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Affine Nichols algebras and quantum affine (super)algebras

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毕业论文题目:　　　　　　　仿射 **Nichols** 代数与仿射量子（超）群

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摘　　要

　　本论文的主题围绕着 Nichols 代数理论。Nichols 代数在近20年的 Hopf 代数理论中占据了核心地位，它是 Andruskiewitsch-Schneider 对有限维 pointed Hopf 代数的分类工作的最核心工具。同时，对角型（即阿贝尔群上）Nichols 代数包括了 Lusztig 定义的量子群，小量子群的正部，Yamene 定义的（仿射）量子超群的正部等等重要量子群对象。此外，在非阿贝尔群型 Nichols 代数中，还包括了著名的 Fomin-Kirillov 代数等重要研究对象。本论文主要聚焦于对角型 Nichols 代数的研究。

　　在对角型 Nichols 代数理论中，2010年前后 Heckenberger 发现了此类对象具有令人惊讶的 Weyl 群胚与广义根系结构，是对 Kac-Moody 代数的 Weyl 群与根系结构的完美推广，并且 Heckenberger 完整分类了有限情形的 Weyl 群胚与广义根系。自然地，人们可以提出如下后续问题：什么是仿射 Weyl 群胚，什么是仿射广义根系，具有仿射 Weyl 群胚的 Nichols 代数（我们称之为仿射 Nichols 代数）的分类以及它们具有什么样的性质。前两个问题都有明确回答，但是仿射 Nichols 代数的结构与分类是长时间以来的困难问题。仿射 Nichols 代数中包含了经典的仿射量子群的正部，以及已知的仿射量子超群的正部。

　　但是除去这些经典对象，人们对其余的仿射 Nichols 代数的研究几乎无任何进展。在这些对象中，最简单的一个即为仿射量子超群$U_v(A(0,2)^{(4)})$的正部$U_v^+(A(0,2)^{(4)})$ 或者等价地 Nichols 代数$B(\mathbb{V})$。$U_v(A(0,2)^{(4)})$也是仿射量子超群$U_v(A(2m,2n)^{(4)})$ 中最基础也是最重要的一个。仿射李超代数$A(2m,2n)^{(4)}$具有一个不同于其他所有仿射李代数和仿射李超代数的性质：它的虚根δ是奇的并且虚根向量未必交换。对此族对象的已有研究仅限于当v是超越数时，人们知道它的塞尔型 defining relations 与根重数。当v 仅是非单位根时，甚至塞尔型 defining relations 也是未知的。$U_v(A(0,2)^{(4)})$ 和它的独特性质与难点，以及一条低维的典型的虚根向量不交换的关系，被 Yamane 在2018 年巴西的国际数学家大会数学物理卫星会议上首次指出。Yamane 在此会议上提问：在如此复杂的交换关系下，如何构造$U_v(A(0,2)^{(4)})$ 的 Drinfeld 第二实现。本论文对当v是非单位根时$U_v(A(0,2)^{(4)})$ 的结构做了如下结果，包括了对 Yamane 的提问的回答：

1. 本文给出了仿射李超代数$A(0,2)^{(4)}$的 Lyndon word 型根向量以及其所有虚根向量之间的交换关系。

2. 本文确定了$U_v(A(0,2)^{(4)})$在 Chevalley 生成元下的 defining relations ，即其量子塞尔关系，也意味着获得了其 Drinfeld-Jimbo 型实现。

3. 基于 Lyndon word 理论与生成函数工具，本文为$U_v(A(0,2)^{(4)})$构造了具有良好性质的虚根向量，并且运用生成函数给出了这些虚根向量的余乘法的一个重要估计。

4. 利用上述估计，我们决定了 $U_v^+(A(0,2)^{(4)})$ 的根重数并给出了其 Lyndon word 型 PBW 基。

5. 此外，本文刻画了此余乘法的估计与实际的代数交换关系之间的对应，从而具体描述出了所有实根虚根向量之间的交换关系，即构造了 $U_v^+(A(0,2)^{(4)})$ 的 Drinfeld 第二实现，为今后进一步研究一般的 $U_v(A(2m,2n)^{(4)})$ 奠定了基础，也是今后研究此类量子群的表示论的最基本工具。

6. 本文还给出了在单边余理想李代数观点下对仿射量子（超）群 Drinfeld 生成元（虚根向量）的解释。本文证明了对于 $U_v^+(\hat{sl_2})$，即文中的 Nichols 代数 $B(\mathbb{U})$ 的情形，在每一个 $n\delta$ 的虚根空间中，存在一个卓越的元素使得它与某一侧的实根向量共同构成一个左（右）余理想子代数。这个元素正是 $U_v(\hat{sl_2})$ 的由 Drinfeld 构造并用作 Drinfeld 第二实现的生成元的选择。进一步地，本文也研究了在 $U_v^+(A(0,2)^{(4)})$ 情形时类似的单边余理想子代数。本文证明了此时只有在对应偶数倍的 δ 的虚根空间上，存在同样构造的单边余理想子代数，然而在奇数倍 δ 的虚根空间的情形，不可能存在此类余理想子代数。本文也给出了上述所有余理想子代数的一组 PBW 基。

7. 本文给出了 $U_v(A(0,2)^{(4)})$ 的根向量，尤其虚根向量（Drinfeld generators）的余乘法的更精确的估计，是今后计算 $U_v(A(0,2)^{(4)})$ 以及更一般的 $U_v(A(2m,2n)^{(4)})$ 的 R-matrix 的必不可少的工具。

关键词：仿射量子超群；Nichols 代数；Hopf 代数.

THESIS: Affine Nichols algebras and quantum affine (super)algebras

SPECIALIZATION: Fundamental Mathematics

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Abstract

The thesis is about the theory of Nichols algebras. In recent 20 years, the theory of Nichols algebras is in the center of the theory of Hopf algebras. It plays a fundamental role in the classification work on f.d. pointed Hopf algebras by Andruskiewitsch and Schneider. Meanwhile, Nichols algebras of diagonal type (i.e. over abelian groups) include a lot of important objects in the theory of quantum groups, e.g. the positive part of Lusztig's (small) quantum groups, Yamane's (affine) quantum supergroups. Moreover, Nichols algebras over non-abelian groups include the famous Fomin-Kirillov algebras and other important objects in algebra. The thesis mainly focus on Nichols algebras of diagonal type.

In the theory of Nichols algebras of diagonal type, around 2010 Heckenberger found that these objects have surprising structures of Weyl groupoids and generalized root systems, which is a perfect generalization of the Weyl groups and root systems of Kac-Moody algebras. Naturally, people can ask the following questions: What is affine Weyl groupoids, what is affine root systems, what properties do the Nichols algebras with affine Weyl groupoids (we call them affine Nichols algebras) have, what is the classification of affine Nichols algebras. The first two questions have concrete answers, but the structure and classification of affine Nichols algebras are long time difficult questions. Affine Nichols algebras contain the positive parts of the classical affine quantum groups and the known affine quantum supergroups.

However, there were almost no results on the remaining affine Nichols algebras. Among these objects, the easiest one is the positive part of quantum affine superalgebra $U_v(A(0,2)^{(4)})$, or equivalently the affine Nichols algebra $B(\mathbb{V})$. $U_v(A(0,2)^{(4)})$ is the most fundamental one in the family of quantum affine superalgebras $U_v(A(2m,2n)^{(4)})$. The Lie superalgebras $A(2m,2n)^{(4)}$ have a unique property (difficulty), compared to all the other affine Lie algebras and affine Lie superalgebras. That is, the imaginary root δ is odd and the imaginary root vectors do not necessarily commute with each other. The only known results are, when v is transcendental, the Serre type defining relations and the root multiplicities are determined. However, when v is only not a root of unity, even the Serre type defining relations are unknown. The object $U_v(A(0,2)^{(4)})$ together with a concrete non-commutativity relation at a low degree were announced

by Yamane in his presentation at the Workshop on Mathematical Physics, Brazil, ICM2018 Satellite. Yamane asked: The commutation relations are so complicated, how to construct the Drinfeld second realization? The thesis gives the following results on the structure of $U_v(A(0,2)^{(4)})$, when v is not a root of unity, including the answer to Yamane's question.

1. We give the Lyndon word type PBW basis of Lie superalgebra $A(0,2)^{(4)}$, together with the commutation relations among all the imaginary root vectors.

2. When v is not a root of unity, we prove that the defining relations satisfied by the Chevalley generators of $U_v^+(A(0,2)^{(4)})$ or equivalently the corresponding Nichols algebra $B(\mathbb{V})$ are the quantum Serre relations, which means the Drinfeld-Jimbo type realization of $U_v(A(0,2)^{(4)})$ is obtained.

3. Based on Lyndon word theory, we construct distinguished imaginary root vectors and give an important estimation of their comultiplication in the form of generating functions.

4. With this estimation, we determine all the root multiplicities and give a PBW basis of $U_v^+(A(0,2)^{(4)})$.

5. We also describe concretely all the commutation relations among real root vectors and distinguished imaginary root vectors using generating functions, that is, the Drinfeld second realization is obtained for $U_v^+(A(0,2)^{(4)})$, which must be the fundamental tool for studying general $U_v(A(2m,2n)^{(4)})$ and the representation theory of these quantum affine superalgebras.

6. We also give a new point of view for the choice of the Drinfeld generators (imaginary root vectrs), that is, at the root space of every $n\delta$, there exists a distinguished element s.t. this distinguished element with all the real root vectors of one side generate a left (right) coideal subalgebras (of the positive part). This distinguished element is just the Drinfeld generator of the imaginary root $n\delta$, used in the Drinfeld second realization. We also research the corresponding one-sided coideal subalgebras in the case of $U_v(A(0,2)^{(4)})$. We prove that only at the root space of even times δ, there exists the one-sided coideal subalgebra under the same construction. At the root space of odd times δ, there does not exist this kind of one-sided coideal subalgebras. We also give a PBW basis for every one-sided algebras above.

7. We give a more precise estimation of the comultiplication of the root vectors of $U_v(A(0,2)^{(4)})$, especially of the imaginary root vectors, which must be a key for future research on the R-matrix of $U_v(A(0,2)^{(4)})$ and even general $U_v(A(2m,2n)^{(4)})$.

Keywords: Quantum affine superalgebras; Affine Nichols algebras; Hopf algebras.

Chapter 1   Introduction

§1.1   Background and main results

Quantum groups have been introduced formally in [Dri85] and [Jim85] as deformations of the universal enveloping algebra $U(\mathfrak{g})$ of a Kac-Moody algebra $\mathfrak{g}$. Affine quantum groups are also called quantum affine algebras, and the theories of their representations are very rich, with many important applications in many branches of mathematics and mathematical physics. In addition, quantum affine algebras also have the famous Drinfeld second realizations, first appearing in [Dri87] and then [Bec94], [Jin98], [Dam12], [Dam15], [JZ16]. Drinfeld second realizations are very important for the theory of quantum affine algebras because they are crucial for the study of representations; see, e.g. [CP91], [CP98].

Affine Kac-Moody superalgebras are generalizations of affine Kac-Moody algebras, defined and classified by van de Leur in [vdL89]. In this classification, there is a special family named $A(m,n)^{(4)}$. The main feature of $A(m,n)^{(4)}$ is that there exist odd imaginary roots and that the imaginary root vectors do not necessarily commute with each other. In addition, when $m = n$, the cardinality of the Serre-type defining relations is infinite (also in the quantum case) due to the one-dimensional center spanned by the identity matrix. Motivated by the work [AT24] and for the exploration of the quantum case, we give the Lyndon word type PBW basis of $A(0,2)^{(4)}$. We also describe the commutation relations among all the imaginary root vectors. See the following results. See Section 2 for related notations.

Proposition 1.1.1 $\{X_n, Y_n, L_{2n}, M_{2n-1} | n \geq 1\}$ *is a set of PBW generators (root vectors) of* $U^+(A(0,2)^{(4)})$, *with respect to the following order:*

$$
\begin{aligned}
Y_1 > Y_2 &> \cdots > Y_n > \cdots \\
&> M_1 > M_3 > \cdots > M_{2n+1} > \cdots \\
&> L_2 > L_4 > \cdots > L_{2n} > \cdots \\
&> \cdots > X_n > \cdots X_2 > X_1.
\end{aligned}
$$

Proposition 1.1.2 *The commutation relations of the imaginary root vectors* L_{2n} *and* M_{2n+1} *are the following:*

(a) $[L_{2m}, L_{2n}] = 0.$

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(b) $[M_{2m+1}, M_{2n+1}] = 0$ *if* $m+n$ *is odd,* $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+1+1}^2$, *if* $m+n$ *is even.*

(c) $[L_{4n}, M_{2n+1}] = 0$, $[L_{4m+2}, M_{2n+1}] = \frac{1}{2^{2m}} M_{4m+2n+3}$.

Quantum superalgebras and quantum affine superalgebras appeared first in [Yam94] and [Yam99], respectively, which are foundational for these theories. In them, Yamane defined quantum (affine) superalgebras abstractly, i.e., by the radical of a bilinear form, which is similar to the non-super case introduced by [Lus93]. Then Yamane got the defining relations satisfied by the Chevalley generators of affine Lie superalgebras and the corresponding quantum affine superalgebras except the case of $A(m,m)^{(2)}$ and $A(m,m)^{(4)}$, which means the Drinfeld-Jimbo type realization is also obtained for these quantum affine superalgebras.

For quantum affine superalgebras, the Drinfeld second realizations are also important tools for studying representations. Related work appeared in [Yam99], [HSTY08], [Zha14], [BFK24], etc.

Next, we discuss $U_v(A(m,n)^{(4)})$ when $m \neq n$. When v is transcendental, the defining relations satisfied by the Chevalley generators were known in [Yam99], the root multiplicities were known in [EG09], which are the same as the root multiplicities of the corresponding affine Lie superalgebra. However, no explicit form of a PBW basis was given. When v is not a root of unity, the defining relations satisfied by the Chevalley generators, the root multiplicities, PBW basis and the Drinfeld second realization were all unknown.

In the last years, Nichols algebras attracted much interest from many branches of mathematics such as Hopf algebras, quantum groups, algebraic geometry and number theory, conformal field theory, etc. Concretely, Nichols algebras play an important role in the classification of finite-dimensional pointed Hopf algebras; see [AS10]. It is also well known that the theory of Nichols algebras is closely related to the theory of quantum groups. In the finite case, the positive part of a Drinfeld-Jimbo type quantum group is a Nichols algebra when v is not a root of unity; see Chapter 16 of [HS20]. In the affine case, this is also true by [HK07]. If we define a quantum group abstractly following Lusztig and Yamane, i.e., using the radical of the bilinear form, then the positive part is just a Nichols algebra by definition.

In the present paper, we discuss mainly $U_v(\hat{sl}_2)$ and $U_v(A(0,2)^{(4)})$ defined abstractly. $U_v(A(0,2)^{(4)})$ is unavoidable for further research on $U_v(A(m,n)^{(4)})$. $U_v(\hat{sl}_2)$ and $U_v(A(0,2)^{(4)})$ look similar and share some common properties, but are also very different.

The Dynkin diagram of $U_v(\hat{sl}_2)$ is

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$$\circ \Longleftarrow \circ$$
$$\;0 \qquad\quad 1$$

whereas the Dynkin diagram of $U_v(A(0,2)^{(4)})$ is

$$\circ \Longleftarrow \bullet$$
$$\;0 \qquad\quad 1$$

$U_v^+(\hat{sl}_2)$ is isomorphic the Nichols algebra $B(\mathbb{U})$, where $\mathbb{U}$ is a braided vector space of diagonal type with braiding matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & q \end{pmatrix}$$

with respect to a basis $\{x_1, x_2\}$ and $q = v^2$. The isomorphism maps the Chevalley generators E_0, E_1 to x_1 and x_2. The defining relations satisfied by the Chevalley generators or equivalently x_1, x_2 are the quantum Serre relations when q is not a root of unity, for a proof see [HK07].

$U_v^+(A(0,2)^{(4)})$ is isomorphic to the Nichols algebra $B(\mathbb{V})$, where $\mathbb{V}$ is a braided vector space of diagonal type with braiding matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & -q \end{pmatrix}$$

with respect to a basis $\{x_1, x_2\}$ and $q = v^2$. Again, the isomorphism maps the Chevalley generators E_0, E_1 to x_1 and x_2. Conjecturally, the defining relations of $U_v^+(A(0,2)^{(4)})$ or equivalently $B(\mathbb{V})$ are also the quantum Serre relations when q is not a root of unity. In the present paper, this will be proved in the following theorem.

Theorem 1.1.3 *Suppose q is not a root of unity. The defining relations of $B(\mathbb{V})$ satisfied by the Chevalley generators are the quantum Serre relations (2.4), which means that the Drinfeld-Jimbo type realization is obtained for $U_v(A(0,2)^{(4)})$.*

The Nichols algebra $B(\mathbb{V})$ first appeared in Cuntz's classification of rank two affine Nichols algebras [Cun18]. An affine Nichols algebra is a Nichols algebra of diagonal type with an affine Weyl groupoid or equivalently affine Tits arrangement, see [CMW17]. All Nichols algebras in the classification in [Cun18] are quite unexplored except $U_v^+(\hat{sl}_2)$, that is, $B(\mathbb{U})$, which has been very well known. We believe that these Nichols algebras have a Lie theoretic background.

By the reflection theory of Nichols algebras of diagonal type, see [Hec06]; we conclude that $B(\mathbb{U})$ and $B(\mathbb{V})$ share the same real roots, that is, $n\delta + \alpha_1$ and $n\delta + \alpha_2$

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for $n \geq 0$, where $\delta = \alpha_1 + \alpha_2$. The imaginary roots of $B(\mathbb{U})$ are well known as $n\delta$, $n > 0$, and the multiplicities are all 1. The only possible imaginary roots of $B(\mathbb{V})$ are also $n\delta$, but the multiplicities have been, in general, unknown. In this thesis, all the root multiplicities of $B(\mathbb{V})$ are determined and a PBW basis is given.

Theorem 1.1.4 *Suppose q is not a root of unity, the multiplicities of $4n\delta$ and $(2n+1)\delta$ are 1, of $(4n+2)\delta$ is 2, which coincide with the root multiplicities of the Lie superalgebra $A(0,2)^{(4)}$. The set of decreasing ordered products of elements in $\{X_n, Y_n, L_{2n}, M_{2n-1}, n \in \mathbb{N}\}$ forms a PBW basis of $B(\mathbb{V})$. The order in $\{X_n, Y_n, L_{2n}, M_{2n-1}, n \in \mathbb{N}\}$ is*

$$X_1 < X_2 < \cdots < X_n < \cdots$$
$$< \cdots < L_{2n} < L_{2n-2} < L_{2n-4} < \cdots < L_2$$
$$< \cdots < M_{2n-1} < M_{2n-3} < \cdots < M_5 < M_3 < M_1$$
$$< \cdots < Y_n < \cdots < Y_2 < Y_1.$$

For strategic reasons, we use the notation X_1 and Y_1 for the two Chevalley generators E_0 and E_1 of $U_v^+(\hat{sl}_2)$, or equivalently, the two basis vectors x_1 and x_2 of the braided vector space $\mathbb{U}$. Let $[\,,\,]$ be the braided bracket. Let $L_1 = [X_1, Y_1]$. Define

$$X_{n+1} = [X_n, L_1], \ Y_{n+1} = [L_1, Y_n], \ \text{for } n \geq 1.$$

Define $L_n = [X_1, Y_n]$ for all $n \geq 1$. Then, rewriting the relations in the Drinfeld second realization, we obtain the following relations in $B(\mathbb{U})$. Let $\theta = q - q^{-1}$, $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$ and define $L_0 = \frac{1}{\theta}$, define $\bar{L}_n$ for all $n \geq 1$ using the equation

$$\sum_{n=0}^{\infty} \theta L_n u^n = \exp\left(\sum_{n=1}^{\infty} \theta \bar{L}_n u^n\right).$$

Then for all $m, n, k \geq 1$,

(a) $[L_m, L_n] = 0 = [\bar{L}_m, \bar{L}_n]$,

(b) $[X_k, \bar{L}_m] = \frac{[m]_q}{m} X_{k+m}, [\bar{L}_m, Y_k] = \frac{[m]_q}{m} Y_{k+m}$,

(c) $[X_n, X_{n+k}] = -[X_{n+k-1}, X_{n+1}], [Y_{n+k}, Y_n] = -[Y_{n+1}, Y_{n+k-1}]$,

(d) $[X_m, Y_n] = L_{m+n-1}$.

We can define X_n, Y_n, L_n similarly for $B(\mathbb{V})$. It is not difficult to prove that X_n and Y_n are still real root vectors. However, when one looks at the relations in

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imaginary roots spaces, great difficulty appears. At degree 3δ, we have the following relation:

$$L_3 = \frac{\theta - 1}{2}[L_2, L_1] + \theta L_1 L_2 - \theta L_1^3, \qquad (1.1)$$

and L_2, L_1 do not commute any more. Due to this non-commutativity, $U_v(A(0,2)^{(4)})$ is much more complicated than $U_v(\hat{sl}_2)$. The algebra $U_v(A(0,2)^{(4)})$ together with this property (difficulty) and with Equation (1.1) was first announced by Yamane in his presentation at the Workshop on Mathematical Physics, August 10-13, ICM2018 Satellite. Equation (1.1) is Equation (3.2a) in Prop. 3.2.1 of the present paper. In general, it was unknown how to describe similar relations at higher degrees. In the present thesis, we give a solution; see (3.24a) in Prop. 3.2.16. Moreover, we obtain the Drinfeld second realization for $B(\mathbb{V})$ or equivalently $U_v^+(A(0,2)^{(4)})$, see the following theorem from Chapter 7.

Theorem 1.1.5 *The algebra $B(\mathbb{V})$ or equivalently $U_v^+(A(0,2)^{(4)})$ is generated by real root vectors X_n, Y_n for $n > 0$, imaginary root vectors M_{2n+1}, Z_{4n+2}, $\mathscr{L}_{4n+2}$ for $n \geq 0$, $\mathscr{L}_{4n}$ for $n > 0$, with the following defining relations.*

I:

Z_{4n+2}, $\mathscr{L}_{4n}$ *commute with all the imaginary root vectors,* $[\mathscr{L}_{4m+2}, \mathscr{L}_{4n+2}] = 0, \forall m, n \geq 0$.

$[\mathscr{L}_{4m+2}, M_{2n+1}] = \frac{[2m+1]_q}{2^{2m-1}} M_{4m+2n+3}$,

$[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+n+1}^2$ *if $m+n$ is even,* $[M_{2n+1}, M_{2m+1}] = 0$ *if $m+n$ is odd,*

$2\theta i \mathbb{Z}(u) = \tan(2\theta i \mathscr{Z}(u))$.

II:

$$X_k \mathbb{M} = \mathbb{M}_{(1)}(-1)^{d(\mathbb{M}_{(1)}) \cdot (k-1)} f_{X_k}(\mathbb{M}_{(2)}),$$
$$\mathbb{M} Y_k = (-1)^{k \cdot d(\mathbb{M}_{(2)})} g_{Y_k}(\mathbb{M}_{(1)}) \mathbb{M}_{(2)},$$

where

$$\mathbb{M}_{(1)} \otimes \mathbb{M}_{(2)} = (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M}) \frac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}},$$

$$[X_k, Z_{4n+2}] = 2^{2n} \frac{[2n+1]_q}{2n+1} X_{4n+k+2},$$
$$[Z_{4n+2}, Y_k] = 2^{2n} \frac{[2n+1]_q}{2n+1} Y_{4n+k+2},$$

$$X_k \overline{\mathfrak{L}} = \overline{\mathfrak{L}}_{(1)}(-1)^{d(\overline{\mathfrak{L}}_{(1)}) \cdot (k-1)} f_{X_k}(\overline{\mathfrak{L}}_{(2)}),$$

and

$$\overline{\mathfrak{L}}Y_k = (-1)^{k \cdot d(\overline{\mathfrak{L}}_{(2)})} g_{Y_k}(\overline{\mathfrak{L}}_{(1)}) \overline{\mathfrak{L}}_{(2)},$$

where

$$\overline{\mathfrak{L}}_{(1)} \otimes \overline{\mathfrak{L}}_{(2)} := \Delta(\overline{\mathfrak{L}}) = \overline{\mathfrak{L}} \otimes 1 + 1 \otimes \overline{\mathfrak{L}} + 2\Big(\mathfrak{M}(au) + \mathfrak{M}(bu)\Big)$$

and

$$\mathfrak{M}(u) = \frac{\Big(-2\theta\mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{I}} + 2\theta\mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{II}}\Big)\Big(1 + 2\theta\mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{II}} + 2\theta\mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{I}}\Big)}{1 + (2\theta)^2\mathbb{Z} \otimes \mathbb{Z}},$$

$$[X_k, \mathscr{L}_{4n}] = [4n]_q X_{4n+k},$$
$$[\mathscr{L}_{4n}, Y_k] = [4n]_q Y_{4n+k},$$

f, g are defined in Subsection 6.2.

III:

$$[X_{2i+1}, Y_{2n+2-(2i+1)}] = L_{2n+1}, \quad [X_{2i}, Y_{2n+2-2i}] = L'_{2n+1},$$
$$[X_{2i+1}, Y_{2n+3-(2i+1)}] = L_{2n+2}, \quad [X_{2i}, Y_{2n+3-2i}] = L'_{2n+2},$$

together with ‡ in Prop. 7.3.2, and † in Prop. 7.3.1.

We also identify every imaginary root $n\delta$ of $B(\mathbb{U})$ and every imaginary root of $2n\delta$ of $B(\mathbb{V})$ with a left (right) coideal subalgebra which is minimal left(right) coideal subalgebra in the sense of being $\mathbb{N}^2$-graded and intersecting the subalgebra generated by all the imaginary root vectors nontrivially. We also give a more precise estimation of Drinfeld generators of $2n\delta$ for $B(\mathbb{V})$, which must be a key tool for studying R-matrix of $U_q(A(0,2)^{(4)})$ and general $U_q(A(2m,2n)^{(4)})$. We also give a PBW basis of every one-sided coideal subalgebra mentioned above. See Propositions 8.1.7, 8.1.8, 8.2.4, 8.2.5, 8.2.6, 8.2.7 in Chapter 8.

Our main tool is Lyndon word theory, see [Kha99], [Ros99]. There are many excellent applications of the Lyndon word theory in Hopf algebra theory and Lie theory, see [Hec06], [HRZ08], [Ang15], [CHW16], [AT24], [NT24], etc. In terms of Lyndon word theory, if we define the order by $x_1 < x_2$ for the basis of the braided vector space $\mathbb{U}$, then X_n, Y_n, L_n are the super-letters corresponding to the Lyndon words

$$x_1(x_1x_2)^{n-1}, \ (x_1x_2)^{n-1}x_2, \ x_1(x_1x_2)^{n-1}x_2,$$

respectively.

We also use an important subquotient of Nichols algebras of diagonal type that appeared in [GH07] and [AAH19]. The subquotient is a braided Hopf algebra induced by the braided Hopf algebra structure of the initial Nichols algebra. In [AAH19], [AG22], [AG23], the subquotient played a central role in the proof of the conjecture in [AAH21] about the equivalence between the finiteness of the root system and the finiteness of the GK dimension for Nichols algebras of diagonal type. The subquotient is very beneficial in the study of $B(\mathbb{V})$ because it has a very manageable comultiplication, which can be used to detect possible relations in $B(\mathbb{V})$, determine the root multiplicities completely and describe the commutation relations in $U_v^+(A(0,2)^{(4)})$ between real root vectors and imaginary root vectors.

Moreover, in the Drinfeld second realization of $U_v(\hat{sl}_2)$, the loop-like generators of the imaginary roots are described by the exponential series with coefficients in $U_v(\hat{sl}_2)$. In the present paper, this method is also used and is very beneficial. In addition, tangent and some other techniques of generating functions are crucial.

Chapter 2   Preliminaries

In the whole thesis, $q \in \mathbb{C}^{\times}$ and is not a root of unity. Let $\theta = q - q^{-1}$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0$ be the set of natural numbers with 0.

All the brackets $[\,,\,]$ in this thesis are braided brackets introduced in Section 2.2.4.3.

§2.1   Kac-Moody superalgebras

§2.1.1   Contragredient Lie superalgebra

Let $\mathbb{I} = \{1, 2, ..., n\}$ and let $(a_{ij})_{i,j \in \mathbb{I}}$ be a complex matrix of rank l. Let H be a complex vector space of dimension $2n - l$ and H^* its dual space. Given a complex matrix $A = (a_{ij})_{i,j \in \mathbb{I}}$ of rank l, there exist linearly independent sets

$$\pi = \{\alpha_i\}_{i \in \mathbb{I}} \text{ and } \pi^{\vee} = \{h_i\}_{i \in \mathbb{I}} \subseteq H$$

such that $\alpha_j(h_i) = a_{ij}$. Then α_i are called simple roots, h_i are called simple coroots. We set

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i, \ Q_+ = \sum_{i=1}^{n} \mathbb{Z}_+\alpha_i.$$

The lattice Q is called the root lattice.

For $\alpha = \sum_i k_i \alpha_i \in Q$, the number $ht\alpha = \sum_i k_i$ is called the height of α.

For $\tau \subseteq \mathbb{I}$, we define $\tilde{g}(A, \tau)$ to be the Lie superalgebra generated by h and $\{e_i, f_i\}_{i \in \mathbb{I}}$ with defining relations:

$$[e_i, f_j] = \delta_{ij} h_i \text{ for } i, j \in \mathbb{I},$$
$$[h, e_i] = \alpha_i(h)e_i, \ [h, f_i] = -\alpha_i(h)f_i \text{ for } i \in \mathbb{I} \text{ and } h \in H,$$
$$[h, h'] = 0 \text{ for } h, h' \in H,$$
$$\deg(h) = 0 \text{ for } h \in H,$$
$$\deg(e_i) = \deg(f_i) = \bar{0} \text{ for } i \notin \tau,$$
$$\deg(e_i) = \deg(f_i) = \bar{1} \text{ for } i \in \tau.$$

Theorem 2.1.1 (a) *Denote by $\tilde{n}_+$ (resp., $\tilde{n}_-$) the subsuperalgebra in $\tilde{g}(A, \tau)$ generated by $e_1, e_2, ..., e_n$ (resp., $f_1, f_2, ..., f_n$), then*

$$\tilde{g}(A, \tau) = \tilde{n}_+ \oplus H \oplus \tilde{n}_- \quad \text{(direct sum of vector spaces).}$$

(b) $\tilde{n}_+$ and $\tilde{n}_-$ are freely generated by $e_1, e_2, ..., e_n$ and $f_1, f_2, ..., f_n$, respectively.

(c) With respect to H one has the root space decomposition:

$$\tilde{g}(A, \tau) = \bigoplus_{\alpha \in Q_+, \ \alpha \neq 0} \tilde{g}_\alpha \oplus H \oplus (\bigoplus_{\alpha \in Q_+, \ \alpha \neq 0} \tilde{g}_{-\alpha}),$$

where

$$\tilde{g}_\alpha = \{x \in \tilde{g}(A, \tau) | [h, x] = \alpha(h)x \ for \ all \ h \in H\},$$

$dim \ \tilde{g}_\alpha < \infty$ and $\tilde{g}_\alpha \subset \tilde{n}_\pm$ for $\pm \alpha \in Q_+$.

(d) Among the ideals intersecting H trivially there exists a unique maximal ideal r in $\tilde{g}(A, \tau)$. Furthermore,

$$r = (r \cap \tilde{n}_-) \oplus (r \cap \tilde{n}_+) \ (direct \ sum \ of \ ideals).$$

The contragredient Lie superalgebra $g(A, \tau)$ is

$$g(A, \tau) = \tilde{g}(A, \tau)/r.$$

The image of H and the e_i and f_i in this quotient are denoted by the same characters. We call the e_i and f_i the Chevalley generators; the vectors e_i are called positive generators, the vectors f_i are called negative generators. A is called the Cartan matrix of $g(A, \tau)$ and H its Cartan subalgebra.

We also have the root decomposition of $g(A, \tau)$:

$$g(A, \tau) = \bigoplus_{\alpha \in Q} g_\alpha,$$

where $g_\alpha = \{x \in g(A, \tau) | [h, x] = \alpha(h)x \ \text{for all} \ h \in H\}$. If $\alpha \neq 0$ and $g_\alpha \neq 0$, then α is called a root with multiplicity $m(\alpha) = dim \ g_\alpha$. Obviously $m(\alpha) < \infty$. Denote Δ by the set of all roots. Set $\Delta_+ = \Delta \cap Q_+$ and $\Delta_- = \Delta \cap Q_-$, they are called positive roots and negative roots. We have

$$\Delta = \Delta_+ \cup \Delta_-.$$

We also have

$$g(A, \tau) = n_- \oplus H \oplus n_+, \ \text{where} \ n_\pm = \bigoplus_{\alpha \in \Delta_\pm} g_\alpha.$$

Set $\Delta_{\bar{0}} = \{\alpha \in \Delta | g_\alpha \subseteq g(A, \tau)_{\bar{0}}\}$ and $\Delta_{\bar{1}} = \{\alpha \in \Delta | g_\alpha \subseteq g(A, \tau)_{\bar{1}}\}$ the sets of

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even roots and odd roots. Then we have the following splitting

$$\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}.$$

§2.1.2   Kac-Moody algebras and Kac-Moody superalgebras

Definition 2.1.2 *A Lie algebra $g(A) = g(A, \emptyset)$ is called a Kac-Moody algebra if A satisfies the following conditions.*

(1) *$a_{ii} = 2$ for all $i \in \mathbb{I}$;*

(2) *a_{ij} are non-positive integers for $i \neq j$;*

(3) *$a_{ij} = 0$ implies $a_{ji} = 0$.*

The definition of Kac-Moody superalgebra is generalized by Kac-Moody algebra. We call A a generalized Cartan matrix if A satisfies the following conditions for $i, j \in \mathbb{I}$:

(1) $a_{ij} = 0$ implies $a_{ji} = 0$;

(2) if $a_{ii} = 0$, then $i \in \tau$;

(3) if $a_{ii} \neq 0$, then $\frac{2a_{ij}}{a_{ii}}$ is a non-positive integer for $i \neq j$;

(4) if $a_{ii} \neq 0$ and $i \in \tau$, then $\frac{a_{ij}}{a_{ii}}$ is a non-positive integer for $i \neq j$.

