G}/\mathbb{G}_{\widetilde{V}})$ by [35, Example 6.1.9]. Clearly,

$$\dim_{\mathbb{k}}(\widetilde{M}) = \dim_{\mathbb{k}}(H) \cdot \dim_{\mathbb{k}} \mathbb{k}(\mathbb{G}/\mathbb{G}_{\widetilde{V}}).$$

Hence

$$\mathrm{FPdim}(\mathrm{Comod}(\widetilde{M})) = \mathrm{FPdim}(\mathrm{Comod}(H)) \cdot \mathrm{FPdim}(\mathcal{D}). \tag{8.4.3}$$

By [34, Theorem 3.4], there is an exact sequence of finite tensor category

$$\mathrm{Comod}(H) \longrightarrow \mathrm{Comod}(\widetilde{M}) \longrightarrow \mathcal{D}. \tag{8.4.4}$$

Lemma 8.2.4 provides a natural identification of $\mathbb{k}$-algebras

$$\mathrm{H}^\bullet(\mathrm{Comod}(H), 1) = \mathrm{H}^\bullet(\mathrm{Comod}(\widetilde{M}), 1).$$

Since $\mathrm{Comod}(H)$ satisfies **FGC** by Lemma 8.4.3, $\mathrm{H}^\bullet(\mathrm{Comod}(\widetilde{M}), 1)$ is a finitely generated algebra. Moreover, for each $W \in \mathrm{Comod}(\widetilde{M})$, there is an identification of $\mathrm{H}^\bullet(\mathrm{Comod}(\widetilde{M}), 1)$ module via

$$\mathrm{H}^\bullet(\mathrm{Comod}(H), \mathscr{H}^0_{\mathrm{Comod}(\widetilde{M})}(\mathcal{D}, W)) = \mathrm{H}^\bullet(\mathrm{Comod}(\widetilde{M}), W).$$

Since $\mathrm{H}^\bullet(\mathrm{Comod}(H), \mathscr{H}^0_{\mathrm{Comod}(\widetilde{M})}(\mathcal{D}, W))$ is a finitely generated module of $\mathrm{H}^\bullet(\mathrm{Comod}(H), 1)$, $\mathrm{H}^\bullet(\mathrm{Comod}(\widetilde{M}), W)$ is a finitely generated module of $\mathrm{H}^\bullet(\mathrm{Comod}(\widetilde{M}), 1)$. Hence $\mathrm{Comod}(\widetilde{M})$ satisfies **FGC**. Recall that there is a surjection of coquasi-Hopf algebras $\widetilde{M} \to M$, which induces a surjective tensor functor $\mathrm{Comod}(\widetilde{M}) \to \mathrm{Comod}(M)$. Thus $\mathrm{Comod}(M)$ satisfies **FGC** by Lemma 8.2.2. □

**Theorem 8.4.5.** *Let $M$ be a finite-dimensional coradically graded pointed coquasi-Hopf algebra over abelian groups, and $C := \mathrm{Comod}(M)$, then $C$ satisfies **FGC**.*

*Proof.* Since $C \cong \mathrm{Comod}(M)$ for some finite-dimensional coradically graded coquasi-Hopf algebra $M$ over an abelian group, the result follows directly from Theorem 8.3.2 and 8.4.4. □

# Bibliography


[1] J. N. Alonso Alvarez and J. M. Fernández Vilaboa. Cleft extensions in braided categories. *Comm. Algebra*, 28(7):3185–3196, 2000.

[2] Nicolás Andruskiewitsch and Iván Angiono. On finite dimensional Nichols algebras of diagonal type. *Bull. Math. Sci.*, 7(3):353–573, 2017.

[3] Nicolás Andruskiewitsch, Iván Angiono, Agustín García Iglesias, Akira Masuoka, and Cristian Vay. Lifting via cocycle deformation. *J. Pure Appl. Algebra*, 218(4):684–703, 2014.

