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作者姓名: 吴金勇

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指导教师: 丁南庆 教授、刘公祥 教授


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# Classification of Hopf algebras of GK-dimension one

**Jinyong Wu**

**Supervised by**

**Professor Nanqing Ding**

**Professor Gongxiang Liu**

**Fundamental Mathematics**

Department of Mathematics, Nanjing University

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基础数学 专业 **2012** 级博士生姓名: 吴金勇
指导教师(姓名、职称): 丁南庆 教授、刘公祥 教授

# 摘　　要

作为非交换代数群理论的一种自然形式, 无限维 Hopf 代数的研究在近些年取得了实质性的进展. 在众多方面, 无限维 Hopf 代数和有限维 Hopf 代数一样都表现出了它的优美的性质. 在代数群理论中, 一个经典的结果是一维连通代数群只有两类: $k^+$ 和 $k^\times$. 本论文就是在前人的工作基础上完成非交换情形的分类, 即 GK-维数为 1 的素正则 Hopf 代数的分类. 具体来说, Lu-Wu-Zhang [24] 定义了同调积分, 并发起了GK维数为 1 的 Hopf 代数的分类工作. 在此基础上, Brown-Zhang [12] 部分分类了 GK-维数为 1 的素正则 Hopf 代数. 本论文构造了一类新的 Hopf 代数 $D(m, d, \xi)$, 并完成了 GK-维数为 1 的有限生成素正则 Hopf 代数的分类. 进一步地, 详细研究了该类新的 Hopf 代数的性质.

**关键词**: Hopf代数; GK-维数; 同调积分, 余根.

**THESIS:** Classification of Hopf algebras of GK-dimension one

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**POSTGRADUATE:** Jinyong Wu

**MENTOR:** Professor Nanqing Ding and Professor Gongixang Liu

## Abstract

As a natural form of non-commutative algebraic group theory, infinite dimensional Hopf algebras have been studied intensively and substantial progress has been made in classifying infinite dimensional noetherian Hopf algebras of low GK-dimension in recent years. Infinite dimensional Hopf algebras and finite dimensional Hopf algebras share exquisite properties in many aspects. It is well-known that there are only two connected algebraic groups of dimension one: $k^+$ and $k^\times$. This fact makes us believe that there should be a complete classification of affine prime regular Hopf algebras of GK-dimension one. In this thesis, we finish the classification based on previous studies. Concretely, Lu-Wu-Zhang [24] introduced the notion of homological integral of Hopf algebras and initiated the classification of Hopf algebras of GK-dimension one. Brown and Zhang [12] made further efforts in this direction and classified all affine prime regular Hopf algebras $H$ of GK-dimension one under a hypothesis. We construct a new class of Hopf algebras $D(m, d, \xi)$ and finish the classification of affine prime regular Hopf algebras of GK-dimension one. Further, properties of these new Hopf algebras are studied detailed.

**Keywords:** Hopf algebras; GK-dimension; Homological integral; coradical.

# CONTENTS

# Chapter 0 Introduction

## §0.1 Background

The Gelfand-Kirillov dimension (GK-dimension for short), which measures the growth of an algebra, is a very useful and powerful tool for investigating noncommutative algebras. Since the popularization of quantum groups around 1980's, a great number of noncommutative Hopf algebras have been introduced and GK-dimension naturally becomes a good invariant for them. In the past few years, Hopf algebras of finite GK-dimension have been studied intensively [1, 5, 12, 14, 24, 33–37, 40, 41]. Since the structure of a Hopf algebra is very rigid by its definition, it is quite possible to classify certain Hopf algebras with finite GK-dimensions.

In [24], Lu-Wu-Zhang introduced the notion of homological integral of Hopf algebras, which is a generalization of integral of (finite-dimensional) Hopf algebras and turns out to be a very useful tool. Also in the paper, they initiated the classification of affine noetherian Hopf algebras of GK-dimension one and asked the following question: Besides the group algebras $k\mathbb{Z}$, $k\mathbb{D}$ (infinite dimensional dihedral group algebra) and infinite dimensional prime Taft algebras, are there other noetherian affine prime regular Hopf algebras of GK-dimension 1? Liu [22] gave a negative answer to this question by discovering a new class of examples. These examples are built by letting a skew primitive element $y$ 'of finite index of nilpotency $n$' act on $k[x^{\pm 1}]$ by setting $yx = \xi xy$, and forming a 'non-split' extension. Brown and Zhang [12] generalized the Liu's algebras by extending $k[x^{\pm 1}]$ to the group algebra $k(\mathbb{Z} \times \mathbb{Z}_b)$, where $\mathbb{Z}_b = \mathbb{Z}/b\mathbb{Z}$ and $b$ is an arbitrary divisor of $n$. Brown and Zhang [12] classified all prime regular Hopf algebras $H$ of GK-dimension one under the hypothesis:

$$\mathrm{im}(H) = 1 \quad \text{or} \quad \mathrm{im}(H) = \mathrm{io}(H) \qquad (*)$$

(see Section 1.4 for the definition of $\mathrm{im}(H)$ and $\mathrm{io}(H)$). In fact, Brown and Zhang [12] gave four classes of Hopf algebras of GK-dimension one: the coordinate algebras of connected algebraic groups of dimension one, the infinite dihedral group algebra, infinite dimensional Taft algebras and generalized Liu algebras, and proved that all prime regular Hopf algebras $H$ of GK-dimension one satisfying condition $(*)$ belong to these four classes. Naturally, they raised the following open question (Question 7.1 in [12]):

(**Que**) *Does their result still hold without the hypothesis* $(*)$?

In classifying Hopf algebras of GK-dimension two, the property "all algebras of GK-dimension $\leqslant 1$ over an algebraically closed field of characteristic 0 are domains" [14] plays a pivotal role. As far as we know, all known classification results of Hopf algebras of GK-dimension two are given under the condition of the algebras being domains. In the following, Hopf algebras which are domains will be called Hopf domains. Goodearl-Zhang [14] classified all affine Hopf domains of GK-dimension two satisfying the condition $\mathrm{Ext}^1_H({}_Hk, {}_Hk) \neq 0$. For those with vanishing Ext-groups, a new class of interesting examples, denoted by $B(n, \{p_i\}_1^s, q, \{\alpha_i\}_1^s)$, were constructed by Wang-Zhang-Zhuang [34]. They conjectured that these $B(n, \{p_i\}_1^s, q, \{\alpha_i\}_1^s)$ together with Hopf algebras given in [14] exhausted all Hopf domains of GK-dimension two which are finitely generated by grouplike and skew primitive elements. This conjecture is equivalent to the following one. By the work of I. Heckenberger [15], Wang-Zhang-Zhuang [34] defined four supplementary Hopf domains of GK-dimension two (if they exist), denoted by $N_5, N_7, N_{10}$ and $N_{21}$ (see [34, Definition 4.4]). Since no example of supplementary Hopf algebras of GK-dimension two had been constructed, they conjectured that:

(**Conj**) *The algebras* $N_5, N_7, N_{10}$ *and* $N_{21}$ *do not exist.*

It is worth to note that the concept *major skew primitive element*, first introduced in [33], plays a vital role in analyzing skew primitive elements of the case when $\mathrm{GKdim}\, C_0 = 1$ ($C_0$ denotes the coradical of $H$). These four supplementary Hopf domains can induce finite dimensional Nichols algebras of diagonal type. Moreover, these Nichols algebras are of standard type of type $B_2$ and type $G_2$ by I.E. Angiono's work in [7]. It is hoped that we can affirm that whether the conjecture is true or not in the near future by lifting the generators and relations of these Nichols algebras to $N_5, N_7, N_{10}$ and $N_{21}$.

In order to study Hopf algebras $H$ of GK-dimensions three and four, a more restrictive but natural condition was added: $H$ is connected, that is, the coradical of $H$ is 1-dimensional. Here the condition "connected" implies that $H$ is a domain by [34, Lemma 1.8(a)](due to Le Bruyn, unpublished). In [41], Zhuang showed that if a Hopf algebra $H$ is connected, then the associated graded Hopf algebra gr$H$ with respect to the coradical filtration is a commutative Hopf algebra. If $H$ has finite GK-dimension, then gr$H$ is isomorphic to the polynomial ring $k[x_1, \cdots, x_n]$, namely, the regular functions $\mathcal{O}(G)$ on a unipotent group $G$, or equivalently, the graded dual $U(\mathfrak{L})^*$ of the universal enveloping algebra over a graded Lie algebra $\mathfrak{L}$. All connected Hopf algebras with GK-dimension three were classified by Zhuang [41] by analyzing a nice gr$H \cong R\#kG$, where $R$ is a certain graded subalgebra of gr$H$ (see [27, Theorem 3]). In addition, after studying coassociative Lie algebra [36], Wang-Zhang-Zhuang [35] classified all connected Hopf algebras with GK-dimension four.

Recently, Wang-Zhang-Zhuang [37] classified all non-locally PI, pointed Hopf domains of GK-dimension two, which extends some results of Brown, Goodearl and others in an ongoing project to understand all Hopf algebras of low GK-dimension. They also classified all pointed Hopf algebras of rank one (the rank is defined to be $\dim_k(k \otimes_{C_0} C_1) - 1$), which generalizes results of Krop-Radford [19] and Wang-You-Chen [32] which classified Hopf algebras of rank one under extra hypothesis. Brown-Gilmartin-Zhang [9] studied two classes of infinite dimensional Hopf algebras. The first class consists of those Hopf algebras that are connected graded as algebras, and the second class are those Hopf algebras that are connected as coalgebras. Both these classes of Hopf algebras share many features in common with enveloping algebras of finite dimensional Lie algebras. Nevertheless, they construct an example of a Hopf $k$-algebra $H$ of GK-dimension five, which is a connected graded algebra and a connected coalgebra, but not isomorphic as an algebra to $U(\mathfrak{g})$ for any Lie algebra $\mathfrak{g}$.

In another direction, the pointed Hopf algebra domains of finite GK-dimension with generic infinitesimal braiding have been classified by Andruskiewitsch and Schneider [5] and Andruskiewitsch and Angiono [1].

## §0.2 Ideas and main results

Philosophically, prime regular Hopf algebras of finite GK-dimension can be regarded as "noncommutative" counterparts of connected algebraic groups. It is well-known that there are only two connected algebraic groups of dimension one. This fact makes us believe that there should be a complete classification of prime regular Hopf algebras of GK-dimension one. In this thesis, we will go on with Brown-Zhang's works [12] and finish the classification of all prime regular Hopf algebras of GK-dimension one.

As the first step, we construct a new class of prime regular Hopf algebras $D(m, d, \xi)$ of GK-dimension one, which implies that the condition $(*)$ is really necessary for Brown-Zhang's classification.

Secondly, we will prove our main result (see Theorem 4.4.3) which states that our new examples together with the four classes of Hopf algebras given in [12] form a complete list, up to isomorphisms of Hopf algebras, of all prime regular Hopf algebras of GK-dimension one.

The key idea to prove the main result is not complicated: let $H$ be a prime regular Hopf algebra of GK-dimension one which doesn't satisfy the condition $(*)$. From this Hopf algebra $H$, we can construct a Hopf subalgebra $\widetilde{H}$ which will be shown to meet the condition $(*)$. Thus the classification result given in [12] can be applied. At last, we show that $\widetilde{H}$ determines the structure of $H$ entirely.

The precess to realize our idea, which motivates our discovery of new examples, turns out to be much more complicated than our expectation: According to Brown-Zhang's classification result, there is a dichotomy on $\widetilde{H}$: $\widetilde{H}$ is either primitive or group-like (see Definition 4.2.1). When $\widetilde{H}$ is primitive, we find that $H$ must be an infinite dimensional Taft algebra. The difficult part is group-like case. In this case, we gradually realize that there is an essential difference between the situation $\frac{\mathrm{io}(H)}{\mathrm{im}\,(H)} > 2$ and the situation $\frac{\mathrm{io}(H)}{\mathrm{im}\,(H)} = 2$. The last situation becomes very delicate: More generators and subtle relations are allowed to appear. Ultimately, this leads us to find the final missing piece in the puzzle of prime regular Hopf algebras of GK-dimension one.

In practice, the assumption "pointed" is always added when we want to classify Hopf algebras of lower GK-dimensions. As a matter of fact, all known examples are pointed and it is widely believed that, at least for prime regular Hopf algerbas of GK-dimension one, these Hopf algebras should be pointed automatically. Our new examples will change this naive understanding since all the new examples are not pointed!

At the first glance, the definition of $D(m, d, \xi)$ seems complicated and quite different from our familiar examples. In order to understand this new Hopf algebra $D(m, d, \xi)$ better, we determine the coradical filtration of $D(m, d, \xi)$ in the last chapter. We answer a question posed by some experts like Professors Xiaowu Chen, Istvan Heckenberger and others. The question they raised is: What is the coradical of $D(m, d, \xi)$ and is the coradical still a Hopf subalgebra? We find that there is a simple subcoalgebra $C$ of $D(m, d, \xi)$ having dimension $m^2 \neq 1$ ($m > 1$ by definition). This shows that $D(m, d, \xi)$ is not pointed directly. Then we show that this simple subcoalgebra $C$ together with other group-likes generate the coradical (see Theorem 5.1.4 for details). From this, we find that the coradical of $D(m, d, \xi)$ is not a Hopf subalgebra and the coradical filtration is finite. Thus, $D(m, d, \xi)$ is a co-Frobenius Hopf algebra with coradical not a Hopf subalgebra. Since $D(m, d, \xi)$ has nonzero integrals, we investigate whether there is a relationship between homological integrals and the nonzero integrals at the end of this chapter.

## §0.3 Organization

Here are the brief summaries of all chapters.

**Chapter 1: Preliminaries**

In this chapter, we recall some definitions and properties related to GK-dimension, Hopf algebras, PI, Artin-Schelter condition, homological integral, known examples of affine prime regular Hopf algebras of GK-dimension one.

**Chapter 2: Some combinatorial equations**

Some combinatorial equations are collected. These equations turn out to be important for the following analysis.

**Chapter 3: New examples**

In this chapter, we construct a new class of prime regular Hopf algebras $D(m, d, \xi)$ of GK-dimension one and study some properties of them.

**Chapter 4: Classification result**

In this chapter, we define a Hopf subalgebra $\widetilde{H}$ of $H$ and show that $\widetilde{H}$ is a prime regular Hopf algebra satisfying the condition $(*)$. Then we reconstruct $H$ from $\widetilde{H}$. $H$ is shown to be an infinite dimensional Taft algebra when $\widetilde{H}$ is of primitive case. And as the desired conclusion, we show that $H$ is isomorphic to some $D(m, d, \xi)$ when $\widetilde{H}$ is of group-like case. Finally, the classification result, and its proof are also formulated in this chapter.

**Chapter 5: Further research about** $D(m, d, \xi)$

One of the main purpose of this chapter is to determine the coradical filtration of $D(m, d, \xi)$. We answer a question posed by some experts like Professors Xiaowu Chen, Istvan Heckenberger and others. We find that there is a simple subcoalgebra $C$ of $D(m, d, \xi)$ having dimension $m^2 \neq 1$ ($m > 1$ by definition). This immediately shows that $D(m, d, \xi)$ is not pointed. $B(m, \omega, \gamma)$ is a normal Hopf subalgebra of $D(m, d, \xi)$, where $\omega = md$ and $\gamma = \xi^2$, is also showed in this chapter.

# Chapter 1 Preliminaries

Throughout this thesis, $k$ denotes an algebraically closed field of characteristic 0, all vector spaces are over $k$. An algebra $A$ is called *affine* is it is finitely generated as a $k$-algebra. A ring $R$ is called *regular* if it has finite global dimension, and it is *prime* if 0 is a prime ideal.

In this chapter we recall the urgent needs around affine noetherian Hopf algebras for completeness and the convenience of the reader. About general background knowledge, the reader is referred to [18] for Gelfand-Kirillov dimension, [26, 30] for Hopf algebras, [25] for noetherian rings and [11, 12, 24] for exposition about noetherian Hopf algebras. We use the symbols $\Delta, \epsilon$ and $S$ respectively, for the coproduct, counit and antipode of a Hopf algebra and the Sweedler's notation for coproduct $\Delta(h) = \sum h_1 \otimes h_2$ $(h \in H)$ will be used freely. Usually we are working on left modules. Let $A^{op}$ denote the opposite algebra of $A$.

## §1.1 Gelfand-Kirillov dimension

**Definition 1.1.1** *The* Gelfand-Kirillov dimension *(GK-dimension for short) of a $k$-algebra $A$ is*

$$\mathrm{GKdim}(A) = \lim \sup_{V} d_V(n),$$

*where the supremum is taken over all finite-dimensional subspaces $V$ of $A$ and*

$$d_V(n) = \dim_k(\sum_{i=0}^{n} V^i).$$

The GK-dimension can be viewed as a non-commutative analogue of the Krull dimension. In fact, for a finitely generated commutative algebra $A$, the GK-dimension of $A$ equals its Krull dimension. For more properties of the GK-dimension, a comprehensive reference is [18].

**Example 1.1.2** *Let $A$ be a $k$-algebra, and let $B = A[x_1, \cdots, x_n]$. Then*

$$\mathrm{GKdim}(B) = \mathrm{GKdim}(A) + n.$$

**Example 1.1.3** *The* Weyl algebra $A_n = A_n(k)$ *is the ring of polynomials in* $2n$ *variables* $x_1, \cdots, x_n, y_1, \cdots, y_n$ *with coefficients in* $k$*, subject to the relations*

$$x_i x_j = x_j x_i,\ \ y_i y_j = y_j y_i, x_i y_j - y_j x_i = \delta_{ij},$$

*where* $\delta_{ij}$ *is the Kronecker symbol. Observe that*

$$A_{n+1} \cong A_n[y_{n+1}][x_{n+1}; \delta] \quad \textit{with} \quad \delta = \frac{\partial}{\partial y_{n+1}}.$$

*Then*

$$\mathrm{GKdim}(A_{n+1}) = \mathrm{GKdim}(A_n[y_{n+1}]) + 1 = \mathrm{GKdim}(A_n) + 2$$

*and hence*

$$\mathrm{GKdim}(A_n) = 2n.$$

## §1.2 Hopf algebras

**Definition 1.2.1** *A* $k$*-algebra (with unit) is a* $k$*-vector space* $A$ *together with two* $k$*-linear maps,* multiplication $m : A \otimes A \to A$ *and* unit $u : k \to A$*, such that the following diagrams are commutative:*

(a) *associativity*

$$\begin{CD} A \otimes A \otimes A @>{\mathrm{Id} \otimes m}>> A \otimes A \\ @V{m \otimes \mathrm{Id}}VV @VV{m}V \\ A \otimes A @>{m}>> A \end{CD} .$$

(b) *unity*

$$\begin{CD} k \otimes A @>{u \otimes \mathrm{Id}}>> A \otimes A @<{\mathrm{Id} \otimes u}<< A \otimes k \\ @V{=}VV @VV{m}V @VV{=}V \\ k \otimes A @>>> A @<<< A \otimes k \end{CD} .$$

**Definition 1.2.2** *A* $k$*-coalgebra (with counit) is a* $k$*-vector space* $C$ *together with two* $k$*-linear maps,* comultiplication $m : A \to A \otimes A$ *and* counit $\epsilon : A \to k$*, such that the following diagrams are commutative:*

(a) *coassociativity*

$$\begin{CD} C @>{\Delta}>> C \otimes C \\ @V{\Delta}VV @VV{\Delta \otimes \mathrm{Id}}V \\ C \otimes C @>{\mathrm{Id} \otimes \Delta}>> C \otimes C \otimes C \end{CD} .$$

(b) *counity*

$$\begin{array}{ccccc} k\otimes C & \xleftarrow{1\otimes \mathrm{Id}} & C & \xrightarrow{\mathrm{Id}\otimes 1} & C\otimes k\ . \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle =} \\ k\otimes C & \xleftarrow{\epsilon\otimes \mathrm{Id}} & C\otimes C & \xrightarrow{\mathrm{Id}\otimes \epsilon} & C\otimes k \end{array}$$

**Definition 1.2.3** *A $k$-vector space $B$ is a* bialgebra *if $(B, m, u)$ is an algebra, $(B, \Delta, \epsilon)$ is a coalgebra, and either of the following (equivalent) conditions holds:*

(a) *$\Delta$ and $\epsilon$ are algebra morphisms,*

(b) *$m$ and $u$ are coalgebra morphisms.*

**Definition 1.2.4** *Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra. Then $H$ is a* Hopf algebra *if there exists a linear endomorphism $S$ from $H$ to $H$ such that the diagram*

$$\begin{array}{ccccc} H\otimes H & \xleftarrow{\Delta} & H & \xrightarrow{\Delta} & H\otimes H \\ {\scriptstyle \mathrm{Id}\otimes S}\downarrow & & \downarrow{\scriptstyle u\circ\epsilon} & & \downarrow{\scriptstyle S\otimes \mathrm{Id}} \\ H\otimes H & \xrightarrow{m} & H & \xleftarrow{m} & H\otimes H \end{array}$$

*commutes. The endomorphism $S$ is called the* antipode *of $H$. In Sweedler notation, this says that for all $h \in H$,*

$$\epsilon(h) = \sum h_1 S(h_2) = \sum S(h_1)h_2.$$

Let $H$ be a Hopf algebra. An element $g \in H$ is called *grouplike* if $\Delta(g) = g \otimes g$. The set of grouplike elements in $H$ is denoted by $G(H)$. An element $y \in H$ is called $(g, h)$-*primitive* if $g, h \in G(H)$ and $\Delta(y) = y \otimes g + h \otimes y$. We use $P_{g,h}(H)$ to denote the set of $(g, h)$-primitive elements in $H$. We simply write $P(H)$ for $P_{1,1}(H)$, the set of primitive elements in $H$.

The *coradical* $C_0$ of a coalgebra $C$ is defined to be the sum of all simple subcoalgebras of $C$. The coalgebra $C$ is called *cosemisimple* if $C = C_0$, *pointed* if $C_0 = kG(C)$, and *connected* if $C_0$ is one-dimensional.

**Definition 1.2.5** *Let $C$ be a coalgebra and $\{F_n\}_{n\geqslant 0}$ a family of subcoalgebras of $C$. Then $\{F_n\}_{n\geqslant 0}$ is called a* coalgebra filtration *if*

(a) *$F_n \subset F_{n+1}$ and $C = \bigcup_{n=0}^{\infty} F_n$,*

(b) *$\Delta(F_n) \subset \sum_{i=0}^{n} F_i \otimes F_{n-i}$.*

In fact the coradical $C_0$ is the bottom piece of a filtration of $C$. We can build up a coalgebra filtration $\{C_n\}$ of $C$ by defining $C_n$ inductively as follows: for each $n \geqslant 1$, define

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C).$$

This filtration is called the *coradical filtration* of $C$ [26].

## §1.3 Stuffs from ring theory

A polynomial identity ring, or PI ring for short, is defined, loosely, as a ring all of whose elements satisfy some polynomial identity. For example, if $R$ is a commutative ring and let $f(x_1, x_2) = x_1x_2 - x_2x_1$, then $R$ satisfies $f$ (i.e. $f(r_1, r_2) = 0$ for all $r_1, r_2 \in R$). This shows that commutative rings are PI rings. Moreover, any ring finite as a module over its center, such as a matrix ring over a commutative ring, is a PI ring.

• *PI-degree.* Let $R$ be a PI ring. The PI-degree of $R$ is defined to be

$$\text{PI-deg}(R) = \min\{n | R \hookrightarrow M_n(C) \text{ for some commutative ring } C\}$$

(see [25, Chapter 13]). It is not, however, true that every PI ring can be embedded in a matrix ring over a commutative ring. To see this, let $R$ be the *exterior algebra* on a countable dimensional vector space over $k$ with basis $e_1, e_2, \cdots$ . Thus $R$ has generators $e_1, e_2, \cdots$ and relations $e_ie_j = e_je_i$ for all $i, j$ (which imply, of course, $e_i^2 = 0$ for all $i$). $R$ satisfies the identity $[[x, y], z]$ but cannot be embedded in a matrix ring over a commutative ring (see [25, Proposition 13.4.5] for detail).

Let $Z$ be an Ore domain, then the *rank* of a $Z$-module $M$ is defined to be the $Q(Z)$-dimension of $Q(Z) \otimes_Z M$, where $Q(Z)$ is the quotient division ring of $Z$. If $R$ is a semiprime PI ring with center $Z$, then the PI-degree of $R$ equals the square root of the rank of $R$ over $Z$ by [25, Theorem 13.4.2].

Recall that an algebra $A$ is said to be *augmented* if there is an algebra morphism $\epsilon: A \to k$.

• *Artin-Schelter condition.* Let $(A, \epsilon)$ be an augmented noetherian algebra. We say $A$ has finite injective dimension if the injective dimensions of ${}_AA$ and $A_A$, $\text{inj.dim}_AA$ and $\text{inj.dim}A_A$, are both finite. In this case these integers are equal by [39], and we write $d$ for the common value. We say $A$ is regular if it has finite global dimension, gl.dim$A < \infty$. The right global dimension always equals the left global dimension [31, Exercise 4.1.1]; and, when finite, the global dimension equals the injective dimension. Write $k$ for the trivial $A$-module $A/\operatorname{Ker}\epsilon$. Then $A$ is *Artin-Schelter Gorenstein*, we usually abbreviate to *AS-Gorenstein*, if

(AS1) injdim$_A A = d < \infty$,

(AS2) $\dim_k \operatorname{Ext}_A^d({}_Ak,\ {}_AA) = 1$ and $\dim_k \operatorname{Ext}_A^i({}_Ak,\ {}_AA) = 0$ for all $i \neq d$,

(AS3) the right $A$-module versions of (AS1, AS2) hold.

