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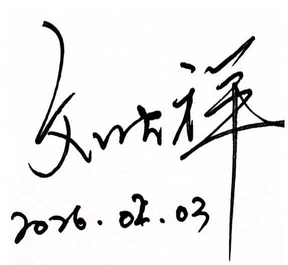

Dissertation for the Doctoral Degree of Science

On the semisimple Hopf algebras $D(m, d, \gamma)/(y)$

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Fundamental Mathematics

School of Mathematics, Nanjing University

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毕业论文题目: 关于半单 **Hopf** 代数 $D(m,d,\gamma)/(y)$ 的研究

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摘 要

本论文旨在研究一类半单 Hopf 代数的表示理论。设 m, d 为两个自然数且满足 $(1+m)d$ 为偶数。令有限群 G 为 $Z_{md}\times Z_m$，其生成元为 x, y，并满足生成关系 $x^{md}=1=y^m, xy=yx$。记 $\Bbbk G$ 为具有通常 Hopf 代数结构的群代数。记 $\Bbbk^G$ 为其对偶 Hopf 代数，即从 G 到 $\Bbbk$ 的函数代数。定义 Hopf 代数 $H(m,d)$ 为 $\Bbbk^G$ 和 $\Bbbk Z_2$ 的阿贝尔扩张类，其中 σ 和 τ 满足 $\sigma(x^iy^j,t,t)=(-1)^{\frac{-i^2-i-2j}{2}}=(-1)^{\frac{(i+2j)(i+2j-1)}{2}}$ 及 $\tau(x^ig^j,x^kg^l,t)=(-1)^{-jk}$. 则 $H(m,d)$ 是半单的 Hopf 代数。

同时，Wu–Liu–Ding 在 [Adv. Math. 296 (2016), 1 – 54] 中引入了 Hopf 代数 $D(m,d,\gamma)$，他们给出了 Gelfand–Kirillov 维数为 1 的素正则 Hopf 代数的完整分类，并构造了这类 Hopf 代数，商 Hopf 代数 $D(m,d,\gamma)/(y)$ 是半单的。Hopf 代数的分类吸引了广泛关注，而我们对半单 Hopf 代数的了解仍然有限。我们给出了半单 Hopf 代数 $D(m,d,\gamma)/(y)$ 正是阿贝尔扩张 $H(m,d)$ 这一猜想的证明。令 $m=2, d=2$, 相应的 $H(2,2)$ 正是 Kashina 在 [J. Algebra, 232(2000), p.617−663] 中定义的 $H_{c:\sigma_0}$。我们首先研究 $H(2,2)$ 上的所有单 Yetter–Drinfel'd 模 V。我们证明了存在 88 个互不同构的单的左 $\mathcal{D}(H(2,2))$-模，其中 32 个是一维的，其余 56 个是二维的。接着，我们分类了所有在 $H(2,2)$ 上的单 Yetter–Drinfel'd 模 V 所对应的有限维 Nichols 代数。我们发现其中存在由 Andruskiewitsch 和 Geraldi [8] 研究的非对角型 Nichols 代数。有限维的对角型 Nichols 代数为 A_1, A_2 或量子平面，而非对角型的为 8 维或 16 维。

关键词: Abelian 扩张，Yetter–Drinfel'd 模，Hopf 代数，Nichols 代数。

THESIS: On the semisimple Hopf algebras $D(m, d, \gamma)/(y)$

SPECIALIZATION: Fundamental Mathematics

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Let m, d be two natural numbers satisfying that $(1+m)d$ is even. Let G be a finite group $Z_{md} \times Z_m$ with generators x, y and generating relations $x^{md} = 1 = y^m, xy = yx$. We denote by $\Bbbk G$ the group algebra with the usual Hopf algebra structure. We denote by $\Bbbk^G$ its dual Hopf algebra, the algebra of functions from G to $\Bbbk$. Let Hopf algebra $H(m, d)$ be the class of Abelian extensions of $\Bbbk Z_2$ by $\Bbbk^G$ with $\sigma(x^i y^j, t, t) = (-1)^{\frac{-i^2-i-2j}{2}} = (-1)^{\frac{(i+2j)(i+2j-1)}{2}}$ and $\tau(x^i g^j, x^k g^l, t) = (-1)^{-jk}$. Then each $H(m, d)$ is a semisimple Hopf algebra.

Meanwhile, Hopf algebras $D(m, d, \gamma)$ were introduced by Wu–Liu–Ding in [Adv. Math. 296 (2016), 1−54], they gave a complete classification of prime regular Hopf algebras of GK-dimension one and constructed these Hopf algebras. The quotient Hopf algebras of $D(m, d, \gamma)/(y)$ are semisimple Hopf algebras. The classification of Hopf algebras have attracted a lot of attention and we still know very little about semisimple Hopf algebras, we would like to study the representations of semisimple Hopf algebras in this thesis. We proved that the semisimple Hopf algebras $D(m, d, \gamma)/(y)$ are exactly the Abelian extensions $H(m, d)$ and $H(2, 2)$ is exactly the Hopf algebra $H_{c:\sigma_0}$ of Kashina in [J. Algebra, 232(2000), pp.617 − 663] . We start with investigating all simple Yetter–Drinfel'd modules V over $H(2, 2)$. We proved that there are 88 non-isomorphic simple left $\mathcal{D}(H(2, 2))$-modules, and 32 of them are one-dimensional and the other 56 ones are two-dimensional. Then we classify all finite-dimensional Nichols algebras of simple Yetter–Drinfel'd modules V over $H(2, 2)$. We find the Nichols algebras of non-diagonal type which were studied by Andruskiewitsch and Geraldi [8] .The finite dimensional Nichols algebras of diagonal type are either A_1, A_2 or quantum planes, and non-diagonal type ones are 8 or 16 dimensional.

Keywords: Abelian extension, Yetter–Drinfel'd modules, Hopf algebras, Nichols algebras.

Chapter 1 Introduction

1.1 Backgrounds

Nichols algebras appear naturally in the classifications of pointed Hopf algebras [11] and have many applications in other fields such as conformal field theory [26], and Lie algebras [4]. Among them, Nichols algebras of diagonal type have attracted a lot of attention and are now well understood [5, 3, 7, 6, 14, 17, 16, 15, 21]. Furthermore, authors are also interested in finite-dimensional Nichols algebras of other types [21, 28, 32]. Meanwhile, the Drinfel'd double of a Hopf algebra was introduced by Drinfel'd in order to study the quasitriangular Hopf algebra. The representation theory of the Drinfel'd double of a Hopf algebra has been studied for many years. The relation between the category of the left-right Yetter–Drinfel'd modules over H and the category of left modules over the Drinfel'd double $\mathcal{D}(H)$ has been considered in the paper [20]. If H is a finite-dimensional Hopf algebra, then these two categories are equivalent.

The Hopf algebras $D(m, d, \gamma)$ was introduced by Wu–Liu–Ding [30], they gave a complete classification prime regular Hopf algebras of GK-dimension one and they constructed these Hopf algebras. Each quotient Hopf algebra $D(m, d, \gamma)/(y)$ is a semisimple Hopf algebra. The classification of Hopf algebras have attracted a lot of attention and we still know very little about semisimple Hopf algebras, we would like to study the representations V of these semisimple Hopf algebras. The finite-dimensional Nichols algebras of the Yetter–Drinfel'd modules V have many new cases, and one can obtain many interesting results, but the complete classification of finite-dimensional Nichols algebras are still under investigation and this will be my ongoing project in the future. And we also would like to consider the following questions:

(1) Our goal is to determine which of our low-dimensional semisimple Hopf algebras coincide with the known examples in the literature.

(2) Is it possible to classify all finite-dimensional Nichols algebras in this special case?

On the other hand, we have another class of semisimple Hopf algebras. Let m, d be two natural numbers satisfying that $(1+m)d$ is even. Let G be the finite groups $Z_{md} \times Z_m$ with generators x, y and generating relations

$$x^{md} = 1, \quad y^m = 1, \quad xy = yx.$$

We denote by $\Bbbk G$ the group algebra with the usual Hopf algebra structure. We denote by $\Bbbk^G$ its dual Hopf algebra, the algebra of functions from G to $\Bbbk$. Let Hopf algebra $H(m,d)$ be the class of Abelian extensions of $\Bbbk Z_2$ by $\Bbbk^G$ in the sense of Masuoka [23] with

$$\sigma(x^i y^j, t, t) = (-1)^{\frac{-i^2-i-2j}{2}} = (-1)^{\frac{(i+2j)(i+2j-1)}{2}}$$

and

$$\tau(x^i g^j, x^k g^l, t) = (-1)^{-jk}.$$

Then each $H(m,d)$ is the semisimple Hopf algebra. We have the following conjecture.

Conjecture 1.1.0.1 (Gongxiang Liu)**.** *Each semisimple Hopf algebra $D(m,d,\gamma)/(y)$ is exactly an Abelian extension $H(m,d)$.*

For question (1), we found an interesting result that when we take $m = 2 = d$, then $D(2,2,\gamma)/(y)$ is the 16-dimensional semisimple Hopf algebra $H_{c:\sigma_0}$ of Kashina [18] (see Table 1.1).

For question (2), the semisimple Hopf algebras in [18] with group-like elements $G(H) = \mathbb{Z}_4 \times \mathbb{Z}_2$ are the following 7 non-trivial ones described by the data $(F, G, \triangleleft, \sigma, \tau)$. The classifications of finite-dimensional Nichols algebras over Hopf algebra of Kashina have attracted a lot of attention and among of them, $H_{b:1}$, $H_{d;-1,1}$ and $H^*_{b:1}$ were studied by Zheng-Gao-Hu-Shi in [32, 33, 34]. However, there are no studies on $H_{c:\sigma_0}$. We start with investigating all simple Yetter–Drinfel'd modules V over $H = H(2,2) (= H_{c:\sigma_0})$.

$H_{a:1}$	$x \triangleleft t = x,\ y \triangleleft t = x^2 y;$ $\sigma(x^i y^j, t \cdot t) = 1,\ \tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$
$H_{a:y}$	$x \triangleleft t = x,\ y \triangleleft t = x^2 y;$ $\sigma(x^i y^j, t \cdot t) = (-1)^j,\ \tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$
$H_{b:y}$	$x \triangleleft t = x^3,\ y \triangleleft t = y;$ $\sigma(x^i y^j, t \cdot t) = (-1)^j,\ \tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$
$H_{b:x^2y}$	$x \triangleleft t = x^3,\ y \triangleleft t = y;$ $\sigma(x^i y^j, t \cdot t) = (-1)^{i+j},\ \tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$
$H_{b:1}$	$x \triangleleft t = x^3,\ y \triangleleft t = y;$ $\sigma(x^i y^j, t \cdot t) = 1,\ \tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$
$H_{c:\sigma_0}$	$x \triangleleft t = xy,\ y \triangleleft t = y;$ $\sigma(x^i y^j, t, t) = (-1)^{\frac{i(i-1)}{2}},\ \tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$
$H_{c:\sigma_1}$	$x \triangleleft t = xy,\ y \triangleleft t = y;$ $\sigma(x^k y^l, t \cdot t) = (-1)^{\frac{k(k-1)}{2}} \mathrm{i}^k,\ \mathrm{i}^2 = -1,$ $\tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$

The semisimple Hopf algebras in [18] with group-like elements $G(H) = \mathbb{Z}_4 \times \mathbb{Z}_2$

1.2 Organization and main results

The thesis is organized as follows.

In Chapter 2, we review some definitions, notations and results related to Yetter–Drinfel'd modules, Drinfel'd double, Nichols algebra and the Hopf algebra H.

The classification of Hopf algebras have attracted a lot of attention and we still know very little about semisimple Hopf algebras, we would like to study the representations of a class of semisimple Hopf algebras in this Chapter 3. We prove that there are 88 non-isomorphic simple left $\mathcal{D}(H_{c:\sigma_0})$-modules, with 32 of them one-dimensional and the remaining 56 two-dimensional. We then classify all finite-dimensional Nichols algebras of simple Yetter–Drinfel'd modules V over $H_{c:\sigma_0}$. Among these, we identify Nichols algebras of non-diagonal type studied by Andruskiewitsch and Geraldi [8], which are of dimension 8, 16, or infinite (see Lemma 3.3.0.4).

In Chapter 4, we study the semisimple Hopf algebras $D(m, d, \gamma)/(y)$. In section 4.1, we introduces the Hopf algebras $D(m, d, \gamma)$, and in section 4.2, we give some results about their quotients $D(m, d, \gamma)/(y)$.

In Chapter 5, we focus on the Abelian extensions $H(m, d)$ in section 5.1. In section 5.2, we will present a proof of a conjecture by Gongxiang Liu (Conjecture 5.0.0.1). We prove that each semisimple Hopf algebra $D(m, d, \gamma)/(y)$ is exactly an Abelian extension $H(m, d)$. In section 5.3, we will provide an example that appears in Chapter 3.

In Chapter 6, we will give the details of the computation for the simple Yetter–Drinfel'd modules. In section 6.1, we will present the one-dimensional Yetter–Drinfel'd modules and in section 6.2, we will present the two-dimensional ones.

Finally, in Chapter 7, we obtain my future research. Based on the results of Chapter 5, we will present some finite-dimensional Nichols algebras over $H(m, d)$.

Throughout the thesis, we work over an algebraically closed field $\Bbbk$ of characteristic zero. All Hopf algebras in this thesis are finite dimensional. Our references for the theory of Hopf algebras are [24, 25]. For a Hopf algebra H, the antipode of H will be denoted by $\mathcal{S}$.

Chapter 2 Preliminaries

In this Chapter, we review some definitions, notations and results related to Yetter–Drinfel'd modules, Drinfel'd double and Nichols algebra.

2.1 Reminders on the Yetter–Drinfel'd modules

All concepts are standard, and one can find the detailed analysis about them in many references (for instance [2, 25]).

Definition 2.1.0.1. *Let H be a Hopf algebra with invertible antipode $\mathcal{S}$,*

(1) A left-right Yetter–Drinfel'd module $(V, \cdot, \rho)$ over H is a left H-module and a right H-comodule $\rho : V \to V \otimes H$ satisfying the compatibility condition

$$\rho(h \cdot v) = h_2 \cdot v_0 \otimes h_3 v_1 \mathcal{S}^{-1}(h_1),$$

for $h \in H, v \in V$. The category of left-right Yetter–Drinfel'd modules over H is denoted by ${}_H\mathcal{YD}^H$.

(2) A left-left Yetter–Drinfel'd module $(V, \cdot, \rho_l)$ is a left H-module and a left H-comodule $\rho_l : V \to H \otimes V$ satisfying the compatibility condition

$$\rho_l(h \cdot v) = h_1 v_{-1} \mathcal{S}(h_3) \otimes h_2 \cdot v_0,$$

for $h \in H, v \in V$. The category of left-left Yetter–Drinfel'd modules over H is denoted by ${}^H_H\mathcal{YD}$.

Let H be a Hopf algebra over $\Bbbk$ with invertible antipode $\mathcal{S}$, the Drinfel'd double $\mathcal{D}(H) = H^{*cop} \otimes H$ is a Hopf algebra with tensor product coalgebra structure and algebra structure defined by

$$(\varphi \otimes h)(\varphi' \otimes h') = \varphi\langle \varphi'_3, h_1\rangle \varphi'_2 \otimes h_2 \langle \varphi'_1, \mathcal{S}^{-1}(h_3)\rangle h',$$

for all $\varphi, \varphi' \in H^*$ and $h, h' \in H$. The following well-known result is due to [20].

Lemma 2.1.0.2. Let H be a finite-dimensional Hopf algebra. Then the category ${}_H\mathcal{YD}^H$ of left-right Yetter–Drinfel'd modules can be identified with the category ${}_{\mathcal{D}(H)}M$ of left modules over the Drinfel'd double $\mathcal{D}(H)$.

Remark 2.1.0.3. (1) Assume $(V, \cdot, \rho)$ is a left-right Yetter–Drinfel'd module over H, where $\rho(v) = v_0 \otimes v_1$ for $v \in V, h \in H$. Let $\rho_l(v) = S(v_1) \otimes v_0$, then $(V, \cdot, \rho_l)$ is a left-left Yetter–Drinfel'd module over H.

(2) The category ${}_H^H\mathcal{YD}$ is a braided monoidal category with braidings

$$c_{V,W}(v \otimes w) = v_{-1} \cdot w \otimes v_0$$

for $V, W \in {}_H^H\mathcal{YD}$, and $v \in V, w \in W$.

2.2 Reminders on the Nichols algebras of vector spaces of dimension 2

Let V be a vector space over $\Bbbk$ and $c \in \mathrm{Aut}(V \otimes V)$. The pair (V, c) is called a braided vector space if c satisfies

$$(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c).$$

and c is called a braiding of the braided vector space (V, c).

Definition 2.2.0.1. Given $\theta \in \mathbb{N}$, set $I_\theta = \{1, 2, \dots \theta\}$. The braided vector space (V, c) of dimension θ is of diagonal type if there exist a basis $\{v_i \mid i \in I_\theta\}$ of V and $q = (q_{ij})_{i,j\in I_\theta} \in \Bbbk^{\theta\times\theta}$ such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i$$

for all $i, j \in I_\theta$. In such case, $q = (q_{ij})_{i,j\in I_\theta}$ is called a braiding matrix of (V, c).

Definition 2.2.0.2. Let $V \in {}^H_H\mathcal{YD}$, a braided $\mathbb{N}$-graded Hopf algebra $R = \oplus_{n\geq 0} R(n)$ in ${}^H_H\mathcal{YD}$ is called Nichols algebra of V if

(1) $\Bbbk \simeq R(0), V \simeq R(1) \in {}^H_H\mathcal{YD}$.

(2) $R(1) = P(R) = \{r \in R \mid \Delta_R(r) = r \otimes 1 + 1 \otimes r\}$.

(3) R is generated as algebra by $R(1)$.

Then, R is denoted by $\mathcal{B}(V) = \oplus_{n\geq 0}\mathcal{B}^n(V)$.

Remark 2.2.0.3. *[10, 27, 28]* The Nichols algebra $\mathcal{B}(V)$ is completely determined by the braided vector space. We recall the following maps:

$$\begin{gathered}\Omega_{n,1} := \mathrm{id} + c_n + c_n c_{n-1} + \cdots + c_n c_{n-1} \cdots c_1 = \mathrm{id} + c_n \Omega_{n-1,1}, \\ \Omega_1 := \mathrm{id}, \quad \Omega_2 := \mathrm{id} + c, \quad \Omega_n := (\Omega_{n-1} \otimes \mathrm{id})\, \Omega_{n-1,1},\end{gathered}$$

where $\Omega_{n,1} \in \mathrm{End}_{\Bbbk}\left(V^{\otimes(n+1)}\right)$, $\Omega_n \in \mathrm{End}_{\Bbbk}\left(V^{\otimes n}\right)$. As a vector space, the Nichols algebra $\mathcal{B}(V)$ is equal to

$$\mathcal{B}(V) = \Bbbk \oplus V \oplus \bigoplus_{n=2}^{\infty} V^{\otimes n} / \ker \Omega_n.$$

Let (V, c) be a finite-dimensional braided vector space. An important problem is to determine the presentation and structure of $\mathcal{B}(V)$, such as determining whether it has finite dimension or finite Gelfand-Kirillov dimension.

Note that there are braided vector spaces (V, c) of dimension 2 not of diagonal type but such that $\mathcal{B}(V)$ are quantum planes. We now give some examples of braided vector space (V, c) of dimension 2, where their Nichols algebras $\mathcal{B}(V)$ are braided Hopf algebras and $\mathcal{B}(V)$ have quadratic relations [8].

(E1) We assume that $k, p, q \neq 0$ and $k^2 \neq pq$. The associated braiding is

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_2} = \begin{pmatrix} k^2 x_1 \otimes x_1 & kq x_2 \otimes x_1 + (k^2 - pq) x_1 \otimes x_2 \\ kp x_1 \otimes x_2 & k^2 x_2 \otimes x_2 \end{pmatrix}.$$

If $k^2 = -1$ or $pq = 1$, then $\mathcal{B}(V)$ have quadratic relations.

(E2) We assume that $k, p, q \neq 0$ and $k^2 \neq pq$. The associated braiding is

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_2} = \begin{pmatrix} k^2x_1 \otimes x_1 & kqx_2 \otimes x_1 + (k^2 - pq)x_1 \otimes x_2 \\ kpx_1 \otimes x_2 & -pqx_2 \otimes x_2 \end{pmatrix}.$$

If $k^2 = -1$ or $pq = 1$, then $\mathcal{B}(V)$ have quadratic relations.

(E3) We assume that $k \neq 0$, and either $p \neq 0$, or $q \neq 0$, or $s \neq 0$. The associated braiding $(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_2}$ is

$$\begin{pmatrix} kx_1 \otimes x_1 & kx_2 \otimes x_1 + qx_1 \otimes x_1 \\ kx_1 \otimes x_2 + px_1 \otimes x_1 & kx_2 \otimes x_2 + sx_1 \otimes x_1 + px_2 \otimes x_1 + qx_1 \otimes x_2 \end{pmatrix}.$$

If $k^2 = 1$, then $\mathcal{B}(V)$ have quadratic relations.

(E4) We assume that $p, q \neq 0$, $p^2 \neq q^2$, $a = p^2 - q^2$ and $b = p^2 + q^2$. The associated braiding $(c(x_i \otimes x_j))_{i,j\in I_2}$ is

$$\begin{pmatrix} (a + 2pq)x_1 \otimes x_1 + ax_2 \otimes x_2 & bx_2 \otimes x_1 + ax_1 \otimes x_2 \\ bx_1 \otimes x_2 + ax_2 \otimes x_1 & (a - 2pq)x_2 \otimes x_2 + ax_1 \otimes x_1 \end{pmatrix}.$$

If $2p^2 = -1$ or $2q^2 = 1$, then $\mathcal{B}(V)$ have quadratic relations.

(E5) We assume that $p, q \neq 0$, and either $\neq q$, or $k \neq 0$. The associated braiding is

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_2} = \begin{pmatrix} px_1 \otimes x_1 & qx_2 \otimes x_1 + (p - q)x_1 \otimes x_2 \\ px_1 \otimes x_2 & -qx_2 \otimes x_2 + kx_1 \otimes x_1 \end{pmatrix}.$$

If $p = -1$ or $q = 1$, then $\mathcal{B}(V)$ have quadratic relations.

(E6) We assume that $k \neq 0$, and either $p \neq 0$, or $q \neq 0$. The associated braiding $(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_2}$ is

$$\begin{pmatrix} k^2x_1 \otimes x_1 & k^2x_2 \otimes x_1 - kpx_1 \otimes x_1 \\ kx_1 \otimes (kx_2 + px_1) & k^2x_2 \otimes x_2 + pqx_1 \otimes x_1 + kqx_1 \otimes x_2 - kqx_2 \otimes x_1 \end{pmatrix}.$$

If $k^4 = 1$, then $\mathcal{B}(V)$ have quadratic relations.

(E7) We assume that $k, p, q \neq 0$. The associated braiding is

$$(c(x_i \otimes x_j))_{i,j\in I_2} = \begin{pmatrix} qx_2 \otimes x_2 & kx_1 \otimes x_2 \\ kx_2 \otimes x_1 & px_1 \otimes x_1 \end{pmatrix}.$$

If $k = -1$ or $pq = 1$, then $\mathcal{B}(V)$ have quadratic relations.

(E8) We assume $k \neq 0$. The associated braiding is

$$(c(x_i \otimes x_j))_{i,j\in I_0} = \begin{pmatrix} kx_1 \otimes x_1 & -kx_2 \otimes x_1 \\ -kx_1 \otimes x_2 & kx_2 \otimes x_2 + kx_1 \otimes x_1 \end{pmatrix}.$$

If $k^2 = 1$, then $\mathcal{B}(V)$ have quadratic relations.

2.3 Reminders on Abelian extensions

We describe Hopf algebra H and give some reminders on Abelian extensions. For Abelian extension, we refer to [1, 12, 23, 29] for related facts.

Definition 2.3.0.1. Let

$$(H) : K \xrightarrow{\iota} H \xrightarrow{\pi} A$$

be a sequence of Hopf algebras and let K^+ be the kernel of the counit of K.

(1) Suppose ι is injective, π is surjective. So that we may regard ι as an inclusion, π as quotient. The sequence (H) is called an extension of A by K if it satisfies

$$HK^+ = \mathrm{Ker}\pi.$$

(2) An extension (H) is called an Abelian extension if K is commutative and A is cocommutative.