[4] Nicolás Andruskiewitsch, Iván Angiono, J. Pevtsova, and S. Witherspoon. Cohomology rings of finite-dimensional pointed Hopf algebras over abelian groups. *Res. Math. Sci.*, 9(1):Paper No. 12, 132, 2022.

[5] Nicolás Andruskiewitsch and Fernando Fantino. On pointed Hopf algebras associated with alternating and dihedral groups. *Rev. Un. Mat. Argentina*, 48(3):57–71 (2008), 2007.

[6] Nicolás Andruskiewitsch and Matías Graña. Braided Hopf algebras over non-abelian finite groups. volume 63, pages 45–78. 1999.

[7] Nicolás Andruskiewitsch and Matías Graña. From racks to pointed Hopf algebras. *Adv. Math.*, 178(2):177–243, 2003.

[8] Nicolás Andruskiewitsch, István Heckenberger, and Hans-Jürgen Schneider. The Nichols algebra of a semisimple Yetter-Drinfeld module. *Amer. J. Math.*, 132(6):1493–1547, 2010.

[9] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Lifting of quantum linear spaces and pointed Hopf algebras of order $p^3$. *J. Algebra*, 209(2):658–691, 1998.

[10] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Finite quantum groups and Cartan matrices. *Adv. Math.*, 154(1):1–45, 2000.

[11] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. Finite quantum groups over abelian groups of prime exponent. *Ann. Sci. École Norm. Sup. (4)*, 35(1):1–26, 2002.

[12] Nicolás Andruskiewitsch and Hans-Jürgen Schneider. On the classification of finite-dimensional pointed Hopf algebras. *Ann. of Math. (2)*, 171(1):375–417, 2010.

[13] Iván Angiono. On Nichols algebras of diagonal type. *J. Reine Angew. Math.*, 683:189–251, 2013.

[14] Iván Angiono, César Galindo, and Mariana Pereira. De-equivariantization of Hopf algebras. *Algebr. Represent. Theory*, 17(1):161–180, 2014.

[15] Iván Angiono and Agustín García Iglesias. Liftings of Nichols algebras of diagonal type II: all liftings are cocycle deformations. *Selecta Math. (N.S.)*, 25(1):Paper No. 5, 95, 2019.

[16] Iván Ezequiel Angiono. A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems. *J. Eur. Math. Soc. (JEMS)*, 17(10):2643–2671, 2015.

[17] Alessandro Ardizzoni and Alice Pavarin. Preantipodes for dual quasi-bialgebras. *Israel J. Math.*, 192(1):281–295, 2012.

[18] Alessandro Ardizzoni and Alice Pavarin. Bosonization for dual quasi-bialgebras and preantipode. *J. Algebra*, 390:126–159, 2013.

[19] Adriana Balan. Yetter-Drinfeld modules and Galois extensions over coquasi-Hopf algebras. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, 71(3):43–60, 2009.

[20] Yuri Bazlov and Arkady Berenstein. Cocycle twists and extensions of braided doubles. In *Noncommutative birational geometry, representations and combinatorics*, volume 592 of *Contemp. Math.*, pages 19–70. Amer. Math. Soc., Providence, RI, 2013.

[21] Petter Andreas Bergh, Julia Yael Plavnik, and Sarah Witherspoon. Support varieties for finite tensor categories: complexity, realization, and connectedness. *J. Pure Appl. Algebra*, 225(9):Paper No. 106705, 21, 2021.

[22] Petter Andreas Bergh, Julia Yael Plavnik, and Sarah Witherspoon. Support varieties for finite tensor categories: the tensor product property. *Ann. Represent. Theory*, 1(4):539–566, 2024.

[23] Yu. N. Bespalov. Crossed modules, quantum braided groups and ribbon structures. *Teoret. Mat. Fiz.*, 103(3):368–387, 1995.

[24] Yu. N. Bespalov. Crossed modules and quantum groups in braided categories. *Appl. Categ. Structures*, 5(2):155–204, 1997.

