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On unimodularity of infinite-dimensional Hopf algebras

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毕业论文题目： On unimodularity of infinite-dimensional Hopf algebras

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摘 要

本文围绕幺模性 (unimodularity) 的定义详细讨论了局部紧拓扑群上的 Haar 积分、有限维 Hopf 代数中的积分以及 AS-Gorenstein 代数上的同调积分，并通过计算一类无穷维 Hopf 代数的同调积分得到关于幺模性在这三类积分中的相似或相异的一些性质。

关键词：Unimodularity；Homological integral；Hopf algebra

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THESIS: On unimodularity of infinite-dimensional Hopf algebras

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ABSTRACT

This paper provides a detailed discussion on the definition of unimodularity, examining Haar integrals on locally compact topological groups, integrals in finite-dimensional Hopf algebras, and homological integrals on AS-Gorenstein algebras. By computing the homological integrals of a class of infinite-dimensional Hopf algebras, we explore the similarities and differences in the properties of unimodularity across these three types of integrals.

KEYWORDS: Unimodularity; Homological integral; Hopf algebra

CONTENTS

Chapter 1 Introduction

Throughout this paper, $\Bbbk$ is assumed to be an algebraically closed field of characteristic 0, and all vector spaces, algebras and Hopf algebras are assumed to be over $\Bbbk$.

Hopf algebras, which naturally unify algebraic structures such as groups, Lie algebras, and quantum groups, have become a central topic in modern mathematics and mathematical physics.

In the finite-dimensional case, the linear dual of a Hopf algebra is again a Hopf algebra, and the theory of integrals plays a crucial role in understanding its structure. For instance, integrals in finite-dimensional Hopf algebras lead to elegant generalizations of classical results such as Maschke's theorem, which characterizes semisimplicity.

However, the situation becomes significantly more complex in the infinite-dimensional setting. Two fundamental challenges arise:

1) Integral Theory in Infinite-Dimensions: While integrals in finite-dimensional Hopf algebras are well understood and yield powerful results, there lacks a natural definition in the infinite-dimensional case.
2) Dual Structures in Infinite-Dimensions: The linear dual of a finite-dimensional Hopf algebra retains a Hopf algebra structure, but this property fails in general for infinite-dimensional Hopf algebras. To address this, Heyneman and Sweedler introduced the concept of the finite dual, which preserves the Hopf algebra structure under certain conditions. Nevertheless, the study of finite duals for general infinite-dimensional Hopf algebras remains highly nontrivial.

To tackle the first problem, Lu, Wu, and Zhang [5] introduced homological integrals for Artin-Schelter Gorenstein (AS-Gorenstein) Hopf algebras, providing a viable generalization of classical integrals in the infinite-dimensional setting. This framework has proven particularly effective in the study of infinite-dimensional Hopf algebras with low Gelfand-Kirillov dimension. Subsequently, Wu, Ding, and Liu [11] applied homological integral theory to classify affine prime regular Hopf algebras of GK-dimension one, demonstrating the utility of this approach in structural classification.

Building on this classification, Li and Liu [4] further investigated the finite duals of these Hopf algebras, explicitly determining their algebraic structure and constructing non-degenerate Hopf pairings between the original Hopf algebras and certain Hopf subalgebras of their finite duals. These results provide deeper insights into the duality properties of infinite-dimensional Hopf algebras.

In this paper, we focus on the homological integrals of the certain Hopf subalgebras constructed in [4]. By computing their left homological integrals, integral order and integral minor, we investigate some interesting properties about unimodularity：

1) Unlike the case of affine prime regular Hopf algebra of GK-dimension one, commutativity is not necessary for unimodularity in general.
2) Similar to the case of finite-dimensional Hopf algebras, for infinite-dimensional Hopf algebras, unimodularity is generally independent of the unimodularity of its dual space.

Chapter 2 Preliminary

2.1 Some basics about Hopf algebra

Definition 2.1. *Let $\Bbbk$ be a field. A $\Bbbk$-algebra (with unit) is a $\Bbbk$-vector space A together with two $\Bbbk$-linear maps, multiplication $m : A \otimes A \to A$, and unit $u : \Bbbk \to A$, such that the following diagrams are commutative:*

a) associativity *b) unit*

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes Id} & A \otimes A \\
{\scriptstyle Id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\
A \otimes A & \xrightarrow[m]{} & A
\end{array}
\qquad
\begin{array}{ccc}
A \otimes A & \xleftarrow{Id \otimes u} & A \otimes \Bbbk \\
{\scriptstyle u \otimes Id}\uparrow & \searrow^{m} & \downarrow{\scriptstyle \eta} \\
\Bbbk \otimes A & \xrightarrow[\eta]{} & A
\end{array}
$$

The Id denotes the identity mapping, and the η denotes the scalar multiplication.

Definition 2.2. *For any $\Bbbk$-space V and W, the twist map $\tau : V \otimes W \to W \otimes V$, is given by $\tau(v \otimes w) = w \otimes v$. Obviously, A is commutative $\iff$ $m \circ \tau = m$ on $A \otimes A$.*

The definition of a coalgebra is made by reversing the arrows in the diagrams in Definition 2.1.

Definition 2.3. *A $\Bbbk$-coalgebra (with counit) is a $\Bbbk$-vector space C together with two $\Bbbk$-linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to \Bbbk$, such that the following diagrams are commutative:*

a) coassociativity *b) counit*

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
{\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes Id} \\
C \otimes C & \xrightarrow[Id \otimes \Delta]{} & C \otimes C \otimes C
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{\otimes 1_k} & C \otimes \Bbbk \\
{\scriptstyle 1_k \otimes}\downarrow & \searrow^{\Delta} & \uparrow{\scriptstyle Id \otimes \epsilon} \\
\Bbbk \otimes C & \xleftarrow[\epsilon \otimes Id]{} & C \otimes C
\end{array}
$$

The two upper maps in b) are given by $c \mapsto 1 \otimes c$, and $c \mapsto c \otimes 1$, for any $c \in C$. We say C is cocommutative if $\tau \circ \Delta = \Delta$.

Definition 2.4. *Let C be any coalgebra, and $c \in C$.*

*a) c is called **group-like** if*

$$\Delta(c) = c \otimes c \quad and \quad \epsilon(c) = 1.$$

The set of group-like elements in C is denoted by $G(C)$.

*b) For $g, h \in G(C)$, c is called g, h-**primitive** if*

$$\Delta(c) = c \otimes g + h \otimes c.$$

*The set of all g, h-primitive elements is denoted by $P_{g,h}(C)$. The elements of $P_{1,1}(C)$ are simply called the **primitive elements** of C, denoted by $P(C)$.*

Definition 2.5. *Let C and D be coalgebras, with comultiplication Δ_C and Δ_D, and counits ϵ_C and ϵ_D, respectively. A coalgebra map $f : C \to D$ is a linear map, such that $\Delta_D \circ f = (f \otimes f)\Delta_C$ and $\epsilon_C = \epsilon_D \circ f$, that means the following diagrams are commutative:*

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow[f \otimes f]{} & D \otimes D \end{array}$$

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ & {\scriptstyle \epsilon_C}\searrow & \downarrow{\scriptstyle \epsilon_D} \\ & & \Bbbk \end{array}$$

Similarly, we have the definition of algebra map.

Definition 2.6. *A $\Bbbk$-space B is a bialgebra if (B, m, u) is an algebra, (B, Δ, ϵ) is a coalgebra, and either of the following (equivalent) conditions holds:*

a) Δ and ϵ are algebra morphisms

b) m and u are coalgebra morphisms.

Theorem 2.7. *Let C be a coalgebra and A an algebra. Then $Hom_{\Bbbk}(C, A)$ becomes an algebra under the convolution product $f * g(c) = m \circ (f \otimes g)(\Delta c), \forall f, g \in Hom_{\Bbbk}(C, A), c \in C$. The unit element in $Hom_{\Bbbk}(C, A)$ is $u\epsilon$.*

Let C be any coalgebra with comultiplication $\Delta : C \to C \otimes C$. The sigma notation for Δ is given as follows: for any $c \in C$, we write

$$\Delta c = c_1 \otimes c_2.$$

The subscripts 1 and 2 are symbolic, and do not indicate particular elements of C, this notation is analogous to the notation used in physics. In these note, we usually simplify the notation by omitting parentheses. In particular, the coassociativity diagram gives that

$$c_1 \otimes c_{2_1} \otimes c_{2_2} = c_{1_1} \otimes c_{1_2} \otimes c_2,$$

this element is written as $c_1 \otimes c_2 \otimes c_3 = \Delta_2(c)$.