Remark 2.3.0.2. *Let G be a finite group and $\Bbbk G$ be the group algebra with the usual Hopf algebra structure. Its dual Hopf algebra $(\Bbbk G)^*$ can be identified with $\Bbbk^G$, the algebra of functions from G to $\Bbbk$. If a Hopf algebra H can fit into an extension $\Bbbk^G \to H \to \Bbbk F$, where G, F are finite groups, then H can be described as a bicrossed product $\Bbbk^G \#_{\sigma,\tau} \Bbbk F$: there exist maps*

$$G \xleftarrow{\triangleleft} G \times F \xrightarrow{\triangleright} F,$$
$$G \times F \times F \xrightarrow{\sigma} \Bbbk^\times,$$
$$G \times G \times F \xrightarrow{\tau} \Bbbk^\times,$$

such that $(F, G, \triangleleft, \triangleright)$ is a matched pair of groups, (σ, τ) is a pair of compatible normal cocycles. That is, $(F, G, \triangleleft, \triangleright, \sigma, \tau)$ satisfies the following conditions,

$$(t \triangleright gg') = (t \triangleright g)((t \triangleleft g) \triangleright g'), \tag{2.1}$$
$$tt' \triangleleft g = (t \triangleleft (t' \triangleright g))(t' \triangleleft g), \tag{2.2}$$
$$\sigma(g \triangleleft t; t', t'')\sigma(g; t, t't'') = \sigma(g; t, t')\sigma(g; tt', t''), \tag{2.3}$$
$$\sigma(1; t, t') = \sigma(g; 1, t') = \sigma(g; t, 1) = 1, \tag{2.4}$$
$$\tau\left(gg', g''; t\right)\tau(g, g'; g'' \triangleright t) = \tau\left(g', g''; t\right)\tau\left(g, g'g''; t\right), \tag{2.5}$$
$$\tau\left(1, g'; t\right) = \tau(g, 1; t) = \tau\left(g, g'; 1\right) = 1, \tag{2.6}$$

and $\sigma\left(gg'; t, t'\right)\tau\left(g, g'; tt'\right)$ equals

$$\sigma\left(g; g' \triangleright t, (g' \triangleleft t) \triangleright t'\right)\sigma\left(g'; t, t'\right)\tau\left(g, g'; t\right)\tau\left(g \triangleleft \left(g' \triangleright t\right), g' \triangleleft t; t'\right), \tag{2.7}$$

for all $g, g', g'' \in G,\ t, t', t'' \in F$. The product, coproduct and antipode of

$$H \simeq \Bbbk^G \#_{\sigma,\tau} \Bbbk F$$

in the basis $\{e_g \# t : g \in G, t \in F\}$ has the form:

$$(e_g \# t) \cdot (e_{g'} \# t') = \delta_{g \triangleleft t, g'}\sigma\left(g; t, t'\right)e_g \# tt', \tag{2.8}$$
$$\Delta\left(e_g \# t\right) = \sum_{g'g''=g}\tau\left(g', g''; t\right)e_{g'} \#\left(g'' \triangleright t\right) \otimes e_{g''} \# t, \tag{2.9}$$
$$\mathcal{S}\left(e_g \# t\right) = \sigma\left(g^{-1}; g \triangleright t, (g \triangleright t)^{-1}\right)^{-1}\tau\left(g^{-1}, g; t\right)^{-1}e_{(g \triangleleft t)^{-1}} \# (g \triangleright t)^{-1}. \tag{2.10}$$

For example, let G be the group

$$\{x, y \mid x^4 = 1 = y^2, xy = yx\}.$$

We denote by $\{e_g\}_{g \in G}$ the dual basis of G in $\Bbbk^G$ and denote by t the generator of $\mathbb{Z}_2$. Then we can obtain $H = H_{c:\sigma_0}$ in [18] as an Abelian extension of the following form

$$\Bbbk^G \xrightarrow{\iota} H \xrightarrow{\pi} \Bbbk\mathbb{Z}_2.$$

equivalently, the Hopf algebra H is determined by $(\mathbb{Z}_2, G, \triangleleft, \triangleright, \sigma, \tau)$ where

$$\triangleright \text{ is a trivial action}, \qquad x \triangleleft t = xy, \qquad y \triangleleft t = y, \tag{2.11}$$
$$\sigma(x^i y^j, t, t) = (-1)^{\frac{i(i-1)}{2}}, \quad \tau(x^i y^j, x^k y^l, t) = (-1)^{jk}. \tag{2.12}$$

We denote by $e_\gamma t$ the $e_\gamma \# t$ for $\gamma \in G$. As an algebra, H is generated by $\{e_g, t\}_{g\in G}$ and Hopf algebra structure of H is given by

$$e_g e_h = \delta_{g,h} e_g,\ t e_g = e_{g \triangleleft t} t,\ t^2 = \sum_{g\in G} \sigma(g,t,t) e_g, \tag{2.13}$$

$$\Delta(e_g) = \sum_{h,k\in G,\ hk=g} e_h \otimes e_k,\ \Delta(t) = [\sum_{g,h\in G} \tau(g,h,t) e_g \otimes e_h](t\otimes t), \tag{2.14}$$

$$\mathcal{S}(t) = \sum_{g\in G} \sigma(g^{-1},t,t)^{-1} \tau(g,g^{-1},t)^{-1} e_{(g\triangleleft t)^{-1}} t,\ \mathcal{S}(e_g) = e_{g^{-1}}, \tag{2.15}$$

$$\epsilon(t) = 1,\ \epsilon(e_g) = \delta_{g,1} 1, \quad \forall g,h \in G. \tag{2.16}$$

Chapter 3 On the Hopf algebra $H_{c:\sigma_0}$

In this Chapter, we start with investigating all simple Yetter–Drinfel'd modules V over $H_{c:\sigma_0}$. We proved that there are 88 non-isomorphic simple left $\mathcal{D}(H_{c:\sigma_0})$-modules, and 32 of them are one-dimensional and the other 56 ones are two-dimensional. Then we classify all finite-dimensional Nichols algebras of simple Yetter–Drinfel'd modules V over $H_{c:\sigma_0}$. We find once again the Nichols algebras of non-diagonal type which were studied by Andruskiewitsch [8] and non-diagonal type ones are 8-, 16- or infinite dimensional.

This Chapter is organized as follows. In section 3, we describe the Drinfel'd double $\mathcal{D}(H_{c:\sigma_0})$ of $H_{c:\sigma_0}$. We devote sections 4 and 5 to giving our main results (see Theorem 3.2.1.3 and Theorem 3.3.0.6).

3.1 The Drinfel'd double of $H_{c:\sigma_0}$

In the following of this chapter, we let H be the Hopf algebra $H_{c:\sigma_0}$ and $G = \{x, y \mid x^4 = 1 = y^2, xy = yx\}$. In this section, we want to describe the Drinfel'd double $\mathcal{D}(H)$. For this purpose, we define the following elements in H^*:

$$\zeta_g : \begin{cases} e_h \mapsto \delta_{g,h} \\ e_h t \mapsto 0, \end{cases} \qquad \chi_h : \begin{cases} e_g \mapsto 0 \\ e_g t \mapsto \delta_{g,h} \end{cases} \qquad (g, h \in G). \tag{3.1}$$

Deducing from the coalgebra structure of H, it is not hard to see that the algebraic structure of H^* is given by

$$\begin{aligned} &\zeta_{x^i y^j}\zeta_{x^k y^l} = \zeta_{x^{i+k} y^{j+l}}, \quad \chi_{x^i y^j}\chi_{x^k y^l} = (-1)^{jk}\chi_{x^{i+j} y^{j+l}}, \\ &\zeta_{x^i y^j}\chi_{x^k y^l} = 0 = \chi_{x^k y^l}\zeta_{x^i y^j}, \quad 1_{H^*} = \varepsilon = \zeta_1 + \chi_1. \end{aligned} \tag{3.2}$$

Since $e_{x^iy^j},\ t\ (0\le i\le 3,\ 0\le j\le 1)$ are the generators of H and $\zeta_{x^iy^j},\ \chi_{x^iy^j}\ (0\le i\le 3,\ 0\le j\le 1)$ are the generators of H^*, the Drinfel'd double $\mathcal{D}(H)=H^{*cop}\bowtie H$ is generated by $\zeta_{x^iy^j}\bowtie 1_H$, $\chi_{x^iy^j}\bowtie 1_H$, $1_{H^{*cop}}\bowtie e_{x^iy^j}$, $1_{H^{*cop}}\bowtie t$, and we abbreviate them by $\zeta_{x^iy^j},\ \chi_{x^iy^i},\ e_{x^iy^j},\ t$.

Theorem 3.1.0.1. As an algebra, $\mathcal{D}(H)$ is generated by $e_{x^iy^j},\ t,\ \zeta_{x^iy^j},\ \chi_{x^iy^j},\ (0\le i\le 3,\ 0\le j\le 1)$ with relations:

$$\begin{aligned}&\zeta_{x^iy^j}\zeta_{x^ky^l}=\zeta_{x^{i+k}y^{j+l}},\chi_{x^iy^j}\chi_{x^ky^l}=(-1)^{jk}\chi_{x^{i+k}y^{j+l}},\\&\zeta_{x^iy^j}\chi_{x^ky^l}=0=\chi_{x^ky^l}\zeta_{x^iy^j},\end{aligned}\tag{3.3}$$

$$1=\sum_{g\in G}e_g,\quad e_ge_{g'}=\delta_{g,g'}e_g,\quad te_g=e_{g\triangleleft t}t,\quad t^2=\sum_g\sigma(g;t,t)e_g,\quad (g,g'\in G),\tag{3.4}$$

$$e_g\chi_h=\chi_he_{gh(h^{-1}\triangleleft t)},\qquad e_g\zeta_h=\zeta_he_g,\quad (g,h\in G),\tag{3.5}$$

$$\begin{aligned}&t\zeta_h=\zeta_{h\triangleleft t}(\sum_{g\in G}\tau\,(h\triangleleft t,g;t)\,\tau(\,(h\triangleleft t)g,h^{-1}\triangleleft t;t)\tau(h\triangleleft t,h^{-1}\triangleleft t;\ t)e_gt),\quad (h\in G),\\&t\chi_h=\chi_{h\triangleleft t}(\sum_{g\in G}\tau\,(h,g;t)\,\tau(hg,h^{-1}\triangleleft t;t)\tau(h\triangleleft t,h^{-1}\triangleleft t;\ t)e_gt),\quad (h\in G).\end{aligned}\tag{3.6}$$

Proof: We only verify two equations of (3.5) and (3.6). Due to the definition, it follows that,

$$hf=(1_{H^{*cop}}\bowtie h)(f\bowtie 1_H)=h_1\rightharpoonup f\leftharpoonup\mathcal{S}^{-1}(h_3)\bowtie h_2=\langle f_1,\mathcal{S}^{-1}(h_3)\rangle\langle f_3,h_1\rangle f_2\bowtie h_2,$$

where $h\in H, f\in H^{*cop}$. Let $k\in H$, it is straightforward to show that

$$\langle\langle f_1,\mathcal{S}^{-1}(h_3)\rangle\langle f_3,h_1\rangle f_2,k\rangle=\langle f,\mathcal{S}^{-1}(h_3)\cdot k\cdot h_1\rangle.$$

Our problem reduces to determine $\mathcal{S}^{-1}(h_3)\cdot k\cdot h_1$ in each case. Assume $\gamma,g,h\in G$ and $k=e_\gamma$ or $e_\gamma t$, after some tedious manipulation by (2.11), it follows that,

$$\mathcal{S}^{-1}(\,e_{g_3}\,)\cdot\ e_\gamma\cdot\ e_{g_1}=\delta_{{g_3}^{-1},\gamma}\,e_\gamma\cdot\ e_{g_1}=\delta_{{g_3}^{-1},\gamma}\delta_{\gamma,g_1}\,e_\gamma,\tag{3.7}$$

$$\mathcal{S}^{-1}(\,e_{g_3}\,t)=\sigma(\,{g_3}^{-1};\ t,\ t)\tau(\,{g_3}^{-1},\ g_3;t)e_{(g_3\triangleleft t)^{-1}}\,t,$$

$$\mathcal{S}^{-1}(\,e_{g_3}t\,)\cdot\ (e_\gamma\ t)\ \cdot\ (e_{g_1}\ t)=\tau(\,{g_3}^{-1},\ g_3;t)\delta_{((g_3\triangleleft t)^{-1})\triangleleft t,\gamma}\delta_{(g_3\triangleleft t)^{-1},g_1}(e_{\gamma\triangleleft t}t),\tag{3.8}$$

$$\Delta(t) = \Delta(1 \cdot t) = \Delta\left(\Sigma_g e_g\, t\right) = \sum_{g_1, g_2 \in G} \tau\left(g_1, g_2; t\right)\left(e_{g_1}\, t\right) \otimes \left(e_{g_2}\, t\right),$$

$$\Delta_2(t) = (\Delta \otimes id)(\Delta(t)) = \sum_{g_1, g_2, g_3 \in G} \tau\left(g_1, g_2; t\right) \tau(g_1 g_2, g_3; t)(e_{g_1}\, t) \otimes (e_{g_2}\, t) \otimes (e_{g_3}\, t). \tag{3.9}$$

Let $k = e_\gamma$, by (3.7), it follows that

$$\begin{aligned}
&\langle \zeta_h, \mathcal{S}^{-1}(\ e_{g_3}\ ) \cdot\ e_\gamma \cdot\ e_{g_1} \rangle = \delta_{g_3{}^{-1},\gamma} \delta_{\gamma, g_1} \delta_{\gamma, h}, \\
&\langle (\zeta_h)_1, \mathcal{S}^{-1}(e_{g_3}) \rangle \langle (\zeta_h)_3, (e_{g_1}) \rangle (\zeta_h)_2 = \zeta_h, \\
&e_g \zeta_h = \zeta_h e_g, \ \ (g_3^{-1} = g_1 = \gamma = h,\ e_{g_2} = e_g).
\end{aligned}$$

It is clear that relation holds for $k = e_\gamma\, t$.

Let $k = e_\gamma\, t$, due to (3.8) and (3.9), it follows that

$$\begin{aligned}
&\langle \chi_h, \tau(g_1, g_2; t) \tau(g_1 g_2, g_3; t) \mathcal{S}^{-1}(\ e_{g_3}\ t\ ) \cdot\ (e_\gamma\, t)\ \cdot\ (e_{g_1}\, t) \rangle \\
&= \tau(g_1, g_2; t) \tau(g_1 g_2, g_3; t) \tau(g_3{}^{-1}, g_3; t) \delta_{((g_3 \triangleleft t)^{-1}) \triangleleft t, \gamma} \delta_{(g_3 \triangleleft t)^{-1}, g_1} \delta_{h \triangleleft t, \gamma}, \\
&\langle (\chi_h)_1, \mathcal{S}^{-1}(e_{g_3}\, t) \rangle \langle (\chi_h)_3, e_{g_1}\, t \rangle (\chi_h)_2 = \chi_{h \triangleleft t}, \\
&t \chi_h = \chi_{h \triangleleft t} (\sum_{g \in G} \tau\left(h, g; t\right) \tau(hg, h^{-1} \triangleleft t; t) \tau(h \triangleleft t, h^{-1} \triangleleft t;\ t)(e_g\, t)), \\
&\ (g_3{}^{-1} \triangleleft t = \gamma \triangleleft t = h = g_1,\ g_2 = g).
\end{aligned}$$

Also, it is not hard to see that relation holds for $k = e_\gamma$. □

3.2 The simple Yetter–Drinfel'd modules V over $H_{c:\sigma_0}$

In the previous section, we already described the algebraic structure of the Drinfel'd double $\mathcal{D}(H)$. In this section, we want to give all simple left $\mathcal{D}(H)$-modules, equivalently, give all simple Yetter–Drinfel'd modules V over H.

3.2.1 The simple left $\mathcal{D}(H_{c:\sigma_0})$-module

Let V be a left $\mathcal{D}(H)$-module, we define $V_\mu = \{v \in V \mid e_h v = \delta_{\mu,h} v,\ \ h \in G\}$ for $\mu \in G$, then V decomposes into a direct sum of V_μ's as vector space. Furthermore, by the relations $\zeta_g\chi_h = 0 = \chi_h\zeta_g$ $(g,h \in G)$ and $1_{H^*} = \varepsilon = \zeta_1 + \chi_1$, it follows that as a vector space V_μ decomposes into a direct sum of $\zeta_1 V_\mu$ and $\chi_1 V_\mu$. Let $S = \{1, y, x^2, x^2y\}$ and $T = \{x, xy, x^3, x^3y\}$ be the subsets of G. By (2.11), it follows that T and S are stable under the action $\triangleleft$ of t. Furthermore, the elements of S are fixed points under the action above. We denote by ξ a primitive 4-th root of 1. We construct the following 32 one-dimensional spaces together with the corresponding $\mathcal{D}(H)$-action:

(1) $V_{1,i}^+$ $(i = 0,1,2,3)$: the action of $\mathcal{D}(H)$ is given by

$$\begin{aligned} &e_{x^ky^l} \mapsto \delta_{x^ky^l,1}, \quad t \mapsto 1, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}, \\ &\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1). \end{aligned}$$

(2) $V_{1,i}^-$ $(i = 0,1,2,3)$: the action of $\mathcal{D}(H)$ is given by

$$\begin{aligned} &e_{x^ky^l} \mapsto \delta_{x^ky^l,1}, \quad t \mapsto -1, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}, \\ &\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1). \end{aligned}$$

(3) $V_{y,i}^+$ $(i = 0,1,2,3)$: the action of $\mathcal{D}(H)$ is given by

$$\begin{aligned} &e_{x^ky^l} \mapsto \delta_{x^ky^l,y}, \quad t \mapsto 1, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}(-1)^l, \\ &\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1). \end{aligned}$$

(4) $V_{y,i}^-$ $(i = 0,1,2,3)$: the action of $\mathcal{D}(H)$ is given by

$$\begin{aligned} &e_{x^ky^l} \mapsto \delta_{x^ky^l,y}, \quad t \mapsto -1, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}(-1)^l, \\ &\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1). \end{aligned}$$

(5) $V_{x^2,i}^+$ $(i = 0,1,2,3)$: the action of $\mathcal{D}(H)$ is given by

$$\begin{aligned} &e_{x^ky^l} \mapsto \delta_{x^ky^l,x^2}, \quad t \mapsto \xi, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}, \\ &\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1). \end{aligned}$$

(6) $V_{x^2,i}^-$ $(i = 0,1,2,3)$: the action of $\mathcal{D}(H)$ is given by

$$\begin{aligned} &e_{x^ky^l} \mapsto \delta_{x^ky^l,x^2}, \quad t \mapsto -\xi, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}, \\ &\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1). \end{aligned}$$

(7) $V^{+}_{x^2y,i}$ $(i = 0, 1, 2, 3)$: the action of $\mathcal{D}(H)$ is given by

$$e_{x^ky^l} \mapsto \delta_{x^ky^l,x^2y}, \quad t \mapsto \xi, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}(-1)^l,$$
$$\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1).$$

(8) $V^{-}_{x^2y,i}$ $(i = 0, 1, 2, 3)$: the action of $\mathcal{D}(H)$ is given by

$$e_{x^ky^l} \mapsto \delta_{x^ky^l,x^2y}, \quad t \mapsto -\xi, \quad \zeta_{x^ky^l} \mapsto \xi^{ik}(-1)^l,$$
$$\chi_{x^ky^l} \mapsto 0, \ (0 \le k \le 3, \ 0 \le l \le 1).$$

(S0) We denote them through a unified way $V^{\pm}_{\mu,i}$ for $\mu \in \{1, \ y, \ x^2, \ x^2y\}, \ 0 \le i \le 3.$

Lemma 3.2.1.1. The above 32 one-dimensional spaces together with the corresponding action are $\mathcal{D}(H)$-modules. Furthermore, all one-dimensional $\mathcal{D}(H)$-modules belong to $(S0)$.

Proof: It is straightforward to show that $V^{\pm}_{\mu,i}$ is a simple module through checking the relations given in Theorem 3.1.0.1. Assume V is a one-dimensional $\mathcal{D}(H)$-module. Take a basis $\{v\}$ of V. Since $1 = \sum_{k,l} e_{x^ky^l}$ is a decomposition of 1 into orthogonal idempotents, it follows that there exists $\mu \in G$ such that $e_{x^ky^l} \cdot v = \delta_{x^ky^l,\mu} v$, equivalently, $v = e_\mu v$ and $V = V_\mu$. Since $te_g v = e_{g \triangleleft t} tv$ $(g \in G)$, we have $t \cdot v \in V_{\mu \triangleleft t}$ and this leads to $\mu \triangleleft t = \mu$, equivalently, $\mu \in S$. Assume $V = \chi_1 V_u$ and $\dim V = 1$. But in this situation, $e_g \chi_h = \chi_h e_{gh(h^{-1} \triangleleft t)}$, it induces $\chi_h v_\mu \in V_{\mu(h \triangleleft t)h^{-1}}$. In particular, $\chi_x v_\mu \in V_{\mu y}$, this contradicts the fact $\dim V = 1$. Thus, we have

$$v = \zeta_1 v, \qquad V = \zeta_1 V_\mu \ (\mu \in S), \qquad \chi_{x^ky^l} \cdot v = 0, \quad t \cdot v = \lambda v \ (\lambda \in \Bbbk). \tag{3.1}$$

According to (7.3), it follows that $\lambda^2 v = t^2 v = \sum_g \sigma(g, t, t) e_g v$. Thus,

$$\lambda^2 = \sigma(\mu, t, t), \quad \lambda = \pm 1 \ (\mu = 1, \ y), \quad \lambda = \pm \xi \ (\mu = x^2, \ x^2y).$$

Assume $\zeta_x v = mv, \ \zeta_y v = nv$, where $m, n \in \Bbbk$. By relation (3.3), it follows that

$$(\zeta_x)^4 = \zeta_{x^4} = \zeta_1, \qquad (\zeta_y)^2 = \zeta_{y^2} = \zeta_1. \tag{3.2}$$

Thus, we have

$$v \overset{(4.1)}{=} \zeta_1 v = \zeta_{x^4} v \overset{(4.2)}{=} (\zeta_x)^4 v = m^4 v, \quad v \overset{(4.1)}{=} \zeta_1 v = \zeta_{y^2} v \overset{(4.2)}{=} (\zeta_y)^2 v = n^2 v. \tag{3.3}$$

According to the relation (7.5), it follows that

$$t\zeta_x v = \zeta_{x\lhd t}(\sum_g \tau\,(x\lhd t, g; t)\,\tau(\,(x\lhd t)g, x^{-1}\lhd t; t)\tau(x\lhd t, x^{-1}\lhd t;\; t)(e_g t))v,$$
$$t\zeta_y v = \zeta_{y\lhd t}(\sum_g \tau\,(y\lhd t, g; t)\,\tau(\,(y\lhd t)g, y^{-1}\lhd t; t)\tau(y\lhd t, y^{-1}\lhd t;\; t)(e_g t))v.$$

Equivalently,

$$n\tau\,(xy, \mu; t)\,\tau((xy)\mu, x^3y; t)\tau(xy, x^3y;\; t) = 1 \tag{3.4}$$

Due to (3.3) (3.4), it follows that

$$\begin{aligned} &m = \xi^i,\ i\in\{0,1,2,3\},\\ &n = 1,\ (\mu = 1,\ x^2),\\ &n = -1,\ (\mu = y,\ x^2y). \end{aligned} \tag{3.5}$$

By the above arguments, it follows that there are exactly 32 non-isomorphic one-dimensional simple left $\mathcal{D}(H)$-modules, as shown above. □

Next, we turn to two-dimensional case. Let $i = 0,1,2,3$, we construct the following 16 two-dimensional spaces $k\langle v,\ w\rangle$ together with the corresponding $\mathcal{D}(H)$-action:

(1) $W^+_{x,i,+}$: Let $0\le k\le 3,\ 0\le l\le 1$, we define $\chi_{x^ky^l}\mapsto 0$,

$$e_{x^ky^l}\mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0\\ 0 & \delta_{x^ky^l,xy}\end{pmatrix}, \qquad t\mapsto \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},$$
$$\zeta_x\mapsto \begin{pmatrix} \xi^i & 0\\ 0 & \xi^i\end{pmatrix}, \qquad \zeta_y\mapsto \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}.$$

(2) $W^+_{x,i,-}$: Let $0\le k\le 3,\ 0\le l\le 1$, we define $\chi_{x^ky^l}\mapsto 0$,

$$e_{x^ky^l}\mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0\\ 0 & \delta_{x^ky^l,xy}\end{pmatrix}, \qquad t\mapsto \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},$$
$$\zeta_x\mapsto \begin{pmatrix} -\xi^i & 0\\ 0 & \xi^i\end{pmatrix}, \qquad \zeta_y\mapsto \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

(3) $W^-_{x^3,i,+}$: Let $0\le k\le 3,\ 0\le l\le 1$, we define $\chi_{x^ky^l}\mapsto 0$,

$$e_{x^ky^l}\mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0\\ 0 & \delta_{x^ky^l,x^3y}\end{pmatrix}, \qquad t\mapsto \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},$$
$$\zeta_x\mapsto \begin{pmatrix} \xi^i & 0\\ 0 & \xi^i\end{pmatrix}, \qquad \zeta_y\mapsto \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}.$$

(4) $W^{-}_{x^3,i,-}$: Let $0 \leq k \leq 3,\ 0 \leq l \leq 1$, we define $\chi_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, x^3} & 0 \\ 0 & \delta_{x^k y^l, x^3 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\zeta_x \mapsto \begin{pmatrix} -\xi^i & 0 \\ 0 & \xi^i \end{pmatrix}, \qquad \zeta_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(S1) We denote them through a unified way $W^{\pm}_{\mu,i,\pm}$ for $\mu \in \{x,\ x^3\},\ 0 \leq i \leq 3$.

Let $i = 0, 1$, we construct the following 8 two-dimensional spaces $k\langle v,\ w\rangle$ together with the corresponding $\mathcal{D}(H)$-action:

(1) $X^{+}_{1,i}$: Let $0 \leq k \leq 3,\ 0 \leq l \leq 1$, we define $\chi_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, 1} & 0 \\ 0 & \delta_{x^k y^l, 1} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\zeta_x \mapsto \begin{pmatrix} \xi^i & 0 \\ 0 & -\xi^i \end{pmatrix}, \qquad \zeta_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) $X^{+}_{y,i}$: Let $0 \leq k \leq 3,\ 0 \leq l \leq 1$, we define $\chi_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, y} & 0 \\ 0 & \delta_{x^k y^l, y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\zeta_x \mapsto \begin{pmatrix} \xi^i & 0 \\ 0 & -\xi^i \end{pmatrix}, \qquad \zeta_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(3) $X^{-}_{x^2,i}$: Let $0 \leq k \leq 3,\ 0 \leq l \leq 1$, we define $\chi_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, x^2} & 0 \\ 0 & \delta_{x^k y^l, x^2} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\zeta_x \mapsto \begin{pmatrix} \xi^i & 0 \\ 0 & -\xi^i \end{pmatrix}, \qquad \zeta_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(4) $X^-_{x^2y,i}$: Let $0 \le k \le 3,\ 0 \le l \le 1$, we define $\chi_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^2y} & 0 \\ 0 & \delta_{x^ky^l,x^2y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\zeta_x \mapsto \begin{pmatrix} \xi^i & 0 \\ 0 & -\xi^i \end{pmatrix}, \qquad \zeta_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(S2) We denote them through a unified way $X^{\pm}_{\mu,i}$ for $\mu \in \{1,\ y,\ x^2,\ x^2y\},\ i = 0, 1.$

Then we construct the following 16 two-dimensional spaces $k\langle v,\ w\rangle$ together with the corresponding $\mathcal{D}(H)$-action:

(1) $Y_{x,1}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -\xi \\ \xi & 0 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) $Y_{x,2}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(3) $Y_{x,3}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -\xi \\ \xi & 0 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(4) $Y_{x,4}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(5) $Y_{x,5}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(6) $Y_{x,6}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(7) $Y_{x,7}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(8) $Y_{x,8}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x} & 0 \\ 0 & \delta_{x^ky^l,xy} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(9) $Y_{x^3,1}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(10) $Y_{x^3,2}$: Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -\xi \\ -\xi & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(11) $Y_{x^3,3}$: Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & \xi \\ \xi & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(12) $Y_{x^3,4}$: Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -\xi \\ -\xi & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(13) $Y_{x^3,5}$: Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(14) $Y_{x^3,6}$: Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(15) $Y_{x^3,7}$: Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(16) $Y_{x^3,8}$: Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^ky^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,x^3} & 0 \\ 0 & \delta_{x^ky^l,x^3y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(S3) We denote them through a unified way $Y_{\mu,j}$ for $\mu \in \{x,\ x^3\},\ 0 \leq j \leq 8.$

Finally, we construct the following 16 two-dimensional spaces $k\langle v,\ w\rangle$ together with the corresponding $\mathcal{D}(H)$-action:

(1) $Z_{1,1}:$ Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,1} & 0 \\ 0 & \delta_{x^ky^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) $Z_{1,2}:$ Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,1} & 0 \\ 0 & \delta_{x^ky^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(3) $Z_{1,3}:$ Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,1} & 0 \\ 0 & \delta_{x^ky^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(4) $Z_{1,4}:$ Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,1} & 0 \\ 0 & \delta_{x^ky^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(5) $Z_{1,5}:$ Let $0 \leq k \leq 3$ and $0 \leq l \leq 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^ky^l} \mapsto \begin{pmatrix} \delta_{x^ky^l,1} & 0 \\ 0 & \delta_{x^ky^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(6) $Z_{1,6}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l,1} & 0 \\ 0 & \delta_{x^k y^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(7) $Z_{1,7}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l,1} & 0 \\ 0 & \delta_{x^k y^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(8) $Z_{1,8}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l,1} & 0 \\ 0 & \delta_{x^k y^l,y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(9) $Z_{x^2,1}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l,x^2} & 0 \\ 0 & \delta_{x^k y^l,x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(10) $Z_{x^2,2}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l,x^2} & 0 \\ 0 & \delta_{x^k y^l,x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -\xi & 0 \\ 0 & -\xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(11) $Z_{x^2,3}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l,x^2} & 0 \\ 0 & \delta_{x^k y^l,x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(12) $Z_{x^2,4}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, x^2} & 0 \\ 0 & \delta_{x^k y^l, x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -\xi & 0 \\ 0 & -\xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(13) $Z_{x^2,5}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, x^2} & 0 \\ 0 & \delta_{x^k y^l, x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} \xi & 0 \\ 0 & -\xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(14) $Z_{x^2,6}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, x^2} & 0 \\ 0 & \delta_{x^k y^l, x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -\xi & 0 \\ 0 & \xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(15) $Z_{x^2,7}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, x^2} & 0 \\ 0 & \delta_{x^k y^l, x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} \xi & 0 \\ 0 & -\xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(16) $Z_{x^2,8}:$ Let $0 \le k \le 3$ and $0 \le l \le 1$, we define $\zeta_{x^k y^l} \mapsto 0$,

$$e_{x^k y^l} \mapsto \begin{pmatrix} \delta_{x^k y^l, x^2} & 0 \\ 0 & \delta_{x^k y^l, x^2 y} \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} -\xi & 0 \\ 0 & \xi \end{pmatrix},$$
$$\chi_x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \chi_y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(S4) We denote them through a unified way $Z_{\mu,j}$ for $\mu \in \{1,\ x^2\},\ 0 \le j \le 8$.