[25] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. *Proc. Nat. Acad. Sci. U.S.A.*, 83(10):3068–3071, 1986.

[26] Alain Bruguières and Sonia Natale. Exact sequences of tensor categories. *Int. Math. Res. Not. IMRN*, (24):5644–5705, 2011.

[27] Daniel Bulacu and Erna Nauwelaerts. Radford's biproduct for quasi-Hopf algebras and bosonization. *J. Pure Appl. Algebra*, 174(1):1–42, 2002.

[28] M. Cuntz. On subsequences of quiddity cycles and Nichols algebras. *J. Algebra*, 502:315–327, 2018.

[29] M. Cuntz, B. Mühlherr, and C. J. Weigel. On the Tits cone of a Weyl groupoid. *Comm. Algebra*, 47(12):5261–5285, 2019.

[30] Jeroen Dello and Yinhuo Zhang. Braided autoequivalences and the equivariant Brauer group of a quasitriangular Hopf algebra. *J. Algebra*, 445:244–279, 2016.

[31] R. Dijkgraaf, V. Pasquier, and P. Roche. Quasi-Hopf algebras, group cohomology and orbifold models [MR1128130 (92m:81238)]. In *Integrable systems and quantum groups (Pavia, 1990)*, pages 75–98. World Sci. Publ., River Edge, NJ, 1992.

[32] Chongying Dong, Siu-Hung Ng, and Li Ren. Orbifolds and minimal modular extensions. *Adv. Math.*, 462:Paper No. 110103, 43, 2025.

[33] V. G. Drinfeld. Quasi-Hopf algebras. *Algebra i Analiz*, 1(6):114–148, 1989.

[34] Pavel Etingof and Shlomo Gelaki. Exact sequences of tensor categories with respect to a module category. *Adv. Math.*, 308:1187–1208, 2017.

[35] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[36] Pavel Etingof and Viktor Ostrik. Finite tensor categories. *Mosc. Math. J.*, 4(3):627–654, 782–783, 2004.

[37] Sergey Fomin and Anatol N. Kirillov. Quadratic algebras, Dunkl elements, and Schubert calculus. In *Advances in geometry*, volume 172 of *Progr. Math.*, pages 147–182. Birkhäuser Boston, Boston, MA, 1999.

[38] Igor B. Frenkel, Yi-Zhi Huang, and James Lepowsky. On axiomatic approaches to vertex operator algebras and modules. *Mem. Amer. Math. Soc.*, 104(494):viii+64, 1993.

[39] Eric Friedlander and Cris Negron. Cohomology for Drinfeld doubles of some infinitesimal group schemes. *Algebra Number Theory*, 12(5):1281–1309, 2018.

[40] Eric M. Friedlander and Cris Negron. Support theory for Drinfeld doubles of some infinitesimal group schemes. *Algebra Number Theory*, 17(1):217–260, 2023.

[41] Victor Ginzburg and Shrawan Kumar. Cohomology of quantum groups at roots of unity. *Duke Math. J.*, 69(1):179–198, 1993.

[42] Christopher Goff, Geoffrey Mason, and Siu-Hung Ng. On the gauge equivalence of twisted quantum doubles of elementary abelian and extra-special 2-groups. *J. Algebra*, 312(2):849–875, 2007.

[43] M. Graña, István Heckenberger, and L. Vendramin. Nichols algebras of group type with many quadratic relations. *Adv. Math.*, 227(5):1956–1989, 2011.

[44] Matías Graña. On Nichols algebras of low dimension. In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 111–134. Amer. Math. Soc., Providence, RI, 2000.

[45] I. Heckenberger and Hans-Jürgen Schneider. Root systems and Weyl groupoids for Nichols algebras. *Proc. Lond. Math. Soc. (3)*, 101(3):623–654, 2010.

[46] István Heckenberger. The Weyl groupoid of a Nichols algebra of diagonal type. *Invent. Math.*, 164(1):175–188, 2006.

