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作 者 姓 名: 周坤

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指 导 教 师: 刘公祥 教授

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Quasitriangular structures on some semisimple Hopf algebras

by Kun Zhou

Supervised by Professor Gongxiang Liu

Fundamental Mathematics

Department of Mathematics, Nanjing University

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基础数学 专业 2017 级博士生姓名: 周坤
指导教师(姓名、职称): 刘公祥 教授

摘 要

拟三角 Hopf 代数是量子群中一类重要的 Hopf 代数，关于它有许多研究的热点，例如拟三角 Hopf 代数分类以及构造流形不变量等。许多关于拟三角 Hopf 代数的课题都有一个共同点，就是它们依赖于拟三角 Hopf 代数的构造，因此我们把拟三角 Hopf 代数的构造作为本论文研究的核心。

已知的拟三角 Hopf 代数构造方法可以分成两种，一种是对于给定的 Hopf 代数求出其上所有拟三角结构，一种是发现新的 Hopf 代数并给出它上面的拟三角结构。一般来说，关于这两种构造的课题都是困难的，即使是最小维数的非平凡极小三角半单 Hopf 代数至今也未被确定。本文探讨的是第一种构造方法。

具体来说，我们探究了两类半单 Hopf 代数上的拟三角结构，一类是由特殊阿贝尔扩张得到的，另一类则是一些极小拟三角 Hopf 代数。对于第一类 Hopf 代数，我们给出了其上所有的拟三角结构。对于第二类 Hopf 代数，我们主要证明了最小维数的非平凡极小三角半单 Hopf 代数为 16 维，并具体给出了一个 16 维的例子。

下面是主要结果的描述。

对于第一类 Hopf 代数 $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$，我们首先证明了 $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ 上的拟三角结构只有两类，一类被称为平凡的，而另一类被称为非平凡的。平凡的拟三角结构是群 G 上的一些双特征，因此容易被给出。之后我们给出了所有非平凡的拟三角结构并且确定了 $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ 上的拟三角结构个数。在给出所有非平凡拟三角结构的过程中，我们探究了拟三角结构的对称性并得到一些命题，而这些命题对于求解非平凡的拟三角结构起到关键作用。

接下来，我们选取第一类 Hopf 代数中的最简单的两族 Hopf 代数进行深入研究，我们证明了这两族 Hopf 代数上的所有拟三角结构都是φ-对称的并且具体给出了其上的所有拟三角结构。

对于第二类 Hopf 代数，也就是被我们称为满秩极小拟三角 Hopf 代数。这类 Hopf 代数是一些极小拟三角 Hopf 代数。我们举例说明了存在 Hopf 代数 H，使得 H 是极小拟三角 Hopf 代数但不是满秩极小拟三角 Hopf 代数。进一步地，我们找出了例子说明存在 Hopf 代数 K，使得 K 是满秩极小拟三角 Hopf 代数但不是极小三角 Hopf 代数。随后，我们讨论了满秩极小拟三角 Hopf 代数的刻画。特别的，我们构造了一族非平凡的满秩极小拟三角 Hopf 代数。

最后，我们构造了一族非平凡极小三角半单 Hopf 代数并给出了其上所有非平凡极小三角结构。作为构造结果的应用，我们证明了最小维数的非平凡极小三角半单 Hopf 代数为 16 维并给出了一个 16 维的例子。

关键词: Hopf 代数; 拟三角结构; Abelian 扩张.

THESIS: Quasitriangular structures on some semisimple Hopf algebras

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Abstract

Quasitriangular Hopf algebras is an important class of Hopf algebras in quantum groups and there are many active studies on them, such as the classification of them and the construction of invariants of manifolds from quasitriangular Hopf algebras etc. Since many topics on quasitriangular Hopf algebras have a common feature, that is depending heavily on the construction of quasitriangular Hopf algebras, we put the exploration of the construction of quasitriangular Hopf algebras as the central work of this thesis.

The known construction methods of quasitriangular Hopf algebras can be divided into two types, one is to find all the quasitriangular structures on a given Hopf algebra, and the other is to discover a new Hopf algebra and give the quasitriangular structures on it. In general, the subjects related to both construction methods all are difficult, even the non-trivial minimal triangular semisimple Hopf algebra with the smallest dimension is not clear for us yet. We study the first construction method in this thesis.

Specifically, we study the quasitriangular structures on two classes of semisimple Hopf algebras, one class of Hopf algebras are obtained by a special kind of abelian extension of Hopf algebras, and the other class of Hopf algebras are some minimal quasitriangular Hopf algebras. For the first class, we give all the quasitriangular structures on them. For the second class, we mainly prove that the smallest dimension among the non-trivial minimal triangular semisimple Hopf algebras is 16 and give an example of 16 dimension Hopf algebra specifically.

The main results are described as follows.

For the first class of Hopf algebras which are $\mathbb{k}^G\#_{\sigma,\tau}\mathbb{k}\mathbb{Z}_2$, we first prove that there are only two forms of quasitriangular structures on them, one is called trivial quasitriangular structures, while the other is called non-trivial quasitriangular structures. The trivial quasitriangular structures are some bicharacters on the group G and thus they are easy to be given. After that we give all non-trivial quasitriangular structures and determine the number of quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. In the process of giving all non-trivial quasitriangular structures, we explore the symmetries of quasitriangular structures and obtain some propositions which play a key role in solving all non-trivial quasitriangular structures.

Next, we select the simplest two families of Hopf algebras from the first class of Hopf algebras for further study. We prove that all quasitriangular structures on these two families of Hopf algebras are φ-symmetric and give all quasitriangular structures on them.

For the second class of Hopf algebras, which we call full rank minimal quasitriangular Hopf algebras. This class of Hopf algebras are some minimal quasitriangular Hopf algebras. We give an example to show that there exists a Hopf algebra H such that H is a minimal quasitriangular Hopf algebra but not a full rank minimal quasitriangular Hopf algebra. Further, we use another example to illustrate that there exists a Hopf algebra K such that K is a full rank minimal quasitriangular Hopf algebra but not a minimal triangular Hopf algebra. Subsequently, we discuss the characterizations of full rank minimal quasitriangular Hopf algebras. In particular, we construct a family of non-trivial full rank minimal quasitriangular Hopf algebras.