Lemma 3.2.1.2. (1) The above 56 two-dimensional spaces $(S1$-$S4)$ together with the corresponding action are $\mathcal{D}(H)$-modules.

(2) The $\mathcal{D}(H)$-modules $(S1$-$S4)$ are irreducible.

Proof:

(1) We only verify the case $(S3)$ since the other cases can be proved similarly. Let $\mu \in \{x,\ x^3\}$. Suppose $v \in \chi_1 V_\mu, w = \chi_x \cdot v$ and $V = \Bbbk\langle v, w\rangle$. Since $e_g\chi_h = \chi_h e_{gh(h^{-1}\triangleleft t)}$, it follows that $w \in \chi_1 V_{\mu y}$. Since $t \cdot v_\mu \in \chi_1 V_{\mu y}$, we can suppose $t \cdot v = \lambda w$. Let $A_S = \Bbbk\langle \chi_s\rangle_{s\in S}$ be an algebra generated by $\{\chi_s \mid s \in S\}$. Due to (3.3), it follows that A_S is a commutative algebra. Since every simple module over A_S is 1-dimensional, we can assume

$$\chi_{x^2} v = \widetilde{m} v, \quad \chi_y v = \widetilde{n} v, \tag{3.6}$$

where $\widetilde{m}, \widetilde{n} \in \Bbbk$. By relation (3.3), it follows that

$$(\chi_{x^2})^2 = \chi_{x^4} = \chi_1, \qquad (\chi_y)^2 = \chi_{y^2} = \chi_1. \tag{3.7}$$

Thus, we have

$$v = \chi_1 v \overset{(4.7)}{=} (\chi_{x^4}) v \overset{(4.7)}{=} (\chi_{x^2})^2 v = \widetilde{m}^2 v, \quad v = \chi_1 v \overset{(4.7)}{=} \chi_{y^2} v \overset{(4.7)}{=} (\chi_y)^2 v = \widetilde{n}^2 v. \tag{3.8}$$

Let $\chi_s v = \beta(s) v$ for $s \in S$ defined in (3.6). Then $(\beta(x^2))^2 = \beta(x^4) = 1,\ (\beta(y))^2 = \beta(y^2) = 1$. Since S is the subgroup of G and $T = Sx$, we have following actions of $\mathcal{D}(H)$:

$$\begin{aligned}
&\chi_{sx}(v) = \tau(x,\ s, t)\beta(s) w = \beta(s) w,\\
&\chi_{sx}(w) = \tau(s,\ x, t)\beta(s)\beta(x^2) v,\\
&\chi_s w = \beta(s)\tau(s, x, t) w, \chi_s v = \beta(s) v,\\
&t \cdot w = t \cdot \chi_x v = \lambda \tau(x\mu y, x^3 y, t)\beta(y)\beta(x^2) v,\\
&t^2 v = \lambda^2 \tau(x\mu y, x^3 y, t)\beta(y)\beta(x^2) v = \sigma(\mu, t, t) v,\\
&\lambda^2 \tau(x\mu y, x^3 y, t)\beta(y)\beta(x^2) = \sigma(\mu, t, t).
\end{aligned} \tag{3.9}$$

Due to (3.8) and (3.9), it follows that

$$\widetilde{m} = \beta(x^2) = \pm 1, \quad \widetilde{n} = \beta(y) = \pm 1, \quad \lambda^2 = \frac{\sigma(\mu, t, t)}{-\beta(y)\beta(x^2)}. \tag{3.10}$$

i.e. λ is determined by $\beta(x^2)$ and $\beta(y)$. Thus, there are 16 two-dimensional left-$\mathcal{D}(H)$modules as $(S3)$.

(a) If $\Bbbk\langle v+\lambda w\rangle$ is a $\mathcal{D}(H)$-submodule of $(S1)$, then $\zeta_{x^k y^l}(v+\lambda w)$ $(0 \le k \le 3,\ 0 \le l \le 1)$ and $t(v+\lambda w)$ belong to $\Bbbk\langle v+\lambda w\rangle$. But

$$\zeta_y(v+\lambda w) = v - \lambda w \text{ or } -v+\lambda w \notin \Bbbk\langle v+\lambda w\rangle,$$

this leads a contradiction. Thus, there are 16 non-isomorphic two-dimensional simple $\mathcal{D}(H)$-modules as $(S1)$.

(b) $(S2)$ follow in a similar manner as (a).

(c) If $\Bbbk\langle v+\lambda w\rangle$ is a $\mathcal{D}(H)$-submodule of $(S3)$, then $\chi_{x^k y^l}(v+\lambda w)$ must belong to $\Bbbk\langle v+\lambda w\rangle$ $(0 \le k \le 3,\ 0 \le l \le 1)$. But $\chi_y(v+\lambda w) \notin \Bbbk\langle v+\lambda w\rangle$ by directly computation. Thus, there are 16 non-isomorphic two-dimensional simple $\mathcal{D}(H)$-modules, as shown in the $(S3)$.

(d) The $\mathcal{D}(H)$-modules in $(S4)$ are simple by the same arguments as (c).

$\square$

With these preparations, we can give the first main result.

Theorem 3.2.1.3. There are exactly 88 non-isomorphic simple Yetter–Drinfel'd modules over H, and 32 of them are one-dimensional and the other 56 ones are two-dimensional.

TABLE 1. Simple Yetter--Drinfel'd modules over H

(S0)	$V^+_{1,0}, V^+_{1,1}, V^+_{1,2}, V^+_{1,3}, V^-_{1,0}, V^-_{1,1}, V^-_{1,2}, V^-_{1,3}$
	$V^+_{y,0}, V^+_{y,1}, V^+_{y,2}, V^+_{y,3}, V^-_{1,0}, V^-_{y,1}, V^-_{y,2}, V^-_{y,3}$
	$V^+_{x^2,0}, V^+_{x^2,1}, V^+_{x^2,2}, V^+_{x^2,3}, V^-_{x^2,0}, V^-_{x^2,1}, V^-_{x^2,2}, V^-_{x^2,3}$
	$V^+_{x^2y,0}, V^+_{x^2y,1}, V^+_{x^2y,2}, V^+_{x^2y,3}, V^-_{x^2y,0}, V^-_{x^2y,1}, V^-_{x^2y,2}, V^-_{x^2y,3}$
(S1)	$W^+_{x,0,-}, W^+_{x,1,-}, W^+_{x,2,-}, W^+_{x,3,-}, W^+_{x,0,+}, W^+_{x,1,+}, W^+_{x,2,+}, W^+_{x,3,+}$
	$W^-_{x^3,0}, W^-_{x^3,1}, W^-_{x^3,2}, W^-_{x^3,3}$, $W^-_{x^3,0}, W^-_{x^3,1}, W^-_{x^3,2}, W^-_{x^3,3}$
$(S2)$	$X^+_{1,0}, X^+_{1,1}, X^+_{y,0}, X^+_{y,1}, X^-_{x^2,0}, X^-_{x^2,1}, X^-_{x^2y,0}, X^-_{x^2y,1}$
$(S3)$	$Y_{x,1}, Y_{x,2}, Y_{x,3}, Y_{x,4}, Y_{x,5}, Y_{x,6}, Y_{x,7}, Y_{x,8},$
	$Y_{x^3,1}, Y_{x^3,2}, Y_{x^3,3}, Y_{x^3,4}, Y_{x^3,5}, Y_{x^3,6}, Y_{x^3,7}, Y_{x^3,8},$
$(S4)$	$Z_{1,1}, Z_{1,2}, Z_{1,3}, Z_{1,4}, Z_{1,5}, Z_{1,6}, Z_{1,7}, Z_{1,8},$
	$Z_{x^2,1}, Z_{x^2,2}, Z_{x^2,3}, Z_{x^2,4}, Z_{x^2,5}, Z_{x^2,6}, Z_{x^2,7}, Z_{x^2,8},$

Proof: By Lemmas 3.2.1.1, 3.2.1.2, there are 32 non-isomorphic one-dimensional and 56 non-isomorphic two-dimensional simple Yetter–Drinfel'd modules over H.
Due to Lemma 2.1.0.2 and $32 \cdot 1^2 + 56 \cdot 2^2 = 256 = \dim \mathcal{D}(H)$, we finish the proof. $\square$

3.2.2 Left-left Yetter–Drinfel'd modules

Lemma 3.2.2.1. Fix a basis of each module in $(S0$-$S4)$, the following table give us braidings of $\mathcal{D}(H)$-modules $(S0$-$S4)$ as left-left Yetter–Drinfel'd modules over H.

TABLE 2. *Braidings of* $\mathcal{D}(H)$*-modules in* $(S0, S1, S2, S4)$

$(S0)$	$c(v \otimes v) = \pm(v \otimes v)$
$(S1)$	$q = \begin{pmatrix} \xi^i & \xi^i \\ -\xi^i & \xi^i \end{pmatrix}, \quad \begin{pmatrix} \xi^i & -\xi^i \\ \xi^i & \xi^i \end{pmatrix} \; (i = 0,\ 1,\ 2,\ 3)$
$(S2)$	$q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$
$(S4)$	$q = \begin{pmatrix} \pm 1 & \mp 1 \\ \pm 1 & \pm 1 \end{pmatrix}, \quad \begin{pmatrix} \pm\xi & \mp\xi \\ \pm\xi & \pm\xi \end{pmatrix},$

TABLE 3. *Braidings of* $\mathcal{D}(H)$*-modules in* $(S3)$

$\mu = x$	$c(v \otimes v) = \pm\xi w \otimes w$ $c(v \otimes w) = \pm\xi v \otimes w$ $c(w \otimes v) = \pm\xi w \otimes v$ $c(w \otimes w) = \mp\xi v \otimes v$	$c(v \otimes v) = \mp w \otimes w$ $c(v \otimes w) = \pm v \otimes w$ $c(w \otimes v) = \pm w \otimes v$ $c(w \otimes w) = \pm v \otimes v$	$c(v \otimes v) = \pm w \otimes w$ $c(v \otimes w) = \pm v \otimes w$ $c(w \otimes v) = \pm w \otimes v$ $c(w \otimes w) = \mp v \otimes v$	$c(v \otimes v) = \mp\xi w \otimes w$ $c(v \otimes w) = \pm\xi v \otimes w$ $c(w \otimes v) = \pm\xi w \otimes v$ $c(w \otimes w) = \pm\xi v \otimes v$
$\mu = x^3$	$c(v \otimes v) = \mp w \otimes w$ $c(v \otimes w) = \mp v \otimes w$ $c(w \otimes v) = \mp w \otimes v$ $c(w \otimes w) = \pm v \otimes v$	$c(v \otimes v) = \mp\xi w \otimes w$ $c(v \otimes w) = \pm\xi v \otimes w$ $c(w \otimes v) = \pm\xi w \otimes v$ $c(w \otimes w) = \pm\xi v \otimes v$	$c(v \otimes v) = \mp\xi w \otimes w$ $c(v \otimes w) = \mp\xi v \otimes w$ $c(w \otimes v) = \mp\xi w \otimes v$ $c(w \otimes w) = \pm\xi v \otimes v$	$c(v \otimes v) = \mp w \otimes w$ $c(v \otimes w) = \pm v \otimes w$ $c(w \otimes v) = \pm w \otimes v$ $c(w \otimes w) = \pm v \otimes v$

Proof: We only verify the case $(S3)$. By the action of $\mathcal{D}(H)$, we can give following H-comodule structure. Then we determine the braidings $c_{V,W}$ for $V, W \in {}^H_H\mathcal{YD}$.

$$
\begin{aligned}
&\rho(v) = v \otimes \Sigma_{s\in S}\beta(s)(e_s t) + w \otimes \Sigma_{s\in S}\beta(s)(e_{xs}t),\\
&\rho(w) = w \otimes \Sigma_{s\in S}\beta(s)\tau(s,x,t)(e_s t) + v \otimes \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}t),\\
&\rho_l(v) = \Sigma_{s\in S}\beta(s)S(e_s t) \otimes v + \Sigma_{s\in S}\beta(s)S(e_{xs}t) \otimes w,\\
&\rho_l(w) = \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_s t) \otimes w + \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}t) \otimes v,\\
&c(v\otimes v) = \Sigma_{s\in S}[\beta(s)\mathcal{S}(e_s t)\cdot v\otimes v] + \Sigma_{s\in S}[\beta(s)\mathcal{S}(e_{xs}t)\cdot v\otimes w] = \Sigma_{s\in S}[\beta(s)\\
&\mathcal{S}(e_{xs}t)\cdot v\otimes w],\\
&c(v\otimes w) = \Sigma_{s\in S}[\beta(s)\mathcal{S}(e_s t)\cdot w\otimes v] + \Sigma_{s\in S}[\beta(s)\mathcal{S}(e_{xs}t)\cdot w\otimes w] = \Sigma_{s\in S}[\beta(s)\\
&\mathcal{S}(e_{xs}t)\cdot w\otimes w],\\
&c(w\otimes v) = \Sigma_{s\in S}[\beta(s)\tau(s,x,t)\mathcal{S}(e_s t)\cdot v\otimes w] + \Sigma_{s\in S}[\beta(s)\beta(x^2)\tau(s,x,t)\\
&S(e_{xs}t)\cdot v\otimes v] = \Sigma_{s\in S}[\beta(s)\beta(x^2)\tau(s,x,t)\mathcal{S}(e_{xs}t)\cdot v\otimes v],\\
&c(w\otimes w) = \Sigma_{s\in S}[\beta(s)\tau(s,x,t)S(e_s t)\cdot w\otimes w] + \Sigma_{s\in S}[\beta(s)\beta(x^2)\tau(s,x,t),\\
&S(e_{xs}t)\cdot w\otimes v] = \Sigma_{s\in S}[\beta(s)\beta(x^2)\tau(s,x,t)\mathcal{S}(e_{xs}t)\cdot w\otimes v].
\end{aligned}
$$

Due to (2.11), we have the following formula which will be used to calculate the braiding.

$$
\begin{array}{ccc}
S(e_1 t) = e_1 t, & S(e_y t) = e_y t, & S(e_{xy}t) = e_{x^3}t,\\
S(e_{x^3}t) = (e_{xy}t), & S(e_x t) = -e_{x^3y}t, & S(e_{x^2}t) = -e_{x^2}t,\\
S(e_{x^2y}t) = -e_{x^2y}t, & S(e_{x^3y}t) = -(e_x t). &
\end{array}
\tag{3.1}
$$

Let $\mu = x$, we get the following braiding:

$$
\begin{aligned}
&c(v\otimes v) = (-e_{x^3y}t + \beta(y)e_{x^3}t + \beta(x^2)e_{xy}t - \beta(x^2y)e_x t)\cdot v\otimes w = \lambda\beta(x^2)w\otimes w,\\
&c(v\otimes w) = (-e_{x^3y}t + \beta(y)e_{x^3}t + \beta(x^2)e_{xy}t - \beta(x^2y)e_x t)\cdot w\otimes w = \lambda v\otimes w,\\
&c(w\otimes v) = (-\beta(x^2)e_{x^3y}t - \beta(x^2y)e_{x^3}t + e_{xy}t + \beta(y)e_x t)\cdot v\otimes v = \lambda w\otimes v,\\
&c(w\otimes w) = (-\beta(x^2)e_{x^3y}t - \beta(x^2y)e_{x^3}t + e_{xy}t + \beta(y)e_x t)\cdot w\otimes v = -\lambda\beta(x^2)v\otimes v.
\end{aligned}
$$

Let $\mu = x^3$, we get the following braiding:

$$
\begin{aligned}
&c(v\otimes v) = (-e_{x^3y}t + \beta(y)e_{x^3}t + \beta(x^2)e_{xy}t - \beta(x^2y)e_x t)\cdot v\otimes w = -\lambda w\otimes w,\\
&c(v\otimes w) = (-e_{x^3y}t + \beta(y)e_{x^3}t + \beta(x^2)e_{xy}t - \beta(x^2y)e_x t)\cdot w\otimes w = -\lambda\beta(x^2)v\otimes w,\\
&c(w\otimes v) = (-\beta(x^2)e_{x^3y}t - \beta(x^2y)e_{x^3}t + e_{xy}t + \beta(y)e_x t)\cdot v\otimes v = -\lambda\beta(x^2)w\otimes v,\\
&c(w\otimes w) = (-\beta(x^2)e_{x^3y}t - \beta(x^2y)e_{x^3}t + e_{xy}t + \beta(y)e_x t)\cdot w\otimes v = \lambda v\otimes v.
\end{aligned}
$$

Thus, we get the braidings of left-left Yetter–Drinfel'd modules over H of table $(S3)$.

□

3.3 Finite dimensional Nichols algebras of V

In the previous section, we already gave all simple Yetter–Drinfel'd modules V over H. In this section, we want to determine all the finite dimensional Nichols algebras of simple Yetter–Drinfel'd modules over H.

First, we give the Nichols algebras of Yetter–Drinfel'd modules that appear in $(S0)$.

Lemma 3.3.0.1. The Nichols algebras of one-dimensional simple Yetter–Drinfel'd modules
$(V^{\pm}_{\mu,i}, c_{\pm 1})$
$(i = 0, 1, 2, 3)$ of H are as follows,

(1) $\mathcal{B}((V^{\pm}_{\mu,i}, c_1)) \simeq \Bbbk[v]$, where $c_1(v \otimes v) = v \otimes v$.

(2) $\mathcal{B}((V^{\pm}_{\mu,i}, c_{-1})) \simeq \Bbbk[v]/\langle v^2 \rangle = \bigwedge \Bbbk\langle v \rangle$, and $\dim(\mathcal{B}((V^{\pm}_{\mu,i}, c_{-1}))) = 2$, where

$$c_{-1}(v \otimes v) = -v \otimes v.$$

Proof: It is well-known that this claim holds. For (2), the Nichols algebra is of type A_1. $\square$

Second, we give the Nichols algebras of Yetter–Drinfel'd modules that appear in $(S1)$. Denote by $(W^{\pm}_{\mu,i,\pm}, c_i)$ $(i = 0, 1, 2, 3)$ the simple Yetter–Drinfel'd modules of H with braiding matrices

$$q_1 = \begin{pmatrix} \xi^i & \xi^i \\ -\xi^i & \xi^i \end{pmatrix}, \quad q_2 = \begin{pmatrix} \xi^i & -\xi^i \\ \xi^i & \xi^i \end{pmatrix}.$$

Lemma 3.3.0.2. Then Nichlos algebras of $(W^{\pm}_{\mu,i,\pm}, c_i)$ $(i = 0, 1, 2, 3)$ are as follows,

(1) If $i = 1$ or 3, $q = q_1$, then $\mathcal{B}((W^{\pm}_{\mu,i,\pm}, c_i)) \simeq T(W^{\pm}_{\mu,i,\pm})/\langle vw \mp \xi^i wv, wv \pm \xi^i vw, v^4, w^4 \rangle$, and $\dim(\mathcal{B}((W^{\pm}_{\mu,i,\pm}, c_i))) = 16$.

(2) If $i = 1$ or 3, $q = q_2$, then $\mathcal{B}((W^{\pm}_{\mu,i,\pm}, c_i)) \simeq T(W^{\pm}_{\mu,i,\pm})/\langle vw \pm \xi^i wv, wv \mp \xi^i vw, v^4, w^4 \rangle$, and $\dim(\mathcal{B}((W^{\pm}_{\mu,i,\pm}, c_i))) = 16$.

(3) If $i = 2$, $q = q_1, q_2$, then Nichols algebras of $(W^{\pm}_{\mu,i,\pm}, c_i)$ are 8 dimensional and of type A_2.

(4) If $i=0,\ q=q_1,q_2$, then $\mathcal{B}((W^{\pm}_{\mu,i,\pm},c_i))$ are infinite dimensional.

Proof: Let $i=1$ or 3, then we have $q_{ij}q_{ji}=1$ for all $i\neq j$, and the order of q_{ii} are all equal to 4. By the result of ([9], [2, Example 27]), we have $\mathcal{B}((W^{\pm}_{\mu,i,\pm},c_j))\simeq T(W^{\pm}_{\mu,i,\pm})/\langle vw-q_{12}wv, wv-q_{21}v, v^4, w^4\rangle$ Furthermore, $\{v^{a_1}w^{a_2}: 0\leq a_k\leq 3\}$ is a basis of $\mathcal{B}(W^{\pm}_{\mu,i,\pm})$ and $\dim(\mathcal{B}((W^{\pm}_{\mu,i,\pm},c_j)))=\Pi_{i=1,2}N^i=16$. For (1), (2), Nichols algebras are quantum planes. (3) follows from ([13, Proposition 2.11]), Nichols algebras are of A_2 type. For (4), it is easy to show that $\mathcal{B}((W^{\pm}_{\mu,i,\pm},c_j))$ are infinite dimensional. $\square$

Next, we give the Nichols algebras of Yetter–Drinfel'd modules that appear in $(S2)$.

Lemma 3.3.0.3. Let $X^{\pm}_{\mu,i}$ $(i=0\ ,1)$ be simple Yetter–Drinfel'd modules of H with braiding matrices

$$q=\begin{pmatrix}\xi^i & \xi^i\\ \xi^i & \xi^i\end{pmatrix}\ (i=0,\ 2).$$

Then Nichlos algebras of $(X^{\pm}_{\mu,i},c_i)$ are as follows

(1) If $i=0$, then $\mathcal{B}((X^{\pm}_{\mu,i},c_i))\simeq S(X^{\pm}_{\mu,i})$.

(2) If $i=2$, then $\mathcal{B}((X^{\pm}_{\mu,i},c_i))\simeq\bigwedge(X^{\pm}_{\mu,i})$, and $\dim(\mathcal{B}((\mathrm{X}^{\pm}_{\mu,\mathrm{i}},\mathrm{c_i})))=4$.

Proof: The claims (1), (2) follow from ([2, example 31]). For (2), the Nichols algebra is of type $A_1\times A_1$ and $\mathcal{B}((X^{\pm}_{\mu,i},c_i))$ is a quantum plane. $\square$

We are in a position to give the Nichols algebras of Yetter–Drinfel'd modules that appear in $(S3)$ now. There are some Nichlos algebras of non-diagonal types which were also studied by Andruskiewitsch [8].

Lemma 3.3.0.4. Let $r,p,m\in\Bbbk$, assume $(Y_{\mu,j},c_{r,p,m})$ $(j=0,1,2,3,4,5,6,7,8\ ,r,p,m\in\Bbbk)$ be simple Yetter–Drinfel'd modules of H with braidings

$$c(v\otimes v)=rw\otimes w, c(v\otimes w)=pv\otimes w,$$
$$c(w\otimes v)=pw\otimes v, c(w\otimes w)=mv\otimes v.$$

The Nichlos algebras of $(Y_{\mu,j}, c_{r,p,m})$ are not of diagonal type and given as follow:

(1) If $p = -1, rm = -1$, then

$$\mathcal{B}((Y_{\mu,j}, c_{r,p,m})) \cong T(Y_{\mu,j})/\langle vw, wv, v^4 + w^4\rangle, \ \dim(\mathcal{B}((Y_{\mu,j}, c_{r,p,m}))) = 8.$$

(2) If $p = 1$, then $\dim(\mathcal{B}((Y_{\mu,j}, c_{r,p,m}))) = \infty$.

(3) If $rm = 1, \ p = \pm i$ and $r = -i$, then

$$\mathcal{B}((Y_{\mu,j}, c_{r,p,m})) \cong T(Y_{\mu,j})/\langle w^2 + iv^2, vwvw, wvwv\rangle, \ \dim(\mathcal{B}((Y_{\mu,j}, c_{r,p,m}))) = 16.$$

(4) If $rm = 1, \ p = \pm i, \ r = i$, then

$$\mathcal{B}((Y_{\mu,j}, c_{r,p,m})) \cong T(Y_{\mu,j})/\langle w^2 - iv^2, vwvw, wvwv\rangle, \ \dim(\mathcal{B}((Y_{\mu,j}, c_{r,p,m}))) = 16.$$

Proof: Since $(Y_{\mu,j}, c_{r,p,m})$ is diagonal type if and only if $p^2 = rm$ ([8, Remark 3.15]), it follows that the braidings are not of diagonal type. (1), (3), (4) are the results in ([8, Proposition 3.16]). It is sufficient to prove (2), let $n \in \mathbb{N}$ and let

$$a_n = \underbrace{v \otimes w \otimes v \otimes w \otimes v \otimes w \otimes v \otimes w \cdots}_{n}$$

be a vector in $\mathcal{B}((Y_{\mu,j}, c_{r,1,m}))$. After direct computation, we have $\Omega_n(a_n) = ka_n$ for some $k \neq 0 \in \Bbbk$. Thus, we have completed the proof. □

Finally, we give the Nichols algebras of Yetter–Drinfel'd modules that appear in $(S4)$.

Lemma 3.3.0.5. Let $(Z_{\mu,j}, c_i)$ $(j = 1,2,3,4,5,6,7,8; \ i = 0,1,2,3)$ be simple Yetter–Drinfel'd modules over H with braiding matrices

$$q = \begin{pmatrix} \xi^i & -\xi^i \\ \xi^i & \xi^i \end{pmatrix}.$$

Then the Nichols algebras of $(Z_{\mu,j}, c_i)$ are as follows,

(1) If $i = 1$or 3, then $\mathcal{B}((Z_{\mu,j}, c_i)) \simeq T(Z_{\mu,j})/\langle vw \pm \xi^i wv, wv \mp \xi^i vw, v^4, w^4\rangle$ and

$$\dim(\mathcal{B}((Z_{\mu,j}, c_i))) = 16.$$

(2) If $i = 2$, then $\mathcal{B}((Z_{\mu,j}, c_i)) \simeq T(Z_{\mu,j})/\langle vw - wv, v^2, w^2\rangle$ and

$$\dim(\mathcal{B}((Z_{\mu,j}, c_i))) = 8.$$

(3) If $i = 0$, then $\mathcal{B}((Z_{\mu,j}, c_i))$ is infinite dimensional.