Definition 2.1.3 *A contragredient Lie superalgebra $g(A, \tau)$ is called a Kac-Moody superalgebra if A is a generalized Cartan matrix.*

§2.1.3   Affine Kac-Moody superalgebra, $A(2m, 2n)^{(4)}$ and its loop realization

Naturally, a Lie superalgebra g has a $\mathbb{Z}$-grading $g = \bigoplus_{n \in \mathbb{Z}} g(n)$ where $g(0)$ is the Cartan part and $g(1)$ is the space of positive simple root vectors.

Definition 2.1.4 [vdL89] *A Kac-Moody superalgebra g is said to be of finite growth if the dimension of $g(n)$ grows not faster than a polynomial in n and it is called affine if it is not of finite dimension but of finite growth.*

Similar to the classical loop realization of (twisted) affine Lie algebras in [Kac90], twisted affine Kac-Moody superalgebras have a completely similar construction, we refer to [vdL89].

Let g be a f.d. simple Kac-Moody superalgebra, let σ be an automorphism of finite order $m \neq 1$. The eigenvalues of σ are of the form ϵ^k where $\epsilon = e^{\frac{2\pi i}{m}}$. Since σ is diagonalizable, its eigenspaces define a $\mathbb{Z}_m$-grading of g:

$$g = \bigoplus_{k \in \mathbb{Z}_m} g(\sigma)_k, \text{ where } g(\sigma)_k = \{x \in g | \sigma(x) = \epsilon^k x\}.$$

Let $(\ ,\ )$ be a non-degenerate supersymmetric invariant bilinear form on g. Since g is simple such a form exists and is unique up to a scalar factor.

Now fix an automorphism σ of g of order $m \neq 1$. We define the subalgebra $L(g, \sigma)$ of $L(g) = L \otimes g$ by

$$L(g, \sigma) = \bigoplus_{k \in \mathbb{Z}} t^k \otimes g(\sigma)_{\bar{k}}.$$

As a twisted analog of $\hat{L}(g)$ we further introduce

$$\hat{L}(g, \sigma) = L(g, \sigma) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with

$$\deg(c) = \deg(d) = \bar{0}, \text{ and}$$
$$[t^k x + \lambda c + \mu d, t^j x + \nu c + \tau d] = t^{k+j}[x, y]_g + \mu j t^j y - \nu k t^k x + k \delta_{k,-j}(x, y)c,$$

where $x, y \in g$; $\lambda, \mu, \nu, \tau \in \mathbb{C}$ and c and d are the central element and the derivation.

When $g = A(2m, 2n) = \mathrm{sl}(2m + 1 | 2n + 1)$, there exists an automorphism σ of order 4 as the following:

$$
\begin{bmatrix}
(a_{p,q})_{p,q} & (b_{p,q})_{p,q} & (b_{p.2n+1})_p \\
\hline
(c_{p,q})_{p,q} & (d_{p,q})_{p,q} & (d_{p,2n+1})_p \\
(c_{2n+1,q})_q & (d_{2n+1,q})_q & d_{2n+1,2n+1}
\end{bmatrix}
\stackrel{\sigma}{\longmapsto}
$$

$$
\begin{bmatrix}
((-1)^{p+q+1}a_{2m+2-q,2m+2-p})_{p,q} & ((-1)^{p+q}c_{2n+1-q,2m+2-p})_{p,q} & ((-1)^{q+1}ic_{2n+1,2m+2-q})_q \\
\hline
((-1)^{p+q+1}b_{2m+2-q,2n+1-p})_{p,q} & ((-1)^{p+q+1}d_{2n+1-q,2n+1-p})_{p,q} & ((-1)^q id_{2n+1,2n+1-q})_q \\
((-1)^{q+1}ib_{2m+2-q,2n+1})_q & ((-1)^{q+1}id_{2n+1-q,2n+1})_q & -d_{2n+1,2n+1}
\end{bmatrix}.
$$

For more detailed information on affine Lie superalgebras $A(2m, 2n)^{(4)}$ we refer to [vdL89] and van de Leur's PhD thesis **Contragredient Lie superalgebras of finite growth**.

Remark 2.1.5 *Affine Lie superalgebras $A(2m, 2n)^{(4)}$ have a unique property (difficulty), compared to all the other affine Lie algebras and affine Lie superalgebras. That*

is, the imaginary root δ is odd and imaginary root vectors do not necessarily commute with each other.

§2.1.4 Affine Kac-Moody superalgebra $A(0,2)^{(4)}$ and its root system

In this thesis we focus on affine Kac-Moody superalgebra $g(A,\tau)$, where A is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

$\mathbb{I} = \{1,2\}$ and $\tau = \{2\}$. It is $A(0,2)^{(4)}$, which is the most fundamental example of the family $A(2m,2n)^{(4)}$. Its Dynkin diagram is

In particular, the order 4 automorphism used in the loop realization is the following:

$$\begin{bmatrix} a_{11} & b_{11} & b_{12} & b_{13} \\ c_{11} & d_{11} & d_{12} & d_{13} \\ c_{21} & d_{21} & d_{22} & d_{23} \\ c_{31} & d_{31} & d_{32} & d_{33} \end{bmatrix} \overset{\sigma}{\mapsto} \begin{bmatrix} -a_{11} & c_{21} & -c_{11} & -ic_{31} \\ -b_{12} & -d_{22} & d_{12} & id_{32} \\ b_{11} & d_{21} & -d_{11} & -id_{31} \\ -ib_{13} & -id_{23} & id_{13} & -d_{33} \end{bmatrix}$$

Here $i^2 = -1$.

Denote the two simple roots (positive) by α_1 and α_2. Then by the loop realization or the reflection theory one can get the positive real roots are $n\alpha_1 + (n+1)\alpha_2$ and $(n+1)\alpha_1 + n\alpha_2$, for $n \geq 0$, the imaginary roots are $n\delta$, where $\delta = \alpha_1 + \alpha_2$. The root multiplicities of $(4n+4)\delta$ and $(2n+1)\delta$ is 1, of $(4n+2)\delta$ is 2.

Meanwhile, the Serre type defining relations is also known in [Yam99]. Denote the two positive Chevalley generators by X_1 and Y_1. The Serre type defining relations are

$$[X_1,[X_1,[X_1,Y_1]]] = 0, \ [Y_1,[Y_1,[Y_1,X_1]]] = 0.$$

§2.1.5 Lyndon word basis of $A(0,2)^{(4)}$ and commutation relations among imaginary root vectors

It is well-known that one can use Lyndon word theory to write root vectors for Lie algebras and Lie superalgebras by a standard way, see [LR95].

Denote $[X_1,Y_1]$ by L_1. Define $X_{n+1} = [X_n,L_1]$, $Y_{n+1} = [L_1,Y_n]$. Define $L_n = [X_1,Y_n]$. To describe the words of X_1 and Y_1, we use 1 and 2 to represent them. Then

X_n, Y_n, L_n correspond to Lyndon words

$$1(12)^{n-1}, \ (12)^{n-1}2, \ 1(12)^{n-1}2.$$

Define $M_1 = L_1$, $M_{2n+1} = [L_2, M_{2n-1}]$. M_{2n+1} corresponds to the Lyndon word

$$(1122)^n 12.$$

We have the following order:

$$Y_1 > Y_2 > \cdots > Y_n > \cdots$$
$$> M_1 > M_3 > \cdots > M_{2n+1} > \cdots$$
$$> L_2 > L_4 > \cdots > L_{2n} > \cdots$$
$$> \cdots > X_n > \cdots X_2 > X_1.$$

Proposition 2.1.6 $\{X_n, Y_n, L_{2n}, M_{2n-1} | n \geq 1\}$ *is a set of PBW generators of* $U^+(A(0,2)^{(4)})$.

Proof: With the algorithm in [LR95], the proof of result is routine by some elementary calculation. $\square$

Proposition 2.1.7 *The commutation relations of the imaginary root vectors* L_{2n} *and* M_{2n+1} *are the following:*

(a) $[L_{2m}, L_{2n}] = 0$.

(b) $[M_{2m+1}, M_{2n+1}] = 0$ *if* $m+n$ *is odd,* $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+1+1}^2$, *if* $m+n$ *is even.*

(c) $[L_{4n}, M_{2n+1}] = 0$, $[L_{4m+2}, M_{2n+1}] = \frac{1}{2^{2m}} M_{4m+2n+3}$.

Proof: It follows from direct computation by the loop realization of $A(0,2)^4$ above, alternatively one can also get then directly from the Serre relations. $\square$

Remark 2.1.8 *This result tells us that the non-commutativity is strongly related to odd imaginary root vectors. Moreover if we want to construct the Drinfeld second realization for the quantum case, one must find distinguished imaginary root vectors having similar commutation relations to the above.*

§2.2 Affine Nichols algebras

In this section we mainly aim to introduce the definition of affine Nichols algebras and the classification of rank 2 case.

§2.2.1 Braided vector spaces and the diagonal type case

Definition 2.2.1 *Suppose V is a vector space, $c \in GL(V \otimes V)$. We call a pair (V, c) is a braided vector space if c satisfies*

$$(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c).$$

Remark 2.2.2 *This equation is introduced independently by C.N. Yang and R.J. Baxter in statistical mechanics and has important applications in many branches of mathematics.*

Let $\mathcal{K}$ be a field. Let V be a s-dimensional vector space with basis $\{x_1, ..., x_s\}$, $\mathbb{I} = \{1, 2, ..., s\}$. Let $\boldsymbol{q} = (q_{ij}) \in (\mathcal{K}^\times)^{\mathbb{I} \times \mathbb{I}}$. Define $c^{\boldsymbol{q}} \in GL(V \otimes V)$ by

$$c^{\boldsymbol{q}}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad i, j \in \mathbb{I}.$$

Then it is easy to see $c^{\boldsymbol{q}}$ satisfies the Yang-Baxter equation.

Definition 2.2.3 *We say a braided vector space (V, c) is of diagonal type if $c = c^{\boldsymbol{q}}$ for some matrix $\boldsymbol{q}$ and in addition $q_{ii} \neq 1$, $i \in \mathbb{I}$.*

Definition 2.2.4 *Let V be an s-dimensional braided vector space of diagonal type. Let $(x_i)_{i \in \mathbb{I}}$ be a basis of V and let $\boldsymbol{q} = (q_{ij})_{i,j \in \mathbb{I}}$ be the braiding matrix with respect to the basis $(x_i)_{i \in \mathbb{I}}$. The Dynkin diagram of V with respect to this basis $(x_i)_{i \in \mathbb{I}}$ is a labeled graph with s vertices. The vertices correspond to $i \in \mathbb{I}$ and are labeled by q_{ii}. For any $1 \leq i, j \leq s$, there is an edge between vertex i and vertex j if and only if $q_{ij} q_{ji} \neq 1$. In this case, $q_{ij} q_{ji}$ is the label of this edge.*

§2.2.2 Braided monoidal categories and Yetter-Drinfeld modules

In this subsection we mainly introduce the category of Yetter-Drinfeld modules, which is a typical example of braided monoidal categories.

§2.2.2.1 Braided monoidal categories

For the definition of a monoidal category, we refer to [EGNO15].

A monoidal category is a collection $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ where

(a) $\mathcal{C}$ is a category;

(b) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor, called the tensor product;

(c) $\mathbf{1} \in \mathcal{C}$ is an object called the unit;

(d) $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is an invertible natural transformation, called the associativity constraint;

(e) $l_X : X \to X \otimes 1$, $r_X : X \to 1 \otimes X$, are invertible natural transformations, called the left and right unit constraints.

They are required to satisfy the pentagon and the triangle axioms as the following:

$$
\begin{array}{ccc}
((X \otimes Y) \otimes Z) \otimes U & \xrightarrow{\ a_{(X \otimes Y),Z,U}\ } & (X \otimes Y) \otimes (Z \otimes U) \\
{\scriptstyle a_{X,Y,Z} \otimes id_U}\downarrow & & \downarrow{\scriptstyle a_{X,Y,Z \otimes U}} \\
(X \otimes (Y \otimes Z)) \otimes U & & X \otimes (Y \otimes (Z \otimes U)) \\
\end{array}
$$

with $a_{X,Y \otimes Z,U}$ and $id_X \otimes a_{Y,Z,U}$ through $X \otimes ((Y \otimes Z) \otimes U)$

and

$$
\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\ a_{X,\mathbf{1},Y}\ } & X \otimes (\mathbf{1} \otimes Y) \\
& {\scriptstyle l_X \otimes id_Y}\nwarrow \quad \nearrow{\scriptstyle id_X \otimes r_Y} & \\
& X \otimes Y & \\
\end{array}
$$

A braided monoidal category is a monoidal category $\mathcal{C}$ provided with a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$, called the braiding, that is required to fulfill the hexagon axioms, meaning the following diagrams commute:

$$
\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\
{\scriptstyle c_{X,Y} \otimes id}\downarrow & & & & \downarrow{\scriptstyle a_{Y,Z,X}} \\
(Y \otimes X) \otimes Z & \xrightarrow{a_{Y,X,Z}} & Y \otimes (X \otimes Z) & \xrightarrow{id \otimes c_{X,Z}} & Y \otimes (Z \otimes X),
\end{array}
$$

$$
\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{a_{X,Y,Z}^{-1}} & (X \otimes Y) \otimes Z & \xrightarrow{c_{X \otimes Y,Z}} & Z \otimes (X \otimes Y) \\
{\scriptstyle id \otimes c_{Y,Z}}\downarrow & & & & \downarrow{\scriptstyle a_{Z,X,Y}^{-1}} \\
X \otimes (Z \otimes Y) & \xrightarrow{a_{X,Y,Z}^{-1}} & (X \otimes Z) \otimes Y & \xrightarrow{c_{X,Z} \otimes id} & (Z \otimes X) \otimes Y,
\end{array}
$$

for all $X, Y, Z \in \mathcal{C}$.

A monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is called strict if the maps $a_{X,Y,Z}, l_X, r_X$ are the identity maps for all $X, Y, Z \in \mathcal{C}$. In this thesis the monoidal categories of interest

are all categories of vector spaces with an additional algebraic structure and with associativity and unit constraints as for vector spaces. We view the category of vector spaces and related monoidal categories as strict monoidal categories.

§2.2.2.2 Yetter-Drinfeld modules

Let H be a Hopf algebra with bijective antipode S.

Definition 2.2.5 *A Yetter-Drinfeld module over H is a vector space V provided with*

(a) *a structure of left H-module $\cdot: H \otimes V \to V$ and*

(b) *a structure of left H-comodule $\delta: V \to H \otimes V$, such that*

for any $h \in H$ and $v \in V$, the following compatibility condition holds:

$$\delta(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)}v_{(0)}.$$

Then we have the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules, with morphisms being linear maps that preserve both the action and the coaction.

Proposition 2.2.6 ${}^H_H\mathcal{YD}$ *is a braided monoidal category with the tensor product of modules and comodules and braiding*

$$c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}, \quad V, W \in {}^H_H\mathcal{YD}, \quad v \in V, \ w \in W,$$
$$c_{W,V}^{-1}(v \otimes w) = w_{(0)} \otimes S^{-1}(w_{(-1)}) \cdot v, \quad V, W \in {}^H_H\mathcal{YD}, \quad v \in V, \ w \in W.$$

§2.2.3 Braided Hopf algebras and Nichols algebras

Let $\mathcal{C}$ be a strict braided monoidal category. For proofs and more details we refer to [HS20].

Proposition 2.2.7 *Let $V \in \mathcal{C}$, and $T(V) = \oplus_{n\geq0}T^n(V)$ the tensor algebra of the vector space V.*

(1) *$T(V)$ is an algebra in $\mathcal{C}$.*

(2) *For any algebra A in $\mathcal{C}$ and any morphism $f : V \to A$ in $\mathcal{C}$, there is exactly one algebra morphism $\phi : T(V) \to A$ in $\mathcal{C}$ extending f.*

Definition 2.2.8 (1) *Let* (A, μ_A, η_A) *and* (B, μ_B, η_B) *be algebras in* $\mathcal{C}$. *Define* $\mu_{A \otimes B}$ *and* $\eta_{A \otimes B}$ *by*

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{id \otimes c_{B,A} \otimes id} (A \otimes A)(B \otimes B) \xrightarrow{\mu_A \otimes \mu_B} A \otimes B$$

$$\mathcal{K} \cong \mathcal{K} \otimes \mathcal{K} \xrightarrow{\eta_A \otimes \eta_B} A \otimes B.$$

Then $(A \otimes B, \mu_{A \otimes B}, \eta_{A \otimes B})$ *is called the tensor product of algebras in* $\mathcal{C}$.

(2) *Let* $(C, \Delta_C, \epsilon_C)$ *and* $(D, \Delta_D, \epsilon_D)$ *be coalgebras in* $\mathcal{C}$. *Define* $\Delta_{C \otimes D}$ *and* $\epsilon_{C \otimes D}$ *by*

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} (C \otimes C) \otimes (D \otimes D) \xrightarrow{id \otimes c_{C,D} \otimes id} (C \otimes D) \otimes (C \otimes D)$$

$$C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} \mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}.$$

Then $(C \otimes D, \Delta_{C \otimes D}, \epsilon_{C \otimes D})$ *is called the tensor product of coalgebras in* $\mathcal{C}$.

Proposition 2.2.9 *Let* A, B, C, D *be algebras in* $\mathcal{C}$.

(1) $(A \otimes B), \mu_{A \otimes B}, \eta_{A \otimes B}$ *is an algebra in* $\mathcal{C}$.

(2) *The canonical isomorphism* $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ *is an isomorphism of algebras in* $\mathcal{C}$.

(3) *Let* $\phi : A \to C$ *and* $\psi : B \to D$ *be morphisms of algebras in* $\mathcal{C}$. *Then* $\phi \otimes \psi : A \otimes B \to C \otimes D$ *is a morphism of algebras in* $\mathcal{C}$.

Proposition 2.2.10 *Let* C, D, E, F *be coalgebras in* $\mathcal{C}$.

(1) $(C \otimes D, \Delta_{C \otimes D}, \epsilon_{C \otimes D})$ *is a coalgebra in* $\mathcal{C}$.

(2) *The canonical isomorphism* $(C \otimes D) \otimes E \cong C \otimes (D \otimes E)$ *is an isomorphisms of coalgebras in* $\mathcal{C}$.

(3) *Let* $\phi : C \to E$ *and* $\psi : D \to F$ *be morphisms of coalgebras in* $\mathcal{C}$. *Then* $\phi \otimes \psi : C \otimes D \to E \otimes F$ *is a morphism of coalgebras in* $\mathcal{C}$.

Proposition 2.2.11 *Let R be an object in $\mathcal{C}$, and let*

$$\mu : R \otimes R \to R, \ \eta : \mathcal{K} \to R, \ \Delta : R \to R \otimes R, \ \epsilon : R \to \mathcal{K}$$

be morphisms in $\mathcal{C}$. Assume that (R, μ, η) is an algebra and (R, Δ, ϵ) is a coalgebra in $\mathcal{C}$. Then the following are equivalent.

(1) Δ and ϵ are morphisms of algebras in $\mathcal{C}$.

(2) μ and η are morphisms of coalgebras in $\mathcal{C}$.

Definition 2.2.12 *Let R be an object in $\mathcal{C}$, and let*

$$\mu : R \otimes R \to R, \ \eta : \mathcal{K} \to R, \ \Delta : R \to R \otimes R, \ \epsilon : R \to \mathcal{K}$$

be morphisms in $\mathcal{C}$. Assume that (R, μ, η) is an algebra and (R, Δ, ϵ) is a coalgebra in $\mathcal{C}$. Then $(R, \mu, \eta, \Delta, \epsilon)$ is a bialgebra in $\mathcal{C}$ if Δ and ϵ are morphisms of algebras in $\mathcal{C}$ or equivalently μ and η are morphisms of coalgebras in $\mathcal{C}$. Moreover, if an bialgebra R admits an antipode S in $\mathcal{C}$, then R is a Hopf algebra in $\mathcal{C}$.

Definition 2.2.13 *Let $V \in \mathcal{C}$, and $T(V)$ the tensor algebra of V in $\mathcal{C}$. Then by the previous proposition, there are uniquely determined algebra morphisms in $\mathcal{C}$*

$$\Delta : T(V) \to T(V) \otimes T(V), \ \epsilon : T(V) \to \mathcal{K}$$

such that

$$\Delta(v) = 1 \otimes v + v \otimes 1, \ \epsilon(v) = 0$$

for all $v \in V$, where $T(V) \otimes T(V)$ is the tensor product of algebras in $\mathcal{C}$.

Definition 2.2.14 *Let $V \in \mathcal{C}$. Let $I(V)$ be the largest coideal of $T(V)$ contained in $\oplus_{n \geq 2} T^n(V)$. The Nichols algebra of V is defined by*

$$B(V) = T(V)/I(V).$$

§2.2.4 Nichols algebras of diagonal type

Definition 2.2.15 *If V is of diagonal type, then we say $B(V)$ is a Nichols algebra of diagonal type.*

Remark 2.2.16 *For every braided vector space of diagonal type $(V, c^{\mathbf{q}})$, we can realize it as a Yetter-Drinfeld module over H, where H is the group algebra of the free abelian group Γ generated by $(g_i)_{i \in \mathbb{I}}$ and $q_{ij} = \chi_j(g_i)$. The comodule structure is $\delta(x_i) = g_i$, the module structure is $g_i \cdot x_j = \chi_j(g_i)$.*

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§2.2.4.1 Examples of of Nichols algebras of diagonal type

We give some examples of Nichols algebras of diagonal type.

Example 2.2.17 *Let V be a braided vector space of dimension 2, of diagonal type with braiding matrix* $\begin{pmatrix} q & q \\ q & q \end{pmatrix}$.

(1) *The case $q = 1$ is not of diagonal type strictly speaking because we require $q_{ii} \neq 1$. Nevertheless, we have $B(V) \cong S(V)$.*

(2) *If $q = -1$, then $B(V) \cong \Lambda(V)$.*

§2.2.4.2 An equivalent definition of Nichols algebra of diagonal type

Let (V, c) be a braided vector space of diagonal type, with basis $\{x_1, x_2, ..., x_s\}$ and braiding matrix $(q_{ij})_{1 \leq i,j \leq s}$, where $q_{ij} = q_{ji}$ for any $1 \leq i, j \leq s$.

Proposition 2.2.18 *There exists a unique bilinear form*
$(\cdot|\cdot) : T(V) \times T(V) \to \mathcal{K}$ *such that $(1|1) = 1$, and:*

$$(x_i|x_j) = \delta_{ij}, \text{ for any } i, j;$$
$$(x|yy') = (x_{(1)}|y)(x_{(2)}|y'), \text{ for any } x, y, y' \in T(V);$$
$$(xx'|y) = (x|y_{(1)})(x'|y_{(2)}), \text{ for any } x, x', y \in T(V).$$

This is a symmetric form, for which we have:

$$(x|y) = 0, \text{ for any } x \in T(V)_g, \ y \in T(V)_h, \ g, h \in \Gamma, \ g \neq h.$$

The radical of this form $\{x \in T(V) : (x|y) = 0, \forall y \in T(V)\}$ coincides with $I(V)$, so $(\cdot|\cdot)$ induces a non-degenerate bilinear form on $B(V) = T(V)/I(V)$.

§2.2.4.3 Braided brackets, Braided Jacobi identity and skew derivations

Let $s \in \mathbb{N}$, $X = \{x_1, ..., x_s\}$. Let $\mathbb{X}$ be the set of words with letters in X. Let V be a s-dimensional vector space with basis $\{x_1, ..., x_s\}$. We can identify $\mathcal{K}\mathbb{X}$ with $T(V)$. Then $T(V)$ is a $\mathbb{Z}^s$-graded algebra determined by $\deg x_i = \alpha_i, 1 \leq i \leq s$, where $\{\alpha_1, ..., \alpha_s\}$ is the canonical basis of $\mathbb{Z}^s$.

Let c be a braiding of V. The braided bracket of $x, y \in T(V)$ is defined by

$$[x, y]_c = m \circ (id - c)(x \otimes y).$$

Assume that (V, c) is of diagonal type with braiding matrix $(q_{ij})_{1 \leq i,i \leq s}$ with respect to the basis $\{x_1, ..., x_s\}$, and let $\chi : \mathbb{Z}^s \times \mathbb{Z}^s \to \mathcal{K}^\times$ be the bicharacter determined by

the condition $\chi(\alpha_i, \alpha_j) = q_{ij}$ for each pair $1 \leq i, j \leq s$. Then for each pair of $\mathbb{Z}^s$-homogeneous elements $u, v \in \mathbb{X}$, $c(u \otimes v) = q_{u,v} v \otimes u$, $q_{u,v} = \chi(\deg u, \deg v) \in \mathcal{K}^\times$. In such case, the "braided Jacobi identity" and skew derivations hold:

$$[[u,v],w] = [u,[v,w]] - \chi(\alpha,\beta)v[u,w] + \chi(\beta,\gamma)[u,w]v,$$
$$[u,vw] = [u,v]w + \chi(\alpha,\beta)v[u,w],$$
$$[uv,w] = \chi(\beta,\gamma)[u,w]v + u[v,w],$$

where u, v, w are homogeneous elements in $T(V)$, α, β, γ are the degrees of u, v, w. If $\chi(\beta,\alpha)\chi(\alpha,\beta) = 1$, then we have

$$[[u,v],w] = [u,[v,w]] - \chi(\alpha,\beta)[v,[u,w]]. \tag{2.1}$$

§2.2.4.4 An important subquotient

Let H be a Hopf algebra with bijective antipode.

Proposition 2.2.19 *Let B be a bialgebra in $^H_H\mathcal{YD}$, let K be a subalgebra of B and let I be a subject of K in $^H_H\mathcal{YD}$, such that it is a coideal of B, an ideal of K and*

$$\Delta(K) \subseteq K \otimes K + I \otimes B.$$

Then K/I inherits a structure of bialgebra in $^H_H\mathcal{YD}$ from B.

Proof: See Prop. 3.8 in [AAH19]. $\qquad\square$

Now we apply this proposition to the following. Let α_1, α_2 be the canonical basis of $\mathbb{Z}^2$. Let $V = V_1 \oplus V_2$ be a direct sum in $^H_H\mathcal{YD}$. Then $B(V)$ has a unique $\mathbb{N}_0^2$-grading (as a Hopf algebra in $^H_H\mathcal{YD}$)

$$B(\mathbb{V}) = \bigoplus_{\alpha \in \mathbb{N}_0^2} B^\alpha(\mathbb{V})$$

such that $\deg V_1 = \alpha_1$ and $\deg V_2 = \alpha_2$. Let $r \in \mathbb{Q}_{\geq 0}$. We set

$$B_{\geq 1} = \bigoplus_{\substack{\alpha = a_1\alpha_1 + a_2\alpha_2 \in \mathbb{N}_0^2: \\ a_1 \geq ra_2}} B^\alpha(\mathbb{V}), \qquad\qquad B_{>1} = \bigoplus_{\substack{\alpha = a_1\alpha_1 + a_2\alpha_2 \in \mathbb{N}_0^2: \\ a_1 > ra_2}} B^\alpha(\mathbb{V}),$$

$$K_{\geq 1} = \{x \in B(\mathbb{V}) \mid \Delta(x) \in B_{\geq 1} \otimes B(\mathbb{V})\}, \qquad K_{>1} = K_{\geq 1} \cap B_{>1}.$$

Lemma 2.2.20 *$K_{\geq 1}$ and $K_{>1}$ have the following properties.*

(1) $K_{\geq 1} \subseteq B_{\geq 1}$.

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(2) $K_{\geq 1}$ is a subalgebra of $B(V)$ in ${}^H_H\mathcal{YD}$.

(3) $\Delta(K_{\geq 1}) \subseteq K_{\geq 1} \otimes K_{\geq 1} + K_{>1} \otimes B(V)$.

(4) $K_{>1}$ is an ideal of $K_{\geq 1}$ and a coideal of $B(V)$ in ${}^H_H\mathcal{YD}$.

Proof: See Prop. 3.9 in [AAH19]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

By this lemma we have the following proposition immediately.

Proposition 2.2.21 *The braided Hopf algebra structure of $B(V)$ induces a braided Hopf algebra structure on $K_{\geq 1}/K_{>1}$.*

For our braided vector space $\mathbb{V}$, we know that $\mathbb{V} = \mathbb{C}x_1 \oplus \mathbb{C}x_2$, so $B(\mathbb{V})$ has a unique $\mathbb{N}_0^2$-grading as a braided Hopf algebra

$$B(\mathbb{V}) = \bigoplus_{\alpha \in \mathbb{N}_0^2} B^\alpha(\mathbb{V})$$

such that $\deg x_1 = \alpha_1$, $\deg x_2 = \alpha_2$. Now set

$$B_{\geq 1} = \bigoplus_{\substack{\alpha = a_1\alpha_1 + a_2\alpha_2 \in \mathbb{N}_0^2: \\ a_1 \geq a_2}} B^\alpha(\mathbb{V}), \qquad\qquad B_{>1} = \bigoplus_{\substack{\alpha = a_1\alpha_1 + a_2\alpha_2 \in \mathbb{N}_0^2: \\ a_1 > a_2}} B^\alpha(\mathbb{V}),$$

$$K_{\geq 1} = \{x \in B(\mathbb{V}) \,|\, \Delta(x) \in B_{\geq 1} \otimes B(\mathbb{V})\}, \qquad K_{>1} = K_{\geq 1} \cap B_{>1}.$$

The subquotient we use is $K_{\geq 1}/K_{>1}$.

§2.2.5 Cartan graphs, Weyl groupoids and root systems over Cartan graphs

In this subsection we introduce Cartan graphs, Weyl groupoids and root systems over Cartan graphs, which are perfect generalizations of the classical Weyl groups and root systems.

§2.2.5.1 Cartan graphs and Weyl groupoids

Let $(\alpha_i)_{i\in\mathbb{I}}$ for the standard basis of $\mathbb{Z}^\mathbb{I}$ for any finite set $\mathbb{I}$.

Definition 2.2.22 *Let $\mathbb{I}$ be an non-empty finite set, $\mathcal{X}$ a non-empty set, and $r : \mathbb{I} \times \mathcal{X} \to \mathcal{X}$, $A : \mathbb{I} \times \mathbb{I} \times \mathcal{X} \to \mathbb{Z}$ maps. For all $i, j \in \mathbb{I}$ and $X \in \mathcal{X}$ we write $r_i(X) = r(i, X)$, $a_{ij}^X = A(i, j, X)$, and $A^X = (a_{ij}^X)_{i,j\in\mathbb{I}}$. The quadruple $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ is called a semi-Cartan graph if for all $X \in \mathbb{X}$, the matrix A^X is a Cartan matrix and if the following holds:*

(CG1) *For all $i \in \mathbb{I}$, $r_i^2 = id$.*

(CG2) *For all $i, j \in \mathbb{I}$ and $X \in \mathcal{X}$, $a_{ij}^X = a_{ij}^{r_i(X)}$.*

For any $X \in \mathcal{X}$ and any $i \in \mathbb{I}$ let

$$s_i^X \in Aut(\mathbb{Z}^{\mathbb{I}}), \ s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i \text{ for all } j \in \mathbb{I}.$$

Example 2.2.23 *Let $\mathbb{I} = \{1, 2\}$, $\mathcal{X} = X_1, X_2$, r_1 be the nontrivial permutation and r_2 is the identity. Let*

$$A^{X_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad A^{X_2} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Then this is a semi-Cartan graph.

Definition 2.2.24 *Let $\mathcal{X}$ be a set and M be a monoid. We denote by $\mathcal{D}(\mathcal{X}, M)$ the category with objects $Ob\mathcal{D}(\mathcal{X}, M) = \mathcal{X}$, and morphisms*

$$Hom(X, Y) = \{(Y, f, X) | f \in M\} \text{ for all } X, Y \in \mathcal{X},$$

where composition of morphism is defined by

$$(Z, g, Y) \circ (Y, f, X) = (Z, gf, X) \text{ for all } X, Y, Z \in \mathcal{X}, \ f, g \in M.$$

Let $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be a semi-Cartan graph. We call the smallest subcategory of $\mathcal{D}(\mathcal{X}, End(\mathbb{Z}^{\mathbb{I}}))$ which contains all morphisms $(r_i(X), s_i^X, X)$ the Weyl groupoid of $\mathcal{G}$. We write $\mathcal{W}(\mathcal{G})$ for this subcategory. The morphisms $(r_i(X), s_i^X, X)$ are usually abbreviated by s_i^X or by s_i, if no confusion is likely.

Remark 2.2.25 *Let $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be a semi-Cartan graph, $X \in \mathcal{X}$ and $i \in \mathbb{I}$. Then it is easy to see $s_i^X = s_i^{r_i(X)}$ and $(s_i)^2 = id$. Therefore s_i^X and $s_i^{r_i(X)}$ are inverse morphisms. Consequently all morphisms of $\mathcal{W}(\mathcal{G})$ are invertible and hence $\mathcal{W}(\mathcal{G})$ is a groupoid.*

Definition 2.2.26 *Let $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be a semi-Cartan graph. For all $X \in \mathcal{X}$, the set*

$$\Delta^{X \ re} = \{w(\alpha_i) \in \mathbb{Z}^{\mathbb{I}} | w \in Hom(\mathcal{W}(\mathcal{G}), X), i \in \mathbb{I}\}$$

is called the set of real roots of $\mathcal{G}$ at X. The real root $\alpha_i, i \in \mathbb{I}$, are called simple. The elements of

$$\Delta_+^{X \ re} = \Delta^{X \ re} \cap \mathbb{N}_0^{\mathbb{I}} \text{ and } \Delta_-^{X \ re} = \Delta^{X \ re} \cap -\mathbb{N}_0^{\mathbb{I}}$$

are called positive and negative, respectively.

The semi-Cartan graph $\mathcal{G}$ is called finite, if $\Delta^{X\ re}$ is finite for all $X \in \mathcal{X}$.