[47] István Heckenberger. Classification of arithmetic root systems. *Adv. Math.*, 220(1):59–124, 2009.

[48] István Heckenberger. Lusztig isomorphisms for Drinfeld doubles of bosonizations of Nichols algebras of diagonal type. *J. Algebra*, 323(8):2130–2182, 2010.

[49] István Heckenberger and Hans-Jürgen Schneider. Nichols algebras over groups with finite root system of rank two I. *J. Algebra*, 324(11):3090–3114, 2010.

[50] István Heckenberger and Hans-Jürgen Schneider. Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid. *Israel J. Math.*, 197(1):139–187, 2013.

[51] István Heckenberger and Hans-Jürgen. Schneider. Yetter-Drinfeld modules over bosonizations of dually paired Hopf algebras. *Adv. Math.*, 244:354–394, 2013.

[52] István Heckenberger and Hans-Jürgen Schneider. *Hopf algebras and root systems*, volume 247 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, [2020] ©2020.

[53] István Heckenberger and L. Vendramin. Nichols algebras over groups with finite root system of rank two III. *J. Algebra*, 422:223–256, 2015.

[54] István Heckenberger and L. Vendramin. A classification of Nichols algebras of semisimple Yetter-Drinfeld modules over non-abelian groups. *J. Eur. Math. Soc. (JEMS)*, 19(2):299–356, 2017.

[55] István Heckenberger and Leandro Vendramin. Nichols algebras over groups with finite root system of rank two II. *J. Group Theory*, 17(6):1009–1034, 2014.

[56] C. E. Hempel. Metacyclic groups. *Comm. Algebra*, 28(8):3865–3897, 2000.

[57] Hua-Lin Huang and Gongxiang Liu. On coquasitriangular pointed Majid algebras. *Comm. Algebra*, 40(10):3609–3621, 2012.

[58] Hua-Lin Huang, Gongxiang Liu, Yuping Yang, and Yu Ye. Finite quasi-quantum groups of diagonal type. *J. Reine Angew. Math.*, 759:201–243, 2020.

[59] Hua-Lin Huang, Gongxiang Liu, Yuping Yang, and Yu Ye. Finite quasi-quantum groups of rank two. *Trans. Amer. Math. Soc. Ser. B*, 8:635–678, 2021.

[60] Hua-Lin Huang, Gongxiang Liu, Yuping Yang, and Yu Ye. On the classification of finite quasi-quantum groups over abelian groups. *Adv. Math.*, 486:Paper No. 110740, 2026.

[61] Hua-Lin Huang, Gongxiang Liu, and Yu Ye. Quivers, quasi-quantum groups and finite tensor categories. *Comm. Math. Phys.*, 303(3):595–612, 2011.

[62] Hua-Lin Huang, Gongxiang Liu, and Yu Ye. Graded elementary quasi-Hopf algebras of tame representation type. *Israel J. Math.*, 209(1):157–186, 2015.

[63] Hua-Lin Huang, Yuping Yang, and Yinhuo Zhang. On nondiagonal finite quasi-quantum groups over finite abelian groups. *Selecta Math. (N.S.)*, 24(5):4197–4221, 2018.

[64] Yi-Zhi Huang. Rigidity and modularity of vertex tensor categories. *Commun. Contemp. Math.*, 10(suppl. 1):871–911, 2008.

[65] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[66] Victor Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[67] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 1998.

[68] Bowen Li and Gongxiang Liu. On gauge equivalence of twisted quantum doubles. *Manuscripta Math.*, 176(3):Paper No. 36, 29, 2025.

[69] Bowen Li and Gongxiang Liu. Reflection of nichols algebras over coquasi-hopf algebras. *arXiv*, 2512.04560, 2025.

[70] Bowen Li and Gongxiang Liu. Cohomology of pointed finite tensor categories. *Comm. Algebra*, 54(4):1814–1823, 2026.