Proof: This claim is similar as Lemma 3.3.0.2. □

With these preparations, we are in a position to classify all finite-dimensional Nichols algebras of simple Yetter–Drinfel'd modules V over H now.

Theorem 3.3.0.6. Let V be the simple Yetter–Drinfel'd modules over H, all finite dimensional Nichols algebras of V are as follows,

(1) Diagonal type: A_1, A_2 or quantum planes.

(2) Non-diagonal type: Nichols algebras $\mathcal{B}(V)$ are 8 or 16 dimensional.

Proof: Let $V \in {}^H_H\mathcal{YD}$ be the braided vector spaces of diagonal type. Due to the Lemmas 3.3.0.1, 3.3.0.2, 3.3.0.3, 3.3.0.5, it follows that Nichols algebras of V are finite dimensional if and only if $\mathcal{B}(V)$ are of Cartan type A_1, A_2 or quantum planes. Let $V \in {}^H_H\mathcal{YD}$ be the braided vector spaces of non-diagonal type, due to Lemma 3.3.0.4, it follows that Nichols algebras of V are finite dimensional if and only if

$$p = -1, \ rm = -1$$

or

$$rm = 1, \ p = \pm i.$$

i.e. Nichols algebras $\mathcal{B}(V)$ are 8 or 16 dimensional. □

3.4 Conclusion

In this Chapter, we proved that there are 88 non-isomorphic simple left $\mathcal{D}(H)$-modules, and 32 of them are one-dimensional and the other 56 ones are two-dimensional. Then we classify all finite-dimensional Nichols algebras of simple Yetter–Drinfel'd modules V over H. We find the Nichols algebras of non-diagonal type which were studied by Andruskiewitsch and Geraldi [8] . There are 8-, 16- or infinite dimensional (see Lemma 3.3.0.4).

In section 5.3, We will show that H is exactly the Hopf algebra $D(m, d, \gamma)/(y)$ of Wu–Liu–Ding in [Adv. Math. 296 (2016), 1−54] with $m = 2$ and $d = 2$, where $D(m, d, \gamma)/(y)$ will be introduced in the Chapter 4.

Chapter 4 On the Hopf algebras $D(m, d, \gamma)/(y)$

The Hopf algebras $D(m, d, \gamma)$ were introduced by Wu–Liu–Ding [30], they gave a complete classification of prime regular Hopf algebras of GK-dimension one and they constructed the Hopf algebras $D(m, d, \gamma)$. As the quotient Hopf algebras of $D(m, d, \gamma)$, we obtain that each

$$D(m, d, \gamma)/(y)$$

is a semisimple Hopf algebra.

4.1 Hopf algebras $D(m, d, \gamma)$

In this section, we recall some well-known facts about the Hopf algebras $D(m, d, \gamma)$. Let m, d be two natural numbers satisfying that $(1 + m)d$ is even and ξ a primitive $2m$-th root of 1. Define

$$\omega := md, \quad \gamma := \xi^2.$$

As an algebra, each $D = D(m, d, \gamma)$ is generated by

$$x^{\pm 1}, g^{\pm 1}, y, u_0, u_1, \cdots, u_{m-1},$$

subject to the following relations

$$xx^{-1} = x^{-1}x = 1, \qquad gg^{-1} = g^{-1}g = 1, \qquad xg = gx, \tag{4.1}$$

$$xy = yx, \qquad yg = \xi gy, \qquad y^m = 1 - x^\omega = 1 - g^m, \tag{4.2}$$

$$xu_i = u_i x^{-1}, \qquad yu_i = \varphi_i u_{i+1} = \xi x^d u_i y, \quad u_i g = \gamma^i x^{-2d} g u_i, \tag{4.3}$$

and

$$u_iu_j=\begin{cases}(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\varphi_i\varphi_{i+1}\cdots\varphi_{m-2-j}y^{i+j}g, & i+j\leqslant m-2,\\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}y^{i+j}g, & i+j=m-1,\\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\varphi_i\cdots\varphi_{m-1}\varphi_0\cdots\varphi_{m-2-j}y^{i+j-m}g, & \text{otherwise},\end{cases}$$

where $\varphi_i=1-\gamma^{-i-1}x^d$ and $0\leqslant i,j\leqslant m-1$.

The coproduct Δ, the counit ϵ and the antipode S of $D(m,d,\gamma)$ are given by

$$\Delta(x)=x\otimes x,\ \ \Delta(g)=g\otimes g,\ \ \Delta(y)=y\otimes g+1\otimes y, \tag{4.4}$$

$$\Delta(u_i)=\sum_{j=0}^{m-1}\gamma^{j(i-j)}u_j\otimes x^{-jd}g^ju_{i-j}; \tag{4.5}$$

$$\epsilon(x)=\epsilon(g)=\epsilon(u_0)=1,\ \ \epsilon(y)=\epsilon(u_s)=0; \tag{4.6}$$

$$S(x)=x^{-1},\ \ S(g)=g^{-1},\ \ S(y)=-yg^{-1}, \tag{4.7}$$

$$S(u_i)=(-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-i-1}u_i, \tag{4.8}$$

for $0\leq i\leq m-1$ and $1\leqslant s\leqslant m-1$.

4.2 The quotient Hopf algebras $D(m,d,\gamma)/(y)$

As the notations in the section 4.1, let m,d be two natural numbers satisfying that $(1+m)d$ is even and ξ a primitive $2m$-th root of 1. Assume

$$\omega:=md,\quad \gamma:=\xi^2.$$

Then each quotient Hopf algebra $D(m,d,\gamma)/(y)$, as an algebra, is generated by

$$x,g,u_0,u_1,\cdots,u_{m-1},$$

subject to the following relations

$$xg=gx,\ x^{\omega}=1,g^m=1, \tag{4.1}$$

$$xu_i=u_ix^{-1},\ u_i=\gamma^{-i}x^du_i,\ u_ig=\gamma^{-i}gu_i, \tag{4.2}$$

$$u_iu_j=\begin{cases}(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\varphi_0\cdots\varphi_{i-1}\cdots\varphi_{m-1}g, & \text{if }\ i+j\equiv 0\ (\mathrm{mod}\ m),\\ 0, & \text{otherwise},\end{cases} \tag{4.3}$$

where $\varphi_i = 1 - \gamma^{-i-1}x^d$ and $0 \leqslant i, j \leqslant m-1$. The coproduct $D(m,d,\gamma)/(y)$, the counit ϵ and the antipode S of $D(m,d,\gamma)/(y)$ are given by

$$\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g,$$
$$\Delta(u_i) = \sum_{j=0}^{m-1} u_j \otimes g^j u_{i-j},$$
$$\epsilon(x) = \epsilon(g) = \epsilon(u_0) = 1, \quad \epsilon(u_i) = 0,$$
$$S(x) = x^{-1}, \quad S(g) = g^{-1},$$
$$S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id + \frac{3}{2}(1-m)d} g^{m-i-1} u_i, \tag{4.4}$$

for $0 \leq i \leq m-1$.

In the following of this thesis, we will denote $D(m,d,\gamma)/(y)$ as D for convenience. Let C be the subspace spanned by

$$\{g^i u_j | 0 \leq i, j \leq m-1\}.$$

Then C is simple coalgebra.

We have the following decomposition of D into simple coalgbras

$$D = \bigoplus_{i=0}^{md-1} \bigoplus_{j=0}^{m-1} kx^i g^j \oplus \bigoplus_{i=0}^{d-1} x^i C.$$

It is not hard to see that the set

$$\{x^i g^j, x^l g^j u_k | 0 \leq i \leq md-1, 0 \leq j \leq m-1, 0 \leq l \leq d-1, 0 \leq k \leq m-1\}$$

is a basis of D. Denote by G be the group of all group-likes of D. Then clearly every element in D can be written uniquely in the following way:

$$f + \sum_{i=0}^{m-1} f_i u_i$$

for $f, f_i \in kG$ and $0 \leq i \leq m-1$.

We use $Z(D)$ to denote the center of D. Next result helps us to determine the center of D.

Lemma 4.2.0.1. The element $e = f + \sum_{i=0}^{m-1} f_i u_i \in Z(D)$ if and only if

$$f, f_0 u_0 \in Z(D)$$

and

$$f_1 = \ldots = f_{m-1} = 0.$$

Proof: The sufficiency is obvious. We just prove the necessity. At first, we show that

$$f_1 = \ldots = f_{m-1} = 0.$$

Otherwise, assume that, say, $f_1 \neq 0$. By assumption, $ge = eg$ which implies that

$$g f_1 u_1 = f_1 u_1 g.$$

By (5.2), $f_1 u_1 g = \gamma^{-1} g f_1 u_1$. So we have

$$\gamma^{-1} g f_1 u_1 = g f_1 u_1$$

which is absurd. Similarly, we have

$$f_2 = \ldots = f_{m-1} = 0.$$

Secondly, let's show that $f \in Z(D)$. Also, by $eu_i = u_i e$ and (4.3) we know that

$$f u_i = u_i f$$

for $0 \leq i \leq m-1$. By definition, f commutes with all elements in G. Therefore, $f \in Z(D)$. Since $e = f + f_0 u_0$ and $e \in Z(D)$, we also obtain that

$$f_0 u_0 \in Z(D).$$

$\square$

Let ζ be a md th root of unity satisfying $\zeta^d = \gamma$. Define

$$1_i^x := \frac{1}{md} \sum_{j=0}^{md-1} \zeta^{-ij} x^j,$$

$$1_k^g := \frac{1}{m} \sum_{j=0}^{m-1} \gamma^{-kj} g^j.$$

for $0 \le i \le md-1$ and $0 \le k \le m-1$. It is well-known that

$$\{1_i^x 1_k^g | 0 \le i \le md-1, 0 \le k \le m-1\}$$

is also a basis of kG. Therefore, one can assume that

$$f = \sum_{i=0}^{md-1} \sum_{j=0}^{m-1} a_{ij} 1_i^x 1_j^g = \sum_{i,j} a_{ij} 1_i^x 1_j^g.$$

For any natural number i, we use i' to denote the remainder of i divided by m.

Lemma 4.2.0.2. Let $f = \sum_{i,j} a_{ij} 1_i^x 1_j^g$ be an element in kG. Then $f \in Z(D)$ if and only if

$$a_{ij} = a_{md-i,j-i'}$$

for all $0 \le i \le md-1, 0 \le j \le m-1$.

Proof: Define

$$1_i^x := \frac{1}{d}(1 + \zeta^{-i}x + \zeta^{-2i}x^2 + \ldots + \zeta^{-(d-1)i}x^{d-1})$$

for $0 \le i \le md-1$. For any $0 \le k \le m-1$, it is not hard to see that the elements in

$$\{1_i^x | i \equiv k \pmod m\}$$

are linear independent. Direct computation shows that

$$1_i^x u_k = \begin{cases} 1_i^x u_k, & \text{if } i \equiv k \pmod m, \\ 0, & \text{otherwise,} \end{cases} \tag{4.5}$$

$$u_k 1_i^x = \begin{cases} u_k 1_i^x, & \text{if } i+k \equiv 0 \pmod m, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

and

$$1_{md-i}^x = 1_i^{x^{-1}}.$$

Therefore, we have

$$\begin{aligned} fu_k &= \sum_{i,j} a_{ij} 1_i^x 1_j^g u_k \\ &= \sum_{i,j} a_{ij} 1_i^x u_k 1_{j-k}^g \\ &= \sum_{i \equiv k \pmod m, j} a_{ij} 1_i^x 1_j^g u_k, \end{aligned}$$

and

$$\begin{aligned} u_k f &= \sum_{i,j} a_{ij} u_k 1_i^x 1_j^g \\ &= \sum_{i+k\equiv 0 \pmod{m},j} a_{ij} u_k 1_i^x 1_j^g \\ &= \sum_{i+k\equiv 0 \pmod{m},j} a_{ij} 1_i^{x^{-1}} u_k 1_j^g \\ &= \sum_{i+k\equiv 0 \pmod{m},j} a_{ij} 1_{md-i}^x u_k 1_j^g \\ &= \sum_{i+k\equiv 0 \pmod{m},j} a_{ij} 1_{md-i}^x 1_{j+k}^g u_k. \end{aligned}$$

This means that $fu_k = u_k f$ if and only if

$$a_{k+lm,j} = a_{m(d-l)-k,j-k}$$

for some $0 \leq k \leq m-1$. From this, the proof is completed. $\square$

Assume that $f_0 = \sum_{i,j} b_{ij} 1_i^x 1_j^g$. Using (4.5), we know that

$$f_0 u_0 = \sum_{i\equiv 0 \pmod{m},j} b_{ij} 1_i^x 1_j^g u_0.$$

So we can assume that

$$f_0 = \sum_{i\equiv 0 \pmod{m},j} b_{ij} 1_i^x 1_j^g$$

directly. With this assumption, we have the following result.

Lemma 4.2.0.3. The element $f_0 u_0$ belongs to the center of D if and only if

$$f_0 = \begin{cases} \sum_j b_{0j} 1_0^x 1_j^g, & \textit{if } md \textit{ is odd}, \\ \sum_j b_{0j} 1_0^x 1_j^g + \sum_j b_{\frac{md}{2},j} 1_{\frac{md}{2}}^x 1_j^g, & \textit{if } md \textit{ is even}. \end{cases} \tag{4.7}$$

Proof: From $xf_0u_0 = f_0u_0x$, we have

$$xf_0 = x^{-1} f_0,$$

which implies exactly the equation (4.7). The converse is straightforward. $\square$

Next, we want determine when a central element is idempotent.

Lemma 4.2.0.4. Let $e = f + f_0 u_0$ be an element living in the center $Z(D)$. Then $e^2 = e$ if and only if

$$f = f^2 + f_0^2 u_0^2$$

and

$$f_0 = 2 f f_0.$$

Proof: By Lemma 4.2.0.1, f commutes with $f_0 u_0$ and

$$(f_0 u_0)^2 = f_0^2 u_0^2.$$

From this, the lemma becomes clear. □

Lemma 4.2.0.5. Let $f = \sum_{i,j} a_{ij} 1_i^x 1_j^g$ and $f_0 = \sum_{i,j} b_{ij} 1_i^x 1_j^g$ satisfying

$$e = f + f_0 u_0$$

is a central element. Then e is an idempotent if and only if

$$\begin{aligned} a_{sm,j} &= a_{sm,j}^2 + b_{sm,j}^2 \zeta^{-\frac{1+m}{2} smd} \gamma^j \ (0 \le s \le d-1,\ 0 \le j \le m-1), && (4.8) \\ a_{ij}^2 &= a_{ij} \quad (i \not\equiv 0\ (mod\ m),\ 0 \le j \le m-1), && (4.9) \\ b_{ij} &= 2 a_{ij} b_{ij} \quad (0 \le i \le md-1,\ 0 \le j \le m-1). && (4.10) \end{aligned}$$

Proof: We just translate the equivalent conditions in Lemma 4.2.0.4 into the equalities about coefficients. □

By equation (4.10), we know that $a_{ij} = \frac{1}{2}$ if $b_{ij} \neq 0$. By equation (4.8), we have that

$$b_{sm,j} = \pm \frac{1}{2} \sqrt{\gamma^{-j} \zeta^{\frac{1+m}{2} smd}}$$

if $b_{sm,j} \neq 0$. We use $[.]$ to denote the floor function, i.e. for any rational number t, $\lfloor t \rfloor$ is the biggest integer which is not bigger than t.

Proposition 4.2.0.6. If md is even, then the following is a complete set of primitive central idempotents of D:

$$\begin{aligned}
&\frac{1}{2}1_0^x 1_j^g + \frac{1}{2}\sqrt{\gamma^{-j}}1_0^x 1_j^g u_0,\\
&\frac{1}{2}1_0^x 1_j^g - \frac{1}{2}\sqrt{\gamma^{-j}}1_0^x 1_j^g u_0,\\
&\frac{1}{2}1_{\frac{md}{2}}^x 1_j^g + \frac{1}{2}\sqrt{\gamma^{-j}(-1)^{\frac{1+m}{2}d}}1_{\frac{md}{2}}^x 1_j^g u_0,\\
&\frac{1}{2}1_{\frac{md}{2}}^x 1_j^g - \frac{1}{2}\sqrt{\gamma^{-j}(-1)^{\frac{1+m}{2}d}}1_{\frac{md}{2}}^x 1_j^g u_0,\\
&1_{sm}^x 1_j^g + 1_{(d-s)m}^x 1_j^g, \quad (0 < s \leq d-1, s \neq \frac{d}{2},\ 0 \leq j \leq m-1)\\
&1_{lm+i}^x 1_j^g + 1_{(d-l-1)m+(m-i)}^x 1_{j-i}^g,\\
&(0 \leq l \leq d-1, 0 < i \leq \lfloor \frac{m}{2} \rfloor,\ 0 \leq j \leq m-1).
\end{aligned}$$

If md is odd, then the following is a complete set of primitive central idempotents of D:

$$\begin{aligned}
&\frac{1}{2}1_0^x 1_j^g + \frac{1}{2}\sqrt{\gamma^{-j}}1_0^x 1_j^g u_0,\\
&\frac{1}{2}1_0^x 1_j^g - \frac{1}{2}\sqrt{\gamma^{-j}}1_0^x 1_j^g u_0,\\
&1_{sm}^x 1_j^g + 1_{(d-s)m}^x 1_j^g,\\
&(0 < s \leq d-1,\ 0 \leq j \leq m-1)\\
&1_{lm+i}^x 1_j^g + 1_{(d-l-1)m+(m-i)}^x 1_{j-i}^g,\\
&(0 \leq l \leq d-1, 0 < i \leq \lfloor \frac{m}{2} \rfloor,\ 0 \leq j \leq m-1).
\end{aligned}$$

Proof: According to Lemmas 4.2.0.2-4.2.0.5, we know all above elements are central idempotents. It is easy to find that the sum of these elements is just 1. So to show the result, it is enough to show that they are all primitive central idempotents. We just prove this fact for the case md even since the other case can be proved in the same way. In fact, by definition we can find the elements in the last two lines presented in this proposition can be decomposed into a sum of two idempotents which are not central, and so the simple modules corresponding to these central idempotents have dimension ≥ 2. There are $\frac{(d-2)m}{2} + \frac{(m-1)dm}{2}$ cental idempotents in the last two lines and $4m$ ones in the first two lines. Therefore, all of these idempotents create an ideal with dimension $\geq 4m + 4(\frac{(d-2)m}{2} + \frac{(m-1)dm}{2}) = 2m^2d = \dim D$. This implies they are all primitive. $\square$

Corollary 4.2.0.7. If md is even, then as an algebra D has the following decomposition:

$$D = k^{(4m)} \oplus M_2(k)^{(\frac{m^2d-2m}{2})}.$$

If md is odd, then as an algebra D has the following decomposition:

$$D = k^{(2m)} \oplus M_2(k)^{(\frac{m^2d-m}{2})}.$$

4.3 Conclusion

In this Chapter, we recall the definition of each Hopf algebras $D(m, d, \gamma)$ and their quotients. The Hopf algebras $D(m, d, \gamma)$ were introduced by Wu–Ding–Liu [30]. In their work, they gave a complete classification of prime regular Hopf algebras of GK-dimension one and explicitly constructed the algebras $D(m, d, \gamma)$. By taking quotients of $D(m, d, \gamma)$, we obtain the family of Hopf algebras

$$D(m, d, \gamma)/(y).$$

We study the properties of these quotient Hopf algebras. We will prove that each quotient Hopf algebra $D(m, d, \gamma)/(y)$ can be constructed by Abelian extension in next Chapter.

Chapter 5 The Abelian extensions $H(m, d)$

Recall that each Hopf algebras $D(m, d, \gamma)/(y)$ in section 4.2 is semisimple. We consider the following questions in this chapter:

1) Can each $D(m, d, \gamma)/(y)$ be constructed by an Abelian extension?
2) Can we classify all simple Yetter-Drinfeld modules V over $D(m, d, \gamma)/(y)$?
3) Can we classify all finite-dimensional Nichols algebras over $D(m, d, \gamma)/(y)$?

Let m, d be natural numbers such that $(1 + m)d$ is even. Let G be the finite abelian group

$$Z_{md} \times Z_m$$

generated by elements x and y with relations

$$x^{md} = 1, \quad y^m = 1, \quad xy = yx.$$

We denote by $\Bbbk G$ the group algebra of G, equipped with the usual Hopf algebra structure, and by $\Bbbk^G$ its dual Hopf algebra, i.e., the algebra of functions from G to $\Bbbk$.

Let $H(m, d)$ be the class of Hopf algebras defined as Abelian extensions as follows, Each $H(m, d)$ is constructed as

$$D \simeq \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2,$$

which is determined by the data $(\mathbb{Z}_2,\ G,\ \triangleleft,\ \triangleright,\ \sigma,\ \tau)$ given by

$$\begin{aligned}
&\triangleright \text{ is a trivial action,}\\
&g \triangleleft t = g,\\
&x \triangleleft t = x^{-1}g^{-1},\\
&\tau(x^i g^j,\ x^k g^l,\ t) = \gamma^{-jk},\\
&\sigma(x^i g^j,\ t,\ t) = \gamma^{\frac{-i^2+i+2(m-1)j}{2}}.
\end{aligned}$$

We can show that each $H(m,d)$ is a semisimple Hopf algebra.

For the question 1), we have the following conjecture.

Conjecture 5.0.0.1 (Gongxiang Liu)**.** Each semisimple Hopf algebra $D(m,d,\gamma)/(y)$ is exactly constructed by an Abelian extension $H(m,d)$.

We will prove this conjecture in section 5.1 and section 5.2. We will give an important example in section 5.3. For the question 2) and 3), we will present some results in Chapter 7.

5.1 Abelian extension

As the same notation in section 4.2, we denote by D the Hopf algebra

$$D = D(m,d,\gamma)/(y).$$

It is straightforward to verify that the elements

$$x^i g^j,\ x^l g^j u_k$$

with $0 \le i \le md-1$, $0 \le j \le m-1$, $0 \le l \le d-1$ and $0 \le k \le m-1$, which form a $\Bbbk$-basis of D. Denote by G the group of all group-like elements of D:

$$G = \mathbb{Z}_{md} \times \mathbb{Z}_m = \langle\, x, g \mid x^{md} = 1,\ g^m = 1,\ xg = gx \,\rangle.$$

For each $x^i g^j \in G$, recall that we have

$$e_{x^i g^j} = 1_i^x 1_j^g.$$

It is clear that the set

$$\{\, e_{x^i g^j} \mid 0 \le i \le md-1,\ 0 \le j \le m-1 \,\}$$

forms a dual basis of G. Let K be their linear span. Then we can obtain that

$$K \cong \Bbbk^G = (\Bbbk G)^*.$$

Now we define the following elements of D:

$$t = \sum_{i+j=m-1} g^i u_j, \qquad A_0 = x^d t g^{-1}, \qquad A_1 = gAg^{-1}, \tag{5.1}$$

$$A_2 = g^2 A g^{-2}, \quad \ldots, \quad A_{m-1} = g^{m-1} A g^{-(m-1)}. \tag{5.2}$$

Let B be the $m \times m$ matrix

$$B = \begin{pmatrix} g^{m-2} & \gamma^2 g^{m-3} & \cdots & \gamma^{2i} g^{m-i-2} & \cdots & \gamma^{2(m-1)} g^{-1} \\ g^{m-2} & \gamma^3 g^{m-3} & \cdots & \gamma^{3i} g^{m-i-2} & \cdots & \gamma^{3(m-1)} g^{-1} \\ \vdots & \vdots & & \vdots & & \\ g^{m-2} & \gamma^{m+1} g^{m-3} & \cdots & \gamma^{(m+1)i} g^{m-i-2} & \cdots & \gamma^{(m+1)(m-1)} g^{-1} \end{pmatrix}.$$

Proposition 5.1.0.1. The determinant of B is given by

$$\mid B \mid = \begin{cases} \displaystyle\prod_{1 \le j < i \le m-1} \left(\gamma^{i+1} - \gamma^{j+1}\right), & \text{if } m \text{ is odd,} \\ g^{\frac{m}{2}} \displaystyle\prod_{1 \le j < i \le m-1} \left(\gamma^{i+1} - \gamma^{j+1}\right), & \text{if } m \text{ is even.} \end{cases}$$

Proof: We express the entries of B as follows. The rows are indexed by $r = 1, \ldots, m$ and the columns are indexed by $c = 1, \ldots, m$, respectively. The (r, c)-entry of B is given by

$$B_{r,c} = \gamma^{(r+1)(c-1)}\, g^{\,m-c-1}.$$

Indeed, for $c = 1$ this yields g^{m-2}, and for $c = m$ we obtain g^{-1}; the exponent of γ gives $\gamma^{(r+1)(c-1)}$, reproducing the pattern $\gamma^{2i}, \gamma^{3i}, \ldots, \gamma^{(m+1)i}$ when $i = c - 1$.

Next, factor the column-dependent power of g from each column: from column c factor out g^{m-c-1}. This gives

$$\det B = \left(\prod_{c=1}^{m} g^{\,m-c-1} \right) \det\!\left(\gamma^{(r+1)(c-1)}\right)_{1 \le r,c \le m}.$$

The total exponent of g is

$$\sum_{c=1}^{m}(m-c-1)=\sum_{k=0}^{m-1}(k-1)=\frac{m(m-1)}{2}-m=\frac{m(m-3)}{2},$$

so the prefactor equals $g^{\frac{m(m-3)}{2}}$. The remaining determinant is of Vandermonde type. Setting $\alpha_r=\gamma^{r+1}$ for $r=1,\dots,m$, the matrix $(\gamma^{(r+1)(c-1)})$ consists of the column vectors $(1,\alpha_r,\alpha_r^2,\dots,\alpha_r^{m-1})_r$. Therefore,

$$\det\left(\gamma^{(r+1)(c-1)}\right)=\prod_{1\le j<i\le m}(\alpha_i-\alpha_j)=\prod_{1\le j<i\le m}\left(\gamma^{i+1}-\gamma^{j+1}\right).$$

Combining both factors, we obtain

$$\det B=g^{\frac{m(m-3)}{2}}\prod_{1\le j<i\le m}\left(\gamma^{i+1}-\gamma^{j+1}\right).$$

Finally, we simplify the power of g using $g^m=1$. If m is odd, then $\frac{m(m-3)}{2}$ is divisible by m, and hence $g^{\frac{m(m-3)}{2}}=1$. If m is even, write $m=2k$; then

$$g^{\frac{m(m-3)}{2}}=g^{k(2k-3)}=(g^k)^{2k-3}=g^k,$$

since $2k-3$ is odd and $g^{2k}=1$. Thus $g^{\frac{m(m-3)}{2}}=g^{m/2}$ when m is even. Consequently, we obtain that

$$\det B=\begin{cases}\displaystyle\prod_{1\le j<i\le m}\left(\gamma^{i+1}-\gamma^{j+1}\right), & \text{if } m \text{ is odd,}\\ \displaystyle g^{m/2}\prod_{1\le j<i\le m}\left(\gamma^{i+1}-\gamma^{j+1}\right), & \text{if } m \text{ is even.}\end{cases}$$

□

Remark 5.1.0.2. If γ is a root of unity whose order is less than m, then there exist indices i,j such that the determinant vanishes. In particular, when $\gamma=1$ (order 1) and $m\ge 2$, the determinant is zero. When $\gamma=-1$ (order 2) and $m\ge 3$, the determinant is zero. However, for the example $H(2,2)$ in section 5.3, the determinant is non-zero due to $m=2$. Furthermore, in all our examples, since each γ is primitive root of unity, each determinant of B is non-zero.