Finally, we construct a family of non-trivial minimal triangular semisimple Hopf algebras and give all non-trivial minimal triangular structures on them. As an application, we prove that the smallest dimension among non-trivial minimal triangular semisimple Hopf algebras is 16 and give an example of 16 dimension Hopf algebra.

Keywords: Hopf algebras; Quasitriangular structures; Abelian extension.

Chapter 1 Introduction

§1.1 Background

Quasitriangular Hopf algebras were introduced by Drinfel'd [11] to give solutions to the quantum Yang-Baxter equations. Quasitriangular Hopf algebras are the Hopf algebras whose finite-dimensional representations form a braided rigid tensor category, which naturally relates them to low dimensional topology(see [27], [34], [24], [66]). Moreover, Drinfel'd proved that any finite dimensional Hopf algebra can be embedded into a finite dimensional quasitriangular Hopf algebra, which we now call its quantum double. Since quasitriangular Hopf algebras, especially the triangular ones are close to groups and Lie algebras, they are more tractable than that of general Hopf algebras and hence they can be used to support a testing ground for general Hopf algebraic ideas, methods and conjectures. Quasitriangular Hopf algebras, even if the triangular ones are far from known well. To our knowledge, many authors have studied quasitriangular Hopf algebras from the following different aspects in recent years.

(1) Classification of quastiriangular Hopf algebras with given dimension, especially the triangular ones;

(2) Construct quasitriangular structures on known Hopf algebras or give method to construct new quasitriangular Hopf algebras;

(3) Using quasitriangular Hopf algebras to construct topological invariants;

(4) Study special quasitriangular Hopf algebras, such as ribbon Hopf algebras and modular Hopf algebras;

(5) Explore braided tensor categories which are the categorical version of quasitriangular Hopf algebras;

Because the problems (3)-(5) heavily depends on the answers of the problem (1)-(2), we mainly focus on problems (1)-(2). The problem (1) is an important and basic problem in Hopf algebras. However, this intriguing problem turns out to be extremely hard and it is still widely open. Fortunately, there are two classes of quasitriangular

Hopf algebras that are relatively well understood and they are semisimple triangualar ones and pointed triangular ones. The semisimple triangular Hopf algebras over $\mathbb{k}$(and cosemisimple if the characteristic of $\mathbb{k}$ is positive) are completed classified in [15], [14]. The key theorem about such Hopf algebras states that each of them is obtained by twisting a group algebra of a finite group [15, Theorem 2.1]. So far, all known triangular Hopf algebras in characteristic 0 have Chevally property-namely, the Jacobson radical is Hopf ideal. Naturally, we can ask the following question

Question 1.1.1 *Does there exists a triangular Hopf algebra in characteristic* 0 *which has no Chevalley property?*

Furthermore, we find that there are many quasitriangular Hopf algebras such that they are not triangular(see Remark 7.2.5), such as the 8-dimension Kac algebra K_8 but they are full rank minimal quasitriangular Hopf algebras(see Section 8.1 for definition), then we pose the following question

Question 1.1.2 *Give classification for* full rank *minimal quasitriangular Hopf algebras with given dimension?*

For the problem (2), all the quasitriangular structures on some well known Hopf algebras are gotten, such as all quasitriangular structures on group algebras are determined in [10] and all quasitriangular structures on small quantum groups $U_q(sl_n)'$ are given in [20]. In [21], S. Gelaki asked if there is a non-trivial minimal triangular semisimple Hopf algebra? Then he and P. Etingof constructed a series of minimal triangular semisimple Hopf algebra in [12]. Their method is to construct twists iteratively and give some minimal triangular structures on some semisimple Hopf algebras. And their results depends on solutions of set theoretic of Yang-Baxter equations. Inspired by [21] and [12], we can naturally ask the following problems.

Question 1.1.3 *Whether there are other ways to give series of minimal and triangular semisimple Hopf algebras? and if so, give all minimal triangular structures on them?*

Question 1.1.4 *What is smallest dimension among non-trivial minimal triangular semisimple Hopf algebras? Then give all minimal triangular structures on a smallest one?*

Also there are many other authors, such as [25], [53], [55], [76] have studied the problem (2). We note that quasitriangular structures on pointed Hopf algebras have been studied by many authors over past decades, such as [20], [55], [57], but few authors study the quasitriangular structures on semisimple Hopf algebras in recent years. In [63], D. E. Radford proved that the number of quasitriangular structures on a semisimple Hopf algebra is finite and hence it is hoped that we can not only calculate the quasitriangular structures on semisimple Hopf algebras but also study the number of quasitriangular structures. For example, in 2011 S. Natale [53] proved that there is no quasitriangular structure on some semisimple Hopf algebras which comes from some special abelian extensions. She mainly used the conclusion that full fusion subcategories of $\operatorname{Rep} D^{\omega}(G)$ are determined in [52, Theorem 5.1]. For us, we not only want to get more abundant quasitriangular structures on semisimple Hopf algebras, but also want to study the relationship between the number of quasitriangular structures on a given semisimple Hopf algebra H and the H itself. So we want to know more information about quasitriangular structures of a given Hopf algebra and it's better to determine all quasitriangular structures, and we believe that it will greatly help us to learn quasitriangular Hopf algebras. Because it is known that there are many quasitriangular structures on finite abelian groups and their quasitriangular structures are easy to obtained by their bicharacters, so we choose simplest non-trivial semisimple Hopf algebras $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ which come from the following abelian extension to study the problem (2)

$$\Bbbk^G \xrightarrow{\iota} A \xrightarrow{\pi} \Bbbk\mathbb{Z}_2,$$

here we assume that G is a finite abelian group to make things easier. Naturally, we can ask the following question

Question 1.1.5 *Give quasitriangular structures on* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$*?*

Since we feel interested in the number of quasitriangular structures on a given semisimple Hopf algebra, we also pose the following question.