If, further, gl.dim$A = d$, then $A$ is called Artin-Schelter regular, usually shortened to AS-regular.

Many Hopf algebras associated to classical and quantum groups are AS-Gorenstein and AS-regular [8, 10]. The following result is the combination of [38, Theorem 0.1] and [38, Theorem 0.2 (1)], which shows that a large number of Hopf algebras are AS-Gorenstein.

**Lemma 1.3.1** *Each affine noetherian PI Hopf algebra is AS-Gorenstein.*

## §1.4 Homological integrals

The concept *homological integral*, first introduced in [24, Definition 1.1], can be defined for an AS-Gorenstein augmented algebra.

**Definition 1.4.1** *[12, Definition 1.3] Let $(A, \epsilon)$ be a noetherian augmented algebra and suppose that $A$ is AS-Gorenstein of injective dimension $d$. Any non-zero element of the one-dimensional $A$-bimodule* $\operatorname{Ext}_A^d({}_Ak,\ {}_AA)$ *is called a* left homological integral *of $A$. We write $\int_A^l = \operatorname{Ext}_A^d({}_Ak,\ {}_AA)$. Any non-zero element in $\operatorname{Ext}_{A^{op}}^d(k_A, A_A)$ is called a* right homological integral *of $A$. We write $\int_A^r = \operatorname{Ext}_{A^{op}}^d(k_A, A_A)$. By abusing the language we also call $\int_A^l$ and $\int_A^r$ the left and the right homological integrals of $A$ respectively.*

Homological integrals exist only for AS-Gorenstein Hopf algebras. Hence free Hopf algebras (of at least two variables) and universal enveloping algebras of infinite dimensional Lie algebras do not have homological integrals. By Lemma 1.3.1, each affine noetherian PI Hopf algebra has homological integrals.

When a Hopf algebra $H$ is finite dimensional, then homological integrals agree with the classical integrals [26, Definition 2.1.1] in the following way: the (classical) left integral is an $H$-subbimodule of $H$; and it is identified with the left homological integral $\mathrm{Hom}_H(k, H)$ via the natural homomorphism

$$\mathrm{Hom}_H(\epsilon, H) : \mathrm{Hom}_H(k, H) \to \mathrm{Hom}_H(H, H) \cong H.$$

The same holds for the right integral.

Note that both $\int_H^l$ and $\int_H^r$ are 1-dimensional $H$-bimodules. As a left $H$-module, $\int_H^l \cong k$, but as a right $H$-module, $\int_H^l$ may not be isomorphic to $k$. A similar comment applies to $\int_H^r$. We Say $H$ is *unimodular* if $\int_H^l$ is isomorphic to $k$ as $H$-bimodules. In this case, $\int_H^l \cong \int_H^r$ as $H$-bimodules. The unimodular property means that

$$hx = xh = \epsilon(h)x$$

for all $h \in H$ and $x \in \int_H^l$. When $H$ is finite dimensional, this definition agrees with the calssical definition in [26, Page.17].

The following lemma provides a useful method to calculate homological integrals.

**Lemma 1.4.2** *[24, Lemma 2.6] Let $H$ be an AS-Gorenstein Hopf algebra and let $x$ be a normal nonzero- divisor of $H$ such that $(x)$ is a Hopf ideal of $H$. Suppose that $\tau$ is the algebra automorphism of $H$ such that $xh = \tau(h)x$ for all $h \in H$.*

(a) *$H' := H/(x)$ is an AS-Gorenstein Hopf algebra.*

(b) *$\int_H^l \cong (\int_{H'}^l)^{\tau^{-1}}$ as right $H$-modules.*

(c) *If $x$ is central, then $\int_H^l \cong \int_{H'}^l$.*

• *Winding automorphisms.* Let $H$ be an affine noetherian PI Hopf algebra. By Lemma 1.3.1, it is AS-Gorenstein and thus has left homological integrals $\int_H^l$. Let $\pi : H \to H/\mathrm{r.ann}(\int_H^l)$ be the canonical algebra homomorphism, where $\mathrm{r.ann}(\int_H^l)$ denotes the set of right annihilators of $\int_H^l$ in $H$. We write $\Xi_\pi^l$ for the *left winding automorphism* of $H$ associated to $\pi$, namely

$$\Xi_\pi^l(a) := \sum \pi(a_1)a_2 \quad \text{for } a \in H.$$

Similarly we use $\Xi_\pi^r$ for the right winding automorphism of $H$ associated to $\pi$, that is,

$$\Xi_\pi^r(a) := \sum a_1\pi(a_2) \quad \text{for } a \in H.$$

Let $G_\pi^l$ and $G_\pi^r$ be the subgroups of $\mathrm{Aut}_{k\text{-alg}}(H)$ generated by $\Xi_\pi^l$ and $\Xi_\pi^r$, respectively.

• *Integral order and integral minor.* With the same notions as above, the *integral order* $\mathrm{io}(H)$ of $H$ is defined by the order of the group $G_\pi^l$ :

$$\mathrm{io}(H) := |G_\pi^l|. \tag{1.1}$$

As noted in [12, Lemma 2.1], we always have $|G_\pi^l| = |G_\pi^r|$. So the above definition is independent of the choice of $G_\pi^l$ or $G_\pi^r$. In addition, if $H$ is prime regular of GK-dimension one, then [24, Theorem 7.1] implies

$$\text{PI-deg}(H) = \mathrm{io}(H).$$

The *integral minor* of $H$, denoted by $\mathrm{im}(H)$, is defined by

$$\mathrm{im}(H) := |G_\pi^l / G_\pi^l \cap G_\pi^r|. \tag{1.2}$$

**Remark 1.4.3** *Crudely speaking,* $\mathrm{io}(H)$ *is a measure of the commutativity of* $H$ *and* $\mathrm{im}(H)$ *is a measure of the cocommutativity of* $H$*. In fact, for a prime regular Hopf algebra* $H$ *of GK-dimension one, we have* $\mathrm{io}(H) = 1$ *if and only if* $H$ *is commutative* (see *[24, Corollary 7.8]*) *and* $\mathrm{im}(H) = 1$ *if and only if* $H$ *is cocommutative* (see *[12, Section 4]*).

• *Strongly graded and bigraded properties.* Let $H$ be a prime regular Hopf algebra of GK-dimension one. By [12, Theorem 2.5], $|G_\pi^l|$ is finite, say $n$. Therefore, the character group $\widehat{G_\pi^l} := \mathrm{Hom}_{k\text{-alg}}(kG_\pi^l, k)$ of $G_\pi^l$ is isomorphic to $G_\pi^l$. Similarly, the character group $\widehat{G_\pi^r}$ of $G_\pi^r$ is isomorphic to $G_\pi^r$.

Fix a primitive $n$th root $\zeta$ of 1 in $k$, and define $\chi \in \widehat{G_\pi^l}$ and $\eta \in \widehat{G_\pi^r}$ by setting

$$\chi(\Xi_\pi^l) = \zeta \quad \text{and} \quad \eta(\Xi_\pi^r) = \zeta.$$

Thus $\widehat{G_\pi^l} = \{\chi^i | 0 \leqslant i \leqslant n-1\}$ and $\widehat{G_\pi^r} = \{\eta^j | 0 \leqslant j \leqslant n-1\}$. For each $0 \leqslant i, j \leqslant n-1$, let

$$H_i^l := \{a \in H | \Xi_\pi^l(a) = \chi^i(\Xi_\pi^l)a\}$$

and

$$H_j^r := \{a \in H | \Xi_\pi^r(a) = \eta^j(\Xi_\pi^r)a\}.$$

The following lemma comes from [12, Theorem 2.5 (b)]. A graded $A = \bigoplus_{0\leqslant i\leqslant n-1} A_i$ is strongly graded means that $A_iA_j = A_{i+j}$ for all $0 \leqslant i, j \leqslant n-1$, where $i+j$ is interpreted mod $n$.

**Lemma 1.4.4** (1) *$H = \bigoplus_{\chi^i\in \widehat{G_\pi^l}} H_i^l$ is strongly $\widehat{G_\pi^l}$-graded.*

(2) *$H = \bigoplus_{\eta^j\in \widehat{G_\pi^r}} H_j^r$ is strongly $\widehat{G_\pi^r}$-graded.*

It is clear that $\Xi_\pi^l\Xi_\pi^r = \Xi_\pi^r\Xi_\pi^l$, so $H_i^l$ is stable under the action of $G_\pi^r$. Consequently, the $\widehat{G_\pi^l}$- and $\widehat{G_\pi^r}$-gradings on $H$ are *compatible* in the sense that

$$H_i^l = \bigoplus_{0\leqslant j\leqslant n-1} (H_i^l \cap H_j^r) \quad \text{and} \quad H_j^r = \bigoplus_{0\leqslant i\leqslant n-1} (H_i^l \cap H_j^r)$$

for all $i, j$. Then $H$ is a bigraded algebra:

$$H = \bigoplus_{0\leqslant i,j\leqslant n-1} H_{ij}, \tag{1.3}$$

where $H_{ij} = H_i^l \cap H_j^r$. And we write $H_0 = H_{00}$ for convenience.

For later use, we collect some properties about $H$ which are [12, Proposition 2.1 (c)(e)] and [12, Theorem 2.5 (f)].

**Lemma 1.4.5** *Let $H$ be an affine prime regular Hopf algebra of GK-dimensional one. Then*

(a) *$\Delta(H_i^l) \subseteq H_i^l \otimes H$ and $\Delta(H_j^r) \subseteq H \otimes H_j^r$; thus $H_i^l$ is a right coideal of $H$ and $H_j^r$ is a left coideal of $H$;*

(b) *$\Xi_\pi^r S = S(\Xi_\pi^l)^{-1}$, where $(\Xi_\pi^l)^{-1} = \Xi_{\pi\circ S}^l$.*

(c) *$H_0^l, H_0^r$ and $H_0$ are affine commutative Dedekind domains with $H_0^l \cong H_0^r$.*

Note that only (c) needs all the hypotheses of $H$, (a) and (b) hold if $H$ has finite integral order.

**Remark 1.4.6** (1) *By [28, 29], prime affine algebras of GK-dimension one are noetherian and PI automatically. So* "noetherian" *and* "PI" *do not appear in the title of the classification.*

(2) *If $H$ is an affine prime regular Hopf algebra of GK-dimensional one, then* gl.dim$H = 1$. *Indeed, assume that* gl.dim$H = d$. *Wu and Zhang [38] proved that every noetherian affine PI Hopf algebra is Cohen-Macaulay and this forces $d = 1$.*

## §1.5 Known examples

The following examples come from [12].

• *Connected algebraic groups of dimension one.* It is well-known that there are precisely two connected algebraic groups of dimension one (see, say [16, Theorem 20.5]) over an algebraically closed field $k$. Therefore, there are precisely two commutative $k$-affine domains of GK-dimension one which admit a structure of Hopf algebra, namely $H_1 = k[x]$ and $H_2 = k[x^{\pm 1}]$. For $H_1$, $x$ is a primitive element, and for $H_2$, $x$ is a group-like element. Commutativity and cocommutativity imply that $\text{io}(H_i) = \text{im}(H_i) = 1$ for $i = 1, 2$.

• *Infinite dihedral group algebra.* Let $\mathbb{D}$ denote the infinite dihedral group $\langle g, x | g^2 = 1, gxg = x^{-1} \rangle$. Both $g$ and $x$ are group-like elements in the group algebra $k\mathbb{D}$. By cocommutativity, $\text{im}(k\mathbb{D}) = 1$. To compute the homological integral, we note that $x - 1$ is a normal element of $H$ since

$$(x-1)g = (-gx^{-1})(x-1).$$

Using this fact and Lemma 1.4.2 one sees that as a right $H$-module,

$$\int_{k\mathbb{D}}^{l} \cong k\mathbb{D}/\langle x-1, g+1 \rangle.$$

This implies $\text{io}(k\mathbb{D}) = 2$.

• *Infinite dimensional Taft algebras.* Let $n$ and $t$ be integers with $n > 1$ and $0 \leqslant t \leqslant n-1$. Fix a primitive $n$th root $\xi$ of 1. Let $H(n, t, \xi)$ be the algebra generated by $x$ and $g$ subject to the relations

$$g^n = 1 \quad \text{and} \quad xg = \xi g x.$$

Then $H(n, t, \xi)$ is a Hopf algebra with coalgebra structure given by

$$\Delta(g) = g \otimes g,\ \epsilon(g) = 1 \quad \text{and} \quad \Delta(x) = x \otimes g^t + 1 \otimes x,\ \epsilon(x) = 0,$$

and with

$$S(g) = g^{-1} \quad \text{and} \quad S(x) = -xg^{-t}.$$

Note that $x$ is a normal nonzero-divisor of $H(n, t, \xi)$. Then $H' := H/(x)$ is commutative, which implies that $\int_{H'}^{l} \cong \int_{H'}^{r} \cong k$ as $H'$-bimidules. Applying Lemma 1.4.2, we have

$$\int_{H}^{l} \cong H/\langle x, g - \xi^{-1} \rangle,$$

and the corresponding homomorphism $\pi$ yields left and right winding automorphisms

$$\Xi_{\pi}^{l} : \begin{cases} x \longmapsto x, \\ g \longmapsto \xi^{-1} g, \end{cases} \quad \text{and} \quad \Xi_{\pi}^{r} : \begin{cases} x \longmapsto \xi^{-t} x, \\ g \longmapsto \xi^{-1} g. \end{cases}$$

So that $G_{\pi}^{l} = \langle \Xi_{\pi}^{l} \rangle$ and $G_{\pi}^{r} = \langle \Xi_{\pi}^{r} \rangle$ have order $n$. If $\gcd(n, t) = 1$, then $G_{\pi}^{l} \cap G_{\pi}^{r} = \{1\}$ and [12, Propositon 3.3] implies that there exists a primitive $n$th root $\eta$ of 1 such that $H(n, t, \xi) \cong H(n, 1, \eta)$ as Hopf algebras. If $\gcd(n, t) \neq 1$, let $m := n/\gcd(n, t)$, then $G_{\pi}^{l} \cap G_{\pi}^{r} = \langle (\Xi_{\pi}^{l})^{m} \rangle$.

Thus we have $\mathrm{io}(H(n, t, \xi)) = n$ and $\mathrm{im}(H(n, t, \xi)) = m$ for any $t$. In particular, $\mathrm{im}(H(n, 0, \xi)) = 1$, $\mathrm{im}(H(n, 1, \xi)) = n$ and $\mathrm{im}(H(n, t, \xi)) = m = n/t$ when $t|n$.

Now assume $t|n$ and $m = n/t$, and let $H := H(n, t, \xi)$, we calculate the homogeneous parts $H_{i}^{l}$, $H_{j}^{r}$ and $H_{ij}$ for our later arguments. The gradings start from fixing a primitive $n$th root $\zeta$ of 1. We choose $\zeta = \xi^{-1}$. By the expressions of $\Xi_{\pi}^{l}$ and $\Xi_{\pi}^{r}$, it is not difficult to find that

$$H_{i}^{l} = k[x]g^{i} \quad \text{and} \quad H_{j}^{r} = k[xg^{-t}]g^{j} \tag{1.4}$$

for all $0 \leqslant i, j \leqslant n - 1$. Thus we have

$$H_{00} = k[x^{m}] \quad \text{and} \quad H_{i,i+jt} = k[x^{m}]x^{j}g^{i} \tag{1.5}$$

for all $0 \leqslant i \leqslant n - 1, 0 \leqslant j \leqslant m - 1$. Moreover we can see that

$$H_{ij} = 0 \ \text{ if } \ i - j \not\equiv 0 \ (\mathrm{mod}\ t)$$

for all $0 \leqslant i, j \leqslant n - 1$.

• *Generalized Liu algebras.* Let $n$ and $\omega$ be positive integers. The generalized Liu algebra, denoted by $B(n,\omega,\gamma)$, is generated by $x^{\pm 1}, g$ and $y$, subject to the relations

$$\begin{cases} xx^{-1} = x^{-1}x = 1, \quad xg = gx, \quad xy = yx, \\ yg = \gamma gy, \\ y^n = 1 - x^{\omega} = 1 - g^n, \end{cases}$$

where $\gamma$ is a primitive $n$th root of 1. The comultiplication, counit and antipode of $B(n,\omega,\gamma)$ are given by

$$\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g, \quad \Delta(y) = y \otimes g + 1 \otimes y,$$

$$\epsilon(x) = 1, \quad \epsilon(g) = 1, \quad \epsilon(y) = 0,$$

and

$$S(x) = x^{-1}, \quad S(g) = g^{-1} \quad S(x) = -yg^{-1}.$$

Let $B := B(n,\omega,\gamma)$. Since $y$ is a normal nonzero-divisor of $B$ and $(y)$ is a Hopf ideal of $B$. Denote by $B' := B/(y)$. Then $B' \cong k[x^{\pm 1}]$ and hence

$$\int_{B'}^{l} \cong B/\langle y, x-1, g-1 \rangle \cong k$$

as $B'$-bimodules. Using Lemma 1.4.2, we get

$$\int_{B}^{l} = B/\langle y, x-1, g-\gamma^{-1} \rangle.$$

The corresponding homomorphism $\pi$ yields left and right winding automorphisms

$$\Xi_\pi^l : \begin{cases} x \longmapsto x, \\ g \longmapsto \gamma^{-1}g, \\ y \longmapsto y, \end{cases} \quad \text{and} \quad \Xi_\pi^r : \begin{cases} x \longmapsto x, \\ g \longmapsto \gamma^{-1}g, \\ y \longmapsto \gamma^{-1}y. \end{cases}$$

Clearly these automorphisms have order $n$ and $G_\pi^l \cap G_\pi^r = \{1\}$, whence $\mathrm{io}(B) = \mathrm{im}(B) = n$.

Choosing $\zeta = \gamma^{-1}$ for defining gradings of $B$, we can see

$$B_i^l = k\langle x^{\pm 1}, y \rangle g^i \quad \text{and} \quad B_j^r = k\langle x^{\pm 1}, yg^{-1} \rangle g^j \tag{1.6}$$

for all $0 \leqslant i, j \leqslant n-1$. Thus

$$B_0 = k[x^{\pm 1}] \quad \text{and} \quad B_{ij} = k[x^{\pm 1}]y^{j-i}g^i, \tag{1.7}$$

where $j - i$ is interpreted mod $n$.

Now, the main results of [12] can be formulated in the following form (for details, see [12, Proposition 3.1, Theorems 4.1 and 6.1]).

**Theorem 1.5.1** *Assume $H$ is an affine prime regular Hopf algebra of GK-dimensional one.*

(a) *If* $\mathrm{io}(H) = \mathrm{im}(H) = 1$, *then* $H \cong k[x]$ *or* $H \cong k[x^{\pm 1}]$;

(b) *If* $\mathrm{io}(H) = n > 1$, $\mathrm{im}(H) = 1$, *then* $H \cong H(n, 0, \xi)$ *or* $H \cong k\mathbb{D}$;

(c) *If* $\mathrm{io}(H) = \mathrm{im}(H) = n$, *then* $H \cong H(n, 1, \xi)$ *or* $H \cong B(n, \omega, \gamma)$.

In [12, Theorem 6.1], the case (c) in the above theorem is expressed in a more general and convenient form. For our purpose, we state the general form as follows.

**Lemma 1.5.2** *Let $H$ be an affine prime regular Hopf algebra of GK-dimension one. Assume that there exists an algebra homomorphism $\mu : H \to k$ such that $n :=$* $\text{PI-deg}(H) = |G^l_\mu|$ *and $G^l_\mu \cap G^r_\mu = \{1\}$, where $G^l_\mu$ and $G^r_\mu$ are the groups of left and right winding automorphisms associated to $\mu$. Then $H$ is isomorphic as a Hopf algebra either to the Taft algebra $H(n, 1, \xi)$ or to the generalized Liu algebra $B(n, \omega, \gamma)$. As a consequence,* $\mathrm{io}(H) = \mathrm{im}(H)$.

# Chapter 2 Some combinatorial equations

We collect some combinatorial equations in this chapter. These equations turn out to be important for the following analysis.

We introduce the quantum binomial coefficients $\binom{n}{l}_q$ first. For any positive integers $n, l$ with $0 \leqslant l \leqslant n$, any $0 \neq q \in k$,

$$\binom{n}{l}_q := \frac{(n)_q!}{(l)_q!(n-l)_q!},$$

where $(n)_q! = \prod_{j=1}^{n}(j)_q$ and $(j)_q = \sum_{i=0}^{j-1} q^i$.

Throughout this chapter, $m, d$ are two natural numbers, $\gamma$ is an $m$th primitive root of 1 and $\xi$ an element in $k$ satisfying $\xi^m = -1$. For each $i \in \mathbb{Z}$, $\phi_i$ is a polynomial defined by

$$\phi_i := 1 - \gamma^{-i-1}x^d.$$

Take $t$ to be an arbitrary integer, define $\bar{t}$ to be the unique element in $\{0, 1, \ldots, m-1\}$ satisfying $\bar{t} \equiv t \pmod{m}$. Then we have

$$\phi_t = \phi_{\bar{t}}$$

since $\gamma^m = 1$.

With this observation, we can use

$$]s, t[$$

to denote the resulted polynomial by omitting all items *from* $\phi_{\bar{s}}$ *to* $\phi_{\bar{t}}$ in $\phi_0\phi_1 \cdots \phi_{m-1}$, that is

$$]s,t[ = \begin{cases} \phi_{\bar{t}+1} \cdots \phi_{m-1}\phi_0 \cdots \phi_{\bar{s}-1}, & \text{if } \bar{t} \geqslant \bar{s} \\ 1, & \text{if } \bar{s} = \bar{t}+1 \\ \phi_{\bar{t}+1} \cdots \phi_{\bar{s}-1}, & \text{if } \bar{s} \geqslant \bar{t}+2. \end{cases} \tag{2.1}$$

For example, $]-1,-1[ = ]m-1, m-1[ = \phi_0\phi_1 \cdots \phi_{m-2}$.

To study equations with omitting items, the following formula is useful for us.

**Lemma 2.0.3** *[17, Proposition IV.2.7.] Fix an invertible element $q$ of the field $k$. For any scalar $a$ we have*

$$(a-z)(a-qz)\cdots(a-q^{n-1}z)=\sum_{l=0}^{n}(-1)^l\binom{n}{l}_q q^{\frac{l(l-1)}{2}}a^{n-l}z^l.$$

**Lemma 2.0.4** *With notions defined as above, we have*

$$\sum_{j=0}^{m-1}\gamma^{-j}\ ]j-1,j-1[\ = mx^{(m-1)d}.$$

***Proof.*** By Lemma 2.0.3, $\phi_0\cdots\phi_{m-1}=(1-x^{dm})$. Note that

$$\begin{aligned}\sum_{j=0}^{m-1}\ ]j-1,j-1[\ &= \sum_{j=0}^{m-1}1+\gamma^{-j}x^d+\gamma^{-2j}x^{2d}+\cdots+\gamma^{-(m-1)j}x^{(m-1)d}\\ &= m.\end{aligned}$$

So

$$\begin{aligned}&\sum_{j=0}^{m-1}\ ]j-1,j-1[\ -\sum_{j=0}^{m-1}\gamma^{-j}x^d\ ]j-1,j-1[\\ &=\sum_{j=0}^{m-1}(1-\gamma^{-j}x^d)\ ]j-1,j-1[\\ &=\sum_{j=0}^{m-1}\phi_0\phi_1\cdots\phi_{m-1}\\ &=m(1-x^{md}).\end{aligned}$$

Therefore, $\sum_{j=0}^{m-1}\gamma^{-j}\ ]j-1,j-1[\ = mx^{(m-1)d}$. □

If we omit two items, then we have

**Lemma 2.0.5**

$$\sum_{j=0}^{m-1}\gamma^{-j}\ ]j-2,j-1[\ = 0.$$

***Proof.*** By Lemma 2.0.3,

$$\begin{aligned}
]j-2,j-1[ &= (1-\gamma^{-j-1}x^d)(1-\gamma^{-(j+1)-1}x^d)\cdots(1-\gamma^{-(m+j-3)-1}x^d)\\
&= \sum_{l=0}^{m-2}(-1)^l\binom{m-2}{l}_{\gamma^{-1}}\gamma^{-\frac{l(l-1)}{2}}(\gamma^{-j-1}x^d)^l\\
&= \sum_{l=0}^{m-2}(-1)^l\binom{m-2}{l}_{\gamma^{-1}}\gamma^{-\frac{l(l+1)}{2}-lj}x^{ld}.
\end{aligned}$$

So, to get the result, it is sufficient to show that

$$\sum_{j=0}^{m-1}\gamma^{-j}\gamma^{-lj}=\sum_{j=0}^{m-1}\gamma^{-(l+1)j}=0$$

for all $0\leqslant l\leqslant m-2$. Since $1\leqslant l+1\leqslant m-1$, it is clear that $\sum_{j=0}^{m-1}\gamma^{-(l+1)j}=0$. Thus the conclusion is established. □

As a direct consequence of this lemma, we get the following basic observation.

**Corollary 2.0.6**

$$\sum_{j=0}^{m-1}\xi^{2j}\gamma^{-2j}\,]j-2,j-1[=0$$

*if and only if* $\xi^2=\gamma$, *where* $\xi$ *is an element in* $k$ *satisfying* $\xi^m=-1$.