Theorem 5.1.0.3. Each Hopf algebra D is generated by $x^{\pm1}$, $g^{\pm1}$, and t.

Proof: From the relations $x^d u_i=\gamma^i u_i$ and $\gamma^i g^{-1}u_i=u_i g^{-1}$, we can define the generates A_i for $i=0,1,2,\dots,m-1$,

$$A_0=x^d t g^{-1}=\sum_{i=0}^{m-1}\gamma^{2i}g^{m-i-2}u_i.$$

and a sequence of similar elements:

$$A_i = g^i A_0 g^{-i} = \sum_{i=0}^{m-1} \gamma^{(i+2)i} g^{m-i-2} u_i, \qquad i = 0, 1, \ldots, m-1.$$

Let B denote the $m \times m$ matrix considered in Proposition 5.1.0.1. The elements $A_0, A_1, \ldots, A_{m-1}$ satisfy

$$\begin{cases} A_0 = x^d t g^{-1} = \displaystyle\sum_{i=0}^{m-1} \gamma^{2i}\, g^{m-i-2}\, u_i, \\ \\ A_k = g^k A_0 g^{-k} = g A_{k-1} g^{-1} = \displaystyle\sum_{i=0}^{m-1} \gamma^{(k+2)i}\, g^{m-i-2}\, u_i, \qquad k = 1, \ldots, m-1. \end{cases} \tag{5.3}$$

These relations are precisely equivalent to

$$(u_0, u_1, \ldots, u_{m-1})^T = B^{-1}(A_0, A_1, A_2, \ldots, A_{m-1})^T. \tag{5.4}$$

Since each A_k $(k = 0, \ldots, m-1)$ can be expressed in terms of $x^{\pm 1}$, $g^{\pm 1}$, and t, it follows from equation (5.4) and the definition of B in Proposition 5.1.0.1 that every u_i lies in the subalgebra generated by these elements. Therefore, D is generated by $x^{\pm 1}$, $g^{\pm 1}$, and t as an algebra. □

Example 5.1.0.4. Let $m = 3$, $d = 2$ and $\xi^6 = 1 = \gamma^3$, then $D = \langle x, g, u_0, u_1, u_2 \rangle$ with following relations,

$$\begin{aligned} &x^6 = 1 = g^3, u_0^2 = \tfrac{1}{3} x^{-2d} \varphi_0 \varphi_1 g, \\ &u_0 u_1 = 0 = u_1 u_0, u_0 u_2 = 0 = u_2 u_0, u_2^2 = 0, \\ &u_1 u_2 = \tfrac{1}{\gamma} x^{-2d} \varphi_1 \varphi_2 g, u_2 u_1 = (-1)\xi^{-1} \tfrac{1}{\gamma^2} x^{-2d} \varphi_2 \varphi_0 g, \\ &\varphi_0 = 1 - \gamma^{-1} x^d, \varphi_1 = 1 - \gamma^{-2} x^d, \varphi_2 = 1 - x^d, \\ &x u_i = u_i x^{-1}, u_0 = x^d u_0, \quad \gamma u_1 = x^d u_1, \\ &\gamma^2 u_2 = x^d u_2, \quad u_0 g = g u_0, \quad \gamma u_1 g = g u_1, \gamma^2 u_2 g = g u_2. \end{aligned}$$

Let $t = g^2 u_0 + g u_1 + u_2$, $A = x^d t g^{-1}$, $A_1 = g^{-1} t$, $A_2 = t g^{-1}$, i.e.

$$\begin{aligned} A &= x^d t g^{-1} = g u_0 + \gamma^2 u_1 + \gamma g^2 u_2 \\ A_1 &= g A g^{-1} = g^{-1} t = g u_0 + u_1 + g^2 u_2 \\ A_2 &= g^2 A g^{-2} = t g^{-1} = g u_0 + \gamma u_1 + \gamma^2 g^2 u_2 \end{aligned}$$

We have:

$$\begin{aligned} u_0 &= \tfrac{1}{3} g \left(t g^{-1} + x^d t g^{-1} + g^{-1} t \right) \\ u_1 &= \tfrac{1}{3} \left(\gamma^2 t g^{-1} + \gamma x^d t g^{-1} + g^{-1} t \right) \\ u_2 &= \tfrac{1}{3} g \left(\gamma t g^{-1} + \gamma^2 x^d t g^{-1} + g^{-1} t \right) \end{aligned} \qquad \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} g & g & g \\ \gamma & 1 & \gamma^2 \\ \gamma^2 g & g & \gamma g \end{pmatrix} \begin{pmatrix} A \\ A_1 \\ A_2 \end{pmatrix}$$

Thus, we have $D = \langle x, g, t\rangle$ with

$$x^6 = 1, \quad g^3 = 1.$$

Recall that $\varphi_i = 1 - \gamma^{-i-1}x^d$ for $0 \leqslant i \leqslant m-1$.

Proposition 5.1.0.5. We have

$$\varphi_i \cdot \cdots \cdot \varphi_{m+i-2} = \begin{cases} m, & \text{if } i - p \equiv 0 \pmod{m}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

Proof: By the definition of

$$e_{x^p g^q} \ (0 \le p \le md-1, \ 0 \le q \le m-1)$$

and by the relation

$$\zeta^d = \gamma = \xi^2, \tag{5.6}$$

we can obtain that

$$x^{nd} = \sum_p \sum_q \gamma^{np} e_{x^p g^q} \tag{5.7}$$

for all n in $\mathbb{Z}$, and

$$x^{-\frac{1+m}{2}d} g^{m-1} = \sum_p \sum_q \xi^{-p(1+m)} \gamma^{m-1} e_{x^p g^q}. \tag{5.8}$$

By the definition of φ_i, we obtain that $\varphi_i \cdot \cdots \cdot \varphi_{m+i-2}$ is equals to

$$(1 - \gamma^{-(i+1)}x^d)(1 - \gamma^{-(i+2)}x^d) \cdots (1 - \gamma^{-(m+i-1)}x^d). \tag{5.9}$$

By the equation (5.7) and (5.8), we obtain that the equation (5.9) is exactly the equation

$$\prod_{k=1}^{m-1} \left(\sum_{p=0}^{md-1} \sum_{q=0}^{m-1} (1 - \gamma^{-(i+k)}\gamma^p) e_{x^p g^q} \right). \tag{5.10}$$

By the equation (4.3), we get that equation (5.10) is equals to the equation:

$$\sum_p \sum_q [(1 - \gamma^{-(i+1)}\gamma^p)(1 - \gamma^{-(i+2)}\gamma^p) \cdots (1 - \gamma^{-(m+i-1)}\gamma^p)] e_{x^p g^q} \tag{5.11}$$

This equation (5.11) is exactly the equation:

$$\sum_p \sum_q \gamma^{(p-i)(m-1)}[(1+\gamma^{i-p}+(\gamma^{i-p})^2 \cdots (\gamma^{i-p})^{m-1}]e_{x^p g^q}. \tag{5.12}$$

Thus, it is easy to show that

$$\varphi_i \cdots\cdots \varphi_{m+i-2} = \begin{cases} m, & \text{if } i-p \equiv 0 \ (\text{mod } m), \\ 0, & \text{otherwise.} \end{cases} \tag{5.13}$$

□

Theorem 5.1.0.6. As an algebra, D is generated by the elements

$$e_{x^p g^q}(0 \le p \le md-1,\ 0 \le q \le m-1)$$

together with t, subject to the following relations:

$$e_{x^p g^q}\, e_{x^{p'} g^{q'}} = (\delta_{x^p g^q,\, x^{p'} g^{q'}})e_{x^p g^q}, \tag{5.14}$$

$$\sum_{p=0}^{md-1} \sum_{q=0}^{m-1} e_{x^p g^q} = 1, \tag{5.15}$$

$$t^2 = \sum_{p=0}^{md-1} \sum_{q=0}^{m-1} \gamma^{\frac{-p^2+p+2(m-1)q}{2}}\ e_{x^p g^q} \tag{5.16}$$

$$t\, e_x = e_{x^{-1} g^{-1}}\, t, \tag{5.17}$$

$$t\, e_g = e_g\, t. \tag{5.18}$$

Proof: We verify a part of the relations. First, we prove equation (5.16). Let

$$\lambda(i) = \frac{1}{m}(-1)^{m-i}\, \xi^{-2i(i-1)}\, (\xi^{m-i})^2.$$

It is easy to see that

$$\lambda(i) = \lambda(i'), \quad i \equiv i' \quad (\text{mod } m). \tag{5.19}$$

Thus, we can compute t^2 as follows:

$$t^2 = \sum_{i=0}^{m-1} (g^{m-1-i} u_i) \left(\sum_{l=0}^{m-1} g^{m-1-l} u_l \right) \tag{5.20}$$

$$\overset{(4.3)}{=} \sum_{i=0}^{m-1} g^{m-1-i} u_i\, g^{(m-1)-(m-i)} u_{m-i} \tag{5.21}$$

$$\overset{(4.2)}{=} \sum_{i=0}^{m-1} \gamma^{-i(i-1)} g^{m-2} u_i u_{m-i}. \tag{5.22}$$

By the equation (4.3), we get equation (5.22) is the eqaution

$$\sum_{i=0}^{m-1}(-1)^{m-i}\gamma^{-i(i-1)}\,\xi^{-(m-i)}\,\gamma^{\frac{(m-i)(m-i+1)}{2}}\,\frac{1}{m}\,\varphi_i\cdots\varphi_{m+i-2}\,x^{-\frac{1+m}{2}d}g^{m-1}. \tag{5.23}$$

By the equation (4.5), we get the equation (5.23) is exactly the equation

$$\sum_{i=0}^{m-1}\lambda(i)\,\varphi_i\cdots\varphi_{m+i-2}\,x^{-\frac{1+m}{2}d}g^{m-1}. \tag{5.24}$$

By the equation (5.19), one can get the equation (5.24) is equal to

$$\sum_{p=0}^{md-1}\sum_{q=0}^{m-1}\lambda(i)\,\xi^{-p(1+m)}\,\gamma^{m-1}e_{x^p g^q}. \tag{5.25}$$

$$=\sum_{p=0}^{md-1}\sum_{q=0}^{m-1}\xi^{p-p^2}\,\gamma^{m-1}e_{x^p g^q} \tag{5.26}$$

$$=\sum_{p=0}^{md-1}\sum_{q=0}^{m-1}\gamma^{\frac{-p^2+p+2(m-1)q}{2}}\,e_{x^p g^q}. \tag{5.27}$$

Therefore, equation (5.16) holds as claimed.
Next, we verify equation (5.17). Recall that $e_x = e_{x^1g^0} = 1_1^x 1_0^g$ and $e_g = 1_0^x 1_1^g$, where

$$1_i^x = \frac{1}{4}\sum_{j=0}^{3}\zeta^{-ij}x^j, \qquad 1_k^g = \frac{1}{2}\sum_{j=0}^{1}\gamma^{-kj}g^j.$$

Thus, we have the following equation:

$$te_x = \frac{1}{m^2d}t(\sum_{j=0}^{md-1}\zeta^{-j}x^j)(\sum_{l=0}^{m-1}g^l) = \frac{1}{m^2d}\sum_{j=0}^{md-1}(\zeta^{-j}x^{-j})t(\sum_{i=0}^{m-1}g^l) \tag{5.28}$$

$$=\frac{1}{m^2d}\sum_{j=0}^{md-1}(\zeta^{-j}x^{-j})(\sum_{l=0}^{m-1}x^{-ld}g^l)t \tag{5.29}$$

$$=\frac{1}{m^2d}\sum_{j=0}^{md-1}[\sum_{l=0}^{m-1}(\zeta^j x^j)(x^{-ld}g^l)]t \tag{5.30}$$

$$=\frac{1}{m^2d}\sum_{j=0}^{md-1}(\sum_{i=0}^{m-1}\zeta^j x^{j-ld}g^l)t \tag{5.31}$$

$$=\frac{1}{m^2d}\sum_{j=0}^{nd-1}(\sum_{l=0}^{m-1}\zeta^{j-ld}x^{j-ld}\zeta^{ld}g^l)t \tag{5.32}$$

$$\tag{5.33}$$

This is exactly the equation:

$$\frac{1}{m^2d}(\sum_{j=0}^{md-1}\zeta^{j-ld}x^{j-ld})(\sum_{l=0}^{m-1}\gamma^l g^l)t \tag{5.34}$$

$$=\frac{1}{m^2d}(\sum_{j=0}^{m^d-1}\zeta^j x^j)(\sum_{l=0}^{m-1}\gamma^l g^l)t \tag{5.35}$$

$$=\frac{1}{m^2d}(\sum_{j=0}^{m^d-1}\zeta^{-(md-1)j}x^j)(\sum_{l=0}^{m-1}\gamma^{-(m-1)l}g^l)t \tag{5.36}$$

$$=(e_{x^{-1}g^{-1}})t. \tag{5.37}$$

Finally, equation (5.18) follows by a similar argument:

$$te_g=\frac{1}{m^2d}t(\sum_{j=0}^{md-1}x^j)(\sum_{l=0}^{m-1}\gamma^{-l}g^l) \tag{5.38}$$

$$=\frac{1}{m^2d}(\sum_{j=0}^{md-1}x^{-j})t(\sum_{l=0}^{m-1}\gamma^{-l}g^l) \tag{5.39}$$

$$=\frac{1}{m^2d}(\sum_{j=0}^{md-1}x^j)(\sum_{l=0}^{m-1}\gamma^{-l}x^{-ld}g^l) \tag{5.40}$$

$$=\frac{1}{m^2d}(\sum_{j=0}^{md-1}x^j)(\sum_{l=0}^{m-1}\gamma^{-l}g^l) \tag{5.41}$$

$$=e_g t. \tag{5.42}$$

□

Proposition 5.1.0.7. The coproduct of t is given by

$$\Delta(t)=[\sum_{k'=0}^{md-1}\sum_{l'=0}^{m-1}\gamma^{-kl'}\gamma^{(m-2)l}e_{x^{k'}y^{l'}}\otimes\sum_{k=0}^{md-1}\sum_{l=0}^{m-1}\gamma^{(m-2)l}e_{x^ky^l}](t\otimes t). \tag{5.43}$$

Proof: Let $0\le i\le m-1$, we know that u_i is equal to the following equation:

$$\frac{1}{m}g^{i+2}(\gamma^{-2i}A_0+\gamma^{-3i}A_1+\gamma^{-4i}A_2+\cdots\gamma^{-(m+1)i}A_{m-1}) \tag{5.44}$$

The equation (5.44) is exactly the equation (5.45) and equation (5.46)

$$\frac{1}{m}(\gamma^{-2i}x^{2d}g^{i-1} + \gamma^{-3i}x^{3d}g^{i-1} + \cdots + \gamma^{-(m+1)i}x^{(m+1)d}g^{i-1})t, \tag{5.45}$$

$$\frac{1}{m}(\gamma^{-2i}x^{2d}g^{i-1} + \gamma^{-3i}x^{3d}g^{i-1} + \cdots + \gamma^{-mi}x^{md}g^{i-1} + \gamma^{-(m+1)i}x^{(m+1)d}g^{i-1})t. \tag{5.46}$$

Thus, the each element $g^{m-1-i+j}u_{i-j}$ is equal to

$$\frac{1}{m}(\gamma^{2(j-i)}x^{2d}g^{m-2} + \cdots + \gamma^{m(j-i)}x^{md}g^{m-2} + \gamma^{(m+1)(j-i)}x^{(m+1)d}g^{m-2})t. \tag{5.47}$$

Let $p = i - j$, we can get the equation (5.48)

$$\begin{aligned} &\frac{1}{m}(\gamma^{-2p}x^{2d}g^{m-2} + \gamma^{-3p}x^{3d}g^{m-2} + \gamma^{-4p}x^{4d}g^{m-2} + \\ &\cdots + \gamma^{-mp}x^{md}g^{m-2} + \gamma^{-(m+1)p}x^{(m+1)d}g^{m-2})t. \end{aligned} \tag{5.48}$$

This is exactly the equation (5.49)

$$\frac{1}{m}[\sum_{k=0}^{md-1}\sum_{l=0}^{m-1}\gamma^{(m-2)l}(1 + \gamma^{(k-p)} + \gamma^{(k-p)^2} + \gamma^{(k-p)^3} + \ldots \gamma^{(k-p)^{m-1}})e_{x^k y^l}t. \tag{5.49}$$

Thus, we get

$$g^{m-1-i+j}u_{i-j} = \sum_{(k \equiv p \text{ mod } m)}\sum_{l=0}^{m-1}\gamma^{(m-2)l}e_{x^k y^l}t. \tag{5.50}$$

Based on the equation (5.50), we obtain that

$$\begin{aligned}
\Delta(t) &= \sum_{i=0}^{m-1} \Delta\left(g^{m-1-i}u_i\right) && (5.51)\\
&= \sum_{i=0}^{m-1}(\sum_{j=0}^{m-1} g^{m-1-i}u_j \otimes g^{m-1-i+j}u_{i-j}) && (5.52)\\
&= \sum_{i=0}^{m-1}(\sum_{j=0}^{m-1} g^{m-1-(j+i-j)}u_j \otimes g^{m-1-(i-j)}u_{i-j}) && (5.53)\\
&= \sum_{i=0}^{m-1}(\sum_{j=0}^{m-1} g^{j-i}(\sum_{j=0}^{m-1} g^{m-1-j}u_j \otimes g^{m-1-(i-j)}u_{i-j}) && (5.54)\\
&\overset{p=i-j}{=} (\sum_{j=0}^{m-1} g^{-p}(\sum_{j=0}^{m-1} g^{m-1-j}u_j \otimes g^{m-1-p}u_p) && (5.55)\\
&= \sum_{p} g^{-p}t \otimes g^{m-1-p}u_p && (5.56)\\
&= \sum_{p}[\sum_{k'=0}^{md-1}\sum_{l'=0}^{m-1} \gamma^{-pl'} e_{x^{k'}y^{l'}}t \otimes \sum_{(k\equiv p \bmod m)} \sum_{l=0}^{m-1} \gamma^{(m-2)l} e_{x^k y^l} t] && (5.57)\\
&= \sum_{p}[\sum_{k'=0}^{md-1}\sum_{l'=0}^{m-1} \gamma^{-pl'} e_{x^{k'}y^{l'}} \otimes \sum_{(k\equiv p \bmod m)} \sum_{l=0}^{m-1} \gamma^{(m-2)l} e_{x^k y^l}](t\otimes t) && (5.58)\\
& && (5.59)
\end{aligned}$$

Let $p = k$, we get that the equation $\Delta(t)$ is equals to the equation (5.60):

$$\Delta(t) = [\sum_{k'=0}^{md-1}\sum_{l'=0}^{m-1} \gamma^{-kl'}\gamma^{(m-2)l} e_{x^{k'}y^{l'}} \otimes \sum_{k=0}^{md-1}\sum_{l=0}^{m-1} \gamma^{(m-2)l} e_{x^k y^l}](t\otimes t) \tag{5.60}$$

$\square$

Let t be the generator of $\Bbbk\mathbb{Z}_2$, assume $\triangleleft$ and $\triangleright$ are the actions: $G \overset{\triangleleft}{\leftarrow} G \times \Bbbk\mathbb{Z}_2 \overset{\triangleright}{\rightarrow} \Bbbk\mathbb{Z}_2$. Then $(\Bbbk\mathbb{Z}_2,\ G,\ \triangleleft,\ \triangleright)$ is a matched pair of groups and $(\sigma,\ \tau)$ is a pair of compatible normal cocycles of $(\Bbbk\mathbb{Z}_2,\ G,\ \triangleleft,\ \triangleright)$.

5.2 The proof of the Conjecture 5.0.0.1

Proof: Since $\dim D/\dim \Bbbk^G = 2$ is the smallest prime divisor of $\dim D$, it follows from [22] that F is normal. A finite-dimensional Hopf algebra of prime dimension p is isomorphic to the group algebra $\Bbbk\mathbb{Z}_p$ [31]. Therefore, we have $F \simeq \Bbbk\mathbb{Z}_2$. Based on the Section 2.3 and Section 5.1, we obtain that each Hopf algebra D is an Abelian extension of $\Bbbk\mathbb{Z}_2$ by $\Bbbk^G$. Equivalently, D is the Hopf algebra $H(m, d)$, constructed as the Abelian extension

$$D \simeq \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2,$$

which is determined by the data $(\mathbb{Z}_2,\, G,\, \triangleleft,\, \triangleright,\, \sigma,\, \tau)$ given by

$$x \triangleleft t = x^{-1}g^{-1}, \quad g \triangleleft t = g, \tag{5.1}$$

$$\tau(x^i g^j,\, x^k g^l,\, t) = \gamma^{-jk}, \tag{5.2}$$

$$\sigma(x^i g^j,\, t,\, t) = \gamma^{\frac{-i^2+i+2(m-1)j}{2}}. \tag{5.3}$$

$\square$

5.3 An example $H(2, 2)$

Let $m = 2 = d$, the Hopf algebra $H(2, 2) = D/(y)$ is generated by x, g, u_0, u_1, subject to the following relations

$$\begin{aligned}
&x^4 = 1 = g^2,\\
&xg = gx,\\
&xu_i = u_i x^{-1},\\
&u_i = \gamma^{-i}x^2 u_i \quad (\gamma = -1),\\
&u_i g = \gamma^{-i} g u_i \quad (\gamma = -1),\\
&u_0^2 = \tfrac{1}{2}(xg + x^3 g),\\
&u_1^2 = -\tfrac{\gamma}{2\xi}(xg - x^3 g) \quad (\gamma = -1),\\
&u_0 u_1 = 0 = u_1 u_0,
\end{aligned}$$

where ξ is a primitive $2m$-th root of 1 and $\zeta = i$ is a primitive dm-th root of 1 (in this example, we fix $\xi = -i$ and $\zeta = i$).

Let $t = gu_0 + u_1$, then the Hopf algebra H is generated by x, g, t satisfying the relations

$$\begin{aligned}
&x^4 = 1 = g^2,\\
&tx = x^{-1}t,\\
&tg = (x^2 g)t.
\end{aligned}$$

The coproducts of the generators are given by

$$\Delta(x) = x \otimes x,\ \Delta(g) = g \otimes g,$$

$$\Delta(u_0) = u_0 \otimes u_0 + u_1 \otimes gu_1,$$
$$\Delta(u_1) = u_0 \otimes u_1 + u_1 \otimes gu_0.$$

Since Δ is an algebra homomorphism, we have

$$\Delta(t) = \Delta(gu_0) + \Delta(u_1) = \Delta(g)\Delta(u_0) + \Delta(u_1).$$

Step 1. We compute $\Delta(gu_0)$:

$$\begin{aligned}\Delta(gu_0) &= (g \otimes g)(u_0 \otimes u_0 + u_1 \otimes gu_1)\\ &= gu_0 \otimes gu_0 + gu_1 \otimes g^2 u_1.\end{aligned}$$

Since $g^2 = 1$, this simplifies to

$$\Delta(gu_0) = gu_0 \otimes gu_0 + gu_1 \otimes u_1.$$

Step 2. Adding $\Delta(u_1)$, we obtain

$$\Delta(t) = gu_0 \otimes gu_0 + gu_1 \otimes u_1 + u_0 \otimes u_1 + u_1 \otimes gu_0. \tag{EDt}$$

Using the relations $u_1 = -x^2 u_1$ and $g^2 = 1$, one checks that the operator

$$\frac{1}{2}\big((1+g) \otimes 1 + (1-g) \otimes x^2\big)$$

acts on $t \otimes t$ by transforming the terms $gu_0 \otimes u_1$ and $u_1 \otimes u_1$ into $gu_1 \otimes u_1$ and $u_0 \otimes u_1$, respectively.

Step 3. Observe that

$$t \otimes t = (gu_0 + u_1) \otimes (gu_0 + u_1)$$
$$= gu_0 \otimes gu_0 + gu_0 \otimes u_1 + u_1 \otimes gu_0 + u_1 \otimes u_1.$$

Using the relations

$$u_1 = -x^2 u_1, \qquad gu_1 = -x^2 gu_1,$$

which hold in the case $m = d = 2$, the expression in (EDt) can be rewritten as

$$\Delta(t) = \frac{1}{2}\Big((1+g) \otimes 1 + (1-g) \otimes x^2\Big)(t \otimes t).$$

Therefore, the coproduct of t is given by

$$\Delta(t) = \frac{1}{2}((1+g) \otimes 1 + (1-g) \otimes x^2)(t \otimes t),$$

$$\Delta(x) = x \otimes x,\ \Delta(g) = g \otimes g,$$

We compute the action of t on the primitive idempotent e_x. Since $m = d = 2$. We have $md = 4$ and $\zeta = i$ is a primitive 4-th root of unity. Since $\gamma = \zeta^d = \zeta^2 = -1$. For $0 \le i \le 3$ and $0 \le k \le 1$, recall that

$$1_i^x = \frac{1}{4}\sum_{j=0}^{3} \zeta^{-ij} x^j, \qquad 1_k^g = \frac{1}{2}\sum_{j=0}^{1} \gamma^{-kj} g^j.$$

The idempotent e_x. Since $e_x = e_{x^1 g^0} = 1_1^x 1_0^g$, we compute

$$\begin{aligned} 1_1^x &= \frac{1}{4}\big(1 + \zeta^{-1}x + \zeta^{-2}x^2 + \zeta^{-3}x^3\big) \\ &= \frac{1}{4}\big(1 - ix - x^2 + ix^3\big), \end{aligned}$$

where we use $\zeta^{-1} = -i$, $\zeta^{-2} = -1$, and $\zeta^{-3} = i$. Moreover,

$$1_0^g = \frac{1}{2}(1 + g).$$

Hence

$$e_x = \frac{1}{8}\big(1 - ix - x^2 + ix^3\big)(1 + g).$$

The idempotent $e_{x^3 g}$. Since $e_{x^3 g} = e_{x^3 g^1} = 1_3^x 1_1^g$, we have

$$\begin{aligned} 1_3^x &= \frac{1}{4}\big(1 + \zeta^{-3}x + \zeta^{-6}x^2 + \zeta^{-9}x^3\big) \\ &= \frac{1}{4}\big(1 + ix - x^2 - ix^3\big), \end{aligned}$$

using $\zeta^{-3} = i$, $\zeta^{-6} = -1$, and $\zeta^{-9} = -i$. Furthermore,

$$1_1^g = \frac{1}{2}(1 - g).$$

Therefore,

$$e_{x^3 g} = \frac{1}{8}\big(1 + ix - x^2 - ix^3\big)(1 - g).$$

We obtain that

$$\begin{aligned}
te_x &= t\frac{1}{8}\left(1+i^{-1}x+i^{-2}x^2+i^{-3}x^3\right)(1+x^2g)\\
&= \frac{1}{8}\left(1+i^3x^3+i^2x^2+ix\right)(1+x^2g)t\\
&= \frac{1}{8}\left(1+i^3x^3+i^2x^2+ix+xg^2-ixg-g+ix^3g\right)t\\
&= \frac{1}{8}\left(1+ix+i^2x^2+i^3x^3\right)(1-g)t\\
&= \frac{1}{8}\left(1+ix-x^2-ix^3\right)(1-g)t\\
&= e_{x^3g}t,
\end{aligned}$$

where the fourth equality follows from a direct regrouping of terms using $x^4=1$ and $g^2=1$. Thus, we can also obtain that $x \triangleleft t = x^{md-1}g^{-1}$.