Definition 2.8. *Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra. Then H is a Hopf algebra if there exists an element $S \in Hom_{\Bbbk}(H, H)$, which is an inverse to Id_H under the convolution $*$. S is called an antipode for H. Note that in sigma notation, S satisfies*

$$\sum (Sh_1)h_2 = \epsilon(h)1_H = \sum h_1(Sh_2), \forall h \in H.$$

We give some relevant examples of Hopf algebras.

Example 2.9. Let G be a group and $\Bbbk$ a field. Then $\Bbbk G$ becomes a Hopf algebra with comultiplication, counit and antipode given by:

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1},$$

for all $g \in G$, extended linearly to all of $\Bbbk G$.

Example 2.10. Assume that $char(\Bbbk) \neq 2$. Let H be the algebra generated by c and x with relations:

$$c^2 = 1, \quad x^2 = 0, \quad xc = -cx$$

Then H becomes a Hopf algebra with comultiplication, counit and antipode given by:

$$\Delta(c) = c \otimes c, \quad \Delta(x) = c \otimes x + x \otimes 1,$$
$$\epsilon(c) = 1, \quad \epsilon(x) = 0,$$
$$S(c) = c^{-1}, \quad S(x) = -cx.$$

H is called Sweedler's 4-dimensional Hopf algebra, and it was the first example of a non-commutative and non-cocommutative Hopf algebra.

Definition 2.11. *A map $f : H \to K$ of Hopf algebras is a Hopf morphism, if it is a bialgebra morphism and $f(S_H h) = S_K f(h), \forall h \in H$.*

Definition 2.12. *Let H be a Hopf algebra with multiplication μ, comultiplication Δ, counit ϵ, antipode S, and unit η. A subspace $I \subseteq H$ is called a Hopf ideal if:*

a) I is a two-sided ideal of H:

$$\mu(I \otimes H) \subseteq I \quad \textit{and} \quad \mu(H \otimes I) \subseteq I.$$

b) I is a coideal of H:

$$\Delta(I) \subseteq I \otimes H + H \otimes I \quad \textit{and} \quad \epsilon(I) = 0.$$

c) The antipode S preserves I:

$$S(I) \subseteq I.$$

The quotient H/I then inherits a Hopf algebra structure from H.

2.2 Finite duals for Hopf algebras

When studying an algebra structure, it is always effective to consider the linear dual of it. It is known that when $(H, m, u, \Delta, \epsilon, S)$ is a finite-dimensional Hopf algebra, its linear dual $(H^*, \Delta^*, \epsilon^*, m^*, u^*, S^*)$ is also a Hopf algebra and has the same dimension as well as similar or dual properties with H.

However, the linear dual of an infinite-dimensional Hopf algebra fails to have Hopf algebra structure in general because the algebra structure of a bialgebra may not introduce a canonical coalgebra structure on its linear dual.

In [2], Heyneman and Sweedler described the **finite dual** of an infinite-dimensional Hopf algebra H, which is not defined on the entire dual space H^*, but on a certain subspace of it:

Definition 2.13. *Let H be a Hopf algebra over $\Bbbk$ and denote its dual space by H^*. The finite dual of H is defined as*

$$H^\circ := \{f \in H^* \mid f \text{ vanishes on a cofinite ideal } I\}$$

It is well known that H° has a Hopf algebra structure naturally. Similar to the finite-dimensional case, there are several relationships between H and H°. However, for an infinite-dimensional Hopf algebra, its finite dual might be hard to be studied easily. For examples and properties of the finite dual, we refer to Chapter 9 in [6].

2.3 Gelfand-Kirillov dimension

The Gelfand-Kirillov dimension, GK-dimension for short, becomes a powerful tool to study noncommutative algebras, especially for those with infinite dimensions.

Definition 2.14. *The Gelfand-Kirillov dimension (GK-dimension for short) of a $\Bbbk$-algebra A is*

$$\mathrm{GKdim}(A) = \limsup_{V} d_V(n),$$

where the supremum is taken over all finite-dimensional subspaces V of A and

$$d_V(n) = \dim_k\left(\sum_{i=0}^{n} V^i\right).$$

The GK-dimension can be viewed as a non-commutative analogue of the Krull dimension. In fact, for a finitely generated commutative algebra A, the GK-dimension of A equals its Krull dimension.

Example 2.15. GK-dimension of $\Bbbk[x_1, x_2, \cdots, x_n] = n$.

2.4 Stuff from ring theory

In this paper, a ring R is called **regular** if it has finite global dimension, and it is **prime** if 0 is a prime ideal.

• *PI ring*. Let R be an associative ring (not necessarily commutative). If there exists a nonzero polynomial $f(x_1, x_2, \cdots, x_n)$ (with coefficients in $\Bbbk$) such that for all n-tuples $(r_1, r_2, \cdots, r_n) \in R^n$, $f(r_1, r_2, \cdots, r_n) = 0$, then R is called a **PI ring** (Polynomial Identity Ring).

• *Artin-Schelter condition*. Recall that an algebra A is said to be **augmented** if there is an algebra morphism $\epsilon : A \to \Bbbk$. Let (A, ϵ) be an augmented noetherian algebra. Then A is **Artin-Schelter Gorenstein**, we usually abbreviate to **AS-Gorenstein**, if

(AS1) $\operatorname{injdim}_A A = d < \infty$,

(AS2) $\dim_\Bbbk \operatorname{Ext}^d_A({}_A\Bbbk, {}_AA) = 1$ and $\dim_\Bbbk \operatorname{Ext}^i_A({}_A\Bbbk, {}_AA) = 0$ for all $i \neq d$,

(AS3) the right A-module versions of (AS1, AS2) hold.

Chapter 3 Integrals

In this chapter, we recall the basic definitions and related properties of Integrals on locally compact topological groups, bialgebras, finite-dimensional Hopf algebras and infinite-dimensional Hopf algebras. Throughout this chapter, we focus on **unimodularity** of the objects above, and an the end of this chapter we raise a question about it .

3.1 Haar integral on locally compact topological groups

In abstract harmonic analysis theory, the extension of Fourier analysis to compact non-Abelian groups was made possible by the Peter-Weyl theorem. Crucially, the theorem revealed that the key requirement for Fourier representations is not the finiteness of the group, but the existence of an invariant integral that assigns a finite volume to the group. In this case Haar integral plays an important role in it.

In the case of non-abelian groups, it is necessary to distinguish left and right invariance. For instance, an integral on a topological group G is left invariant if

$$\int_G f(ax)\, dx = \int_G f(x)\, dx$$

for all $a \in G$.

Definition 3.1. *Let G be a locally compact group, A left (resp.right) Haar measure on G is a nonzero Radon measure μ on G satisfies $\mu(xE)=\mu(E)$ (resp. $\mu(Ex)=\mu(E)$) for every Borel set $E \subseteq G$ and every $x \in G$.*

Let $C_c(G)$ be the space of compactly supported continuous complex-valued functions *on G, the Haar integral for $f \in C_c(G)$ is defined by*

$$I(f) = \int_G f \, d\mu.$$

It is easy to see that the Haar integral associated to left (resp.right) Haar measure is left (resp.right) invariant. Once we get a left Haar measure μ on group G, let $\tilde{\mu}(E) = \mu(E^{-1})$, then $\tilde{\mu}$ is a right Haar measure on G. From this point of view, it is of little importance whether one chooses to study left or right Haar measure.

For second countable groups, the existence of Haar measure was first proved by Haar [1] and the uniqueness was first proved by von Neumann [7]. The first systematic treatment of analysis on locally compact groups using Haar measure was given by Weil [10], who showed that the countability assumptions were unnecessary :

Theorem 3.2. *Every locally compact group G possesses a left Haar measure.*

Theorem 3.3. *If ν and μ are left Haar measures on G, there exists $c \in (0, \infty)$ such that $\mu = c\nu$*

Here we give some examples:

Example 3.4. $dx/|x|$ is a Haar measure on the multiplicative group $\mathbb{R} \setminus \{0\}$.

Example 3.5. $dxdy/(x^2 + y^2)$ is a Haar measure on the multiplicative group $\mathbb{C} \setminus \{0\}$.