For any $X \in \mathcal{X}$ and $i, j \in \mathbb{I}$ let

$$m_{ij}^X = |\Delta^{X\ re} \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)|.$$

We say that $\mathcal{G}$ is a Cartan graph if the following hold.

(CG3) For all $X \in \mathcal{X}$, the set $\Delta^{X\ re}$ consists of positive and negative roots.

(CG4) Let $X \in \mathcal{X}$ and $i, j \in \mathbb{I}$. If $m_{ij}^X < \infty$, then $(r_i r_j)^{m_{ij}^X(X)} = X$.

§2.2.5.2 Root systems

Let $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be a Cartan graph.

Definition 2.2.27 *For all $X \in \mathcal{X}$ let R^X be a subset of $\mathbb{Z}^{\mathbb{I}}$ with the following properties.*

(1) $0 \notin R^X$ and $\alpha_i \in R^X$ for all $X \in \mathcal{X}$ and $i \in \mathbb{I}$.

(2) $R^X \subseteq \mathbb{N}_0^{\mathbb{I}} \cup -\mathbb{N}_0^{\mathbb{I}}$ for all $X \in \mathcal{X}$.

(3) For any $X \in \mathcal{X}$ and $i \in \mathbb{I}$, $s_i^X(R^X) = R^{r_i(X)}$.

Then we say the pair $(\mathcal{G}, (R^X)_{X \in \mathcal{X}})$ is a root system over $\mathcal{G}$. A root system over $\mathcal{G}$ is said to be reduced if for all $X \in \mathcal{X}$ and $\alpha \in R^X$ the root α and $-\alpha$ are the only rational multiples of α in R^X. A root system over $\mathcal{G}$ is finite if R^X is a finite set for all $X \in \mathcal{X}$.

Example 2.2.28 *The pair $(\mathcal{G}, (\Delta^{X\ re})_{X \in \mathcal{X}})$ is a root system over $\mathcal{G}$. Indeed, Axioms (1) and (3) follow from the definition of $\Delta^{X\ re}$ for all $X \in \mathcal{X}$, and (2) follows from (CG3). The root system $(\mathcal{G}, (\Delta^{X\ re})_{X \in \mathcal{X}})$ is reduced.*

§2.2.6 Lyndon word theory and root systems of Nichols algebras of diagonal type

We follow the form of Lyndon word theory in [HZ18], for more details and a more general case see [GH07].

Follow the notations in Section 1.1. Define an ordering on X by $x_1 < x_2 < ... < x_s$, consider the lexicographic ordering on $\mathbb{X}$. Let $\mathbb{X}^{\times}$ be the set of nonempty words in $\mathbb{X}$.

Definition 2.2.29 *A word $u \in \mathbb{X}^{\times}$ is a Lyndon word if for any decomposition $u = vw$, $u, v \in \mathbb{X}^{\times}$, we have $u < w$.*

Example 2.2.30 *Let $X = \{1, 2, 3\}$ with the usual ordering. Then 12 is a Lyndon word, since $12 < 2$. Also 12122 and 132 are Lyndon words. But $11, 1212, 21, 121$ are not Lyndon words. The Lyndon words of length ≤ 3 are*

$$1, 2, 3, 12, 13, 23, 112, 113, 122, 123, 132, 133, 223, 233.$$

Proposition 2.2.31 *A word $u \in \mathbb{X}^\times$ is a Lyndon word if and only if either $u \in X$, or there exist Lyndon words v, w such that $v < w$ and $u = vw$. Moreover if w is a Lyndon word and $w = uv$ such that $u \in \mathbb{X}^\times$, v is a Lyndon word and $|v|$ is maximal with these properties, then u is also a Lyndon word and $u < uv < v$.*

Definition 2.2.32 *Any word u of length at least two has a unique decomposition into the product of two Lyndon words $u = vw$ where the length of w is maximal. This decomposition is called Shirshov decomposition.*

Example 2.2.33 *Let $X = \{1, 2, 3\}$. Then 123123233 is a Lyndon word and the Shirshov decomposition of this word is $(123)(123233)$.*

Let $\mathbb{L}$ be the set of Lyndon words.

Theorem 2.2.34 *Any $w \in \mathbb{X}$ can be written uniquely as $w = u_1 u_2 \cdots u_m$, where $m \in \mathbb{N}_0$, $u_1, ..., u_m \in \mathbb{L}$ and $u_m \leq ... \leq u_2 \leq u_1$. Moreover u_m is the lexicographically smallest proper end on w.*

Example 2.2.35 $1231233123122123 = (1231233)(123)(122123)$.

Recall that we identify $\mathcal{K}\mathbb{X}$ with $T(V)$. For a Lyndon word u, we denote $[u] \in T(V)$ inductively as follows:

(1) $[u] = u$, if $u \in X$,

(2) $[u] = [[v], [w]]$ where $u = vw$ is the Shirshov decomposition, if $|u| \geq 2$. $[\ ,\ ]$ is the braided bracket defined in Section 1.2.4.

We call $[u]$ super-letters. Then the total ordering on $\mathbb{X}$ induces a total ordering on the set of super-letters.

Lemma 2.2.36 *Let $w \in \mathbb{L}$. Then $[w] = w+$ linear combination of terms w' where $|w'| = |w|$ and $w' > w$.*

Theorem 2.2.37 *The set*

$$\{[u_1]^{k_1}[u_2]^{k_2}\cdots[u_m]^{k_m}|m \in \mathbb{N}_0, u_i \in \mathbb{L} \text{ for all } i,$$
$$u_1 > u_2 > \cdots u_m, k_1, ..., k_m \geq 0\}$$

is a vector space basis of $T(V)$.

Proof: By Thm. 2.2.34 and Lemma 2.2.36. □

Proposition 2.2.38 *Let $u, v \in \mathbb{L}$. Assume that $u < v$. Then*

$$[u][v] - \chi(deg\ u, deg\ v)[v][u] \in \sum_{w=u_1u_2\cdots u_k,\ u_1\geq u_2\geq\cdots\geq u_k\geq uv} [u_1]\cdots[u_k].$$

If $w = u_1u_2\cdots u_k$ for $u_1 \geq u_2 \geq \cdots \geq u_k$ and $u_1, ..., u_k \in \mathbb{L}$ for some $k \in \mathbb{N}$, define $[w]$ by $[u_1][u_2]\cdots[u_k]$. Denote $M_{>u}$ by the vector space spanned by all $[w] \in T(V)$ such that $w = v_1v_2\cdots v_k$, $v_1 \geq v_2 \geq \cdots \geq v_k > u$, $v_1, ..., v_k \in \mathbb{L}$ for some $k \in \mathbb{N}$.

Proposition 2.2.39 *Let $u \in \mathbb{L}$. Then*

$$\Delta([u]) - 1 \otimes [u] - [u] \otimes 1 \in T(V) \otimes M_{>u}.$$

Proof: We refer to the proof of Prop. 3.6 in [GH07]. □

For $\alpha \in \mathbb{Z}^s$, let $o_\alpha \in \mathbb{N} \cup \infty$ be the order of $\chi(\alpha, \alpha)$. Let $O_\alpha = \{1, o_\alpha, \infty\}$.

Definition 2.2.40 *Let $w \in \mathbb{X}^\times$. We say that $[w]$ is a root vector candidate if $w = v^k$ for some Lyndon word v and $k \in O_{deg\ v}\backslash\{\infty\}$.*

Definition 2.2.41 *A root vector candidate $[w]$ is called a root vector of $B(V)$ if $[w] \in B(V)$ is not a linear combination of elements of the form $[v_k]^{m_k}\cdots[v_1]^{m_1}$, where $k, m_k, ..., m_1 \in \mathbb{N}$ and $[v_1], ..., [v_k]$ are root vector candidates or equivalently root vectors with $v_k > \cdots > v_1 > w$.*

Theorem 2.2.42 *Let $\boldsymbol{R} \subseteq \mathbb{X}^\times$ such that $w \in \boldsymbol{R}$ if and only if $[w]$ is a root vector. Then the elements*

$$[v_k]^{m_k}\cdots[v_1]^{m_1},\ k \in \mathbb{N},\ v_1, ..., v_k \in \boldsymbol{R},\ v_1 < v_2 < \cdots < v_k,$$
$$\forall 1 \leq i \leq k,\ 0 \leq m_i < O_{deg\ v_i}\backslash\{\infty\}, \tag{2.2}$$

form a vector space basis of $B(V)$.

Definition 2.2.43 *Define*

$$\Delta_+ = \{deg\ u | u \in \boldsymbol{R}\}$$

of positive roots of $B(V)$ and the root system

$$\Delta = \Delta_+ \cup -\Delta_-$$

of $B(V)$. For any $\alpha \in \Delta_+$, the number of $u \in \boldsymbol{R}$ with $deg\ u = \alpha$ is called the multiplicity of α.

§2.2.7 Differential operators, reflections and Weyl groupoids of Nichols algebras of diagonal type

In this subsection we introduce the reflection theory of Nichols algebras of diagonal type, constructed by Heckenberger.

§2.2.7.1 Differential operators

We refer to Chapter 7 of [HS20].

Definition 2.2.44 *Define right and left differential operators $\partial_i^R, \partial_i^L : B(\mathbb{V}) \to B(\mathbb{V}), 1 \leq i \leq s$ by:*

(1) $\partial_i^R(1) = \partial_i^L(1) = 0$, $\partial_i^R(x_j) = \partial_i^L(x_j) = \delta_{ij}$ *for all $1 \leq i, j \leq s$.*

(2) $\partial_i^R(xy) = x\partial_i^R(y) + \partial_i^R(x)\chi(\alpha_i, \beta)y$ *for all $1 \leq i, j \leq s$ and $x, y \in B(\mathbb{V})$, y is $\mathbb{N}_0^s$-homogeneous with degree β.*

(3) $\partial_i^L(xy) = \partial_i^L(x)y + \chi(\alpha, \alpha_i)x\partial_i^L(y)$ *for all $1 \leq i, j \leq s$ and $x, y \in B(\mathbb{V})$, x is $\mathbb{N}_0^s$-homogeneous with degree α.*

Proposition 2.2.45 *For $x \in B(\mathbb{V})$ and x is not a constant, in $B(\mathbb{V})$ we have*

$$x = 0 \Leftrightarrow \partial_i^R(x) = 0\ \forall 1 \leq i \leq s,$$
$$\Leftrightarrow \partial_i^L(x) = 0\ \forall 1 \leq i \leq s.$$

§2.2.7.2 Reflections

Let V be a braided vector space of diagonal type with bicharacter χ, realized in $_H^H\mathcal{YD}$ for group algebra H of some abelian group. Let $i \in \mathbb{I} = \{1, 2, ..., s\}$ and let

$$h_i = \min\{m' \geq 1 | 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^{m'-1} = 0\} \cup \{\infty\}.$$

Let $\mathbb{K}_i$ be the subalgebra of $B(V)$ generated by the elements $(ad_c x_i)^m(x_j)$, $m \geq 0$, $j \neq i$, where

$$(ad_c x_i)(y) = x_i y - (g_i \cdot y)x_i.$$

Then we have the following Lemma,

Lemma 2.2.46 *If $h_i = \infty$, then $B(V) \cong \mathbb{K}_i \otimes \mathcal{K}[x_i]$, and if $h_i \in \mathbb{N}$, then $B(V) \cong \mathbb{K}_i \otimes \mathcal{K}[x_i]/(x_i^{h_i})$ as $\mathbb{Z}^s$-graded objects in ${}^H_H\mathcal{YD}$.*

Proposition 2.2.47 *For all $i \in \mathbb{I}$, $\mathbb{K}_i = \partial_i^R$ is a coideal subalgebra of $B(V)$.*

Proposition 2.2.48 *Let $i \in \mathbb{I}$. The following are equivalent.*

(1) *The algebra $\mathbb{K}_i$ is finite generated.*

(2) *For all $j \in \mathbb{I}, j \neq i$, there exists $m_{ij} \geq 0$ such that $(ad_c x_i)^{m_{ij}+1}(x_j) = 0$.*

(3) *For all $j \in \mathbb{I}, j \neq i$, there exists $m_{ij} \geq 0$ such that*

$$1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^{m_{ij}} \ or \ q_{ii}^{m_{ij}} q_{ij} q_{ji} = 1.$$

For all $i \neq j$, m_{ij} in (2) and m_{ij} in (3) can be chosen to be the same.

Definition 2.2.49 *Let $i \in \mathbb{I}$. We say that $B(V)$ is i-finite, if $\mathbb{K}_i$ is a finitely generated algebra. In this case, let*

$$a_{ij}^{\chi} = -min\{m \in \mathbb{N}_0 | 1 + q_{ii} + q_{ii}^2 + \cdots + q_{ii}^m = 0 \ or \ q_{ii}^m q_{ij} q_{ji} = 1\}$$

and $a_{ii}^{\chi} = 2$.

Note that $\mathcal{K}\partial_i^L = (\mathcal{K}x_i)^* \in {}^H_H\mathcal{YD}$.

Theorem 2.2.50 *Let $i \in \mathbb{I}$. Let B_i be the subalgebra of $End(B(V))$ generated by the endomorphisms*

$$\partial_i^L \ and \ L_x : y \mapsto xy, \quad x \in \mathbb{K}_i.$$

Then the elements $(\partial_i^L)^m L_x$, where $m \geq 0$ and $x \in \mathbb{K}_i$, spans the vector space B_i. If $B(V)$ is i-finite, then B_i is generated as an algebra by $V_i \in {}^H_H\mathcal{YD}$, where

$$V_i = \mathcal{K}\partial_i^L \bigoplus \oplus_{j \neq i} \mathcal{K}(ad_c x_i)^{-a_{ij}^{\chi}}(x_j).$$

In this case, $B_i \cong B(V_i)$ as graded braided Hopf algebras in ${}^H_H\mathcal{YD}$.

Theorem 2.2.51 *Let $i \in \mathbb{I}$. Assume that $B(V)$ is i-finite. Let s_i^{χ} defined by $s_i^{\chi}(\alpha_j) = \alpha_j - a_{ij}^{\chi}(\alpha_i)$ for all $j \in \mathbb{I}$. Then $s_i^{\chi}(\Delta(B(V))) = \Delta(B(V_i))$.*

Now we calculate the bicharacter corresponding to V_i. For all $w \in \mathrm{Aut}(\mathbb{Z}^n)$, let $w^*\chi$ be the bicharacter on $\mathbb{Z}^n$ defined by

$$w^*\chi(\alpha, \beta) = \chi(w^{-1}(\alpha), w^{-1}(\beta)).$$

Proposition 2.2.52 *Let $i \in \mathbb{I}$. Assume that $B(V)$ is i-finite, and let V_i be as in Theorem 2.2.50.*

Let $\boldsymbol{\chi}$ denote the set of bicharacters on $\mathbb{Z}^n$. For all $i \in \mathbb{I}$ let

$$r_i : \boldsymbol{\chi} \to \boldsymbol{\chi}, \quad r_i(\bar{\chi}) = \begin{cases} (s_i^{\bar{\chi}})^*\bar{\chi} & \text{if } \bar{\chi} \text{ is } i\text{-finite}, \\ \bar{\chi} & \text{otherwise.} \end{cases} \tag{2.3}$$

Then $r_i^2 = \mathrm{id}$, and hence the map r_i are bijections for all $i \in \mathbb{I}$. Let $\mathcal{G}$ be the subgroup

$$\mathcal{G} = \langle r_i | i \in \mathbb{I} \rangle$$

of the group of bijections of χ. Let $\mathcal{G}(\chi) = \{r(\chi) | r \in \mathcal{G}\}$ be the $\mathcal{G}$-orbit of χ in $\boldsymbol{\chi}$.

Theorem 2.2.53 *Assume that $\bar{\chi}$ is i-finite for all $\bar{\chi} \in \mathcal{G}(\chi)$ and $i \in \mathbb{I}$. Let $\mathcal{X} = \mathcal{G}(\chi), r_i : \mathcal{X} \to \mathcal{X}$ for $i \in \mathbb{I}$ as in (1.3), and $A^{\bar{\chi}} = (a_{ij}^{\bar{\chi}})_{i,j \in \mathbb{I}}$ for $\bar{\chi} \in \mathcal{X}$ such that $a_{ii}^{\bar{\chi}} = 2$ for all $i \in \mathbb{I}$ and*

$$a_{ij}^{\bar{\chi}} = -min\{m \in \mathbb{N}_0 | (1 + \bar{q}_{ii} + \cdots + \bar{q}_{ii}^m)(\bar{q}_{ii}^m \bar{q}_{ij} \bar{q}_{ji} - 1) = 0\}$$

for all $i, j \in \mathbb{I}, i \neq j$, where $\bar{q}_{kl} = \bar{\chi}(\alpha_k, \alpha_l)$ for all $k, l \in \mathbb{I}$. Then $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r_i, A)$ is a Cartan graph. Then $\mathcal{R} = \mathcal{R}(\mathcal{G}, (\Delta^{\bar{\chi}})_{\bar{\chi} \in A})$ is a root system of type $\mathcal{G}$, where $\Delta^{\bar{\chi}}$ is the root system for $\bar{\chi}$ defined in Def. 2.2.43.

Definition 2.2.54 *Let (V, c) be a finite dimensional braided vector space of diagonal type and let $\boldsymbol{q} = (q_{ij})_{i,j \in \mathbb{I}}$ be a braiding matrix of V. We say that $\boldsymbol{q}$ is of Cartan type if there exist a Cartan matrix $A = (a_{ij})_{i,j \in \mathbb{I}}$ such that for all $i, j \in \mathbb{I}$,*

$$q_{ij} q_{ij} = q_{ii}^{a_{ij}}, \quad \text{where } 0 \leq -a_{ij} \leq ord(q_{ii}) \text{ if } i \neq j.$$

Proposition 2.2.55 *Assume that V is of Cartan type with bicharacter χ, then $B(V)$ is i-finite for all $i \in \mathbb{I}$ and $a_{ij}^{\chi} = a_{ij}$. In addition, for all $i \in \mathbb{I}$, V_i with bicharacter χ' is also of Cartan type and $a_{ij}^{\chi'} = a_{ij}$.*

Example 2.2.56 *Let V be a braided vector space of dimension 2, od diagonal type with*

braiding matrix $\begin{pmatrix} q & q \\ q & q \end{pmatrix}$. *If q is a primitive Nth root of unity, then V is of Cartan type* $\begin{pmatrix} 2 & 2-N \\ 2-N & 2 \end{pmatrix}$. *Thus if $N = 3$, then is of Cartan type A_2 and dim $B(V)=27$.*

Example 2.2.57 *Assume that $s = 2, q_{11} = q_{22} = -1$ and $q_{12} = q_{21} = q$, when $q \in \mathcal{K}^\times, q^2 \neq 1$. Let $\chi_1 = r_1(\chi), \chi_2 = r_2(\chi)$. Explicit calculations show that*

$$A^{\chi_1} = A^{\chi_2} = A^\chi = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

$$(r_1(\chi)(\alpha_i, \alpha_j))_{i,j=1,2} = \begin{pmatrix} -1 & -q^{-1} \\ -q^{-1} & q^2 \end{pmatrix},$$

$$(r_2(\chi)(\alpha_i, \alpha_j))_{i,j=1,2} = \begin{pmatrix} q^2 & -q^{-1} \\ -q^{-1} & -1 \end{pmatrix},$$

and $r_2 r_1(\chi) = r_1(\chi), r_1 r_2(\chi) = r_2(\chi)$. Hence $\mathcal{X} = \{\chi, r_1(\chi), r_2(\chi)\}$. This example corresponds to the Lie superalgebra $sl(2|1)$ with three different choices of the Cartan subalgebra.

§2.2.8 Affine Nichols algebras

Now we are ready to define affine Nichols algebras.

Definition 2.2.58 *Let $\mathcal{G} = \mathcal{G}(\mathbb{I}, \mathcal{X}, r, A)$ be a Cartan graph. Assume there exist a root system R over $\mathcal{G}$. Let $X \in \mathcal{X}$ be an object. Assume that there is exactly one imaginary root in R_+^X and that there is no root system of type $\mathcal{G}$ with more than one positive imaginary root at X. Then we call $\mathcal{G}$ affine Cartan graph and the Weyl groupoid $\mathcal{W}(\mathcal{G})$ an affine Weyl groupoid. An affine Nichols algebra is a Nichols algebra of diagonal type with an affine Weyl groupoid.*

We talk about rank 2 case. Let V be a two dimensional braided vector space of diagonal type with a braiding matrix $(q_{ij})_{1 \leq i,j \leq 2}$. Define the triple $\boldsymbol{q}$ by $\boldsymbol{q} = (q_1, q, q_2)$ by $q_{11} = q_1, q_{22}, q = q_{12}q_{21}$. We say $\boldsymbol{q}$ is an affine triple if $B(V)$ is an affine Nichols algebra.

Theorem 2.2.59 *If $\boldsymbol{q} = (q_1, q, q_2)$ is an affine triple, then up to permutations of labels, its Dynkin diagram* $\overset{q_1 \quad q \quad q_2}{\circ\!\!-\!\!-\!\!-\!\!-\!\!\circ}$ *is one of those listed in Figure 2.1 in the next page.*

	generalized Dynkin diagram	parameters	period
1	$\zeta \quad \zeta \quad \zeta$ $\circ\!-\!\!-\!\!-\!\circ$	$\zeta \in \mu_3$	(2)
2	$\zeta^2 \; \zeta^5 \; \zeta^2$ $\circ\!-\!\!-\!\!-\!\circ$	$\zeta \in \mu_6$	(2)
3	$\zeta \quad \zeta^4 \quad \zeta^4$ $\circ\!-\!\!-\!\!-\!\circ$	$\zeta \in \mu_6$	(2)
4	$\zeta^4 \; \zeta \quad \zeta^2 \; \zeta^2 \; \zeta^3 \; \zeta^2 \; \zeta^2 \; \zeta \quad \zeta^4$ $\circ \; \circ \;\; \circ\!-\!\!-\!\circ \;\; \circ\!-\!\!-\!\circ$	$\zeta \in \mu_6$	(2)
5	$\zeta \quad \zeta^{10} \; \zeta^4$ $\circ\!-\!\!-\!\!-\!\circ$	$\zeta \in \mu_{12}$	(2)
6	$\zeta \quad \zeta^4 \quad \zeta^4$ $\circ\!-\!\!-\!\!-\!\circ$	$\zeta \in \mu_5$	$(1,4)$
7	$\zeta^4 \; \zeta^4 \; \zeta$ $\circ\!-\!\!-\!\!-\!\circ$	$\zeta \in \mu_8$	$(1,4)$
8	$\zeta \quad \zeta^9 \quad \zeta^4$ $\circ\!-\!\!-\!\!-\!\circ$	$\zeta \in \mu_{10}$	$(1,4)$
9	$\zeta \quad \zeta^{10} \; \zeta^9 \; \zeta^9 \; \zeta^8 \; \zeta^4$ $\circ\!-\!\!-\!\circ \; \circ\!-\!\!-\!\circ$	$\zeta \in \mu_{12}$	$(2,3,1,3)$
10	$\zeta \quad \zeta^8 \; \zeta^6 \; \zeta^6 \; \zeta^4 \; \zeta^3 \; \zeta^3 \; \zeta^2 \; \zeta^9$ $\circ\!-\!\!-\!\circ \; \circ\!-\!\!-\!\circ \; \circ\!-\!\!-\!\circ$ $\zeta^9 \; \zeta^4 \; \zeta^6 \; \zeta^6 \; \zeta^8 \; \zeta^7$ $\circ\!-\!\!-\!\circ \; \circ\!-\!\!-\!\circ$	$\zeta \in \mu_{12}$	$(4,1,3,3,1)$
11	$\zeta \quad \zeta^{12} \; \zeta^9 \; \zeta^9 \; \zeta^6 \; \zeta^4$ $\circ\!-\!\!-\!\circ \; \circ\!-\!\!-\!\circ$	$\zeta \in \mu_{18}$	$(6,1,3,1)$
12	$q \quad q^{-2} \; q$ $\circ\!-\!\!-\!\!-\!\circ$	$q \in \mathbb{C}^\times \backslash \{\pm 1\}$	(2)
13	$q \quad q^{-2} \; -q$ $\circ\!-\!\!-\!\!-\!\circ$	$q \in \mathbb{C}^\times \backslash \{\pm 1\}$	(2)
14	$q \quad q^{-4} \; q^4$ $\circ\!-\!\!-\!\!-\!\circ$	$q \in \mathbb{C}^\times$, $q \notin \mu_3$, $q \notin \mu_4$, $q \neq \pm 1$	$(1,4)$

Figure 2.1 Generalized Dynkin diagrams of affine bicharacters of rank two.

§2.2.9 Braided vector space $\mathbb{V}$, the Nichols algebra $B(\mathbb{V})$ and the root system of $B(\mathbb{V})$

Next we consider the following rank 2 affine Nichols algebra $B(\mathbb{V})$ in Figure 2.1.

Let $\mathbb{V}$ be a 2-dimensional braided vector space of diagonal type with braiding matrix $\begin{pmatrix} q & q^{-1} \\ q^{-1} & -q \end{pmatrix}$, q is not a root of unity. $B(\mathbb{V})$ is the main research object in this thesis. We will prove $B(\mathbb{V})$ is isomorphic to $U_v(A(0,2)^{(4)})^+$ with $q = v^2$.

It is easy to see $B(\mathbb{V})$ is of affine Cartan type with Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.
Then the real roots of $B(\mathbb{V})$ are just the positive real roots of $\hat{sl}_2$, namely $(n+1)\alpha_1 + n\alpha_2$ and $n\alpha_1 + (n+1)\alpha_2$ for $n \geq 0$. $B(\mathbb{V})$ also has imaginary roots. The only possible imaginary roots are $n\delta$, $\delta = \alpha_1 + \alpha_2$ and $n \geq 1$ but the multiplicities are in general unknown.

Note that if β is an imaginary root, then we have $\chi(\beta, \alpha)\chi(\alpha, \beta) = 1$ for all $\alpha \in \mathbb{N}_0^2$, and hence (1.2.1) holds.

Remark 2.2.60 *For $U_v(\hat{sl}_2)^+$ when v is not a root of unity, the corresponding Nichols algebra is $B(V)$, where V has braiding matrix $\begin{pmatrix} q & q^{-1} \\ q^{-1} & q \end{pmatrix}$ and $q = v^2$.*

§2.3 Quantum affine (super)algebras $U_v(\hat{sl}_2)$ and $U_v(A(0,2)^{(4)})$

In this section we first introduce the definition of quantum affine algebra $U_v(\hat{sl}_2)$ when v is not a root of unity. We give two realizations of $U_v(\hat{sl}_2)$: The Drinfeld-Jimbo type realization and the Drinfeld second realization. Then we introduce the quantum affine superalgebra $U_v(A(0,2)^{(4)})$.

§2.3.1 $U_v(\hat{sl}_2)$

Let $v \in \mathbb{C}^\times$ is assumed not to be a root of unity. For any integer $r > 0$ define

$$[r]_v = \frac{v^r - v^{-r}}{v - v^{-1}}.$$

Definition 2.3.1 *The quantum affine algebra $U_v(\hat{sl}_2)$ is the associative algebra over $\mathbb{C}$ with generators $e_i^\pm, K_i^{\pm 1}, i = 0, 1$, and the following relations:*

$$K_i K_i^{-1} = K_i^{-1} K_i = 1,$$
$$K_0 K_1 = K_1 K_0,$$
$$K_i e_i^{\pm} K_i^{-1} = v^{\pm 2} e_i^{\pm},$$
$$K_i e_j^{\pm} K_i^{-1} = v^{\mp 2} e_j^{\pm}, i \neq j,$$
$$[e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{v - v^{-1}},$$
$$[e_0^{\pm}, e_1^{\mp}] = 0,$$
$$(e_i^{\pm})^3 e_j^{\pm} - [3]_v (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_v e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \ (i \neq j).$$

Moreover, $U_v(\hat{sl}_2)$ is a Hopf algebra over $\mathbb{C}$ with comultiplication

$$\Delta(e_i^+) = e_i^+ \otimes K_i + 1 \otimes e_i^+,$$
$$\Delta(e_i^{-1}) = e_i^{-1} \otimes 1 + K_i^{-1} \otimes e_i^{-1},$$
$$\Delta(K_i) = K_i \otimes K_i,$$

and antipode

$$S(K_i) = K_i^{-1}, \ S(e_i^+) = -e_i^+ K_i^{-1}, \ S(e_i^{-1}) = -K_i e_i^{-1}.$$

Then we introduce the famous Drinfeld second realization.

Definition 2.3.2 *The quantum affine algebra $U_v(\hat{sl}_2)$ is isomorphic to the associative algebra over $\mathbb{C}$ with generators $x_k^{\pm}(k \in \mathbb{Z}), h_k(k \in \mathbb{Z} - 0), K_{\pm 1}$, central elements $C^{\pm 1}$ and the following relations:*

$$CC^{-1} = C^{-1} C = KK^{-1} = K^{-1} K = 1,$$
$$[h_k, h_l] = \delta_{k,-l} \frac{1}{k} [2k]_v \frac{C^k - C^{-k}}{v - v^{-1}},$$
$$K h_k = h_k K,$$
$$K x_k^{\pm} K^{-1} = v^{\pm 2} x_k^{\pm},$$
$$[h_k, x_l^{\pm}] = \pm \frac{1}{k} [2k]_v C^{\mp (\frac{1}{2})(k+|k|)} x_{k+l}^{\pm},$$
$$x_{k+1}^{\pm} x_l^{\pm} - v^{\pm 2} x_l^{\pm} x_{k+l}^{\pm} = v^{\pm 2} x_k^{\pm} x_{l+1}^{\pm} - x_{l+1}^{\pm} x_k^{\pm},$$
$$[x_k^+, x_l^-] = \frac{1}{v - v^{-1}} (C^{k-1} \psi_{k+l} - \phi_{k+l}),$$

where the ψ_k and ϕ_k are defined by the following equalities of formal power series:

$$\sum_{k=0}^{\infty} \psi_k u^k = K \exp((v - v^{-1}) \sum_{k=1}^{\infty} h_k u^k),$$

$$\sum_{k=0}^{\infty} \psi_{-k} u^{-k} = K^{-1} \exp((v - v^{-1}) \sum_{k=1}^{\infty} h_{-k} u^{-k}).$$

The isomorphism between these two realizations is given by:

$$K_0 \mapsto CK^{-1}, \ K_1 \mapsto K, \ e_1^{\pm} \mapsto x_0^{\pm},$$

$$e_0^{\pm} \mapsto x_1^- K^{-1}, \ e_0^- \mapsto C^{-1} K x_{-1}^+.$$

It is well known that the positive part of $U_v(\hat{sl}_2)$ is isomorphic to $B(\mathbb{U})$. Then we can rewrite the relations in the Drinfeld second realization in $B(\mathbb{V})$ as following.

For degree reason We give new notation X_1 and Y_1 of the two Chevalley generators E_0 and E_1 of $U_v(\hat{sl}_2)^+$ or equivalently the two base vectors x_1 and x_2 of the braided vector space $\mathbb{U}$. Let $[\ ,\ ]$ be the braided bracket. Define $L_1 = [X_1, Y_1]$. Define

$$X_n = [\cdots [X_1, L_1], L_1], \cdots], L_1], Y_n = [L_1, [\cdots, [L_1, [L_1, Y_1] \cdots],$$

where L_1 appear $n - 1$ times for both cases. Define $L_n = [X_1, Y_n]$. Then rewrite the relations in the Drinfeld second realization, we have the following relations hold in $B(\mathbb{U})$:

Let $\theta = q - q^{-1}$. Define

$$\sum_{n=0}^{\infty} \theta L_n u^n = \exp(\sum_{n=1}^{\infty} \theta \bar{L}_n u^n),$$

then $\forall m, n, k > 0$,

(a) $[L_m, L_n] = 0 = [\bar{L}_m, \bar{L}_n]$,

(b) $[X_k, \bar{L}_m] = \frac{[m]}{m} X_{k+m}, [\bar{L}_m, Y_k] = \frac{[m]}{m} Y_{k+m}$,

(c) $[X_n, X_{n+k}] = -[X_{n+k-1}, X_{n+1}], [Y_{n+k}, Y_n] = -[Y_{n+1}, Y_{n+k-1}]$,

(d) $[X_m, Y_n] = L_{m+n-1}$.

§2.3.2 $\quad U_v(A(0,2)^{(4)})$

$U_v(A(0,2)^{(4)})$ is defined by Hiroyuki Yamane in [Yam99]. In this paper, first Yamane defined the quantum affine superalgebras abstractly, i.e. by the radical of

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the standard bilinear form. Note that under this definition, the positive part of $U_v(A(0,2)^{(4)})$ is just $B(\mathbb{V})$ by definition.

Then Yamane determined the defining relations satisfied by the Chevalley generators of all the quantum affine superalgebras except $A(m,m)^{(2)}$ and $A(m,m)^{(4)}$ case. This means if we view v as an element in $\mathbb{C}$, then when v is transcendental, the defining relations satisfied by the Chevalley generators of $U_v(A(0,2)^{(4)})^+$ is

$$[E_0, [E_0, [E_0, E_1]]] = 0, \ [[[E_0, E_1], E_1], E_1] = 0, \tag{2.4}$$

where E_0 and E_1 are the Chevalley generators. In other words, the defining relations of $B(\mathbb{V})$ satisfied by the Chevalley generators are exactly these quantum Serre relations. Moreover, the root multiplicities of $U_v(A(0,2)^{(4)})$ are also known by [EG09], when v is transcendental. However, no PBW basis was given till now.

In addition, when v is not a root of unity, the defining relations satisfied by the Chevalley generators, the root multiplicities, PBW basis are all unknown. The reason is at $A(0,2)^{(4)}$ and the quantum case, there exist odd imaginary roots and imaginary root vectors do not necessarily commute with each other, which is introduced in the following several paragraphs. In the following parts of this thesis, we overcome this difficulty. For $U_v^+(A(0,2)^{(4)})$ or equivalently the Nichols algebra $B(\mathbb{V})$, we give all the commutation relations among the distinguished imaginary root vectors and give the defining relations satisfied by the Chevalley generators, determine the root multiplicities, and give a PBW basis.