We are going to computer te_g. Recall that $x^4=1$, $g^2=1$, $\gamma=-1$, and

$$x^2u_0 = u_0, \qquad x^2u_1 = -u_1.$$

The element t is given by $t = gu_0+u_1$. The primitive idempotent corresponding to $g\in G$ is

$$e_g = 1_0^x1_1^g = \frac{1}{8}(1+x+x^2+x^3)(1-g).$$

We compute te_g directly. First,

$$\begin{aligned}
te_g &= (gu_0+u_1)\frac{1}{8}(1+x+x^2+x^3)(1-g)\\
&= \frac{1}{8}(1+x^{-1}+x^{-2}+x^{-3})(gu_0+u_1)(1-g)\\
&= \frac{1}{8}(1+x+x^2+x^3)(gu_0+u_1)(1-g),
\end{aligned}$$

where we used $x^4=1$ in the last equality. Next,

$$\begin{aligned}
(gu_0+u_1)(1-g) &= gu_0-u_0+u_1-u_1g\\
&= gu_0-u_0+u_1-x^2gu_1.
\end{aligned}$$

Using the relation $x^2u_1=-u_1$, we obtain

$$x^2gu_1 = gx^2u_1 = -gu_1,$$

and hence

$$(gu_0+u_1)(1-g) = gu_0-u_0+u_1+gu_1.$$

Adding this equation into the previous expression yields

$$\begin{aligned} te_g &= \frac{1}{8}(1+x+x^2+x^3)(gu_0 - u_0 + u_1 + gu_1) \\ &= \frac{1}{8}(1+x+x^2+x^3)(1-g)(gu_0 + u_1) \\ &= e_g\, t. \end{aligned}$$

This equality relies essentially on the relation $x^2u_1 = -u_1$. Therefore, we have

$$g \triangleleft t = g.$$

Recall that

$$t = gu_0 + u_1.$$

We compute t^2 directly:

$$\begin{aligned} t^2 &= (gu_0 + u_1)(gu_0 + u_1) \\ &= g^2u_0^2 + gu_0u_1 + u_1gu_0 + u_1^2. \end{aligned}$$

Using the relations

$$g^2 = 1, \qquad u_0u_1 = u_1u_0 = 0,$$

this reduces to

$$t^2 = u_0^2 + u_1^2.$$

By the defining multiplication rules of the u_i in the case $m = d = 2$, we have

$$u_0^2 = \frac{1}{2}(xg + x^3g), \qquad u_1^2 = -\frac{\gamma}{2\xi}(xg - x^3g).$$

Since $\gamma = -1$, it follows that

$$u_1^2 = \frac{1}{2\xi}(xg - x^3g).$$

Therefore,

$$\begin{aligned} t^2 &= \frac{1}{2}(xg + x^3g) + \frac{1}{2\xi}(xg - x^3g) \\ &= \frac{1}{2}\left(1 + \xi^{-1}\right)xg + \frac{1}{2}\left(1 - \xi^{-1}\right)x^3g. \end{aligned}$$

Equivalently, expressing t^2 in the idempotent basis

$$\{\, e_{x^pg^q} \mid 0 \le p \le 3,\ 0 \le q \le 1 \,\},$$

we claim that

$$t^2 = \sum_{p=0}^{3}\sum_{q=0}^{1} \gamma^{\frac{-p^2-p-2q}{2}}\, e_{x^pg^q},$$

Recall that we are given parameters $m = d = 2$, so we assume $\zeta = i$ (a primitive 4th root of unity) and $\xi = -i$, and $\gamma = \zeta^d = \zeta^2 = -1$. We are going to compare the following two equations:

$$t^2 = \sum_{p=0}^{3} \sum_{q=0}^{1} \gamma^{\frac{-p^2-p-2q}{2}} e_{x^p g^q},$$

and

$$t^2 = \frac{1}{2}(1+\xi^{-1})xg + \frac{1}{2}(1-\xi^{-1})x^3 g,$$

where $\xi = -i$ is a primitive 4-th root of unity and

$$e_{x^i g^j} = 1_i^x 1_j^g = \left(\frac{1}{4}\sum_{k=0}^{3} \zeta^{-ik} x^k\right)\left(\frac{1}{2}\sum_{l=0}^{1} \gamma^{-jl} g^l\right).$$

Explicit computation of the right-hand side

First, expand $e_{x^p g^q}$:

$$e_{x^p g^q} = \frac{1}{8}\sum_{a=0}^{3}\sum_{b=0}^{1} \zeta^{-pa}\gamma^{-qb} x^a g^b.$$

Thus,

$$\sum_{p,q} \gamma^{\frac{-p^2-p-2q}{2}} e_{x^p g^q} = \frac{1}{8}\sum_{p=0}^{3}\sum_{q=0}^{1} \gamma^{\frac{-p^2-p-2q}{2}} \sum_{a=0}^{3}\sum_{b=0}^{1} \zeta^{-pa}\gamma^{-qb} x^a g^b.$$

Interchange sums:

$$= \frac{1}{8}\sum_{a=0}^{3}\sum_{b=0}^{1}\left(\sum_{p=0}^{3} \zeta^{-pa}\gamma^{\frac{-p^2-p}{2}} \sum_{q=0}^{1} \gamma^{-q(b+1)}\right) x^a g^b,$$

since $\gamma^{\frac{-p^2-p-2q}{2}} = \gamma^{\frac{-p^2-p}{2}}\gamma^{-q}$ and $\gamma^{-qb}\gamma^{-q} = \gamma^{-q(b+1)}$. Now compute the inner sum over q. Since $\gamma = -1$,

$$\sum_{q=0}^{1} \gamma^{-q(b+1)} = \sum_{q=0}^{1} (-1)^{-q(b+1)} = \begin{cases} 2 & \text{if } (-1)^{b+1} = 1, \\ 0 & \text{if } (-1)^{b+1} = -1. \end{cases}$$

Thus, the sum is nonzero only when $b+1$ is even, i.e., $b = 1$. For $b = 1$, it equals 2. Hence,

$$\sum_{p,q} \gamma^{\frac{-p^2-p-2q}{2}} e_{x^p g^q} = \frac{1}{8}\sum_{a=0}^{3}\left(\sum_{p=0}^{3} \zeta^{-pa}\gamma^{\frac{-p^2-p}{2}}\right)\cdot 2 \cdot x^a g = \frac{1}{4}\sum_{a=0}^{3} S(a) x^a g,$$

where $S(a) = \sum_{p=0}^{3} \zeta^{-pa}\gamma^{\frac{-p^2-p}{2}}$.

Evaluation of $S(a)$

With $\zeta = i$ and $\gamma = -1$, we have $\gamma^{\frac{-p^2-p}{2}} = (-1)^{\frac{-p^2-p}{2}} = (-1)^{-\frac{p(p+1)}{2}} = (-1)^{\frac{p(p+1)}{2}}$. Compute for $p = 0, 1, 2, 3$:

$$\begin{aligned}
p = 0 : &\quad \frac{0(0+1)}{2} = 0 \Rightarrow (-1)^0 = 1, \quad \zeta^{-0a} = 1, \\
p = 1 : &\quad \frac{1 \cdot 2}{2} = 1 \Rightarrow (-1)^1 = -1, \quad \zeta^{-a} = i^{-a}, \\
p = 2 : &\quad \frac{2 \cdot 3}{2} = 3 \Rightarrow (-1)^3 = -1, \quad \zeta^{-2a} = i^{-2a} = (-1)^{-a} = (-1)^a, \\
p = 3 : &\quad \frac{3 \cdot 4}{2} = 6 \Rightarrow (-1)^6 = 1, \quad \zeta^{-3a} = i^{-3a} = i^{-a} i^{-2a} = i^{-a}(-1)^a.
\end{aligned}$$

Thus,

$$S(a) = 1 - i^{-a} - (-1)^a + i^{-a}(-1)^a = (1 - (-1)^a)(1 - i^{-a}).$$

If a is even, $(-1)^a = 1$ so $S(a) = 0$. If a is odd, $(-1)^a = -1$, so $1 - (-1)^a = 2$, and

$$S(a) = 2(1 - i^{-a}).$$

Compute for odd a:

$$\begin{aligned}
a = 1 : &\quad i^{-1} = -i \Rightarrow 1 - i^{-1} = 1 + i, \quad S(1) = 2(1 + i), \\
a = 3 : &\quad i^{-3} = i^{-1} i^{-2} = (-i)(-1) = i \Rightarrow 1 - i^{-3} = 1 - i, \quad S(3) = 2(1 - i).
\end{aligned}$$

Therefore,

$$\frac{1}{4} \sum_{a=0}^{3} S(a) x^a g = \frac{1}{4} \big(2(1+i)xg + 2(1-i)x^3 g\big) = \frac{1}{2}(1+i)xg + \frac{1}{2}(1-i)x^3 g.$$

Comparison with t^2

Since $\xi = -i$, we have:

$$t^2 = \frac{1}{2}(1 + \xi^{-1})xg + \frac{1}{2}(1 - \xi^{-1})x^3 g = \sum_{p=0}^{3} \sum_{q=0}^{1} \gamma^{\frac{-p^2-p-2q}{2}} e_{x^p g^q}.$$

Finally, if we compare the above formulas for the product and coproduct in $H(2,2)$ with the corresponding formulas for $k^G\#_{\sigma,\tau}kF$ which are giving as follows:

$$\begin{aligned}
&e_g e_h = \delta_{g,h} e_g,\ te_g = e_{g\triangleleft t}t,\ t^2 = \sum_{g\in G}\sigma(g,t,t)e_g,\\
&\Delta(e_g) = \sum_{h,k\in G,\ hk=g} e_h\otimes e_k,\ \Delta(t) = [\sum_{g,h\in G}\tau(g,h,t)e_g\otimes e_h](t\otimes t),\\
&\mathcal{S}(t) = \sum_{g\in G}\sigma(g^{-1},t,t)^{-1}\tau(g,g^{-1},t)^{-1}e_{(g\triangleleft t)^{-1}}t,\ \mathcal{S}(e_g) = e_{g^{-1}},\\
&\epsilon(t) = 1,\ \epsilon(e_g) = \delta_{g,1}1, \quad \forall g,h\in G.
\end{aligned} \tag{5.1}$$

and using a surjective map

$$\begin{aligned}
&\pi : H \to kZ_2 = \Bbbk\langle 1,\omega\rangle;\\
&t \mapsto \omega, e_{x^p g^q} \mapsto \varepsilon(e_{x^p g^q}) = \delta_{e_{x^p g^q},1}\cdot 1,\\
&(u_1 \mapsto g, u_0 \mapsto 0).
\end{aligned} \tag{5.2}$$

We deduce

$$x \triangleleft t = x^{-1}g^{-1}, \quad g\triangleleft t = g, \tag{5.3}$$

$$\tau(x^i g^j, x^k g^l, t) = (-1)^{-jk}, \tag{5.4}$$

$$\sigma(x^i g^j, t, t) = (-1)^{\frac{-i^2-i-2j}{2}} = (-1)^{\frac{(i+2j)(i+2j-1)}{2}}. \tag{5.5}$$

If we replace g by x^2y, then the Hopf algebra H is determined by $(Z_2, G, \triangleleft, \triangleright, \sigma, \tau)$, where $\triangleright$ is a trivial action,

$$x\triangleleft t = x^{-1}g^{-1} = x^{-1}x^2y = xy,\ y\triangleleft t = y,$$

$$\sigma(x^i y^j, t, t) = (-1)^{\frac{i(i-1)}{2}},$$

$$\tau(x^i y^j, x^k y^l, t) = (-1)^{jk}.$$

Thus, the Hopf algebra $H(2,2)$ is the $H_{c:\sigma_0}$ of Kashina [18]. We have studied the Hopf algebra $H_{c:\sigma_0}$ in the chapter 3.

5.4 More examples

Case $m = 2$ and d is even

Let $m = 2$, $x^{2d} = 1$, $\zeta^{2d} = 1$, $\gamma^2 = 1$, $\zeta^d = \gamma = \xi^2$, $\gamma^{-1} = \zeta^{-d}$. Since $m = 2$ and d is even, we have $(1+m)d$ is even. Each Hopf algebra

$$H = D(2,d,\xi)/(y)$$

is generated by x, g, u_0, u_1, subject to the following relations

$$x^{2d} = 1 = g^2, xg = gx; \tag{5.1}$$

$$xu_i = u_i x^{-1},\ u_i = (-1)^{-i} x^2 u_i,\ u_i g = (-1)^i g u_i; \tag{5.2}$$

$$u_0^2 = \frac{1}{2}(x^{-\frac{3d}{2}}(1+x^d)), u_1^2 = \frac{1}{2i}(x^{-\frac{3d}{2}}(1-x^d)), u_0u_1 = 0 = u_1u_0. \tag{5.3}$$

Let $t = gu_0 + u_1$, we have

$$u_1 = \frac{1}{2}(t - x^2 t), \quad u_0 = \frac{1}{2}(tg + x^2 tg).$$

Then the Hopf algebra $D(2, d, \xi)/(y)$ is also generated by x, g, t satisfying the relations

$$\begin{aligned} &x^{2d} = 1 = g^2, \\ &tx = x^{-1}t, tg = (x^{-d}g)t, \\ &t^2 = u_0^2 + u_1^2, \\ &\Delta(t) = \tfrac{1}{2}((1+g)\otimes 1 + (1-g)\otimes x^d)(t\otimes t), \\ &te_x = (e_{x^{2d-1}g})t, \quad te_g = e_g t. \end{aligned} \tag{5.4}$$

If we compare the formulas for the product and coproduct in $H(2, d)$ with some corresponding formulas for $k^G\#_{\sigma,\tau}kF$, we deduce

$$x \triangleleft t = x^{2d-1}g^{-1}, \quad g \triangleleft t = g, \tag{5.5}$$

$$\tau(x^i g^j, x^k g^l, t) = (-1)^{-jk}, \tag{5.6}$$

$$\sigma(x^i g^j, t, t) = (-1)^{\frac{(i+2j)(i+2j-1)}{2}}. \tag{5.7}$$

These formulae complete the description of H as a a bicrossed product $k^G\#_{\sigma,\tau}kF$.

Case $m = 3$ and $d = 2$

Let $m = 3$, $d = 2$ and $\gamma^3 = 1$, $\xi^2 = \gamma = \zeta^2$, Following the Example 5.1.0.4, we have $D = \langle x, g, u_0, u_1, u_2\rangle = \langle x, g, t\rangle$ with

$$x^6 = 1, \quad g^3 = 1 \quad \text{and} \quad t = g^2u_0 + gu_1 + u_2.$$

Since $m = 3$, $d = 2$, $\xi^6 = 1$, $\gamma^3 = 1$, and generators x, g, u_0, u_1, u_2. Using the multiplication and commutation relations

$$u_0 g = gu_0, \quad \gamma u_1 g = gu_1, \quad \gamma^2 u_2 g = gu_2,$$

and the non-zero products

$$u_0^2 = \frac{1}{3}x^{-4}\varphi_0\varphi_1 g, \quad u_1u_2 = \frac{1}{\gamma}x^{-4}\varphi_1\varphi_2 g, \quad u_2u_1 = -\frac{1}{\gamma^2}\xi^{-1}x^{-4}\varphi_2\varphi_0 g,$$

where the index p is taken modulo 6, q modulo 3. The φ_i factors are

$$\varphi_0 = 1 - \gamma^{-1}x^2, \quad \varphi_1 = 1 - \gamma^{-2}x^2, \quad \varphi_2 = 1 - \gamma^{-3}x^2.$$

We compute

$$\begin{aligned} t^2 &= (g^2u_0 + gu_1 + u_2)^2 \\ &= (g^2u_0)^2 + (gu_1)u_2 + u_2(gu_1) \\ &= gu_0^2 + \frac{1}{\gamma^2}u_1u_2g - \frac{1}{\gamma}\xi^{-1}u_2u_1g \\ &= \frac{1}{3}x^{-4}\varphi_0\varphi_1g^2 + \frac{1}{\gamma^2}x^{-4}\varphi_1\varphi_2g^2 - \frac{1}{\gamma}\xi^{-1}x^{-4}\varphi_2\varphi_0g^2. \end{aligned}$$

Let $e_{x^pg^q} = \frac{1}{6}\sum_{j=0}^{5}\sum_{l=0}^{2}\zeta^{-pj}\gamma^{-ql}x^jg^l$ with $\zeta^6 = 1$ and $\gamma^3 = 1$. Then t^2 can be expressed as a linear combination of $e_{x^pg^q}$:

$$t^2 = \sum_{p=0}^{5}\sum_{q=0}^{2} c_{p,q}\, e_{x^pg^q},$$

where the coefficients $c_{p,q}$ are obtained as $(-1)^{\frac{(p+2q)(p+2q-1)}{2}}$.

The coproduct is given by

$$\Delta(t) = \Delta(g^2u_0 + gu_1 + u_2) = \Delta(g^2)\Delta(u_0) + \Delta(g)\Delta(u_1) + \Delta(u_2).$$

Using

$$\Delta(g^k) = g^k \otimes g^k, \quad \Delta(u_i) = \sum_{j=0}^{2} u_j \otimes g^ju_{i-j},$$

we compute each term:

$$\begin{aligned} \Delta(g^2u_0) &= (g^2 \otimes g^2)(u_0 \otimes u_0 + u_1 \otimes gu_2 + u_2 \otimes g^2u_1) \\ &= g^2u_0 \otimes g^2u_0 + g^2u_1 \otimes g^3u_2 + g^2u_2 \otimes g^4u_1 \\ &= g^2u_0 \otimes g^2u_0 + g^2u_1 \otimes u_2 + g^2u_2 \otimes gu_1. \end{aligned}$$

Similarly,

$$\begin{aligned} \Delta(gu_1) &= gu_0 \otimes gu_1 + gu_1 \otimes g^2u_0 + gu_2 \otimes g^3u_2 \\ &= gu_0 \otimes gu_1 + gu_1 \otimes g^2u_0 + gu_2 \otimes u_2, \\ \Delta(u_2) &= u_0 \otimes u_2 + u_1 \otimes gu_1 + u_2 \otimes g^2u_0. \end{aligned}$$

Adding all terms, we obtain

$$\Delta(t) = \Delta(g^2u_0) + \Delta(gu_1) + \Delta(u_2) = (g^2 \otimes g^2)(u_0 \otimes u_0 + u_1 \otimes gu_2 + u_2 \otimes g^2u_1 +$$
$$(g \otimes g)(u_0 \otimes u_1 + u_1 \otimes gu_0 + u_2 \otimes g^2u_2) + u_0 \otimes u_2 + u_1 \otimes gu_1 + u_2 \otimes g^2u_0$$
$$= t \otimes g^2u_0 + g^2t \otimes gu_1 + gt \otimes u_2.$$

On the other hand, we have

$$t \otimes t = (g^2u_0 + gu_1 + u_2) \otimes (g^2u_0 + gu_1 + u_2)$$
$$= g^2u_0 \otimes g^2u_0 + g^2u_0 \otimes gu_1 + g^2u_0 \otimes u_2 + gu_1 \otimes g^2u_0 + gu_1 \otimes gu_1 + gu_1 \otimes u_2 +$$
$$u_2 \otimes g^2u_0 + u_2 \otimes gu_1 + u_2 \otimes u_2.$$

In practice, one can factor out $t \otimes t$ and write

$$\Delta(t) = \Big[\sum_{g,h\in G} \tau(g,h,t)e_g \otimes e_h\Big](t \otimes t),$$

where $\tau(g,h,t)$ are determined by the multiplication relations of u_iu_j and the commutation relations with g and x. There are six non-zero terms in the sum corresponding to the indices satisfying $i + j \equiv 0 \pmod 3$. And in this example, we have

$$\tau(x^ig^j, x^kg^l, t) = (-1)^{-jk}.$$

5.5 Conclusion

In this Chapter, we study the Hopf algebras $H(m,d)$. First, we explain what these Abelian extensions are. Then, we prove a conjecture by Gongxiang Liu (Conjecture 5.0.0.1). This proof is important and we will use this result later in Chapter 7 to help us classify the Yetter–Drinfeld modules over $H(m,d)$. To make things clear, we also work through a full example for the algebra $H(2,2)$. This show how to proof of the conjecture by Gongxiang Liu.

In this chapter, we focus on presenting the computational process for $H(2,2)$ because it plays an important role in Chapter 3 and in the proof of the conjecture, and we need to clarify that it is the 16-dimensional Hopf algebra $H_{c:\sigma_0}$. Great care must be taken in choosing the generators and parameters, which is also why we emphasize providing the detailed computations. The computations for $H(2,d)$ and $H(3,2)$ follow similarly, we have omitted some of the details.

Chapter 6 The detailed calculation of $H_{c:\sigma_0}$

Let H be the Hopf algebra $H_{c:\sigma_0}$ in this Chapter. We will present the detailed computation for the Yetter–Drinfel'd modules of H in Chapter 3.

Let $V = \oplus_{u\in G} V_u$ be the k-vector space, where $V_\mu = \{v \in V \mid e_h v = \delta_{\mu,h} v\}$ defined in Chapter 3. Recall that by the relations $te_g v_u = e_{g\triangleleft t} t v_u$, we have $t \cdot v_u \in V_{u\triangleleft t}$. Let $\zeta_g v = \alpha(g)v$, where $\alpha : kG \to k$ is an algebra map since $\zeta_g \zeta_{g'} = \zeta_{gg'}$. By the relations $\zeta_{x^i y^j}\chi_{x^k y^l} = 0$ and $1_{H^*} = \varepsilon = \zeta_1 + \chi_1$, we have the decomposition of V_u for $u \in G(H)$:

$$V_u = \zeta_1 V_u \oplus \chi_1 V_u.$$

Let $S = \{1, y, x^2, x^2 y\}$ and $T = \{x, xy, x^3, x^3 y\}$, we have $G = T \cup S$.

Remark 6.0.0.1. Let V_u $(u \in G)$ be the $\mathcal{D}(H)$-modules. By $e_g V_u = \delta_{g,u} V_u$, it is clear that they are pairwise non-isomorphic.

Let $\alpha, \alpha', \beta : G \to k$ be characters of G such that

$$\zeta_g V_u = \alpha(g) V_u, \qquad \zeta_g V_{uy} = \alpha'(g) V_{uy}.$$

Let $\chi_s V_u = \beta(s) V_u$ for $s \in S$. Assume ξ is a primitive 4-th root of 1.

6.1 The one-dimensional Yetter–Drinfel'd modules

$\mathcal{D}(H)$ has 32 non-isomorphic one-dimensional irreducible representations as the following.

left action	$\zeta_1 V_1$	$\zeta_1 V_{x^2}$	$\zeta_1 V_{x^2 y}$	$\zeta_1 V_y$
t	$\lambda = \pm 1$	$\lambda = \pm \mathrm{i}$	$\lambda = \pm \mathrm{i}$	$\lambda = \pm 1$
ζ_x	$\alpha(x) = \xi$	$\alpha(x) = \xi$	$\alpha(x) = \xi$	$\alpha(x) = \xi$
ζ_y	$\alpha(y) = 1$	$\alpha(y) = 1$	$\alpha(y) = -1$	$\alpha(y) = -1$
e_g	$\delta_{g,1}$	δ_{g,x^2}	$\delta_{g,x^2 y}$	$\delta_{g,y}$

Then $(\zeta_1 V_u, \rho_s) \in {}^H_H\mathcal{YD}$ with its module structure, comodule structure and braiding given by

$\zeta_1 V_1$	$\zeta_1 V_{x^2}$	$\zeta_1 V_{x^2y}$	$\zeta_1 V_y$
$c(v\otimes v)=v\otimes v$	$c(v\otimes v)=\xi^2 v\otimes v$	$c(v\otimes v)=-\xi^2 v\otimes v$	$c(v\otimes v)=-v\otimes v$

Firstly, we assume $V=\zeta_1 V_u$ and fix $u\in S$.

If $h\in S$, we have $t\cdot v_u\in V_u$. Assume that $t\cdot v_\mu=\lambda v_\mu$ when we let dim(V) $=1$. According to $\lambda^2 v_\mu=t^2 v_\mu=\sum_g \sigma(g,t,t)e_g v_\mu$. we know $\lambda^2=\sigma(u,t,t)$. Let $\zeta_g v=\alpha(g)v$, where $\alpha: G\to k$ is a characters of G.

In this situation, By $t\zeta_h v_\mu=\zeta_{h\triangleleft t}(\sum_g \tau\,(h\triangleleft t,g;t)\,\tau(\,(h\triangleleft t)g,h^{-1}\triangleleft t;t)\tau(h\triangleleft t,h^{-1}\triangleleft t;\ t)(e_g\#t))v_\mu$, we get

$$t\zeta_h v_\mu=\lambda\tau\,(h,\mu;t)\,\tau(\,h\mu,h;t)\tau(h,h;\ t)\zeta_h v_\mu. \tag{6.1}$$

In particular, $\alpha(y)=\tau\,(y,\mu;t)\,\tau(\,y\mu,y;t)\tau(y,y;\ t)\alpha(y)$ and $\tau\,(y,\mu;t)\,\tau(\,y\mu,y;t)\tau(y,y;\ t)=1$ for $u\in S$.

If $h\in T$, we have $\lambda^2=-1$, $h\triangleleft t=hy$ and $tv_\mu\in V_{\mu\triangleleft t}=V_u$. By $t\zeta_h v_\mu=\zeta_{h\triangleleft t}(\sum_g \tau\,(h\triangleleft t,g;t)\,\tau(\,(h\triangleleft t)g,h^{-1}\triangleleft t;t)\tau(h\triangleleft t,h^{-1}\triangleleft t;\ t)(e_g\#t))v_\mu$, we see that

$$t\zeta_h v_\mu=\lambda\tau\,(hy,\mu;t)\,\tau(\,hy\mu,h^{-1}y;t)\tau(hy,h^{-1}y;\ t)\zeta_{hy}v_\mu. \tag{6.2}$$

In particular, $\alpha(x)=\tau\,(xy,\mu;t)\,\tau(\,xy\mu,x^3y;t)\tau(xy,x^3y;\ t)\alpha(x)\alpha(y)$ for $u\in S$. Thus, we have the the following equation:

$$\zeta_x v_1=\zeta_{xy}v_1,\zeta_x v_y=-\zeta_{xy}v_y,\zeta_x v_{x^2}=\zeta_{xy}v_{x^2},\zeta_x v_{x^2y}=-\zeta_{xy}v_{x^2y}. \tag{6.3}$$

By 6.3 it follows that $\alpha(y)=1$ when $u=1,x^2$; $\alpha(y)=-1$, when $u=y,x^2y$.