***Proof.*** Let $\theta=\xi^2\gamma^{-2}$, then $\theta$ is an $m$th root of 1. By the proof of the above lemma, it is sufficient to verify that $\theta\gamma^{-s}\neq 1$ for all $0\leqslant s\leqslant m-2$. It follows that $\theta=\gamma^{-1}$, i.e., $\xi^2=\gamma$. □

**Lemma 2.0.7** *Fix* $i$ *such that* $1\leqslant i\leqslant m-1$ *and let* $1\leqslant i'\leqslant i$. *Then*

$$\sum_{j=0}^{m-1}\gamma^{-i'j}\,]j-1-i,j-1[=0$$

.

***Proof.*** Using Lemma 2.0.3, we have

$$\begin{aligned}
&]j-1-i, j-1[\\
&= \phi_j \cdots \phi_{m-1}\phi_0 \cdots \phi_{j-i-2}\\
&= (1-\gamma^{-j-1}x^d)(1-\gamma^{-(j+1)-1}x^d)\cdots(1-\gamma^{-(m+j-i-2)-1}x^d)\\
&= \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l-1)}{2}} (\gamma^{-j-1}x^d)^l\\
&= \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-lj} x^{ld}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{j=0}^{m-1} \gamma^{-i'j}\, ]j-1-i, j-1[\\
&= \sum_{j=0}^{m-1} \gamma^{-i'j} \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-lj} x^{ld}\\
&= \sum_{l=0}^{m-1-i} (-1)^l \binom{m-1-i}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}} x^{ld} \sum_{j=0}^{m-1} \gamma^{-(i'+l)j}.
\end{aligned}$$

Since $0 \leqslant l \leqslant m-1-i$, $1 \leqslant i' \leqslant i'+l \leqslant m-1-i+i' \leqslant m-1$. Then we have $\sum_{j=0}^{m-1} \gamma^{-(i'+l)j} = 0$ for all $0 \leqslant l \leqslant m-1-i$ and $1 \leqslant i' \leqslant i$. This ends the proof. □

The next technical result is also needed.

**Lemma 2.0.8** *Let $0 \leqslant t \leqslant i+j \leqslant m-1$, $0 \leqslant l \leqslant m-1-i-j$ and let $q$ be a primitive $m$th root of 1. Then*

$$\begin{aligned}
&q^{\frac{(l+t)(l+t+1)}{2}+t(i+j-t)} \cdot (-1)^{l+t} \binom{m-1-t}{l}_q \binom{m-1+t-i-j}{l+t}_q\\
&= \binom{i+j}{t}_q \binom{m-1-i-j}{l}_q.
\end{aligned}$$

***Proof.*** Since

$$\binom{m-1-t}{l}_q \binom{m-1+t-i-j}{l+t}_q = \frac{(m-1-t)!_q}{(l)!_q(m-1-t-l)!_q} \cdot \frac{(m-1+t-i-j)!_q}{(l+t)!_q(m-1-l-i-j)!_q}$$

and

$$\binom{i+j}{t}_q \binom{m-1-i-j}{l}_q = \frac{(i+j)!_q}{(t)!_q(i+j-t)!_q} \cdot \frac{(m-1-i-j)!_q}{(l)!_q(m-1-l-i-j)!_q},$$

we have

$$\begin{aligned}
&\frac{\binom{m-1-t}{l}_q \binom{m-1+t-i-j}{l+t}_q}{\binom{i+j}{t}_q \binom{m-1-i-j}{l}_q} \\
&= \frac{(m-l-t)_q(m-l+1-t)_q \cdots (m-1-t)_q}{(t+1)_q(t+2)_q \cdots (t+l)_q} \\
&\quad \cdot \frac{(m-i-j)_q(m-i-j+1)_q \cdots (m-i-j+t-1)_q}{(i+j-t+1)_q(i+j-t+2)_q \cdots (i+j)_q}.
\end{aligned}$$

Note that for every number $c$, $0 \leqslant c \leqslant m-1$,

$$\begin{aligned}
(m-c)_q = 1+q+\cdots+q^{m-1-c} &= -(q^{m-c}+q^{m-c+1}\cdots+q^{m-1}) \\
&= -q^{m-c}(1+q+\cdots+q^{c-1}) = -q^{m-c}(c)_q.
\end{aligned}$$

Thus

$$\frac{(m-l-t)_q(m-l+1-t)_q \cdots (m-1-t)_q}{(t+1)_q(t+2)_q \cdots (t+l)_q} = (-1)^l q^{-\frac{l(1+l+2t)}{2}}$$

and

$$\frac{(m-i-j)_q(m-i-j+1)_q \cdots (m-i-j+t-1)_q}{(i+j-t+1)_q(i+j-t+2)_q \cdots (i+j)_q} = (-1)^t q^{-\frac{t(1-t+2(i+j))}{2}}.$$

Therefore,

$$\frac{\binom{m-1-t}{l}_q \binom{m-1+t-i-j}{l+t}_q}{\binom{i+j}{t}_q \binom{m-1-i-j}{l}_q} = (-1)^{l+t} q^{-\frac{l(1+l+2t)+t(1-t+2(i+j))}{2}} = (-1)^{l+t} q^{-\frac{(l+t)(l+t+1)}{2}-t(i+j-t)}.$$

This completes the proof. □

# Chapter 3 New examples

In this chapter, we will introduce a new class of algebras $D(m,d,\xi)$ and show that these algebras are prime regular Hopf algebras of GK-dimension one. Some properties about $D(m,d,\xi)$ are established, in particular, we will show that $D(m,d,\xi)$ is not a pointed Hopf algebra.

## §3.1 Definition of $D(m,d,\xi)$

As before, let $m, d$ be two natural numbers satisfying that $(1+m)d$ is even and $\xi$ a primitive $2m$th root of 1. Define

$$\omega := md, \quad \gamma := \xi^2.$$

• *The algebra structure.* As an algebra, $D(m,d,\xi)$ is generated by $x^{\pm 1}, g^{\pm 1}, y, u_0, u_1, \cdots, u_{m-1}$, subject to the following relations

$$xx^{-1} = x^{-1}x = 1,\ gg^{-1} = g^{-1}g = 1,\ xg = gx,\ xy = yx,\ yg = \gamma gy,\ y^m = 1-x^{\omega} = 1-g^m, \tag{3.1}$$

$$xu_i = u_i x^{-1},\ yu_i = \phi_i u_{i+1} = \xi x^d u_i y,\ u_i g = \gamma^i x^{-2d} g u_i, \tag{3.2}$$

$$u_i u_j = \begin{cases} (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\phi_{i+1}\cdots\phi_{m-2-j}y^{i+j}g, & \text{if } i+j \leqslant m-2,\\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}y^{i+j}g, & \text{if } i+j = m-1,\\ (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\phi_i\cdots\phi_{m-1}\phi_0\cdots\phi_{m-2-j}y^{i+j-m}g, & \text{otherwise,} \end{cases} \tag{3.3}$$

where $\phi_i = 1-\gamma^{-i-1}x^d$ and $0 \leqslant i,j \leqslant m-1$.

Since $\gamma = \xi^2$, the expression $(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}$ in Equation (3.3) equals $(-1)^{-j}\xi^{j^2}$. We still use this expression $(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}$ because it is convenient for the further computations involving coproduct and antipode.

To give a unified expression for the last terrible relation (3.3), we have the following observations. On one hand, as observed at the beginning of Section 3, if we still define $\phi_t = 1-\gamma^{-t-1}x^d$ for any $t \in \mathbb{Z}$, then

$$\phi_t = \phi_{\bar{t}},$$

where $\bar{t} \equiv t \pmod m$. For any $i, j \in \mathbb{Z}$, we have

$$]{-1-j}, i-1[ = \begin{cases} \phi_{\bar{i}} \cdots \phi_{m-1}\phi_0 \cdots \phi_{m-2-\bar{j}}, & \text{if } \bar{i}+\bar{j} \geqslant m \\ 1, & \text{if } \bar{i}+\bar{j} = m-1 \\ \phi_{\bar{i}} \cdots \phi_{m-2-\bar{j}}, & \text{if } \bar{i}+\bar{j} \leqslant m-2. \end{cases}$$

by (2.1).

For the convenience of our later computations, the next notion is also useful for us,

$$[s,t] := \begin{cases} \phi_{\bar{s}}\phi_{\bar{s}+1} \cdots \phi_{\bar{t}}, & \text{if } \bar{t} \geqslant \bar{s} \\ 1, & \text{if } \bar{s} = \bar{t}+1 \\ \phi_{\bar{s}} \cdots \phi_{m-1}\phi_0 \cdots \phi_{\bar{t}}, & \text{if } \bar{s} \geqslant \bar{t}+2. \end{cases}$$

In fact, $[s,t]$ can be considered as the resulted polynomial (except the case $\bar{s} = \bar{t}+1$) by preserving all items *from* $\phi_{\bar{s}}$ *to* $\phi_{\bar{t}}$ in $\phi_0\phi_1 \cdots \phi_{m-1}$. So, by definition, we have

$$[i, m-2-j] = \,]{-1-j}, i-1[. \tag{3.4}$$

On the other hand, we find that

$$(-1)^{-km-j}\xi^{-km-j}\gamma^{\frac{(km+j)(km+j+1)}{2}} = (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \tag{3.5}$$

for any $k \in \mathbb{Z}$. Therefore, if we define

$$u_s := u_{\bar{s}},$$

then the relation (3.3) can be replaced by

$$\begin{aligned} u_i u_j &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \,]{-1-j}, i-1[\, y^{\overline{i+j}} g \\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \,[i, m-2-j]\, y^{\overline{i+j}} g \end{aligned} \tag{3.6}$$

for all $i, j \in \mathbb{Z}$.

We give a bigrading on this algebra for use later. Define the following two algebra automorphisms of $D(m, d, \xi)$:

$$\Xi_\pi^l : \begin{cases} x \longmapsto x, \\ y \longmapsto y, \\ g \longmapsto \gamma^{-1} g, \\ u_i \longmapsto \xi^{-1} u_i, \end{cases} \quad \text{and} \quad \Xi_\pi^r : \begin{cases} x \longmapsto x, \\ y \longmapsto \gamma^{-1} y, \\ g \longmapsto \gamma^{-1} g, \\ u_i \longmapsto \xi^{-(2i+1)} u_i. \end{cases}$$

It is straightforward to show that $\Xi_\pi^l$ and $\Xi_\pi^r$ are indeed algebra automorphisms of $D(m,d,\xi)$ and these automorphisms have order $2m$ by noting that $\xi$ is a primitive $2m$th root of 1 and $u_i \neq 0$ in $D(m,d,\xi)$ for all $i$ (if one $u_i = 0$ in $D(m,d,\xi)$, then $y^{\overline{i+j}}g = 0$ by (3.3), which is absurd). Choosing $\zeta = \xi^{-1}$, define

$$D_i^l := \{h \in D(m,d,\xi) | \Xi_\pi^l(h) = \zeta^i h\}, \quad D_j^r := \{h \in D(m,d,\xi) | \Xi_\pi^r(h) = \zeta^j h\}$$

for $0 \leqslant i, j \leqslant 2m-1$. Direct computations show that

$$D_i^l = \begin{cases} k\langle x^{\pm 1}, y\rangle g^{\frac{i}{2}}, & i = \text{even},\\ \sum_{s=0}^{m-1} k[x^{\pm 1}]g^{\frac{i-1}{2}}u_s, & i = \text{odd}, \end{cases}$$

and

$$D_j^r = \begin{cases} k\langle x^{\pm 1}, yg^{-1}\rangle g^{\frac{j}{2}}, & j = \text{even},\\ \sum_{s=0}^{m-1} k[x^{\pm 1}]g^s u_{\frac{j-1}{2}-s}, & j = \text{odd}. \end{cases}$$

Therefore

$$D_{ij} := D_i^l \cap D_j^r = \begin{cases} k[x^{\pm 1}]y^{\frac{j-i}{2}}g^{\frac{i}{2}}, & i,j = \text{even},\\ k[x^{\pm 1}]g^{\frac{i-1}{2}}u_{\frac{j-i}{2}}, & i,j = \text{odd},\\ 0, & \text{otherwise}. \end{cases} \tag{3.7}$$

Since $\sum_{i,j} D_{ij} = D(m,d,\xi)$, we have

$$D(m,d,\xi) = \bigoplus_{i,j=0}^{2m-1} D_{ij} \tag{3.8}$$

which is a bigrading on $D(m,d,\xi)$ automatically.

Let $D := D(m,d,\xi)$, then $D \otimes D$ is graded naturally by inheriting the grading defined above. In particular, for any $h \in D \otimes D$, we use

$$h_{(s_1,t_1)\otimes(s_2,t_2)}$$

to denote the homogeneous part of $h$ in $D_{s_1,t_1} \otimes D_{s_2,t_2}$. This notion will be used freely in the proof of Proposition 3.2.1.

• *The coalgebra structure and the antipode.* The coproduct $\Delta$, the counit $\epsilon$ and the antipode $S$ of $D(m,d,\xi)$ are given by

$$\Delta(x) = x \otimes x, \quad \Delta(g) = g \otimes g, \quad \Delta(y) = y \otimes g + 1 \otimes y,$$

$$\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j};$$

$$\epsilon(x) = \epsilon(g) = \epsilon(u_0) = 1, \ \ \epsilon(y) = \epsilon(u_s) = 0;$$

$$S(x) = x^{-1}, \ \ S(g) = g^{-1}, \ \ S(y) = -yg^{-1},$$

$$S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id+\frac{3}{2}(1-m)d} g^{m-i-1} u_i, \tag{3.9}$$

for $0 \leq i \leq m-1$ and $1 \leqslant s \leqslant m-1$.

Since $g^m = x^{md}$ and (3.5) , the definition about $S(u_i)$ still holds for any integer $i$, that is, (3.9) can be replaced in a more convenient way:

$$S(u_s) = (-1)^s \xi^{-s} \gamma^{-\frac{s(s+1)}{2}} x^{sd+\frac{3}{2}(1-m)d} g^{m-s-1} u_s \tag{3.10}$$

for all $s \in \mathbb{Z}$.

**Remark 3.1.1** *Recall that $\xi^2 = \gamma$. It is not hard to see that the subalgebra of $D(m,d,\xi)$ generated by $x^{\pm 1}, g^{\pm 1}, y$ is exact the generalized Liu algebra $B(m,\omega,\gamma)$. Indeed, by Equations (3.7) and (3.8), $B_{ij} = D_{2i,2j}$ for all $0 \leqslant i,j \leqslant m-1$. By definition, $D(m,d,\xi)$ is affine. Moreover, $D(m,d,\xi)$ is a finitely generated $k[x^{\pm 1}]$-module by Equation (3.7). Thus $D(m,d,\xi)$ has GK-dimension one. At the same time, note that $Z = k[z_s | z_s := x^s + x^{-s}, s \in \mathbb{N}_0]$ lies in the center of $D(m,d,\xi)$, which implies that $D(m,d,\xi)$ is PI. In one word, $D(m,d,\xi)$ is affine, PI and has GK-dimension one, and contains $B(m,\omega,\gamma)$ as a Hopf subalgebra.*

## §3.2 $D(m,d,\xi)$ is a Hopf algebra

The main aim of this section is to show that $D(m,d,\xi)$ is indeed a Hopf algebra.

**Proposition 3.2.1** *The algebra $D(m,d,\xi)$ defined above is a Hopf algebra.*

*Proof.* The proof is standard but not easy. For completeness and the convenience of the reader, we give the proof here. As usual, we decompose the proof into several steps. Since the subalgebra generated by $x^{\pm 1}, y, g$ is just the generalized Liu algebra $B(m,\omega,\gamma)$, which is a Hopf algebra already, we only need to verify the related relations in $D(m,d,\xi)$ where $u_i$ are involved.

• *Step* 1 ($\Delta$ and $\epsilon$ are algebra homomorphisms).

First of all, it is clear that $\epsilon$ is an algebra homomorphism. Since $x$ and $g$ are group-like elements, the verifications of $\Delta(x)\Delta(u_i) = \Delta(u_i)\Delta(x^{-1})$ and $\Delta(u_i)\Delta(g) = \gamma^i\Delta(x^{-2d})\Delta(g)\Delta(u_i)$ are simple and so they are omitted.

(1) *The proof of* $\Delta(\phi_i)\Delta(u_{i+1}) = \Delta(y)\Delta(u_i) = \xi\Delta(x^d)\Delta(u_i)\Delta(y)$.

By definition $\Delta(u_i) = \sum_{j=0}^{m-1}\gamma^{j(i-j)}u_j \otimes x^{-jd}g^j u_{i-j}$ for all $0 \leqslant i \leqslant m-1$, we have

$$\begin{aligned}\Delta(\phi_i)\Delta(u_{i+1}) &= (1\otimes 1 - \gamma^{-i-1}x^d\otimes x^d)\sum_{j=0}^{m-1}\gamma^{j(i+1-j)}u_j\otimes x^{-jd}g^j u_{i+1-j}\\ &= \sum_{r=0}^{m-1}\gamma^{r(i+1-r)}u_r\otimes x^{-rd}g^r u_{i+1-r} - \sum_{l=0}^{m-1}\gamma^{l(i+1-l)-i-1}x^d u_l\otimes x^{(1-l)d}g^j u_{i+1-l}.\end{aligned}$$

And

$$\begin{aligned}\Delta(y)\Delta(u_i) &= (y\otimes g + 1\otimes y)\sum_{j=0}^{m-1}\gamma^{j(i-j)}u_j\otimes x^{-jd}g^j u_{i-j}\\ &= \sum_{l=0}^{m-1}\gamma^{l(i-l)}yu_l\otimes x^{-ld}g^{l+1}u_{i-l} + \sum_{r=0}^{m-1}\gamma^{r(i-r)}u_r\otimes yx^{-rd}g^r u_{i-r}\\ &= \sum_{l=0}^{m-1}\gamma^{l(i-l)}u_{l+1}\otimes x^{-ld}g^{l+1}u_{i-l} - \sum_{l=0}^{m-1}\gamma^{l(i-l)-l-1}x^d u_{l+1}\otimes x^{-ld}g^{l+1}u_{i-l}\\ &\quad + \sum_{r=0}^{m-1}\gamma^{r(i-r+1)}u_r\otimes x^{-rd}g^r u_{i+1-r} - \sum_{r=0}^{m-1}\gamma^{(r-1)(i+1-r)}u_r\otimes x^{(1-r)d}g^r u_{i+1-r}\\ &= \sum_{r=0}^{m-1}\gamma^{r(i-r+1)}u_r\otimes x^{-rd}g^r u_{i+1-r} - \sum_{l=0}^{m-1}\gamma^{l(i-l)-l-1}x^d u_{l+1}\otimes x^{-ld}g^{l+1}u_{i-l}.\end{aligned}$$

Hence $\Delta(\phi_i)\Delta(u_{i+1}) = \Delta(y)\Delta(u_i)$. Similarly,

$$\begin{aligned}
&\xi\Delta(x^d)\Delta(u_i)\Delta(y)\\
&= \xi\cdot(x^d\otimes x^d)(\sum_{j=0}^{m-1}\gamma^{j(i-j)}u_j\otimes x^{-jd}g^ju_{i-j})(y\otimes g+1\otimes y)\\
&= \sum_{s=0}^{m-1}\gamma^{s(i-s)}yu_s\otimes x^{(1-s)d}g^su_{i-s}g+\sum_{t=0}^{m-1}\gamma^{t(i-t)}x^du_t\otimes x^{-td}g^tyu_{i-t}\\
&= \sum_{s=0}^{m-1}\gamma^{(s+1)(i-s)}u_{s+1}\otimes x^{-(s+1)d}g^{s+1}u_{i-s}-\sum_{s=0}^{m-1}\gamma^{(s+1)(i-s-1)}x^du_{s+1}\otimes x^{-(s+1)d}g^{s+1}u_{i-s}\\
&\quad+\sum_{t=0}^{m-1}\gamma^{t(i-t)}x^du_t\otimes x^{-td}g^tu_{i+1-t}-\sum_{t=0}^{m-1}\gamma^{(t-1)(i-t)-1}x^du_t\otimes x^{(1-t)d}g^tu_{i+1-t}\\
&= \sum_{s=0}^{m-1}\gamma^{(s+1)(i-s)}u_{s+1}\otimes x^{-(s+1)d}g^{s+1}u_{i-s}-\sum_{t=0}^{m-1}\gamma^{(t-1)(i-t)-1}x^du_t\otimes x^{(1-t)d}g^tu_{i+1-t},
\end{aligned}$$

which equals $\Delta(\phi_i)\Delta(u_{i+1})$ clearly.

(2) *The proof of* $\Delta(u_iu_j) = \Delta(u_i)\Delta(u_j)$.

We have that

$$\begin{aligned}
\Delta(u_i)\Delta(u_j) &= \sum_{s=0}^{m-1}\gamma^{s(i-s)}u_s\otimes x^{-sd}g^su_{i-s}\cdot\sum_{t=0}^{m-1}\gamma^{t(j-t)}u_t\otimes x^{-td}g^tu_{j-t}\\
&= \sum_{t=0}^{m-1}\sum_{s=0}^{m-1}\gamma^{s(i-s)}u_s\gamma^{(t-s)(j-t+s)}u_{t-s}\otimes x^{-sd}g^su_{i-s}x^{-(t-s)d}g^{t-s}u_{j-t+s}\\
&= \sum_{t=0}^{m-1}\sum_{s=0}^{m-1}\gamma^{(t-s)(j-t+s)+(i-s)t}u_su_{t-s}\otimes x^{-td}g^tu_{i-s}u_{j-t+s}.
\end{aligned}$$

By the bigrading given in (3.8), we can find that for each $0\leqslant t\leqslant m-1$,

$$\sum_{s=0}^{m-1}\gamma^{(t-s)(j-t+s)+(i-s)t}u_su_{t-s}\otimes x^{-td}g^tu_{i-s}u_{j-t+s}\in D_{2,2+2t}\otimes D_{2+2t,2+2(i+j)},$$

where the suffixes in $D_{2,2+2t}\otimes D_{2+2t,2+2(i+j)}$ are interpreted mod $2m$.