Question 1.1.6 *Is there any relationship between the number of quasitriangular structures on a given semisimple Hopf algebra and its dimension?*

In addition, we are interested in invariants of quasitriangular Hopf algebras. In [62], D. E. Radford showed that the antipode of a quasitriangular Hopf algebra is inner and thus the antipode can give an invariant for identify non-quasitriangualrity. Even more exciting result has been gotten in [54] which states that there no exist non-trivial quasitriangular Hopf algebra with odd and square free dimension. These interesting results inspire us to put forward the following question.

Question 1.1.7 *Find more invariants of Hopf algebras to identify the quasitriangularity?*

Finally, because we are very interested in analytical version of Hopf algebras, i.e the Kac algebras, we follow the step in [1] to explore whether a semimiple Hopf algebra over complex field is Kac algebra and let's repose the following question.

Question 1.1.8 *Is a quasitriangular semisimple Hopf algebra over complex field must be Kac algebra?*

Based on the above questions 1.1.1-1.1.8, we realize that the construction of quasitriangular Hopf algebras plays an important role in answering them, i.e some interesting examples of quasitriangular Hopf algebras may support good ideas to solve these problems, and it may even disprove certain conjectures. Therefore, the questions 1.1.3-1.1.6 are focused in the thesis, while the other questions will be discussed in the future.

§1.2 Main results

As we mentioned in the previous subsection, we mainly discuss the construction of quasitriangular structures in the thesis. To solve the problems 1.1.3-1.1.6, we first consider the problem 1.1.5. To answer the problem 1.1.5, we give the following Theorems 1.2.1-1.2.3.

Theorem 1.2.1 *Let R be a general solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and let $(\alpha_{ij},\beta_i,\gamma_i,\delta)_{1\leq i,j\leq n}$ be a quadruple which is defined as follows*

$$\alpha_{ij} := w^1(s_i,s_j),\ \beta_i := w^2(s_i,a),\ \gamma_i := w^3(a,s_i),\ \delta := w^4(a,a),$$

where $w^i(1\leq i\leq 4)$ are associated functions of R. Then the above quadruple satisfies the conditions (i)-(v) *of Proposition 6.1.10*

Conversely, we have the following theorem.

Theorem 1.2.2 *Given a quadruple* $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq n}$ *satisfying conditions* (i)-(v) *of Proposition 6.1.10, then there exists a unique general solution* R *for* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ *such that the following equations*

$$w^1(s_i, s_j) = \alpha_{ij},\ w^2(s_i, a) = \beta_i,\ w^3(a, s_i) = \gamma_i,\ w^4(a, a) = \delta,$$

where $w^i(1 \leq i \leq 4)$ *are associated functions of* R.

Then we give a necessary and sufficient condition for the existence of a special solution on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ as follows

Theorem 1.2.3 *There exists a quasitriangular structure for* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ *if and only if there exists a quadruple* $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq n}$ *satisfies conditions (i)-(vi) of Proposition 6.2.1.*

So we have answered the question 1.1.5. For the problem 1.1.6, we use the following Theorem 1.2.4 to give a answer to special case. Let T_Q be the set of trivial quasitriangular structures of $\Bbbk^G\#\Bbbk\mathbb{Z}_2$. Then we have

Theorem 1.2.4 *Let* m *be the number of quasitriangular structures of* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$*, then we have* $m \in \{0, |T_Q|, 2|T_Q|\}$*. Moreover, if* $G = \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_r}$ *and* $m \neq 0$ *then the number* m *is a factor of* $2|G|^r$*.*

Later we use the following Theorem 1.2.5 to answer the problem 1.1.3. Let n be an odd number and let $T'_n := \{\text{minimal triangular structures on } H^n_{b:y}\ \}$, then we have

Theorem 1.2.5 *We have the following one-one correspondence:*

$$T'_n \xleftrightarrow{1-1} \{(\alpha, \beta, \omega^k, \delta) \in \Bbbk^4 |\ \alpha^2 = \beta^2 = \delta^2 = 1,\ k \in \mathbb{N} \textit{ and } (k^2, n)|k\}.$$

Finally, we have the following result to answer the problem 1.1.4.

Theorem 1.2.6 *The 16 dimensional Hopf algebra* $H^1_{b:y}$ *is a Hopf algebra with smallest dimension among non-trivial semisimple minimal triangular Hopf algebras.*

§1.3 Organization

In this section, we give an outline of this dissertation.

This dissertation is divided into eight chapters, each of which is subdivided into sections.

In Chapter 1, we provide the research background and main results.

In Chapter 2, we give a preparation of the following chapters.

In Chapter 3, we prove that the quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ has only two forms, one is called trivial while the other is called non-trivial. The trivial quasitriangular structures are easy to give, but the non-trivial quasitriangular structures are difficult to know. Therefore, we give the necessary conditions for the existence of non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. As an application of these results, we give all the quasitriangular structures on H_{2n^2}.

In Chapter 4, we consider how to simplify the calculations of quasitriangular structures and hence the concept of symmetry of quasitriangular structures on any Hopf algebra is introduced, and then some propositions about symmetry that are useful for computing quasitriangular structures are given. Then we apply these conclusions to the special case $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

In Chapter 5, we showed that the non-trivial quasitriangular structures are in one-one correspondence to some special functions on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, which we call quasitriangular functions on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. After that, we focus on quasitriangular functions and give a criterion for determining when a function is a quasitriangular function on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

In Chapter 6, we use the one-one correspondence about quasitriangular functions which was proved in Chapter 5 to get a division-like operation on quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. So we analogize the solution of linear equations and introduce the concepts of general solutions and a special solution of quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Naturally, we reduce the problem of solving the non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ into finding all general solutions and giving a special solution. At last, we give all the general solutions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and get a necessary and sufficient condition for the existence of a special solution.