Since $\alpha: kG\to k$ is an algebra map it follow that $\alpha^4=1$. Let ξ be a 4-th root of unity, then there are 32 one-dimension irreducible representations as above table in this case.

Secondly, If $V=\chi_1 V_u$ and dim $V=1$. But in this situation, $e_g\chi_h=\chi_h e_{gh(h^{-1}\triangleleft t)}$, it follows that $\chi_h v_\mu\in V_{\mu(h\triangleleft t)h^{-1}}$. In particular, when $h\in T$, $\chi_x v_\mu\in V_{\mu y}$, this is a contradiction to dimV $=1$. When $V=\zeta_1 V_u$ and $u\in T$, $t\cdot v\in V_{uy}$, it also contradicts to dim $V=1$.

Since V is a one-dimensional, we must have that $\rho(v)=\Sigma_{h_i}v\otimes h_i$ with $h_i\in H$. Since $f\cdot v=\Sigma_{h_i}<f,h_i>v$ for all $f\in H^*$, it follows that $\rho(v)=v\otimes\Sigma_{h\in G}\alpha(h)e_h$ and $\rho_s(v)=\Sigma_{h\in G}\alpha(h)S(e_h)\otimes v$. By $c(v\otimes v)=\Sigma_{h\in G}\alpha(h)S(e_h)\cdot v\otimes v$, we can see that the braiding of left-left Yetter-Drinfeld module $\zeta_1 V_u$ is $c(v\otimes w)=\pm v\otimes v$.

6.2 The two-dimensional Yetter–Drinfel'd modules

Let $u\in T$, $v_u\in\zeta_1 V_\mu$, then $\mathcal{D}(H)$ has 16 non-isomorphic two-dimension irreducible representations:

left action	$V =< v_x, v_{xy} >$	$V =< v_x, v_{xy} >$	$< v_{x^3}, v_{x^3y} >$	$< v_{x^3}, v_{x^3y} >$
t	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
ζ_x	$\alpha(x) = \xi = -\alpha'(x)$	$\alpha(x) = \xi = \alpha'(x)$	$\alpha(x^3) = \xi = -\alpha'(x^3)$	$\alpha(x^3) = \xi = \alpha'(x^3)$
ζ_y	$\alpha(y) = 1, \alpha'(y) = -1$	$\alpha(y) = -1, \alpha(y) = 1$	$\alpha(y) = 1, \alpha'(y) = -1$	$\alpha(y) = -1, \alpha(y) = 1$
e_g	$\begin{pmatrix} \delta_{g,x} & 0 \\ 0 & \delta_{g,xy} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x} & 0 \\ 0 & \delta_{g,xy} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x^3} & 0 \\ 0 & \delta_{g,x^3y} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x^3} & 0 \\ 0 & \delta_{g,x^3y} \end{pmatrix}$

Then $(\zeta_1 V_u, \rho_s) \in {}^H_H\mathcal{YD}$ with its module structure, comodule structure and braiding given by

$V =< v_x, v_{xy} >$	$V =< v_x, v_{xy} >$	$< v_{x^3}, v_{x^3y} >$	$< v_{x^3}, v_{x^3y} >$
$c(v \otimes v) = \xi^3 v \otimes v$	$c(v \otimes v) = \xi^3 v \otimes v$	$c(v \otimes v) = \xi v \otimes v$	$c(v \otimes v) = \xi v \otimes v$
$c(v \otimes w) = \xi^3 w \otimes v$	$c(v \otimes w) = -\xi^3 w \otimes v$	$c(v \otimes w) = \xi w \otimes v$	$c(v \otimes w) = -\xi w \otimes v$
$c(w \otimes v) = -\xi^3 v \otimes w$	$c(w \otimes v) = \xi^3 v \otimes w$	$c(w \otimes v) = -\xi v \otimes w$	$c(w \otimes v) = \xi v \otimes w$
$c(w \otimes w) = \xi^3 w \otimes w$	$c(w \otimes w) = \xi^3 w \otimes w$	$c(w \otimes w) = \xi w \otimes w$	$c(w \otimes w) = \xi w \otimes w$

Firstly, we see that $\chi_h V = 0$. If $h \in T$, since $te_g = e_{g \triangleleft t} t$ and $e_g \zeta_h = \zeta_h e_g$, we have $w = tv_u = v_{\mu y} \in V_{uy}$ and $\zeta_h v_u \in V_u$. Let $\zeta_g v = \alpha(g) v, \zeta_g w = \alpha'(g) w$, where $\alpha, \alpha' : G \to k$ are characters of G. Let $v = v_u, w = t \cdot v$ and $V =< v, w >$ as a linear space.

Since

$$t\zeta_h v_\mu = \zeta_{h \triangleleft t}(\sum_g \tau\,(h \triangleleft t, g; t)\,\tau(h \triangleleft t)g, h^{-1} \triangleleft t; t)\tau(h \triangleleft t, h^{-1} \triangleleft t;\ t)(e_g \cdot t))v_\mu$$

then we see that

$$t\zeta_h v_\mu = \tau(hy, \mu y; t)\tau(h\mu, h^{-1}y; t)\tau(hy, h^{-1}y;\ t)\zeta_{hy} w. \tag{6.1}$$

Since $hy \in T$, we can let $h = hy$. Thus, we have:

$$\zeta_h(w) = \tau(h, \mu y; t)\tau(hy\mu, h^{-1}; t)\tau(h, h^{-1};\ t)t\zeta_{hy} \cdot v. \tag{6.2}$$

In particular, we have

$$\alpha'(x) = \tau(xy\mu, x^3; t)\alpha(x)\alpha(y). \tag{6.3}$$

If $h \in S$, we have $tv_u = v_{\mu y} \in V_{uy}$, $\zeta_h v_u \in V_u$ by $te_g = e_{g \triangleleft t} t$ and $e_g \zeta_h = \zeta_h e_g$. Since $t\zeta_h v_\mu = \zeta_{h \triangleleft t}(\sum_g \tau\,(h \triangleleft t, g; t)\,\tau(\ (h \triangleleft t)g, h^{-1} \triangleleft t; t)\tau(h \triangleleft t, h^{-1} \triangleleft t;\ t)(e_g \cdot t))v_\mu$ it follows that $t\zeta_h v_\mu = \tau(h, \mu y; t)\tau(hy\mu, h^{-1}; t)\tau(h, h^{-1};\ t)\zeta_h v_{\mu y}$. Thus, we have

$$\alpha'(h) = \tau(h, \mu y; t)\tau(hy\mu, h^{-1}; t)\tau(h, h^{-1};\ t)\alpha(h).$$

In particular, we see that

$$\alpha'(y) = \tau(y, \mu y; t)\alpha(y) = -\alpha(y). \tag{6.4}$$

By (6.3)(6.4) it follows that

$$\alpha(y) = -\alpha'(y) = \pm 1,$$

$$\alpha'(x) = -\alpha(x)\alpha(y),\ \alpha'(x)\alpha'(y) = \alpha(x),$$

$$\alpha'(x^3) = -\alpha(x^3)\alpha(y), \alpha'(x^3)\alpha'(y) = \alpha(x^3)$$

Since $t^2 v_\mu = \sum_g \sigma(g, t, t) e_g v_\mu$ it follows that $t \cdot w = \sigma(u, t, t)v$. Then the action of $\mathcal{D}(H)$ on V can be defined as following:

$$t \cdot v = w,\ t \cdot w = \sigma(u, t, t)v,\ \zeta_h v = \alpha(h)v, \tag{6.5}$$

$$\zeta_h \cdot w = \tau(h, \mu y; t)\tau(hy\mu, h^{-1}; t)\tau(h, h^{-1};\ t)\alpha(hy)w.\ \ (\forall h \in T) \tag{6.6}$$

$$\zeta_h \cdot w = \tau(h, \mu y; t)\tau(hy\mu, h^{-1}; t)\tau(h, h^{-1};\ t)\alpha(h)w\ \ (\forall h \in S). \tag{6.7}$$

Since α, α' are the characters it follows that $\alpha^4(x) = 1 = {\alpha'}^4(x), \alpha^2(y) = 1 = {\alpha'}^2(y)$. If $< v + \lambda w >$ is a $\mathcal{D}(H)$-submodule of $V =< v, w >$, we see that $\zeta_h(v + \lambda w) = \alpha(h)v + \alpha'(h)w$ and $t(v + \lambda w) = w + \lambda\sigma(u, t, t)v$ are belonging to $< v + \lambda w >$. Then we have $\frac{\lambda}{\lambda\alpha'(h)} = \frac{1}{\alpha(h)}$ and $\lambda\sigma(u, t, t) = 1$. This means $\alpha(h) = \alpha'(h)$ for all $h \in G$. But $\alpha(y) = -\alpha'(y) \neq 0$ in above representations, so there are 16 two-dimension irreducible representations as above table. Firstly, we have the right comodule structure

$$\rho(v) = v \otimes \Sigma_{h\in G}\alpha(h)e_h, \rho(w) = w \otimes \Sigma_{h\in G}\alpha'(h)e_h,$$

$$\rho_s(v) = \Sigma_{h\in G}\alpha(h)S(e_h) \otimes v, \rho(w) = \Sigma_{h\in G}\alpha'(h)S(e_h) \otimes w.$$

We only only verify the braiding of left-left Yetter-Drinfeld module $\zeta_1 V_u, u = x^3$.

$$c(v \otimes v) = \Sigma_{h\in G}\alpha(h)S(e_h) \cdot v \otimes v = \xi v \otimes v,$$

$$c(v \otimes w) = \Sigma_{h\in G}\alpha(h)S(e_h) \cdot w \otimes v = \xi w \otimes v,$$

$$c(w \otimes v) = \Sigma_{h\in G}\alpha'(h)S(e_h) \cdot v \otimes w = -\xi v \otimes w,$$

$$c(w \otimes w) = \Sigma_{h\in G}\alpha'(h)S(e_h) \cdot w \otimes w = \xi w \otimes w.$$

Let $\mu \in S$, $v_u \in \zeta_1 V_\mu$, then $\mathcal{D}(H)$ has 8 non-isomorphic two-dimension irreducible representations $(i = 0, 1)$:

left action	$V =< v_1, tv_1 >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = -1$	$V =< v_y, tv_y >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = 1$	$V =< v_{x^2}, tv_{x^2} >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = -1$	$V =< v_{x^2y}, tv_{x^2y} >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = 1$
t	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
ζ_x	$\begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha'(x) \end{pmatrix}$	$\begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha'(x) \end{pmatrix}$	$\begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha'(x) \end{pmatrix}$	$\begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha'(x) \end{pmatrix}$
ζ_y	$\begin{pmatrix} \alpha(y) & 0 \\ 0 & \alpha(y) \end{pmatrix}$	$\begin{pmatrix} \alpha(y) & \\ 0 & \alpha(y) \end{pmatrix}$	$\begin{pmatrix} \alpha(y) & 0 \\ 0 & \alpha(y) \end{pmatrix}$	$\begin{pmatrix} \alpha(y) & 0 \\ 0 & \alpha(y) \end{pmatrix}$
e_g	$\begin{pmatrix} \delta_{g,1} & 0 \\ 0 & \delta_{g,1} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,y} & 0 \\ 0 & \delta_{g,y} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x^2} & 0 \\ 0 & \delta_{g,x^2} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x^2y} & 0 \\ 0 & \delta_{g,x^2y} \end{pmatrix}$

$(\zeta_1 V_u, \rho_s) \in {}^H_H\mathcal{YD}$ with its module structure, comodule structure and braiding given by

$V =< v_1, tv_1 >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = -1$	$V =< v_y, tv_y >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = 1$	$V =< v_{x^2}, tv_{x^2} >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = -1$	$V =< v_{x^2y}, tv_{x^2y} >$ $\alpha'(x) = \xi^i = -\alpha(x), \alpha(y) = 1$
$c(v \otimes v) = v \otimes v$	$c(v \otimes v) = v \otimes v$	$c(v \otimes v) = \xi^2 v \otimes v$	$c(v \otimes v) = \xi^2 v \otimes v$
$c(v \otimes w) = w \otimes v$	$c(v \otimes w) = w \otimes v$	$c(v \otimes w) = \xi^2 w \otimes v$	$c(v \otimes w) = \xi^2 w \otimes v$
$c(w \otimes v) = v \otimes w$	$c(w \otimes v) = v \otimes w$	$c(w \otimes v) = \xi^2 v \otimes w$	$c(w \otimes v) = \xi^2 v \otimes w$
$c(w \otimes w) = w \otimes w$	$c(w \otimes w) = w \otimes w$	$c(w \otimes w) = \xi^2 w \otimes w$	$c(w \otimes w) = \xi^2 w \otimes w$

If dimV > 1, then $t \cdot v$ must can not belong to $< v >$. That is $t \cdot v \in \zeta_1 V_u$ but $t \cdot v \notin < v >$. Then we assume $V =< v, w = t \cdot v >$. By $\zeta_g \zeta_h = \zeta_{gh}$, $\zeta_1 \zeta_g = \zeta_g$, $\zeta_g \zeta_{g^{-1}} = \zeta_1$, we can see that $\{e_g\}_{g \in G}$ is isomorphic to G as abelian groups. Thus, we can let $\zeta_g v = \alpha(g) \cdot v$ and $\zeta_g w = \alpha'(g) \cdot w$, $\alpha, \alpha' : G \to k$ are characters of G. Then we can see that

$$t \cdot v = w, \ t \cdot w = \sigma(u, t, t)v, \zeta_h v = \alpha(h) \cdot v, \zeta_h w = \alpha'(h) \cdot w; \tag{6.8}$$

$$\zeta_h \cdot w = \zeta_h t \cdot v = \tau(h, u, t)\tau(hu, h^{-1}, t)\tau(h, h^{-1}, t)t\zeta_{h \triangleleft t} v; \tag{6.9}$$

$$\alpha'(h) = \alpha(h \triangleleft t)\tau(h, u, t)\tau(hu, h^{-1}, t)\tau(h, h^{-1}, t). \tag{6.10}$$

By (6.10) it follows that $\alpha'(y) = \alpha(y)$. and $\alpha'(x) = \alpha(xy)$ when $u = 1, x^2$ and $\alpha'(y) = \alpha(y)$, $\alpha'(x) = -\alpha(xy)$ when $u = y$ or $u = x^2 y$.

For $u = 1$ or y, it is easy to see that $< v + w >$, $< v - w >$ are submodules of $< v_u, tv_u >$, when $\alpha(x) = \alpha'(x)$, $\alpha(y) = \alpha'(y)$. For $u = x^2$ or $x^2 y$, it is easy to see that $< v + \mathrm{i}w >$, $< v - \mathrm{i}w >$ are submodules of $< v_u, tv_u >$, when $\alpha(x) = \alpha'(x)$, $\alpha(y) = \alpha'(y)$.

$$\rho(v) = v \otimes \Sigma_{h \in G} \alpha(h) e_h, \rho(w) = w \otimes \Sigma_{h \in G} \alpha'(h) e_h.$$

$$\rho_s(v) = \Sigma_{h \in G} \alpha(h) S(e_h) \otimes v, \rho_s(w) = \Sigma_{h \in G} \alpha'(h) S(e_h) \otimes w.$$

We only only verify the braiding of left-left Yetter-Drinfeld module $\zeta_1 V_u$, $u = x^2 y$.

$$c(v \otimes v) = \Sigma_{h \in G} \alpha(h) S(e_h) \cdot v \otimes v = \xi^2 v \otimes v,$$

$$c(v \otimes w) = \Sigma_{h \in G} \alpha(h) S(e_h) \cdot w \otimes v = \xi^2 w \otimes v,$$

$$c(w \otimes v) = \Sigma_{h \in G} \alpha'(h) S(e_h) \cdot v \otimes w = \xi^2 v \otimes w,$$

$$c(w \otimes w) = \Sigma_{h \in G} \alpha'(h) S(e_h) \cdot w \otimes w = \xi^2 w \otimes w.$$

Remark 6.2.0.1.

$$S(e_1 \# t) = e_1 \# t, S(e_y \# t) = e_y \# t, S(e_{x^2} \# t) = -e_{x^2} \# t, S(e_{x^2 y} \# t) = -e_{x^2 y} \# t,$$

$$S(e_x \# t) = -e_{x^3 y} \# t, S(e_{xy} \# t) = e_{x^3} \# t, S(e_{x^3} \# t) = (e_{xy} \# t), S(e_{x^3 y} \# t) = -(e_x \# t)$$

Let $\mu \in T$, if $v_u \in \chi_1 V_\mu$, then $\mathcal{D}(H)$ has 16 non-isomorphic two-dimension irreducible representations:

left action	$V =< v_x, v_{xy} >$	$V =< v_x, v_{xy} >$	$V =< v_x, v_{xy} >$	$V =< v_x, v_{xy} >$
t	$\begin{pmatrix} 0 & -\lambda \\ \lambda = \pm \mathrm{i} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda \\ \lambda = \pm 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda \\ \lambda = \pm 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\lambda \\ \lambda = \pm \mathrm{i} & 0 \end{pmatrix}$
χ_x	$\begin{pmatrix} 0 & \beta(x^2) = 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \beta(x^2) = -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \beta(x^2) = 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \beta(x^2) = -1 \\ 1 & 0 \end{pmatrix}$
e_g	$\begin{pmatrix} \delta_{g,x} & 0 \\ 0 & \delta_{g,xy} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x} & 0 \\ 0 & \delta_{g,xy} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x} & 0 \\ 0 & \delta_{g,xy} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x} & 0 \\ 0 & \delta_{g,xy} \end{pmatrix}$

left action	$< v_{x^3}, v_{x^3y} >$	$< v_{x^3}, v_{x^3y} >$	$< v_{x^3}, v_{x^3y} >$	$< v_{x^3}, v_{x^3y} >$
t	$\begin{pmatrix} 0 & -\lambda \\ \lambda = \pm 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda \\ \lambda = \pm \mathrm{i} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \lambda \\ \lambda = \pm \mathrm{i} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\lambda \\ \lambda = \pm 1 & 0 \end{pmatrix}$
χ_x	$\begin{pmatrix} 0 & \beta(x^2) = 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \beta(x^2) = -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \beta(x^2) = 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \beta(x^2) = -1 \\ 1 & 0 \end{pmatrix}$
e_g	$\begin{pmatrix} \delta_{g,x^3} & 0 \\ 0 & \delta_{g,x^3y} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x^3} & 0 \\ 0 & \delta_{g,x^3y} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x^3} & 0 \\ 0 & \delta_{g,x^3y} \end{pmatrix}$	$\begin{pmatrix} \delta_{g,x^3} & 0 \\ 0 & \delta_{g,x^3y} \end{pmatrix}$

Then $(\chi_1 V_u, \rho_s) \in {}^H_H\mathcal{YD}$ with new type braiding given by

$(V_{x,\pm i,\pm i,\pm i,\mp i}, c)$, $(v = v_x, w = \chi_x v_x)$	$(V_{x,\mp,\pm,\pm,\pm}, c)$,$(v = v_x, w = \chi_x v_x)$	$(V_{x,\pm,\pm,\pm,\mp}, c)$,$(v = v_x, w = \chi_x v_x)$	$(V_{x,\mp i,\pm i,\pm i,\pm i}, c)$,$(v = v_x, w = \chi_x v_x)$
$c(v \otimes v) = \pm \mathrm{i} w \otimes w$ $c(v \otimes w) = \pm \mathrm{i} v \otimes w$ $c(w \otimes v) = \pm \mathrm{i} w \otimes v$ $c(w \otimes w) = \mp \mathrm{i} v \otimes v$	$c(v \otimes v) = \mp w \otimes w$ $c(v \otimes w) = \pm v \otimes w$ $c(w \otimes v) = \pm w \otimes v$ $c(w \otimes w) = \pm v \otimes v$	$c(v \otimes v) = \pm w \otimes w$ $c(v \otimes w) = \pm v \otimes w$ $c(w \otimes v) = \pm w \otimes v$ $c(w \otimes w) = \mp v \otimes v$	$c(v \otimes v) = \mp \mathrm{i} w \otimes w$ $c(v \otimes w) = \pm \mathrm{i} v \otimes w$ $c(w \otimes v) = \pm \mathrm{i} w \otimes v$ $c(w \otimes w) = \pm \mathrm{i} v \otimes v$

$(V_{x^3,\mp,\mp,\mp,\pm}, c)$, $(v = v_{x^3}, w = \chi_x v_{x^3})$	$(V_{x^3,\mp i,\pm i,\pm i,\pm i}, c)$, $(v = v_{x^3}, w = \chi_x v_{x^3})$	$(V_{x^3,\mp i,\mp i,\mp i,\pm i}, c)$, $(v = v_{x^3}, w = \chi_x v_{x^3})$	$(V_{x^3,\mp,\pm,\pm,\pm}, c)$, $(v = v_{x^3}, w = \chi_x v_{x^3})$
$c(v \otimes v) = \mp w \otimes w$ $c(v \otimes w) = \mp v \otimes w$ $c(w \otimes v) = \mp w \otimes v$ $c(w \otimes w) = \pm v \otimes v$	$c(v \otimes v) = \mp \mathrm{i} w \otimes w$ $c(v \otimes w) = \pm \mathrm{i} v \otimes w$ $c(w \otimes v) = \pm \mathrm{i} w \otimes v$ $c(w \otimes w) = \pm \mathrm{i} v \otimes v$	$c(v \otimes v) = \mp \mathrm{i} w \otimes w$ $c(v \otimes w) = \mp \mathrm{i} v \otimes w$ $c(w \otimes v) = \mp \mathrm{i} w \otimes v$ $c(w \otimes w) = \pm \mathrm{i} v \otimes v$	$c(v \otimes v) = \mp w \otimes w$ $c(v \otimes w) = \pm v \otimes w$ $c(w \otimes v) = \pm w \otimes v$ $c(w \otimes w) = \pm v \otimes v$

By $e_g \chi_h = \chi_h e_{gh(h^{-1} \triangleleft t)}$ it follows that $\chi_h v_u \in V_{uy},\ \forall h \in T$ and $\chi_h v_u \in V_u,\ \forall h \in S$, Let $v = v_u, w = \chi_x v_u, V =< v, w >$, Since $t \cdot v_u \in V_{uy}$, we can suppose $t \cdot v = \lambda w$. Let $\beta : G \to k$ be a character and $\chi_s v = \beta(s) v$ for $s \in S$. Since abelian group S is the subgroup of G and $T = Sx$, we can see that

$$(\beta(x))^4 = \beta(x^4) = 1, (\beta(y))^2 = \beta(y^2) = 1,$$

$$\chi_{sx}(v) = \tau(x,\ s, t)\beta(s)w = \beta(s)w$$

$$\chi_{sx}(w) = \tau(s,\ x, t)\beta(s)\beta(x^2)v$$

$$\chi_s w = \beta(s)\tau(s, x, t)w, \chi_s v = \beta(s)v$$

$$t \cdot w = t \cdot \chi_x v = \lambda \tau(xuy, x^3y, t)\beta(y)\beta(x^2)v$$

$$t^2 v = t \cdot (t \cdot v) = t \cdot (\lambda w) = \lambda^2 \tau(xuy, x^3y, t)\beta(y)\beta(x^2)v = \sigma(u, t, t)v$$

$$\lambda^2 \tau(xuy, x^3y, t)\beta(y)\beta(x^2) = \sigma(u, t, t)$$

We see that

$$\beta(x) = \xi, \beta(x^2) = \beta(y) = \pm 1, \lambda^2 = \frac{\sigma(u,t,t)}{-\beta(y)\beta(x^2)} = \pm 1, \tag{6.11}$$

If $< v + \lambda w >$ is a $\mathcal{D}(H)$-submodule of $V =< v, w >$, we see that

$$\chi_y(v + \lambda w) = \beta(y)v - \lambda\beta(y)w \in < v + \lambda w > . \tag{6.12}$$

Then we have $\beta(y) = 0$, this is a contradiction to $\beta^2(y) = 1$. Thus, there are 16 two-dimension irreducible representations as above.

$$\rho(v) = v \otimes \Sigma_{s\in S}\beta(s)(e_s\#t) + w \otimes \Sigma_{s\in S}\beta(s)(e_{xs}\#t),$$

$$\rho(w) = w \otimes \Sigma_{s\in S}\beta(s)\tau(s,x,t)(e_s\#t) + v \otimes \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}\#t).$$

$$\rho_s(v) = \Sigma_{s\in S}\beta(s)S(e_s\#t) \otimes v + \Sigma_{s\in S}\beta(s)S(e_{xs}\#t) \otimes w,$$

$$\rho_s(w) = \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_s\#t) \otimes w + \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}\#t) \otimes v.$$

$$\mathcal{S}^{-1}(\ e_g\#t) = \sigma(\ g^{-1};\ t,\ t)\tau(\ g^{-1},\ g; t)e_{(g\triangleleft t)^{-1}}\#t$$

$$c(v\otimes v) = \Sigma_{s\in S}\beta(s)S(e_s\#t)\cdot v\otimes v + \Sigma_{s\in S}\beta(s)S(e_{xs}\#t)\cdot v\otimes w = \Sigma_{s\in S}\beta(s)S(e_{xs}\#t)\cdot v\otimes w,$$

$$c(v\otimes w) = \Sigma_{s\in S}\beta(s)S(e_s\#t)\cdot w\otimes v + \Sigma_{s\in S}\beta(s)S(e_{xs}\#t)\cdot w\otimes w = \Sigma_{s\in S}\beta(s)S(e_{xs}\#t)\cdot w\otimes w,$$

$$c(w \otimes v) = \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_s\#t)\cdot v\otimes w + \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}\#t)\cdot v\otimes v = \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}\#t)\cdot v\otimes v,$$

$$c(w\otimes w) = \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_s\#t)\cdot w\otimes w + \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}\#t)\cdot w\otimes v = \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}\#t)\cdot w\otimes v$$

Let $u = x$, we get the following braiding:

$$c(v \otimes v) = (-e_{x^3y}t + \beta(y)e_{x^3}t + \beta(x^2)e_{xy}t - \beta(x^2y)e_xt)\cdot v\otimes w = \lambda\beta(x^2)w\otimes w,$$

$$c(v \otimes w) = (-e_{x^3y}t + \beta(y)e_{x^3}t + \beta(x^2)e_{xy}t - \beta(x^2y)e_xt)\cdot w\otimes w = \lambda v\otimes w,$$

$$c(w \otimes v) = (-\beta(x^2)e_{x^3y}t - \beta(x^2y)e_{x^3}t + e_{xy}t + \beta(y)e_xt)\cdot v\otimes v = \lambda w\otimes v,$$

$$c(w \otimes w) = (-\beta(x^2)e_{x^3y}t - \beta(x^2y)e_{x^3}t + e_{xy}t + \beta(y)e_xt)\cdot w\otimes v = -\lambda\beta(x^2)v\otimes v.$$