Note that

$$u_su_{t-s} = (-1)^{-(t-s)}\xi^{-(t-s)}\gamma^{\frac{(t-s)(t-s+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\,[s, m-2-t+s]\,y^tg$$

and

$$u_{i-s}u_{j-t+s} = (-1)^{-(j-t+s)}\xi^{-(j-t+s)}\gamma^{\frac{(j-t+s)(j-t+s+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\,[i-s, m-2+t-j-s]\,y^{\overline{i+j-t}}g.$$

Using Lemma 2.0.3, we get

$$
\begin{aligned}
[s, m-2-t+s] &= (1-\gamma^{-s-1}x^d)(1-\gamma^{-s-2}x^d)\cdots(1-\gamma^{-(m-2-t+s)-1}x^d) \\
&= \sum_{l=0}^{m-1-t} (-1)^l \binom{m-1-t}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l-1)}{2}} (\gamma^{-s-1}x^d)^l \\
&= \sum_{l=0}^{m-1-t} (-1)^l \binom{m-1-t}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-sl} x^{ld},
\end{aligned}
$$

and

$$
\begin{aligned}
&[i-s, m-2+t-j-s] \\
&= (1-\gamma^{-(i-s)-1}x^d)(1-\gamma^{-(i-s+1)-1}x^d)\cdots(1-\gamma^{-(i-s+m-2-\overline{i+j-t})-1}x^d) \\
&= \sum_{r=0}^{m-1-\overline{i+j-t}} (-1)^r \binom{m-1-\overline{i+j-t}}{r}_{\gamma^{-1}} \gamma^{-\frac{r(r-1)}{2}} (\gamma^{s-i-1}x^d)^r \\
&= \sum_{r=0}^{m-1-\overline{i+j-t}} (-1)^r \binom{m-1-\overline{i+j-t}}{r}_{\gamma^{-1}} \gamma^{-\frac{r(r+1)}{2}+(s-i)r} x^{rd}.
\end{aligned}
$$

Then for each $0 \leqslant t \leqslant m-1$,

$$
\begin{aligned}
&\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} \\
&= \sum_{s=0}^{m-1} \gamma^{(t-s)(j-t+s)+(i-s)t} u_s u_{t-s} \otimes x^{-td} g^t u_{i-s} u_{j-t+s} \\
&= \sum_{s=0}^{m-1} \gamma^{(t-s)(j-t+s)+(i-s)t} (-1)^{-(t-s)} \xi^{-(t-s)} \gamma^{\frac{(t-s)(t-s+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \, [s, m-2-t+s] \, y^t g \\
&\qquad \otimes x^{-td} g^t (-1)^{-(j-t+s)} \xi^{-(j-t+s)} \gamma^{\frac{(j-t+s)(j-t+s+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \, [i-s, m-2+t-j-s] \, y^{\overline{i+j-t}} g \\
&= (-1)^{-j} \xi^{-j} \frac{1}{m^2} \Big( \sum_{s=0}^{m-1} \gamma^{\frac{j^2+j}{2}+(i-s)t-t(i+j-t)} \, [s, m-2-t+s] \, \otimes x^{-td} \, [i-s, m-2+t-j-s] \, \Big) \\
&\qquad (x^{-\frac{1+m}{2}d} y^t g \otimes x^{-\frac{1+m}{2}d} y^{\overline{i+j-t}} g^{t+1}) \\
&= (-1)^{-j} \xi^{-j} \frac{1}{m^2} \sum_{l=0}^{m-1-t} \sum_{r=0}^{m-1-\overline{i+j-t}} \gamma^{\frac{j(j+1)-l(l+1)-r(r+1)}{2}-t(j-t)-ir} (-1)^{l+r} \binom{m-1-t}{l}_{\gamma^{-1}} \\
&\qquad \binom{m-1-\overline{i+j-t}}{r}_{\gamma^{-1}} \sum_{s=0}^{m-1} \gamma^{(r-l-t)s} \cdot (x^{ld} \otimes x^{(r-t)d}) \cdot (x^{-\frac{1+m}{2}d} y^t g \otimes x^{-\frac{1+m}{2}d} y^{\overline{i+j-t}} g^{t+1}).
\end{aligned}
$$

Meanwhile, $u_i u_j = (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\ [i, m-2-j]\ y^{\overline{i+j}}g$. Since

$$\begin{aligned}\Delta(y^{\overline{i+j}}) &= (1\otimes y + y\otimes g)^{\overline{i+j}}\\ &= \sum_{t=0}^{\overline{i+j}} \binom{\overline{i+j}}{t}_{\gamma^{-1}} (1\otimes y)^{\overline{i+j}-t}\cdot (y\otimes g)^t\\ &= \sum_{t=0}^{\overline{i+j}} \binom{\overline{i+j}}{t}_{\gamma^{-1}} y^t \otimes y^{\overline{i+j}-t}g^t\end{aligned}$$

and

$$\begin{aligned}&\Delta([i, m-2-j])\\ =\ &(1\otimes 1 - \gamma^{-i-1}x^d\otimes x^d)(1\otimes 1-\gamma^{-i-2}x^d\otimes x^d)\cdots(1\otimes 1-\gamma^{-(i+m-2-\overline{i+j})-1}x^d\otimes x^d)\\ =\ &\sum_{l=0}^{m-1-\overline{i+j}} (-1)^l \binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l-1)}{2}}(\gamma^{-i-1}x^d\otimes x^d)^l\\ =\ &\sum_{l=0}^{m-1-\overline{i+j}} (-1)^l \binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-il}\cdot x^{ld}\otimes x^{ld},\end{aligned}$$

we get

$$\begin{aligned}\Delta(u_iu_j) &= \Delta((-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\ [i,m-2-j]\ y^{\overline{i+j}}g)\\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}\Delta(x^{-\frac{1+m}{2}d})\Delta(\ [i,m-2-j]\ )\Delta(y^{\overline{i+j}})\Delta(g)\\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}\sum_{l=0}^{m-1-\overline{i+j}}(-1)^l\binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}}\gamma^{-\frac{l(l+1)}{2}-il}\sum_{t=0}^{\overline{i+j}}\binom{\overline{i+j}}{t}_{\gamma^{-1}}\\ &\qquad (x^{-\frac{1+m}{2}d}\otimes x^{-\frac{1+m}{2}d})\cdot(x^{ld}\otimes x^{ld})\cdot(y^t\otimes y^{\overline{i+j-t}}g^t)\cdot(g\otimes g)\\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}\sum_{t=0}^{\overline{i+j}}\cdot\sum_{l=0}^{m-1-\overline{i+j}}(-1)^l\binom{\overline{i+j}}{t}_{\gamma^{-1}}\binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}}\gamma^{-\frac{l(l+1)}{2}-il}\\ &\qquad (x^{ld}\otimes x^{ld})\cdot(x^{-\frac{1+m}{2}d}y^tg\otimes x^{-\frac{1+m}{2}d}y^{\overline{i+j-t}}g^{t+1}).\end{aligned}$$

For each $0 \leqslant t \leqslant \overline{i+j}$, $(x^{ld} \otimes x^{ld}) \cdot (x^{-\frac{1+m}{2}d} y^t g \otimes x^{-\frac{1+m}{2}d} y^{\overline{i+j-t}} g^{t+1}) \in D_{2,2+2t} \otimes D_{2+2t,2+2(i+j)}$ for any $l$. So,

$$\begin{aligned}
&\Delta(u_i u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} \\
&= (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} \sum_{l=0}^{m-1-\overline{i+j}} (-1)^l \binom{\overline{i+j}}{t}_{\gamma^{-1}} \binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}} \gamma^{-\frac{l(l+1)}{2}-il} \\
&\quad (x^{ld} \otimes x^{ld}) \cdot (x^{-\frac{1+m}{2}d} y^t g \otimes x^{-\frac{1+m}{2}d} y^{\overline{i+j-t}} g^{t+1}).
\end{aligned}$$

By the graded structure of $D \otimes D$, $\Delta(u_i)\Delta(u_j) = \Delta(u_i u_j)$ if and only if

$$\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} = 0 \tag{3.11}$$

for all $\overline{i+j} + 1 \leqslant t \leqslant m-1$ and

$$\Delta(u_i u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} = \Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} \tag{3.12}$$

for all $0 \leqslant t \leqslant \overline{i+j}$.

By the expression of $\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))}$, we can find that it is zero if $\sum_{s=0}^{m-1} \gamma^{(r-l-t)s} = 0$. Note that in the case of $\overline{i+j} + 1 \leqslant t \leqslant m-1$, $m-1-\overline{i+j-t} = t-1-\overline{i+j}$. So $0 \leqslant r \leqslant t-1-\overline{i+j}$ and thus $1-m \leqslant r-l-t \leqslant -1-\overline{i+j}$. This means that in this case we always have

$$\sum_{s=0}^{m-1} \gamma^{(r-l-t)s} = 0,$$

which implies (3.11).

Now let $0 \leqslant t \leqslant \overline{i+j}$. Then $1-m \leqslant r-l-t \leqslant m-1-\overline{i+j-t}-t < m$. As discussed above, $\Delta(u_i)\Delta(u_j)_{(2,2+2t)\otimes(2+2t,2+2(i+j))} = 0$ if $r-l-t \neq 0$. So we only need to verify the case when $r = l+t$. At this time, $0 \leqslant l \leqslant m-1-\overline{i+j}$. Then (3.12) holds if and only if

$$\begin{aligned}
&\gamma^{-\frac{(l+t)(l+t+1)}{2}-t(i+j-t)} \cdot (-1)^{l+t} \binom{m-1-t}{l}_{\gamma^{-1}} \binom{m-1-\overline{i+j-t}}{l+t}_{\gamma^{-1}} \\
&= \binom{\overline{i+j}}{t}_{\gamma^{-1}} \binom{m-1-\overline{i+j}}{l}_{\gamma^{-1}},
\end{aligned}$$

which is just Lemma 2.0.8 by setting $q=\gamma^{-1}$.

- *Step* 2 (Coassociative and couint).

  Indeed, for each $0\leqslant i\leqslant m-1$

$$\begin{aligned}(\Delta\otimes \mathrm{Id})\Delta(u_i)&=(\Delta\otimes \mathrm{Id})(\sum_{j=0}^{m-1}\gamma^{j(i-j)}u_j\otimes x^{-jd}g^ju_{i-j})\\&=\sum_{j=0}^{m-1}\gamma^{j(i-j)}(\sum_{s=0}^{m-1}\gamma^{s(j-s)}u_s\otimes x^{-sd}g^su_{j-s})\otimes x^{-jd}g^ju_{i-j}\\&=\sum_{j,s=0}^{m-1}\gamma^{j(i-j)+s(j-s)}u_s\otimes x^{-sd}g^su_{j-s}\otimes x^{-jd}g^ju_{i-j},\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Id}\otimes\Delta)\Delta(u_i)&=(\mathrm{Id}\otimes\Delta)(\sum_{s=0}^{m-1}\gamma^{s(i-s)}u_s\otimes x^{-sd}g^su_{i-s})\\&=\sum_{s=0}^{m-1}\gamma^{s(i-s)}u_s\otimes(\sum_{t=0}^{m-1}\gamma^{t(i-s-t)}x^{-sd}g^su_t\otimes x^{-sd}g^sx^{-td}g^tu_{i-s-t})\\&=\sum_{s,t=0}^{m-1}\gamma^{s(i-s)+t(i-s-t)}u_s\otimes x^{-sd}g^su_t\otimes x^{-(s+t)d}g^{(s+t)}u_{i-s-t}.\end{aligned}$$

It is not hard to see that $(\Delta\otimes \mathrm{Id})\Delta(u_i)=(\mathrm{Id}\otimes\Delta)\Delta(u_i)$ for all $0\leqslant i\leqslant m-1$. The verification of $(\epsilon\otimes \mathrm{Id})\Delta(u_i)=(\mathrm{Id}\otimes\epsilon)\Delta(u_i)=u_i$ is easy and it is omitted.

- *Step* 3 (Antipode is an algebra anti-homomorphism).

  Because $x$ and $g$ are group-like elements, we only check

$$S(u_{i+1})S(\phi_i)=S(u_i)S(y)=\xi S(y)S(u_i)S(x^d)$$

and

$$S(u_iu_j)=S(u_j)S(u_i)$$

here.

(1) *The proof of* $S(u_{i+1})S(\phi_i)=S(u_i)S(y)=\xi S(y)S(u_i)S(x^d)$.

Since $u_iS(\phi_j)=\phi_ju_i$ for all $i,j$,

$$S(u_{i+1})S(\phi_i)=(-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d+\frac{3}{2}(1-m)d}g^{m-i-2}u_{i+1}S(\phi_i)=\phi_iS(u_{i+1}).$$

Through direct calculation, we have

$$\begin{aligned}
S(u_i)S(y) &= (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-i-1}u_i\cdot(-yg^{-1})\\
&= (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{i(i+1)}{2}}x^{(i-1)d+\frac{3}{2}(1-m)d}g^{m-i-1}yu_ig^{-1}\\
&= \phi_i\cdot(-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d+\frac{3}{2}(1-m)d}g^{m-i-2}u_{i+1}\\
&= \phi_iS(u_{i+1}),
\end{aligned}$$

and

$$\begin{aligned}
\xi S(y)S(u_i)S(x^d) &= -\xi yg^{-1}\cdot(-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-i-1}u_ix^{-d}\\
&= (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d+\frac{3}{2}(1-m)d}g^{m-i-2}yu_i\\
&= \phi_iS(u_{i+1}).
\end{aligned}$$

(2) *The proof of* $S(u_iu_j)=S(u_j)S(u_i)$.

Define $\overline{\phi_s}:=1-\gamma^{-s-1}x^{-d}$ for all $s\in\mathbb{Z}$. Using this notion,

$$x^d\overline{\phi_s}=x^d(1-\gamma^{-s-1}x^{-d})=-\gamma^{-s-1}(1-\gamma^{-(m-s-2)-1}x^d)=-\gamma^{-s-1}\phi_{m-s-2}.$$

And so

$$\begin{aligned}
S(u_iu_j) &= S((-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^{-\frac{1+m}{2}d}\ [i,m-2-j]\ y^{\overline{i+j}}g)\\
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}S(g)S(y^{i+j})S([i,m-2-j])S(x^{-\frac{1+m}{2}d})\\
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}g^{-1}(-yg^{-1})^{\overline{i+j}}S([i,m-2-j])x^{\frac{1+m}{2}d}\\
&= (-1)^{\overline{i+j}-j}\xi^{-j}\gamma^{\frac{j(j+1)+(\overline{i+j})(\overline{i+j}+1)}{2}}\frac{1}{m}x^{\frac{1+m}{2}d}S([i,m-2-j])y^{\overline{i+j}}g^{-(\overline{i+j}+1)}\\
&= (-1)^{m-1-j}\xi^{-j}\gamma^{\frac{j(j+1)+(\overline{i+j})(\overline{i+j}+1)+(m-1-\overline{i+j})(-m-2i+\overline{i+j})}{2}}\frac{1}{m}x^{\frac{1+m}{2}d-(m-1-\overline{i+j})d}\\
&\qquad [j,m-2-i]y^{\overline{i+j}}g^{-(\overline{i+j}+1)}\\
&= (-1)^{-j}\xi^{-j}\gamma^{\frac{j^2+j}{2}+i(\overline{i+j}+1)}\frac{1}{m}x^{\frac{1+m}{2}d-(m-1-\overline{i+j})d}\ [j,m-2-i]\ y^{\overline{i+j}}g^{-(\overline{i+j}+1)}\\
&= (-1)^{j}\xi^{-j}\gamma^{\frac{j^2+j}{2}+i(i+j+1)}\frac{1}{m}x^{\frac{3-m}{2}d+(i+j)d}\ [j,m-2-i]\ y^{\overline{i+j}}g^{-(i+j+1)},
\end{aligned}$$

$$
\begin{aligned}
& S(u_j)S(u_i) \\
&= (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} x^{jd+\frac{3}{2}(1-m)d} g^{m-j-1} u_j \cdot (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id+\frac{3}{2}(1-m)d} g^{m-i-1} u_i \\
&= (-1)^{i+j} \xi^{-i-j} \gamma^{-\frac{i(i+1)+j(j+1)}{2}} x^{(j-i)d} g^{m-j-1} u_j g^{m-i-1} u_i \\
&= (-1)^{i+j} \xi^{-i-j} \gamma^{-\frac{i(i+1)+j(j+1)}{2}-j(i+1)} x^{(i+j+2)d} g^{-(i+j+2)} u_j u_i \\
&= (-1)^{j} \xi^{-2i-j} \gamma^{-\frac{j(j+1)}{2}-j(i+1)+(i+j)(i+j+2)} \frac{1}{m} x^{-\frac{1+m}{2}d+(i+j+2)d} \; [i, m-2-j] \; y^{\overline{i+j}} g^{-(i+j+1)} \\
&= (-1)^{j} \xi^{-j} \gamma^{\frac{j^2+j}{2}+i(i+j+1)} \frac{1}{m} x^{\frac{3-m}{2}d+(i+j)d} \; [i, m-2-j] \; y^{\overline{i+j}} g^{-(i+j+1)}.
\end{aligned}
$$

The proof is done.

- *Step* 4 ($(S * \mathrm{Id})(u_i) = (\mathrm{Id} * S)(u_i) = \epsilon(u_i)$).

  In fact,

$$
\begin{aligned}
(S * \mathrm{Id})(u_0) &= \sum_{j=0}^{m-1} S(\gamma^{-j^2} u_j) x^{-jd} g^j u_{-j} \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} x^{jd+\frac{3}{2}(1-m)d} g^{m-j-1} u_j x^{-jd} g^j u_{-j} \\
&= x^{\frac{3}{2}(1-m)d} g^{m-1} \Big(\sum_{j=0}^{m-1} (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} u_j u_{-j}\Big) \\
&= x^{\frac{3}{2}(1-m)d} g^{m-1} \Big(\sum_{j=0}^{m-1} \gamma^{-j} \frac{1}{m} x^{-\frac{1+m}{2}d} \; [j, m-2+j] \; g\Big) \\
&= \frac{1}{m} x^{(1-m)d} \Big(\sum_{j=0}^{m-1} \gamma^{-j} \; ]j-1, j-1[ \;\Big) \\
&= 1 \quad (\text{ by Lemma 2.0.4}) \\
&= \epsilon(u_0).
\end{aligned}
$$

And,

$$
\begin{aligned}
(\mathrm{Id} * S)(u_0) &= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j S(x^{-jd} g^j u_{-j}) \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j S(u_{-j}) S(g^j) x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j (-1)^{-j} \xi^j \gamma^{\frac{j(-j+1)}{2}} x^{-jd+\frac{3}{2}(1-m)d} g^{m+j-1} u_{-j} g^{-j} x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{-j^2} u_j (-1)^{-j} \xi^j \gamma^{\frac{j-j^2}{2}+j^2} x^{\frac{3}{2}(1-m)d} g^{m-1} u_{-j} \\
&= x^{\frac{1-m}{2}d} g^{m-1} \sum_{j=0}^{m-1} (-1)^{-j} \xi^j \gamma^{-\frac{j^2+j}{2}} u_j u_{-j} \\
&= \frac{1}{m} \sum_{j=0}^{m-1} \xi^{2j} \gamma^{-j}\ [j, m-2+j] \\
&= \frac{1}{m} \cdot \sum_{j=0}^{m-1}\ ]j-1, j-1[ \\
&= 1 = \epsilon(u_0) \quad (\text{by the proof of Lemma 2.0.4})
\end{aligned}
$$

For $1 \leqslant i \leqslant m-1$,

$$
\begin{aligned}
(S * \mathrm{Id})(u_i) &= \sum_{j=0}^{m-1} \gamma^{j(i-j)} S(u_j) x^{-jd} g^j u_{i-j} \\
&= \sum_{j=0}^{m-1} \gamma^{j(i-j)} (-1)^j \xi^{-j} \gamma^{-\frac{j(j+1)}{2}} x^{jd+\frac{3}{2}(1-m)d} g^{m-1-j} u_j x^{-jd} g^j u_{i-j} \\
&= x^{\frac{3}{2}(1-m)d} g^{m-1} (\sum_{j=0}^{m-1} (-1)^j \xi^{-j} \gamma^{ij-\frac{j(j+1)}{2}} u_j u_{i-j}) \\
&= (-1)^{-i} \xi^{-i} \gamma^{\frac{i(i+3)}{2}} \frac{1}{m} x^{(1-m)d} y^i (\sum_{j=0}^{m-1} \gamma^{-j}\ [j, m-2-i+j]) \\
&= (-1)^{-i} \xi^{-i} \gamma^{\frac{i(i+3)}{2}} \frac{1}{m} x^{(1-m)d} y^i (\sum_{j=0}^{m-1} \gamma^{-j}\ ]j-1-i, j-1[) \\
&= 0 \quad (\text{by Lemma 2.0.7}) \\
&= \epsilon(u_i),
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{Id} * S)(u_i) &= \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j S(u_{i-j}) g^{-j} x^{jd} \\
&= \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j (-1)^{i-j} \xi^{j-i} \gamma^{-\frac{(i-j)(i-j+1)}{2}} x^{(i-j)d+\frac{3}{2}(1-m)d} g^{m+j-i-1} u_{i-j} g^{-j} x^{jd} \\
&= x^{id+\frac{1-m}{2}d} g^{m-1-i} (\sum_{j=0}^{m-1} (-1)^{i-j} \xi^{j-i} \gamma^{-\frac{i(i+1)+j(j+1)}{2}} u_j u_{i-j}) \\
&= \xi^{-2i} \gamma^{i(i+1)} \frac{1}{m} x^{(i-m)d} y^i g^{m-i} (\sum_{j=0}^{m-1} \xi^{2j} \gamma^{-(i+1)j} [j, m-2-i+j]) \\
&= \xi^{-2i} \gamma^{i(i+1)} \frac{1}{m} x^{(i-m)d} y^i g^{m-i} (\sum_{j=0}^{m-1} \gamma^{-ij} \ [j, m-2-i+j]) \\
&= \xi^{-2i} \gamma^{i(i+1)} \frac{1}{m} x^{(i-m)d} y^i g^{m-i} (\sum_{j=0}^{m-1} \gamma^{-ij} \ ]j-1-i, j-1[) \\
&= 0 \quad (\text{ by Lemma 2.0.7}) \\
&= \epsilon(u_i).
\end{aligned}
$$

By steps 1, 2, 3, 4, $D(m, d, \xi)$ is a Hopf algebra. □

## §3.3 Properties of $D(m, d, \xi)$

For short, let $D := D(m, d, \xi)$. For this Hopf algebra, the following observation is not hard.

**Lemma 3.3.1** *$D(m, d, \xi)$ is PI, affine and has GK-dimension one.*

***Proof.*** See Remark 3.1.1. □

**Lemma 3.3.2** gl.dim$D(m, d, \xi) = 1$.

***Proof.*** Now we find that $D = \bigoplus_{0 \leqslant i \leqslant 2m-1} D_i^l$ satisfies all conditions stated in [12, Proposition 5.1] and thus by [12, Proposition 5.1(a)] every nonzero homogeneous element is regular (and thus a nonzero-divisor). In particular, $y \in D_0^l$ is a regular element of $D$. It is not hard to see that $(y)$ is a Hopf ideal. Let $D' := D/(y)$. Then $D'$ is a finite dimensional Hopf algebra. We will show that $D'$ is semisimple at first. For notational convenience, the images of $x, g, u_i$ in $D'$ are still written as $x, g$ and $u_i$. One can check that

$$\int_{D'}^{l} := \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j + \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0$$

is a non-zero left integral of $D'$. Indeed, it is not hard to see that $x\int_{D'}^{l} = g\int_{D'}^{l} = \int_{D'}^{l}$ and the following relations

$$x^{md} = g^m = 1,\ u_j u_0 = 0,\ u_i = \gamma^{-i}x^d u_i,\ xu_i = u_i x^{-1},\ u_i g = \gamma^i x^{-2d} g u_i$$

$$u_0^2 = \frac{1}{m} x^{-\frac{1+m}{2}d}\phi_0\cdots\phi_{m-2}g$$

hold in $D'$ for all $0 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant m-1$. Here we only explain the equation $u_i = \gamma^{-i}x^d u_i$ since the others are clear. Since $(1-\gamma^{-i}x^d)u_i = \phi_{i-1}u_i = yu_{i-1} = 0$ in $D'$, we have $u_i = \gamma^{-i}x^d u_i$. Therefore,

$$\begin{aligned} u_0 \cdot \int_{D'}^{l} &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0 + \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0^2 \\ &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j u_0 + \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} x^i g^j \\ &= \epsilon(u_0)\cdot \int_{D'}^{l}, \end{aligned}$$

and

$$\begin{aligned} u_s \cdot \int_{D'}^{l} &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} \gamma^{sj} x^i g^j u_s \\ &= \sum_{i=0}^{md-1}\sum_{j=0}^{m-1} \gamma^{sj} x^i g^j \gamma^{-s} x^d u_s \\ &= \gamma^{-s}\sum_{i=0}^{md-1}\sum_{j=0}^{m-1} \gamma^{sj} x^i g^j u_s, \end{aligned}$$

which implies that $\sum\limits_{i=0}^{md-1}\sum\limits_{j=0}^{m-1} \gamma^{sj}x^i g^j u_s = 0$, and so $u_s \cdot \int_{D'}^{l} = 0 = \epsilon(u_s)\int_{D'}^{l}$ for all $1 \leqslant s \leqslant m-1$. Clearly, $\epsilon(\int_{D'}^{l}) = 2m^2 d \neq 0$. So $D'$ is semisimple.

Secondly, let $M$ be the trivial $D$-module $k$. By [23] or [10, Corollary 1.4], it is enough to show that $\mathrm{p.dim}_D M$=1. Since $y$ is a regular element (i.e., not a zero divisor) of $D$ and $yM = 0$, $M$ cannot be a submodule of a free $D$-module, which implies $\mathrm{p.dim}_D M \neq 0$. Since $D'$ is semisimple, we get $\mathrm{p.dim}_{D'} M = 0$. By standard method of "change of rings" (see e.g., [20, Lemma 5.26]), $\mathrm{p.dim}_D M$=1. □

**Lemma 3.3.3** $\mathrm{io}(D) = 2m, \mathrm{im}(D) = m.$

***Proof.*** By the proof of Lemma 3.3.2 $D'$ is semisimple, then $D'$ is unimodular. Note that the left and right homological integrals agree with the classical left and right integrals respectively when the Hopf algebra is of finite dimensional. So $D'$ is also unimodular for homological integrals, that is, the left homological integral of $D'$, also denoted by $\int_{D'}^l$, is isomorphic to $k$ as $D'$-bimodule. Thus we have $\int_{D'}^l = D'/(x-1, g-1, u_0-1, u_1, \cdots, u_{m-1})$. Using [24, Lemma 2.6],

$$\int_D^l = (\int_{D'}^l)^{\tau^{-1}} = D/(y, x-1, g-\gamma^{-1}, u_0-\xi^{-1}, u_1, u_2, \cdots, u_{m-1}),$$

where $\tau$ is the algebra automorphism of $D$ such that $yh = \tau(h)y$ for all $h \in D$. The corresponding homomorphism $\pi$ yields left and right winding automorphisms

$$\Xi_\pi^l : \begin{cases} x \longmapsto x, \\ y \longmapsto y, \\ g \longmapsto \gamma^{-1}g, \\ u_i \longmapsto \xi^{-1}u_i, \end{cases} \quad \text{and} \quad \Xi_\pi^r : \begin{cases} x \longmapsto x, \\ y \longmapsto \gamma^{-1}y. \\ g \longmapsto \gamma^{-1}g, \\ u_i \longmapsto \xi^{-(2i+1)}u_i. \end{cases}$$

Clearly these automorphisms have order $2m$ and $G_\pi^l \cap G_\pi^r = \langle (\Xi_\pi^l)^m \rangle$. Then we have $\mathrm{io}(D) = 2m$ and $\mathrm{im}(D) = m$ by $|G_\pi^l \cap G_\pi^r| = 2$. □

**Remark 3.3.4** *These winding automorphisms are just the automorphisms constructed in Section 3.1 and the corresponding gradings on D have been exhibited there.*

**Lemma 3.3.5** $D(m, d, \xi)$ *is prime.*

***Proof.*** We know that $G_\pi^l$ is a finite abelian group acting faithfully on $D(m,d,\xi)$ (see the proof of Lemma 3.3.3). Moreover, the $\widehat{G_\pi^l}$-grading is strong and $D_0^l = k[x^{\pm 1}, y]$ is a commutative domain, which shows that $D$ meets the initial conditions of [12, Proposition 5.1]. It follows that PI-deg$(D) \leqslant \text{io}(D) = 2m$. Since $D$ is regular, we have io$(D) \leqslant$ PI-deg$(D/P_0)$ by [24, Lemma 5.3], where $P_0$ is the minimal prime ideal of $D$ contained in Ker$(\epsilon)$. It is clear that PI-deg $(D/P_0) \leqslant$ PI-deg$(D)$. So PI-deg$(D) =$ io$(D) = 2m$ and thus [12, Proposition 5.1 (d)] applied. Therefore $D$ is prime. □

The next proposition is a direct consequence of Lemmas 3.3.1, 3.3.2, 3.3.3 and 3.3.5.

**Proposition 3.3.6** *$D(m,d,\xi)$ is an affine prime regular Hopf algebra of GK-dimension one with* io$(D) = 2m$, im$(D) = m$.