In Chapter 7, we discuss some interesting quasitriangular structures which were called φ-symmetric quasitriangular structures. We give a simple necessary and sufficient condition for the existence of a φ-symmetric quasitriangular structure. Then we investigate two special classes of Hopf algebras $K(8n, \sigma, \tau)$ and $A(8n, \sigma, \tau)$ belonging to $\mathbb{k}^G \#_{\sigma,\tau} \mathbb{k}\mathbb{Z}_2$. And we proved that $\mathbb{k}^G \#_{\sigma,\tau} \mathbb{k}\mathbb{Z}_2$ always has either $K(8n, \sigma, \tau)$ as its quotient or $A(8n, \sigma, \tau)$ as its quotient. Moreover, we showed that all quasitriangular structures on these two class Hopf algebras are φ-symmetric. Later all non-trivial quasitriangular structures on $K(8n, \sigma, \tau)$ and $A(8n, \sigma, \tau)$ are given.

In Chapter 8, we provide a system method to construct series of special minimal quasitriangular Hopf algebras which include triangular minimal quasitriangular Hopf algebras. The concept of full rank minimal quasitriangular Hopf algebra is introduced and a series of full rank minimal quasitriangular Hopf algebras are constructed. Then the construction of minimal triangular semisimple Hopf algebras is discussed and all minimal triangular quasitriangular structures on a family of Hopf algebras $H_{b:y}^n$ have been obtained. Finally, a smallest dimension Hopf algebra among non-trivial minimal triangular semisimple Hopf algebras has been given and all its minimal triangular quasitriangular structures are determined.

Chapter 2 Preliminaries

In this chapter, we mainly review some preliminaries used in this paper. It mainly includes the definitions and some basic conclusions of Abelian extension and quasitriangular Hopf algebras.

§2.1 The definition of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

In this section, we recall the definition of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, and then we give some examples of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ for guiding our further research.

Definition 2.1.1 *A short exact sequence of Hopf algebras is a sequence of Hopf algebras and Hopf algebra maps*

$$K \xrightarrow{\iota} H \xrightarrow{\pi} A \tag{2.1}$$

such that

(i) ι *is injective,*

(ii) π *is surjective,*

(iii) $\ker(\pi) = HK^+$, K^+ *is the kernel of the counit of* K.

In this situation it is said that H is an extension of A by K [45, Definiton 1.4]. An extension (2.1) above such that K is commutative and A is cocommutative is called *abelian.* In this paper, we only study the following special abelian extensions

$$\Bbbk^G \xrightarrow{\iota} H \xrightarrow{\pi} \Bbbk\mathbb{Z}_2,$$

where G is a finite abelian group. Abelian extensions were classified by Masuoka (see [45, Proposition 1.5]), and the above H can be expressed as $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ which is defined as follows.

Let $\mathbb{Z}_2 = \{1, x\}$ be the cyclic group of order 2 and let G be a finite group. To give the description of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we need the following data

(i) $\triangleleft : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ is an injective group homomorphism.

(ii) $\sigma : G \to \Bbbk^{\times}$ is a map such that $\sigma(g \triangleleft x) = \sigma(g)$ for $g \in G$ and $\sigma(1) = 1$.

(iii) $\tau : G \times G \to \Bbbk^{\times}$ is a unital 2-cocycle and satisfies that $\sigma(gh)\sigma(g)^{-1}\sigma(h)^{-1} = \tau(g,h)\tau(g \triangleleft x, h \triangleleft x)$ for $g, h \in G$.

The aim of (i) is to avoid making a commutative algebra (in such case all quasitriangular structures are given by bicharacters and thus is known).

Definition 2.1.2 [1, Section 2.2] *As an algebra, the Hopf algebra $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ is generated by $\{e_g, x\}_{g \in G}$ satisfying*

$$e_g e_h = \delta_{g,h} e_g,\ x e_g = e_{g \triangleleft x} x,\ x^2 = \sum_{g \in G} \sigma(g) e_g, \quad g, h \in G.$$

The coproduct, counit and antipode are given by

$$\Delta(e_g) = \sum_{h,k \in G,\ hk=g} e_h \otimes e_k,\ \Delta(x) = [\sum_{g,h \in G} \tau(g,h) e_g \otimes e_h](x \otimes x),$$
$$\epsilon(x) = 1,\ \epsilon(e_g) = \delta_{g,1} 1,$$
$$\mathcal{S}(x) = \sum_{g \in G} \sigma(g)^{-1} \tau(g, g^{-1})^{-1} e_{g \triangleleft x} x,\ \mathcal{S}(e_g) = e_{g^{-1}},\ g \in G.$$

The following are some examples of $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ and we will discuss them in next sections.

Example 2.1.3 Let $n \in \mathbb{N}$ and assume that w is a primitive nth root of 1 in $\Bbbk$. Then *the generalized Kac-Paljutkin algebra H_{2n^2}* [58, Section 2.2] belongs to $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$. By definition, the data $(G, \triangleleft, \sigma, \tau)$ of H_{2n^2} is given by the following way

(i) $G = \mathbb{Z}_n \times \mathbb{Z}_n = \langle a, b | a^n = b^n = 1, ab = ba \rangle$ and $a \triangleleft x = b, b \triangleleft x = a$.

(ii) $\sigma(a^i b^j) = w^{ij}$ for $1 \leq i, j \leq n$.

(iii) $\tau(a^i b^j, a^k b^l) = (w)^{jk}$ for $1 \leq i, j, k, l \leq n$.

Among of them, if we take $n = 2$ then the resulting Hopf algebra is just the well-known Kac-Paljutkin 8-dimensional algebra K_8. That's the reason why we call H_{2n^2} the generalized Kac-Paljutkin algebra.

Example 2.1.4 Let n be a natural number. A Hopf algebra H belonging to $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is denoted by $K(8n,\sigma,\tau)$ if the data $(G,\triangleleft,\sigma,\tau)$ of H satisfies

(i) $G=\mathbb{Z}_{2n}\times\mathbb{Z}_2=\langle a,b|a^{2n}=b^2=1,ab=ba\rangle$;

(ii) $a\triangleleft x=ab, b\triangleleft x=b$.