By the reflection theory of Nichols algebras of diagonal type we conclude that $B(\mathbb{U})$ and $B(\mathbb{V})$ share the same real roots, that is, $n\delta + \alpha_1$ and $n\delta + \alpha_2$ for $n \geq 0$, where $\delta = \alpha_1 + \alpha_2$. The imaginary roots of $B(\mathbb{U})$ are well known to be $n\delta$, $n > 0$, and the multiplicities are all 1. The only possible imaginary roots of $B(\mathbb{V})$ are also $n\delta$, but the multiplicities are in general unknown.

One can define X_n, Y_n, L_n in the same way for $B(\mathbb{V})$. It is easy to prove X_n and Y_n are still real root vectors. However, when one looks at the relations in the imaginary root space, a big difficulty appears. At degree 3δ, we have the following relations:

$$L_3 = \frac{\theta - 1}{2}[L_2, L_1] + \theta L_1 L_2 - \theta L_1^3, \tag{2.5}$$

and L_2, L_1 do not commute any more. In general, it was unknown how to describe these relations. In the present thesis, we give a solution.

Chapter 3   The Nichols algebra $B(\mathbb{V})$ and the relations

In this chapter we aim to introduce the affine Nihcols algbera $B(\mathbb{V})$ and determine the main commutation relations in $B(\mathbb{V})$.

§3.1   Some basic definitions and properties in $B(\mathbb{V})$

Let X_n, Y_n be the real root vector candidates corresponding to the Lyndon word

$$x_1(x_1 x_2)^{n-1} \text{ and } (x_1 x_2)^{n-1} x_2.$$

Define L_n by $[X_1, Y_n]$, L'_n by $[X_2, Y_{n-1}]$ (it is convenient to define $L_1 = L'_1 = [X_1, Y_1]$ and $L_0 = L'_0 = \frac{1}{\theta}$). Then L_n is the imaginary root vector candidate at $n\delta$ corresponding to the Lyndon word

$$x_1(x_1 x_2)^{n-1} x_2.$$

Define $\hat{L}_{n+1} = [L_n, L_1]$, $\tilde{L}_n = \frac{1}{2}(L_n + L'_n)$ (it is convenient to define $\tilde{L}_n = 0$ for $n < 0$).

Define $M_{2n+1} = \mathrm{ad}^n_{L_2}(L_1)$, where $\mathrm{ad}_X(Y)$ means $[X, Y]$ for $X, Y \in B(\mathbb{V})$. Then M_{2n+1} is the imaginary root vector candidate at $(2n+1)\delta$ corresponding to the Lyndon word

$$(x_1 x_1 x_2 x_2)^n x_1 x_2.$$

We have the following order in terms of super-letters

$$X_1 < X_2 < \cdots < X_n < \cdots$$
$$< \cdots < L_n < L_{n-1} < L_{n-2} < \cdots < L_2$$
$$< \cdots < M_{2n-1} < M_{2n-3} < \cdots < M_5 < M_3 < M_1$$
$$< \cdots < Y_n < \cdots < Y_2 < Y_1.$$

Lemma 3.1.1 *If X_n and Y_n are not 0 in $B(\mathbb{V})$, then they are root vectors at corresponding degrees.*

Proof: Note that X_n and Y_n are the biggest super-letters at corresponding degrees. Then it follows from the definition of root vectors and degree reason.   □

Lemma 3.1.2 *In $B(\mathbb{V})$ we have*

$$\Delta(Y_n) \equiv Y_n \otimes 1 + 1 \otimes Y_n \mod B_{\geq 1} \otimes B(\mathbb{V})$$

and we have $X_n \in K_{\geq 1}$, $L_n \in K_{\geq 1}$, $Y_n \notin K_{\geq 1}$, $X_n \in K_{>1}$, $L_n \notin K_{>1}$.

Proof: $\Delta(Y_n)$ follows from Prop. 2.2.39. Since $L_n = [X_1, Y_n]$, by the expression of $\Delta(Y_n)$, we have $L_n \in K_{\geq 1}$. In particular $L_1 \in K_{\geq 1}$, then immediately $X_n \in K_{\geq 1}$. Other conclusions are obvious. $\qquad\square$

Lemma 3.1.3 *The following basic relations hold in $B(\mathbb{V})$.*

(a) $[X_n, L_1^2] = X_{n+2}$ *and* $[L_1^2, Y_n] = Y_{n+2}$.

(b) $L'_{2n+1} = L_{2n+1} + [L_{2n}, L_1]$, $L'_{2n+2} = L_{2n+2} - [L_{2n+1}, L_1]$.

(c) *Fix n, if $\forall i \leq n$, $[L_{i-2}, L_1^2] = 0$, then $[X_{2k+1}, Y_{n-2k}] = L_n$, $\forall 0 \leq k \leq \frac{n-1}{2}$, $k \in \mathbb{N}_0$ and $[X_{2k+2}, Y_{n-(2k+1)}] = L'_n$, $\forall 0 \leq k \leq \frac{n-2}{2}$, $k \in \mathbb{N}_0$.*

(d) $[M_{2n+1}, M_{2n-1}] = [L_2, M_{2n-1}^2]$ *for all $n \geq 1$. If $[L_2, M_{2n-1}^2] = 0$, then $[M_{2n+3}, M_{2n-1}] = -2M_{2n+1}^2$.*

Proof: It follows from the braided Jacobi identity, skew derivations and definitions directly. $\qquad\square$

§3.2 Main relations in $B(\mathbb{V})$

§3.2.1 Relations in root space with degree $\leq 4\delta$

We have known quantum Serre relations hold in $B(\mathbb{V})$:

$$[X_1, X_2] = [Y_2, Y_1] = 0.$$

In degree $\leq 4\delta$, we have the following relations. These relations are prepared to obtain the central elements in the next subsection and as a starting point for the induction proof of Prop. 3.2.17.

Proposition 3.2.1 *In $B(\mathbb{V})$, the following relations hold.*

$$[X_1, L_2] = (q+2)X_3 - \theta L_1 X_2, \tag{3.1a}$$

$$[X_1, L_2'] = qX_3 - \theta L_1 X_2, \tag{3.1b}$$

$$[L_2, Y_1] = qY_3 + \theta Y_2 L_1, \tag{3.1c}$$

$$[L_2', Y_1] = (q-2)Y_3 + \theta Y_2 L_1, \tag{3.1d}$$

$$2L_3 - \theta L_1 L_2' = \theta L_2 L_1 - [L_2, L_1], \tag{3.2a}$$

$$2L_3' - \theta L_2' L_1 = \theta L_1 L_2 + [L_2, L_1], \tag{3.2b}$$

$$\tilde{L}_3 = \frac{1}{2}\theta L_1 \tilde{L}_2 + \frac{1}{2}\theta \tilde{L}_2 L_1, \tag{3.2c}$$

$$[X_2, L_2] = qX_4 + \theta L_1 X_3, \tag{3.3a}$$

$$[X_2, L_2'] = (q-2)X_4 + \theta L_1 X_3, \tag{3.3b}$$

$$[L_2, Y_2] = (q+2)Y_4 - \theta Y_3 L_1, \tag{3.3c}$$

$$[L_2', Y_2] = qY_4 - \theta Y_3 L_1, \tag{3.3d}$$

$$[L_2, L_1^2] = 0 \ ([M_3, M_1] = 0), \tag{3.4a}$$

$$[L_3, L_1^2] = 0. \tag{3.4b}$$

Proof: $[X_1, L_2'] = [X_1, [X_2, Y_1]] = qX_2 L_1 + q^{-1}L_1 X_2 = qX_3 - \theta L_1 X_2$. Then $[X_1, L_2] = (q+2)X_3 - \theta L_1 X_2$. Now we have proved (3.1a), (3.1b). Similarly, (3.1c), (3.1d) also hold.

On one hand,

$$[[X_1, L_2], Y_1] = [(q+2)X_3 - \theta L_1 X_2, Y_1] \tag{3.5}$$
$$= (q+2)L_3 - \theta(L_1 L_2' - q^{-1}Y_2 X_2).$$

On the other hand,

$$[[X_1, L_2], Y_1] = [X_1, [L_2, Y_1]] - [L_2, L_1] \tag{3.6}$$
$$= [X_1, qY_3 + \theta Y_2 L_1] - [L_2, L_1]$$

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$$= qL_3 + \theta(L_2L_1 + q^{-1}Y_2X_2) - [L_2, L_1].$$

Comparing (3.5) and (3.6) we get (3.2a) and equivalently (3.2b). (3.2c) is the sum of (3.2a) and (3.2b).

Using $[X_1, \quad]$ to act on (3.2b) we can get

$$(q + q^{-1} + 1)[X_2, L_2] = q(q + q^{-1} + 1)X_4 + \theta(q + q^{-1} + 1)L_1X_3.$$

Since q is not a root of unity, we get (3.3a) and then immediately (3.3b). Similarly, we get (3.3c) and (3.3d).

Using $[\quad, Y_2]$ act on (3.1a), by similar calculation we can get

$$(q + 2)[L_2, L_1^2] = \theta L_1 L_3' - \theta L_3 L_1. \tag{3.7}$$

Using $[\quad, Y_1]$ act on (3.3a) we can get

$$q[L_2, L_1^2] = \theta L_3' L_1 - \theta L_1 L_3 - [L_2, L_2'],$$

i.e.,

$$(q - 2)[L_2, L_1^2] = \theta L_3' L_1 - \theta L_1 L_3. \tag{3.8}$$

Taking the sum of (3.7) and (3.8) we can get

$$2q[L_2, L_1^2] = \theta[L_2, L_1^2].$$

Since q is not a root of unity, we have $[L_2, L_1^2] = 0$ or equivalently $[M_3, M_1] = 0$, which is (3.4a). Then (3.4b) holds by (3.2a). $\qquad\square$

Remark 3.2.2 *We will use the technique in this proof frequently later, to produce new relations in imaginary root space from known relations.*

Remark 3.2.3 *Compared to the case of $U_v(\hat{sl}_2)$, (3.2a) tells us that L_3 is no longer a root vector in terms of Lyndon word theory. We will see later that for $n > 1$, L_{2n+1} are also no longer root vectors.*

§3.2.2 Central elements in $K_{\geq 1}/K_{>1}$

We will first prove that M_{2m+1}^2, $\forall m \geq 0$ commute with L_n, $\forall n > 0$ in $B(\mathbb{V})$.

Lemma 3.2.4 *Suppose there is a number sequence $\{a_n\}_{n\geq 1}$ in $\mathbb{C}$ and $a_{n+2} = Aa_{n+1} + Ba_n$, for $A, B \in \mathbb{C}$. Suppose the roots x_1, x_2 of $x^2 - Ax - B = 0$ are different, then*

$$a_n = \frac{x_2^{n-1}(a_2 - x_1 a_1) - x_1^{n-1}(a_2 - x_2 a_1)}{x_2 - x_1}.$$

The following proposition plays an important role in this paper. There are several very useful corollaries after that. Moreover, in the next subsection, we will write this proposition and its corollaries in the form of generating functions, which will be crucial for the determination of the root multiplicities and the construction of the Drinfeld second realization.

Proposition 3.2.5 *For $k \geq 0$, we have the following relations in $B(\mathbb{V})$.*

$$\tilde{L}_{4k+1} = \frac{1}{2}\theta L_1 \tilde{L}_{4k} + \frac{1}{2}\theta \tilde{L}_{4k} L_1 + \frac{1}{4}[\tilde{L}_{4k-2}, M_3]. \tag{3.9a}$$

$$[\tilde{L}_{4k}, L_1] = \frac{1}{2}\theta M_3 \tilde{L}_{4k-2} + \frac{1}{2}\theta \tilde{L}_{4k-2} M_3 + \frac{1}{4}[\tilde{L}_{4k-4}, M_5]. \tag{3.9b}$$

$$[\tilde{L}_{4k+2}, L_1] = \frac{1}{2}\theta M_3 \tilde{L}_{4k} + \frac{1}{2}\theta \tilde{L}_{4k} M_3 + \frac{1}{4}[\tilde{L}_{4k-2}, M_5]. \tag{3.9c}$$

$$\tilde{L}_{4k+3} = \frac{1}{2}\theta L_1 \tilde{L}_{4k+2} + \frac{1}{2}\theta \tilde{L}_{4k+2} L_1 + \frac{1}{4}[\tilde{L}_{4k}, M_3]. \tag{3.9d}$$

$$[\tilde{L}_n, L_1^2] = 0 = [L_n, L_1^2] \ for \ n = 4k, 4k+1, 4k+2, 4k+3. \tag{3.9e}$$

$$[\tilde{L}_2, \tilde{L}_{4k}] = 0 \ and \ [\tilde{L}_2, \tilde{L}_{4k+2}] = 0. \tag{3.9f}$$

Proof: We have known it holds for $k = 0$ from Prop. 3.2.1. Suppose it holds for k, we consider the case of k+1. We denote anti bracket by $[\ ,\ ]'$, i.e.,

$$[x, y]' = m \circ (id + c)(x \otimes y).$$

We have known $[L_i, L_1^2] = 0$ for $i \leq 4k+3$ from induction hypothesis. Then by Lemma 3.1.3, $[X_3, Y_{4k+3}] = L_{4k+5}$ and $[X_2, Y_{4k+3}] = L'_{4k+4}$. Using $[\ , Y_{4k+3}]$ act on (3.1a), on one hand we have

$$[[X_1, L_2], Y_{4k+3}] = [(q+2)X_3 - \theta L_1 X_2, Y_{4k+3}] \tag{3.10}$$
$$= (q+2)L_{4k+5} - \theta(L_1 L'_{4k+4} - q^{-1}Y_{4k+4}X_2).$$

On the other hand, from Lemma 3.1.3 (a), (3.1c) and induction hypothesis we have

$$[L_2, Y_{4k+3}] = qY_{4k+5} + \theta Y_{4k+4}L_1,$$

then

$$[[X_1, L_2], Y_{4k+3}] = [X_1, qY_{4k+5} + \theta Y_{4k+4}L_1] - [L_2, L_{4k+3}] \tag{3.11}$$
$$= qL_{4k+5} + \theta(L_{4k+4}L_1 + q^{-1}Y_{4k+4}X_2) - [L_2, L_{4k+3}].$$

Comparing (3.10) and (3.11) we get

$$2L_{4k+5} = \theta L_1 L'_{4k+4} + \theta L_{4k+4}L_1 - [L_2, L_{4k+3}]. \tag{3.12}$$

Similarly, using $[\quad, Y_{4k+2}]$ act on (3.3a), on one hand we have

$$[[X_2, L_2], Y_{4k+2}] = [qX_4 + \theta L_1 X_3, Y_{4k+2}] \tag{3.13}$$
$$= qL'_{4k+5} + \theta(L_1 L_{4k+4} + q^{-1}Y_{4k+3}X_3).$$

On the other hand we have

$$[[X_2, L_2], Y_{4k+2}] = [X_2, (q+2)Y_{4k+4} - \theta Y_{4k+3}L_1] - [L_2, L'_{4k+3}] \tag{3.14}$$
$$= (q+2)L'_{4k+5} - \theta(L'_{4k+4}L_1 - q^{-1}Y_{4k+3}X_3) - [L_2, L'_{4k+3}].$$

Comparing (3.13) and (3.14) we get

$$2L'_{4k+5} = \theta L'_{4k+4}L_1 + \theta L_1 L_{4k+4} + [L_2, L'_{4k+3}]. \tag{3.15}$$

Taking the sum of (3.12) and (3.15) we get

$$\tilde{L}_{4k+5} = \frac{1}{2}\theta L_1 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4}L_1 + [L_2, [L_{4k+2}, L_1]]$$
$$= \frac{1}{2}\theta L_1 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4}L_1 + [L_2, [\tilde{L}_{4k+2}, L_1]]$$
$$= \frac{1}{2}\theta L_1 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4}L_1 + [\tilde{L}_{4k+2}, M_3].$$

Then (3.9a) holds.

Taking the difference of (3.12) and (3.15) we get

$$[L_{4k+4}, L_1] = [\tilde{L}_{4k+4}, L_1] = [L_2, \tilde{L}_{4k+3}] = [L_2, \frac{1}{2}\theta L_1 \tilde{L}_{4k+2} + \frac{1}{2}\theta \tilde{L}_{4k+2}L_1 + \frac{1}{4}[\tilde{L}_{4k}, M_3]]$$
$$\tag{3.16}$$
$$= \frac{1}{2}\theta M_3 \tilde{L}_{4k+2} + \frac{1}{2}\theta \tilde{L}_{4k+2}M_3 + \frac{1}{4}[\tilde{L}_{4k}, M_5].$$

Then (3.9b) holds.

Now we have

$$
\begin{aligned}
[\tilde{L}_{4k+4}, L_1^2] =&[[\tilde{L}_{4k+4}, L_1], L_1] \\
=&\frac{1}{2}\theta[M_3, [\tilde{L}_{4k+2}, L_1]]' + \frac{1}{4}[[\tilde{L}_{4k}, M_5], L_1] \\
=&(\frac{1}{2}\theta)^2[M_3^2, \tilde{L}_{4k}] + \frac{1}{2}\theta\frac{1}{4}[M_3, [\tilde{L}_{4k-2}, M_5]]' - \frac{1}{2}[\tilde{L}_{4k}, M_3^2] \\
&- \frac{1}{4}[M_5, \frac{1}{2}\theta M_3\tilde{L}_{4k-2} + \frac{1}{2}\theta\tilde{L}_{4k-2}M_3 + \frac{1}{4}[\tilde{L}_{4k-4}, M_5]] \\
=&(\frac{1}{2}\theta)^2[M_3^2, \tilde{L}_{4k}] - \frac{1}{2}[\tilde{L}_{4k}, M_3^2] - (\frac{1}{4})^2[\tilde{L}_{4k-4}, M_5^2] \\
=&(-\frac{1}{4}\theta^2 - \frac{1}{2})[\tilde{L}_{4k}, M_3^2] - (\frac{1}{4})^2[\tilde{L}_{4k-4}, M_5^2].
\end{aligned}
$$

(for $k = 0$, we have $[\tilde{L}_4, L_1^2] = \frac{1}{2}\theta[M_3, M_3]' = 0$.)

Define A$=-\frac{1}{4}\theta^2 - \frac{1}{2}$, B$=-(\frac{1}{4})^2$, we have

$$
[\tilde{L}_{4k+4}, L_1^2] = A[\tilde{L}_{4k}, M_3^2] + B[\tilde{L}_{4k-4}, M_5^2].
$$

Similarly we have

$$
[\tilde{L}_{4k}, M_3^2] = A[\tilde{L}_{4k-4}, M_5^2] + B[\tilde{L}_{4k-8}, M_7^2] = \cdots
$$

and the last one is

$$
[\tilde{L}_8, M_{2k-1}^2] = A[\tilde{L}_4, M_{2k+1}^2].
$$

Note that

$$
[\tilde{L}_4, M_{2k+1}^2] = [[\tilde{L}_4, M_{2k+1}], M_{2k+1}] = [\frac{1}{2}\theta M_{2k+3}\tilde{L}_2 + \frac{1}{2}\theta\tilde{L}_2 M_{2k+3}, M_{2k+1}] = 0.
$$

So now we have proved $[\tilde{L}_{4k+4}, L_1^2] = 0 = [L_{4k+4}, L_1^2]$ and moreover $[\tilde{L}_{4k+4-4i}, M_{2i+1}^2] = 0$ for $i \geq 0$. Then automatically $[\tilde{L}_{4k+5}, L_1^2] = 0 = [L_{4k+5}, L_1^2]$.

Using induction hypothesis and the same technique in the proof of (3.9a), we can get

$$
[L_2, L_{4k+4}] = \theta L_1 L'_{4k+5} - \theta L_{4k+5} L_1 \tag{3.17}
$$

and

$$
[L_2, L'_{4k+4}] = \theta L'_{4k+5} L_1 - \theta L_1 L_{4k+5}. \tag{3.18}
$$

Since we have known $[L_{4k+4}, L_1^2] = 0 = [\tilde{L}_{4k+4}, L_1^2]$, then take the sum of (3.17) and

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(3.18) we get $[L_2, \tilde{L}_{4k+4}] = 0 = [\tilde{L}_2, \tilde{L}_{4k+4}]$.

Also using the induction hypothesis and the same technique in the proof of (3.9a) and (3.9b), we have (3.9c), (3.9d) hold for k+1.

The last and most difficult part is to prove $[\tilde{L}_{4k+6}, L_1^2] = 0$.

Note that we have known

$$[\tilde{L}_{4k+6}, L_1] = [L_2, \tilde{L}_{4k+5}] = \frac{1}{2}\theta M_3 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4} M_3 + \frac{1}{4}[\tilde{L}_{4k+2}, M_5], \qquad (3.19)$$

then we have the following two arguments about $[\tilde{L}_{4k+6}, L_1^2]$.

Argument 1: By (3.19),

$$
\begin{aligned}
[\tilde{L}_{4k+6}, L_1^2] =& [[\tilde{L}_{4k+6}, L_1], L_1] \\
=& \frac{1}{2}\theta[M_3, [\tilde{L}_{4k+4}, L_1]]' + \frac{1}{4}[[\tilde{L}_{4k+2}, M_5], L_1] \\
=& (\frac{1}{2}\theta)^2[M_3^2, \tilde{L}_{4k+2}] + \frac{1}{2}\theta\frac{1}{4}[M_3, [\tilde{L}_{4k}, M_5]]' + \frac{1}{4}[\tilde{L}_{4k+2}, -2M_3^2] - \frac{1}{4}[M_5, [\tilde{L}_{4k+2}, L_1]] \\
=& -\frac{1}{4}(\theta^2 + 2)[\tilde{L}_{4k+2}, M_3^2] + \frac{1}{2}\theta\frac{1}{4}[M_3, [\tilde{L}_{4k}, M_5]]' \\
& -\frac{1}{4}\frac{1}{2}\theta(-1)[M_3, [M_5, \tilde{L}_{4k}]]' - (\frac{1}{4})^2[\tilde{L}_{4k-2}, M_5^2],
\end{aligned}
$$

i.e.,

$$[\tilde{L}_{4k+6}, L_1^2] = A[\tilde{L}_{4k+2}, M_3^2] + B[\tilde{L}_{4k-2}, M_5^2].$$

Similarly we have

$$[\tilde{L}_{4k+2}, M_3^2] = A[\tilde{L}_{4k-2}, M_5^2] + B[\tilde{L}_{4k-6}, M_7^2] = \cdots$$

and the last one is

$$[\tilde{L}_6, M_{2k+1}^2] = -\frac{1}{4}(\theta^2 + 3)[\tilde{L}_2, M_{2k+3}^2].$$

So we have

$$[\tilde{L}_{4k+6}, L_1^2] = a_{k+2}[\tilde{L}_2, M_{2k+3}^2],$$

where the sequence of number $a_n, n \geq 1$ satisfies

$$a_1 = 1, a_2 = -\frac{1}{4}(\theta^2 + 3), a_{n+2} = Aa_{n+1} + Ba_n.$$

Argument 2: Similarly to the proof of $[L_2, L_1^2] = 0$, we have

$$(q + 2)[L_{4k+6}, L_1^2] = \theta L_1 L_{4k+7}' - \theta L_{4k+7} L_1 - [L_2, L_{4k+6}], \qquad (3.20)$$

$$q[L_{4k+6}, L_1^2] = \theta L_{4k+7}' L_1 - \theta L_1 L_{4k+7} - [L_2, L_{4k+6}'] \qquad (3.21)$$

then take the sum of (3.20) and (3.21) we have

$$[\tilde{L}_{4k+6}, L_1^2] = \theta L_1 \tilde{L}_{4k+7} - \theta \tilde{L}_{4k+7} L_1 - \frac{1}{2}[L_2, [\tilde{L}_{4k+5}, L_1]]. \qquad (3.22)$$

We have known

$$\tilde{L}_{4k+7} = \frac{1}{2}\theta L_1 \tilde{L}_{4k+6} + \frac{1}{2}\theta \tilde{L}_{4k+6} L_1 + \frac{1}{4}[\tilde{L}_{4k+4}, M_3]. \qquad (3.23)$$

So by (3.22) (3.23) and (3.19) we have

$$
\begin{aligned}
[\tilde{L}_{4k+6}, L_1^2] =& \theta \frac{1}{2}\theta[L_1^2, \tilde{L}_{4k+6}] + \theta \frac{1}{4}[L_1, [\tilde{L}_{4k+4}, M_3]]' - \frac{1}{2}([\tilde{L}_{4k+6}, L_1^2] + [\tilde{L}_{4k+5}, M_3]) \\
=& -\frac{1}{2}(\theta^2 + 1)[\tilde{L}_{4k+6}, L_1^2] + \theta \frac{1}{4}[L_1, [\tilde{L}_{4k+4}, M_3]]' \\
& -\frac{1}{2}[\frac{1}{2}\theta L_1 \tilde{L}_{4k+4} + \frac{1}{2}\theta \tilde{L}_{4k+4} L_1 + \frac{1}{4}[\tilde{L}_{4k+2}, M_3], M_3]
\end{aligned}
$$

i.e.,

$$(3 + \theta^2)[\tilde{L}_{4k+6}, L_1^2] = -\frac{1}{4}[\tilde{L}_{4k+2}, M_3^2],$$

i.e.,

$$(3 + \theta^2)[\tilde{L}_{4k+6}, L_1^2] = -\frac{1}{4}a_{k+1}[\tilde{L}_2, M_{2k+3}^2].$$

In summary we have

$$[\tilde{L}_{4k+6}, L_1^2] = a_{k+2}[\tilde{L}_2, M_{2k+3}^2],$$
$$(3 + \theta^2)[\tilde{L}_{4k+6}, L_1^2] = -\frac{1}{4}a_{k+1}[\tilde{L}_2, M_{2k+3}^2].$$

So if $\begin{vmatrix} 1 & a_{k+2} \\ 3 + \theta^2 & -\frac{1}{4}a_{k+1} \end{vmatrix} \neq 0$, we have $[\tilde{L}_{4k+6}, L_1^2] = 0$ and then $[L_{4k+6}, L_1^2] = 0$. We use Lemma 3.2.4. Here $a_1 = 1$, $a_2 = -\frac{1}{4}(\theta^2 + 3)$, $a_{n+2} = Aa_{n+1} + Ba_n$ where $A = -\frac{1}{4}\theta^2 - \frac{1}{2}$, $B = -(\frac{1}{4})^2$. Then by Lemma 3.2.4 we have

$$a_n = (-\frac{1}{4})^{n-1}[2n - 1]_q.$$

Then we have

$$\begin{vmatrix} 1 & a_{k+2} \\ 3 + \theta^2 & -\frac{1}{4}a_{k+1} \end{vmatrix} = -(-\frac{1}{4})^{k+1}(q + q^{-1})[2k + 4]_q \neq 0,$$

when q is not a root of unity. Hence $[\tilde{L}_{4k+6}, L_1^2] = 0$ and moreover we know that

$[\tilde{L}_{4k+6-4i}, M^2_{2i+1}] = 0$ for $i \geq 0$. Then automatically $[\tilde{L}_{4k+7}, L^2_1] = 0 = [L_{4k+7}, L^2_1]$ and similar to the proof of $[L_2, \tilde{L}_{4k+4}] = 0 = [\tilde{L}_2, \tilde{L}_{4k+4}]$ before, we can get $[L_2, \tilde{L}_{4k+6}] = 0 = [\tilde{L}_2, \tilde{L}_{4k+6}]$. Now (3.9e) and (3.9f) holds. $\qquad\square$

Remark 3.2.6 *Note that in the proof of* (3.9c), (3.9d) *and* $[\tilde{L}_2, \tilde{L}_{4k+2}] = 0$, *we get five formulas completely similar to* (3.12), (3.15), (3.16), (3.17), (3.18) *corresponding to different indices. We will not distinguish them by new labels when we refer to them later.*

Corollary 3.2.7 $[M_{2m+1}, M_{2n+1}] = 0$ *if $m+n$ is odd,* $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M^2_{m+n+1}$ *if $m+n$ is even, and M^2_{2n+1} commute with all L_m for all $m, n \geq 0$.*

Proof: By an easy induction on $m + n$ we can get the formula of $[M_{2m+1}, M_{2n+1}]$. Then we can see $[L_1, M^2_{2n+1}] = 0$ for all n. In the proof of Prop. 3.2.5, we know M^2_{2n+1} commutes with all $\tilde{L}_{2m}$. Then by Prop. 3.2.5, M^2_{2n+1} commute with all $\tilde{L}_m$ and this is equivalent to M^2_{2n+1} commute with all L_m. $\qquad\square$

Remark 3.2.8 *Note that in the proofs of Prop. 3.2.1, Prop. 3.2.5 and Corollary 3.2.7, we only use quantum Serre relations.*

Corollary 3.2.9 *The comultiplication formulas of L_n and L'_n in $K_{\geq 1}/K_{>1}$ are as the following.*

$$\Delta(L_{2n}) = \sum_{i=0}^{n} \theta L_{2n-2i} \otimes L_{2i} - \sum_{i=0}^{n-1} \theta L'_{2n-1-2i} \otimes L_{1+2i},$$

$$\Delta(L'_{2n}) = \sum_{i=0}^{n} \theta L'_{2n-2i} \otimes L'_{2i} - \sum_{i=0}^{n-1} \theta L_{2n-1-2i} \otimes L'_{1+2i},$$

$$\Delta(L_{2n+1}) = \sum_{i=0}^{n} \theta L_{2n+1-2i} \otimes L_{2i} + \sum_{i=0}^{n} \theta L'_{2n-2i} \otimes L_{1+2i},$$

$$\Delta(L'_{2n+1}) = \sum_{i=0}^{n} \theta L'_{2n+1-2i} \otimes L'_{2i} + \sum_{i=0}^{n} \theta L_{2n-2i} \otimes L'_{1+2i}.$$

Proof: By L^2_1 commutes with L_n and an easy induction, we can get the expressions of $\Delta(L_{2n})$ and $\Delta(L_{2n+1})$. Then we can get the expressions of $\Delta(L'_{2n})$ and $\Delta(L'_{2n+1})$ by Lemma 3.1.3 (b). $\qquad\square$

Remark 3.2.10 *Comparing to the formulas in Corollary 3.2.9, we recall that the comultiplication formula in the subquotient $K_{\geq 1}/K_{>1}$ of L_n in $U_v^+(\hat{sl}_2)$ is*

$$\Delta(L_n) = \sum_{i=0}^{n} \theta L_{n-i} \otimes L_i,$$

corresponding to Prop. 4.3 in [CP91].

Corollary 3.2.11 *L_n and L'_n are not 0 and linearly independent in $B(\mathbb{V})$, for $n \geq 0$.*

Proof: By Corollary 3.2.9 and an easy induction. $\qquad\square$

Lemma 3.2.12 *We have the following results using differential operators.*

(a) $\partial_2^R(L_1^2) = -q^{-1}\theta X_2$, $\partial_2^R(Y_{2n+1}) = \theta L'_{2n}$, $\partial_2^R(Y_{2n}) = -\theta L'_{2n-1}$, $\partial_2^R(L_{2n}) = -\theta(X_1 L'_{2n-1} - q^{-2}L'_{2n-1}X_1)$, $\partial_2^R(L_{2n+1}) = \theta(X_1 L'_{2n} - q^{-2}L'_{2n}X_1)$.

(b) $\partial_1^L(L_1^2) = q^{-1}\theta Y_2$, $\partial_1^L(X_{2n+1}) = \theta L_{2n}$, $\partial_1^L(X_{2n}) = \theta L'_{2n-1}$, $\partial_1^L(L'_{2n}) = \theta(L'_{2n-1}Y_1 + q^{-2}Y_1 L'_{2n-1})$, $\partial_1^L(L_{2n+1}) = \theta(L_{2n}Y_1 - q^{-2}Y_1 L_{2n})$.

Proof: By Lemma 3.1.3, Prop. 3.2.5 and induction. $\qquad\square$

Corollary 3.2.13 *In terms of the Lyndon word, X_n and Y_n are the root vectors at the corresponding degrees.*

Proof: By Lemma 3.1.1, Lemma 3.2.12, Corollary 3.2.11. $\qquad\square$

Corollary 3.2.14 *As an algebra $K_{\geq 1}/K_{>1}$ is isomorphic to the subalgebra of $B(\mathbb{V})$ generated by L_n for $n > 0$. Then M_{2m+1}^2 are central elements in $K_{\geq 1}/K_{>1}$.*

Proof: By Corollary 3.2.13 we have X_n and Y_n are real root vectors, then the only possible imaginary root vectors are L_n for $n \geq 1$ and their iterated commutators according to the Lyndon word theory. By the expression of $\Delta(Y_n)$ in Lemma 3.1.2, we get $K_{\geq 1}$ is the subalgebra generated by X_n and L_n and $K_{\geq 1} = K_{>1} \oplus K_{=1}$ where $K_{=1}$ is the subalgebra generated by L_n and $K_{>1}$ is the subalgebra $K_{\geq 1} \cap B_{>1}$. This means that as an algebra $K_{\geq 1}/K_{>1}$ is isomorphic to the subalgebra in $B(\mathbb{V})$ generated by L_n. $\qquad\square$

Remark 3.2.15 *In summary, $K_{\geq 1}/K_{>1}$ is a braided Hopf algebra whose algebra structure is nothing new, but has a manageable coalgebra structure.*

§3.2.3 Relations described by generating functions

In this subsection we will describe the relations in $B(\mathbb{V})$ using generating functions, which will be powerful in proofs of main theorems in this paper.

Let $\theta' := q + q^{-1}$, $a = \sqrt{\frac{q}{2}}$, $b = i\sqrt{\frac{q^{-1}}{2}}$.

Let $a_1 = 1$, $a_2 = \frac{1}{2}\theta$, $a_{n+2} = \frac{1}{2}\theta a_{n+1} + \frac{1}{4}a_n$, then by Lemma 3.2.4 we have

$$a_n = \frac{(\frac{1}{2}q)^n - (-\frac{1}{2}q^{-1})^n}{\frac{1}{2}(q + q^{-1})}.$$

First, we introduce three fundamental generating functions in this paper.