**Proposition 3.3.7** *$D$ is not pointed.*

***Proof.*** Let $f: D \to D' = D/(y)$ be the canonical Hopf epimorphism. We need to show that $u_i \neq 0$ in $D'$ for all $i$ firstly. Since $y$ is a normal element of $D$, $u_i = 0$ in $D'$ forces $u_i = yz_i$ for some $z_i$ in $D$. By the bigrading structure of $D$ and the fact the $y$ is nonzero-divisor, one can see that $z_i = \alpha_i u_{i-1}$ for some $\alpha_i \in D_{00} = k[x^{\pm 1}]$. Therefore,

$$u_i = yz_i = y\alpha_i u_{i-1} = \alpha_i y u_{i-1} = \alpha_i \phi_{i-1} u_i,$$

which contradicts the fact that $\phi_{i-1} = 1 - \gamma^{-i}x^d$ is not invertible in $[x^{\pm 1}]$. Secondly, by [26, Corollary 5.3.5], $D'$ is pointed if $D$ is pointed. We will show that $D'$ is not pointed. Otherwise, it is semisimple and pointed and so it is cosemisimple and pointed by [21]. This implies that $D'$ is cocommutative which is absurd. Therefore, $D$ is not pointed. □

A direct consequence of this proposition is that $D(m,d,\xi)$ is not isomorphic to any one of Hopf algebras listed in Section 1.5.

**Corollary 3.3.8** *As a Hopf algebra, $D(m,d,\xi)$ is not isomorphic to any one of algebras listed in Section 1.5.*

***Proof.*** All of Hopf algebras given in Section 1.5 are pointed while $D(m,d,\xi)$ is not. □

**Remark 3.3.9** *In practice, the assumption "pointed" is always added when we want to classify Hopf algebras of lower GK-dimensions. As a matter of fact, all known examples are pointed and it is widely believed that, at least for prime regular Hopf algerbas of GK-dimension one, these Hopf algebras should be pointed automatically. Our new examples will change this naive understanding since all the new examples are not pointed!*

# Chapter 4 Classification results

In this chapter, $H$ always denotes a prime regular Hopf algebra of GK-dimension one with $\mathrm{io}(H) = n > \mathrm{im}(H) = m > 1$ and let $t := n/m$.

In Section 4.1, the definition of the Hopf subalgebra $\widetilde{H}$ and some basic properties of $\widetilde{H}$ are given. In particular, we show that $\widetilde{H}$ is a prime regular Hopf algebra satisfying the condition $(*)$. Thus $\widetilde{H}$ is either primitive or group-like. Then we can reconstruct $H$ from $\widetilde{H}$. In details, the general relations between $\widetilde{H}$ and $H$ are built in Section 4.2. Section 4.3 is designed to analyse the primitive case and $H$ is shown to be an infinite dimensional Taft algebra. We consider the group-like case in the section 4.4, and as the desired conclusion, we show that $H$ is isomorphic to some $D(m, d, \xi)$. Finally, the classification result, and its proof are also formulated in this section.

## §4.1 The Hopf subalgebra $\widetilde{H}$

The aim of this section is to construct a Hopf subalgebra $\widetilde{H}$ of $H$ satisfying the conditions of Lemma 1.5.2.

Recall that the letter $\zeta$ denotes the primitive $n$th root of 1 in the definition of the bigrading of $H$ given in Section 1.4:

$$H = \bigoplus_{0 \leqslant i,j \leqslant n-1} H_{ij},$$

where $H_{ij} = H_i^l \cap H_j^r$.

It is not hard to find that [12, Lemma 6.3] still holds in our case and so we state it without proof.

**Lemma 4.1.1** *Let* $H = \bigoplus_{0 \leqslant i,j \leqslant n-1} H_{ij}$. *Then*

(a) $S(H_i^l) = H_{-i}^r$ *and* $S(H_{ij}) = H_{-j,-i}$ (*where the suffixes are interpreted* mod $n$).

(b) *If* $i \neq j$, *then* $\epsilon(H_{ij}) = 0$.

(c) $\epsilon(H_{ii}) \neq 0$.

**Lemma 4.1.2** *Let* $\Xi_\pi^l$ *and* $\Xi_\pi^r$ *be the left and right winding automorphisms defined as in Section 1.4 respectively. Then* $(\Xi_\pi^l)^m = (\Xi_\pi^r)^m$.

***Proof.*** Since $G_\pi^l$ and $G_\pi^r$ are cyclic groups, $G_\pi^l \cap G_\pi^r$ is a cyclic subgroup of both $G_\pi^l$ and $G_\pi^r$. By $\mathrm{im}(H) = |G_\pi^l / G_\pi^l \cap G_\pi^r| = m$, the order of $G_\pi^l \cap G_\pi^r$ is $t$. But the order $t$ cyclic subgroup of $G_\pi^l$ and $G_\pi^r$ is just $\langle (\Xi_\pi^l)^m \rangle$ and $\langle (\Xi_\pi^r)^m \rangle$. So there exists an integer $s, 1 \leqslant s \leqslant t-1$, such that $(\Xi_\pi^l)^m = (\Xi_\pi^r)^{sm}$. Now we show that $s = 1$. Since $\epsilon(H_{11}) \neq 0$ by Lemma 4.1.1, we have $H_{11} \neq 0$. Then there exists $0 \neq x \in H_{11} = \{x \in H | \Xi_\pi^l(x) = \zeta x, \Xi_\pi^r(x) = \zeta x\}$, so

$$\zeta^m x = (\Xi_\pi^l)^m(x) = (\Xi_\pi^r)^{sm}(x) = \zeta^{sm} x.$$

Therefore $\zeta^m = \zeta^{sm}$, and hence $s = 1$. □

**Lemma 4.1.3** *For every $j$ with $1 \leqslant j \leqslant n-1$, $H_{0j} \neq 0$ if and only if $j \equiv 0 \pmod t$ for all $0 \leqslant j \leqslant n-1$.*

***Proof.*** Let $0 \neq x \in H_{0j}$. Note that $H_{0j} = \{x \in H | \Xi_\pi^l(x) = x, \Xi_\pi^r(x) = \zeta^j x\}$. By the above lemma, we have

$$x = (\Xi_\pi^l)^m(x) = (\Xi_\pi^r)^m(x) = \zeta^{jm} x.$$

This implies $\zeta^{jm} = 1$. So we get $j \equiv 0 \pmod t$. That is, if $j \not\equiv 0 \pmod t$ then $H_{0j} = 0$. Therefore we can write

$$H_0^l = \bigoplus_{0 \leqslant j \leqslant m-1} H_{0,jt}.$$

Now it remains to show that each $H_{0,jt} \neq 0$ for all $0 \leqslant j \leqslant m-1$.

Let $s$ be the minimum positive integer such that $H_{0,st} \neq 0$. Such $s$ must exist. Indeed, if there is no such $s$, that is, $H_{0,jt} = 0$ for all $1 \leqslant j \leqslant m-1$. Then $H_0^l = H_0$. Dually, $H_0^r = H_0 = H_0^l$. Therefore $H_0^l$ is a Hopf subalgebra of $H$ by Lemma 1.4.5 again. By the proofs of [12, Propositions 4.2 and 4.3], we see that this can happen only in the following cases: $H$ is isomorphic as a Hopf algebra either to the Taft algebra $H(n, 0, \xi)$ or to the infinite dihedral group algebra $k\mathbb{D}$. In either case, $\mathrm{im}(H) = 1$. This contradicts the hypothesis $\mathrm{im}(H) > 1$. Next, we show that $s$ must be a factor of $m$, that is, $s|m$. If not, there exists a positive number $a$ such that $sa < m$ and $s(a+1) > m$. Since $H_0^l$ is a domain, $0 \neq (H_{0,st})^{a+1} \subset H_{0,st(a+1)-mt}$ and thus $s(a+1) - m < s$ is a smaller number such that $H_{0,(s(a+1)-m)t} \neq 0$ which contradicts to the minimality of

$s$. Therefore, $H_0^l = \bigoplus_{0\leqslant j\leqslant \frac{m}{s}-1} H_{0,jst}$ (by using the minimality of $s$ again) and thus to show the result it is enough to show that $s = 1$.

We claim that

$$H = \bigoplus_{0\leqslant i\leqslant n-1, 0\leqslant j\leqslant \frac{m}{s}-1} H_{i,i+jst}. \qquad (\diamond)$$

At first, since $H_{ii} \neq 0$ for all $i$ by Lemma 4.1.1 and every nonzero homogeneous element is nonzero-divisor by [12, Proposition 5.1(a)], we have $H_{i,i+jst} \supseteq H_{ii}H_{0,jst} \neq 0$ for all $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant \frac{m}{s} - 1$. Secondly, we show that $H_{ik} \neq 0$ if and only if $st|(i-k)$. We already know the "if" part and we only need to show the "only if" part. In fact, suppose $H_{ik} \neq 0$ and let $0 \neq x \in H_{ik} = \{h \in H | \Xi_\pi^l(h) = \zeta^i h, \Xi_\pi^r(h) = \zeta^k h\}$. Applying Lemma 4.1.2, we have

$$\zeta^{im} x = (\Xi_\pi^l)^m(x) = (\Xi_\pi^r)^m(x) = \zeta^{km} x.$$

That is, $\zeta^{im} = \zeta^{km}$ and so $i - k \equiv 0 \pmod t$. So it is harmless to assume that $k = i + at$ for $a$ a positive number and we need to show that $s|a$. Applying [12, Proposition 5.1(a)] again, $H_{0,at} \supseteq H_{i,i+at}H_{n-i,n-i} \neq 0$ and thus $s|a$ since we already know that $H_0^l = \bigoplus_{0\leqslant j\leqslant \frac{m}{s}-1} H_{0,jst}$. This proves the equation $(\diamond)$. By the equation $(\diamond)$, it is not hard to see that $\mathrm{im}(H) = m/s$. Therefore, $s = 1$. □

The proof of this lemma indeed implies the following result.

**Proposition 4.1.4** *For every $i, j$ with $1 \leqslant i, j \leqslant n-1$, $H_{ij} \neq 0$ if and only if $i - j \equiv 0 \pmod t$ for all $0 \leqslant i, j \leqslant n-1$ .*

This proposition also implies that

$$H_{it}^l = \bigoplus_{0\leqslant j\leqslant m-1} H_{it,jt} \quad \text{and} \quad H_{jt}^r = \bigoplus_{0\leqslant i\leqslant m-1} H_{it,jt}$$

for all $0 \leqslant i, j \leqslant m-1$. Thus we can define $\widetilde{H}$ as follows:

$$\widetilde{H} := \bigoplus_{0\leqslant i,j\leqslant m-1} H_{it,jt} = \bigoplus_{0\leqslant i\leqslant m-1} H_{it}^l = \bigoplus_{0\leqslant j\leqslant m-1} H_{jt}^r. \qquad (4.1)$$

**Lemma 4.1.5** *$\widetilde{H}$ is a Hopf subalgebra of $H$.*

***Proof.*** We need to verify the algebra structure, coalgebra structure and the antipode condition of $\widetilde{H}$. Clearly, by the bigraded structure of $H$, it is easy to see that $\widetilde{H}$ is an algebra. Since each $H_i^l$ is a right coideal of $H$ and $H_j^r$ is a left coideal of $H$ for all $0 \leqslant i, j \leqslant n-1$ (by Lemma 1.4.5), $\Delta(H_{it,jt}) \subseteq \sum_{k,l} H_{kt}^l \otimes H_{lt}^r \subseteq \widetilde{H} \otimes \widetilde{H}$. So $\widetilde{H}$ is a coalgebra. Note that $S(H_{it,jt}) = H_{-jt,-it} \subset \widetilde{H}$ by Lemma 4.1.1, thus $\widetilde{H}$ is a Hopf subalgebra of $H$. □

**Lemma 4.1.6** *$\widetilde{H}$ is regular and* gl.dim$(\widetilde{H}) = 1$.

***Proof.*** Deducing from the fact that $H = \bigoplus_{0 \leqslant i \leqslant n-1} H_i^l = \bigoplus_{0 \leqslant j \leqslant n-1} H_j^r$ is a strongly graded algebra, we get a new grading for $H$:

$$H = \bigoplus_{0 \leqslant i \leqslant t-1} \widetilde{H} H_i^l = \bigoplus_{0 \leqslant i \leqslant t-1} \widetilde{H}(H_1^l)^i.$$

Since $H_1^l$ is an affine invertible $H_0^l$-bimodule, $\widetilde{H} H_i^l$ is an affine invertible $\widetilde{H}$-bimodule. Therefore, $H$ is a projective $\widetilde{H}$-module and $\widetilde{H}$ is a direct summand of $H$ as an $\widetilde{H}$-module. Thus, if $V$ is any $\widetilde{H}$-module, then

$$\text{p.dim}_{\widetilde{H}}(V) \leqslant \text{p.dim}_{\widetilde{H}}(H \otimes_{\widetilde{H}} V) \leqslant \text{p.dim}_H(H \otimes_{\widetilde{H}} V),$$

where the second inequality holds because $H$ is a projective $\widetilde{H}$-module. Therefore, gl.dim$(\widetilde{H}) \leqslant 1$. Since $\widetilde{H}$ is not semisimple (otherwise, it is finite dimensional), gl.dim$\widetilde{H} = 1$. □

**Lemma 4.1.7** *$\widetilde{H}$ is prime and PI-deg$(\widetilde{H})$=$m$.*

***Proof.*** Note that $\widetilde{H} = \bigoplus_{0 \leqslant i \leqslant m-1} H_{it}^l$. Therefore, the lemma follows directly from [12, Corollary 5.1 (b)]. □

**Proposition 4.1.8** *Let $\widetilde{H}$ be the algebra constructed as* (4.1). *Then it is isomorphic as a Hopf algebra either to the Taft algebra $H(m, 1, \xi)$ or to the generalized Liu algebra $B(m, \omega, \gamma)$. As a consequence,* io$(\widetilde{H})$ = im$(\widetilde{H}) = m$.

***Proof.*** By the above three lemmas, $\widetilde{H}$ is an affine prime regular Hopf algebra, and it is clear that $\widetilde{H}$ is of GK-dimension one. Denote the restriction of the actions of $\Xi_\pi^l$ and $\Xi_\pi^r$ to $\widetilde{H}$ by $\Gamma^l$ and $\Gamma^r$, respectively. Since $\widetilde{H} = \bigoplus_{0 \leqslant i \leqslant m-1} H_{it}^l$, we can see that for each $0 \leqslant i \leqslant m-1$ and any $0 \neq x \in H_{it}^l$,

$$(\Gamma^l)^m(x) = \zeta^{itm} x = x.$$

This implies that the group $\langle \Gamma^l \rangle$ has order $m$. Similarly, $|\langle \Gamma^r \rangle| = m$. We claim that

$$\langle \Gamma^l \rangle \cap \langle \Gamma^r \rangle = 1.$$

In fact, if $(\Gamma^l)^i = (\Gamma^r)^j$ for some $0 \leqslant i, j \leqslant m-1$. Choose $0 \neq x \in H_{tt}$, we find

$$\zeta^{ti} x = (\Gamma^l)^i(x) = (\Gamma^r)^j(x) = \zeta^{tj} x$$

which implies $i = j$. Let $0 \neq y \in H_{0,t}$, then

$$y = (\Gamma^l)^i(y) = (\Gamma^r)^j(y) = \zeta^{tj} y$$

forces $j = 0$. Thus we get $i = j = 0$, i.e., $\langle \Gamma^l \rangle \cap \langle \Gamma^r \rangle = 1$. Therefore, Lemma 1.5.2 is applied. So, $\mathrm{io}(\widetilde{H}) = \mathrm{im}(\widetilde{H}) = m$ and $\widetilde{H}$ is isomorphic as a Hopf algebra either to the Taft algebra $H(m, 1, \xi)$ or to the generalized Liu algebra $B(m, \omega, \gamma)$. □

## §4.2 From $\widetilde{H}$ to $H$

This section is to build some general relations between $\widetilde{H}$ and $H$. Our final aim is to show that $\widetilde{H}$ can determine the structures of $H$ entirely. We start with the following definition.

**Definition 4.2.1** *We call $H$ is* primitive (*respectively,* group-like) *if $\widetilde{H}$ is primitive, i.e., $\widetilde{H} \cong H(m, 1, \xi)$ (respectively, if $\widetilde{H}$ is group-like, i.e., $\widetilde{H} \cong B(m, \omega, \gamma)$).*

By Proposition 4.1.8, $H$ is either primitive or group-like. And $\widetilde{H} = \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt}$ is bigraded automatically by definition. Meanwhile, $\widetilde{H}$ has left integrals $\int_{\widetilde{H}}^l$ and from which we also can construct another bigrading on $\widetilde{H}$ just as in Section 1.4.

**Lemma 4.2.2** *Under a suitable choice of $\zeta$, these two bigradings on $\widetilde{H}$ are the same.*

***Proof.*** The proof is indeed implicit in the arguments given in [12, Section 6]. For completeness, we prove it here. We start with a general form. Let $A$ be a prime regular Hopf algebra of GK-dimension one with an algebra homomorphism $\mu : A \to k$ (note that $\mu$ need not be the algebra homomorphism $\pi : A \to A/\text{r.ann}(\int_A^l))$. Denote the groups of the left and right winding automorphisms $\Xi_\mu^l$ and $\Xi_\mu^r$ associated to $\mu : A \to k$ by $G_\mu^l$ and $G_\mu^r$ respectively. Fix a primitive $n$th root $\zeta$ of 1, and define $\chi \in \widehat{G_\mu^l}$ and $\eta \in \widehat{G_\mu^r}$ by setting

$$\chi(\Xi_\mu^l) = \zeta \quad \text{and} \quad \eta(\Xi_\mu^r) = \zeta.$$

Assume that $G_\mu^l$ and $G_\mu^r$ satisfy

$$|G_\mu^l| = \text{PI-deg}(A) = n \quad \text{and} \quad G_\mu^l \cap G_\mu^r = \{1\} \qquad (\star).$$

Then there is a bigrading on

$$A = \bigoplus_{0 \leqslant i,j \leqslant n-1} A_{ij},$$

where $A_{ij} = \{a \in A | \Xi_\mu^l(a) = \zeta^i a,\ \Xi_\mu^r(a) = \zeta^j a\}$. It is proved in [12, Section 6] that $A \cong H(n, 1, \xi)$ or $A \cong B(n, \omega, \gamma)$ and under a suitable choice of $\zeta$, $A_{ij}$ is exactly $H(n, 1, \xi)_{ij}$ or $B(n, \omega, \gamma)_{ij}$, where $H(n, 1, \xi)_{ij}$ and $B(n, \omega, \gamma)_{ij}$ are homogeneous components of the bigrading constructed in Section 1.4. Now the proof is completed by noting that the algebra homomorphism $\mu = \pi|_{\widetilde{H}} : \widetilde{H} \to k$ (where $\pi : H \to H/\text{r.ann}(\int_H^l))$ induces the left and right winding automorphisms satisfying the condition $(\star)$ by Proposition 4.1.8. □

**Remark 4.2.3** *By Lemma 4.2.2, we can freely use the calculation results of $H_{ij}$ to*

$$\widetilde{H} := \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt}$$

*as in Section 1.5. That is, if $H$ is primitive, then we can assume that $H_{it,jt} = k[x^m]x^{j-i}g^i$; and if $H$ is group-like, then we can take that $H_{it,jt} = k[x^{\pm 1}]y^{j-i}g^i$.*

Now we denote the fraction field of $H_0^l$ by $Q_0^l$. Brown and Zhang [12, Section 5] showed that there is a delicate $\widehat{G_\pi^l}$-action on $Q_0^l$ defined as follows: For each $\chi^i \in \widehat{G_\pi^l}$ and $a \in Q_0^l$,

$$\kappa_i(a) := u_i a u_i^{-1}.$$

Here, to avoid confusion, we denote the automorphism of $Q_0^l$ corresponding to $\chi^i \in \widehat{G_\pi^l}$ by $\kappa_i$. And since $H_i^l$ is an invertible $H_0^l$-module, $Q_0^l H_i^l = Q_0^l u_i$ for any non-zero element $u_i \in H_i^l$. Brown and Zhang proved that this action is independent of the choice of the non-zero element $u_i$ and clearly satisfies

$$xa = \kappa_i(a)x \tag{4.2}$$

for $x \in H_i^l$ and $a \in H_0^l$. Furthermore, [12, Proposition 5.1 (b)] implies that the restriction of $\kappa_i$ to $H_0^l$ is still an automorphism and we denote this restriction by $\kappa_i^l$ for convenience. In a special situation, the following lemma is implicit in [12, Subsection 6.2].

**Lemma 4.2.4** *$K^l := \{\kappa_i^l\}_{0 \leqslant i \leqslant n-1}$ is a cyclic group.*

***Proof.*** We only need to show $\kappa_i^l = (\kappa_1^l)^i$ for all $i$. Let $x_1, x_1'$ be any non-zero elements of $H_1^l$, for any $a \in H_0^l$,

$$x_1 x_1' a = x_1 \kappa_1^l(a) x_1' = \kappa_1^l(\kappa_1^l(a)) x_1 x_1'.$$

Applying the strongly graded property of $H$, it follows that $\kappa_2^l = (\kappa_1^l)^2$. Similarly, $\kappa_i^l = (\kappa_1^l)^i$ for all $2 \leqslant i \leqslant n-1$. So $K^l$ is a cyclic group. □

Meanwhile, $G_\pi^r$ acts on $H_0^l$ as explained in Section 1.4. We denote by $\rho^l$ the resulting map from $G_\pi^r$ to $\mathrm{Aut}(H_0^l)$. Let $P_\pi^l := \langle \rho^l(G_\pi^r), K^l \rangle \subseteq \mathrm{Aut}(H_0^l)$. By [12, Proposition 5.2], $P_\pi^l$ is abelian and $P_\pi^l = K^l$. So the cyclic group $\rho^l(G_\pi^r)$ is a subgroup of $K^l$. By [12, Proposition 5.1], $Z(H) \subseteq H_0^l$. Similarly, $Z(H) \subseteq H_0^r$. Therefore, $Z(H) \subseteq H_0$ and thus $H_0$ is a $Z(H)$-module naturally. One of our key observations is the following lemma.

**Lemma 4.2.5** *$H_0$ is a torsion-free $Z(H)$-module of rank at least $t$.*

***Proof.*** Since PI-deg$(H) = \mathrm{io}(H) = n$, $K := \{\kappa_i\}_{0 \leqslant i \leqslant n-1}$ acts faithfully on $Q_0^l$ by [12, Proposition 5.1]. So it is easy to see that $K^l$ acts faithfully on $H_0^l$. Then we have a $\widehat{K^l}$-grading on $H_0^l$:

$$H_0^l = \bigoplus_{\chi^i \in \widehat{K^l}} (H_0^l)_i^\kappa,$$

where $(H_0^l)_i^\kappa = \{x \in H_0^l \mid g(x) = \chi^i(g)x, \forall g \in K^l\}$.

Note that the grading of $H_0^l$ induced by the action of $\rho^l(G_\pi^r)$ is just $H_0^l = \bigoplus_{0\leqslant i\leqslant m-1} H_{0,it}$. Since $\rho^l(G_\pi^r)$ is a subgroup of the cyclic group $K^l$, $H_0^l = \bigoplus_{\chi^i \in \widehat{K^l}} (H_0^l)_i^\kappa$ is a refinement of $H_0^l = \bigoplus_{0\leqslant i\leqslant m-1} H_{0,it}$. Therefore

$$H_0 = (H_0^l)_0^\kappa \bigoplus (H_0^l)_m^\kappa \bigoplus \cdots \bigoplus (H_0^l)_{(t-1)m}^\kappa.$$

Because of the faithful action of $K^l$ on $H_0^l$, each $(H_0^l)_i^\kappa$ is torsion-free on $(H_0^l)_0^\kappa$ of rank at least one. By [12, Proposition 5.1], $(H_0^l)_0^\kappa = Z(H)$. Thus $H_0$ is torsion-free over $Z(H)$ of rank at least $t$. □

With this observation, we have the following conclusion.

**Proposition 4.2.6** *Each homogeneous component $H_{i,i+jt}$ of $H = \bigoplus_{0\leqslant i\leqslant n-1, 0\leqslant j\leqslant m-1} H_{i,i+jt}$ is a free $H_0$-module of rank one on both sides.*

***Proof.*** We work with left modules. By [12, Proposition 5.1], each $H_i^l$ is a torsion-free $H_0^l$-module. So the rank of $H_i^l$ on $H_0^l$ is at least one. By Proposition 4.1.4, each homogeneous $H_{i,i+jt}$ is a non-zero $H_0$-module, then it is torsion-free of rank at least one. Since $\widetilde{H}$ is isomorphic as a Hopf algebra either to the Taft algebra $H(m, 1, \xi)$ or to the generalized Liu algebra $B(m, \omega, \gamma)$, $H_0$ is isomorphic either to $k[x^m]$ or to $k[x^{\pm 1}]$. Both of them are principal ideal domains, so each $H_{i,i+jt}$ is free on $H_0$ of rank at least one which implies the rank of $H$ over $H_0$ is not smaller than $nm$. By Lemma 4.2.5, the rank of $H_0$ over $Z(H)$ is at least $t$. Therefore,

$$\mathrm{rank}_{Z(H)} H \geqslant nmt = n^2.$$

Recall that the PI-degree of $H$ is $n$ and equals the square root of the rank of $H$ over $Z(H)$. So the rank of $H_0$ over $Z(H)$ is $t$ and each $H_{i,i+jt}$ is a free $H_0$-module of rank one. □

Since $H_0 = k[x^m]$ or $H_0 = k[x^{\pm 1}]$, there is a generating set $\{u_{i,i+jt} | 0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant m-1\}$ of these $H_{i,i+jt}$ satisfying

$$u_{00} = 1 \quad \text{and} \quad H_{i,i+jt} = u_{i,i+jt} H_0 = H_0 u_{i,i+jt}.$$

So, $H$ can be written as

$$H = \bigoplus_{\substack{0\leqslant i\leqslant n-1 \\ 0\leqslant j\leqslant m-1}} H_0 u_{i,i+jt} = \bigoplus_{\substack{0\leqslant i\leqslant n-1 \\ 0\leqslant j\leqslant m-1}} u_{i,i+jt} H_0. \tag{4.3}$$

With these preparations, we can analyse the structure of $H$ further by dichotomy now: either $\widetilde{H}$ is $H(m,1,\xi)$, the primitive case; or $\widetilde{H}$ is $B(m,\omega,\gamma)$, the group-like case.