If we take $n=1$ and let $\sigma(a^ib^j)=(-1)^{(i-j)j}$ and $\tau(a^ib^j,a^kb^l)=(-1)^{j(k-l)}$ for $1\leq i,j,k,l\leq 2$, then we can easily check that the resulting 8-dimensional Hopf algebra is just the Kac-Paljutkin 8-dimensional algebra K_8. Therefore, we give another kind of generalization of K_8.

Example 2.1.5 Let $n\in\mathbb{N}$ such that $n\geq 2$ and assume that ζ is a primitive $2n$th root of 1. A Hopf algebra H belonging to $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is denoted by $K(8n,\zeta)$ if the data $(G,\triangleleft,\sigma,\tau)$ of H satisfies the following conditions

(i) $G=\mathbb{Z}_{2n}\times\mathbb{Z}_2=\langle a,b|a^{2n}=b^2=1,ab=ba\rangle$ and $a\triangleleft x=ab, b\triangleleft x=b$.

(ii) $\sigma(a^ib^j)=(-1)^{\frac{i(i-1)}{2}}\zeta^i$ for $1\leq i\leq 2n$ and $1\leq j\leq 2$.

(iii) $\tau(a^ib^j,a^kb^l)=(-1)^{jk}$ for $1\leq i,k\leq 2n$ and $1\leq j,l\leq 2$.

This recover some familiar examples of semisimple Hopf algebras. For example, $K(16,\zeta)$ is the 16 dimensional Hopf algebra $H_{c:\sigma_1}$ in [30, Section 3.1]. Moreover, it can be seen that $K(8n,\zeta)$ belongs to $K(8n,\sigma,\tau)$.

Example 2.1.6 Let n be a natural number. A Hopf algebra H belonging to $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is denoted by $A(8n,\sigma,\tau)$ if the data $(G,\triangleleft,\sigma,\tau)$ of H satisfies

(i) $G=\mathbb{Z}_{4n}=\langle a|a^{4n}=1\rangle$;

(ii) $a\triangleleft x=a^{2n+1}$.

In fact, non-trivial Hopf algebra $A(8n,\sigma,\tau)$ exists. For example we can make $\sigma(a^i)=1$ and $\tau(a^i,a^j)=(-1)^{ij}$ for $1\leq i,j\leq 4n$, then we get a non-trivial Hopf algebra $A(8n,\sigma,\tau)$.

Example 2.1.7 [30, Section 3.1] The 16 dimensional semisimple Hopf algebra $H_{b:y}$ belongs to $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, and the data $(G,\triangleleft,\sigma,\tau)$ of $H_{b:y}$ is given as follows

(i) $G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a, b | a^4 = b^2 = 1, ab = ba \rangle$ and $a \triangleleft x = a^3, b \triangleleft x = b$.

(ii) $\sigma(a^i b^j) = (-1)^j,\ 1 \leq i \leq 4,\ 1 \leq j \leq 2$.

(iii) $\tau(a^i b^j, a^k b^l) = (-1)^{jk},\ 1 \leq i, k \leq 4,\ 1 \leq j, l \leq 2$.

Example 2.1.8 Let $n \in \mathbb{N}$. A Hopf algebra H belonging to $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ is denoted by $H^n_{b:y}$ if the data $(G, \triangleleft, \sigma, \tau)$ of H satisfies the following conditions

(i) $G = \mathbb{Z}_{4n} \times \mathbb{Z}_{2n} = \langle a, b | a^{4n} = b^{2n} = 1, ab = ba \rangle$ and $a \triangleleft x = a^{2n+1}, b \triangleleft x = b$.

(ii) $\sigma(a^i b^j) = (-1)^j$ for $1 \leq i \leq 4n$ and $1 \leq j \leq 2n$.

(iii) $\tau(a^i b^j, a^k b^l) = (-1)^{jk}$ for $1 \leq i, k \leq 4n$ and $1 \leq j, l \leq 2n$.

If $n = 1$, then $H^1_{b:y}$ is the 16 dimensional Hopf algebra $H_{b:y}$ in [30, Section 3.1]. The following example will be used to show that there exist Hopf algebras $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ such that they admit no quasitriangular structure.

Example 2.1.9 Let n be an odd number and let i be a primitive 4th root of 1. A Hopf algebra H belonging to $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ is denoted by A_{32n^2} if the data $(G, \triangleleft, \sigma, \tau)$ of H satisfies the following conditions

(i) $G = \mathbb{Z}_{4n} \times \mathbb{Z}_{4n} = \langle a, b | a^{4n} = b^{4n} = 1, ab = ba \rangle$ and $a \triangleleft x = a^{2n+1}, b \triangleleft x = b$;

(ii) $\sigma(g) = 1$ for $g \in G$;

(iii) $\tau(a^i b^j, a^k b^l) = (\mathrm{i})^{jk}$ for $1 \leq i, k \leq 4n$ and $1 \leq j, l \leq 4n$.

§2.2 Quasitriangular structures on Hopf algebras

In this section, we review the definition of quasitriangular structures on Hopf algebras and give some basic results about quasitriangular structures.

The definition of the Hopf algebra can be found in [59]. Recall that a quasitriangular Hopf algebra is a pair (H, R) where H is a Hopf algebra and $R = \sum R^{(1)} \otimes R^{(2)}$ is an invertible element in $H \otimes H$ such that

(i) $(\Delta \otimes \mathrm{Id})(R) = R_{13} R_{23}$ and $(\mathrm{Id} \otimes \Delta)(R) = R_{13} R_{12}$.

(ii) $\Delta^{op}(h) R = R \Delta(h)$ for $h \in H$.

Here by definition $R_{12} = \sum R^{(1)} \otimes R^{(2)} \otimes 1$, $R_{13} = \sum R^{(1)} \otimes 1 \otimes R^{(2)}$ and $R_{23} = \sum 1 \otimes R^{(1)} \otimes R^{(2)}$. The element R is called a universal $\mathcal{R}$-matrix of H or a quasitriangular structure on H. If R is a universal $\mathcal{R}$-matrix of H such that $RR_{21} = 1 (R_{21} := \tau(R))$, then we call R is a triangular structure on H.