Let

$$\widetilde{\mathbb{L}}(u) = \sum_{n=0}^{\infty} \theta \tilde{L}_{2n} u^{2n},$$

$$\mathbb{M}(u) = \sum_{n=0}^{\infty} M_{2n+1} u^{2n+1},$$

$$\mathbb{Z}(u) = \sum_{n=0}^{\infty} M_{2n+1}^2 (-1)^n u^{4n+2} = \mathbb{M}(u)^2 \text{ (by Corollary 3.2.7)}.$$

For convenience of notation we denote $\mathbb{M}(hu), \mathbb{Z}(hu)$ by $\mathbb{M}(h), \mathbb{Z}(h)$ respectively for $h \in \mathbb{C}^{\times}$.

Also for convenience of notation, we introduce

$$A(u) := \sum_{n=0}^{\infty} \theta a_{n+1} M_{2n+1} u^{2n+1} = \frac{2\theta}{\theta'}\big(a\mathbb{M}(a) - b\mathbb{M}(b)\big),$$

$$B(u) := \sum_{n=1}^{\infty} \theta a_n M_{2n+1} u^{2n+1} = \frac{2\theta}{\theta'}\big(a^{-1}\mathbb{M}(a) - b^{-1}\mathbb{M}(b)\big),$$

$$A'(u) := \frac{1}{i}A(iu), \; B'(u) := iB(iu).$$

Sometimes we just denote $A(u), B(u)$ by A, B respectively.

Proposition 3.2.16 *We have the following relations in the form of series.*

$$A(u)\widetilde{\mathbb{L}}(u) = \widetilde{\mathbb{L}}(u)A'(u) = \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} u^{2n+1}. \tag{3.24a}$$

$$B(u)\widetilde{\mathbb{L}}(u) = \widetilde{\mathbb{L}}(u)B'(u) = \sum_{n=1}^{\infty} \theta \hat{L}_{2n+1} u^{2n+1}. \tag{3.24b}$$

$$[A, B] = 0 = [A', B']. \tag{3.24c}$$

$$A^2 = \frac{2\theta}{\theta'}(q\mathbb{Z}(a) + q^{-1}\mathbb{Z}(b)), \frac{1}{4}B^2 = -\frac{2\theta}{\theta'}(q\mathbb{Z}(b) + q^{-1}\mathbb{Z}(a)). \tag{3.24d}$$

$$\frac{1}{2}[A(u), L_1 u] = \frac{2\theta}{\theta'}(\mathbb{Z}(a) - \mathbb{Z}(b)) = \frac{1}{\theta}(\frac{1}{4}B^2 + A^2). \tag{3.24e}$$

Proof: (3.24a), (3.24b) follow from Prop. 3.2.5 and (3.24c), (3.24d), (3.24e) follow from Corollary 3.2.7. $\qquad\qquad\square$

§3.2.4 More fundamental relations

In this section, we will use differential operators to get the expressions of

$$[X_1, L_{2n}], \; [X_1, L'_{2n}], \; [X_2, L_{2n}], \; [X_2, L'_{2n}]$$

and

$$[L_{2n}, Y_1], \; [L'_{2n}, Y_1], \; [L_{2n}, Y_2], \; [L'_{2n}, Y_2].$$

In general, by Lemma 3.1.3 (a) and L_1^2 lies in the center of the subalgebra generated by all L_n, one can get $[X_k, L_{2n}]$ for $k \geq 3$ and others immediately.

Define two sequences of polynomials P_n and Q_n:

$$P_0 = 1, P_n = q^n + q^{n-1} + (-1)^{n-1}P_{n-1},$$
$$Q_0 = 1, Q_n = q^n - q^{n-1} + (-1)^n Q_{n-1}.$$

We have

$$P_n = 2q^{2\lfloor\frac{n}{2}\rfloor}\frac{1 - (-q^{-2})^{\lfloor\frac{n}{2}\rfloor+1}}{1 + q^{-2}} + (-1)^{n+1}q^n,$$

$$Q_n = (-1)^n 2q^{2\lfloor\frac{n}{2}\rfloor}\frac{1 - (-q^{-2})^{\lfloor\frac{n}{2}\rfloor+1}}{1 + q^{-2}} + (-1)^{n+1}q^n,$$

$$P_{2n} = Q_{2n}.$$

The next proposition is for the proof of $[\tilde{L}_{2n}, \tilde{L}_{2m}] = 0, \forall n, m \geq 1$ in Prop. 3.2.21.

Proposition 3.2.17 *We have the following relations in $B(\mathbb{V})$.*

$$[X_1, L_{2n}] = \sum_{i=0}^{n-1} P_{2n-1-2i}\theta L_{2i}X_{2n+1-2i} - \sum_{i=0}^{n-1} P_{2n-2-2i}\theta L'_{2i+1}X_{2n-2i}, \tag{3.25a}$$

$$[X_1, L'_{2n}] = \sum_{i=0}^{n-1} q^{2n-1-2i}\theta L'_{2i}X_{2n+1-2i} - \sum_{i=0}^{n-1} q^{2n-2-2i}\theta L_{2i+1}X_{2n-2i}, \tag{3.25b}$$

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$$[X_2, L_{2n}] = \sum_{i=0}^{n-1} q^{2n-1-2i}\theta L_{2i}X_{2n+2-2i} + \sum_{i=0}^{n-1} q^{2n-2-2i}\theta L'_{2i+1}X_{2n-2i+1}, \qquad (3.25c)$$

$$[X_2, L'_{2n}] = \sum_{i=0}^{n-1} Q_{2n-1-2i}\theta L'_{2i}X_{2n+2-2i} + \sum_{i=0}^{n-1} Q_{2n-2-2i}\theta L_{2i+1}X_{2n-2i+1}, \qquad (3.25d)$$

$$[L_{2n}, Y_1] = \sum_{i=0}^{n-1} q^{2n-1-2i}\theta Y_{2n+1-2i}L_{2i} + \sum_{i=0}^{n-1} q^{2n-2-2i}\theta Y_{2n-2i}L_{2i+1}, \qquad (3.25e)$$

$$[L'_{2n}, Y_1] = \sum_{i=0}^{n-1} Q_{2n-1-2i}\theta Y_{2n+1-2i}L'_{2i} + \sum_{i=0}^{n-1} Q_{2n-2-2i}\theta Y_{2n-2i}L'_{2i+1}, \qquad (3.25f)$$

$$[L_{2n}, Y_2] = \sum_{i=0}^{n-1} P_{2n-1-2i}\theta Y_{2n+2-2i}L_{2i} - \sum_{i=0}^{n-1} P_{2n-2-2i}\theta Y_{2n-2i+1}L_{2i+1}, \qquad (3.25g)$$

$$[L'_{2n}, Y_2] = \sum_{i=0}^{n-1} q^{2n-1-2i}\theta Y_{2n+2-2i}L'_{2i} - \sum_{i=0}^{n-1} q^{2n-2-2i}\theta Y_{2n-2i+1}L'_{2i+1}. \qquad (3.25h)$$

Proof: We use induction. By Prop. 3.2.1 we know the proposition holds for $n = 1$. Suppose the proposition holds for n, we consider the case of $n + 1$.

We have $[L_{2n+1}, L_1^2] = 0$. Hence in $B(\mathbb{V})$, by Lemma 3.2.12 we have

$$\begin{aligned}
0 &= \partial_2^R([L_{2n+1}, L_1^2]) \\
&= L_{2n+1}(-q^{-1}\theta X_2) + \partial_2^R(L_{2n+1})L_1^2 - (L_1^2\partial_2^R(L_{2n+1}) + q^{-1}\theta X_2 L_{2n+1}) \\
&= -q^{-1}\theta[X_2, L_{2n+1}] + [\partial_2^R(L_{2n+1}), L_1^2],
\end{aligned}$$

i.e.,

$$[X_2, L_{2n+1}] = q[[X_1, L'_{2n}] + q^{-1}\theta L'_{2n}X_1, L_1^2],$$

again by Lemma 3.2.12.

Using induction hypothesis (3.25b), we have

$$[X_2, L_{2n+1}] = \sum_{i=0}^{n} q^{2n-2i}\theta L'_{2i}X_{2n+3-2i} - \sum_{i=0}^{n-1} q^{2n-1-2i}\theta L_{2i+1}X_{2n-2i+2}. \qquad (3.26)$$

Then by (3.26),

$$\begin{aligned}
[X_1, L'_{2n+2}] &= [X_1, [X_2, Y_{2n+1}]] = q[X_2, L_{2n+1}] - \theta L_{2n+1}X_2 \\
&= \sum_{i=0}^{n} q^{2n-2i+1}\theta L'_{2i}X_{2n+3-2i} - \sum_{i=0}^{n-1} q^{2n-2i}\theta L_{2i+1}X_{2n-2i+2} - \theta L_{2n+1}X_2
\end{aligned}$$

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$$= \sum_{i=0}^{n} q^{2n-2i+1} \theta L'_{2i} X_{2n+3-2i} - \sum_{i=0}^{n} q^{2n-2i} \theta L_{2i+1} X_{2n-2i+2}.$$

Since $[X_1, L_{2n+2}] = [X_1, L'_{2n+2}] + [X_1, [L_{2n+1}, L_1]]$, so we need to calculate $[X_1, L_{2n+1}]$.

$$\begin{aligned}
[X_1, L_{2n+1}] =& [X_1, L'_{2n+1}] - [X_1, [L_{2n}, L_1]] \\
=& [X_1, [X_2, Y_{2n}]] - [[X_1, L_{2n}], L_1] + [X_2, L_{2n}] \\
=& (q+1)[X_2, L_{2n}] + \theta L_{2n} X_2 - [[X_1, L_{2n}], L_1] \\
=& \sum_{i=0}^{n-1} (q^{2n-2i} + q^{2n-1-2i}) \theta L_{2i} X_{2n+2-2i} \\
&+ \sum_{i=0}^{n-1} (q^{2n-1-2i} + q^{2n-2-2i}) \theta L'_{2i+1} X_{2n-2i+1} \\
&+ \theta L_{2n} X_2 - \left(\sum_{i=0}^{n-1} P_{2n-1-2i} \theta (L_{2i} X_{2n+2-2i} + \hat{L}_{2i+1} X_{2n+1-2i}) \right. \\
&\left. - \sum_{i=0}^{n-1} P_{2n-2-2i} \theta (L'_{2i+1} X_{2n-2i+1} - \hat{L}_{2i+2} X_{2n-2i}) \right),
\end{aligned}$$

by (3.25a), (3.25c). By the definition of P_n, we have

$$\begin{aligned}
[X_1, L_{2n+1}] =& \sum_{i=0}^{n-1} P_{2n-2i} \theta L_{2i} X_{2n+2-2i} - \sum_{i=1}^{n} P_{2n-2i} \theta \hat{L}_{2i} X_{2n+2-2i} + \theta L_{2n} X_2 \quad (3.27) \\
&+ \sum_{i=0}^{n-1} P_{2n-1-2i} \theta L'_{2i+1} X_{2n+1-2i} - \sum_{i=0}^{n-1} P_{2n-1-2i} \theta \hat{L}_{2i+1} X_{2n+1-2i} \\
=& \sum_{i=0}^{n} P_{2n-2i} \theta L'_{2i} X_{2n+2-2i} + \sum_{i=0}^{n-1} P_{2n-1-2i} \theta L_{2i+1} X_{2n+1-2i}.
\end{aligned}$$

Then

$$\begin{aligned}
[X_1, L_{2n+2}] &= [X_1, L'_{2n+2}] + [X_1, [L_{2n+1}, L_1]] \\
&= [X_1, L'_{2n+2}] + [[X_1, L_{2n+1}], L_1]] + [X_2, L_{2n+1}] \\
&= (q+1)[X_2, L_{2n+1}] - \theta L_{2n+1} X_2 + [[X_1, L_{2n+1}], L_1]].
\end{aligned}$$

By (3.26) and (3.27) and similar calculation, we have

$$[X_1, L_{2n+2}] = \sum_{i=0}^{n} P_{2n+1-2i} \theta L_{2i} X_{2n+3-2i} - \sum_{i=0}^{n} P_{2n-2i} \theta L'_{2i+1} X_{2n+2-2i}.$$

$[X_2, L_{2n+2}]$, $[X_2, L'_{2n+2}]$, $[L_{2n+2}, Y_1]$, $[L'_{2n+2}, Y_1]$, $[L_{2n+2}, Y_2]$, $[L'_{2n+2}, Y_2]$ follow from a completely similar argument, we omit the details. $\qquad\square$

Remark 3.2.18 *Note that in Prop. 3.2.17, we use the abstract definition of a Nichols algebra, that is, we use differential operators to derive certain relations. It would be very tedious if we only use quantum Serre relations. However, one can prove that these relations can also be derived only by quantum Serre relations, see the proof of Theorem 6.0.5.*

Equation $[\tilde{L}_{2m}, \tilde{L}_{2n}] = 0$ is equivalent to

$$[L_{2m}, L_{2n}] + [L'_{2m}, L_{2n}] + [L_{2m}, L'_{2n}] + [L'_{2m}, L'_{2n}] = 0.$$

So we need to know the expressions of each summand.

Lemma 3.2.19 *Suppose $[\tilde{L}_{2k-2}, \tilde{L}_{2N}] = 0 \ \forall 2 \le k \le n$ and $\forall N > 0$, then*

$$[\tilde{L}_{2k-1}, \hat{L}_{2N+1}] = [\hat{L}_{2k-1}, \tilde{L}_{2N+1}] \ \forall 1 \le k \le n \ and \ \forall N > 0.$$

Proof: We only need to prove

$$[\tilde{L}_{2n-1}, [\tilde{L}_{2N}, L_1]] = [[\tilde{L}_{2n-2}, L_1], \tilde{L}_{2N+1}].$$

By Prop. 3.2.5,

$$
\begin{aligned}
\text{LHS} &= \left[\frac{1}{2}\theta L_1 \tilde{L}_{2n-2} + \frac{1}{2}\theta \tilde{L}_{2n-2} L_1 + \frac{1}{4}[\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N}, L_1] \right] \\
&= \frac{1}{2}\theta \left[L_1, [\tilde{L}_{2n-2}, [\tilde{L}_{2N}, L_1]] \right]' + \frac{1}{4}\left[[\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N}, L_1] \right].
\end{aligned}
$$

$$
\begin{aligned}
\text{RHS} &= \left[[\tilde{L}_{2n-2}, L_1], \frac{1}{2}\theta L_1 \tilde{L}_{2N} + \frac{1}{2}\theta \tilde{L}_{2N} L_1 + \frac{1}{4}[\tilde{L}_{2N-2}, M_3] \right] \\
&= -\frac{1}{2}\theta \left[L_1, [[\tilde{L}_{2n-2}, L_1], \tilde{L}_{2N}] \right]' + \frac{1}{4}\left[[\tilde{L}_{2n-2}, L_1], [\tilde{L}_{2N-2}, M_3] \right]
\end{aligned}
$$

Since $[\tilde{L}_{2n-2}, \tilde{L}_{2N}] = 0$, we only need to prove

$$\left[[\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N}, L_1] \right] = \left[[\tilde{L}_{2n-2}, L_1], [\tilde{L}_{2N-2}, M_3] \right].$$

By using the expressions of $[\tilde{L}_{2N}, L_1]$ and $[\tilde{L}_{2n-2}, L_1]$ in Prop. 3.2.5 and similar calculation, we get that we only need to prove

$$\left[[\tilde{L}_{2n-4}, M_3], [\tilde{L}_{2N-4}, M_5]\right] = \left[[\tilde{L}_{2n-6}, M_5], [\tilde{L}_{2N-2}, M_3]\right].$$

Repeating these steps this lemma will be proved. $\square$

Lemma 3.2.20 *Suppose* $[\tilde{L}_{2k}, \tilde{L}_{2N}] = 0$ *for all* $k \leq n-1$ *and for all* $N > 0$, *then* $[\tilde{L}_{2n}, \tilde{L}_{2N}] = 0$ *for all* $N > 0$ *is equivalent to*

$$-[[\tilde{L}_{2n-2}, \tilde{L}_{2N+1}], L_1] + [[\tilde{L}_{2N+2}, \tilde{L}_{2n-3}], L_1] = 0 \text{ for all } N > 0.$$

Proof: Considering $[[X_1, L_{2n}], Y_{2N}]$, using the same technique as Prop. 3.2.5, we can get

$$[L_{2n}, L_{2N}] = \sum_{i=0}^{n-1} P_{2n-1-2i} \theta [L_{2n+2N-2i}, L_{2i}]$$
$$+ \sum_{i=0}^{n-1} P_{2n-2-2i} \theta (L'_{2i+1} L'_{2n+2N-2i-1} - L_{2n+2N-2i-1} L_{2i+1}).$$

Note that in this calculation we do not need to care about terms with form $Y_a L_b X_c$ or $Y_a L'_b X_c$. All terms of this form would be canceled by Corollary 3.2.11 and Corollary 3.2.13. One can also verify this directly.

Similarly considering $[[X_1, L'_{2n}], Y_{2N}]$ we can get

$$[L'_{2n}, L_{2N}] = \sum_{i=0}^{n-1} q^{2n-1-2i} \theta [L_{2n+2N-2i}, L'_{2i}]$$
$$+ \sum_{i=0}^{n-1} q^{2n-2-2i} \theta (L_{2i+1} L'_{2n+2N-1-2i} - L_{2n+2N-1-2i} L'_{2i+1}).$$

Similarly considering $[[X_2, L_{2n}], Y_{2N-1}]$ we can get

$$[L_{2n}, L'_{2N}] = \sum_{i=0}^{n-1} q^{2n-1-2i} \theta [L'_{2n+2N-2i}, L_{2i}]$$
$$+ \sum_{i=0}^{n-1} q^{2n-2-2i} \theta (L'_{2n+2N-1-2i} L_{2i+1} - L'_{2i+1} L_{2n+2N-1-2i}).$$

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Similarly considering $[[X_2, L'_{2n}], Y_{2N-1}]$ we can get

$$[L'_{2n}, L'_{2N}] = \sum_{i=0}^{n-1} Q_{2n-1-2i}\theta[L'_{2n+2N-2i}, L'_{2i}]$$
$$+ \sum_{i=0}^{n-1} Q_{2n-2-2i}\theta(L'_{2n+2N-1-2i}L'_{2i+1} - L_{2i+1}L_{2n+2N-1-2i}).$$

Then by the recursive construction of P_n, Q_n and $[\tilde{L}_{2k}, \tilde{L}_{2N}] = 0 \ \forall N > 0, \ k \leq n-1$, we have

$$[L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}]$$
$$= \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i})\theta([L_{2n+2N-2i}, L_{2i}] - [L'_{2n+2N-2i}, L'_{2i}]$$
$$+ \sum_{i=0}^{n-1} P_{2n-2-2i}\theta([L'_{2i+1}, L'_{2n+2N-2i-1}] - [L_{2i+1}, L_{2n+2N-2i-1}])$$
$$+ \sum_{i=0}^{n-1} q^{2n-2-2i}\theta([L_{2i+1}, L'_{2n+2N-2i-1}] - [L'_{2i+1}, L_{2n+2N-2i-1}]).$$

We have the following identities by definition and Lemma 3.2.19:

$$[L_{2m}, L_{2n}] - [L'_{2m}, L'_{2n}] = [\hat{L}_{2m}, \tilde{L}_{2n}] + [\tilde{L}_{2m}, \hat{L}_{2n}], \ \forall m, n \geq 0.$$
$$[L'_{2i+1}, L'_{2n+2N-2i-1}] - [L_{2i+1}, L_{2n+2N-2i-1}]$$
$$= [\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}] + [\hat{L}_{2i+1}, \tilde{L}_{2n+2N-2i-1}] = 2[\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}],$$
$$[L_{2i+1}, L'_{2n+2N-2i-1}] - [L'_{2i+1}, L_{2n+2N-2i-1}]$$
$$= [\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}] - [\hat{L}_{2i+1}, \tilde{L}_{2n+2N-2i-1}] = 0, \ \forall 0 \leq i \leq n-1.$$

Then by the above identities

$$[L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}]$$
$$= \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i})\theta([\hat{L}_{2n+2N-2i}, \tilde{L}_{2i}] + [\tilde{L}_{2n+2N-2i}, \hat{L}_{2i}])$$
$$+ 2\sum_{i=0}^{n-1} P_{2n-2-2i}\theta[\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}]$$
$$= \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i})\theta\left(-[\tilde{L}_{2n+2N-2i-1}, \hat{L}_{2i+1}]\right)$$

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$$+ [L_1, [\tilde{L}_{2n+2N-2i-1}, \tilde{L}_{2i}]] + [[\tilde{L}_{2n+2N-2i}, \tilde{L}_{2i-1}], L_1] + [\tilde{L}_{2i-1}, \hat{L}_{2n+2N-2i+1}]\Big)$$

$$+ 2 \sum_{i=0}^{n-1} P_{2n-2-2i} \theta [\tilde{L}_{2i+1}, \hat{L}_{2n+2N-2i-1}].$$

By the construction of P_n, we get

$$P_{2n-2-2i} = q^{2n-2-2i} - q^{2n-4-2i} - P_{2n-4-2i}.$$

Then

$$[L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}]$$
$$= \sum_{i=0}^{n-1} (P_{2n-2-2i} + q^{2n-2-2i}) \theta ([L_1, [\tilde{L}_{2n+2N-2i-1}, \tilde{L}_{2i}]] + [[\tilde{L}_{2n+2N-2i}, \tilde{L}_{2i-1}], L_1]).$$

When q is not a root of unity, it is easy to see $(P_{2n-2-2i} + q^{2n-2-2i})\theta$ is not 0. Then by induction hypothesis we can see

$$[L_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L_{2N}] + [L'_{2n}, L'_{2N}] = 0$$

is equivalent to

$$-[[\tilde{L}_{2n-2}, \tilde{L}_{2N+1}], L_1] + [[\tilde{L}_{2N+2}, \tilde{L}_{2n-3}], L_1] = 0$$

for $\forall N \geq 0$. $\qquad\square$

Define
$$D_{2k} = -[[\tilde{L}_{2k}, \tilde{L}_{2N+1}], L_1] + [[\tilde{L}_{2N+2}, \tilde{L}_{2k-1}], L_1].$$

Note that D_{2k} is related to N, but we omit it, because when we deal with D_{2k} later, N is always fixed.

Define $\bar{L}_{2n}$ by the following formal power series:

$$\theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n} = \exp(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n}).$$

Define $C_k(P(u))$ by the coefficient of u^k in the power series $P(u)$.

Proposition 3.2.21 *There exists a series of number R_{2k} such that $[\bar{L}_{2k}, L_1] = R_{2k} M_{2k+1}$*

and $[\tilde{L}_{2m}, \tilde{L}_{2n}] = 0$ *for all* $k, m, n \geq 1$. R_{2k} *is defined by*

$$\theta R_{2k} = C_{2k-2}\left(\frac{1}{2}\theta\right) + C_{2k-2}\left(\frac{1}{2}\theta\exp(\sum_{n=1}^{k-1}\theta R_{2n}u^{2n})\right)$$
$$+ C_{2k-4}\left(\frac{1}{4}(\exp(\sum_{n=1}^{k-1}\theta R_{2n}u^{2n}) - 1)\right) - C_{2k}\left(\exp(\sum_{n=1}^{k-1}\theta R_{2n}u^{2n})\right).$$

Moreover, we have $R_{4k} = 0$ *and* $R_{4k+2} = \frac{1}{2^{2k}}\frac{[2k+1]_q}{2k+1}$.

Proof: We will prove this proposition by induction on n that $[\bar{L}_{2n}, L_1] = R_{2n}M_{2n+1}$ for some R_{2n} and $[\tilde{L}_{2n}, \tilde{L}_{2N}] = 0$ for $\forall N \geq 0$.

For $n = 1$, we have know in Section 1 that $[\bar{L}_2, L_1] = M_3$ which means $R_2 = 1$ and coincide with the formula of R_{2k} when $k = 1$, and $[\tilde{L}_2, \tilde{L}_{2N}] = 0$ for $\forall N \geq 0$. Now suppose this proposition holds for $n \leq k$, i.e we have known $[\bar{L}_{2n}, L_1] = R_{2n}M_{2n+1}$ and $[\tilde{L}_{2n}, \tilde{L}_{2N}] = 0$ for $n \leq k$ and $\forall N \geq 0$, we consider $n = k + 1$.

First we will prove $[\bar{L}_{2k+2}, L_1] = R_{2k+2}M_{2k+3}$ for some R_{2k+2}.

By Prop. 3.2.5 we know

$$\theta\sum_{n=0}^{\infty}[\tilde{L}_{2n}u^{2n}, L_1] = \frac{1}{2}\theta M_3\theta\sum_{n=0}^{\infty}\tilde{L}_{2n-2}u^{2n}$$
$$+ \frac{1}{2}\theta\theta\sum_{n=0}^{\infty}\tilde{L}_{2n-2}u^{2n}M_3 + \frac{1}{4}[\theta\sum_{n=0}^{\infty}\tilde{L}_{2n-4}u^{2n}, M_5]$$

i.e.,

$$\theta\sum_{n=0}^{\infty}[\tilde{L}_{2n}u^{2n}, L_1] = \frac{1}{2}\theta u^2 M_3\theta\sum_{n=0}^{\infty}\tilde{L}_{2n-2}u^{2n-2}$$
$$+ \frac{1}{2}\theta u^2\theta\sum_{n=0}^{\infty}\tilde{L}_{2n-2}u^{2n-2}M_3 + \frac{1}{4}u^4[\theta\sum_{n=0}^{\infty}\tilde{L}_{2n-4}u^{2n-4}, M_5]$$

i.e.,

$$[\exp(\theta\sum_{n=1}^{\infty}\bar{L}_{2n}u^{2n}), L_1] = \frac{1}{2}\theta u^2 M_3\exp(\theta\sum_{n=1}^{\infty}\bar{L}_{2n}u^{2n}) \tag{3.28}$$
$$+ \frac{1}{2}\theta u^2\exp(\theta\sum_{n=1}^{\infty}\bar{L}_{2n}u^{2n})M_3 + \frac{1}{4}u^4[\exp(\theta\sum_{n=1}^{\infty}\bar{L}_{2n}u^{2n}), M_5].$$

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Note that
$$\exp(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n}) = \exp(\theta \sum_{n=2}^{\infty} \bar{L}_{2n} u^{2n})\exp(\theta \bar{L}_2 u^2)$$

, we denote $\exp(\theta \sum_{n=2}^{\infty} \bar{L}_{2n} u^{2n})$ by $\exp(2)$ and in general $\exp(\theta \sum_{n=k}^{\infty} \bar{L}_{2n} u^{2n})$ by $\exp(k)$ for convenience. It is easy to get

$$\exp(\theta \bar{L}_{2n} u^2) L_1 = \exp(ad_{\theta \bar{L}_{2n}} u^{2n}) \circ L_1 \exp(\theta \bar{L}_{2n} u^2).$$

Denote $ad_{\theta \bar{L}_{2n}} u^{2n}$ by $\mathcal{M}_{2n}$ for further convenience, where $ad_{\theta \bar{L}_{2n}}$ means the operator $[\theta \bar{L}_{2n}, \quad]$. Now we analyze (3.28).

$$\text{LHS} = [\exp(2), L_1]\exp(\theta \bar{L}_2 u^2) + \exp(2)\left(\exp(\mathcal{M}_2) - 1\right) \circ L_1 \exp(\theta \bar{L}_2 u^2).$$

Keeping doing similar calculation under our induction hypothesis, we will get

$$\text{LHS} = \left([\exp(k+1), L_1] + \exp(k+1)\left(\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) - 1\right) \circ L_1\right) \cdot \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}).$$

Similarly, we can get

$$\text{RHS} = \left(\frac{1}{2}\theta u^2 M_3 \exp(k+1) + \frac{1}{2}\theta u^2 \exp(k+1)\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) \circ M_3\right.$$
$$\left. + \frac{1}{4} u^4 \left([\exp(k+1), M_5] + \exp(k+1)(\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) - 1) \circ M_5\right)\right) \cdot \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}).$$

Canceling the ending terms of both sides and comparing the coefficient of u^{2k+2}, we will get

$$\theta[\bar{L}_{2k+2}, L_1] + C_{2k+2}\left((\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) - 1) \circ L_1\right)$$
$$= C_{2k}\left(\frac{1}{2}\theta M_3\right) + C_{2k}\left(\frac{1}{2}\theta\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) \circ M_3\right) + C_{2k-2}\left(\frac{1}{4}(\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) - 1) \circ M_5\right),$$

Hence we have $\theta[\bar{L}_{2k+2}, L_1] = R_{2k+2} M_{2k+3}$, where

$$R_{2k+2} = C_{2k}\left(\frac{1}{2}\theta\right) + C_{2k}\left(\frac{1}{2}\theta\exp(\sum_{n=1}^{k} \theta R_{2n} u^{2n})\right)$$

$$+ C_{2k-2} \left(\frac{1}{4} \big(\exp(\sum_{n=1}^{k} \theta R_{2n} u^{2n}) - 1 \big) \right) - C_{2k+2} \left(\exp(\sum_{n=1}^{k} \theta R_{2n} u^{2n}) \right).$$

Then we need to prove $D_{2k} = 0$, for any $N \geq 0$. Fix N. Considering the formal power series:

$$\theta \sum_{n=0}^{\infty} D_{2n} u^{2n}.$$

$$
\theta \sum_{n=0}^{\infty} D_{2n} u^{2n} = - \left[[\theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n}, \tilde{L}_{2N+1}], L_1 \right] + \left[[\tilde{L}_{2N+2}, \frac{1}{2} \theta u^2 L_1 \theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n} \right.
$$
$$
\left. + \frac{1}{2} \theta u^2 \theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n} L_1 + \frac{1}{4} u^4 [\theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n}, M_3]], L_1 \right]
$$
$$
= - [A, L_1] + \left[[\tilde{L}_{2N+2}, B], L_1 \right].
$$

Now we calculate A and B. The calculation is similar to the previous. We have

$$A = A_{2k+2} \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}),$$

where

$$A_{2k+2} = [\exp(k+1), \tilde{L}_{2N+1}] + \exp(k+1) \left(\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) - 1 \right) \circ \tilde{L}_{2N+1}.$$

$$B = B_{2k+2} \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}),$$

where

$$
B_{2k+2} = \frac{1}{2} \theta u^2 L_1 \exp(k+1) + \frac{1}{2} \theta u^2 \exp(k+1) \exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) \circ L_1
$$
$$
+ \frac{1}{4} u^4 \left([\exp(k+1), M_3] + \exp(k+1) \big(\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) - 1 \big) \circ M_3 \right).
$$

Then

$$
\theta \sum_{n=0}^{\infty} D_{2n} u^{2n} = - \left[A_{2k+2} \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right] + \left[[\tilde{L}_{2N+2}, B_{2k+2} \prod_{n=1}^{k} \exp(\theta \overline{L}_{2n} u^{2n})], L_1 \right]
$$

$$= - \left[A_{2k+2} \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right] + \left[[\tilde{L}_{2N+2}, B_{2k+2}] \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right]$$

$$= \left[(-A_{2k+2} + [\tilde{L}_{2N+2}, B_{2k+2}]) \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}), L_1 \right].$$

We need to prove that the coefficient of u^{2k} in $\theta \sum_{n=0}^{\infty} D_{2n} u^{2n}$ is 0.

It is enough to prove $C_{2k} \left([\Phi \prod_{n=1}^{k} \exp(\theta \bar{L}_{2n} u^{2n}), L_1] \right) = 0$, where

$$\Phi = - \exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) \circ \tilde{L}_{2N+1} + \left[\tilde{L}_{2N+2}, \frac{1}{2} \theta u^2 L_1 \right.$$

$$\left. + \frac{1}{2} \theta u^2 \exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) \circ L_1 + \frac{1}{4} u^4 \left(\exp(\sum_{n=1}^{k} \mathcal{M}_{2n}) - 1 \right) \circ M_3 \right].$$

We claim that $\Phi_{2i} := C_{2i}(\Phi) = 0$ for $i \leq k$.

We have $\Phi_{2i} = \phi_{2i} \left(\frac{1}{2} \theta M_{2i+1} \tilde{L}_{2N} + \frac{1}{2} \theta \tilde{L}_{2N} M_{2i+1} + \frac{1}{4} [\tilde{L}_{2N-2}, M_{2i+3}] \right)$, where

$$\phi_{2i} = C_{2i} \left(- \exp(\sum_{n=1}^{k} \theta R_{2n} u^{2n}) + \frac{1}{2} \theta u^2 + \frac{1}{2} \theta u^2 \exp(\sum_{n=1}^{k} \theta R_{2n} u^{2n}) \right.$$

$$\left. + \frac{1}{4} u^4 (\exp(\sum_{n=1}^{k} \theta R_{2n} u^{2n}) - 1) \right).$$

Then $\Phi_{2i} = 0$ for $i \leq k$ holds by this and the induction hypothesis of R_{2i} for $i \leq k$.

It is not difficult to get the general formula of R_{2m}, we omit the details. $\qquad \square$

Corollary 3.2.22 *We have the following commutation relations in $B(\mathbb{V})$.*

(a) $[\tilde{L}_{2m}, \tilde{L}_{2n}] = 0 = [\bar{L}_{2m}, \bar{L}_{2n}]$.

(b) $[M_{2m+1}, M_{2n+1}] = 0$ *if $m+n$ is odd,* $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2 M_{m+n+1}^2$ *if $m+n$ is even.*

(c) $[\bar{L}_{2m}, M_{2n+1}] = R_{2m} M_{2m+2n+1}$, *where $R_{4m} = 0$ and $R_{4m+2} = \frac{1}{2^{2m}} \frac{[2m+1]_q}{2m+1}$.*

By these relations, the only possible imaginary root vectors are L_{4n} at $4n\delta$, L_{4n+2} and M_{2n+1}^2 at $(4n+2)\delta$, M_{2n+1} at $(2n+1)\delta$.