[71] Bowen Li and Gongxiang Liu. Reflection theory of nichols algebras over coquasi-hopf algebras with bijective antipode. *arXiv*, 2602.07946, 2026.

[72] Gongxiang Liu and Siu-Hung Ng. On total Frobenius-Schur indicators. In *Recent advances in representation theory, quantum groups, algebraic geometry, and related topics*, volume 623 of *Contemp. Math.*, pages 193–213. Amer. Math. Soc., Providence, RI, 2014.

[73] Daowei Lu, Xiaohui Zhang, and Dingguo Wang. More properties of yetter-drinfeld category over dual quasi-hopf algebras. *arXiv*, 1512.01357.

[74] S. Majid. Free braided differential calculus, braided binomial theorem, and the braided exponential map. *J. Math. Phys.*, 34(10):4843–4856, 1993.

[75] Shahn Majid. Tannaka-Kreĭn theorem for quasi-Hopf algebras and other results. In *Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990)*, volume 134 of *Contemp. Math.*, pages 219–232. Amer. Math. Soc., Providence, RI, 1992.

[76] Geoffrey Mason. The quantum double of a finite group and its role in conformal field theory. In *Groups '93 Galway/St. Andrews, Vol. 2*, volume 212 of *London Math. Soc. Lecture Note Ser.*, pages 405–417. Cambridge Univ. Press, Cambridge, 1995.

[77] Geoffrey Mason and Siu-Hung Ng. Group cohomology and gauge equivalence of some twisted quantum doubles. *Trans. Amer. Math. Soc.*, 353(9):3465–3509, 2001.

[78] M. Mastnak, J. Pevtsova, P. Schauenburg, and S. Witherspoon. Cohomology of finite-dimensional pointed Hopf algebras. *Proc. Lond. Math. Soc. (3)*, 100(2):377–404, 2010.

[79] Ehud Meir. Geometric perspective on Nichols algebras. *J. Algebra*, 601:390–422, 2022.

[80] Alexander Milinski and Hans-Jürgen Schneider. Pointed indecomposable Hopf algebras over Coxeter groups. In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 215–236. Amer. Math. Soc., Providence, RI, 2000.

[81] Cris Negron and Julia Plavnik. Cohomology of finite tensor categories: duality and Drinfeld centers. *Trans. Amer. Math. Soc.*, 375(3):2069–2112, 2022.

[82] Siu-Hung Ng and Peter Schauenburg. Central invariants and higher indicators for semisimple quasi-Hopf algebras. *Trans. Amer. Math. Soc.*, 360(4):1839–1860, 2008.

[83] Van C. Nguyen, Xingting Wang, and Sarah Witherspoon. Finite generation of some cohomology rings via twisted tensor product and Anick resolutions. *J. Pure Appl. Algebra*, 223(1):316–339, 2019.

[84] Florin Panaite and Freddy Van Oystaeyen. Quasi-Hopf algebras and the centre of a tensor category. In *Hopf algebras and quantum groups (Brussels, 1998)*, volume 209 of *Lecture Notes in Pure and Appl. Math.*, pages 221–235. Dekker, New York, 2000.

[85] Bernardo Uribe. On the classification of pointed fusion categories up to weak Morita equivalence. *Pacific J. Math.*, 290(2):437–466, 2017.

# 博士期间科研成果

1 Bowen Li and Gongxiang Liu, On gauge equivalence of twisted quantum doubles, Manuscripta Math. **176** (2025), no. 3, Paper No. 36, 29 pp.; MR4911811.
2 Bowen Li and Gongxiang Liu, Cohomology of pointed finite tensor categories, Comm. Algebra **54** (2026), no. 4, 1814–1823; MR5027894.
3 Bowen Li and Gongxiang Liu, Reflection Theory of Nichols Algebras over Coquasi-Hopf Algebras with Bijective Antipode. arXiv:2602.07946.
4 Bowen Li and Gongxiang Liu, Reflection of Nichols Algebras over Coquasi-Hopf Algebras. arXiv:2512.04560.