The following lemma is well-known.

Lemma 2.2.1 [59, Proposition 12.2.11] *Let H be a Hopf algebra and $R \in H \otimes H$. For $f \in H^*$, if we denote $l(f) := (f \otimes \mathrm{Id})(R)$ and $r(f) := (\mathrm{Id} \otimes f)(R)$, then the following statements are equivalent*

(i) $(\Delta \otimes \mathrm{Id})(R) = R_{13}R_{23}$ *and* $(\mathrm{Id} \otimes \Delta)(R) = R_{13}R_{12}$.

(ii) $l(f_1)l(f_2) = l(f_1f_2)$ *and* $r(f_1)r(f_2) = r(f_2f_1)$ *for* $f_1, f_2 \in H^*$.

Let $\mathrm{C}(H^*) := \langle \chi_V |\ V$ is a finite dimensional representation of $H \rangle$ as vector space, where χ_V means the character of V, then we have

Lemma 2.2.2 *If (H, R) be a quasitriangular Hopf algebra, then $C(H^*)$ is a commutative subalgebra of H^*.*

Proof: Let V, W be two finite dimensional representations of H, then it is known that $\tau \circ R : V \otimes W \to W \otimes V$ is H-module map. Therefore we have $\chi_V \chi_W = \chi_W \chi_V$.

Chapter 3 Two forms of universal $\mathcal{R}$-matrices

In this chapter, we will prove that for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ there are at most two forms of universal $\mathcal{R}$-matrices. Then we give some necessary conditions for the existence of nontrivial forms. Using these necessary conditions, we determine all quasitriangular structures on generalized Kac-Paljutkin algebras H_{2n^2}(see Example 2.1.3).

§3.1 Forms of universal $\mathcal{R}$-matrices

In this section, we will prove that for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ there are at most two forms of universal $\mathcal{R}$-matrices.

The following lemma shows that the algebra structure of the dual Hopf algebra of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is very simple

Lemma 3.1.1 *Denote the dual basis of $\{e_g, e_gx\}_{g\in G}$ by $\{E_g, X_g\}_{g\in G}$, that is, $E_g(e_h) = \delta_{g,h}$, $E_g(e_hx) = 0$, $X_g(e_h) = 0$, $X_g(e_hx) = \delta_{g,h}$ for $g,h \in G$. Then the following equations hold in the dual Hopf algebra $(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$:*

$$E_gE_h = E_{gh},\ E_gX_h = X_hE_g = 0,\ X_gX_h = \tau(g,h)X_{gh},\ \ g,h \in G.$$

Proof: Direct computations show that

$$E_gE_h(e_k) = E_{gh}(e_k) = \delta_{gh,k},\ \ E_gE_h(e_kx) = E_{gh}(e_kx) = 0$$

for $g,h,k \in G$. As a result, we have $E_gE_h = E_{gh}$. Similarly, one can get the last two equations. □

Let $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ as before. Recall the sets S, T we have defined in Section 1.2, they are defined as follows

$$S := \{g \mid g \in G,\ g \triangleleft x = g\},\ \ T := \{g \mid g \in G,\ g \triangleleft x \neq g\}.$$

A very basic observation is:

Lemma 3.1.2 *We have $S \subseteq TT$ where $TT = \{gh \mid g,h \in T\}$.*

Proof: Clearly, for $s \in S, t \in T$, we have $ts \in T$. From the Definition 2.1.2 we know that the action $\triangleleft$ is injective, therefore $T \neq \emptyset$. Let $t \in T$ and it is obvious that $S = t(t^{-1}S)$ and hence $S \subseteq TT$. $\square$

With the help of S, T, we find that

Lemma 3.1.3 *Let $w^1 : G \times G \to \Bbbk$, $w^2 : G \times G \to \Bbbk$, $w^3 : G \times G \to \Bbbk$, $w^4 : G \times G \to \Bbbk$ be four maps and define R as follows*

$$\begin{aligned} R: = &\sum_{g,h \in G} w^1(g,h) e_g \otimes e_h + \sum_{g,h \in G} w^2(g,h) e_g x \otimes e_h + \\ &\sum_{g,h \in G} w^3(g,h) e_g \otimes e_h x + \sum_{g,h \in G} w^4(g,h) e_g x \otimes e_h x. \end{aligned}$$

If R satisfies $\Delta^{op}(e_g)R = R\Delta(e_g)$ for $g \in G$, then

(i) $w^2(t,g) = 0,\ t \in T, g \in G.$

(ii) $w^3(g,t) = 0,\ t \in T, g \in G.$

(iii) $w^4(s,t) = w^4(t,s) = 0,\ s \in S, t \in T.$

Proof: Because we have assumed that G is an abelian group, we get $\Delta^{op}(e_g) = \Delta(e_g)$. Since $\Delta^{op}(e_g)R = R\Delta(e_g)$ for $g \in G$ by the condition, we know $\Delta(e_g)R = R\Delta(e_g)$. Observe that $\{e_g, e_g x\}_{g \in G}$ is a linear basis for $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ and if we compare the two sides of the equation $\Delta(e_g)R = R\Delta(e_g)$ then we obtain the following equations

$$\Delta(e_g)[\sum_{h,k \in G} w^2(h,k) e_h x \otimes e_k] = [\sum_{h,k \in G} w^2(h,k) e_h x \otimes e_k]\Delta(e_g), \tag{3.1}$$

$$\Delta(e_g)[\sum_{h,k \in G} w^3(h,k) e_h \otimes e_k x] = [\sum_{h,k \in G} w^3(h,k) e_h \otimes e_k x]\Delta(e_g), \tag{3.2}$$

$$\Delta(e_g)[\sum_{h,k \in G} w^4(h,k) e_h x \otimes e_k x] = [\sum_{h,k \in G} w^4(h,k) e_h x \otimes e_k x]\Delta(e_g). \tag{3.3}$$

Firstly, we analyze equation (3.1) as follows

$$\Delta(e_g)[\sum_{h,k\in G} w^2(h,k)e_hx\otimes e_k] = \sum_{\substack{h,k\in G\\ hk=g}} w^2(h,k)e_hx\otimes e_k, \tag{3.4}$$

$$[\sum_{h,k\in G} w^2(h,k)e_hx\otimes e_k]\Delta(e_g) = \sum_{\substack{h,k\in G\\ hk=g}} w^2(h\triangleleft x,k)e_{h\triangleleft x}x\otimes e_k. \tag{3.5}$$

Note that if $h\in T, k\in G$ such that $hk=g$, then $e_hx\otimes e_k$ will appear in (3.4) while not in (3.5). As a result $w^2(h,k)=0$ for $h\in T, k\in G$ and thus (i) has been proved.