Example 3.6. $|detT|^{-n}dT$ is a left and right Haar measure on the group $GL(n, \mathbb{R})$, where dT is Lebesgue measure on the vector space of all real $n \times n$ matrices.

Example 3.7. The **ax+b group** G is the group of all affine transformation $x \to ax + b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$. On G, $dadb/a^2$ is a left Haar measure and $dadb/a$ is a right Haar measure.

In Example 3.7, we find that the left Haar measure and the right Haar measure are not necessarily the same. To investigate the extent to which the left Haar measure fails to be right-invariant, **modular function** is defined.

Let G be a locally compact group with left Haar measure λ. If, for $x \in G$, we define

$\lambda_x(E) = \lambda(Ex)$, then λ_x is again a left Haar measure. By the uniqueness Theorem 3.3, there is a number $\Delta(x) > 0$ such that $\lambda_x = \Delta(x)\lambda$, and $\Delta(x)$ is independent of the original choice of λ. The function $\Delta : G \to (0, \infty)$ thus defined is called the **modular function** of G.

G is called **unimodular** if $\Delta \equiv 1$, that is, if left Haar measure is also right Haar measure. Unimodularity is a useful property in a number of respects. Obviously Abelian groups and discrete groups are unimodular, but many others are too. Here are some classes of examples:

Proposition 3.8. *If G is compact, then G is unimodular.*

Proposition 3.9. *If G/[G,G] is compact, then G is unimodular.*

As a consequence of Proposition 3.9, one can see that every connected semi-simple Lie group G is unimodular. More generally, one has the following result:

Proposition 3.10. *Let G be a Lie group and* $\mathfrak{g}$ *the Lie algebra of G, then G is unimodular if and only if*

$$|det\ Ad(g)| = 1 \quad \textit{for all } g \in G,$$

where Ad is the adjoint representation. If G is connected, this is equivalent to requiring

$$tr\ ad(X) = 0 \ \ \textit{for all } X \in \mathfrak{g},$$

where ad is the adjoint representation of the $\mathfrak{g}$.

It follows easily from Proposition 3.10 that every connected nilpotent Lie group and connected reductive Lie group is unimodular. The simplest example of a non-unimodular group is the $ax + b$ group.

3.2 Integrals in finite-dimensional Hopf algebras

Before introducing integrals in finite-dimensional Hopf algebras, we recall the definition of integrals for bialgebras.

Let H be a bialgebra. Then its dual space H^* has an algebra structure which is dual to the coalgebra structure on H. The multiplication is given by the convolution product. To simplify notation, if $h^*, g^* \in H^*$ we will denote the product of h^*, g^* in H^* by h^*g^*.

Definition 3.11. *A map $T \in H^*$ is called a left (resp.right) integral of the bialgebra H if $h^*T = h^*(1)T$ (resp.$Th^* = h^*(1)T$) for all $h^* \in H^*$. The set of left (resp.right) integrals of H is denoted by $\int_H^l$ (resp.$\int_H^r$).*

Following example is the motivating influence for the terminology "integral".

Example 3.12. Let G be a compact topological group, and let H be the Hopf algebra of continuous complex-valued representative functions; that is,

$$H = \{f \in (\mathbb{C}G)^\circ | f : G \to \mathbb{C} \text{ } is \text{ } continuous\}$$

Let μ denote Haar measure on G, and consider the Haar integral $\int_G f(x)d\mu(x)$. It follows that the map from H to $\mathbb{C}$ given by $f \mapsto \int_G f(x)d\mu(x)$ is an integral in the sense of definition 3.11.

When H is a finite-dimensional Hopf algebra, there is still another way to work with integrals. We recall that there is an isomorphism of algebras $\phi : H \to H^{**}$ defined by

$$\phi(h)(h^*) = h^*(h) \quad \text{for any } h \in H, h^* \in H^*.$$

Then it makes sense to talk about left integrals for the Hopf algebra H^*, these being elements in H^{**}.

Since ϕ is bijective, there exists a nonzero element $h \in H$ such that $\phi(h) \in H^{**}$ is a left integral for H^*. As any element in H^{**} is of the form $\phi(l)$ with $l \in H$, this means that for any $l \in H$ we have

$$\phi(l)\phi(h) = \phi(l)(1_{H^*})\phi(h).$$

But $\phi(l)\phi(h) = \phi(lh)$ (since ϕ is a morphism of algebras) and $\phi(l)(1_{H^*}) = \phi(l)(\epsilon) = \epsilon(l)$, hence the condition that $\phi(h)$ is a left integral for H^* is equivalent to

$$lh = \epsilon(l)h \quad \text{for any } l \in H.$$

The above discussion gives the definition of integrals in finite-dimensional Hopf algebras:

Definition 3.13. *Let H be a finite-dimensional Hopf algebra. A left (resp.right) integral in H is an element $t \in H$ for which $ht = \epsilon(h)t$ (resp. $th = \epsilon(h)t$) for all $h \in H$.*

Remark 3.14. *Left integrals in H are in fact left integrals for H^*.*

Left and right integrals in finite-dimensional Hopf algebras were introduced by Larson and Sweedler [3] where they established the existence and uniqueness in the theorem of Section 2 of that paper:

Theorem 3.15. *There exists nonzero left and right integral in any finite-dimensional Hopf algebra,and moreover, the subspace they span has dimension 1, i.e.,*

$$\int_H^l \cong \Bbbk \cong \int_H^r .$$

Similar to the case of Haar measure on locally compact topological group, we call a Hopf algebra H **unimodular** if

$$\int_H^l = \int_H^r .$$

Example 3.16. Let G be a finite group, then $t = \sum_{g \in G} g$ is a left (and right) integral in Hopf algebra $\Bbbk G$, which means that $\Bbbk G$ is unimodular.

Example 3.17. Let H denote Sweedler's 4-dimensional Hopf algebra described in Example 2.10 . Then $x + cx$ is a left integral in H, and $x - cx$ is a right integral in H, which means that H is not an unimodular Hopf algebra.

An important application of integrals in finite-dimensional Hopf algebras is the following result, proved by Larson and Sweedler, known as Maschke's theorem:

Theorem 3.18. *Let H be a finite-dimensional Hopf algebra. Then H is a semisimple algebra if and only if $\epsilon(\int_H^l) \neq 0$.*

Remark 3.19. *If G is finite group, and $H = \Bbbk G$, then from example 3.16 we saw that $t = \sum_{g \in G} g$ is a left integral in H. Then $\epsilon(t) = |G|1_{\Bbbk}$, where $|G|$ is the order of the group G. Then theorem 3.18 shows that the Hopf algebra $\Bbbk G$ is semisimple if and only if $|G|1_{\Bbbk} \neq 0$, hence if and only if $char(\Bbbk)$ does not divide the order of the group G. This is the well-known Maschke's theorem for groups.*

Similarly to the modular function of a locally compact topological group, we make some comments on the relationship between left and right integrals in finite-dimensional Hopf algebras.

Let H be a finite-dimensional Hopf algebra, for any $0 \neq t \in \int_H^l$, also $th \in \int_H^l$ for any $h \in H$. Since $\int_H^l$ is one-dimensional, it follows that $th = \alpha_{H^*}(h)t$, for some $\alpha_{H^*} \in \Bbbk$. Moreover, clearly $\alpha_{H^*} \in H^*$ and so is a group-like element of H^*. Finally, if we had begun with some $0 \neq t' \in \int_H^r$, then $ht' = \alpha_{H^*}^{-1}(h)t'$. The element $\alpha_{H^*} \in H^*$ constructed above is called the **distinguished group-like element** of H^*. Since H is finite-dimensional, we also have the distinguished group-like element of $H \cong H^{**}$, denoted by α_H. Clearly we have:

Theorem 3.20. *H is unimodular $\iff \alpha_{H^*} = \epsilon \iff \alpha_H = 1$.*

Corollary 3.21. *If H is semisimple, then H is unimodular.*

Remark 3.22. *The converse of Corollary 3.21 is false.*

A classical result proved by Larson and Radford in [3] states that if H is of characteristic 0, then

$$H \text{ is semisimple } \iff H^* \text{ is semisimple.}$$

One may ask whether H is unimodular is equivalent to H^* is unimodular, unfortunately, this is false in general, but it can be true under additional conditions (see proposition 10 of [8]).