# 致谢

时间一晃而过，转眼间我即将离开南京大学，离开这个我在这里求学了九年的地方。还记得 2021 年本科毕业的那个夏天，当时在写致谢时候我觉得来日方长，所以并未多写，准备等到写博士论文再细细写来，但时间飞速流逝，确实到了该好好告别感谢的时候了。

首先，我要感谢我的导师刘公祥老师，初识刘老师是于高等代数的课上，还是 2017 年的时候，刘老师讲课严谨，但是又深入浅出，同时还富有幽默感，高代每两周会有一次四节连上的课时，但刘老师总能提起我们的兴趣，同时也会照顾我们的感受，合理安排四节课的内容不让我们感觉压力过大。当时第一个学期的期中满分 120 的卷子我只考了 81 分，还记得从刘老师手里接过分数条时的沮丧，但是他鼓励我们考的不好不要泄气，告诫我们要掌握好基础知识，所以后来我高代倒也没有落下，算是挽救了一下数分拉低的绩点。后来是大二的时候，我还在仙林，每周三下午和几位同学一起去鼓楼上刘老师的群表示论，其实当时我都没有考虑好要不要分流的时候选择数学与应用数学，我非常担心我能不能学好，有没有能力从事数学研究，刘老师的课给了我一定的信心，同时让我对代数也有了一定的好感度，所以纠结了许久后，我最后还是选择了数学与应用数学。大三下面临保研，同时又疫情在家，几乎没有任何信息的来源，因为我的绩点并不是很高，所以留在南大是最优选，我想要直博，并且我当时心里导师的选择也只有刘老师。当时其实并不清楚刘老师的研究方向，而且几门分析的课程也学的挺好的，但是我其实完全没考虑过分析方向，现在看来是挺奇怪的，可能这就是刘老师的个人魅力吧，在一次线上和刘老师聊了后，他就直接收了我，保研这段经历就顺利地过去了。大四回校后，刘老师鼓励我多读书，多学习新的知识开拓视野，到了本科毕业论文选题的时候，刘老师也给我们充分的自由，然后我在美国数学会网站上搜索了一通，相中了 Andruskewitch 和 Schneider 的那篇分类点 Hopf 代数的文章，发表在 Annals 上，刘老师也鼓励我好好读这篇文章，现在想来这个行为就跟一口吃成个胖子差不多，当时也没有系统读任何一本 Hopf 代数的书籍，就直接去看了文章，好在时间够多，花了几个月也啃了个大概出来，借此机会，刘老师也给了我关于 coquasi-Hopf 代数的提升问题，惭愧的是，一直到今天，我仍旧没有做出来这个问题。开始参与讨论班并且讲讨论班之后，我接触了更多的知识，也很感谢刘老师这几年的讨论班设置，不仅让我们学习了 Hopf 代数与张量范畴，还有关于三角范畴以及 VOA 的相关内容，虽然暂时用不上，但是让我看到了张量范畴和其他分支的联系。刘老师非常强调例子，他常常在一个很抽象的定义给出后问我们要例子，遗憾的是大多时候并不能给出，但他的话深刻烙印在了我的脑子里，此时此刻我的心里也有一个理解程度比较高的关于 coquasi-Hopf 代数的例子，算是不负他的一遍遍教导。博一和博二的前半年我碌碌无为，当时也没有上路子，很多文章读了但是跟没读一样，我当时也不怎么会抠细节，经常有些不显然的话语会被忽略，刘老师很认真地带着我读了 Etingof 和 Gelaki 关于素数阶循环群上 quasi-Hopf 代数的提升问题，在被一遍遍拷问中我意识到了之前读文章的方式存在很大问题。转机是 2022 年的冬天，刘老师发给了我他和他的合作者们刚刚新鲜出炉的论文初稿，他们的目标是去证明一类非对角型的 Nichols 代数是无限维的，从而完全分类有限维余根分次的点 coquasi-Hopf 代数。我在研读的时候，发现了他们证明中有一些计算错误，导致当时他们的证明方法失效了，然后我就想能不能补上这个漏洞，可惜的是到最后我也没有用他们的方法给出一个补证，但是在刘老师的指导下，我给出了一个新的证明，以此为基础，我做了一些计算性的工作，最后写出了第一篇文章。不久之后，我又写出了第二篇文章，但写完之后，进度便陷入停滞状态，整个博三一年，我没有做出任何工作，虽然学到了一些新知识，但我就像一只埋头乱撞的苍蝇，找不到题目去做，幸好刘老师帮我停下了这个状态，把根系的课题塞给了我做，在边学边做的过程中，我似乎也知道了哪些问题是能做的值得做的，很感谢刘老师给了我这个问题，由它以及它衍生的一系列问题，都是很值得去进一步探索的。回首望去，在南大这九年，刘老师从头至尾一直都在，我万分感谢刘老师对我的付出和栽培。