## §4.3 Primitive case

If $H$ is primitive, $\widetilde{H} = H(m,1,\xi)$. We will prove $H$ is isomorphic as a Hopf algebra to $H(n,t,\theta)$, for $\theta$ some primitive $n$th root of 1. Recall that

$$\widetilde{H} = H(m,1,\xi) = k\langle g, x | g^m = 1, xg = \xi gx\rangle.$$

Note that $H = \bigoplus_{0\leqslant i\leqslant n-1, 0\leqslant j\leqslant m-1} H_{i,i+jt}$, $\widetilde{H} = \bigoplus_{0\leqslant i,j\leqslant m-1} H_{it,jt}$ and $H_{it,jt} = k[x^m]x^{j-i}g^i$ (the index $j-i$ is interpreted mod $m$). In particular, $H_0 = k[x^m]$, $H_{0,jt} = k[x^m]x^j$ and $H_{tt} = k[x^m]g$.

By Lemma 4.1.1, $\epsilon(u_{11}) \neq 0$. Multiplied with a suitable scalar, we can assume that $\epsilon(u_{11}) = 1$ throughout this section.

**Lemma 4.3.1** *Let $u := u_{11}$. Then $H_1^l = H_0^l u$, $H = \bigoplus_{0\leqslant i\leqslant t-1} \widetilde{H} u^i$ and $u$ is invertible.*

***Proof.*** By the bigraded structure of $H$, we have

$$H_{0t}H_{11} \subseteq H_{1,1+t}, \quad H_{0,(m-1)t}H_{1,1+t} \subseteq H_{11},$$

which imply

$$H_{0t}H_{0,(m-1)t}H_{1,1+t} \subseteq H_{0t}H_{11} \subseteq H_{1,1+t}.$$

Since $H_{0t}H_{0,(m-1)t} = x^m H_0$ is a maximal ideal of $H_0 = k[x^m]$ and $H_{1,1+t}$ is a free $H_0$-module of rank one (by Proposition 4.2.6 ), $H_{0t}H_{0,(m-1)t}H_{1,1+t}$ is a maximal $H_0$-submodule of $H_{1,1+t}$. Thus $H_{0t}H_{11} = H_{0t}H_{0,(m-1)t}H_{1,1+t} = x^m H_{1,1+t}$ or $H_{0t}H_{11} = H_{1,1+t}$.

If $H_{0t}H_{11} = x^m H_{1,1+t}$, then $xu_{11} = x^m\alpha(x^m)u_{1,1+t}$ for some polynomial $\alpha(x^m) \in k[x^m]$. So

$$x(u_{11} - x^{m-1}\alpha(x^m)u_{1,1+t}) = 0.$$

Therefore, $x^m(u_{11} - x^{m-1}\alpha(x^m)u_{1,1+t}) = 0$. Note that each homogenous $H_{i,i+jt}$ is a torsion-free $H_0$-module, so

$$u_{11} = x^{m-1}\alpha(x^m)u_{1,1+t}.$$

By assumption, $\epsilon(u_{11}) = 1$. But, by definition, $\epsilon(x) = 0$. This is impossible. So $H_{0t}H_{11} = H_{1,1+t}$ which implies that $H_{0,t}u_{11} = H_{1,1+t}$.

Similarly, we can show that $H_{0,it}u_{11} = H_{1,1+it}$ for $0 \leqslant i \leqslant m-1$. Thus $H_1^l = H_0^l u_{11}$. Since $H = \bigoplus_{0\leqslant i\leqslant n-1} H_i^l$ is strongly graded, $u_{11}$ is invertible and $H_i^l = H_0^l u_{11}^i$ for all $0 \leqslant i \leqslant n-1$. Let $u := u_{11}$, then we have

$$H = \bigoplus_{0\leqslant i\leqslant t-1} \widetilde{H}u^i.$$

□

We are in a position to determine the structure of $H$ now.

**Proposition 4.3.2** *With above notations, we have*

$$u^t = g, \quad xu = \theta ux,$$

*where $\theta$ is a primitive $n$th root of* 1.

***Proof.*** By $H_{0t}u = uH_{0t}$, there exists a polynomial $\beta(x^m) \in k[x^m]$ such that

$$xu = ux\beta(x^m).$$

Then

$$xu^t = u^t x\beta'(x^m)$$

for some polynomial $\beta'(x^m) \in k[x^m]$ induced by $\beta(x^m)$. Since $u^t$ is invertible and $u^t \in H_{t,t} = k[x^m]g$, $u^t = ag$ for $0 \neq a \in k$. By assumption, $\epsilon(u) = 1$ and thus $a = 1$. Therefore, $u^t = g$. Since $xg = \xi gx$, we have $\beta'(x^m) = \xi$. Then it is easy to see that $\beta(x^m) = \theta \in k$ with $\theta^t = \xi$. Of course, $\theta^n = 1$.

The last job is to show that $\theta$ is a primitive $n$th root of 1. Indeed, assume $\theta$ is a primitive $n'$th root of 1. Then $Z(H) = k[x^{n'}]$, the center of $H$. Recall that the PI-degree of $H$ equals the square root of the rank of $H$ over $Z(H)$. So the equalities

$$n^2 = nm\,\mathrm{rank}_{Z(H)}H_0 = nmn'/m$$

hold. That is, $n' = n$ and $\theta$ is a primitive $n$th root of 1. □

**Proposition 4.3.3** *The element $u$ is a group-like element of $H$.*

***Proof.*** First of all $H_0^r \cong k[x] \cong H_0^l$. Then $H_0^r \otimes H_0^l \cong k[x,y]$ and the only invertible elements in $H_0^r \otimes H_0^l$ are nonzero scalars in $k$. Since $\Delta(u)$ and $u \otimes u$ are invertible, $\Delta(u)(u \otimes u)^{-1}$ is invertible (and hence a scalar). Thus $u$ must be grouplike by noting that $\epsilon(u) = 1$. □

The next theorem follows from Lemma 4.3.1 and Propositions 4.3.2, 4.3.3 directly.

**Theorem 4.3.4** *Let $H$ be an affine prime regular Hopf algebra of GK-dimension one satisfying $\mathrm{io}(H) = n > \mathrm{im}(H) = m > 1$. If $H$ is primitive, then $H$ is isomorphic as a Hopf algebra to an infinite dimensional Taft algebra.*

## §4.4 Group-like case

If $H$ is group-like, then $\widetilde{H} = B(m, \omega, \gamma)$ for $\gamma$ a primitive $m$th root of 1 and $\omega$ a positive integer. As usual, the generators of $B(m, \omega, \gamma)$ are denoted by $x^{\pm 1}, y$ and $g$. By Remark 4.2.3, we can assume that $\widetilde{H} = \bigoplus_{0 \leqslant i,j \leqslant m-1} H_{it,jt}$ with

$$H_{it,jt} = k[x^{\pm 1}]y^{j-i}g^i$$

(the index $j - i$ is interpreted mod $m$). In particular, $H_0 = k[x^{\pm 1}]$, $H_{0,jt} = k[x^{\pm 1}]y^j$ and $H_{t,t} = k[x^{\pm 1}]g$.

Set $u_i := u_{1,1+it} (0 \leqslant i \leqslant m-1)$ for convenience. By the structure of the bigrading of $H$, we have

$$yu_0 = \phi_0 u_1,\ yu_1 = \phi_1 u_2, \cdots, yu_{m-2} = \phi_{m-2} u_{m-1},\ yu_{m-1} = \phi_{m-1} u_0 \qquad (4.4)$$

and

$$u_0 y = \varphi_0 u_1,\ u_1 y = \varphi_1 u_2, \cdots, u_{m-2} y = \varphi_{m-2} u_{m-1},\ u_{m-1} y = \varphi_{m-1} u_0 \qquad (4.5)$$

for some polynomials $\phi_i, \varphi_i \in k[x^{\pm 1}], 0 \leqslant i \leqslant m-1$. With these notions and the equality $y^m = 1 - x^\omega$, we find that

$$(1 - x^\omega) u_0 = y^m u_0 = \phi_0 \phi_1 \cdots \phi_{m-1} u_0 \qquad (4.6)$$

and

$$u_0(1-x^{\omega}) = u_0 y^m = \varphi_0\varphi_1\cdots\varphi_{m-1}u_0. \tag{4.7}$$

**Proposition 4.4.1** *There is no group-like affine prime regular Hopf algebra of GK-dimension one $H$ satisfying $\mathrm{io}(H) = n > \mathrm{im}(H) = m > 1$ and $n/m > 2$.*

***Proof.*** Since $u_iH_0 = H_0u_i$, we have

$$u_ix = \alpha_i(x^{\pm 1})u_i \quad \text{and} \quad u_ix^{-1} = \beta_i(x^{\pm 1})u_i$$

for some $\alpha_i(x^{\pm 1}), \beta_i(x^{\pm 1}) \in k[x^{\pm 1}]$. From

$$u_i = u_ixx^{-1} = \alpha_i(x^{\pm 1})u_ix^{-1} = \alpha_i(x^{\pm 1})\beta_i(x^{\pm 1})u_i,$$

we get $\alpha_i(x^{\pm 1})\beta_i(x^{\pm 1}) = 1$ and thus $\alpha_i(x^{\pm 1}) = \lambda_i x^{a_i}$ for some $0 \neq \lambda_i \in k, 0 \neq a_i \in \mathbb{Z}$.

Note that $u_i^t \in H_{t,(1+it)t} = k[x^{\pm 1}]y^{\bar{it}}g$, where $\bar{it} \equiv it \pmod m$. So we have $u_i^t = \gamma_i(x^{\pm 1})y^{\bar{it}}g$ for some $\gamma_i(x^{\pm 1}) \in k[x^{\pm 1}]$. Hence $u_i^t$ commutes with $x$. Applying $u_ix = \lambda_i x^{a_i}u_i$ to $u_i^tx = xu_i^t$, we get $\lambda_i^{\sum_{s=0}^{t-1} a_i^s} = 1$ and $x^{a_i^t} = x$. If $t$ is odd, $a_i = 1$ and if $t$ is even, then $a_i$ is either 1 or $-1$.

Now we consider the special case $i = 0$. By $\epsilon(xu_0) = \epsilon(u_0x) \neq 0$, we find that $\lambda_0 = 1$.

If $a_0 = 1$, that is $u_0x = xu_0$, then we will see $u_ix = xu_i$ for all $0 \leqslant i \leqslant m-1$. In fact, by

$$\phi_0xu_1 = x\phi_0u_1 = xyu_0 = yxu_0 = yu_0x = \phi_0u_1x,$$

we have $u_1x = xu_1$ since $H_{1,1+t}$ is a torsion-free $H_0$-module. Similarly, $u_ix = xu_i$ for all $0 \leqslant i \leqslant m-1$. Then by the strongly graded structure $u_{i,i+jt} \in H_i^l = (H_1^l)^i$ and $x$ is commutative with $H_1^l$, it is not hard to see that $u_{i,i+jt}x = xu_{i,i+jt}$ for all $0 \leqslant i \leqslant n-1, 0 \leqslant j \leqslant m-1$. Therefore the center $Z(H) \supseteq H_0 = k[x^{\pm 1}]$. By [12, Lemma 5.2], $Z(H) \subseteq H_0$ and thus $Z(H) = H_0 = k[x^{\pm 1}]$. This implies that

$$\mathrm{rank}_{Z(H)}H = \mathrm{rank}_{H_0}H = nm < n^2,$$

which contradicts the fact: the PI-degree of $H$ is $n$ and equals the square root of the rank of $H$ over $Z(H)$.

If $a_0 = -1$, that is $u_0 x = x^{-1} u_0$, we can deduce that $u_{i,i+jt} x = x^{-1} u_{i,i+jt}$ for all $0 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m-1$ by using the parallel proof of the case $a_0 = 1$. For $s \in \mathbb{N}$, let $z_s := x^s + x^{-s}$. Define $k[z_s | s \geq 0]$ to be the subalgebra of $k[x^{\pm 1}]$ generated by all $z_s$. Note that $k[x^{\pm 1}]$ has rank 2 over $k[z_s | s \geqslant 1]$. Thus $Z(H) \supseteq k[z_s | s \geq 0]$. Using [12, Lemma 5.2] again, we have $Z(H) = k[z_s | s \geq 0]$. Hence

$$\text{rank}_{Z(H)} H = 2nm \neq n^2$$

since $n/m > 2$ by assumption. This contradicts the fact that the PI-deg$H = n$ again.

Combining these two cases, we get the desired result. □

We turn now to consider the case: $\text{io}(H) = 2\,\text{im}(H) = 2m$. In this case, $t = 2$. As discussed in the proof of Proposition 4.4.1, if such $H$ exists then the following relations

$$u_i x = x^{-1} u_i \quad (0 \leqslant i \leqslant m-1) \tag{4.8}$$

hold in $H$. Using these relations and (4.6), we have

$$\varphi_0 \cdots \varphi_{m-1} = 1 - x^{-\omega}. \tag{4.9}$$

To determine the structure of $H$, we need to give some harmless assumptions on the choice of $u_i$ $(0 \leqslant i \leqslant m-1)$: (1) We assume $\epsilon(u_0) = 1$; (2) Let $\xi_s := e^{\frac{2s\pi i}{\omega}}$ and thus $1 - x^\omega = \Pi_{s \in S}(1 - \xi_s x)$, where $S := \{0, 1, \cdots, \omega - 1\}$. Since

$$\phi_0 \cdots \phi_{m-1} = y^m = 1 - x^\omega \tag{4.10}$$

by (4.6), we have

$$\phi_i = k_i x^{c_i} \Pi_{s \in S_i}(1 - \xi_s x),$$

where $k_i \in k$, $c_i$ is an integer and $S_i$ is a subset of $S$. The second assumption is: Take the $u_i$'s such that $\phi_i = \prod_{s \in S_i}(1 - \xi_s x)$. Due to Equation (4.10), this assumption can be realized; (3) By the strongly graded structure of $H$, the equality $H_2^l = H_0^l g$ and the fact that $g$ is invertible in $H$, we can take $u_{j,j+2i}$ such that

$$u_{j,j+2i} = \begin{cases} g^{\frac{j-1}{2}} u_i & \text{if } j \text{ is odd,} \\ y^i g^{\frac{j}{2}} & \text{if } j \text{ is even,} \end{cases}$$

for all $2 \leqslant j \leqslant 2m-1$. In the rest of this section, we always make these assumptions.

We still need two notations, which appeared in the proof of Proposition 3.2.1. For a polynomial $f = \sum a_i x^{b_i} \in k[x^{\pm 1}]$, we denote by $\bar{f}$ the polynomial $\sum a_i x^{-b_i}$. Then by (4.8), we have $fu_i = u_i\bar{f}$ and $u_i f = \bar{f}u_i$ for all $0 \leqslant i \leqslant m-1$. For any $h \in H \otimes H$, we use

$$h_{(s_1,t_1)\otimes(s_2,t_2)}$$

to denote the homogeneous part of $h$ in $H_{s_1,t_1} \otimes H_{s_2,t_2}$. Both these notations will be used frequently in the proof of the next proposition.

**Proposition 4.4.2** *Let $H$ be a prime regular Hopf algebra with $\widetilde{H} = B(m,\omega,\gamma)$. Assume that $\mathrm{io}(H) = 2\,\mathrm{im}(H) > 2$, then as a Hopf algebra,*

$$H \cong D(m,d,\xi)$$

*constructed as in Section 3.1, where $m$ divides $\omega$ and 2 divides $d(m+1)$.*

*Proof.* We divide the proof into several steps.

*Claim 1. We have $m|\omega$ and $yu_i = \phi_i u_{i+1} = \xi x^d u_i y$ for $d = \frac{\omega}{m}$ and some $\xi \in k$ satisfying $\xi^m = -1$.*

*Proof of Claim 1:* By associativity of the multiplication, we have many equalities:

$$\begin{aligned} yu_0y^{m-1} &= \phi_0\varphi_1\varphi_2\cdots\varphi_{m-1}u_0 \\ &= \varphi_0\phi_1\varphi_2\cdots\varphi_{m-1}u_0 \\ &\cdots \\ &= \varphi_0\varphi_1\varphi_2\cdots\phi_{m-1}u_0, \end{aligned}$$

which imply that $\phi_i\varphi_j = \varphi_i\phi_j$ for all $0 \leqslant i,j \leqslant m-1$. Using associativity again, we have

$$
\begin{aligned}
y^m u_0 y^{m(m-1)} &= (1-x^\omega)u_0(1-x^\omega)^{m-1} = -x^\omega(1-x^{-\omega})^m u_0 \\
&= -x^\omega(\varphi_0\varphi_1\varphi_2\cdots\varphi_{m-1})^m u_0 \\
&= (\phi_0\varphi_1\varphi_2\cdots\varphi_{m-1})^m u_0 \\
&= (\varphi_0\phi_1\varphi_2\cdots\varphi_{m-1})^m u_0 \\
&\cdots \\
&= (\varphi_0\varphi_1\varphi_2\cdots\phi_{m-1})^m u_0,
\end{aligned}
$$

where the fourth "=", for example, is gotten in the following way: We multiply $u_0$ by one $y$ from left side at first, then multiply it with $y^{m-1}$ from right side, then continue the procedures above. From these equalities, we have

$$\phi_i^m = -x^\omega \varphi_i^m$$

for all $0 \leqslant i \leqslant m-1$. So $\phi_i = \xi_i x^d \varphi_i$ where $d = \frac{\omega}{m}$ and $\xi_i \in k$ satisfying $\xi_i^m = -1$. Applying $\phi_i\varphi_j = \varphi_i\phi_j$, we can see $\xi_i = \xi_j$ for all $0 \leqslant i, j \leqslant m-1$, and we write $\xi := \xi_i$. Therefore we have $yu_i = \xi x^d u_i y$, where $d = \omega/m$ is an integer. □

In the following of the proof, $d$ is fixed to be the number $\omega/m$.

*Claim 2. We have $u_i g = \lambda_i x^{-2d} g u_i$ for $\lambda_i = \pm\gamma^i$ and $0 \leqslant i \leqslant m-1$.*

*Proof of Claim 2:* Since $g$ is invertible in $H$, $u_i g = \psi_i g u_i$ for some invertible $\psi_i \in k[x^{\pm 1}]$. Then $u_i g^m = \psi_i^m g^m u_i$ yields $\psi_i^m = x^{-2\omega}$. So $\psi_i = \lambda_i x^{-2d}$ for $\lambda_i \in k$ with $\lambda_i^m = 1$. Our last task is to show that $\lambda_i = \pm\gamma^i$. To show this, we need a preparation, that is, we need to show that $u_i u_j \neq 0$ for all $i, j$. Otherwise, assume that there exist $i_0, j_0 \in \{0, \dots, m-1\}$ such that $u_{i_0} u_{j_0} = 0$. Using Claim 1, we can find that $u_{i_0} u_j \equiv 0$ and $u_i u_{j_0} \equiv 0$ for all $i, j$. Let $(u_{i_0})$ and $(u_{j_0})$ be the ideals generated by $u_{i_0}$ and $u_{j_0}$ respectively. Then it is not hard to find that $(u_{i_0})(u_{j_0}) = 0$ which contradicts $H$ being prime. So we always have

$$u_i u_j \neq 0 \tag{4.11}$$

for all $0 \leqslant i, j \leqslant m-1$.

Applying this observation, we have $0 \neq u_i^2 \in H_{2,2+4i} = k[x^{\pm 1}]y^{2i}g$, $u_i^2 g = \psi_i \overline{\psi_i} g u_i^2 = \gamma^{2i} g u_i^2$. Thus $\psi_i = \pm\gamma^i x^{-2d}$ which implies that $\lambda_i = \pm\gamma^i$. The proof is ended. □

We can say more about $\lambda_i$ at this stage. By $0 \neq u_i u_j g = \gamma^{i+j} g u_i u_j$, we know that $\psi_i = \gamma^i x^{-2d}$ for all $i$ or $\psi_i = -\gamma^i x^{-2d}$ for all $i$. So

$$\lambda_i = \gamma^i \quad \text{or} \quad \lambda_i = -\gamma^i \tag{4.12}$$

for all $0 \leqslant i \leqslant m-1$. In fact, we will show that $\psi_i = \gamma^i x^{-2d}$ for all $i$ later.

*Claim 3. For each* $0 \leqslant i \leqslant m-1$, *there are* $f_{ij}, h_{ij} \in k[x^{\pm 1}]$ *with* $h_{ij}$ *monic such that*

$$\Delta(u_i) = \sum_{j=0}^{m-1} f_{ij} u_j \otimes h_{ij} g^j u_{i-j}, \tag{4.13}$$

*where the following* $i-j$ *is interpreted* mod $m$.

*Proof of Claim 3:* Since $u_i \in H_{1,1+2i}$, $\Delta(u_i) \in H_1^l \otimes H_{1+2i}^r$. Noting that $H_1^l = \bigoplus_{j=0}^{m-1} H_0 u_j$ and $H_{1+2i}^r = \bigoplus_{s=0}^{m-1} H_0 g^s u_{i-s}$, we can write

$$\Delta(u_i) = \sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (u_j \otimes g^s u_{i-s}),$$

where $F_{js}^i \in H_0 \otimes H_0$. Then we divide the proof into two steps.

• *Step* 1 ($\Delta(u_i) = \sum_{0 \leqslant j \leqslant m-1} F_{jj}^i (u_j \otimes g^j u_{i-j})$).

Recall that $u_i g = \lambda_i x^{-2d} g u_i$, where $\lambda_i$ is either $\gamma^i$ for all $i$ or $-\gamma^i$ for all $i$. The equations

$$\begin{aligned}\Delta(u_i g) = \Delta(u_i)\Delta(g) &= \sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (u_j \otimes g^s u_{i-s})(g \otimes g) \\ &= \sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (\lambda_j x^{-2d} g u_j \otimes \lambda_{i-s} x^{-2d} g^{s+1} u_{i-s}) \\ &= \sum_{0 \leqslant j,s \leqslant m-1} \lambda_j \lambda_{i-s} (x^{-2d} g \otimes x^{-2d} g) F_{js}^i (u_j \otimes g^s u_{i-s})\end{aligned}$$

and

$$\begin{aligned}\Delta(\lambda_i x^{-2d} g u_i) &= \lambda_i (x^{-2d} g \otimes x^{-2d} g)\Delta(u_i)\Delta(g) \sum_{0 \leqslant j,s \leqslant m-1} F_{js}^i (u_j \otimes g^s u_{i-s}) \\ &= \sum_{0 \leqslant j,s \leqslant m-1} \lambda_i (x^{-2d} g \otimes x^{-2d} g) F_{js}^i (u_j \otimes g^s u_{i-s})\end{aligned}$$

imply that $\lambda_i = \lambda_j \lambda_{i-s}$ for all $j, s$. If $\lambda_i = -\gamma^i$ for all $i$, then we have $-\gamma^i = \lambda_i = \lambda_j \lambda_{i-s} = \gamma^{j+i-s}$. This implies $j = s \pm m/2$. Applying $(\epsilon \otimes \mathrm{Id})$ to $\Delta(u_i)$,

$$(\epsilon \otimes \mathrm{Id})\Delta(u_i) = (\epsilon \otimes \mathrm{Id})(F_{0,\, m/2}^i) g^{m/2} u_{i-m/2} \neq u_i,$$

which is absurd. If $\lambda_i = \gamma^i$ for all $i$, then $\gamma^i = \lambda_i = \lambda_j \lambda_{i-s} = \gamma^{j+i-s}$. This implies $j = s$ (which is compatible with the equality $(\epsilon \otimes \mathrm{Id})\Delta(u_i) = u_i$). So we get $F^i_{js} \neq 0$ only if $j = s$ and $\lambda_i = \gamma^i$ for all $i$. Thus we have $\Delta(u_i) = \sum_{0 \leqslant j \leqslant m-1} F^i_{jj}(u_j \otimes g^j u_{i-j})$ for all $i$.

• *Step* 2 (There exist $f_{ij}, h_{ij} \in H_0$ with $h_{ij}$ monic such that $F^i_{jj} = f_{ij} \otimes h_{ij}$ for $0 \leqslant i, j \leqslant m-1$).