Similar to the case of Lie groups, there is a criterion (with respect to the adjoint action) for unimodularity of finite-dimensional Hopf algebras:

Theorem 3.23. *Let $a \triangleright h = \sum a_1 h S(a_2)$ and $h \triangleleft a = \sum S(a_1) h a_2$ denote the left and right adjoint action of H on itself respectively for $a, h \in H$, let Λ be a non-zero left integral in H and p be a non-zero left integral in H^*, when dim$(H)1 \neq 0$, we have:*

$$H \text{ is unimodular} \iff p(\Lambda \triangleright 1) \neq 0$$

and

$$H^* \text{ is unimodular} \iff p(1 \triangleleft \Lambda) \neq 0$$

3.3 Homological integrals on AS-Gorenstein algebras

Integrals have been playing an important role in the studies of finite-dimensional Hopf algebras. As a conditional generalization of integrals in infinite-dimensional cases, **Homological integral**, first introduced in [5], has been widely used in the research of infinite-dimensional Hopf algebras of low GK-dimensions.

Definition 3.24. *Let (A, ϵ) be a noetherian augmented algebra and suppose that A is AS-Gorenstein of injective dimension d. Any non-zero element of the one-dimensional A-bimodule $\mathrm{Ext}^d_A({}_A\Bbbk, {}_AA)$ is called a **left homological integral** of A. We write $\int^l_A = \mathrm{Ext}^d_A({}_A\Bbbk, {}_AA)$. Any non-zero element in $\mathrm{Ext}^d_{A^{op}}(\Bbbk_A, A_A)$ is called a **right homological integral** of A. We write $\int^r_A = \mathrm{Ext}^d_{A^{op}}(\Bbbk_A, A_A)$. By abusing the language we also call $\int^l_A$ and $\int^r_A$ the left and the right homological integrals of A respectively.*

• *Winding automorphisms.* Let H be a Hopf algebra which has left homological integrals $\int^l_H$. Let $\pi : H \to H/\mathrm{r.ann}(\int^l_H)$ be the canonical algebra homomorphism, where $\mathrm{r.ann}(\int^l_H)$ denotes the set of right annihilators of $\int^l_H$ in H. We write Ξ^l_π for the *left* *winding automorphism* of H associated to π, namely

$$\Xi_\pi^l(a) := \sum \pi(a_1)a_2 \qquad \text{for } a \in H.$$

Similarly we use Ξ_π^r for the right winding automorphism of H associated to π, that is,

$$\Xi_\pi^r(a) := \sum a_1\pi(a_2) \qquad \text{for } a \in H.$$

Let G_π^l and G_π^r be the subgroups of $\mathrm{Aut}_{\Bbbk\text{-alg}}(H)$ generated by Ξ_π^l and Ξ_π^r, respectively.

• *Integral order and integral minor.* With the same notions as above, the *integral order* $\mathrm{io}(H)$ of H is defined by the order of the group G_π^l :

$$\mathrm{io}(H) := |G_\pi^l|.$$

we always have $|G_\pi^l| = |G_\pi^r|$. So the above definition is independent of the choice of G_π^l or G_π^r. The *integral minor* of H, denoted by $\mathrm{im}(H)$, is defined by

$$\mathrm{im}(H) := |G_\pi^l/G_\pi^l \cap G_\pi^r|.$$

Crudely speaking, $\mathrm{io}(H)$ is a measure of the commutativity of H and $\mathrm{im}(H)$ is a measure of the cocommutativity of H.

Homological integrals exist only for AS-Gorenstein Hopf algebras. Hence free Hopf algebras (of at least two variables) and universal enveloping algebras of infinite-dimensional Lie algebras do not have homological integrals.

When a Hopf algebra H is finite-dimensional, then homological integrals agree with the classical integrals in the following way: the (classical) left integral is an H-subbimodule of H; and it is identified with the left homological integral $\mathrm{Hom}_H(\Bbbk, H)$

via the natural homomorphism

$$\mathrm{Hom}_H(\epsilon, H) : \mathrm{Hom}_H(\Bbbk, H) \to \mathrm{Hom}_H(H, H) \cong H.$$

The same holds for the right integral. Note that both $\int_H^l$ and $\int_H^r$ are 1-dimensional H-bimodules. As a left H-module, $\int_H^l \cong \Bbbk$, but as a right H-module, $\int_H^l$ may not be isomorphic to $\Bbbk$. A similar comment applies to $\int_H^r$.

Definition 3.25. *Let H be a Hopf algebra with $\int_H^l$ and $\int_H^r$. We say H is unimodular if $\int_H^l$ is isomorphic to $\Bbbk$ as H-bimodules.*

The unimodular property means that

$$hx = xh = \epsilon(h)x$$

for all $h \in H$ and $x \in \int_H^l$. When H is finite-dimensional, this definition agrees with the classical definition.

Proposition 3.26. *Suppose H is noetherian. The following are equivalent:*

1. *H is unimodular.*
2. *$\int_H^r \cong \Bbbk$ as H-bimodules.*
3. *$\int_H^l \cong \int_H^r$ as H-bimodules.*
4. *$io(H) = 1$.*

Theorem 3.18 can be stated alternatively: a finite-dimensional Hopf algebra H is semisimple (i.e., has global dimension 0) if and only if $\epsilon(\int_H^l) \neq 0$. The term $\epsilon(\int_H^l)$ makes sense in the infinite-dimensional case in the following way. The counit $\epsilon : H \to \Bbbk$ induces an H-bimodule homomorphism, which is also denoted by ϵ,

$$\epsilon : \int_H^l = \mathrm{Ext}_H^d({}_H\Bbbk, {}_HH) \to \mathrm{Ext}_H^d({}_H\Bbbk, {}_H\Bbbk).$$

The original purpose for defining homological integral is to generalize Theorem 3.18 for infinite-dimensional Hopf algebras, especially noetherian affine PI Hopf algebras. Then it is natural to ask if the condition $\epsilon(\int_H^l) \neq 0$ stated above is equivalent to H having finite global dimension. The answer is 'No' as Example 3.2 in [5] shows. But there still exists an analogue of Corollary 3.21:

Theorem 3.27. *If* $\epsilon(\int_H^l) \neq 0$*, then H is unimodular.*

One important application of homological integral is to classify Hopf algebras of GK-dimensional one. In Sections 6 and 7 of [5], an amount of effort has been made to investigate the homological integrals and related homological properties of affine prime regular Hopf algebras of GK-dimension one. In 2016, Wu, Ding and Liu [11] gave a complete classification of affine prime regular Hopf algebras of GK-dimension one, where previous work in [5] plays an important role in it.

One interesting property of prime regular Hopf algebras of GK-dimensional one is:

Proposition 3.28. *H is unimodular if and only if H is commutative.*

This is an elegant and useful result that one may wish still work in general. However, our computation in Chapter 5 gives a negative answer.

Chapter 4 Some known examples

4.1 Affine prime regular Hopf algebra of GK-dimension one

In this section, we recall the classification result on affine prime regular Hopf algebras of GK-dimension one given in [11] .

Lemma 4.1. *Any prime regular Hopf algebra of GK-dimension one must be isomorphic to one of the following:*

1) *Connected algebraic groups of dimension one:* $\Bbbk[x]$ *and* $\Bbbk[x^{\pm 1}]$*;*
2) *Infinite dihedral group algebra* $\Bbbk\mathbb{D}_\infty$*;*
3) *Infinite dimensional Taft algebras* $T_\infty(n, v, \xi)$*, where* n, v *are integers satisfying* $0 \leqslant v \leqslant n-1$*, and* ξ *is a primitive nth root of* 1*;*
4) *Generalized Liu's algebras* $B(n, \omega, \gamma)$*, where n,* ω *are positive integers and* γ *is a primitive nth root of* 1*;*
5) *The Hopf algebras* $D(m, d, \xi)$*, where* m, d *are positive integers satisfying* $(1+m)d$ *is even and* ξ *is a primitive* $2m$*th root of* 1.