Proof: These relations hold immediately by Prop. 3.2.21 and Prop. 3.2.5. Then by induction on degree we can get these possible imaginary root vectors. $\qquad \square$

Remark 3.2.23 *Notice that these commutation relations correspond perfectly with the commutation relations among imaginary root vectors for the Lie superalgebra case, see Prop. 2.1.7.*

Remark 3.2.24 *Note that these commutation relations between imaginary root vectors are derived from Prop. 3.2.5, Corollary 3.2.7, Prop. 3.2.17. If we prove that Prop. 3.2.17 can also be derived only from quantum Serre relations, then so are these commutation relations between imaginary root vectors.*

Chapter 4   Comultiplication of $K_{\geq 1}/K_{>1}$

This section is about the comultiplication of $B(\mathbb{V})$'s subquotient $K_{\geq 1}/K_{>1}$, which is the most important part of this paper. The total determination of the root multiplicities and the construction of the Drinfeld second realization of $U_v^+(A(0,2)^{(4)})$ all rely on this. In this section, we also get some partial information about the commutation relations between X_1 and the distinguished imaginary root vectors with manageable comultiplication in $K_{\geq 1}/K_{>1}$.

§4.1 $\Delta(\mathbb{M}(u))$ and $\Delta(\mathbb{Z}(u))$

Proposition 4.1.1

$$\Delta(\mathbb{M}(u)) = \Big(\mathbb{M}(u) \otimes 1 + 1 \otimes \mathbb{M}(u)\Big)\frac{1}{1 - 2\theta\mathbb{M}(u) \otimes \mathbb{M}(u)},$$

$$\Delta(\mathbb{Z}(u)) = \Big(\mathbb{Z}(u) \otimes 1 + 1 \otimes \mathbb{Z}(u)\Big)\frac{1}{1 + (2\theta^2)\mathbb{Z}(u) \otimes \mathbb{Z}(u)}.$$

Proof: We have

$$\Delta(L_2) = L_2 \otimes 1 - \theta L_1 \otimes L_1 + 1 \otimes L_2, \ \ \Delta(L_1) = L_1 \otimes 1 + 1 \otimes L_1. \tag{4.1}$$

Recall that $M_{2n+1} = [L_2, M_{2n-1}]$. With (4.1) and Corollary 3.2.7, it is not difficult to get $\Delta(\mathbb{M}(u))$. Since $\mathbb{M}(u)^2 = \mathbb{Z}(u)$, we have

$$
\begin{aligned}
\Delta(\mathbb{Z}(u)) &= (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M})\frac{1}{1 - 2\theta\mathbb{M} \otimes \mathbb{M}}(\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M})\frac{1}{1 - 2\theta\mathbb{M} \otimes \mathbb{M}}\\
&= (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M})^2\frac{1}{1 + 2\theta\mathbb{M} \otimes \mathbb{M}}\frac{1}{1 - 2\theta\mathbb{M} \otimes \mathbb{M}}\\
&= (\mathbb{M}^2 \otimes 1 + 1 \otimes \mathbb{M}^2)\frac{1}{1 + (2\theta)^2\mathbb{M}^2 \otimes \mathbb{M}^2}\\
&= (\mathbb{Z} \otimes 1 + 1 \otimes \mathbb{Z})\frac{1}{1 + (2\theta)^2\mathbb{Z} \otimes \mathbb{Z}}.
\end{aligned}
$$

$\square$

Define $\mathcal{Z}(u)$ by the following power series

$$\arctan(2\theta i\mathbb{Z}(u)) = 2\theta i\mathcal{Z}(u).$$

We denote $\mathcal{Z}(u)$ by $\sum_{n=0}^{\infty} Z_{4n+2}u^{4n+2}$.

Proposition 4.1.2 $\mathcal{Z}(u)$ *is primitive in the subquotient* $K_{\geq 1}/K_{>1}$ *and the leading term of* $[X_1, Z_{4n+2}]$ *is* $2^{2n}\frac{[2n+1]_q}{2n+1}X_{4n+3}$, *which means that* Z_{4n+2} *is not* 0 *in* $B(\mathbb{V})$.

Proof: $\arctan(x) = \frac{i}{2}\log(\frac{i+x}{i-x})$. Hence we just need to prove that $\frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}}$ is group-like.

$$\Delta(\frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}}) = \frac{1+2\theta\frac{\mathbb{Z}\otimes 1+1\otimes\mathbb{Z}}{1+(2\theta)^2\mathbb{Z}\otimes\mathbb{Z}}}{1-2\theta\frac{\mathbb{Z}\otimes 1+1\otimes\mathbb{Z}}{1+(2\theta)^2\mathbb{Z}\otimes\mathbb{Z}}} = \frac{(1+2\theta\mathbb{Z}\otimes 1)(1+1\otimes 2\theta\mathbb{Z})}{(1-2\theta\mathbb{Z}\otimes 1)(1-1\otimes 2\theta\mathbb{Z})} = \frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}}\otimes\frac{1+2\theta\mathbb{Z}}{1-2\theta\mathbb{Z}}.$$

By induction, it is easy to get the leading term of $[X_1, 2\theta(-1)^k M_{2k+1}^2]$ is $2\theta 2^{2k}X_{4k+3}$. Then the coefficient we want is

$$\frac{1}{2\theta i}C_{4n+2}\Big(\arctan(\sum_{k=0}^{\infty}2\theta i 2^{2k}u^{4k+2})\Big).$$

Let $x = 2u^2$, then we need to calculate

$$\frac{1}{2\theta i}C_{4n+2}\Big(\arctan(\theta i\frac{x}{1-x^2})\Big).$$

$$\frac{d}{dx}\arctan(\theta i\frac{x}{1-x^2}) = i\Big(\frac{q}{1-q^2x^2} - \frac{q^{-1}}{1-q^{-2}x^2}\Big) = iq\sum_{n=0}^{\infty}q^{2n}x^{2n} - iq^{-1}\sum_{n=0}^{\infty}q^{-2n}x^{2n},$$

Taking integral we get

$$\arctan(\theta i\frac{x}{1-x^2}) = i\sum_{n=0}^{\infty}\frac{q^{2n+1}}{2n+1}x^{2n+1} - i\sum_{n=0}^{\infty}\frac{q^{-(2n+1)}}{2n+1}x^{2n+1},$$

then immediately we have that the coefficient we want is $2^{2n}\frac{[2n+1]_q}{2n+1}$. $\quad\square$

§4.2 $\quad \Delta(\overline{\mathbb{L}})$

Denote $\sum_{n=1}^{\infty}\theta\bar{L}_{2n}u^{2n}$ by $\overline{\mathbb{L}}$, i.e., we have

$$\exp(\overline{\mathbb{L}}) = \widetilde{\mathbb{L}}.$$

Define $*(u) = \Delta(\widetilde{\mathbb{L}}(u))$.

Proposition 4.2.1

$$*(u) = \left(1 - A \otimes A + \frac{1}{2}B \otimes \frac{1}{2}B \right.$$

$$+ \left(\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B \right) \otimes \left(\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B \right) \right)$$

$$\cdot \exp\left(\overline{\mathbb{L}}(u) \otimes 1 + 1 \otimes \overline{\mathbb{L}}(u) \right)$$

$$:= \#(u) \cdot \exp\left(\overline{\mathbb{L}}(u) \otimes 1 + 1 \otimes \overline{\mathbb{L}}(u) \right).$$

and also

$$*(u) = \exp\left(\overline{\mathbb{L}}(u) \otimes 1 + 1 \otimes \overline{\mathbb{L}}(u) \right) \cdot \left(1 - A' \otimes A' + \frac{1}{2}B' \otimes \frac{1}{2}B' \right.$$

$$+ \left(\theta^{-1}(\frac{1}{4}B'^2 + A'^2) + A' \cdot \frac{1}{2}B' \right) \otimes \left(\theta^{-1}(\frac{1}{4}B'^2 + A'^2) + A' \cdot \frac{1}{2}B' \right) \right)$$

$$:= \exp\left(\overline{\mathbb{L}}(u) \otimes 1 + 1 \otimes \overline{\mathbb{L}}(u) \right) \cdot \#'(u).$$

Proof: By Corollary 3.2.9, in the subquotient, we have $\Delta(\tilde{L}_{2n}) =$

$$\tilde{L}_{2n} \otimes 1 - \theta\tilde{L}_{2n-1} \otimes \tilde{L}_1 + \theta\tilde{L}_{2n-2} \otimes \tilde{L}_2 - \cdots + \theta\tilde{L}_2 \otimes \tilde{L}_{2n-2} - \theta\tilde{L}_1 \otimes \tilde{L}_{2n-1} + 1 \otimes \tilde{L}_{2n}$$

$$+ \theta\frac{1}{2}\hat{L}_{2n-2} \otimes \frac{1}{2}\hat{L}_2 + \theta\frac{1}{2}\hat{L}_{2n-3} \otimes \frac{1}{2}\hat{L}_3 + \cdots + \theta\frac{1}{2}\hat{L}_3 \otimes \frac{1}{2}\hat{L}_{2n-3} + \theta\frac{1}{2}\hat{L}_2 \otimes \frac{1}{2}\hat{L}_{2n-2},$$

i.e., $*(u) =$

$$\sum_{n=0}^{\infty} \theta\tilde{L}_{2n}u^{2n} \otimes \sum_{n=0}^{\infty} \theta\tilde{L}_{2n}u^{2n} - \sum_{n=0}^{\infty} \theta\tilde{L}_{2n+1}u^{2n+1} \otimes \sum_{n=0}^{\infty} \theta\tilde{L}_{2n+1}u^{2n+1}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2}\theta\hat{L}_{2n}u^{2n} \otimes \sum_{n=1}^{\infty} \frac{1}{2}\theta\hat{L}_{2n}u^{2n} + \sum_{n=1}^{\infty} \frac{1}{2}\theta\hat{L}_{2n+1}u^{2n+1} \otimes \sum_{n=1}^{\infty} \frac{1}{2}\theta\hat{L}_{2n+1}u^{2n+1}.$$

Then by (3.24a), (3.24b), (3.24e) in Prop. 3.2.16, it is easy to get the conclusion. $\square$

Lemma 4.2.2

$$\#(u)\#'(iu) = \left(1 + (2\theta)^2\mathbb{Z}(a) \otimes \mathbb{Z}(a) \right) \cdot \left(1 + (2\theta)^2\mathbb{Z}(b) \otimes \mathbb{Z}(b) \right).$$

Proof: By definitions of A, A' and B, B',

$$\#(u)\#'(iu) = \left(1 - A \otimes A + \frac{1}{2}B \otimes \frac{1}{2}B + \left(\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B\right)\right.$$

$$\left.\otimes\left(\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B\right)\right)$$

$$\cdot \left(1 + A \otimes A - \frac{1}{2}B \otimes \frac{1}{2}B + \left(-\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B\right)\right.$$

$$\left.\otimes\left(-\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B\right)\right).$$

Omitting a long but purely elementary calculation using (3.24c), (3.24d) in Prop. 3.2.16, we get the conclusion. $\qquad\square$

Lemma 4.2.3 *The leading term of* $[X_1, \bar{L}_{2n}]$ *is* $\frac{(q^{2n}+(-1)^n)(q^n+(-1)^{n+1})}{nq^{n-1}(q^2-1)}X_{2n+1}$.

Proof: We have $\log(1 + \theta\sum_{n=1}^{\infty}\tilde{L}_{2n}u^{2n}) = \theta\sum_{n=1}^{\infty}\bar{L}_{2n}u^{2n}$. By Prop. 3.2.17, we get the leading term of $[X_1, \theta\tilde{L}_{2n}u^{2n}]$ is

$$\theta\left(q^{2n-2}\frac{1 - (-q^{-2})^n}{1 + q^{-2}} + q^{2n-1}\right)X_{2n+1}u^{2n}$$

$$= \frac{\theta}{1 + q^{-2}}\sum_{n=1}^{\infty}\left((q^{-1} + q^{-2} + q^{-3})(q^2u^2)^n - q^{-2}(-u^2)^n\right)X_{2n+1}.$$

Then we have the leading term of $[X_1, \bar{L}_{2n}]$ is

$$\theta^{-1}C_{2n}\left(\log\left(1 + \frac{\theta}{1 + q^{-2}}\sum_{n=1}^{\infty}\left((q^{-1} + q^{-2} + q^{-3})(q^2u^2)^n - q^{-2}(-u^2)^n\right)\right)\right)X_{2n+1},$$

i.e.,

$$\theta^{-1}C_{2n}\left(\log\left(1 + \frac{\theta}{1 + q^{-2}}\left((q^{-1} + q^{-2} + q^{-3})q^2u^2\frac{1}{1 - q^2u^2} - q^{-2}(-u^2)\frac{1}{1 + u^2}\right)\right)\right)X_{2n+1}.$$

Then by using the same technique as in Prop. 4.1.2 we have the leading term of $[X_1, \bar{L}_{2n}]$ is

$$\frac{(q^{2n} + (-1)^n)(q^n + (-1)^{n+1})}{nq^{n-1}(q^2 - 1)}X_{2n+1}.$$

$\qquad\square$

Define $\mathring{L}_{4n}$ by the following, recall $a = \sqrt{\frac{q}{2}}$, $b = i\sqrt{\frac{q^{-1}}{2}}$,

$$\sum_{n=1}^{\infty} 2\theta \mathring{L}_{4n} u^{4n} = \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} + \frac{1}{2}\log\left(1 - (2\theta)^2 \mathbb{Z}(a)^2\right) + \frac{1}{2}\log\left(1 - (2\theta)^2 \mathbb{Z}(b)^2\right).$$

Proposition 4.2.4 $\mathring{L}_{4n}$ *is primitive in the subquotient* $K_{\geq 1}/K_{>1}$. *The leading term of* $[X_1, \mathring{L}_{4n}]$ *is* $\frac{[4n]_q}{4n} X_{4n+1}$, *which means* $\mathring{L}_{4n}$ *is not 0 in* $B(\mathbb{V})$ *when* q *is not a root of unity.*

Proof: Recall $\Delta(\widetilde{\mathbb{L}}(u)) = *(u)$, $\Delta(\tilde{L}(iu)) = *(iu)$, so $\Delta(\sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n}) = \log(*(u) * (iu))$.

$$*(u) \cdot *(iu) = \#(u) \cdot \exp\left(\mathbb{L}(u) \otimes 1 + 1 \otimes \overline{\mathbb{L}}(u)\right) \cdot \exp\left(\bar{L}(iu) \otimes 1 + 1 \otimes \bar{L}(iu)\right) \cdot \#'(iu)$$

$$= \#(u) \cdot \exp\left(\sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} \otimes 1 + 1 \otimes \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n}\right) \cdot \#'(iu),$$

then

$$\log(*(u) * (iu)) = \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} \otimes 1 + 1 \otimes \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} + \log(\#(u)\#'(iu)),$$

since $\bar{L}_{4n}$ commute with all M_{2m+1} by Corollary 3.2.22. Then by Lemma 4.2.2 and Prop. 4.1.1 we can get that $\mathring{L}_{4n}$ is primitive immediately.

For the leading term of $[X_1, \mathring{L}_{4n}]$, by Lemma 4.2.3 we have that the leading term of $[X_1, \bar{L}_{2n}]$ is $\frac{(q^{2n}+(-1)^n)(q^n+(-1)^{n+1})}{nq^{n-1}(q^2-1)} X_{2n+1}$. Recall that the leading term of $[X_1, M_{2n+1}^2]$ is $(-1)^n 2^{2n} X_{4n+3}$. Then we have the leading term of $[X_1, \mathring{L}_{4n}]$ is

$$\frac{1}{2\theta}\left(2\theta \frac{(q^{4n}+1)(q^{2n}-1)}{2nq^{2n-1}(q^2-1)} + C_{4n}\left(\frac{1}{2}\log\left(1 - (2\theta \frac{a^2 u^2}{1 - 4a^4 u^4})^2\right)\right)\right.$$
$$\left. + C_{4n}\left(\frac{1}{2}\log\left(1 - (2\theta \frac{b^2 u^2}{1 - 4b^4 u^4})^2\right)\right)\right) X_{4n+1}.$$

Then by the same technique as in Prop. 4.1.2 we can prove that the leading term of $[X_1, \mathring{L}_{4n}]$ is $\frac{[4n]_q}{4n} X_{4n+1}$. $\qquad\square$

However, for the comultiplication of $\bar{L}_{4n+2}$, a similar calculation does not work since $\bar{L}_{4n}$ is central in $K_{\geq 1}/K_{>1}$ but $\bar{L}_{4n+2}$ is not. We introduce new generating functions.

Define

$$\mathbb{M}_{\mathrm{I}}(u) = \sum_{n=0}^{\infty} M_{4n+1} u^{4n+1}, \mathbb{M}_{\mathrm{II}}(u) = \sum_{n=0}^{\infty} M_{4n+3} u^{4n+3}.$$

, sometimes we just use $\mathbb{M}_{\mathrm{I}}$ and $\mathbb{M}_{\mathrm{II}}$ for convenience.

Define

$$\overline{\mathfrak{L}} = \sum_{n=0}^{\infty} \theta(4n+2) \bar{L}_{4n+2} u^{4n+2}.$$

Proposition 4.2.5 *In the subquotient $K_{\geq 1}/K_{>1}$, we have*

$$\Delta(\overline{\mathfrak{L}}) = \overline{\mathfrak{L}} \otimes 1 + 1 \otimes \overline{\mathfrak{L}} + 2\Big(\mathfrak{M}(au) + \mathfrak{M}(bu)\Big)$$

where

$$\mathfrak{M}(u) = \frac{\Big(-2\theta \mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{I}} + 2\theta \mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{II}}\Big)\Big(1 + 2\theta \mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{II}} + 2\theta \mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{I}}\Big)}{1 + (2\theta)^2 \mathbb{Z} \otimes \mathbb{Z}}.$$

Proof: We know

$$\overline{\mathbb{L}} = \log(\widetilde{\mathbb{L}}).$$

Since $[\tilde{L}_{2n}, \tilde{L}_{2m}] = 0$ for any $m, n \in \mathbb{N}$, using $u \cdot \frac{d}{du}$ act on it, we have

$$\sum_{n=1}^{\infty} \theta 2n \bar{L}_{2n} u^{2n} = \frac{u \cdot \frac{d}{du} \widetilde{\mathbb{L}}}{\widetilde{\mathbb{L}}}.$$

Then

$$\Delta(u \cdot \frac{d}{du}\overline{\mathbb{L}}) = \sum_{n=1}^{\infty} \theta 2n \Delta(\bar{L}_{2n}) u^{2n} = \frac{u \cdot \frac{d}{du} \Delta(\widetilde{\mathbb{L}})}{\Delta(\widetilde{\mathbb{L}})}.$$

We view $\Delta(\widetilde{\mathbb{L}})^{-1}$ as right inverse. We have known $\Delta(\widetilde{\mathbb{L}}) = \#(u)\exp\overline{\mathbb{L}} \otimes \exp\overline{\mathbb{L}}$ by Prop. 4.2.1.

$$\Delta(u \cdot \frac{d}{du}\overline{\mathbb{L}})$$
$$= \big(u \cdot \frac{d}{du}\#(u)\exp\overline{\mathbb{L}} \otimes \exp\overline{\mathbb{L}} + \#(u)\exp\overline{\mathbb{L}} \otimes \exp\overline{\mathbb{L}}(u \cdot \frac{d}{du}\overline{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du}\overline{\mathbb{L}}))$$
$$\quad \cdot \exp(-\overline{\mathbb{L}}) \otimes \exp(-\overline{\mathbb{L}})\#(u)^{-1}$$
$$= u \cdot \frac{d}{du}\#(u) \cdot \#(u)^{-1} + \#(u)(u \cdot \frac{d}{du}\overline{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du}\overline{\mathbb{L}})\#(u)^{-1}$$
$$= (u \cdot \frac{d}{du}\overline{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du}\overline{\mathbb{L}}) + u \cdot \frac{d}{du}\#(u) \cdot \#(u)^{-1}$$

$$- [(u \cdot \frac{d}{du}\overline{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du}\overline{\mathbb{L}}), \#(u)]\#(u)^{-1}.$$

Let Ω be the operator $u \cdot \frac{d}{du} - [u \cdot \frac{d}{du}\overline{\mathbb{L}} \otimes 1 + 1 \otimes u \cdot \frac{d}{du}\overline{\mathbb{L}}, \quad]$.
Then we just need to calculate

$$\big(\Omega \circ \#(u)\big) \cdot \#(u)^{-1}. \tag{$\diamond$}$$

The calculation of $\big(\Omega \circ \#(u)\big) \cdot \#(u)^{-1}$ is very long but elementary, using the commutation relations between different $\mathbb{M}$. The result of $\Delta(\overline{\mathfrak{L}})$ is a part of $\Delta(u \cdot \frac{d}{du}\overline{\mathbb{L}})$ by degree. We will not write down all the details, but we will write down two representative parts of $(\diamond)$, see Prop. 4.2.7 and Prop. 4.2.8 below. $\qquad\square$

The main tool is the following lemma, in which the commutation relations of various generating functions with $M_{2n+1}, n \geq 0$ as coefficients are described. Recall that $\mathbb{M}(u) = \mathbb{M}_{\mathrm{I}}(u) + \mathbb{M}_{\mathrm{II}}(u)$, we define $\mathbb{M}_{\mathrm{I}}(u) - \mathbb{M}_{\mathrm{II}}(u)$ by $\overline{\mathbb{M}}(u)$ and denote $\mathbb{M}(hu)$, $\mathbb{M}_{\mathrm{I}}(hu)$, $\mathbb{M}_{\mathrm{II}}(hu)$, $\overline{\mathbb{M}}(hu)$ by $\mathbb{M}(h)$, $\mathbb{M}_{\mathrm{I}}(h)$, $\mathbb{M}_{\mathrm{II}}(h)$, $\overline{\mathbb{M}}(h)$ for convenience, where $h \in \mathbb{C}^{\times}$.

Lemma 4.2.6 *We have the following relations in $B(\mathbb{V})$.*

$$[\mathbb{M}_{\mathrm{I}}(x), \mathbb{M}_{\mathrm{II}}(y)] = 0 \text{ for any } x, y \in \mathbb{C}, \tag{4.2a}$$

$$[\mathbb{M}(a), \mathbb{M}(b)] = [\overline{\mathbb{M}}(a), \overline{\mathbb{M}}(b)] = \frac{-2}{\theta i}\big(\mathbb{Z}(a) + \mathbb{Z}(b)\big), \tag{4.2b}$$

$$[\mathbb{M}(a), \overline{\mathbb{M}}(b)] = [\overline{\mathbb{M}}(a), \mathbb{M}(b)] = \frac{-2}{\theta' i}\big(\mathbb{Z}(a) - \mathbb{Z}(b)\big), \tag{4.2c}$$

$$u \cdot \frac{d}{du}\mathbb{Z}(u) = 2(\mathbb{M}_{\mathrm{I}}(u)^2 - \mathbb{M}_{\mathrm{II}}(u)^2) = [\mathbb{M}(u), \overline{\mathbb{M}}(u)]. \tag{4.2d}$$

Proof: (4.2a) follows from Corollary 3.2.22 (b). (4.2b) and (4.2c) follow from (4.2a) and direct computation using Corollary 3.2.22 (b). (4.2d) also follows from direct computation using Corollary 3.2.22 (b). $\qquad\square$

Recall that by Lemma 4.2.2

$$\#(u)\#'(iu) = \big(1 + (2\theta)^2\mathbb{Z}(a) \otimes \mathbb{Z}(a)\big) \cdot \big(1 + (2\theta)^2\mathbb{Z}(b) \otimes \mathbb{Z}(b)\big).$$

Denote $(1 + (2\theta)^2\mathbb{Z}(a) \otimes \mathbb{Z}(a)) \cdot (1 + (2\theta)^2\mathbb{Z}(b) \otimes \mathbb{Z}(b))$ by $\mathcal{C}$. Then $(\diamond)$ is

$$\frac{\Omega\#(u) \cdot \#'(iu)}{\mathcal{C}},$$

where

$$\#(u) = \left(1 - A \otimes A + \frac{1}{2}B \otimes \frac{1}{2}B \right.$$

$$\left. + (\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B) \otimes (\theta^{-1}(\frac{1}{4}B^2 + A^2) + A \cdot \frac{1}{2}B) \right)$$

$$\#'(iu) = \left(1 + A \otimes A - \frac{1}{2}B \otimes \frac{1}{2}B \right.$$

$$\left. + (\theta^{-1}(\frac{1}{4}B^2 + A^2) - A \cdot \frac{1}{2}B) \otimes (\theta^{-1}(\frac{1}{4}B^2 + A^2) - A \cdot \frac{1}{2}B) \right).$$

For convenience, we denote $\theta^{-1}(\frac{1}{4}B^2 + A^2)$ by T and denote the operator $u \cdot \frac{d}{du} - [u \cdot \frac{d}{du}\overline{\mathbb{L}}(u), \quad]$ by $\bigstar$.

Proposition 4.2.7

$$\Omega(A \otimes A - \frac{1}{2}B \otimes \frac{1}{2}B)$$
$$= 4\theta\big(\mathbb{M}_{\mathrm{I}}(a) \otimes \mathbb{M}_{\mathrm{I}}(a) - \mathbb{M}_{\mathrm{II}}(a) \otimes \mathbb{M}_{\mathrm{II}}(a) + \mathbb{M}_{\mathrm{I}}(b) \otimes \mathbb{M}_{\mathrm{I}}(b) - \mathbb{M}_{\mathrm{II}}(b) \otimes \mathbb{M}_{\mathrm{II}}(b)\big).$$

Proof: We have

$$\Omega(A \otimes A - \frac{1}{2}B \otimes \frac{1}{2}B) = \bigstar A \otimes A + A \otimes \bigstar A - (\bigstar\frac{1}{2}B \otimes \frac{1}{2}B + \frac{1}{2}B \otimes \bigstar\frac{1}{2}B).$$

First we focus on $\bigstar A$.

Denote $\mathrm{ad}_{L_2(xu)^2}$ by ad_x for $x \in \mathbb{C}$. By Corollary 3.2.22 (c), we have

$$[u \cdot \frac{d}{du}\overline{\mathbb{L}}(u), L_1] = 4(\frac{\mathrm{ad}_a}{1 - \mathrm{ad}_a^2} + \frac{\mathrm{ad}_b}{1 - \mathrm{ad}_b^2}) \circ L_1,$$

which means

$$[u \cdot \frac{d}{du}\overline{\mathbb{L}}(u), A(u)] = 4(\frac{\mathrm{ad}_a}{1 - \mathrm{ad}_a^2} + \frac{\mathrm{ad}_b}{1 - \mathrm{ad}_b^2}) \circ A(u).$$

Meanwhile, it is easy to get

$$u \cdot \frac{d}{du}A(u) = \frac{2\theta}{\theta'}\Big(a\frac{1 + \mathrm{ad}_a}{1 - \mathrm{ad}_a} \circ \mathbb{M}(a) - b\frac{1 + \mathrm{ad}_b}{1 - \mathrm{ad}_b} \circ \mathbb{M}(b)\Big).$$

Then

$$\bigstar A = \left(\frac{1 + \mathrm{ad}_a}{1 - \mathrm{ad}_a} - 4\frac{\mathrm{ad}_a}{1 - \mathrm{ad}_a^2} - 4\frac{\mathrm{ad}_b}{1 - \mathrm{ad}_b^2}\right)\frac{2\theta}{\theta'}a\mathbb{M}(a)$$

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$$- \left(\frac{1 + \mathrm{ad}_b}{1 - \mathrm{ad}_b} - 4 \frac{\mathrm{ad}_a}{1 - \mathrm{ad}_a^2} - 4 \frac{\mathrm{ad}_b}{1 - \mathrm{ad}_b^2} \right) \frac{2\theta}{\theta'} b \mathbb{M}(b).$$

We also have

$$a\mathbb{M}(a) = \frac{a^2}{1 - \mathrm{ad}_a} \circ L_1 u, \ \ b\mathbb{M}(b) = \frac{b^2}{1 - \mathrm{ad}_b} \circ L_1 u.$$

Then we have

$$\bigstar A(u) = A'(u) - \frac{2\theta}{\theta'} \left(\frac{4\mathrm{ad}_b}{1 - \mathrm{ad}_b^2} \frac{a^2}{1 - \mathrm{ad}_a} \circ L_1 u - \frac{4\mathrm{ad}_a}{1 - \mathrm{ad}_a^2} \frac{b^2}{1 - \mathrm{ad}_b} \circ L_1 u \right)$$

$$= A'(u) + \frac{4}{\theta'} (a^{-1} \mathbb{M}_{\mathrm{I}}(a) - b^{-1} \mathbb{M}_{\mathrm{I}}(b)).$$

Similarly we have

$$\bigstar \frac{1}{2} B(u) = -\frac{1}{2} B'(u) + \frac{8}{\theta'} (a\mathbb{M}_{\mathrm{II}}(a) - b\mathbb{M}_{\mathrm{II}}(b)).$$

Note that $\mathbb{M}_{\mathrm{I}}(u) = \frac{\mathbb{M}(u) + \overline{\mathbb{M}}(u)}{2}$, $\mathbb{M}_{\mathrm{II}}(u) = \frac{\mathbb{M}(u) - \overline{\mathbb{M}}(u)}{2}$. Then we have

$$\bigstar A = \frac{2}{\theta'} \left(a\theta' \overline{\mathbb{M}}(a) + b\theta' \overline{\mathbb{M}}(b) + a^{-1} \mathbb{M}(a) - b^{-1} \mathbb{M}(b) \right)$$

$$\bigstar B = \frac{2}{\theta'} \left(-a^{-1} \theta' \overline{\mathbb{M}}(a) - b^{-1} \theta' \overline{\mathbb{M}}(b) + 4a\mathbb{M}(a) - 4b\mathbb{M}(b) \right)$$

$$A = \frac{2\theta}{\theta'} \left(a\mathbb{M}(a) - b\mathbb{M}(b) \right), \ \ \frac{1}{2} B = \frac{\theta}{\theta'} \left(a^{-1} \mathbb{M}(a) - b^{-1} \mathbb{M}(b) \right).$$

By direct computation with these formulas, we get the expression of $\Omega(A \otimes A - \frac{1}{2} B \otimes \frac{1}{2} B)$. $\qquad \qquad \square$

Proposition 4.2.8

$$\left(\bigstar (T + A \cdot \frac{1}{2} B) \otimes (T + A \cdot \frac{1}{2} B) + (T + A \cdot \frac{1}{2} B) \otimes \bigstar (T + A \cdot \frac{1}{2} B) \right)$$

$$\cdot (T - A \cdot \frac{1}{2} B) \otimes (T - A \cdot \frac{1}{2} B)$$

$$= (2\theta)^4 \Big(\frac{1}{2} u \cdot \frac{d}{du} \mathbb{Z}(b) \otimes \mathbb{Z}(b) \cdot \mathbb{Z}(a) \otimes \mathbb{Z}(a) + \frac{1}{2} u \cdot \frac{d}{du} \mathbb{Z}(a) \otimes \mathbb{Z}(a) \cdot \mathbb{Z}(b) \otimes \mathbb{Z}(b)$$

$$+ 2\mathbb{M}_{\mathrm{I}}(b)\mathbb{M}_{\mathrm{II}}(b) \otimes \mathbb{Z}(b) \cdot \mathbb{Z}(a) \otimes \mathbb{Z}(a) + 2\mathbb{M}_{\mathrm{I}}(a)\mathbb{M}_{\mathrm{II}}(a) \otimes \mathbb{Z}(a) \cdot \mathbb{Z}(b) \otimes \mathbb{Z}(b) \Big)$$

$$+ \ symmetric \ part.$$

Remark 4.2.9 $\frac{1}{2} u \cdot \frac{d}{du} \mathbb{Z}(b) \otimes \mathbb{Z}(b) \cdot \mathbb{Z}(a) \otimes \mathbb{Z}(a)$, $\frac{1}{2} u \cdot \frac{d}{du} \mathbb{Z}(a) \otimes \mathbb{Z}(a) \cdot \mathbb{Z}(b) \otimes \mathbb{Z}(b)$ and the symmetric part correspond to the result of $u \cdot \frac{d}{du}$ acting on $\Delta(\sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n})$, which

is known in the proof of Prop. 4.2.4.

Proof: By Lemma 4.2.6 and Prop. 3.2.16 , we have the following relations

$$T + A \cdot \frac{1}{2}B = -\frac{4\theta}{\theta'}\mathbb{Z}(b) - \frac{2\theta^2}{\theta'}i\mathbb{M}(b)\mathbb{M}(a),$$

$$T - A \cdot \frac{1}{2}B = \frac{4\theta}{\theta'}\mathbb{Z}(a) + \frac{2\theta^2}{\theta'}i\mathbb{M}(b)\mathbb{M}(a),$$

$$\bigstar(T + A \cdot \frac{1}{2}B) = \frac{2\theta}{\theta'}\Big(2\overline{\mathbb{M}}(a)\mathbb{M}(a) - 2\overline{\mathbb{M}}(b)\mathbb{M}(b) - \frac{2\theta}{\theta'}\mathbb{Z}(a)$$
$$+ \frac{2\theta}{\theta'}\mathbb{Z}(b) - i\theta\mathbb{M}(b)\overline{\mathbb{M}}(a) - i\theta\overline{\mathbb{M}}(b)\mathbb{M}(a)\Big),$$

$$T^2 - (A \cdot \frac{1}{2}B)^2 = -4\theta^2\mathbb{Z}(a)\mathbb{Z}(b).$$

Then this proposition follows from direct computation. $\square$

Chapter 5   A PBW basis of $B(\mathbb{V})$

In this chapter, we will determine the multiplicities of all the roots and give a PBW basis of $B(\mathbb{V})$. The main tool is the subquotient $K_{\geq 1}/K_{>1}$. In detail, the key step is to use the primitive element at certain degree in $K_{\geq 1}/K_{>1}$. In the case of $U_v(\hat{sl}_2)$, the Drinfeld generators of the imaginary roots are exactly these primitive elements.

§5.1   Root vectors at $(2n+1)\delta$

By Corollary 3.2.22, we know that the only possible root vector at $(2n+1)\delta$ is M_{2n+1}.