We replace $F^i_{jj}$ by $F^i_j$ for convenience. Since

$$\begin{aligned}(\Delta \otimes \mathrm{Id})\Delta(u_i) &= (\Delta \otimes \mathrm{Id})(\sum_{0 \leqslant j \leqslant m-1} F^i_j(u_j \otimes g^j u_{i-j}))\\ &= \sum_{0 \leqslant j \leqslant m-1} (\Delta \otimes \mathrm{Id})(F^i_j)(\sum_{0 \leqslant s \leqslant m-1} F^j_s(u_s \otimes g^s u_{j-s}) \otimes g^j u_{i-j})\\ &= \sum_{0 \leqslant j,s \leqslant m-1} (\Delta \otimes \mathrm{Id})(F^i_j)(F^j_s \otimes 1)(u_s \otimes g^s u_{j-s} \otimes g^j u_{i-j})\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)\Delta(u_i) &= (\mathrm{Id} \otimes \Delta)(\sum_{0 \leqslant j \leqslant m-1} F^i_j(u_j \otimes g^j u_{i-j}))\\ &= \sum_{0 \leqslant j \leqslant m-1} (\mathrm{Id} \otimes \Delta)(F^i_j)(u_j \otimes (\sum_{0 \leqslant s \leqslant m-1} F^{i-j}_s(g^j u_s \otimes g^{j+s} u_{i-j-s}))\\ &= \sum_{0 \leqslant j,s \leqslant m-1} (\mathrm{Id} \otimes \Delta)(F^i_s)(1 \otimes F^{i-s}_{j-s})(u_s \otimes g^s u_{j-s} \otimes g^j u_{i-j}),\end{aligned}$$

we have

$$(\Delta \otimes \mathrm{Id})(F^i_j)(F^j_s \otimes 1) = (\mathrm{Id} \otimes \Delta)(F^i_s)(1 \otimes F^{i-s}_{j-s}) \tag{4.14}$$

for all $0 \leqslant i, j, s \leqslant m-1$.

Begin with the case $i = j = s = 0$. Let $F^0_0 = \sum_{p,q} k_{pq} x^p \otimes x^q$. Comparing equation

$$\begin{aligned}(\Delta \otimes \mathrm{Id})(F^0_0)(F^0_0 \otimes 1) &= (\sum_{p,q} k_{pq} x^p \otimes x^p \otimes x^q)(\sum_{p',q'} k_{p'q'} x^{p'} \otimes x^{q'} \otimes 1)\\ &= (\sum_{p,q,p',q'} k_{pq} k_{p'q'} x^{p+p'} \otimes x^{p+q'} \otimes x^q)\end{aligned}$$

and equation

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)(F^0_0)(1 \otimes F^0_0) &= (\sum_{p,q} k_{pq} x^p \otimes x^q \otimes x^q)(\sum_{p',q'} k_{p'q'} 1 \otimes x^{p'} \otimes x^{q'})\\ &= (\sum_{p,q,p',q'} k_{pq} k_{p'q'} x^p \otimes x^{q+p'} \otimes x^{q+q'}),\end{aligned}$$

one can see that $p = q = 0$ by comparing the degrees of $x$ in these two expressions. Then $F_0^0 = 1 \otimes 1$ by applying $(\epsilon \otimes \mathrm{Id})\Delta$ to $u_0$. Next, consider the case $j = s = 0$. Write $F_0^i = \sum_{p,q} k_{pq} x^p \otimes x^q$. Similarly, we have $F_0^i = x^{a_i} \otimes 1$ for some $a_i \in \mathbb{Z}$ by the equation

$$(\Delta \otimes \mathrm{Id})(F_0^i)(F_0^0 \otimes 1) = (\mathrm{Id} \otimes \Delta)(F_0^i)(1 \otimes F_0^i).$$

Finally, write $F_j^i = \sum_{p,q} k_{pq} x^p \otimes x^q$ and consider the case $s = 0$. Let $F_0^i = x^{a_i} \otimes 1$ and $F_0^j = x^{a_j} \otimes 1$. The equation

$$(\sum_{p,q} k_{pq} x^{p+a_j} \otimes x^p \otimes x^q) = (\Delta \otimes \mathrm{Id})(F_j^i)(F_0^j \otimes 1) = (\mathrm{Id} \otimes \Delta)(F_0^i)(1 \otimes F_j^i) = (\sum_{p,q} k_{pq} x^{a_i} \otimes x^p \otimes x^q)$$

shows that $p = a_i - a_j$, that is, $F_j^i = x^{c_{ij}} \otimes \beta_{ij}$ some $c_{ij} \in \mathbb{Z}, \beta_{ij} \in H_0$.

By steps 1 and 2, $F_j^i$ can be written as $f_{ij} \otimes h_{ij}$ with $h_{ij}$ monic after multiplying suitable scalar, where $f_{ij}, h_{ij} \in k[x^{\pm 1}]$. That is,

$$\Delta(u_i) = \sum_{j=0}^{m-1} f_{ij} u_j \otimes h_{ij} g^j u_{i-j},$$

where $f_{ij}, h_{ij} \in k[x^{\pm 1}]$ with $h_{ij}$ monic. □

Since $\lambda_i = \gamma^i$ for all $i$ has been shown above, we can improve Claim 2 as

*Claim 2'. We have* $u_i g = \gamma^i x^{-2d} g u_i$ *for* $0 \leqslant i \leqslant m-1$.

By Claim 2', we have a unified formula in $H$: For all $s \in \mathbb{Z}$,

$$u_i g^s = \gamma^{is} x^{-2sd} g^s u_i. \tag{4.15}$$

*Claim 4. We have* $\phi_i = 1 - \gamma^{-i-1} x^d$ *for* $0 \leqslant i \leqslant m-1$.

*Proof of Claim 4:* By Claim 3, there are polynomials $f_{0j}, h_{0j}, f_{1j}, h_{1j}$, such that

$$\Delta(u_0) = u_0 \otimes u_0 + f_{01} u_1 \otimes h_{01} g u_{m-1} + \cdots + f_{0,m-1} u_{m-1} \otimes h_{0,m-1} g^{m-1} u_1,$$

$$\Delta(u_1) = f_{10} u_0 \otimes u_1 + u_1 \otimes h_{11} g u_0 + \cdots + f_{1,m-1} u_{m-1} \otimes h_{1,m-1} g^{m-1} u_2.$$

Firstly, we will show $\phi_0 = 1 - \gamma^{-1} x^d$ by considering the equations

$$\Delta(y u_0)_{11 \otimes 13} = \Delta(\xi x^d u_0 y)_{11 \otimes 13} = \Delta(\phi_0 u_1)_{11 \otimes 13}.$$

Direct computations show that

$$\begin{aligned}\Delta(yu_0)_{11\otimes 13} &= u_0 \otimes yu_0 + yf_{0,m-1}u_{m-1} \otimes gh_{0,m-1}g^{m-1}u_1 \\ &= u_0 \otimes \phi_0 u_1 + f_{0,m-1}\phi_{m-1}u_0 \otimes x^{md}h_{0,m-1}u_1, \\ \Delta(\xi x^d u_0 y)_{11\otimes 13} &= \xi x^d u_0 \otimes x^d u_0 y + \xi x^d f_{0,m-1}u_{m-1}y \otimes x^d h_{0,m-1}g^{m-1}u_1 g \\ &= x^d u_0 \otimes \phi_0 u_1 + f_{0,m-1}\phi_{m-1}u_0 \otimes \gamma x^{(m-1)d}h_{0,m-1}u_1.\end{aligned}$$

Owing to $\Delta(yu_0)_{11\otimes 13} = \Delta(\xi x^d u_0 y)_{11\otimes 13}$,

$$(1-x^d)u_0 \otimes \phi_0 u_1 + f_{0,m-1}\phi_{m-1}u_0 \otimes (x^d-\gamma)x^{(m-1)d}h_{0,m-1}u_1 = 0.$$

Thus we can assume $\phi_0 = c_0(x^d-\gamma)x^{(m-1)d}h_{0,m-1}$ for some $0 \neq c_0 \in k$. Then $1-x^d = -c_0^{-1}f_{0,m-1}\phi_{m-1}$. Therefore,

$$\begin{aligned}\Delta(yu_0)_{11\otimes 13} &= u_0 \otimes \phi_0 u_1 - c_0(1-x^d)u_0 \otimes \frac{1}{c_0}\frac{x^d}{x^d-\gamma}\phi_0 u_1 \\ &= u_0 \otimes (1-\frac{x^d}{x^d-\gamma})\phi_0 u_1 + x^d u_0 \otimes \frac{x^d}{x^d-\gamma}\phi_0 u_1 \\ &= u_0 \otimes (\frac{-\gamma}{x^d-\gamma})\phi_0 u_1 + x^d u_0 \otimes \frac{x^d}{x^d-\gamma}\phi_0 u_1,\end{aligned}$$

where $\frac{1}{x^d-\gamma}\phi_0$ is understood as $c_0 x^{(m-1)d}h_{0,m-1}$. Note that $\Delta(\phi_0 u_1)_{11\otimes 13} = \Delta(\phi_0)(f_{10}u_0 \otimes u_1)$. Comparing the first components of $\Delta(yu_0)_{11\otimes 13}$ and $\Delta(\phi_0 u_1)_{11\otimes 13}$, we get $\phi_0 = 1+\theta x^d$ for some $\theta \in k$. Then it is not hard to see that $f_{10} = 1, f_{0,m-1} = \gamma^{-1}, h_{0,m-1} = x^{-(m-1)d}$ and $\theta = -\gamma^{-1}$. So $\phi_0 = 1-\gamma^{-1}x^d$.

Secondly, we want to determine $\phi_s$ for $s \geqslant 1$. To attack this, we will prove the fact

$$f_{j0} = h_{j0} = 1 \tag{4.16}$$

for all $0 \leqslant j \leqslant m-1$ at the same time. We proceed by induction. We already know that $f_{00} = h_{00} = f_{10} = h_{10} = 1$. Assume that $f_{i,0} = h_{i,0} = 1$ now. Similarly, direct computations show that

$$\begin{aligned}\Delta(yu_i)_{11\otimes(1,3+2i)} &= u_0 \otimes yu_i + yf_{i,m-1}u_{m-1} \otimes gh_{i,m-1}g^{m-1}u_{i+1} \\ &= u_0 \otimes \phi_i u_{i+1} + f_{i,m-1}\phi_{m-1}u_0 \otimes x^{md}h_{i,m-1}u_{i+1}, \\ \Delta(\xi x^d u_i y)_{11\otimes(1,3+2i)} &= \xi x^d u_0 \otimes x^d u_i y + \xi x^d f_{i,m-1}u_{m-1}y \otimes x^d h_{i,m-1}g^{m-1}u_{i+1}g \\ &= x^d u_0 \otimes \phi_i u_{i+1} + f_{i,m-1}\phi_{m-1}u_0 \otimes \gamma^{i+1}x^{(m-1)d}h_{i,m-1}u_{i+1}.\end{aligned}$$

By $\Delta(yu_i)_{11\otimes(1,3+2i)} = \Delta(\xi x^d u_i y)_{11\otimes(1,3+2i)}$,

$$(1-x^d)u_0 \otimes \phi_i u_{i+1} + f_{i,m-1}\phi_{m-1}u_0 \otimes (x^d - \gamma^{i+1})x^{(m-1)d}h_{i,m-1}u_{i+1} = 0.$$

Thus we can assume $\phi_i = c_i(x^d - \gamma^{i+1})x^{(m-1)d}h_{i,m-1}$ for some $0 \neq c_i \in k$. Then $1 - x^d = -c_i^{-1}f_{i,m-1}\phi_{m-1}$. Therefore

$$\begin{aligned}\Delta(yu_i)_{11\otimes(1,3+2i)} &= u_0 \otimes \phi_i u_{i+1} - c_i(1-x^d)u_0 \otimes \frac{1}{c_i}\frac{x^d}{x^d-\gamma^{i+1}}\phi_i u_{i+1} \\ &= u_0 \otimes (\frac{-\gamma^{i+1}}{x^d-\gamma^{i+1}})\phi_i u_{i+1} + x^d u_0 \otimes \frac{x^d}{x^d-\gamma^{i+1}}\phi_i u_{i+1}.\end{aligned}$$

Note that $\Delta(\phi_i u_{i+1})_{11\otimes(1,3+2i)} = \Delta(\phi_i)(f_{i+1,0}u_0 \otimes h_{i+1,0}u_{i+1})$. Comparing the first components of $\Delta(yu_i)_{11\otimes(1,3+2i)}$ and $\Delta(\phi_i u_{i+1})_{11\otimes(1,3+2i)}$, we get $\phi_i = 1 - \gamma^{-i-1}x^d$ similarly. And it is not hard to see that $f_{i+1,0} = h_{i+1,0} = 1$ and $f_{i,m-1} = \gamma^{-i-1}, h_{i,m-1} = x^{-(m-1)d}$. So we prove that $f_{i+1,0} = h_{i+1,0} = 1$ at the same time. □

*Claim 5. The coproduct of $H$ is given by*

$$\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)}u_j \otimes x^{-jd}g^j u_{i-j}$$

*for $0 \leqslant i \leqslant m-1$.*

*Proof of Claim 5:* By Claim 3, $\Delta(u_i) = \sum_{j=0}^{m-1} f_{ij}u_j \otimes h_{ij}g^j u_{i-j}$. So, to show this claim, it is enough to determine the explicit form of every $f_{ij}$ and $h_{ij}$. By (4.16), $f_{i,0} = h_{i,0} = 1$. We will prove that $f_{ij} = \gamma^{j(i-j)}$ and $h_{ij} = x^{-jd}$ for all $0 \leqslant i, j \leqslant m-1$ by induction. So it is enough to show that $f_{i,j+1} = \gamma^{(j+1)(i-j-1)}$ and $h_{i,j+1} = x^{-(j+1)d}$ under the hypothesis of $f_{ij} = \gamma^{j(i-j)}$ and $h_{ij} = x^{-jd}$. In fact,

$$\begin{aligned}\Delta(yu_i)_{(1,3+2j)\otimes(3+2j,3+2i)} &= yf_{ij}u_j \otimes gh_{ij}g^j u_{i-j} + f_{i,j+1}u_{j+1} \otimes yh_{i,j+1}g^{j+1}u_{i-j-1} \\ &= f_{ij}yu_j \otimes h_{ij}g^{j+1}u_{i-j} + f_{i,j+1}u_{j+1} \otimes \gamma^{j+1}h_{i,j+1}g^{j+1}yu_{i-j-1}, \\ \Delta(\xi x^d u_i y)_{(1,3+2j)\otimes(3+2j,3+2i)} &= \xi x^d f_{ij}u_j y \otimes x^d h_{ij}g^j u_{i-j}g + \xi x^d f_{i,j+1}u_{j+1} \otimes x^d h_{i,j+1}g^{j+1}u_{i-j-1}y \\ &= f_{ij}yu_j \otimes \gamma^{i-j}x^{-d}h_{ij}g^{j+1}u_{i-j} + x^d f_{i,j+1}u_{j+1} \otimes h_{i,j+1}g^{j+1}yu_{i-j-1}.\end{aligned}$$

By $\Delta(yu_i)_{(1,3+2j)\otimes(3+2j,3+2i)} = \Delta(\xi x^d u_i y)_{(1,3+2j)\otimes(3+2j,3+2i)}$,

$$f_{ij}yu_j \otimes (1-\gamma^{i-j}x^{-d})h_{ij}g^{j+1}u_{i-j} = (x^d - \gamma^{j+1})f_{i,j+1}u_{j+1} \otimes h_{i,j+1}g^{j+1}yu_{i-j-1}.$$

By induction, we have

$$\begin{aligned}&\gamma^{j(i-j)}(1-\gamma^{-j-1}x^d)u_{j+1}\otimes(x^d-\gamma^{i-j})x^{-(j+1)d}g^{j+1}u_{i-j}\\&=(x^d-\gamma^{j+1})f_{i,j+1}u_{j+1}\otimes(1-\gamma^{-i+j}x^d)h_{i,j+1}g^{j+1}u_{i-j}.\end{aligned}$$

This implies that $h_{i,j+1}=x^{-(j+1)d}$ and $f_{i,j+1}=\gamma^{i-j}\gamma^{-j-1}\gamma^{j(i-j)}=\gamma^{(j+1)(i-j-1)}$. $\square$

*Claim 6. For $0\leqslant i,j\leqslant m-1$, the multiplication between $u_i$ and $u_j$ satisfies that*

$$u_iu_j=(-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\;[i,m-2-j]\;y^{\overline{i+j}}g$$

*for some $a\in\mathbb{Z}$ and $i+j$ is interpreted* mod $m$.

*Proof of Claim 6:* We need to consider the relation between $u_0^2$ and $u_ju_{m-j}$ for all $1\leqslant j\leqslant m-1$ at first.

By definition, $x^d\overline{\phi_s}=-\gamma^{-s-1}\phi_{m-s-2}$ for all $s$. Then

$$\begin{aligned}y^mu_0^2&=\xi^{m-j}x^{(m-j)d}y^ju_0y^{m-j}u_0=\xi^{m-j}x^{(m-j)d}\phi_0\cdots\phi_{j-1}u_j\phi_0\cdots\phi_{m-j-1}u_{m-j}\\&=\xi^{m-j}x^{(m-j)d}\phi_0\cdots\phi_{j-1}\cdot\overline{\phi_0}\cdots\overline{\phi_{m-j-1}}u_ju_{m-j}\\&=(-1)^{m-j}\xi^{m-j}\gamma^{-\frac{(m-j)(m-j+1)}{2}}\phi_0\cdots\phi_{m-2}\cdot\phi_{j-1}u_ju_{m-j}.\end{aligned}$$

So

$$\phi_{m-1}u_0^2=(-1)^{m-j}\xi^{m-j}\gamma^{-\frac{(m-j)(m-j+1)}{2}}\phi_{j-1}u_ju_{m-j}. \tag{4.17}$$

Since $u_0^2,u_ju_{m-j}\in H_{22}=k[x^{\pm1}]g$, we may assume $u_0^2=\alpha_0g,u_ju_{m-j}=\alpha_jg$ for some $\alpha_0,\alpha_j\in k[x^{\pm1}]$ for all $1\leqslant j\leqslant m-1$.

Then Equation (4.17) implies $\alpha_0=\alpha\phi_0\cdots\phi_{m-2}$ for some $\alpha\in k[x^{\pm1}]$. We claim $\alpha$ is invertible. Indeed, by $\phi_{m-1}\alpha_0=(-1)^{m-j}\xi^{m-j}\gamma^{-\frac{(m-j)(m-j+1)}{2}}\phi_{j-1}\alpha_j$, we have

$$\alpha_j=(-1)^{j-m}\xi^{j-m}\gamma^{\frac{(m-j)(m-j+1)}{2}}\alpha\;]j-1,j-1[.$$

Then

$$H_{11}\cdot H_{11}+\sum_{j=1}^{m-1}H_{1,1+2j}\cdot H_{1,1+2(m-j)}\subseteq\alpha H_{22}.$$

By the strong grading of $H$,

$$H_{22}=H_{11}\cdot H_{11}+\sum_{j=1}^{m-1}H_{1,1+2j}\cdot H_{1,1+2(m-j)},$$

which shows that $\alpha$ must be invertible. Since $\epsilon(\alpha_0) = 1$ and $\epsilon(\phi_0 \cdots \phi_{m-2}) = m$, we may assume $\alpha_0 = \frac{1}{m} x^a \phi_0 \cdots \phi_{m-2}$ for some integer $a$. Thus

$$\begin{aligned} u_j u_{m-j} &= (-1)^{j-m} \xi^{j-m} \gamma^{\frac{(m-j)(m-j+1)}{2}} \frac{1}{m} x^a \, ]j-1, j-1[ \, g \\ &= (-1)^j \xi^j \gamma^{\frac{-j(-j+1)}{2}} \frac{1}{m} x^a \, ]j-1, j-1[ \, g. \end{aligned}$$

*Case* 1. If $0 \leqslant i+j \leqslant m-2$, then

$$\begin{aligned} y^{i+j} u_0^2 &= \xi^j x^{jd} y^i u_0 y^j u_0 \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} u_i \phi_0 \cdots \phi_{j-1} u_j \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} \cdot \overline{\phi_0} \cdots \overline{\phi_{j-1}} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{i-1} \cdot \phi_{m-1-j} \cdots \phi_{m-2} u_i u_j. \end{aligned}$$

So

$$u_i u_j = (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^a \phi_i \cdots \phi_{m-2-j} y^{i+j} g.$$

*Case* 2. If $i+j = m-1$, then

$$\begin{aligned} y^{i+j} u_0^2 &= \xi^j x^{jd} y^i u_0 y^j u_0 \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} u_i \phi_0 \cdots \phi_{j-1} u_j \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} \cdot \overline{\phi_0} \cdots \overline{\phi_{j-1}} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{i-1} \cdot \phi_{m-1-j} \cdots \phi_{m-2} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{m-2} u_i u_j. \end{aligned}$$

So

$$u_i u_j = (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^a y^{i+j} g.$$

*Case* 3. If $m \leqslant i+j \leqslant 2m-2$, then

$$\begin{aligned} y^{i+j} u_0^2 &= \xi^j x^{jd} y^i u_0 y^j u_0 \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} u_i \phi_0 \cdots \phi_{i-1} u_j \\ &= \xi^j x^{jd} \phi_0 \cdots \phi_{i-1} \cdot \overline{\phi_0} \cdots \overline{\phi_{j-1}} u_i u_j \\ &= (-1)^j \xi^j \gamma^{-\frac{j(j+1)}{2}} \phi_0 \cdots \phi_{i-1} \cdot \phi_{m-1-j} \cdots \phi_{m-2} u_i u_j. \end{aligned}$$

So

$$u_i u_j = (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\phi_i\cdots\phi_{m-1}\phi_0\cdots\phi_{m-2-j}y^{i+j-m}g.$$

Using the notations introduced in Section 3.1, we have a unified expression:

$$\begin{aligned} u_i u_j &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\ [i, m-2-j]\ y^{\overline{i+j}}g \\ &= (-1)^{-j}\xi^{-j}\gamma^{\frac{j(j+1)}{2}}\frac{1}{m}x^a\ ]-1-j, i-1[\ y^{\overline{i+j}}g \end{aligned}$$

for all $i, j$. □

*Claim 7. We have $\xi^2 = \gamma$, $a = -\frac{1+m}{2}d$ and*

$$S(u_i) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id+\frac{3}{2}(1-m)d}g^{m-1-i}u_i$$

*for $0 \leqslant i \leqslant m-1$.*

*Proof of Claim 7:* Since $S(H_{ij}) = H_{-j,-i}$, $S(u_0) = hg^{m-1}u_0$ for some $h \in k[x^{\pm 1}]$. Combining

$$\begin{aligned} S(yu_0) &= S(u_0)S(y) = hg^{m-1}u_0\cdot(-yg^{-1}) = -\xi x^{-d}hg^{m-1}yu_0g^{-1} \\ &= -\xi x^{-d}\phi_0 hg^{m-1}u_1g^{-1} = -\xi\gamma^{-1}x^d\phi_0hg^{m-2}u_1 \end{aligned}$$

with

$$S(yu_0) = S(\phi_0u_1) = S(u_1)S(\phi_0) = \phi_0S(u_1),$$

we get $S(u_1) = -\xi\gamma^{-1}x^dhg^{m-2}u_1$. The computation above tells us that we can prove that

$$S(u_i) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id}hg^{m-1-i}u_i$$

by induction. In fact, assume that $S(u_i) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id}hg^{m-1-i}u_i$, by combining

$$\begin{aligned} S(yu_i) &= S(u_i)S(y) = (-1)^i\xi^{-i}\gamma^{-\frac{i(i+1)}{2}}x^{id}hg^{m-1-i}u_i(-yg^{-1}) \\ &= \phi_i\cdot(-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d}hg^{m-2-i}u_{i+1} \end{aligned}$$

with

$$S(yu_i) = S(\phi_iu_{i+1}) = S(u_{i+1})S(\phi_i) = \phi_iS(u_{i+1}),$$

we find that $S(u_{i+1}) = (-1)^{i+1}\xi^{-(i+1)}\gamma^{-\frac{(i+1)(i+2)}{2}}x^{(i+1)d}hg^{m-2-i}u_{i+1}$.

In order to determine the relationship between $\xi$ and $\gamma$, we consider the equality $(\mathrm{Id} * S)(u_1) = 0$. By computation,

$$
\begin{aligned}
(\mathrm{Id} * S)(u_1) &= \sum_{j=0}^{m-1} \gamma^{j(1-j)} u_j S(x^{-jd} g^j u_{1-j}) \\
&= \sum_{j=0}^{m-1} \gamma^{j(1-j)} u_j \cdot (-1)^{1-j} \xi^{j-1} \gamma^{-\frac{(1-j)(2-j)}{2}} x^{(1-j)d} h g^{m-2+j} u_{1-j} g^{-j} x^{jd} \\
&= \sum_{j=0}^{m-1} (-1)^{1-j} \xi^{j-1} \gamma^{j(1-j)-\frac{(1-j)(2-j)}{2}} x^{(2j-1)d} \overline{h} u_j g^{m-2+j} u_{1-j} g^{-j} \\
&= \sum_{j=0}^{m-1} (-1)^{1-j} \xi^{j-1} \gamma^{-\frac{(1-j)(2-j)}{2}-2j} x^{(3-2m)d} \overline{h} g^{m-2} u_j u_{1-j} \\
&= \frac{1}{m} \xi^{-2} x^{(3-2m)d+a} \overline{h} g^{m-1} (\sum_{j=0}^{m-1} \xi^{2j} \gamma^{-2j} \, ]j-2, j-1[),
\end{aligned}
$$

where Equation (4.15) is used. Thus

$$
(\mathrm{Id} * S)(u_1) = 0 \Leftrightarrow \sum_{j=0}^{m-1} \xi^{2j} \gamma^{-2j} \, ]j-2, j-1[ \, = 0.
$$

This forces $\xi^2 = \gamma$ by Corollary 2.0.6.