Detailed structures and left homological integrals (as a right module) of them are recalled as follows:

Definition 4.2. *It is well-known that there are precisely two commutative* $\Bbbk$*-affine domains of GK-dimension one which admit a structure of Hopf algebra, namely* $H_1 = \Bbbk[x]$ *and* $H_2 = \Bbbk[x^{\pm 1}]$*. For* H_1*, x is a primitive element, and for* H_2*, x is a group-like element.*

Computation shows that

$$\int_{H_i}^{l} \cong \Bbbk \quad \text{and} \quad \mathrm{io}(H_i) = \mathrm{im}(H_i) = 1 \quad \text{for} \quad i = 1, 2,$$

which implies that both $\Bbbk[x]$ and $\Bbbk[x^{\pm 1}]$ are unimodular.

Definition 4.3. *Let $\mathbb{D}$ denote the infinite dihedral group*

$$< g, x | g^2 = 1, \ gxg = x^{-1} > .$$

Both g and x are group-like elements in the group algebra $\Bbbk\mathbb{D}$.

Computation shows that

$$\int_{\Bbbk\mathbb{D}}^{l} \cong \Bbbk\mathbb{D}/(x-1, g+1) \quad \text{and} \quad \mathrm{io}(\Bbbk\mathbb{D}) = 2, \ \mathrm{im}(\Bbbk\mathbb{D}) = 1,$$

which implies that $\Bbbk\mathbb{D}$ is not unimodular.

Definition 4.4. *Let n be a positive integer, $0 \leqslant v \leqslant n-1$, and ξ be a primitive nth root of 1. As an algebra, $T_\infty(n, v, \xi)$ is generated by g and x with relations*

$$g^n = 1, \quad xg = \xi g x.$$

Then $T_\infty(n, v, \xi)$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g^v, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$
$$S(g) = g^{n-1}, \quad S(x) = -\xi^{-v} g^{n-v} x.$$

Computation shows that

$$\int_{T_\infty(n,v,\xi)}^{l} \cong T_\infty(n, v, \xi)/(x, g - \xi^{-1}) \quad \text{and} \quad \mathrm{io}(T_\infty(n, v, \xi)) = n, \ \mathrm{im}(T_\infty(n, v, \xi)) = m,$$

where $m = \frac{n}{\gcd(n,v)}$, which implies that $T_\infty(n, v, \xi)$ is not unimodular.

Definition 4.5. *Let n and ω be positive integers, and γ be a primitive nth root of 1. As an algebra, $B(n, \omega, \gamma)$ is generated by $x^{\pm 1}$, g and y with relations*

$$\begin{cases} xx^{-1} = x^{-1}x = 1, \quad xg = gx, \quad xy = yx, \\ yg = \gamma gy, \\ y^n = 1 - x^{\omega} = 1 - g^n. \end{cases}$$

Then $B(n, \omega, \gamma)$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\begin{aligned} &\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g, \quad \Delta(y) = 1 \otimes y + y \otimes g, \\ &\varepsilon(x) = \varepsilon(g) = 1, \quad \varepsilon(y) = 0, \\ &S(x) = x^{-1}, \quad S(g) = g^{-1}, \quad S(y) = -\gamma^{-1}g^{-1}y. \end{aligned}$$

Computation shows that

$$\int_{B(n,\omega,\gamma)}^{l} \cong B(n, \omega, \gamma)/(y, x - 1, g - \gamma^{-1}) \quad \textit{and} \quad \mathrm{io}(B(n, \omega, \gamma)) = \mathrm{im}(B(n, \omega, \gamma)) = n,$$

which implies that $B(n, \omega, \gamma)$ is not unimodular.

Definition 4.6. *Let m, d be positive integers such that $(1+m)d$ is even and ξ a primitive $2m$th root of unity. Define*

$$\omega := md, \quad \gamma := \xi^2.$$

As an algebra, $D(m,d,\xi)$ is generated by $x^{\pm 1}$, g, y and $u_0, u_1, \cdots, u_{m-1}$ with relations

$$
\begin{gathered}
xx^{-1} = x^{-1}x = 1, \quad gx = xg, \quad yx = xy, \\
yg = \gamma g y, \quad y^m = 1 - x^{\omega} = 1 - g^m, \\
u_i x = x^{-1} u_i, \quad y u_i = \phi_i u_{i+1} = \xi x^d u_i y, \quad u_i g = \gamma^i x^{-2d} g u_i,
\end{gathered}
$$

$$
u_i u_j = \begin{cases}
(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\phi_{i+1}\cdots\phi_{m-2-j}y^{i+j}g & (i+j \leqslant m-2) \\
(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}y^{i+j}g & (i+j = m-1) \\
(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\cdots\phi_{m-1}\phi_0\cdots\phi_{m-2-j}y^{i+j-m}g & (i+j \geqslant m)
\end{cases}
$$

where $\phi_i := 1 - \gamma^{-i-1}x^d$ and $i, j \in m$.

Then $D(m,d,\xi)$ becomes a Hopf algebra with comultiplication, counit and the antipode given by

$$
\begin{gathered}
\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g, \quad \Delta(y) = y \otimes g + 1 \otimes y, \\
\Delta(u_i) = \textstyle\sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j}; \\
\varepsilon(x) = \varepsilon(g) = \varepsilon(u_0) = 1, \quad \varepsilon(y) = \varepsilon(u_l) = 0, \\
S(x) = x^{-1}, \quad S(g) = g^{-1}, \quad S(y) = -yg^{-1} = -\gamma^{-1}g^{-1}y, \\
S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id + \frac{3}{2}(1-m)d} g^{m-i-1} u_i,
\end{gathered}
$$

for $i \in m$ and $1 \leqslant l \leqslant m-1$.

Computation shows that

$$
\int_{D(m,d,\xi)}^{l} \cong D(m,d,\xi)/(y, x-1, g-\gamma^{-1}, u_0 - \xi^{-1}, u_1, u_2, \cdots, u_{m-1}) \quad \text{and}
$$

$$
\mathrm{io}(D(m,d,\xi)) = 2m, \ \mathrm{im}(D(m,d,\xi)) = m,
$$

which implies that $D(m,d,\xi)$ is not unimodular.

4.2 Non-degenerate Hopf pairing

For each Hopf algebra H listed in section 4.1 , [4] computes the finite duals H° of them, which are given by generators and relations. Furthermore, they construct a special kind of Hopf pairing $\langle -,-\rangle : H^\bullet \otimes H \to \Bbbk$ by choosing certain Hopf subalgebra $H^\bullet$ of H°. In this paper, we put our main interests on the $H^\bullet$ for each Hopf algebra H.

Definition 4.7. *Let H and $H^\bullet$ be Hopf algebras. A linear map $\langle -,-\rangle : H^\bullet \otimes H \to \Bbbk$ is called a Hopf pairing (on H), if*

$$\text{(i)}\quad \langle ff', h\rangle = \sum \langle f, h_{(1)}\rangle\langle f', h_{(2)}\rangle, \quad \text{(ii)}\quad \langle f, hh'\rangle = \sum \langle f_{(1)}, h\rangle\langle f_{(2)}, h'\rangle,$$
$$\text{(iii)}\ \langle 1, h\rangle = \varepsilon(h), \qquad \text{(iv)}\quad \langle f, 1\rangle = \varepsilon(f),$$
$$\text{(v)}\quad \langle f, S(h)\rangle = \langle S(f), h\rangle$$

hold for all $f, f' \in H^\bullet$ and $h, h' \in H$. Moreover, it is said to be non-degenerate, if for any $f \in H^\bullet$ and any $h \in H$,

$$\langle f, H\rangle = 0 \textit{ implies } f = 0, \textit{ and } \langle H^\bullet, h\rangle = 0 \textit{ implies } h = 0.$$

Clearly, the definition follows that there are linear maps

$$\alpha : H^\bullet \to H^*,\ f \mapsto \langle f, -\rangle \quad \text{and} \quad \beta : H \to H^{\bullet *},\ h \mapsto \langle -, h\rangle.$$

Furthermore, we know by (ii) that for any $f \in H^\bullet$,

$$M^*(\alpha(f)) = \sum \alpha(f_{(1)}) \otimes \alpha(f_{(2)}) \in H^* \otimes H^*,$$

where M denotes the multiplication on H, and hence the image of α is in fact contained in H°. As a conclusion, Definition follows that there are two maps of Hopf algebras

$$\alpha : H^\bullet \to H^\circ,\ f \mapsto \langle f, -\rangle \quad \text{and} \quad \beta : H \to H^{\bullet\circ},\ h \mapsto \langle -, h\rangle,$$

which are both injective if and only if the Hopf pairing $\langle -,-\rangle$ is non-degenerate.