Let $u = x^3$, we get the following braiding:

$$c(v \otimes v) = (-e_{x^3y}t + \beta(y)e_{x^3}t + \beta(x^2)e_{xy}t - \beta(x^2y)e_xt)\cdot v\otimes w = -\lambda w\otimes w,$$

$$c(v\otimes w)=(-e_{x^3y}t+\beta(y)e_{x^3}t+\beta(x^2)e_{xy}t-\beta(x^2y)e_xt)\cdot w\otimes w=-\lambda\beta(x^2)v\otimes w,$$

$$c(w\otimes v)=(-\beta(x^2)e_{x^3y}t-\beta(x^2y)e_{x^3}t+e_{xy}t+\beta(y)e_xt)\cdot v\otimes v=-\lambda\beta(x^2)w\otimes v,$$

$$c(w\otimes w)=(-\beta(x^2)e_{x^3y}t-\beta(x^2y)e_{x^3}t+e_{xy}t+\beta(y)e_xt)\cdot w\otimes v=\lambda v\otimes v.$$

Let $\mu\in S$ and $v_u\in\chi_1V_\mu$, then $\mathcal{D}(H)$ has 16 non-isomorphic two-dimension irreducible representations:

left action	$V =< v_1, \chi_x v_1 >$ $\lambda=\pm1;\beta(y)=\pm1;\beta(x^2)=\pm1$	$V =< v_{x^2},\ \chi_x v_{x^2} >$ $\lambda=\pm\mathrm{i};\beta(\mathrm{y})=\pm1;\beta(\mathrm{x}^2)=\pm1$
t	$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\beta(y)\end{pmatrix}$	$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\beta(y)\end{pmatrix}$
χ_x	$\begin{pmatrix}0 & \beta(x^2)\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \beta(x^2)\\ 1 & 0\end{pmatrix}$
χ_y	$\begin{pmatrix}\beta(y) & \\ 0 & -\beta(y)\end{pmatrix}$	$\begin{pmatrix}\beta(y) & \\ 0 & -\beta(y)\end{pmatrix}$
e_g	$\begin{pmatrix}\delta_{g,1} & 0\\ 0 & \delta_{g,y}\end{pmatrix}$	$\begin{pmatrix}\delta_{g,x^2} & 0\\ 0 & \delta_{g,x^2y}\end{pmatrix}$

Then $(\chi_1V_u,\rho_s)\in{}^H_H\mathcal{YD}$ with braiding given by

$V =< v_1, \chi_x v_1 >$ $\lambda=\pm1;\beta(y)=\pm1;\beta(x^2)=\pm1$	$V =< v_{x^2},\ \chi_x v_{x^2} >$ $\lambda=\pm\mathrm{i};\beta(\mathrm{y})=\pm1;\beta(\mathrm{x}^2)=\pm1$
$c(v\otimes v)=\pm v\otimes v$ $c(v\otimes w)=\pm w\otimes v$ $c(w\otimes v)=\pm v\otimes w$ $c(w\otimes w)=\mp w\otimes w$	$c(v\otimes v)=\pm\mathrm{i}v\otimes v$ $c(v\otimes w)=\pm\mathrm{i}w\otimes v$ $c(w\otimes v)=\pm\mathrm{i}v\otimes w$ $c(w\otimes w)=\mp\mathrm{i}w\otimes w$

Firstly, we have $t\cdot v_u\in V_u$. By $e_g\chi_h=\chi_he_{gh(h^{-1}\triangleleft t)}$ it follows that $\chi_hv_u\in V_{uy}$, $\forall h\in T$ and $\chi_hv_u\in V_u$, $\forall h\in S$. Let $v=v_u, w=\chi_xv_u, V=<v,w>$ and suppose $t\cdot v=\lambda v$. Let $\beta:G\to k$ be a character and $\chi_sv=\beta(s)v$ for $s\in S$. Since S and and $T=Sx$ are subgroups of G, we can see that:

$$(\beta(x))^4=\beta(x^4)=1,(\beta(y))^2=\beta(y^2)=1,$$

$$\chi_sv=\beta(s)v,\ \chi_sw=\beta(s)\tau(s,x,t)w;$$

$$\chi_{sx}(v)=\beta(s)\tau(s,\ x,t)w,\ \chi_{sx}(w)=\tau(s,\ x,t)\beta(s)\beta(x^2)v;$$

$$t\cdot v=\lambda v,\ t^2v=\lambda^2v=\sigma(u,t,t)v$$

$$t\cdot w=t\cdot\chi_xv=-\lambda\tau(xu,x^3y,t)\beta(y)w=\lambda\beta(y)w$$

The representations as above are two-dimension irreducible representations by the same arguments as (6.12).

It follows that

$$\rho(v) = v \otimes \Sigma_{s\in S}\beta(s)(e_s\#t) + w \otimes \Sigma_{s\in S}\beta(s)\tau(s,x,t)(e_{xs}\#t),$$

$$\rho(w) = w \otimes \Sigma_{s\in S}\beta(s)\tau(s,x,t)(e_s\#t) + v \otimes \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)(e_{xs}\#t).$$

$$\rho_s(v) = \Sigma_{s\in S}\beta(s)S(e_s\#t) \otimes v + \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_{xs}\#t) \otimes w,$$

$$\rho_s(w) = \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_s\#t) \otimes w + \Sigma_{s\in S}\beta(s)\beta(x^2)\tau(s,x,t)S(e_{xs}\#t) \otimes v.$$

We obtain that

$$c(v \otimes v) = \Sigma_{s\in S}\beta(s)S(e_s\#t) \cdot v \otimes v = \lambda v \otimes v,$$

$$c(v \otimes w) = \Sigma_{s\in S}\beta(s)S(e_s\#t) \cdot w \otimes v = \lambda w \otimes v,$$

$$c(w \otimes v) = \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_s\#t) \cdot v \otimes w = \lambda v \otimes w,$$

$$c(w \otimes w) = \Sigma_{s\in S}\beta(s)\tau(s,x,t)S(e_s\#t) \cdot w \otimes w == -\lambda w \otimes w.$$

6.3 Conclusion

In this Chapter, we have presented the detailed calculations of the simple Yetter–Drinfel'd modules. We have classified the one-dimensional and two-dimensional Yetter–Drinfel'd modules and provided explicit descriptions.

To summarize, our main contributions in this Chapter are:

- A complete list of one-dimensional Yetter–Drinfel'd modules.
- A complete list of two-dimensional Yetter–Drinfel'd modules.
- The explicit formulas for the action and coaction in each case.

These calculations serve as a foundation for the main results of the Chapter 3, in particular, for the study of Nichols algebras. We hope that this detailed calculation of $H_{c:\sigma_0}$ will be useful for reading this thesis.

Chapter 7 Future work

The structure and properties of the Hopf algebras $H(m,d)$ have been established in the preceding chapters. This chapter outlines our planned approach for classifying the Yetter-Drinfel'd modules of $H(m,d)$ via Abelian extensions, which motivates the Conjecture 5.0.0.1 proved in Chapter 5. We will state the main results without presenting detailed proofs.

This classification of the Yetter-Drinfel'd modules of $H(m,d)$ is carried out as follows. In section 7.1, we discuss the Drinfel'd double of $H(m,d)$. Based on this, in section 7.2, we introduce the classification of the simple Yetter-Drinfel'd modules over $H(m,d)$. Finally, some Nichols algebras associated with these simple modules are studied in section 7.3. However, the finite-dimensional Nichols algebras of these Yetter-Drinfel'd modules V have many new cases, and we have considered many interesting results, but the complete classification of finite-dimensional Nichols algebras are still under investigation, we will obtain the complete classification in the future.

7.1 The Drinfel'd double of $H(m,d)$

Let G be the finite abelian group

$$Z_{md} \times Z_m$$

generated by elements x and y with relations

$$x^{md} = 1, \quad g^m = 1, \quad xg = gx.$$

Let $g, h \in G$, recall that we define the following elements in $H^* = H(m,d)^*$ in section 3.1,

$$\zeta_g : \begin{cases} e_h \mapsto \delta_{g,h} \\ e_h\#t \mapsto 0, \end{cases} \qquad \chi_h : \begin{cases} e_g \mapsto 0 \\ e_g\#t \mapsto \delta_{g,h}. \end{cases} \tag{7.1}$$

Firstly, we take a look at the Hopf algebra structure of H^* with the base $\zeta_{x^iy^j}, \chi_{x^ky^l}$, $(0 \le i, k \le md-1, 0 \le j, l \le m-1)$. Deducing from the coalgebra structure of H, we obtain

$$\begin{aligned}
\zeta_{x^iy^j}\zeta_{x^ky^l} &= \zeta_{x^{i+k}y^{j+l}}, \\
\chi_{x^iy^j}\chi_{x^ky^l} &= \gamma^{-jk+(m-2)l}\chi_{x^{i+j}y^{j+l}}, \\
\zeta_{x^iy^j}\zeta_{x^ky^l} &= 0 = \zeta_{x^ky^l}\zeta_{x^iy^j}. \\
1_{H^*} &= \varepsilon = \zeta_1 + \chi_1.
\end{aligned} \tag{7.2}$$

Then $e_{x^iy^j}, t\,(0 \le i \le md-1, 0 \le j \le m)$ are the generators of $H(m,d)$ and $\zeta_{x^iy^j}, \chi_{x^iy^j}$ $(0 \le i \le md-1, 0 \le j \le m)$ are the generators of H^*. The Drinfel'd double $\mathcal{D}(H) = H^{*cop} \bowtie H$ is generated by $\zeta_{x^iy^j} \bowtie 1_H$, $\chi_{x^iy^j} \bowtie 1_H$, $1_{H^{*cop}} \bowtie e_{x^iy^j}$, $1_{H^{*cop}} \bowtie t$, and abbreviate them by $\zeta_{x^iy^j}, \chi_{x^iy^i}, e_{x^iy^j}, t$. Then $\zeta_{x^iy^j}e_{x^ky^l}$ means $(\zeta_{x^iy^j} \bowtie 1_H)(1_{H^{*cop}} \bowtie e_{x^ky^l})$. It is clear that $1_{\mathcal{D}(H)} = 1_{(H)^{*cop}} \bowtie 1_H$.

Let $H = H(m,d)$, then we can generalize Theorem 3.1.0.1 to the setting of H.

Theorem 7.1.0.1. $\mathcal{D}(H)$ is generated by $e_{x^iy^j}, t, \zeta_{x^iy^j}, \chi_{x^iy^j}$ $(0 \le i \le md-1, 0 \le j \le m)$ with relations:

$$\begin{aligned}
\zeta_{x^iy^j}\zeta_{x^ky^l} &= \zeta_{x^{i+k}y^{j+l}}, \\
\chi_{x^iy^j}\chi_{x^ky^l} &= \gamma^{-jk+(m-2)l}\chi_{x^{i+k}y^{j+l}}, \\
\zeta_{x^iy^j}\chi_{x^ky^l} &= 0 = \chi_{x^ky^l}\zeta_{x^iy^j},
\end{aligned} \tag{7.3}$$

$$\begin{aligned}
&1 = \textstyle\sum_{\gamma \in G} e_\gamma, \quad e_\gamma e_{\gamma'} = \delta_{\gamma,\gamma'}e_\gamma\,(\gamma, \gamma' \in G), \\
&te_g = e_{g \triangleleft t}t, \quad t^2 = \textstyle\sum_g \sigma(g; t, t)e_g,
\end{aligned} \tag{7.4}$$

$$e_g\chi_h = \chi_h e_{gh(h^{-1} \triangleleft t)}, e_g\zeta_h = \zeta_h e_g, \tag{7.5}$$

$$\begin{aligned}
t\zeta_h &= \zeta_{h\triangleleft t}(\textstyle\sum_g \tau\,(h \triangleleft t, g; t)\,\tau(\,(h \triangleleft t)g, h^{-1} \triangleleft t; t)\tau(h \triangleleft t, h^{-1} \triangleleft t;\ t)(e_g\#t)), \\
t\chi_h &= \chi_{h\triangleleft t}(\textstyle\sum_g \tau\,(h, g; t)\,\tau(hg, h^{-1} \triangleleft t; t)\tau(h \triangleleft t, h^{-1} \triangleleft t;\ t)(e_g\#t)).
\end{aligned} \tag{7.6}$$

Proof: The proof is same as Theorem 3.1.0.1. □

7.2 The simple Yetter–Drinfel'd modules V over $H(m,d)$

Let G be the finite abelian group

$$Z_{md} \times Z_m$$

generated by elements x and g with relations

$$x^{md} = 1, \quad g^m = 1, \quad xg = gx.$$

Based on the Conjucture 5.0.0.1 and its proof, we obtained that

$$H(m,d)=\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2.$$

We denote by $D(H)$ its Drinfel'd double $D(H(m,d))$. Let S be a subset of G that is closed under the action $\triangleleft t$, and let $T=G\setminus S$. Then $G=T\cup S$.

In this section, we denote $x\otimes y\otimes z$ by $x\cdot y\cdot z$. Let (V,c) be a braided vector space. Assume $\{v_i|\ 1\le i\le n\}$ is a basis of V. Recall that in Chapter 5,

$$\begin{aligned}
&\triangleright \text{ is a trivial action,}\\
&g\triangleleft t=g,\\
&x\triangleleft t=x^{-1}g^{-1},\\
&\tau(x^ig^j,\,x^kg^l,\,t)=\gamma^{-jk},\\
&\sigma(x^ig^j,\,t,\,t)=\gamma^{\frac{-i^2+i+2(m-1)j}{2}}.
\end{aligned}$$

We will abbreviate $\sigma(x^ig^j)$ for $\sigma(x^ig^j,\,t,\,t)$ (resp. $\tau(x^ig^j,\,x^kg^l)$). We define a map $\eta:G\times G\to k$ by requiring that $\eta(s_1,s_2)=1$ for all $s_1,s_2\in S$. Assume $s\in S$ and $\lambda\in\Bbbk$, let

$$\alpha:G\to\Bbbk$$

be an algebra map defined by $\zeta_g\cdot 1=\alpha(g)1$. We are now ready to generalize Theorem 3.2.1.3 to the setting of $H(m,d)$.

Theorem 7.2.0.1. All non-isomorphic simple modules over $D(H)$ are giving as following,

(1) **One-dimensional case**: Assume $s\in S$ and $\lambda\in\Bbbk$ such that $\lambda^2=1$. we obtain one-dimensional simple modules $V_{s,\lambda,\alpha}=\Bbbk$ over $D(H)$, where the actions of $D(H)$ on $V_{s,\lambda,\alpha}$ is given by

$$e_g\cdot 1=\delta_{g,s}1,\ x\cdot 1=\lambda 1,\ \zeta_g\cdot 1=\alpha(g)1,\ \chi_g\cdot 1=0,$$

for $g\in G$.

(2) **Two-dimensional case**: Assume $s\in S,\ t\in T$ and assume $\alpha:G\to\Bbbk$ and $\beta:\langle\chi_s\rangle_{s\in S}\to\Bbbk$ are algebra maps, where $\langle\chi_S\rangle_{s\in S}$ is viewed as an algebra, then the two-dimensional simple modules $V_{s,\alpha},\ V_{s,\beta},\ V_{t,\alpha}\ V_{t,\beta}$ over $D(H)$ are obtained as follows:

(a) $V_{s,\alpha}=\Bbbk^2=<v^1,v^2>$ and actions of $D(H)$ on $V_{s,\alpha}$ is given by

$$e_g\cdot v^i=\delta_{g,s},\ i\in\{1,2\},\ x\cdot v^1=v^2,\ x\cdot v^2=\sigma(s)v^1,$$

$$\zeta_g\cdot v^1=\alpha(g)v^1,\ \zeta_g\cdot v^2=\eta(g,s)\alpha(g\triangleleft t)v^2,\ \chi_g\cdot v^i=0,\ i\in\{1,2\}\text{ for }g\in G$$

(b) $V_{s,\beta} = \Bbbk^2 =< v^1, v^2 >$ and actions of $D(H)$ on $V_{s,\beta}$ is given by

$$e_g \cdot v^1 = \delta_{g,s},\ e_g \cdot v^2 = \delta_{g,sb},\ x \cdot v^1 = \lambda v^1,\ x \cdot v^2 = \mu v^2,\ \zeta_g \cdot v^i = 0,\ i \in \{1,2\},$$

$$\chi_{s_1} \cdot v^1 = \beta(s_1)v^1,\ \chi_{s_1} \cdot v^2 = \eta(s_1, a)\beta(s_1)v^2,\ \chi_a \cdot v^1 = v^2,\ \chi_a \cdot v^2 = \tau(a,a)\beta(a^2)v^1$$

where $a \in S, b \in T$ and $\lambda, \mu \in \Bbbk$ such that $\lambda^2 = \sigma(s),\ \frac{\mu}{\lambda} = \lambda(a,s)\tau(a,b)^{-1}\beta(b)$.

(c) $V_{t,\alpha} = \Bbbk^2 =< v^1, v^2 >$ and actions of $D(H)$ on $V_{t,\alpha}$ is given by

$$e_g \cdot v^1 = \delta_{g,t},\ e_g \cdot v^2 = \delta_{g,tb}\,(b \in T),\ x \cdot v^1 = \lambda v^2,\ x \cdot v^2 = \sigma(t)v^1,$$

$$\zeta_g \cdot v^1 = \alpha(g)v^1,\ \zeta_g \cdot v^2 = \eta(g,t)\alpha(g \triangleleft t)v^2,\ \chi_g \cdot v^i = 0,\ i \in \{1,2\},\ g \in G,$$

(d) $V_{t,\beta} = \Bbbk^2 =< v^1, v^2 >$ and actions of $D(H)$ on $V_{t,\beta}$ is given by

$$e_g \cdot v^1 = \delta_{g,t},\ e_g \cdot v^2 = \delta_{g,tb},\ x \cdot v^1 = \lambda v^2,\ x \cdot v^2 = \sigma(t)v^1,$$

$$\zeta_g . v^i = 0,\ i \in \{1,2\},$$

$$\chi_s \cdot v^1 = \beta(s)v^1,\ \chi_s \cdot v^2 = \eta(x \triangleleft t, s)\beta(s)v^2,\ \chi_a \cdot v^1 = \lambda v^2,\ \chi_a \cdot v^2 = \mu v^1$$

where $a \in S, b \in T$ and $\lambda, \mu \in \Bbbk$ such that

$$\frac{\lambda}{\mu} = \eta(t \triangleleft x, b)\beta(b)\lambda(a, t \triangleleft x)\tau(a,b)^{-1},\ \lambda\mu = \tau(a,a)\beta(a^2).$$

Proof: The proof is similar to that of Theorem 3.2.1.3, so we omit the details here. □

We denote $x \otimes y \otimes z$ by $x \cdot y \cdot z$. Let (V, c) be a braided vector space. Assume $\{v_i | 1 \leq i \leq n\}$ is a basis of V. Let $\lambda, s, p, q \in \Bbbk$. We define two dimensional braided vector space (V, c) as follows:

$$c(v_1 \otimes v_1) = \lambda v_2 \otimes v_2, c(v_1 \otimes v_2) = pv_1 \otimes v_2,$$
$$c(v_2 \otimes v_1) = qv_2 \otimes v_1, c(v_2 \otimes v_2) = sv_1 \otimes v_1.$$

Then the braided vector space (V, c) is called of (λ, s, p, q)-type. If $p = q$, for simplicity, we denote the above braided vector space as $V(\lambda, s, p)$ and let $v := v_1, w := v_2$.

Remark 7.2.0.2. By Theorem 7.2.0.1, we know that all braided vector spaces arising from simple Yetter-Drinfeld modules over $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ are of the form described above.

7.3 Some Nichols algebras over $H(m, d)$

In this section, we are going to determine some finite dimensional Nichols algebra of simple Yetter-Drinfeld modules over $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. However, the finite-dimensional Nichols algebras of these Yetter-Drinfel'd modules V have many new cases, and we have considered many interesting results, but the complete classification of finite-dimensional Nichols algebras are still under investigation, we will obtain the complete classification in the future.

Recall that we denote $x \otimes y \otimes z$ by $x \cdot y \cdot z$ and denote (λ, s, p, q) by the braided vector space of (λ, s, p, q)-type. If $p = q$, for simplicity, we denote the above braided vector space as $V(\lambda, s, p)$ and let $v := v_1, w := v_2$.

Theorem 7.3.0.1. If $q = 1$ or q is not root of unity, then $\mathcal{B}((V, \lambda, s, p)) = \infty$.

Proof: Let $n \in \mathbb{N}$. Consider the vector $a_n := u_1 ... u_n$, where $u_{2i-1} = v$ and $u_{2i} = w$ for $1 \leq i \leq [\frac{n}{2}]$. Then $\Omega_n(a_n) = \mu_n a_n$ for some $\mu_n \neq 0 \in \Bbbk$. Thus we have completed the proof. $\square$

Theorem 7.3.0.2. Let (V, c) be a braided vector space of (λ, s, p, q)-type. Then the Nichols algebra

a) $\mathcal{B}((V, \lambda, s, -1)) < \infty$ if and only if λs is root of unity.

b) If λs is primitive n-th root of unity, then $\mathcal{B}((V, \lambda, s, -1)) \cong k\langle v, w \mid vw = wv = 0, v^{2n+1} = w^{2n+1} = 0, v^{2n} + w^{2n} = 0\rangle$ and dim$\mathcal{B}((V, \lambda, s, -1)) = 4n$.

Proof: Let $c_0 = 1, c_i = 1 - (\lambda s)^i, (i \geq 1)$. We have:

$$\begin{gathered}
\Omega_2(v \cdot v) = (id + c)(v \cdot v) = (v \cdot v + \lambda w \cdot w),\\
\Omega_2(w \cdot w) = (id + c)(w \cdot w) = (w \cdot w + s v \cdot v),\\
\Omega_2(v \cdot w) = 0, \quad \Omega_2(w \cdot v) = 0,\\
\Omega_3(v \cdot v \cdot v) = (1 - \lambda s) v \cdot v \cdot v, \quad \Omega_3(w \cdot w \cdot w) = (1 - \lambda s) w \cdot w \cdot w,\\
\Omega_4(v \cdot v \cdot v \cdot v) = v \cdot v \cdot v \cdot v - \lambda^2 w \cdot w \cdot w \cdot w),
\end{gathered}$$

Furthermore, it follows from

$$\begin{gathered}
v^{2n} \cdot w = (-\lambda)^n w^{2n+1},\\
v^{2n} \cdot v = v^{2n+1},\\
v^{2n+1} \cdot v = v^{2n+2} + (-1)^n \lambda^{n+1} w^{2n+2},
\end{gathered}$$

$$
\begin{aligned}
v^{2n+1} \cdot w &= 0, \\
w^{2n} \cdot v &= (-s)^n v^{2n+1}, \\
w^{2n} \cdot w &= w^{2n+1}, \\
w^{2n+1} \cdot w &= w^{2n+2} + (-1)^n s^{n+1} v^{2n+2}, \\
w^{2n+1} \cdot v &= 0.
\end{aligned}
$$

and by induction on n, we have:

$$
\begin{aligned}
\Omega_{2n}(v^{2n}) &= c_0 c_1 \dots c_{n-1}[v^{2n} + (-1)^{n-1}\lambda^n w^{2n}], \\
\Omega_{2n}(w^{2n}) &= c_0 c_1 \dots c_{n-1}[w^{2n} + (-1)^{n-1} s^n v^{2n}], \\
\Omega_{2n+1}(v^{2n+1}) &= c_0 c_1, \dots, c_n v^{2n+1}, \\
\Omega_{2n+1}(w^{2n+1}) &= c_0 c_1, \dots, c_n w^{2n+1},
\end{aligned}
$$

We can see $\Omega_{2n+1}(v^{2n+1}) = 0$ if and only if $c_n = 0$ (i.e.$(\lambda s)^n = 0$), and we can also get $\Omega_{2n}(v^{2n} + w^{2n}) = 0$ from $(\lambda s)^n = 0$. This finish the claim (a) and (b). □

Lemma 7.3.0.3. Let $n \in \mathbb{N}$. Assume $A = \langle x, y |\ x^2 = \alpha y^2 \rangle$ as algebra. If $a = a_1 ... a_{2n+1}$ where $a_i \in \{x, y\}$ for $1 \le i \le 2n+1$, then $a = \beta xyx ... a_{n+2} ... a_{2n+1}$ or $a = \beta yxy ... a_{n+2} ... a_{2n+1}$ for some $\beta \in \Bbbk$.

Proof: We prove inductively for n. It is obvious for $n = 1$. Assume $n = k$ holds. If $a = a_1 ... a_{2k+3}$, then we can assume $a_1 ... a_{2k+1} = xyx ... a_{k+2} ... a_{2k+1}$ or $a = \beta yxy ... a_{k+2} ... a_{2k+1}$ by our induction assumption. Since the reason of symmetry, we can assume $a_1 ... a_{2k+1} = \beta xyx ... a_{k+2} ... a_{2k+1}$. If $a_{k+1} = x$, since $x^2 = \alpha y^2$, we only need to show the case $a_{k+2} ... a_{2k+3} = xyx ... a_{2k+3}$, where a_i and a_{i+1} are different symbol in $\{x, y\}$. Using $x^2 = \alpha y^2$ again, we get $a = \beta xyx ... a_{k+3} ... a_{2k+3}$. Similarly, one can prove that $a = \beta yxy ... a_{k+3} ... a_{2k+3}$ if $a_{k+1} = x$.
□

As an application of above lemma, we have

Theorem 7.3.0.4. If q is n-th primitive root of 1 and $\lambda s = 1$, then $\mathcal{B}((V, \lambda, s, p)) < \infty$.

Proof: Directly, we have $\Omega_2(w^2 - sv^2) = 0$. Thus $\mathcal{B}((V, \lambda, s, p))$ is a quotient algebra of $A = \langle x, y |\ x^2 = sy^2 \rangle$. Next, we claim that $\mathcal{B}((V, \lambda, s, p))^{\otimes(2n-1)} = 0$. Let $u = u_1 ... u_n$ and $t = t_1 ... t_n$, where $u_1 = v$ (resp. $t_1 = w$) and u_i, u_{i+1} (resp. t_i, t_{i+1}) are different symbol in $\{v, w\}$. Then we have $\Omega_n(u) = \Omega_n(t) = 0$. Thus $u = t = 0$. Let $a = a_1 ... a_{2n-1}$, where $a_i \in \{v, w\}$. Since Lemma 7.3.0.3, we have $a = \beta_1 u a_{n+1} ... a_{2n-1}$ or $a = \beta_2 t a_{n+1} ... a_{2n-1}$ for some $\beta_1, \beta_2 \in \Bbbk$. Thus $a = 0$. This implies $\mathcal{B}((V, \lambda, s, p))^{\otimes(2n-1)} = 0$. □

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