Proposition 5.1.1 *When q is not a root of unity, M_{2n+1} for $n \geq 0$ are root vectors.*

Proof: Let $\mathbb{O}_{2n+1}$ be the subalgebra generated by $\{M_{2i+1}, 0 \leq i \leq n\}$. By Prop. 4.1.1 we can get

$$\Delta(M_{2n+1}) \in M_{2n+1} \otimes 1 + 1 \otimes M_{2n+1} + 2\theta L_1^2 \otimes M_{2n-1} - 2\theta M_{2n-1} \otimes L_1^2 + \mathbb{O}_{2n-3} \otimes \mathbb{O}_{2n-3}.$$

We use Theorem 2.2.42. M_{2n+1} is a root vector if and only if M_{2n+1} is a linear combination of $L_1^{k_1} M_3^{k_2} M_5^{k_3} \cdots M_{2n-1}^{k_n}$. By the above formula, the comultiplication of M_{2n+1} has term $2\theta L_1^2 \otimes M_{2n-1} - 2\theta M_{2n-1} \otimes L_1^2$ but the comultiplication of $L_1^2 M_{2n-1}$ can only have term $L_1^2 \otimes M_{2n-1} + M_{2n-1} \otimes L_1^2$. Hence, by induction we can see that M_{2n+1} must be a root vector when q is not a root of unity. $\qquad\square$

§5.2   Root vectors at $(4n+2)\delta$ and $4n\delta$

§5.2.1 M_{2n+1}^2

Since M_{2n+1} is a root vector and $\chi((2n+1)\delta, (2n+1)\delta) = -1$, we should consider M_{2n+1}^2.

Proposition 5.2.1 *M_{2n+1}^2 are root vectors when q is not a root of unity.*

Proof: We use Theorem 2.2.42. Suppose M_{2n+1}^2 is not a root vector. Then the only possibility is M_{2n+1}^2 is a linear combination of products of M_{2k+1}^2 in $B(\mathbb{V})$, where $k \leq n - 1$. Equivalently, Z_{4n+2} is a linear combination of products of Z_{4k+2} where

$k \le n - 1$, denoted this linear combination by Π. Then $Z_{4n+2} - \Pi$ is 0 in $B(\mathbb{V})$ and must be primitive in $K_{\ge 1}/K_{>1}$. Then it is easy to see the only possibility is $\Pi = 0$, since Z_{4k+2} are primitive in $K_{\ge 1}/K_{>1}$ and are root vectors in $B(\mathbb{V})$ for $k \le n - 1$. Then Z_{4n+2} is 0 in $B(\mathbb{V})$, but by Prop. 4.1.2, we have known Z_{4n+2} is not 0 in $B(\mathbb{V})$, contradiction. $\qquad\square$

§5.2.2 $\quad L_{2n}$

Proposition 5.2.2 L_{2n} *are root vectors for* $n \ge 1$ *when* q *is not a root of unity.*

Proof: We prove this by induction on n. We still use Theorem 2.2.42.

L_2 is a root vector because $[L_2, L_1] = M_3$ and M_3 is a root vector.

For L_4, suppose L_4 is not a root vector, then $L_4 = \sum k_{i_1,i_2,j_1} L_1^{i_1} M_3^{i_2} L_2^{j_1}$, or equivalently $\bar{L}_4 = \sum k'_{i_1,i_2,j_1} L_1^{i_1} M_3^{i_2} \bar{L}_2^{j_1}$. By $\Delta(\bar{L}_4)$ in the proof of Prop. 4.2.4 and L_2 is a root vector, $\bar{L}_2^2$ and $\bar{L}_1^2 \bar{L}_2$ cannot appear in this linear combination. Then $\bar{L}_4$ can only be a linear combination of $L_1 M_3$ and L_1^4. Then using $\text{ad}_{\bar{L}_2}$ to act on $\bar{L}_4$, we get $M_3^2 + L_1 M_5 = 0$, a contradiction. Then $\bar{L}_4$ can only equal kL_1^4, or equivalently $\mathring{L}_4 = k(L_1^2)^2$. Since $\mathring{L}_4$ and L_1^2 are primitive in $K_{\ge 1}/K_{>1}$, then the only possibility is $\mathring{L}_4 = 0$, but according to Prop. 4.2.4 $\mathring{L}_4$ is not 0. Then L_4 must be a root vector.

Suppose it holds for n, we consider $n + 1$.

If n is even, then $2(n + 1) = 4m + 2$ for $m = \frac{n}{2}$. Suppose L_{4m+2} is not a root vector, then

$$L_{4m+2} = \sum k_{i_1,i_2,\ldots,i_{2m+1},j_1,\ldots,j_{2m}} L_1^{i_1} M_3^{i_2} \cdots M_{4m+1}^{i_{2m+1}} L_2^{j_1} \cdots L_{4m}^{j_{2m}},$$

or equivalently

$$\bar{L}_{4m+2} = \sum k'_{i_1,i_2,\ldots,i_{2m+1},j_1,\ldots,j_{2m}} L_1^{i_1} M_3^{i_2} \cdots M_{4m+1}^{i_{2m+1}} \bar{L}_2^{j_1} \cdots \bar{L}_{4m}^{j_{2m}}.$$

By Prop. 4.2.5 and induction hypothesis, $\bar{L}_2, ..., \bar{L}_{4m}$ cannot appear. Then

$$\bar{L}_{4m+2} = \sum k''_{i_1,i_2,\ldots,i_{2m+1}} L_1^{i_1} M_3^{i_2} \cdots M_{4m+1}^{i_{2m+1}}.$$

By Corollary 3.2.22 (c), $[\bar{L}_{4m+2}, L_1] = R_{4m+2} M_{4m+3}$. When q is not a root of unity, we know that R_{4m+2} is not 0 and M_{4m+3} is a root vector, contradiction. Then L_{4m+2} must be a root vector.

If n is odd, then $2(n + 1) = 4m$ for $m = \frac{n+1}{2}$. Suppose L_{4m} is not a root vector, then

$$L_{4m} = \sum k_{i_1,i_2,\ldots,i_{2m},j_1,\ldots,j_{2m-1}} L_1^{i_1} M_3^{i_2} \cdots M_{4m-1}^{i_{2m}} L_2^{j_1} \cdots L_{4m-2}^{j_{2m-1}}$$

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or equivalently

$$\mathring{L}_{4m} = \sum k'_{i_1,i_2,...,i_{2m},j_1,...,j_{2m-1}} L_1^{i_1} M_3^{i_2} \cdots M_{4m-1}^{i_{2m}} \bar{L}_2^{j_1} \cdots \bar{L}_{4m-2}^{j_{2m-1}}.$$

For the same reason as last paragraph, $\bar{L}_2, ..., \bar{L}_{4m-2}$ cannot appear. Use $\mathrm{ad}_{\bar{L}_2}$ act on $\mathring{L}_{4m}$, similar to the case of L_4 we get

$$\mathring{L}_{4m} = \sum k''_{i_1,i_2,...,i_m} L_1^{i_1} M_3^{i_2} \cdots M_{2m-1}^{i_m},$$

where $i_1, i_2, ..., i_m$ must be even. Then equivalently

$$\mathring{L}_{4m} = \sum k'''_{i_1,i_2,...,i_m} (Z_2)^{\frac{i_1}{2}} (Z_6)^{\frac{i_2}{2}} \cdots (Z_{4m-2})^{\frac{i_m}{2}}.$$

Since $\mathring{L}_{4m}, Z_2, Z_6, ..., Z_{4m-2}$ are primitive in $K_{\geq 1}/K_{>1}$, the only possibility is $\mathring{L}_{4m} = 0$, but by Prop. 4.2.4 $\mathring{L}_{4m}$ is not 0. Then L_{4m} must be a root vector. $\square$

§5.3   A PBW basis

Now we have the following theorem immediately.

Theorem 5.3.1 *Suppose q is not a root of unity, the multiplicities of $4n\delta$ and $(2n+1)\delta$ are 1, of $(4n+2)\delta$ is 2, which coincide with the root multiplicities of the Lie superalgebra $A(0,2)^{(4)}$. The set of decreasing ordered products of elements in $\{X_n, Y_n, L_{2n}, M_{2n-1}, n \in \mathbb{N}\}$ forms a PBW basis of $B(\mathbb{V})$. The order in $\{X_n, Y_n, L_{2n}, M_{2n-1}, n \in \mathbb{N}\}$ is*

$$X_1 < X_2 < \cdots < X_n < \cdots$$
$$< \cdots < L_{2n} < L_{2n-2} < L_{2n-4} < \cdots < L_2$$
$$< \cdots < M_{2n-1} < M_{2n-3} < \cdots < M_5 < M_3 < M_1$$
$$< \cdots < Y_n < \cdots < Y_2 < Y_1.$$

Chapter 6   The defining relations of $B(\mathbb{V})$ satisfied by the Chevalley generators

We first consider the commutation relations in $B(\mathbb{V})$ between real root vectors. By definition, $[X_m, Y_n]$ is just L_{m+n-1} or L'_{m+n-1} so we do not need to care about this case. We only care about the relations between X_n and X_m, Y_n and Y_m. First, we recall the $U_v^+(\hat{sl_2})$ case because they have some common properties.

Proposition 6.0.1 *In $U_v^+(\hat{sl_2})$ when v is not a root of unity, we have*

$$[X_n, X_{n+r}] = -[X_{n+r-1}, X_{n+1}] \ \text{and} \ [Y_{n+r}, Y_n] = -[Y_{n+1}, Y_{n+r-1}]$$

for $n, r > 0$. Moreover, these two relations can be derived from quantum Serre relations.

Proof: They are part of the relations in the Drinfeld second realization; see [CP91]. The second conclusion can be derived from [HK07], or follows from a similar proof to Lemma 6.0.2 below. $\qquad\qquad\square$

Lemma 6.0.2 *We have the following relations about X_n and Y_n in $B(\mathbb{V})$.*

$$[X_n, X_{n+1}] = 0, \ \text{and} \ \ [Y_{n+1}, Y_n] = 0,$$

for $n \geq 1$. Moreover, these relations are derived by quantum Serre relations.

Proof: We deal with the first one, the second one is completely similar. In fact we claim:

$$[X_n, X_{n+2}] = (-1)^{n+1}[X_{n+1}, X_{n+1}], \tag{6.1a}$$
$$[X_n, X_{n+3}] = -[X_{n+2}, X_{n+1}], \tag{6.1b}$$
$$[X_{n+1}, X_{n+2}] = 0, \tag{6.1c}$$

for $n > 0$. We prove this claim by induction on n. First, $[X_1, X_2] = 0$ holds, which is one of the quantum Serre relations.
$n = 1:$
$$0 = [[X_1, X_2], L_1] = [X_1, X_3] - [X_2, X_2],$$

then (6.1a) holds. Then

$$0 = [[X_1, X_3] - [X_2, X_2], L_1] = [X_1, X_4] + [X_2, X_3] - ([X_2, X_3] - [X_3, X_2]).$$

i.e.,

$$[X_1, X_4] = -[X_3, X_2],$$

which is (6.1b). We also have

$$
\begin{aligned}
0 =& [[X_1, X_2], L_2] \\
=& [X_1, qX_4 + \theta L_1 X_3] + [(q+2)X_3 - \theta L_1 X_2, X_2] \\
=& q[X_1, X_4] + \theta(X_2 X_3 + L_1[X_1, X_3]) + (q+2)[X_3, X_2] - \theta(L_1[X_2, X_2] - qX_3 X_2)
\end{aligned}
$$

which implies $[X_2, X_3] = 0$, which is (6.1c).

For general case, the induction step follows from a completely similar argument to the case of $n = 1$, using a fact that

$$[X_{2k+1}, L_2] = (q+2)X_{2k+3} - \theta L_1 X_{2k+2}$$

and

$$[X_{2k+2}, L_2] = qX_{2k+4} + \theta L_1 X_{2k+3}.$$

Note that all of these formulas come from quantum Serre relations. $\qquad\square$

Then the following corollary holds immediately.

Corollary 6.0.3 *All $[X_n, X_m]$ and $[Y_n, Y_m]$ can be expressed as a linear combination of decreasing product of root vectors by quantum Serre relations.*

Remark 6.0.4 *Note that the commutation relations between X_n and Y_m come from the definition of L_{m+n-1} and L'_{m+n-1} and Prop. 3.2.5, which is derived by quantum Serre relations. Then we get that the commutation relations between real root vectors are all come from quantum Serre relations.*

Theorem 6.0.5 *Suppose q is not a root of unity. The defining relations of $B(\mathbb{V})$ satisfied by the Chevalley generators are the quantum Serre relations (2.4), which means that the Drinfeld-Jimbo type realization is obtained for $U_v(A(0, 2)^{(4)})$.*

Proof: We have determined all the root vectors of $B(\mathbb{V})$ in Theorem 5.3.1. Then it is equivalent to prove that all the commutation relations between these root vectors can be derived from quantum Serre relations. All of these relations are in Section 2 and

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the beginning of Section 5. As stated in Remark 3.2.8, 3.2.18, 3.2.24, 6.0.4, we get these commutation relations almost completely from quantum Serre relations, except Prop. 3.2.17.

We prove that all the commutation relations in Prop. 3.2.17 can be derived from quantum Serre relations. We only prove the first half because the other half is completely similar. We need to show that the expressions of $[X_2, L_{2n}]$ and $[X_2, L_{2n+1}]$ for $n \geq 1$ in Prop. 3.2.17, which are obtained by applying differential operators, can be derived only by quantum Serre relations. Note that relations in Prop. 3.2.5 are all come from quantum Serre relations, and we know (3.15):

$$2L'_{2n+1} = \theta L'_{2n} L_1 + \theta L_1 L_{2n} + [L_2, L'_{2n-1}]$$

holds. Applying ad_{X_1} to this equation, we will get the result of $(q + q^{-1})[X_2, L_{2n}]$ and since q is not a root of unity, we will get the expression of $[X_2, L_{2n}]$ in terms of ordered products of root vectors, repeatedly using induction hypothesis and the following known relations derived by quantum Serre relations: (3.12), (3.15), (3.16), (3.17), (3.18) and Corollary 3.2.7. On the other hand, also by these relations, it is enough to get the expression of $[X_2, L_{2n}]$ in terms of ordered products of root vectors from

$$[X_2, L_{2n}] = q^{2n-1} X_{2n+2} + q^{2n-2} \theta L'_1 X_{2n+1} + q^{2n-3} \theta L_2 X_{2n} + \cdots + q\theta L_{2n-2} X_4 + \theta L'_{2n-1} X_3.$$

Then these two expressions must be the same. This means the expression of $[X_2, L_{2n}]$ in Prop. 3.2.17 also comes from quantum Serre relations. The case of $[X_2, L_{2n+1}]$ is completely similar. $\qquad \square$

Chapter 7 The Drinfeld second realization of $U_v^+(A(0,2)^{(4)})$

We give the Drinfeld second realization of $B(\mathbb{V})$ or $U_v^+(A(0,2)^{(4)})$, that is, choosing distinguished root vectors and writing all the commutation relations among them. The following three sections are denoted by "I&I","I&R", "R&R", where "I","R" means distinguished imaginary root vectors and real root vectors, respectively.

§7.1 I&I

Proposition 7.1.1 *The following commutation relations hold in $B(\mathbb{V})$.*

(a) $\mathring{L}_{4n+4}$ *and* Z_{4n+2} *commute with all imaginary root vectors,* $[\bar{L}_{4m+2}, \bar{L}_{4n+2}] = 0$, $\forall m, n \geq 0$.

(b) $[M_{2m+1}, M_{2n+1}] = 0$ *if* $m+n$ *is odd,* $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+n+1}^2$ *if* $m+n$ *is even.*

(c) $[(4m+2)\bar{L}_{4m+2}, M_{2n+1}] = \frac{1}{2^{2m-1}}[2m+1]_q M_{4m+2n+3}$.

Proof: This holds immediately by Corollary 3.2.22. $\qquad\qquad\qquad\square$

§7.2 I&R

First, we show that the comultiplication in $K_{\geq 1}/K_{>1}$ is closely related to these relations.

Denote the subalgebra generated by $L_n, n \geq 1$ in $B(\mathbb{V})$ by $\mathcal{L}$. We identify $\mathcal{L}$ with the subquotient $K_{\geq 1}/K_{>1}$. Denote the subalgebra generated by $X_n, n \geq 1$, $Y_n, n \geq 1$ by $\mathcal{X}$ and $\mathcal{Y}$. We have $B(\mathbb{V}) \cong \mathcal{Y} \otimes \mathcal{L} \otimes \mathcal{X}$, as vector spaces.

Let $T \in \mathcal{L}$ and suppose that T is homogeneous. Define $d(T)$ by $\deg T = d(T)\delta$. Define $f_{X_k}(1) = X_k$ and $f_{X_k}(T)$, $T \in \mathcal{L}$ by the leading term of $[X_k, T]$, i.e., some scale of $X_{k+d(T)}$. Define $g_{Y_k}(1) = Y_k$ and $g_{Y_k}(T)$, $T \in \mathcal{L}$ by the leading term of $[T, Y_k]$, i.e., some scale of $Y_{k+d(T)}$.

Remark 7.2.1 *It is very easy to determine f_{X_k} g_{Y_k} precisely for any $k \geq 1$ by Prop. 3.2.17, Lemma 3.1.3 (a) and L_1^2 lies in the center of $\mathcal{L}$.*

Lemma 7.2.2 $\Delta(X_n) \in X_n \otimes 1 + 1 \otimes X_n + B(\mathbb{V}) \otimes \mathcal{YL}$, *i.e., no X_i appears in the right tensor factor for $i < n$.*

Proof: This follows from the comultiplication of the super-letters, that is, Prop. 2.2.39. $\qquad\square$

Lemma 7.2.3 *Let $T \in \mathcal{L}$. $\Delta(T) \in T_{(1)} \otimes T_{(2)}$ in the subquotient $+ \mathcal{LX} \otimes \mathcal{YL}$.*

Proof: This also follows from Prop. 2.2.39. $\qquad\square$

Proposition 7.2.4 *Let $T \in \mathcal{L}$. We have*

$$X_k T = T_{(1)}(-1)^{d(T_{(1)}) \cdot (k-1)} f_{X_k}(T_{(2)}),$$

and

$$TY_k = (-1)^{k \cdot d(T_{(2)})} g_{Y_k}(T_{(1)}) T_{(2)}.$$

Proof: We only prove the first one because the second one is similar. It is easy to see for $T \in \mathcal{L}$,

$$X_k T = \sum_{i=1}^{k+d(T)} k_i T_i X_i, \text{ for some } T_i \in \mathcal{L}. \tag{7.1}$$

Consider the terms of type $B(\mathbb{V}) \otimes X_i$ for some i, in the comultiplication of (7.1).

We use the two lemmas above. For LHS, all terms of this type are

$$T_{(1)}(-1)^{d(T_{(1)}) \cdot (k-1)} \otimes f_{X_k}(T_{(2)}).$$

For RHS, all terms of this type are $k_i T_i \otimes X_i$. Then we have

$$X_k T = T_{(1)}(-1)^{d(T_{(1)}) \cdot (k-1)} f_{X_k}(T_{(2)}).$$

$\qquad\square$

For M_{2n+1}, it is more convenient to write the relations in the language of generating functions. Recall that

$$\mathbb{M}(u) = \sum_{n=0}^{\infty} M_{2n+1} u^{2n+1}.$$

Proposition 7.2.5

$$X_k \mathbb{M} = \mathbb{M}_{(1)} (-1)^{d(\mathbb{M}_{(1)}) \cdot (k-1)} f_{X_k}(\mathbb{M}_{(2)}),$$

and

$$\mathbb{M} Y_k = (-1)^{k \cdot d(\mathbb{M}_{(2)})} g_{Y_k}(\mathbb{M}_{(1)}) \mathbb{M}_{(2)},$$

for any $k \geq 1$, where

$$\mathbb{M}_{(1)} \otimes \mathbb{M}_{(2)} = (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M}) \frac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}}.$$

Proof: This follows from Prop. 7.2.4 and $\mathbb{M}_{(1)} \otimes \mathbb{M}_{(2)}$ is known in the proof of Prop. 4.1.1. $\qquad \square$

Recall that
$$\arctan(2\theta i \mathbb{Z}(u)) = 2\theta i \mathcal{Z}(u).$$

We denote $\mathcal{Z}(u)$ by $\sum_{n=0}^{\infty} Z_{4n+2} u^{4n+2}$.

Proposition 7.2.6 *In $B(\mathbb{V})$ we have*

$$[X_k, Z_{4n+2}] = 2^{2n} \frac{[2n+1]_q}{2n+1} X_{4n+k+2},$$
$$[Z_{4n+2}, Y_k] = 2^{2n} \frac{[2n+1]_q}{2n+1} Y_{4n+k+2}.$$

for any $k \geq 1$ and $n \geq 0$.

Proof: The first formula follows from Prop. 4.1.1 and 7.2.4, and Z_{4n+2} lie in the center of $K_{\geq 1}/K_{>1}$ immediately. The second is completely similar. $\qquad \square$

Proposition 7.2.7 *In $B(\mathbb{V})$, we have the following relations.*

$$[X_k, \mathring{L}_{4n}] = \frac{[4n]_q}{4n} X_{4n+k},$$
$$[\mathring{L}_{4n}, Y_k] = \frac{[4n]_q}{4n} Y_{4n+k}.$$

Proof: This follows from Prop. 4.2.4 and 7.2.4 immediately. $\qquad \square$

For $(4n+2)\bar{L}_{4n+2}$, it is more convenient to write the relations in the language of generating functions. Recall that

$$\overline{\mathfrak{L}} = \sum_{n=0}^{\infty} \theta(4n+2)\bar{L}_{4n+2}u^{4n+2},$$

$$\mathbb{M}_{\mathrm{I}}(u) = \sum_{n=0}^{\infty} M_{4n+1}u^{4n+1}, \mathbb{M}_{\mathrm{II}}(u) = \sum_{n=0}^{\infty} M_{4n+3}u^{4n+3}.$$

$$\mathbb{Z}(u) = \sum_{n=0}^{\infty} M_{2n+1}^2(-1)^n u^{4n+2} = \mathbb{M}(u)^2$$

Proposition 7.2.8

$$X_k\overline{\mathfrak{L}} = \overline{\mathfrak{L}}_{(1)}(-1)^{d(\overline{\mathfrak{L}}_{(1)})\cdot(k-1)}f_{X_k}(\overline{\mathfrak{L}}_{(2)}),$$

and

$$\overline{\mathfrak{L}}Y_k = (-1)^{k\cdot d(\overline{\mathfrak{L}}_{(2)})}g_{Y_k}(\overline{\mathfrak{L}}_{(1)})\overline{\mathfrak{L}}_{(2)},$$

where

$$\overline{\mathfrak{L}}_{(1)} \otimes \overline{\mathfrak{L}}_{(2)} = \Delta(\overline{\mathfrak{L}}) = \overline{\mathfrak{L}} \otimes 1 + 1 \otimes \overline{\mathfrak{L}} + 2\Big(\mathfrak{M}(au) + \mathfrak{M}(bu)\Big)$$

and

$$\mathfrak{M}(u) = \frac{\Big(-2\theta\mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{I}} + 2\theta\mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{II}}\Big)\Big(1 + 2\theta\mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{II}} + 2\theta\mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{I}}\Big)}{1 + (2\theta)^2\mathbb{Z} \otimes \mathbb{Z}}.$$

Proof: This follows from Prop. 7.2.4 and 4.2.5. $\qquad\qquad\square$

§7.3   R&R

§7.3.1   Relations between X_n, and between Y_n

Define a sequence of number $s_0 = 1$, $s_1 = -1$, $s_2 = -1$, $s_3 = 1$ and $s_n = s_{n+4}$ for $n \geq 0$. Define a map $P : \mathbb{N} \to \{0,1\} \subset \mathbb{N}$ by $P(n) = 0$ if n is even, $P(n) = 1$ if n is odd.

Proposition 7.3.1 *In $B(\mathbb{V})$ the following relations hold for $n, k \geq 1$.*

$$[X_n, X_{n+2k-1}] = -[X_{n+2k-2}, X_{n+1}], \tag{$\dagger$}$$

$$[X_n, X_{n+2k}] = s_{P(n+1)}[X_{n+2k-1}, X_{n+1}] + 2\sum_{i=2}^{k} s_{P(n+1)+i-1}[X_{n+2k-i}, X_{n+i}]$$

$$[Y_{n+2k-1}, Y_n] = -[Y_{n+1}, Y_{n+2k-2}],$$

$$[Y_{n+2k}, Y_n] = s_{P(n)}[Y_{n+1}, Y_{n+2k-1}] + 2\sum_{i=2}^{k} s_{P(n)+i-1}[Y_{n+i}, Y_{n+2k-i}]$$

Proof: These formulas hold by applying ad_{L_1} to the formulas in Lemma 6.0.2 repeatedly. $\qquad\square$

§7.3.2   The relations between X_n and Y_m

First we have

$$[X_{2i+1}, Y_{2n+2-(2i+1)}] = L_{2n+1}, \ [X_{2i}, Y_{2n+2-2i}] = L'_{2n+1},$$
$$[X_{2i+1}, Y_{2n+3-(2i+1)}] = L_{2n+2}, \ [X_{2i}, Y_{2n+3-2i}] = L'_{2n+2}$$

However we do not use L_n or L'_n as generators, so we should use our generators to generate L_n or L'_n.

We have

$$\sum_{n=1}^{\infty} 2\theta \mathring{L}_{4n} u^{4n} = \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} + \frac{1}{2}\log\left(1 - (2\theta)^2 \mathbb{Z}(a)^2\right) + \frac{1}{2}\log\left(1 - (2\theta)^2 \mathbb{Z}(b)^2\right),$$

$$= \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} - \log \circ \cos(2\theta i \mathcal{Z}(a)) - \log \circ \cos(2\theta i \mathcal{Z}(b)),$$

, and $\mathring{L}_{4n}$, $\mathcal{Z}(u)$, $\bar{L}_{4n+2}$ are the generators we choose, then we get $\overline{\mathbb{L}}(u)$ from our generators. By

$$\widetilde{\mathbb{L}}(u) = \exp(\overline{\mathbb{L}}(u))$$

, we can get $\widetilde{\mathbb{L}}(u)$ from our generators.

Proposition 7.3.2 *In $B(\mathbb{V})$ we have*

$$\sum_{n=0}^{\infty} \theta L_{2n+1} = \frac{\theta}{\theta'}\left((2a - a^{-1})\mathbb{M}(a) - (2b - b^{-1})\mathbb{M}(b)\right)\widetilde{\mathbb{L}}(u), \qquad (\ddagger)$$

$$\sum_{n=0}^{\infty} \theta L'_{2n+1} = \frac{\theta}{\theta'}\left((2a + a^{-1})\mathbb{M}(a) - (2b + b^{-1})\mathbb{M}(b)\right)\widetilde{\mathbb{L}}(u),$$

$$\sum_{n=0}^{\infty} \theta L_{2n} = \left(1 - \frac{4\theta}{\theta'}\mathbb{Z}(b) - \frac{2\theta^2}{\theta'}i\mathbb{M}(b)\mathbb{M}(a)\right)\widetilde{\mathbb{L}}(u),$$

$$\sum_{n=0}^{\infty} \theta L'_{2n} = \left(1 + \frac{4\theta}{\theta'}\mathbb{Z}(b) + \frac{2\theta^2}{\theta'}i\mathbb{M}(b)\mathbb{M}(a)\right)\widetilde{\mathbb{L}}(u).$$

Proof: By Lemma 3.1.3, we get

$$\sum_{n=0}^{\infty} \theta L_{2n+1} = \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} - \frac{1}{2} \sum_{n=0}^{\infty} \theta \hat{L}_{2n+1}, \quad \sum_{n=0}^{\infty} \theta L'_{2n+1} = \sum_{n=0}^{\infty} \theta \tilde{L}_{2n+1} + \frac{1}{2} \sum_{n=0}^{\infty} \theta \hat{L}_{2n+1},$$

$$\sum_{n=1}^{\infty} \theta L_{2n} = \sum_{n=1}^{\infty} \theta \tilde{L}_{2n} + \frac{1}{2} \sum_{n=1}^{\infty} \theta \hat{L}_{2n}, \quad \sum_{n=1}^{\infty} \theta L'_{2n} = \sum_{n=1}^{\infty} \theta \tilde{L}_{2n} - \frac{1}{2} \sum_{n=1}^{\infty} \theta \hat{L}_{2n}.$$

Meanwhile, by Lemma 4.2.6 in Appendix A, we have

$$[\mathbb{M}(a), \mathbb{M}(b)] = \frac{-2}{\theta i}(\mathbb{Z}(a) + \mathbb{Z}(b)),$$

then the proposition holds by direct computation using Prop. 3.2.16. □

§7.4   The Drinfeld second realization

We summarize all the above results as a theorem.

Let $\theta = q - q^{-1}$, $\theta' = q + q^{-1}$, $a = \sqrt{\frac{q}{2}}$, $b = i\sqrt{\frac{q^{-1}}{2}}$.

We choose X_n, Y_n for $n > 0$, M_{2n+1}, Z_{4n+2}, $(4n+2)\bar{L}_{4n+2}$, $(4n+4)\mathring{L}_{4n+4}$ for $n \geq 0$ as root vectors at corresponding root. For simplicity of notation, we denote $(4n+2)\bar{L}_{4n+2}$ by $\mathscr{L}_{4n+2}$, $4n\mathring{L}_{4n}$ by $\mathscr{L}_{4n}$. We introduce the following generating functions with root vectors as coefficients.

$$\mathbb{M}(u) := \sum_{n=0}^{\infty} M_{2n+1} u^{2n+1} = \sum_{n=0}^{\infty} M_{4n+1} u^{4n+1} + \sum_{n=0}^{\infty} M_{4n+3} u^{4n+3} := \mathbb{M}_{\mathrm{I}}(u) + \mathbb{M}_{\mathrm{II}}(u),$$

$$\mathcal{Z} := \sum_{n=0}^{\infty} Z_{4n+2} u^{4n+2},$$

$$\overline{\mathfrak{L}} := \sum_{n=0}^{\infty} (4n+2)\theta\bar{L}_{4n+2} u^{4n+2} = \sum_{n=0}^{\infty} \theta\mathscr{L}_{4n+2} u^{4n+2}, \quad \mathring{\mathfrak{L}} := \sum_{n=1}^{\infty} 4n\theta\mathring{L}_{4n} u^{4n} = \sum_{n=1}^{\infty} \theta\mathscr{L}_{4n} u^{4n}.$$

We use $\int u^{-1}$ to denote the inverse operator of $u \cdot \frac{d}{du}$.

We define two new generating functions as follows.

$$\mathbb{Z}(u) := \sum_{n=0}^{\infty} M_{2n+1}^2 (-1)^n u^{4n+2},$$

$$\widetilde{\mathbb{L}}(u) := \exp\left(\int u^{-1}\overline{\mathfrak{L}} + \int u^{-1}\mathring{\mathfrak{L}} + \frac{1}{2}\log \circ \cos(2\theta i \mathcal{Z}(a)) + \frac{1}{2}\log \circ \cos(2\theta i \mathcal{Z}(b))\right).$$

Theorem 7.4.1 *The algebra $B(\mathbb{V})$ or equivalently $U_v^+(A(0,2)^{(4)})$ is generated by real root vectors X_n, Y_n for $n > 0$, imaginary root vectors M_{2n+1}, Z_{4n+2}, $\mathscr{L}_{4n+2}$ for $n \geq 0$,*

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$\mathscr{L}_{4n}$ for $n > 0$, with the following defining relations.

I:

Z_{4n+2}, $\mathscr{L}_{4n}$ commute with all the imaginary root vectors, $[\mathscr{L}_{4m+2}, \mathscr{L}_{4n+2}] = 0, \forall m, n \geq 0$.

$[\mathscr{L}_{4m+2}, M_{2n+1}] = \dfrac{[2m+1]_q}{2^{2m-1}} M_{4m+2n+3}$,

$[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2 M_{m+n+1}^2$ if $m + n$ is even, $[M_{2n+1}, M_{2m+1}] = 0$ if $m + n$ is odd,

$2\theta i \mathbb{Z}(u) = \tan(2\theta i \mathcal{Z}(u))$.

II:

$$X_k \mathbb{M} = \mathbb{M}_{(1)}(-1)^{d(\mathbb{M}_{(1)}) \cdot (k-1)} f_{X_k}(\mathbb{M}_{(2)}),$$
$$\mathbb{M} Y_k = (-1)^{k \cdot d(\mathbb{M}_{(2)})} g_{Y_k}(\mathbb{M}_{(1)}) \mathbb{M}_{(2)},$$

where
$$\mathbb{M}_{(1)} \otimes \mathbb{M}_{(2)} = (\mathbb{M} \otimes 1 + 1 \otimes \mathbb{M}) \dfrac{1}{1 - 2\theta \mathbb{M} \otimes \mathbb{M}},$$

$$[X_k, Z_{4n+2}] = 2^{2n} \dfrac{[2n+1]_q}{2n+1} X_{4n+k+2},$$
$$[Z_{4n+2}, Y_k] = 2^{2n} \dfrac{[2n+1]_q}{2n+1} Y_{4n+k+2},$$

$$X_k \overline{\mathfrak{L}} = \overline{\mathfrak{L}}_{(1)}(-1)^{d(\overline{\mathfrak{L}}_{(1)}) \cdot (k-1)} f_{X_k}(\overline{\mathfrak{L}}_{(2)}),$$

and
$$\overline{\mathfrak{L}} Y_k = (-1)^{k \cdot d(\overline{\mathfrak{L}}_{(2)})} g_{Y_k}(\overline{\mathfrak{L}}_{(1)}) \overline{\mathfrak{L}}_{(2)},$$

where
$$\overline{\mathfrak{L}}_{(1)} \otimes \overline{\mathfrak{L}}_{(2)} := \Delta(\overline{\mathfrak{L}}) = \overline{\mathfrak{L}} \otimes 1 + 1 \otimes \overline{\mathfrak{L}} + 2 \Big(\mathfrak{M}(au) + \mathfrak{M}(bu) \Big)$$

and
$$\mathfrak{M}(u) = \dfrac{\Big(-2\theta \mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{I}} + 2\theta \mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{II}} \Big) \Big(1 + 2\theta \mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{II}} + 2\theta \mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{I}} \Big)}{1 + (2\theta)^2 \mathbb{Z} \otimes \mathbb{Z}},$$

$$[X_k, \mathscr{L}_{4n}] = [4n]_q X_{4n+k},$$
$$[\mathscr{L}_{4n}, Y_k] = [4n]_q Y_{4n+k},$$

f, g are defined in Subsection 6.2.