In [4], non-degenerate Hopf pairing on $\Bbbk\mathbb{D}_\infty$, $T_\infty(n, v, \xi)$, $B(n, \omega, \gamma)$ and $D(m, d, \xi)$ are constructed as follows, respectively. (The certain relations of generators as well as Hopf algebra structure are listed in Chapter 5)

Proposition 4.8. *For each affine prime regular Hopf algebra H of GK-dimension one,* *we can construct a Hopf algebra* $H^\bullet$ *and a non-degenerate Hopf pairing* $\langle -,- \rangle$: $H^\bullet \otimes H \to \Bbbk$ *as follows. Specifically, keeping the notions used in [4], we have:*

1) *The evaluation* $\langle -,- \rangle : (\Bbbk\mathbb{D}_\infty)^\bullet \otimes \Bbbk\mathbb{D}_\infty \to \Bbbk$ *is a non-degenerate Hopf pairing, where*

$$\begin{aligned}(\Bbbk\mathbb{D}_\infty)^\bullet &= \Bbbk\{\zeta_1 E_2^s,\ \chi_1 E_2^s \mid s \in \mathbb{N}\}\\ &= \Bbbk\{(\zeta_1 - \chi_1)^k E_2^s \mid k \in 2,\ s \in \mathbb{N}\} \ \subseteq\ (\Bbbk\mathbb{D}_\infty)^\circ.\end{aligned}$$

2) *The evaluation* $\langle -,- \rangle : T_\infty(n,v,\xi)^\bullet \otimes T_\infty(n,v,\xi) \to \Bbbk$ *is a non-degenerate Hopf pairing, where*

$$T_\infty(n,v,\xi)^\bullet \ =\ \Bbbk\{\omega^j E_2^s E_1^l \mid j \in n,\ s \in \mathbb{N},\ l \in m\} \subseteq\ T_\infty(n,v,\xi)^\circ,$$

and $m = \frac{n}{\gcd(n,v)}$.

3) *The evaluation* $\langle -,- \rangle : B(n,\omega,\gamma)^\bullet \otimes B(n,\omega,\gamma) \to \Bbbk$ *is a non-degenerate Hopf pairing, where*

$$B(n,\omega,\gamma)^\bullet \ =\ \Bbbk\{\psi_{1,\gamma}^j E_2^s E_1^l \mid j \in n,\ s \in \mathbb{N},\ l \in n\} \subseteq\ B(n,\omega,\gamma)^\circ,$$

4) *The evaluation* $\langle -,- \rangle : D(m,d,\xi)^\bullet \otimes D(m,d,\xi) \to \Bbbk$ *is a non-degenerate Hopf pairing, where*

$$D(m,d,\xi)^\bullet \ =\ \Bbbk\{\zeta_{1,\gamma}^j E_2^s E_1^l,\ \chi_{1,\gamma}^j E_2^s E_1^l \mid i \in \omega,\ j \in m,\ s \in \mathbb{N},\ l \in m\} \subseteq D(m,d,\xi)^\circ.$$

Chapter 5 Homological integrals of $H^\bullet$

In this chapter, we list the Hopf algebra structure of $H^\bullet$ defined in section 4.2, and compute $\int^l_{H^\bullet}$ (as a right $H^\bullet$-module), $\mathrm{io}(H^\bullet)$, $\mathrm{im}(H^\bullet)$, respectively.

5.1 Preparations

First, we should verify that all $H^\bullet$ are AS-Gorenstein so that they do have homological integrals. There is a key observation in [4]:

Proposition 5.1. *For affine prime regular Hopf algebras H of GK-dimension one, consider Hopf algebras $H^\bullet$ constructed in Proposition 4.8. We have*

1) *All the Hopf algebras $H^\bullet$ have GK-dimension one.*
2) *All the Hopf algebras $H^\bullet$ are noetherian.*
3) *The Hopf algebra $(\Bbbk\mathbb{D}_\infty)^\bullet$ is regular while $T_\infty(n, \upsilon, \xi)^\bullet$, $B(n, \omega, \gamma)^\bullet$ and $D(m, d, \xi)^\bullet$ are not when $n, m \geqslant 2$.*

[9] proved that all affine algebras of GK-dimension one are PI, and [12] proved that all noetherian PI Hopf algebras are AS-Gorenstein. As a result, all $H^\bullet$ do have homological integrals.

The following lemma provides a useful method to calculate homological integrals.

Lemma 5.2. *Let H be an AS-Gorenstein Hopf algebra and let x be a normal non-zero-divisor of H such that (x) is a Hopf ideal of H. Suppose that τ is the algebra automorphism of H such that $xh = \tau(h)x$ for all $h \in H$.*

1. *$H' := H/(x)$ is an AS-Gorenstein Hopf algebra.*

2. $\int_H^l \cong (\int_{H'}^l)^{\tau^{-1}}$ *as right* H*-modules.*
3. *If* x *is central, then* $\int_H^l \cong \int_{H'}^l$.

We end this section by listing the well-known *quantum binomial coefficients* for a parameter $q \in \Bbbk^*$, which is defined as

$$\binom{l}{k}_q := \frac{l!_q}{k!_q(l-k)!_q}$$

for integers $l \geqslant k \geqslant 0$, where $l!_q := 1_q 2_q \cdots l_q$ and $l_q := 1 + q + \cdots + q^{l-1}$.

5.2 $(\Bbbk\mathbb{D}_\infty)^\bullet$

As an algebra, $(\Bbbk\mathbb{D}_\infty)^\bullet$ is generated by ζ_1, χ_1, E_2 with relations

$$\zeta_1\zeta_1 = \zeta_1, \quad \chi_1\chi_1 = \chi_1, \quad \zeta_1\chi_1 = \chi_1\zeta_1 = 0, \quad \zeta_1 + \chi_1 = (\zeta_1 - \chi_1)^2 = 1,$$
$$E_2\zeta_1 = \zeta_1 E_2, \quad E_2\chi_1 = \chi_1 E_2.$$

Then $(\Bbbk\mathbb{D}_\infty)^\bullet$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\Delta(\zeta_1) = \zeta_1 \otimes \zeta_1 + \chi_1 \otimes \chi_1, \quad \Delta(\chi_1) = \zeta_1 \otimes \chi_1 + \chi_1 \otimes \zeta_1,$$
$$\Delta(E_2) = (\zeta_1 - \chi_1) \otimes E_2 + E_2 \otimes 1,$$
$$\varepsilon(\zeta_1) = 1, \quad \varepsilon(\chi_1) = \varepsilon(E_2) = 0,$$
$$S(\zeta_1) = \zeta_1, \quad S(\chi_1) = \chi_1, \quad S(E_2) = -(\zeta_1 - \chi_1)E_2.$$

Note that E_2 is normal, central, and (E_2) is a Hopf ideal of $(\Bbbk\mathbb{D}_\infty)^\bullet$, then by lemma 5.2, we have

$$\int_{(\Bbbk\mathbb{D}_\infty)^\bullet}^l \cong \int_{(\Bbbk\mathbb{D}_\infty)^\bullet/(E_2)}^l \cong \int_{\Bbbk\mathbb{Z}_2}^l,$$

since $\Bbbk\mathbb{Z}_2$ is of finite dimensional, it is unimodular for homological integrals,that is,

$\int_{\Bbbk\mathbb{Z}_2}^{l} \cong \Bbbk$ as $\Bbbk\mathbb{Z}_2$-bimodule. Hence as a right H-module,

$$\int_{(\Bbbk\mathbb{D}_\infty)^\bullet}^{l} \cong (\Bbbk\mathbb{D}_\infty)^\bullet/(E_2, \zeta_1 - \chi_1 - 1),$$

the corresponding homomorphism π yields left and right winding automorphisms

$$\Xi_\pi^l = \Xi_\pi^r = \mathrm{Id}_{(\Bbbk\mathbb{D}_\infty)^\bullet},$$

So that G_π^l and G_π^r have order 1, hence

$$\mathrm{io}((\Bbbk\mathbb{D}_\infty)^\bullet) = \mathrm{im}((\Bbbk\mathbb{D}_\infty)^\bullet) = 1,$$

which implies that $\Bbbk\mathbb{D}_\infty)^\bullet$ is unimodular.