Next, we will show $h = x^{\frac{3}{2}(1-m)d}$ by the equations

$$
(S * \mathrm{Id})(u_0) = (\mathrm{Id} * S)(u_0) = 1.
$$

Indeed,

$$\begin{aligned}
(S*\mathrm{Id})(u_0) &= \sum_{j=0}^{m-1} S(\gamma^{-j^2}u_j)x^{-jd}g^j u_{-j}\\
&= \sum_{j=0}^{m-1} \gamma^{-j^2}(-1)^j\xi^{-j}\gamma^{-\frac{j(j+1)}{2}}x^{jd}hg^{m-j-1}u_jx^{-jd}g^ju_{-j}\\
&= hg^{m-1}(\sum_{j=0}^{m-1}(-1)^j\xi^{-j}\gamma^{-\frac{j(j+1)}{2}}u_ju_{-j})\\
&= hg^{m-1}(\sum_{j=0}^{m-1}(-1)^j\xi^{-j}\gamma^{-\frac{j(j+1)}{2}}(-1)^{j-m}\xi^{j-m}\gamma^{\frac{1}{2}(m-j)(m-j+1)}\frac{1}{m}x^a\ ]j-1,j-1[\ g)\\
&= \frac{1}{m}x^ahg^m(\sum_{j=0}^{m-1}(-1)^{-m}\xi^{-m}\gamma^{\frac{m(m+1)}{2}-j}\ ]j-1,j-1[)\\
&= \frac{1}{m}x^ahg^m(\sum_{j=0}^{m-1}\gamma^{-j}\ ]j-1,j-1[)\\
&= x^{(2m-1)d+a}h \quad (\text{ by Lemma 2.0.4}),
\end{aligned}$$

$$\begin{aligned}
(\mathrm{Id}*S)(u_0) &= \sum_{j=0}^{m-1}\gamma^{-j^2}u_j\cdot S(x^{-jd}g^ju_{-j})\\
&= \sum_{j=0}^{m-1}\gamma^{-j^2}u_j\cdot S(u_{-j})S(g^j)x^{jd}\\
&= \sum_{j=0}^{m-1}\gamma^{-j^2}u_j\cdot(-1)^{-j}\xi^j\gamma^{\frac{j(1-j)}{2}}x^{-jd}hg^{m+j-1}u_{-j}g^{-j}x^{jd}\\
&= \sum_{j=0}^{m-1}\gamma^{-j^2}u_j\cdot(-1)^{-j}\xi^j\gamma^{\frac{j(1-j)}{2}+j^2}hg^{m-1}u_{-j}\\
&= x^{(2-2m)d}\overline{h}g^{m-1}(\sum_{j=0}^{m-1}(-1)^{-j}\xi^j\gamma^{\frac{j(1-j)}{2}-j}u_ju_{-j})\\
&= \frac{1}{m}x^{(2-m)d+a}\overline{h}(\sum_{j=0}^{m-1}\xi^{2j}\gamma^{-j}\ ]j-1,j-1[)\\
&= x^{(2-m)d+a}\overline{h} \quad (\text{ by the proof of Lemma 2.0.4}).
\end{aligned}$$

So, $(S*\mathrm{Id})(u_0) = (\mathrm{Id}*S)(u_0) = 1$ implies $h = x^{(1-2m)d-a} = x^{(2-m)d+a}$. Thus $h =$

$x^{\frac{3}{2}(1-m)d}$ and $a = -\frac{1+m}{2}d$. Therefore, for $0 \leqslant i \leqslant m-1$,

$$S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} x^{id+\frac{3}{2}(1-m)d} g^{m-1-i} u_i.$$

□

From Claim 7, we find that $2|(1+m)d$ and we can improve Claim 6 as the following form:

*Claim 6'. For $0 \leqslant i, j \leqslant m-1$, the multiplication between $u_i$ and $u_j$ satisfies that*

$$u_i u_j = (-1)^{-j} \xi^{-j} \gamma^{\frac{j(j+1)}{2}} \frac{1}{m} x^{-\frac{1+m}{2}d} \, [i, m-2-j] \, y^{\overline{i+j}} g$$

*where $i+j$ is interpreted* mod $m$.

We can prove Proposition 4.4.2 now. By Claims 1,2',3,4,5,6' and 7, we have a natural surjective Hopf homomorphism

$$f: \; D(m, d, \xi) \to H, \; x \mapsto x, \; y \mapsto y, \; g \mapsto g, \; u_i \mapsto u_i$$

for $0 \leqslant i \leqslant m-1$. It is not hard to see that $f|_{D_{ij}}: \; D_{ij} \to H_{ij}$ is an isomorphism of $k[x^{\pm 1}]$-modules for $0 \leqslant i, j \leqslant 2m-1$. So $f$ is an isomorphism. □

We conclude this paper by giving the classification of prime regular Hopf algebras of GK-dimension one.

**Theorem 4.4.3** *Let $H$ be a prime regular Hopf algebra of GK-dimension one. Then it is isomorphic to one of the following:*

(1) *the Hopf algebras listed in Section 1.5;*

(2) *the Hopf algebras constructed in Section 3.1.*

***Proof.*** By Theorem 1.5.1, we only need to consider the case $\mathrm{io}(H) > \mathrm{im}(H) > 1$. In this case, $\widetilde{H}$ can be constructed. By Proposition 4.1.8, $\widetilde{H}$ is either primitive or group-like. If $\widetilde{H}$ is primitive, then $H$ is isomorphic to an infinite dimensional Taft algebra by Theorem 4.3.4. If $\widetilde{H}$ is group-like, owing to Proposition 4.4.1 there is no such $H$ satisfying $\frac{\mathrm{io}(H)}{\mathrm{im}(H)} > 2$. Moreover, if $\frac{\mathrm{io}(H)}{\mathrm{im}(H)} = 2$ then $H$ is one of the Hopf algebras constructed in Section 3.1 by Proposition 4.4.2. □

# Chapter 5 Further research about $D(m,d,\xi)$

At the first glance, the definition of $D(m,d,\xi)$ seems complicated and quite different from our familiar examples. As a matter of fact, the main aim of this chapter is just to answer a question posed by some experts like Professors Xiaowu Chen, Istvan Heckenberger and others, and from which we hope that we can understand this new Hopf algebra $D(m,d,\xi)$ better. The question they raised is: What is the coradical of $D(m,d,\xi)$ and is the coradical still a Hopf subalgebra? One of the main results of this chapter is to determine the coradical filtration of $D(m,d,\xi)$.

On the other hand, another motivation is to find new examples of co-Frobenius Hopf algebras (a Hopf algebra is called co-Frobenius if it has a nonzero integral), which was studied intensively [2–4]. In particular, Andruskiewitsch and Dăscălescu [4] proved that a Hopf algebra with a finite coradical filtration is co-Frobenius and conjectured that a co-Frobenius Hopf algebra has finite coradical filtration. Recently, Andruskiewitsch, Cuadra and Etingof proved this conjecture, see [3, Theorem 1.2]. However, examples of co-Frobenius Hopf algebras do not abound in the literature and coradicals of almost all known examples are Hopf subalgebras. Our result in this chapter implies that $D(m,d,\xi)$ is a co-Frobenius Hopf algebra while its coradical is not a Hopf subalgebra.

At first, we find that there is a simple subcoalgebra $C$ of $D(m,d,\xi)$ having dimension $m^2 \neq 1$ ($m > 1$ by definition). This immediately shows that $D(m,d,\xi)$ is not pointed. Then we show that this simple subcoalgebra $C$ together with other group-likes generate the coradical (see Theorem 5.1.4 for details). From this, we find that the coradical of $D(m,d,\xi)$ is not a Hopf subalgebra and the coradical filtration is finite. Thus, $D(m,d,\xi)$ is a co-Frobenius Hopf algebra with coradical not a Hopf subalgebra. Since $D(m,d,\xi)$ has nonzero integrals, we investigate whether there is a relationship between homological integrals and the nonzero integrals at the end of this chapter. The last section of this chapter is devoted to showing that $B(m,\omega,\gamma)$ is a normal Hopf subalgebra of $D(m,d,\xi)$, where $\omega = md$ and $\gamma = \xi^2$.

## §5.1 The coradical filtration of $D(m,d,\xi)$

If $C$ is a coalgebra, the *coradical* of $C$, denoted by $C_0$, is the sum of all simple subcoalgebras of $C$. Then $C = \cup C_i$ with $C_i = \wedge^{i+1} C_0 = \Delta^{-1}(C \otimes C_{i-1} + C_0 \otimes C)$, which gives $C$ the *coradical filtration.*

**Lemma 5.1.1** *$C := \mathrm{span}\{(x^{-d}g)^i u_j | 0 \leqslant i, j \leqslant m-1\}$ is a subcoalgebra of $D(m,d,\xi)$.*

***Proof.*** The verification is standard and the result follows from $\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j}$ and the fact that $x$, $g$ are group-like elements with $xg = gx, x^{md} = g^m$ directly. □

**Proposition 5.1.2** *$C$ is a simple subcoalgebra of $D(m,d,\xi)$.*

***Proof.*** It is sufficient to show that the $k$-linear dual $C^* := \mathrm{Hom}_k(C,k)$ is a simple algebra. In fact, we will see that $C^*$ is the matrix algebra of order $m$. Denote by $f_{ij} := ((x^{-d}g)^i u_j)^*$, that is, $\{f_{ij} | 0 \leqslant i, j \leqslant m-1\}$ is the dual basis of the basis $\{(x^{-d}g)^i u_j | 0 \le i, j \le m-1\}$ of $C$. We prove the proposition by two steps: firstly, we study the multiplication of the dual basis; secondly, we construct an algebraic isomorphism from $C^*$ to the matrix algebra of order $m$.

*Step 1.* Since

$$\begin{aligned}
(f_{i_1,j_1} * f_{i_2,j_2})((x^{-d}g)^i u_j) &= m(f_{i_1,j_1} \otimes f_{i_2,j_2})(\Delta((x^{-d}g)^i u_j)) \\
&= m(f_{i_1,j_1} \otimes f_{i_2,j_2})(\sum_{s=0}^{m-1} \gamma^{s(j-s)} (x^{-d}g)^i u_s \otimes (x^{-d}g)^{i+s} u_{j-s}) \\
&= \sum_{s=0}^{m-1} \gamma^{s(j-s)} f_{i_1,j_1}((x^{-d}g)^i u_s) f_{i_2,j_2}((x^{-d}g)^{i+s} u_{j-s})
\end{aligned}$$

one can see that $(f_{i_1,j_1} * f_{i_2,j_2})((x^{-d}g)^i u_j) \neq 0$ if and only if $i_1 = i, j_1 = s, i_2 = i+s$ and $j_2 = j - s$ for some $0 \leqslant s \leqslant m-1$. This forces $i_1 + j_1 = i_2, i = i_1$ and $j = j_1 + j_2$. So we have

$$f_{i_1,j_1} * f_{i_2,j_2} = \begin{cases} \gamma^{j_1 j_2} f_{i_1, j_1+j_2}, & \text{if } i_1 + j_1 = i_2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

*Step 2.* Set $M = M_m(k)$ and let $E_{ij}$ be the matrix units ( that is, the matrix with 1 is in the (i, j) entry and 0 elsewhere) for $0 \leqslant i, j \leqslant m-1$. Now we claim that

$$\varphi : C^* \to M, f_{ij} \mapsto \gamma^{ij} E_{i,i+j}$$

is an algebraic isomorphism (the index $i+j$ in $E_{i+j}$ is interpreted mod $m$). It is sufficient to verify that $\varphi$ is an algebraic map. In fact,

$$\begin{aligned}
\varphi(f_{i_1,j_1})\varphi(f_{i_2,j_2}) &= \gamma^{i_1 j_1} E_{i_1,i_1+j_1} \gamma^{i_2 j_2} E_{i_2,i_2+j_2} \\
&= \begin{cases} \gamma^{i_1 j_1 + i_2 j_2} E_{i_1,i_2+j_2}, & \text{if } i_1 + j_1 = i_2, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \gamma^{i_1 j_1 + i_2 j_2 - i_1(j_1+j_2)} \varphi(f_{i_1,j_1+j_2}), & \text{if } i_1 + j_1 = i_2, \\ 0, & \text{otherwise}, \end{cases} \\
&= \begin{cases} \varphi(f_{i_1,j_1+j_2}), & \text{if } i_1 + j_1 = i_2, \\ 0, & \text{otherwise}, \end{cases} \\
&= \varphi(f_{i_1,j_1} * f_{i_2,j_2}).
\end{aligned}$$

So $\varphi$ is an algebraic map and the proof is completed. □

**Lemma 5.1.3** *$D(m, d, \xi)$ has a k-linear basis*

$$\{x^i g^j y^t, x^i g^j u_s | i \in \mathbb{Z}, 0 \leqslant j, t, s \leqslant m-1\}$$

***Proof.*** This holds by (3.8) directly: $D(m, d, \xi) = \bigoplus_{i,j=0}^{2m-1} D_{ij}$, where

$$D_{ij} = \begin{cases} k[x^{\pm 1}] y^{j-i} g^i, & i, j = \text{even}, \\ k[x^{\pm 1}] g^{\frac{i-1}{2}} u_{\frac{j-i}{2}}, & i, j = \text{odd}, \\ 0, & \text{otherwise}. \end{cases}$$

□

Set $F := \{x^i g^j | i \in \mathbb{Z}, 0 \leqslant j \leqslant m-1\}$. Define

$$\begin{aligned} kF\text{-span}\{1, C\} &= (\bigoplus_{i\in\mathbb{Z},\, 0\leqslant j\leqslant m-1} x^i g^j) \oplus (\bigoplus_{i\in\mathbb{Z},\, 0\leqslant j\leqslant m-1} x^i g^j C) \\ &= (\bigoplus_{i\in\mathbb{Z},\, 0\leqslant j\leqslant m-1} x^i g^j) \oplus (\bigoplus_{i\in\mathbb{Z},\, 0\leqslant j,s\leqslant m-1} x^i g^j u_s) \end{aligned}$$

as a vector space.

**Theorem 5.1.4** *The coradical of $D(m, d, \xi)$ is $D_0 = kF$-span$\{1, C\}$.*

***Proof.*** Since $x$ and $g$ are grouplike elements, $x^i g^j$ spans a simple subcoalgebra, for each $i \in \mathbb{Z}, 0 \leqslant j \leqslant m-1$. Since $C$ is a simple subcoalgebra by Proposition 5.1.2, $x^i g^j C$ is also a simple subcoalgebra for each $i, j$. So $kF$-span$\{1, C\} \subseteq D_0$. To prove $kF$-span$\{1, C\} = D_0$ , it is sufficient to show that

$$\bigcup_n \wedge^n kF\text{-span}\{1, C\} = D$$

by [26, Lemma 5.3.4]. Since $\Delta(y) = y \otimes g + 1 \otimes y$, we have $y^t \in \wedge^{t+1} kF$-span$\{1, C\}$ for $1 \leqslant t \leqslant m-1$. Considering the basis of $D$ in Lemma 5.1.3, the basis elements not in $kF$-span$\{1, C\}$ are $\{x^i g^j y^t | 1 \leqslant t \leqslant m-1\}$. Thus the equation $\bigcup_n \wedge^n kF$-span$\{1, C\} = D$ holds. So we have $D_0 = kF$-span$\{1, C\}$. □

**Corollary 5.1.5** *The coradical filtration of $D(m, d, \xi)$ is as follows:*

$$\begin{cases} D_0 = kF\text{-span}\{1, C\}, \\ D_1 = \wedge^2 D_0 = D_0 \oplus (kF\text{-span } y) = kF\text{-span}\{1, C, y\}, \\ D_i = \wedge^{i+1} D_0 = kF\text{-span}\{1, C, y, \cdots, y^i\} \text{ for } 1 \leqslant i \leqslant m-1, \\ D_j = D(m, d, \xi) \text{ for } j \geqslant m-1. \end{cases}$$

**Corollary 5.1.6** *$D(m, d, \xi)$ is a co-Frobenius Hopf algebra with coradical not a Hopf subalgebra.*

***Proof.*** Since the coradical filtration of $D(m, d, \xi)$ is finite by the above corollary, $D(m, d, \xi)$ is a co-Frobenius Hopf algebra by [4, Theorem 2.1]. Combining $u_0, u_1 \in D_0$ and $u_0 u_1 = \frac{1}{m} x^{-\frac{1+m}{2}} yg \notin D_0$ by Equation (3.3), we get $D_0$ is not a subalgebra. Thus $D(m, d, \xi)$ a co-Frobenius Hopf algebra with coradical not a Hopf subalgebra. □

**Proposition 5.1.7** *The Hopf subalgebra generated by the coradical $D_0$ is the Hopf algebra $D(m, d, \xi)$.*

***Proof.*** Denote the Hopf subalgebra generated by the coradical $D_0$ by $H$. Since $u_i \in D_0$ for all $0 \leqslant i \leqslant m-1$ by Theorem 5.1.4, we have $u_0u_1 \in H$. Note that $u_0u_1 = -\frac{1}{m}x^{-\frac{1+m}{2}d}yg$ by Equation (3.3). Since $x$ and $g$ are group-like elements, we have $y \in H$. Thus $H$ contains all the generators of $D(m, d, \xi)$. Therefore the Hopf subalgebra generated by the coradical is exactly the Hopf algebra $D(m, d, \xi)$. □

## §5.2 Integral and homological integral

Recall the definition of integrals first.

**Definition 5.2.1** *[13, 26] Let $H$ be a Hopf algebra. An element $T \in H^*$ is called a left integral on $H$ if for all $f \in H^*$,*

$$fT = f(1)T.$$

*The set of left integrals on $H$ is denoted by $\int^l$. Left integrals of $H^{cop}$ are called right integrals on $H$, and the set of right integrals is denoted by $\int^r$.*

**Remark 5.2.2** *[13] It is clear $T \in H^*$ is a left integral if and only if for all $h \in H, \sum h_1 T(h_2) = T(h)1_H$ and it is a right integral if and only if $\sum T(h_1)h_2 = T(h)1_H$.*

It is well known that if $H$ has a nonzero integral then $\dim_k \int^l = \dim_k \int^r = 1$ and $\int^r = S^*(\int^l) = \int^l \circ S$. Homological integrals have similar properties that $\dim_k \int_H^l = \dim_k \int_H^r = 1$ and $\int_H^r = S(\int_H^l)$.

Since $\int_H^l$ and $\int_H^r$ have dimension one, they can induce $k$-linear maps naturally and thus any homological integral can be regarded as an element in $H^*$. It is fascinating to investigate whether there is a relationship between the integrals and the maps induced by homological integrals.

We work out the maps induced by homological integrals first. In Lemma 3.3.3, we have

$$\int_D^l = D/(y, x-1, g-\gamma^{-1}, u_0 - \xi^{-1}, u_1, u_2, \cdots, u_{m-1}).$$

Then the map induced by $\int_D^l$ is

$$T_D^l := \sum_{i\in\mathbb{Z},0\leqslant j\leqslant m-1} \gamma^{-j}(x^i g^j)^* + \sum_{i\in\mathbb{Z},0\leqslant j\leqslant m-1} \gamma^{-j}\xi^{-1}(x^i g^j u_0)^*.$$

As a consequence,

$$T_D^r := \sum_{i\in\mathbb{Z},0\leqslant j\leqslant m-1} \gamma^{j}(x^i g^j)^* + \sum_{i\in\mathbb{Z},0\leqslant j\leqslant m-1} \gamma^{j}\xi(x^i g^j u_0)^*,$$

by$\int_D^r = S(\int_D^l)$.

Then we turn to nonzero integrals on $D$. Assume $T^l$ is a nonzero left integral on $D$. To determine $T^l$, we only need to apply it to the basis in Lemma 5.1.3. Since $x$ and $g$ are group-like elements, we must have $T^l(x^i g^j) = 0$ by Remark 5.2.2. Deducing from $\Delta(y) = y \otimes g + 1 \otimes y$, one can see that

$$T^l(x^i g^j y^s) = 0$$

for $x^i g^j \neq 1$. Similarly, by $\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j}$, we have

$$T^l(x^i g^j u_t) = 0$$

for all $0 \leqslant t \leqslant m-1$. Then it remains to determine $T^l$'s action on $\{1, y^s | 1 \leqslant s \leqslant m-1\}$. Considering

$$T^l(g^{-1}y) = g^{-1}yT^l(1) + g^{-1}T^l(g^{-1}y) = 0,$$

we get $T^l(1) = 0$. Similarly,

$$T^l(g^{-1}y^s) = \sum_{t=0}^{s} \binom{s}{t}_\gamma g^{-1}y^{s-t}T^l(g^{s-t-1}y^t) = \binom{s}{s-1}_\gamma g^{-1}yT^l(y^{s-1}) = 0$$

implies $T^l(y^{s-1}) = 0$ for all $1 \leqslant s \leqslant m-1$. That is, $T^l(y^t) = 0$ for all $0 \leqslant t \leqslant m-2$. Thus we can write $T^l = (y^{m-1})^*$. As a consequence, we have $T^r = (g^{1-m}y^{m-1})^*$ by $\int^r = \int^l \circ S$.

Dispirited and expected, it seems that there is no direct relationship between integrals and homological integrals.

## §5.3 Normal Hopf subalgebra

Let $\omega = md$ and $\gamma = \xi^2$. By Remark 3.1.1, $B(m,\omega,\gamma)$ is a Hopf subalgebra of $D(m,d,\xi)$. We show that this Hopf subalgebra is normal. We recall the definition of *normal Hopf subalgebra* first.

**Definition 5.3.1** *[26] Let $H$ be any Hopf algebra.*

1) *The* left adjoint action *of $H$ on itself is given by*

$$(\mathrm{ad}_l\, x)(y) = \sum x_1 y S(x_2),$$

*for all $x, y \in H$.*

2) *The* right adjoint action *of $H$ on itself is given by*

$$(\mathrm{ad}_r\, x)(y) = \sum S(x_1) y x_2,$$

*for all $x, y \in H$.*

3) *A Hopf subalgebra $K$ of $H$ is called* normal *if*

$$\textit{both } \ (\mathrm{ad}_l\, H)(K) \subseteq K \ \textit{ and } \ (\mathrm{ad}_r\, H)(K) \subseteq K.$$

**Proposition 5.3.2** *$B(m,\omega,\gamma)$ is a normal Hopf subalgebra of $D(m,d,\xi)$.*

***Proof.*** It is sufficient to verify that

$$(\mathrm{ad}_l\, u_i)(B) \subseteq B \ \text{ and } \ (\mathrm{ad}_r\, u_i)(B) \subseteq B$$

for all $0 \leqslant i \leqslant m-1$. Note that

$$\begin{aligned}
(\mathrm{Id} \otimes S)\Delta(u_i) &= (\mathrm{Id} \otimes S)(\sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes x^{-jd} g^j u_{i-j}) \\
&= \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes S(u_{i-j}) x^{jd} g^{-j} \\
&= \sum_{j=0}^{m-1} (-1)^{i-j} \xi^{j-i} \gamma^{\frac{(i-j)(3j-i-1)}{2}} u_j \otimes x^{(i-j)d + \frac{3}{2}(1-m)d} g^{m-i+j-1} u_{i-j} x^{jd} g^{-j} \\
&= \sum_{j=0}^{m-1} (-1)^{i-j} \xi^{j-i} \gamma^{-\frac{(i-j)(i-j+1)}{2}} u_j \otimes x^{id + \frac{3}{2}(1-m)d} g^{m-i-1} u_{i-j}.
\end{aligned}$$

By Equations (3.2) and (3.3), we have $xu_i = u_i x^{-1}$, $yu_i = \phi_i u_{i+1} = \xi x^d u_i y$, $u_i g = \gamma^i x^{-2d} g u_i$ and $u_i u_j \in B$. We can see that

$$(\mathrm{ad}_l\, u_i)(B) \subseteq B \quad \text{and} \quad (\mathrm{ad}_r\, u_i)(B) \subseteq B$$

hold since $B$ is generated by $x^{\pm 1}, g, y$. This ends the proof. $\square$

It is not hard to see that $\bar{D} := D/DB^+ = D/(x-1, y, g-1)$ is generated by $u_0, \cdots, u_{m-1}$ (also denote the images of $u_i$ in $\bar{D}$ by $u_i$) subject to the relations

$$u_i u_j = \begin{cases} 1, & i = j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

by Equation (3.3). The coproduct, counit and antipode of $\bar{D}$ are given by

$$\Delta(u_i) = \sum_{j=0}^{m-1} \gamma^{j(i-j)} u_j \otimes u_{i-j} \quad \text{for all} \quad 0 \leqslant i \leqslant m-1,$$

$$\epsilon(u_0) = 1, \epsilon(u_1) = 0 \quad \text{for all} \quad 1 \leqslant i \leqslant m-1,$$

and

$$S(u_i) = (-1)^i \xi^{-i} \gamma^{-\frac{i(i+1)}{2}} u_i \quad \text{for all} \quad 0 \leqslant i \leqslant m-1.$$

Moreover, one can see that

$$B \hookrightarrow D \to \bar{D}$$

is a Hopf algebras short exact sequence. In fact, denote by $p : D \to \bar{D}$ the canonical epimorphism. Define

$$l : \bar{D} \to D, u_i \mapsto u_i.$$

Then $l$ is a Hopf morphism and $pl = \mathrm{Id}_{\bar{D}}$.

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E-mail address: wujy1753@126.com

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1. J.Y. Wu, G.X. Liu and N.Q. Ding, Classification of affine prime regular Hopf algebras of GK-dimension one. Adv. Math. 296(2016), 1-54.
2. J.Y. Wu, Note on the coradical filtration of $D(m, d, \xi)$, Comm. Algebra, (to appear).
3. J.Y. Wu and G.X. Liu, Yetter-Drinfeld modules over Kac's 8-dimension algebra and new example of Hopf algebras, (preprint).

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