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也还要感谢那些对我有过帮助的同学。感谢张浩同学，从本科开始就和我一直进行数学上的交流学习，我平时话少，他外向一些说的多一些，从他身上确实了解到了很多新的知识，也和他一起走过了南京的很多地方，我心里一直记着和他一起努力奋斗的时光，研究生他去了清华，我们联系上少了很多，但还是偶尔碰面，他讲我听，他讲他的工作、他的一些独到的理解，他的一些见闻，我很感激能和他做朋友，在此祝他未来一帆风顺。感谢王家文同学，我们虽然是不同的研究方向，但是确实挺合的来，我们一起做学生助理、一起去广东招生，他热爱生活，对待生活乐观向上的态度一直鼓励着我。感谢李康桥师兄，带着我们学了 Hopf 代数以及张量范畴，他对待学术严谨的态度让我印象深刻。感谢李凤昌师兄和刘绵涛师兄，在我刚入门时候提供了很多帮助，以及告诉我很多有用的信息。感谢徐成真同学，虽然我们大多数时候只是游戏上的搭子，但是他的思考问题方式确实让人眼前一亮，让我学到了很多。

感谢南京大学，我很开心，能在这里度过九年，看着学校一点点越来越好，不光是从第四轮学科评估的滑铁卢到第五轮学科评估的大丰收，还有基础设施的翻新完善，信息化也更加全面，希望南大的未来更加光明灿烂。

最后，我想要感谢我的爱人王绎，感谢她对我的付出，对我的理解包容，对我一如既往的支持。感谢命运能让我们相聚在南京大学，彼时她在上海工作，她为了我们，综合考量之后辞去了上海的工作考研回到了南京大学，她作出了巨大牺牲，为了我们之间的关系更加积极稳定付出了很多时间和精力，所以我满怀歉意。我能写出这篇论文以及其他几篇小论文，离不开她对我的付出。我还要感谢我们一起养的猫咪们，分别是臭粑、拉脱肛、小雏菊、三三、烧卖、黑子、杨玉环、苏苏、初音、老婆，还有生活在老家但我们一起救助的猫咪一二，以及带给我们很多快乐的猫咪二宝。我很开心它们能出现在我们的生活里，每只猫都有他们的故事，都有各种奇怪的脾气和喜好，但最重要的是它们能给我们带来快乐，让我们对未来充满希望，所以我谨以此文，献给我的爱人和我们的猫咪。同时我还要感谢我的父母、我的岳父岳母，他们给予了我很多鼓励和建议还有经济上的帮助。

其实在我写下致谢之时，我还不知道我未来会去往哪里，但不会再如从前那般迷茫，长风破浪会有时，直挂云帆济沧海！

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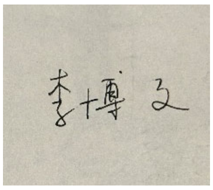
签名：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