5.3 $T_\infty(n, v, \xi)^\bullet$

As an algebra, $T_\infty(n, v, \xi)^\bullet$ is generated by ω, E_1, E_2 with relations

$$\omega^n = 1, \quad E_1^m = 0,$$
$$E_2\omega = \omega E_2, \quad E_1\omega = \xi^v \omega E_1, \quad E_1E_2 = E_2E_1.$$

where n is a positive integer, $0 \leqslant v \leqslant n-1$, and ξ is a primitive nth root of 1 and $m = \frac{n}{\gcd(n,v)}$.

Then, $T_\infty(n, v, \xi)^\bullet$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\Delta(\omega) = \omega \otimes \omega, \quad \Delta(E_1) = 1 \otimes E_1 + E_1 \otimes \omega,$$
$$\Delta(E_2) = 1 \otimes E_2 + E_2 \otimes \omega^m + \sum_{k=1}^{m-1} E_1^{[k]} \otimes \omega^k E_1^{[m-k]},$$
$$\varepsilon(\omega) = 1, \quad \varepsilon(E_1) = \varepsilon(E_2) = 0,$$
$$S(\omega) = \omega^{n-1}, \quad S(E_1) = -\xi^{-v}\omega^{n-1}E_1, \quad S(E_2) = -E_2.$$

where $E_1^{[k]} := \frac{1}{k!_{\xi^v}} E_1^k$ for $1 \leqslant k \leqslant m-1$.

To compute the left homological integral of $T_\infty(n, v, \xi)^\bullet$, recall that for a Hopf algebra H, homological integral is a one-dimensional H-bimodule, we denote the generator by x, then

$$xh = a_h x,$$

where $a_h \in \Bbbk$ for all $h \in H$.

So when an element $h' \in H$ is nilpotent, it is obvious that $a_{h'} = 0$, in this point of view, we have

$$\int_H^l \cong \int_{H/(h')}^l .$$

Hence, since E_1 is nilpotent in $T_\infty(n, v, \xi)^\bullet$, we have

$$\int_{T_\infty(n,v,\xi)^\bullet}^l \cong \int_{T_\infty(n,v,\xi)^\bullet/(E_1)}^l ,$$

Then in $T_\infty(n, v, \xi)^\bullet/(E_1)$, E_2 is normal, central and (E_2) becomes exactly a Hopf ideal in $T_\infty(n, v, \xi)^\bullet/(E_1)$, then similar to the computations in $(\Bbbk\mathbb{D}_\infty)^\bullet$, we have

$$\int_{T_\infty(n,v,\xi)^\bullet/(E_1)}^l \cong \int_{T_\infty(n,v,\xi)^\bullet/(E_1,E_2)}^l \cong \int_{\Bbbk\mathbb{Z}_n}^l \cong \Bbbk \cong T_\infty(n, v, \xi)^\bullet/(E2, E1, \omega - 1),$$

the corresponding homomorphism π yields left and right winding automorphisms

$$\Xi^l_\pi = \Xi^r_\pi = \mathrm{Id}_{T_\infty(n,\upsilon,\xi)^\bullet},$$

which implies that

$$\mathrm{io}(T_\infty(n,\upsilon,\xi)^\bullet)) = \mathrm{im}((T_\infty(n,\upsilon,\xi)^\bullet) = 1,$$

hence $T_\infty(n,\upsilon,\xi)^\bullet$ is unimodular.

Remark 5.3. *As a special case, the connected algebraic groups of dimension one* $H_1 = \Bbbk[x]$ *is equal to* $T_\infty(1,0,1)$. *Therefore,* $(\Bbbk[x])^\bullet$ *is unimodular.*

5.4 $B(n,\omega,\gamma)^\bullet$

As an algebra, $B(n,\omega,\gamma)^\bullet$ is generated by $\psi_{1,\gamma}$, E_1 and E_2 with relations

$$\begin{aligned}
&\psi_{1,\gamma}\psi_{1,\gamma} = \psi_{1,\gamma^2},\ \ \psi_{1,1} = \psi^n_{1,\gamma} = 1,\ \ E_1^n = 0,\\
&E_2\psi_{1,\gamma} = \psi_{1,\gamma}E_2,\ \ E_1\psi_{1,\gamma} = \gamma\psi_{1,\gamma}E_1,\ \ E_1E_2 = E_2E_1 + \frac{1}{n}E_1.
\end{aligned}$$

where n and ω are positive integers, and γ be a primitive nth root of 1.

Then $B(n,\omega,\gamma)^\bullet$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\begin{aligned}
&\Delta(E_1) = 1\otimes E_1 + E_1\otimes\psi_{1,\gamma},\ \ \Delta(\psi_{1,\gamma}) = \psi_{1,\gamma}\otimes\psi_{1,\gamma},\\
&\Delta(E_2) = 1\otimes E_2 + E_2\otimes 1 - \sum_{k=1}^{n-1} E_1^{[k]}\otimes\psi^k_{1,\gamma}E_1^{[n-k]},\\
&\varepsilon(\psi_{1,\gamma}) = 1,\ \ \varepsilon(E_1) = \varepsilon(E_2) = 0,\\
&S(E_1) = -\gamma^{n-1}\psi^{n-1}_{1,\gamma}E_1,\ \ S(E_2) = -E_2,\ \ S(\psi_{1,\gamma}) = \psi_{1,\gamma^{-1}}
\end{aligned}$$

where $E_1^{[k]} := \frac{1}{k!_\gamma}E_1^k$ for $1 \leqslant k \leqslant n-1$.

Notice that E_1 is nilpotent, then same to the case of $T_\infty(n, v, \xi)^\bullet$, we have

$$\int^l_{B(n,\omega,\gamma)^\bullet} \cong \int^l_{B(n,\omega,\gamma)^\bullet/(E_1)},$$

then the rest computations are exactly the same as the case of $T_\infty(n, v, \xi)^\bullet$. As a result, we have

$$\int^l_{B(n,\omega,\gamma)^\bullet} \cong \Bbbk \cong B(n, \omega, \gamma)^\bullet/(E_1, E_2, \omega - 1),$$

and

$$\mathrm{io}(B(n, \omega, \gamma)^\bullet)) = \mathrm{im}((B(n, \omega, \gamma)^\bullet) = 1,$$

which implies that $B(n, \omega, \gamma)^\bullet$ is unimodular.

Remark 5.4. *As a special case, the connected algebraic groups of dimension one* $H_2 = \Bbbk[x^{\pm 1}]$ *is equal to* $B(1, 0, 1)$*. Therefore,* $(\Bbbk[x^{\pm 1}])^\bullet$ *is unimodular.*

5.5 $D(m, d, \xi)^\bullet$

As an algebra, $D(m, d, \xi)^\bullet$ is generated by $\zeta_{1,\gamma}$, $\chi_{1,\gamma}$, E_1, E_2 with relations

$$\begin{aligned}
&\zeta_{1,\gamma}\zeta_{1,\gamma} = \zeta_{1,\gamma^2}, \quad \chi_{1,\gamma}\chi_{1,\gamma} = \chi_{1,\gamma^2}, \\
&\zeta_{1,\gamma}\chi_{1,\gamma} = \chi_{1,\gamma}\zeta_{1,\gamma} = 0, \quad \zeta_{1,1} + \chi_{1,1} = 1, \quad E_1^m = \frac{1}{(1-\gamma)^m}\chi_{1,1}, \\
&E_2\zeta_{1,\gamma} = \zeta_{1,\gamma}E_2, \quad E_1\zeta_{1,\gamma} = \gamma\zeta_{1,\gamma}E_1, \\
&E_2\chi_{1,\gamma} = \chi_{1,\gamma}E_2, \quad E_1\chi_{1,\gamma} = \gamma\chi_{1,\gamma}E_1, \\
&E_1E_2 = E_2E_1 + \frac{1}{m}\zeta_{1,1}E_1,
\end{aligned}$$

where m, d are positive integers such that $(1 + m)d$ is even and ξ a primitive $2m$th root of unity, $\gamma = \xi^2$.