III:

$$[X_{2i+1}, Y_{2n+2-(2i+1)}] = L_{2n+1}, \ [X_{2i}, Y_{2n+2-2i}] = L'_{2n+1},$$
$$[X_{2i+1}, Y_{2n+3-(2i+1)}] = L_{2n+2}, \ [X_{2i}, Y_{2n+3-2i}] = L'_{2n+2},$$

together with ‡ in Prop. 7.3.2, and † in Prop. 7.3.1.

Chapter 8   A more precise estimation of the comultiplication of the Drinfeld generators and one-sided coideal subalgebras

In this chapter we introduce more results related to the comultiplication of imaginary root vectors of affine Nichols algebras $B(\mathbb{U})$ and $B(\mathbb{V})$.

§8.1 $B(\mathbb{U})$

First we look at the Nichols algebra $B(\mathbb{U})$, where $\mathbb{U}$ is a braided vector space of diagonal type with the braiding matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & q \end{pmatrix}$$

with respect to a basis $\{x_1, x_2\}$ and q is not a root of unity. It is easy to see that $B(\mathbb{U})$ if of Cartan type with the Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Suppose x_1 has degree α_1, x_2 has degree α_2. Then it is well known that the real roots of $B(\mathbb{U})$ are $(n+1)\alpha_1 + n\alpha_2$ and $n\alpha_1 + (n+1)\alpha_2$ for $n \geq 0$, the imaginary roots are $n\delta$ for $n > 0$. The root multiplicities are all 1.

Then we construct all the root vectors of $B(\mathbb{U})$. For convenience we denote x_1, x_2 by X_1 and Y_1, and all the brackets below are the braided brackets. Define

$$L_1 = [X_1, Y_1], \ X_{n+1} = [X_n, L_1], \ Y_{n+1} = [L_1, Y_n],$$

define

$$L_n = [X_1, Y_n].$$

Then $\{X_n, Y_n, L_n\}$ are the root vectors in terms of Lyndon word theory, they are also used as generators in the Drinfeld second realization of quantum $\hat{sl}_2$. Recall that L_n commutes with each other. Moreover, we have the following lexicographic order induced by $x_1 < x_2$:

$$X_1 < X_2 < \cdots < X_n < \cdots \tag{8.1}$$

$$< \cdots < L_n < \cdots < L_2 < L_1$$
$$< \cdots < Y_n < \cdots < Y_2 < Y_1.$$

Denote the subalgebra generated by L_n for $n > 0$ by $\mathcal{L}$, denote the subalgebras generated by X_n and Y_n for $n > 0$ by $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then we have

$$B(\mathbb{U}) \cong \mathcal{Y} \otimes \mathcal{L} \otimes \mathcal{X}$$

as linear spaces. Recall that $\mathcal{L}$ is just the subquotient $K_{\geq 1}/K_{>1}$, with a braided Hopf algebra structure induced by the braided Hopf algebra structure of $B(\mathbb{U})$.

Moreover, comparing Prop. 2.2.39 in Lyndon word theory, we have the following more precise estimation of the comultiplication of all the root vectors.

Proposition 8.1.1 *In $B(\mathbb{U})$ we have the following descriptions on the comultiplication,*

(a) $\Delta(X_n) \in X_n \otimes 1 + 1 \otimes X_n + \mathcal{X} \otimes \mathcal{Y}\mathcal{L}$,

(b) $\Delta(Y_n) \in Y_n \otimes 1 + 1 \otimes Y_n + \mathcal{L}\mathcal{X} \otimes \mathcal{Y}$,

(c) $\Delta(L_n) \in \sum_{i=0}^{n} \theta L_i \otimes L_{n-i} + \mathcal{L}\mathcal{X} \otimes \mathcal{Y}\mathcal{L}$.

Proof: One can verify these formulas by induction. $\square$

Let $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$, define $\bar{L}_n$ for all $n \geq 1$ using the equation

$$\sum_{n=0}^{\infty} \theta L_n u^n = \exp(\sum_{n=1}^{\infty} \theta \bar{L}_n u^n).$$

Rewriting the relations in the Drinfeld second realization, we have

$$[X_m, \bar{L}_n] = \frac{[n]_q}{n} X_{m+n}, \tag{8.2}$$

$$[\bar{L}_n, Y_m] = \frac{[n]_q}{n} Y_{n+m}. \tag{8.3}$$

Moreover, each $\bar{L}_n$ is primitive in $\mathcal{L}$ (not the full Nichols algebra).

For a homogeneous element $T \in \mathcal{L}$, define a function d by

$$\deg T = d(T)\delta.$$

Lemma 8.1.2 *$\bar{L}_n$ is the only primitive element in $\mathcal{L}$ at the root space of $n\delta$, up to a scale.*

Proof: Indeed $\mathcal{L}$ is the subalgebra generated by all $\bar{L}_n$, and note that all $\bar{L}_n$ commute with each other. Then $\mathcal{L}$ is isomorphic to the universal enveloping algebra of an abelian Lie algebra, then the primitive elements are exactly $\bar{L}_n$ and their linear combination. However we will use the technique of the following proof frequently later, so we write down it. In general, $w \in \mathcal{L}$ is a linear combination of elements in the form of $\bar{L}_1^{k_1} \bar{L}_2^{k_2} \cdots \bar{L}_s^{k_s}$, for $s \in \mathbb{N}$, each $k_i \geq 0$ and $d(\bar{L}_1^{k_1} \bar{L}_2^{k_2} \cdots \bar{L}_s^{k_s}) = d(w)$. Suppose $d(w) = n$ and w is not $\bar{L}_n$ up to a scale. Then we have

$$w - \lambda \bar{L}_n = w_0 \cdot \bar{L}_s^{k_s} + \text{other terms},$$

where λ is some constant, w_0 is not 0, "other terms" are in the form of $\bar{L}_1^{k_1'} \bar{L}_2^{k_2'} \cdots \bar{L}_s^{k_s'}$ and $k_s' < k_s$. Then we look at the comultiplication in $\mathcal{L}$, if w_0 is not a constant, we get the term $w_0 \otimes \bar{L}_s^{k_s}$. Due to the existence of this term and the maximality of k_s, we obtain a contradiction. Then w must be a scale of $\bar{L}_n$. If w_0 is a constant, then due to the existence of the term $\bar{L}_s \otimes \bar{L}_s^{k_s-1}$, we get the same result. $\qquad \square$

Lemma 8.1.3 *Suppose $T \in \mathcal{L}$ and T is not a constant, then we have $[X_1, T] \in \mathcal{LX}$ and $[X_1, T] \notin \mathcal{L}$.*

Proof: We prove by induction on $d(T)$. When $d(T) = 1$, we have $T = \lambda L_1$ for $\lambda \neq 0$, $[X_1, T] = \lambda X_2$. Suppose this proposition holds for T s.t. $d(T) \leq k$, consider T s.t. $d(T) = k + 1$. If T is primitive in $\mathcal{L}$, then T must be $\lambda \bar{L}_{d(T)}$ and obviously $[X_1, T] = \lambda \frac{[d(T)]_q}{d(T)} X_{1+d(T)}$. If T is not primitive in $\mathcal{L}$, then we have $T_{(1)} \otimes T_{(2)} - T \otimes 1 - 1 \otimes T \neq 0$, where $T_{(1)} \otimes T_{(2)}$ means the comultiplication of $\mathcal{L}$ as a braided Hopf algebra. Now suppose $[X_1, T] = 0$, we look at its comultiplication, $\Delta([X_1, T])$ should also be 0. By Prop. 8.1.1, we have that

$$\Delta(T) \in T_{(1)} \otimes T_{(2)} + \mathcal{LX} \otimes \mathcal{YL},$$

then

$$\Delta([X_1, T]) \in [X_1 \otimes 1 + 1 \otimes X_1, T_{(1)} \otimes T_{(2)} + \mathcal{LX} \otimes \mathcal{YL}],$$
$$\in T_{(1)} \otimes [X_1, T_{(2)}] + \mathcal{LX} \otimes B(\mathbb{U}).$$

Since $T_{(1)} \otimes T_{(2)} - T \otimes 1 - 1 \otimes T \neq 0$, by induction hyperthesis we have $\Delta([X_1, T]) \neq 0$, contradiction. Then $[X_1, T]$ must be non-zero. It is obvious that $[X_1, T] \in \mathcal{LX}$. Moreover, since it is non-zero, $[X_1, T] \notin \mathcal{L}$ holds for degree reason. $\qquad \square$

Lemma 8.1.4 *For a homogeneous element $w \in \mathcal{YL}$, if $[X_1, w] \in \mathcal{YL}$, then we have $w \in \mathcal{Y}$.*

Proof: If not, we can write w as the following.

$$w = \sum_{i=1}^{k_0} a_i + \sum_{i=k_0+1}^{k} a_i b_i,$$

where $k \in \mathbb{N}$, $0 \le k_0 \le k-1$, all the a_i for $1 \le i \le k$ are in $\mathcal{Y}$ and are different PBW type elements, and for $k_0 + 1 \le i \le k$, all the $b_i \in \mathcal{L}$ and are not constants. Then

$$
\begin{aligned}
[X_1, w] &= [X_1, \sum_{i=1}^{k_0} a_i + \sum_{i=k_0+1}^{k} a_i b_i] \\
&= \sum_{i=1}^{k_0} [X_1, a_i] + \sum_{i=k_0+1}^{k} [X_1, a_i] b_i + \sum_{i=k_0+1}^{k} \chi(\alpha_1, \deg a_i) a_i [X_1, b_i] \\
&\in \mathcal{YL} + \sum_{i=k_0+1}^{k} \chi(\alpha_1, \deg a_i) a_i [X_1, b_i]
\end{aligned}
$$

Then using Lemma 8.1.3, we get that $[X_1, w] \in \mathcal{YLX}$ and $[X_1, w] \notin \mathcal{YL}$. $\qquad \square$

Proposition 8.1.5 *In $B(\mathbb{U})$ we have*

$$\Delta(\bar{L}_n) \in \bar{L}_n \otimes 1 + 1 \otimes \bar{L}_n + \mathcal{X} \otimes \mathcal{Y}.$$

Proof: It is easy to see that

$$\Delta(\bar{L}_n) \in \bar{L}_n \otimes 1 + 1 \otimes \bar{L}_n + \mathcal{LX} \otimes \mathcal{YL}.$$

For convenience we denote

$$w = \Delta(\bar{L}_n) - \bar{L}_n \otimes 1 - 1 \otimes \bar{L}_n$$

First we are going to prove that

$$w \in \mathcal{LX} \otimes \mathcal{Y}.$$

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If not, we can assume that

$$w = \sum_{i=1}^{k_0} a_i \otimes b_i + \sum_{i=k_0+1}^{k} a_i \otimes b_i,$$

where $k \in \mathbb{N}$, $0 \leq k_0 \leq k-1$, all the a_i for $1 \leq i \leq k$ are in $\mathcal{LX}$ and are different PBW type elements, all the b_i for $1 \leq i \leq k_0$ are in $\mathcal{Y}$, all the b_i for $k_0+1 \leq i \leq k$ are in $\mathcal{YL}$ but not in $\mathcal{Y}$.

Then we take the comultiplication of (8.2).

On the one hand, we have

$$\Delta(\frac{[n]_q}{n}X_{n+1}) = [X_1 \otimes 1 + 1 \otimes X_1, \bar{L}_n \otimes 1 + 1 \otimes \bar{L}_n + w]$$
$$= \frac{[n]_q}{n}X_{n+1} \otimes 1 + 1 \otimes \frac{[n]_q}{n}X_{n+1} + [X_1 \otimes 1, w] + [1 \otimes X_1, w]$$
$$\in \frac{[n]_q}{n}X_{n+1} \otimes 1 + 1 \otimes \frac{[n]_q}{n}X_{n+1} + \mathcal{LX} \otimes \mathcal{YL} + [1 \otimes X_1, w]$$

Then we analyze the term $[1 \otimes X_1, w]$

$$[1 \otimes X_1, w] = [1 \otimes X_1, \sum_{i=1}^{k_0} a_i \otimes b_i] + [1 \otimes X_1, \sum_{i=k_0+1}^{k} a_i \otimes b_i]$$
$$\in \mathcal{LX} \otimes \mathcal{YL} + \sum_{i=k_0+1}^{k} \chi(\alpha_1, \deg a_i)a_i \otimes [X_1, b_i]$$

By Lemma 8.1.4, for each $i \in \{k_0+1, \cdots k\}$, we have $[X_1, b_i] \in \mathcal{YLX}$ and $[X_1, b_i] \notin \mathcal{YL}$.

On the other hand, by Prop. 8.1.1,

$$\Delta(\frac{[n]_q}{n}X_{n+1}) \in \frac{[n]_q}{n}X_{n+1} \otimes 1 + 1 \otimes \frac{[n]_q}{n}X_{n+1} + \mathcal{X} \otimes \mathcal{YL}.$$

Then we obtain a contradiction. This means we have proved that

$$\Delta(\bar{L}_n) \in \bar{L}_n \otimes 1 + 1 \otimes \bar{L}_n + \mathcal{LX} \otimes \mathcal{Y}.$$

Similarly by the same argument based on the comultiplication of (8.3) we can get that

$$\Delta(\bar{L}_n) \in \bar{L}_n \otimes 1 + 1 \otimes \bar{L}_n + \mathcal{X} \otimes \mathcal{LY}.$$

Then we must have

$$\Delta(\bar{L}_n) \in \bar{L}_n \otimes 1 + 1 \otimes \bar{L}_n + \mathcal{X} \otimes \mathcal{Y}.$$

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$\square$

Remark 8.1.6 *Essentially the conclusion of Prop. 8.1.5 is not new. It appeared in the famous paper* [Dam98]. *The similar estimation was done in the whole Hopf algebra, not in the braided Hopf algebra (i.e. the positive part). The difference is very slight. We write down the proof since in the next section we will use the same idea and technique for the positive part of* $U_v(A(0,2)^{(4)})$, *i.e. the Nichols algebra* $B(\mathbb{V})$.

Proposition 8.1.7 *We obtain a series of left coideal subalgebras* $\mathcal{P}_n$ *and right coidel subalgebras* $\mathcal{Q}_n$ *corresponding to* $n\delta$. $\mathcal{P}_n$ *is the subalgebra generated by* $\{\bar{L}_n, Y_1, Y_2, \cdots, Y_n\}$, $\mathcal{Q}_n$ *is the subalgebra generated by* $\{\bar{L}_n, X_1, X_2, \cdots, X_n\}$. *Moreover they are minimal left (right) coideal subalgebras who are* $\mathbb{N}^2$-*graded and intersect the subalgebra generated by all the imaginary root vectors, i.e.,* $\mathcal{L}$ *nontrivially.*

Proof: By Prop. 8.1.1 (b), Prop. 8.1.5, it is easy to see that the subalgebra generated by $\{\bar{L}_n, Y_1, Y_2, \cdots, Y_n\}$ is a left coideal subalgebra. Moreover, by Prop. 8.1.5, $\Delta(\bar{L}_n)$ must contain a term in $\mathbb{C}X_n \otimes Y_1$, and since $\bar{L}_n$ is not zero, this term must also be nonzero. Meanwhile, note that $\Delta(Y_n)$ contains the term $\lambda L_1 \otimes Y_{n-1}$ for $\lambda \in \mathbb{C}^\times$. Then $\mathcal{P}_n$ is minimal. $\square$

Proposition 8.1.8 $\{\bar{L}_n, Y_1, Y_2, \cdots, Y_n, \cdots\}$ $(\{\bar{L}_n, X_1, X_2, \cdots, X_n, \cdots\})$ *is a set of PBW generators of* $\mathcal{P}_n$ $(\mathcal{Q}_n)$, *with respect to the following order:*

$$Y_1 > Y_2 > \cdots > Y_n > \cdots > \bar{L}_n > \cdots > X_n > \cdots > X_2 > X_1.$$

Proof: By (8.2), (8.3) and the fact that they are PBW generators of $B(\mathbb{U})$. $\square$

§8.2 $B(\mathbb{V})$

we recall the Nichols algebra $B(\mathbb{V})$, where $\mathbb{V}$ is a braided vector space of diagonal type with the braiding matrix

$$\begin{pmatrix} q & q^{-1} \\ q^{-1} & -q \end{pmatrix},$$

with respect to a basis x_1, x_2 and q is not a root of unity. Note that $B(\mathbb{V})$ is also of Cartan type with the Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

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Then we get that $B(\mathbb{V})$ has the same root system as $B(\mathbb{U})$.

Recall that in $B(\mathbb{V})$ we still define X_n, Y_n, L_n in the same way as $B(\mathbb{U})$. Still we denote the subalgebras generated by X_n, L_n, Y_n for $n \geq 1$ by $\mathcal{X}$, $\mathcal{L}$, $\mathcal{Y}$, respectively. Then we have

$$B(\mathbb{V}) \cong \mathcal{Y} \otimes \mathcal{L} \otimes \mathcal{X},$$

as linear spaces. $\mathcal{L}$ is still the same as the subquotient $K_{\geq 1}/K_{>1}$, with a braided Hopf algebra structure induced by the braided Hopf algebra structure of $B(\mathbb{V})$.

Similarly we have the following description on the comultiplication.

Proposition 8.2.1 *In $B(\mathbb{V})$ we have the following descriptions on the comultiplication,*

(a) $\Delta(X_n) \in X_n \otimes 1 + 1 \otimes X_n + \mathcal{X} \otimes \mathcal{Y}\mathcal{L}$,

(b) $\Delta(Y_n) \in Y_n \otimes 1 + 1 \otimes Y_n + \mathcal{L}\mathcal{X} \otimes \mathcal{Y}$,

(c) $\Delta(L_n) \in \sum_{i=0}^{n} \theta L_i \otimes L_{n-i} + \mathcal{L}\mathcal{X} \otimes \mathcal{Y}\mathcal{L}$.

Proof: One can verify these formulas by induction. It is a little more complicated then the case of Prop. 8.1.1, but still routine. $\square$

Additionally, we define M_{2n+1} by

$$M_1 = L_1, \text{ and } M_{2n+1} = [L_2, M_{2n-1}] \text{ for } n \geq 1.$$

Recall that we have the following order induced by $x_1 < x_2$:

$$X_1 < X_2 < \cdots < X_n < \cdots$$
$$< \cdots < L_{2n} < L_{2n-2} < L_{2n-4} < \cdots < L_2$$
$$< \cdots < M_{2n-1} < M_{2n-3} < \cdots < M_5 < M_3 < M_1$$
$$< \cdots < Y_n < \cdots < Y_2 < Y_1.$$

The set of decreasing ordered products of elements in

$$\{X_n, Y_n, L_{2n}, M_{2n-1}, n \in \mathbb{N}\}$$

forms a PBW basis of $B(\mathbb{V})$.

Instead of using L_{2n} and M_{2n+1}^2, we recall the following alternative choice of root vectors. They have more manageable commutation relations and comultiplication

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formulas in $\mathcal{L}$. Define L'_n by $[X_2, Y_{n-1}]$. Define $\tilde{L}_n = \frac{1}{2}(L_n + L'_n)$, $L_0 = L'_0 = \tilde{L}_0 = \frac{1}{\theta}$. Define $\bar{L}_{2n}$ by the following formal power series:

$$\theta \sum_{n=0}^{\infty} \tilde{L}_{2n} u^{2n} = \exp(\theta \sum_{n=1}^{\infty} \bar{L}_{2n} u^{2n}).$$

Recall the following two generating functions

$$\mathbb{M}(u) = \sum_{n=0}^{\infty} M_{2n+1} u^{2n+1},$$

$$\mathbb{Z}(u) = \sum_{n=0}^{\infty} M_{2n+1}^2 (-1)^n u^{4n+2} = \mathbb{M}(u)^2.$$

We have

$$\Delta(\mathbb{M}(u)) = \Big(\mathbb{M}(u) \otimes 1 + 1 \otimes \mathbb{M}(u)\Big) \frac{1}{1 - 2\theta\mathbb{M}(u) \otimes \mathbb{M}(u)},$$

$$\Delta(\mathbb{Z}(u)) = \Big(\mathbb{Z}(u) \otimes 1 + 1 \otimes \mathbb{Z}(u)\Big) \frac{1}{1 + (2\theta^2)\mathbb{Z}(u) \otimes \mathbb{Z}(u)}.$$

Let $\mathbb{O}_{2n+1}$ be the subalgebra generated by $\{M_{2i+1}, 0 \leq i \leq n\}$. Then we can get

$$\Delta(M_{2n+1}) \in M_{2n+1} \otimes 1 + 1 \otimes M_{2n+1} + 2\theta L_1^2 \otimes M_{2n-1} - 2\theta M_{2n-1} \otimes L_1^2$$
$$+ \mathbb{O}_{2n-3} \otimes \mathbb{O}_{2n-3}. \tag{8.4}$$

Define $\mathcal{Z}(u)$ by the following power series

$$\arctan(2\theta i \mathbb{Z}(u)) = 2\theta i \mathcal{Z}(u).$$

We denote $\mathcal{Z}(u)$ by $\sum_{n=0}^{\infty} Z_{4n+2} u^{4n+2}$. Then we have

$$\Delta(Z_{4n+2}) = Z_{4n+2} \otimes 1 + 1 \otimes Z_{4n+2}. \tag{8.5}$$

Define $\mathring{L}_{4n}$ by the following, let $a = \sqrt{\frac{q}{2}}$, $b = i\sqrt{\frac{q-1}{2}}$,

$$\sum_{n=1}^{\infty} 2\theta \mathring{L}_{4n} u^{4n} = \sum_{n=1}^{\infty} 2\theta \bar{L}_{4n} u^{4n} + \frac{1}{2}\log\left(1 - (2\theta)^2 \mathbb{Z}(a)^2\right)$$
$$+ \frac{1}{2}\log\left(1 - (2\theta)^2 \mathbb{Z}(b)^2\right).$$

Then
$$\Delta(\mathring{L}_{4n}) = \mathring{L}_{4n} \otimes 1 + 1 \otimes \mathring{L}_{4n}. \tag{8.6}$$

Define
$$\mathbb{M}_{\mathrm{I}}(u) = \sum_{n=0}^{\infty} M_{4n+1} u^{4n+1}, \mathbb{M}_{\mathrm{II}}(u) = \sum_{n=0}^{\infty} M_{4n+3} u^{4n+3}.$$

, sometimes we just use $\mathbb{M}_{\mathrm{I}}$ and $\mathbb{M}_{\mathrm{II}}$ for convenience. Define
$$\overline{\mathfrak{L}} = \sum_{n=0}^{\infty} \theta(4n+2) \bar{L}_{4n+2} u^{4n+2}.$$

To determine the comultiplication of $\bar{L}_{4n+2}$, we need to use generating functions. In the subquotient $K_{\geq 1}/K_{>1}$, we have
$$\Delta(\overline{\mathfrak{L}}) = \overline{\mathfrak{L}} \otimes 1 + 1 \otimes \overline{\mathfrak{L}} + 2\Big(\mathfrak{M}(au) + \mathfrak{M}(bu)\Big) \tag{8.7}$$

where
$$\mathfrak{M}(u) = \frac{\Big(-2\theta\mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{I}} + 2\theta\mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{II}}\Big)\Big(1 + 2\theta\mathbb{M}_{\mathrm{I}} \otimes \mathbb{M}_{\mathrm{II}} + 2\theta\mathbb{M}_{\mathrm{II}} \otimes \mathbb{M}_{\mathrm{I}}\Big)}{1 + (2\theta)^2 \mathbb{Z} \otimes \mathbb{Z}}.$$

In the case of $B(\mathbb{V})$, we need a more delicate description on the subalgebra $\mathcal{L}$.

Denote the subalgebra generated by $\mathring{L}_{4n}$ and $\bar{L}_{4n+2}$ for $n \geq 1$ by $\mathcal{L}_0$, note that it is commutative. Denote the subalgebra generated by M_{2n-1} for $n \geq 1$ by $\mathcal{M}$. Then we have the following linear spaces isomorphism:
$$\mathcal{L} \cong \mathcal{M} \otimes \mathcal{L}_0.$$

Note that by (8.7), we get that
$$\Delta(\bar{L}_{4n+2}) - \bar{L}_{4n+2} \otimes 1 - 1 \otimes \bar{L}_{4n+2} \in \mathcal{M} \otimes \mathcal{M}. \tag{8.8}$$

Denote the linear space spanned by the set
$$\{M_1^{k_1} M_3^{k_3} \cdots M_{2n+1}^{k_{2n+1}} | \text{ for some } n \geq 0 \text{ and } k_1, k_3, \cdots, k_{2n+1} \in \{0, 1\}\}$$

by $\mathcal{M}_M$. Denote the subalgebra generated by Z_{4n+2} for $n \geq 0$ by $\mathcal{M}_Z$. Then we have
$$\mathcal{L} \cong \mathcal{M}_M \otimes \mathcal{M}_Z \otimes \mathcal{L}_0.$$

We also recall the following commutation relations in $\mathcal{L}$.

(a) if m+n is even, then $[M_{2m+1}, M_{2n+1}] = (-1)^{\frac{m-n}{2}} 2M_{m+n+1}^2$, if $m+n$ is odd, then $[M_{2n+1}, M_{2m+1}] = 0$ if $m+n$ is odd,

(b) Z_{4n+2} and $\mathring{L}_{4n}$ lie in the center of $\mathcal{L}$.

(C) $[\bar{L}_{4n+2}, \bar{L}_{4m+2}] = 0$.

(d) $[\bar{L}_{4n+2}, M_{2m+1}] = \frac{1}{2^{2n}} \frac{[2n+1]_q}{2n+1} M_{4n+2m+3}$.

Then we are going to determine all the homogeneous primitive elements in $\mathcal{L}$, corresponding to each $n\delta$.

Proposition 8.2.2 *In the braided Hopf algebra $\mathcal{L}$, the only homogeneous primitive elements are $\mathring{L}_{4n}$ and Z_{4n-2} for $n \geq 1$.*

Proof: Suppose $w \in \mathcal{L}$ is homogeneous and primitive. Denote the natural map $\mathcal{M} \oplus \mathcal{L}_0 \to \mathcal{L}_0$ by π. Similar to Lemma 8.1.2, by (8.6) and (8.8), it is easy to see $w \in \mathcal{M} \oplus \mathcal{L}_0$. Moreover, $\pi(w) \in \mathbb{C}\mathring{L}_{4n}$ or $\mathbb{C}\bar{L}_{4n-2}$ for some $n \geq 1$.

Then we are going to determine w at every $n\delta$.

$(2n+1)\delta$: At first, we know that $L_1 = M_1$ is primitive in $\mathcal{L}$. Then we consider the case that $d(w) \geq 3$. For degree reason, we have $w \in \mathcal{M} \cong \mathcal{M}_M \otimes \mathcal{M}_Z$. Then by a similar argument to the proof of Lemma 8.1.2, we have

$$w \in \lambda M_{2n+1} + \mathcal{M}_M \cdot \mathcal{M}_Z, \text{ where } \lambda \in \mathbb{C}^\times.$$

Recall that

$$\Delta(M_{2n+1}) \in M_{2n+1} \otimes 1 + 1 \otimes M_{2n+1} + 2\theta L_1^2 \otimes M_{2n-1} - 2\theta M_{2n-1} \otimes L_1^2$$
$$+ \mathbb{O}_{2n-3} \otimes \mathbb{O}_{2n-3},$$

meanwhile, note that $Z_2 = L_1^2$, $\bar{\Delta}(M_{2n-1}Z_2)$ contains

$$Z_2 \otimes M_{2n-1} + M_{2n-1} \otimes Z_2,$$

then w can not be primitive. Then we get that at $(2n+1)\delta$, the only primitive element is L_1, up to a scale.

$4n\delta$: Since $\mathring{L}_{4n}$ is primitive, then it is equivalent to determine primitive elements in $\mathcal{M}$ at every $4n\delta$. By a similar argument to the proof of Lemma 8.1.2, it is impossible to find them.

$(4n + 2)\delta$: There are two possibilities, $w \in \mathcal{M}$ or

$$w = \lambda \bar{L}_{4n+2} + w_0 \text{ where } w_0 \in \mathcal{L}_1 \text{ and } \lambda \neq 0.$$

If $w \in \mathcal{M}$, sill by a similar argument to the proof of Lemma 8.1.2, $w = Z_{4n+2}$. Otherwise, consider $[w, L_1]$. Since $\Delta(w) = w \otimes 1 + 1 \otimes w$, $\Delta(L_1) = L_1 \otimes 1 + 1 \otimes L_1$. Then we have $[w, L_1]$ is primitive. Meanwhile,

$$[w, L_1] = \lambda \frac{1}{2^{2n}} \frac{[2n + 1]_q}{2n + 1} M_{4n+3} + [w_0, L_1].$$

By the result of $(2n + 1)\delta$, $[w, L_1]$ can not be primitive, contradiction. Then Z_{4n+2} is the only primitive element at $(4n + 2)\delta$. $\square$

Then we have the following description on the comultiplication of root vectors at $2n\delta$.

Proposition 8.2.3 *In $B(\mathbb{V})$ we have*

$$\Delta(Z_{4n+2}) \in Z_{4n+2} \otimes 1 + 1 \otimes Z_{4n+2} + \mathcal{X} \otimes \mathcal{Y},$$
$$\Delta(\mathring{L}_{4n}) \in \mathring{L}_{4n} \otimes 1 + 1 \otimes \mathring{L}_{4n} + \mathcal{X} \otimes \mathcal{Y}.$$

Proof: Note that in Prop. 8.2.2 we have determined all the primitive elements at $2n\delta$ for $n \geq 1$. By the same arguments to Lemma 8.1.3, Lemma 8.1.4 and Prop. 8.1.5, we can get these results. $\square$

Recall that we have the following commutation relations:

$$[X_k, \mathring{L}_{4n}] = \frac{[4n]_q}{4n} X_{4n+k}, \tag{8.9}$$

$$[\mathring{L}_{4n}, Y_k] = \frac{[4n]_q}{4n} Y_{4n+k}, \tag{8.10}$$

$$[X_k, Z_{4n+2}] = 2^{2n} \frac{[2n + 1]_q}{2n + 1} X_{4n+k+2}, \tag{8.11}$$

$$[Z_{4n+2}, Y_k] = 2^{2n} \frac{[2n + 1]_q}{2n + 1} Y_{4n+k+2}. \tag{8.12}$$

Proposition 8.2.4 *We obtain a series of left (right) coideal subalgebras $\mathcal{P}_{2n}$ $(\mathcal{Q}_{2n})$ corresponding to $2n\delta$.*

$\mathcal{P}_{4n+2}$ $(\mathcal{Q}_{4n+2})$ is the subalgebra generated by the set

$$\{Z_{4n+2}, Y_1, Y_2, \cdots, Y_{4n+2}\} \ (\{Z_{4n+2}, X_1, X_2, \cdots, X_{4n+2}\}).$$

$\mathcal{P}_{4n}$ ($\mathcal{Q}_{4n}$) is the subalgebra generated by the set

$$\{\mathring{L}_{4n}, Y_1, Y_2, \cdots, Y_{4n}\} \ (\{\mathring{L}_{4n}, X_1, X_2, \cdots, X_{4n}\}).$$

Moreover they are minimal left (right) coideal subalgebras who intersect the subalgebra generated by all the imaginary root vectors, i.e., $\mathcal{L}$ nontrivially.

Proof: Similar to the proof of Corollary 8.1.7. $\qquad\qquad\square$

Proposition 8.2.5 $\{\mathring{L}_{4n}, Y_1, Y_2, \cdots, Y_n, \cdots\}$ ($\{\mathring{L}_{4n}, X_1, X_2, \cdots, X_n, \cdots\}$) is a set of PBW generators of $\mathcal{P}_{4n}$ ($\mathcal{Q}_{4n}$), with the following order:

$$Y_1 > Y_2 > \cdots > Y_n > \cdots > \mathring{L}_{4n} > \cdots > X_n > \cdots > X_2 > X_1.$$

$\{Z_{4n+2}, Y_1, Y_2, \cdots, Y_n, \cdots\}$ ($\{Z_{4n+2}, X_1, X_2, \cdots, X_n, \cdots\}$) is a set of PBW generators of $\mathcal{P}_{4n+2}$ ($\mathcal{Q}_{4n+2}$), with the following order:

$$Y_1 > Y_2 > \cdots > Y_n > \cdots > Z_{4n+2} > \cdots > X_n > \cdots > X_2 > X_1.$$

Proof: By (8.9), (8.10), (8.11), (8.12) and the fact that they are PBW generators of $B(\mathbb{V})$. $\qquad\qquad\square$

For the root vector of $(2n+1)\delta$ and the other root vector of $(4n+2)\delta$, there exist some "bigger" one-sided coideal subalgebras corresponding to them.

Proposition 8.2.6 In $B(\mathbb{V})$ we have

$$\Delta(M_{2n+1}) = M_{2n+1} \otimes 1 + 1 \otimes M_{2n+1} + \mathcal{M}\mathcal{X} \otimes \mathcal{Y}\mathcal{M}.$$

Proof: This follows from an induction proof based on the formulas of $\Delta(L_2)$ and $\Delta(L_1)$. $\qquad\qquad\square$

Proposition 8.2.7 In $B(\mathbb{V})$ we have

$$\Delta(\bar{L}_{4n+2}) \in \bar{L}_{4n+2} \otimes 1 + 1 \otimes \bar{L}_{4n+2} + \mathcal{M} \otimes \mathcal{M} + \mathcal{M}\mathcal{X} \otimes \mathcal{Y}\mathcal{M}.$$

Proof: By (8.7), we know that

$$\Delta(\bar{L}_{4n+2}) \in \bar{L}_{4n+2} \otimes 1 + 1 \otimes \bar{L}_{4n+2} + \mathcal{M} \otimes \mathcal{M} + \mathcal{L}\mathcal{X} \otimes \mathcal{Y}\mathcal{L}.$$

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Then we need to give a more precise estimation on the last term. If not, regarding Prop. 8.2.6 and

$$[\bar{L}_{4n+2}, L_1] = \frac{1}{2^{2n}} \frac{[2n+1]_q}{2n+1} M_{4n+3}, \tag{8.13}$$

it is easy to obtain a contradiction by comparing the comultiplication of both sides of (8.13). □

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