Similarly, for equation (3.2), there are the following equations

$$\Delta(e_g)[\sum_{h,k\in G} w^3(h,k)e_h\otimes e_kx] = \sum_{\substack{h,k\in G\\ hk=g}} w^3(h,k)e_h\otimes e_kx, \tag{3.6}$$

$$[\sum_{h,k\in G} w^3(h,k)e_h\otimes e_kx]\Delta(e_g) = \sum_{\substack{h,k\in G\\ hk=g}} w^3(h,k\triangleleft x)e_h\otimes e_{k\triangleleft x}x. \tag{3.7}$$

Observe that if $h\in G, k\in T$ such that $hk=g$, then $e_h\otimes e_kx$ will appear in (3.6) while not in (3.7). Therefore $w^3(h,k)=0$ for $h\in G, k\in T$ and so (ii) is proved.

For equation (3.3), we obtain the following equations

$$\Delta(e_g)[\sum_{h,k\in G} w^4(h,k)e_hx\otimes e_kx] = \sum_{\substack{h,k\in G\\ hk=g}} w^4(h,k)e_hx\otimes e_kx, \tag{3.8}$$

$$[\sum_{h,k\in G} w^4(h,k)e_hx\otimes e_kx]\Delta(e_g) = \sum_{\substack{h,k\in G\\ hk=g}} w^4(h\triangleleft x,k\triangleleft x)e_{h\triangleleft x}\otimes e_{k\triangleleft x}x. \tag{3.9}$$

Note that if $h\in S, k\in T$, then $e_hx\otimes e_kx$ and $e_kx\otimes e_hx$ will appear in (3.8) and not in (3.9). This implies that $w^4(h,k)=0$ for $h\in S, k\in T$. Similarly, one can find that $w^4(h,k)=0$ for $h\in T, k\in S$. Therefore (iii) has been proved. □

Lemma 3.1.4 *Let R be the element given in Lemma 3.1.3 and assume that $(\Delta\otimes \mathrm{Id})(R) = R_{13}R_{23}$, $(\mathrm{Id}\otimes\Delta)(R) = R_{13}R_{12}$. Then the following equations hold*

(i) $w^2(s_1,s_2)=w^3(s_1,s_2)=w^4(s_1,s_2)=0,\ s_1,s_2\in S.$

(ii) $w^1(g,t_2)w^4(t_1,t_2)=0,\ g\in G,\ t_1,t_2\in T.$

(iii) $w^1(t_1,g)w^4(t_1,t_2)=0,\ g\in G,\ t_1,t_2\in T.$

Proof: We have known $l(X_g)l(X_h)=l(X_gX_h)$ for $g,h\in G$ due to Lemma 2.2.1. Let $s\in S$ and we can find $t_1,t_2\in T$ such that $t_1t_2=s$ because of Lemma 3.1.2 and hence the following equation holds

$$l(X_{t_1}X_{t_2})=\tau(t_1,t_2)l(X_{t_1t_2})=\tau(t_1,t_2)[\sum_{g\in G}w^2(t_1t_2,g)e_g+\sum_{s\in S}w^4(t_1t_2,s)e_sx].$$

At the same time,

$$\begin{aligned}l(X_{t_1})l(X_{t_2})&=(\sum_{t\in T}w^4(t_1,t)e_tx)(\sum_{t\in T}w^4(t_2,t)e_tx)\\&=\sum_{t\in T}w^4(t_1,t)w^4(t_2,t\triangleleft x)e_tx^2\\&=\sum_{t\in T}w^4(t_1,t)w^4(t_2,t\triangleleft x)\sigma(t)e_t.\end{aligned}$$

Since $l(X_{t_1})l(X_{t_2})=l(X_{t_1}X_{t_2})$, we get that $w^4(s,s')=w^2(s,s')=0$ for $s'\in S$ and thus $w^4(s,s')=w^2(s,s')=0$ for $s,s'\in S$. Similarly by $r(X_{t_1})r(X_{t_2})=r(X_{t_2}X_{t_1})$ one can get that $w^3(s,s')=0$ for $s,s'\in S$. Therefore, (i) is proved.

It remains to show (ii) and (iii). We have known $l(E_g)l(X_{t_1})=0$ due to Lemma 3.1.1. However a direct computation shows that $l(E_g)l(X_{t_1})=\sum_{t\in T}w^1(g,t)w^4(t_1,t)e_tx$. Therefore $w^1(g,t)w^4(t_1,t)=0$ for $g\in G,t_1,t\in T$. Similarly, by $r(E_g)r(X_{t_1})=0$ we get that $w^1(t,g)w^4(t,t_1)=0$ for $g\in G,t_1,t\in T$. These are exactly (ii), (iii). □

The following proposition shows that universal $\mathcal{R}$-matrices of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ has only two possible forms.