Then, $D(m, d, \xi)^\bullet$ becomes a Hopf algebra with comultiplication, counit and antipode given by

$$\Delta(E_1) = 1 \otimes E_1 + E_1 \otimes (\zeta_{1,\gamma} + \xi\chi_{1,\gamma}),$$

$$\Delta(E_2) = (\zeta_{1,1} - \chi_{1,1}) \otimes E_2 + E_2 \otimes 1 - \sum_{k=1}^{m-1} (\zeta_{1,1} - \chi_{1,1}) E_1^{[k]} \otimes (\zeta_{1,\gamma} + \xi\chi_{1,\gamma})^{-m+k} E_1^{[m-k]},$$

$$\Delta(\zeta_{1,\gamma}) = \zeta_{1,\gamma} \otimes \zeta_{1,\gamma}$$

$$\Delta(\chi_{1,\gamma}) = \zeta_{1,\gamma} \otimes \chi_{1,\gamma} - \theta_0 \sum_{k=1}^{m-1} \theta_1 \cdots \theta_{k-1} \zeta_{1,\gamma} E_1^{[k]} \otimes \xi^k \chi_{1,\gamma}^{k+1} E_1^{[m-k]} + \chi_{1,\gamma} \otimes \zeta_{1,\gamma^{-1}}$$
$$- \theta_0 \sum_{k=1}^{m-1} \gamma^{-(m-k)} \theta_1 \cdots \theta_{m-k-1} \chi_{1,\gamma} E_1^{[k]} \otimes \zeta_{1,\gamma}^{k+1} E_1^{[m-k]},$$

$$\varepsilon(E_1) = \varepsilon(E_2) = 0, \quad \varepsilon(\zeta_{1,\gamma}) = 1, \quad \varepsilon(\chi_{1,\gamma}) = 0,$$

$$S(E_1) = -\gamma^{-1}(\zeta_{1,\gamma^{-1}} + \xi^{-1}\chi_{1,\gamma^{-1}})E_1,$$

$$S(E_2) = -\zeta_{1,1}E_2 + \chi_{1,1}E_2 + \frac{1-m}{2m}\chi_{1,1}, \quad S(\zeta_{1,\gamma}) = \zeta_{1,\gamma^{-1}}, \quad S(\chi_{1,\gamma}) = \gamma^{-1}\chi_{1,\gamma^{-1}},$$

where $E_1^{[k]} := \frac{1}{k!_\gamma} E_1^k$, and $\theta_0 = \frac{1-\gamma}{1/m}$, $\theta_k = \frac{1-\gamma^{k+1}}{1-\gamma^k}$ $(1 \leqslant k \leqslant m-1)$, we remark that $(\zeta_{1,\gamma} + \xi\chi_{1,\gamma})^m = 1$.

To compute homological integral of $D(m,d,\xi)^{\bullet}$, notice that E_1 is a normal non-zero-divisor and (E_1) is a hopf ideal of $D(m,d,\xi)^{\bullet}$, then by lemma 5.2, we have

$$\int_{D(m,d,\xi)^{\bullet}}^{l} \cong \left(\int_{D(m,d,\xi)^{\bullet}/(E_1)}^{l}\right)^{\tau^{-1}},$$

where

$$\tau : \begin{cases} E_1 \longmapsto E_1, \\ E_2 \longmapsto E_2 + \frac{1}{m}\zeta_{1,1}, \\ \zeta_{1,\gamma} \longmapsto \gamma\zeta_{1,\gamma}, \\ \chi_{1,\gamma} \longmapsto \gamma\chi_{1,\gamma}, \end{cases}$$

is the algebra automorphsim of $D(m,d,\xi)^{\bullet}$ such that $E_1 h = \tau(h) E_1$ for all $h \in D(m,d,\xi)^{\bullet}$.

then similar to the computations in $T_\infty(n, v, \xi)^\bullet$, we have

$$\int^l_{D(m,d,\xi)^\bullet/(E_1)} \cong \Bbbk \cong D(m, d, \xi)^\bullet/(E_1, E_2, \chi_{1,\gamma}, \zeta_{1,\gamma} - 1),$$

thus

$$\begin{aligned}\int^l_{D(m,d,\xi)^\bullet} &\cong (D(m, d, \xi)^\bullet/(E_1, E_2, \chi_{1,\gamma}, \zeta_{1,\gamma} - 1))^{\tau^{-1}} \\ &\cong D(m, d, \xi)^\bullet/(E_1, E_2 + \frac{1}{m}, \chi_{1,\gamma}, \zeta_{1,\gamma} - \gamma^{-1}).\end{aligned}$$

The corresponding homomorphism π yields left and right winding automorphisms

$$\Xi^l_\pi : \begin{cases} E_1 \longmapsto E_1, \\ E_2 \longmapsto E_2 - \frac{1}{m}, \\ \zeta_{1,\gamma} \longmapsto \gamma^{-1}\zeta_{1,\gamma}, \\ \chi_{1,\gamma} \longmapsto \gamma^{-1}\chi_{1,\gamma}, \end{cases} \quad \text{and} \quad \Xi^r_\pi : \begin{cases} E_1 \longmapsto \gamma^{-1}E_1, \\ E_2 \longmapsto E_2 - \frac{\zeta_{1,\gamma}-\chi_{1,\gamma}}{m}, \\ \zeta_{1,\gamma} \longmapsto \gamma^{-1}\zeta_{1,\gamma}, \\ \chi_{1,\gamma} \longmapsto \gamma\chi_{1,\gamma}, \end{cases}$$

which implies that

$$\mathrm{io}(D(m, d, \xi)^\bullet) = \mathrm{im}(D(m, d, \xi)^\bullet) = \infty,$$

hence $D(m, d, \xi)^\bullet$ is not unimodular.

5.6 Conclusions

From the computations above, we find that $T_\infty(n, v, \xi)^\bullet$ and $B(n, \omega, \gamma)^\bullet$ are unimodular while they are non-commutative, which gives a negative answer to the question raised at the end of Chapter 3, this observation may implies that commutativity may not be the crucial key to indicate unimodularity, which is much similar to the case of the Haar measure on locally compact group where Lie groups that are close to being Abelian (i.e., nilpotent) or far from being Abelian (i.e., semisimple) are unimodular.

Furthermore, for H listed in Section 4.1, once we identify $H^\bullet$ as some kind of dual space of H, then we find that whether H is unimodular is equivalent to its dual space is unimodular remains negative in infinite-dimensional cases. And these four pairs of examples demonstrate that for AS-Gorenstien Hopf algebra H, all three scenarios are possible:

1) Both H and its dual space are unimodular.
2) H is unimodular while its dual space is not.
3) Neither H nor its dual space is unimodular.

5.7 Further questions

(a) Is there a similar criterion (with respect to the adjoint action) for unimodularity of infinite-dimensional AS-Gorenstein Hopf algebras?

(b) Are there any other similar or dual properties between the homological integrals of H and $H^{\bullet}$?

参考文献

[1] HAAR A. Der Massbegriff in der Theorie der kontinuierlichen Gruppen[J]. Annals of Mathematics, 1933, 34(1): 147-169.

[2] HEYNEMAN R G, SWEEDLER M E. Affine Hopf algebras, I[J]. Journal of Algebra, 1969, 13(2): 192-241.

[3] LARSON R G, SWEEDLER M E. An associative orthogonal bilinear form for Hopf algebras[J]. American Journal of Mathematics, 1969, 91(1): 75-94.

[4] LI K, LIU G. The finite duals of affine prime regular Hopf algebras of GK-dimension one[J]. ArXiv preprint arXiv:2103.00495, 2021.

[5] LU D M, WU Q S, ZHANG J. Homological integral of Hopf algebras[J]. Transactions of the American Mathematical Society, 2007, 359(10): 4945-4975.

[6] MONTGOMERY S. Hopf algebras and their actions on rings[M]. American Mathematical Soc., 1993.

[7] NEUMANN J V. The uniqueness of Haar's measure[J]. Математический сборник, 1936, 1(5): 721-734.

[8] RADFORD D E. The trace function and Hopf algebras[J]. Journal of Algebra, 1994, 163(3): 583-622.

[9] SMALL L, STAFFORD J, WARFIELD JR R. Affine algebras of Gelfand-Kirillov dimension one are PI[C]//Mathematical Proceedings of the Cambridge Philosophical Society: vol. 97. Cambridge University Press: 407-414.

[10] WEIL A. L'intégration dans les groupes topologiques et ses applications[M]. Hermann, Paris, 1940.

[11] WU J, LIU G, DING N. Classification of affine prime regular Hopf algebras of GK-dimension one[J]. Advances in Mathematics, 2016, 296: 1-54.

[12] WU Q S, ZHANG J. Noetherian PI Hopf algebras are Gorenstein[J]. Transactions of the American Mathematical Society, 2003: 1043-1066.

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