Proposition 3.1.5 *Let R be the element given in Lemma 3.1.3 and assume that it is a universal $\mathcal{R}$-matrix of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Then R must belong to one of the following two cases:*

(i) $R=\sum\limits_{g,h\in G}w^1(g,h)e_g\otimes e_h;$

(ii) $R=\sum\limits_{s_1,s_2\in S}w^1(s_1,s_2)e_{s_1}\otimes e_{s_2}+\sum\limits_{s\in S,t\in T}w^2(s,t)e_sx\otimes e_t+\sum\limits_{t\in T,s\in S}w^3(t,s)e_t\otimes e_sx+\sum\limits_{t_1,t_2\in T}w^4(t_1,t_2)e_{t_1}x\otimes e_{t_2}x.$

Proof: Owing to Lemmas 3.1.3 and 3.1.4, we can assume that R has the following form:

$$R = \sum_{g,h\in G} w^1(g,h)e_g \otimes e_h + \sum_{s\in S,t\in T} w^2(s,t)e_s x \otimes e_t + \\ \sum_{t\in T,s\in S} w^3(t,s)e_t \otimes e_s x + \sum_{t_1,t_2\in T} w^4(t_1,t_2)e_{t_1}x \otimes e_{t_2}x.$$

If $w^4(t_1,t_2) = 0$ for all $t_1,t_2 \in T$, then $l(X_{t_1}) = l(X_{t_2}) = 0$. Using Lemma 3.1.1 we know that $l(X_{t_1})l(X_{t_2}) = l(X_{t_1}X_{t_2})$ and as a result $l(X_{t_1}X_{t_2}) = 0$ for all $t_1,t_2 \in T$. For $s \in S$, we can take $t_1,t_2 \in T$ such that $s = t_1t_2$. Hence we have that $l(X_{t_1}X_{t_2}) = \tau(t_1,t_2)(\sum_{t\in T} w^2(s,t)e_t) = 0$ which implies that $w^2(s,t) = 0$ for $s \in S,\ t \in T$. Similarly, by $r(X_{t_1}) = r(X_{t_2}) = 0$ and $r(X_{t_2}X_{t_1}) = \sum_{t\in T} \tau(t_2,t_1)w^3(t,s)e_t$, we have $w^3(t,s) = 0$ for $s \in S, t \in T$. Since $w^2(s,t) = w^3(t,s) = 0$ for $s \in S, t \in T$, we know that $R = \sum_{g,h\in G} w^1(g,h)e_g \otimes e_h$ and therefore we get the first case.

If there are $t_0, t_0' \in T$ such that $w^4(t_0,t_0') \neq 0$, then we will show that $w^1(t,g) = w^1(g,t) = 0$ for all $g \in G, t \in T$. For any $g \in G$, we have $w^1(g,t_0')w^4(t_0,t_0') = 0$ by (ii) of Lemma 3.1.4 and as a result $w^1(g,t_0') = 0$. Since R is invertible and $(e_t \otimes e_{t_0'})R = w^4(t,t_0')e_t x \otimes e_{t_0'}x$, we know that $w^4(t,t_0') \neq 0$ for $t \in T$. Next, we use (ii) and (iii) of Lemma 3.1.4 repeatedly. We have $w^1(t,g)w^4(t,t_0') = 0$ due to (iii) of Lemma 3.1.4. Thus $w^1(t,g) = 0$ for $t \in T, g \in G$. Since R is invertible and $(e_{t_1} \otimes e_{t_2})R = w^4(t_1,t_2)e_{t_1}x \otimes e_{t_2}x$ for $t_1,t_2 \in T$, we get that $w^4(t_1,t_2) \neq 0$ for $t_1,t_2 \in T$. Because $w^1(g,t)w^4(t_1,t) = 0$ by (ii) of Lemma 3.1.4, we know that $w^1(g,t) = 0$ for $g \in G, t \in T$ and hence we get the second case. $\square$

Remark 3.1.6 For simple, we will call a universal $\mathcal{R}$-matrix R in case (i) (resp. case (ii)) of Proposition 3.1.5 by a *trivial* (resp. *non-trivial*) quasitriangular structure.

§3.2 Universal $\mathcal{R}$-matrices of H_{2n^2}

To determine all universal $\mathcal{R}$-matrices of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we give necessary conditions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ preserving a non-trivial quasitriangular structure firstly. Then we give all universal $\mathcal{R}$-matrices of H_{2n^2} by using these necessary conditions. For any finite set X, we use $|X|$ to denote the number of elements in X.

Proposition 3.2.1 *If there is a non-trivial quasitriangular structure on* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$*, then*

(i) $|S| = |T|$;

(ii) *there is* $b \in S$ *such that* $b^2 = 1$ *and* $t \triangleleft x = tb$ *for* $t \in T$;

(iii) $\tau(s_1, s_2) = \tau(s_2, s_1),\ s_1, s_2 \in S$;

Proof: Assume that R is a non-trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, then we have $l(E_{t_1})l(E_{t_2}) = \sum_{s\in S} w^3(t_1, s)w^3(t_2, s)\sigma(s)e_s$ for $t_1, t_2 \in T$. In this situation, we claim that $TT = S$. In fact, suppose that there are $t_1, t_2 \in T$ satisfying $t_1t_2 \in T$. Then it is easy to see that $l(E_{t_1t_2}) = \sum_{s\in S} w^3(t_1t_2, s)e_sx$ which contradicts to the fact $l(E_{t_1})l(E_{t_2}) = l(E_{t_1t_2})$ (Lemma 2.2.1). Thus we have $TT = S$. Take a $t \in T$. We get that $tT \subseteq S$ and thus $|T| \le |S|$. Since $tS \subseteq T$, $|T| \ge |S|$. As a result we have $|T| = |S|$ and thus (i) has been proved. Next we will show (ii). Take a $t_0 \in T$, then we have $T = t_0S$. Let $t_0 \triangleleft x = t_1$ and denote $b = t_0^{-1}t_1$, then we have $b \in S$ by $TT = S$. Since $t_0S \subseteq T$ and $(t_0s) \triangleleft x = (t_0s)b$, we have $t \triangleleft x = tb$ for $t \in T$. It is easy to know that $b^2 = 1$ since $\triangleleft x$ is a group automorphism with order 2 and thus (ii) has been proved. Now let's show (iii). Assume that R is a non-trivial quasitrianglar structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, then we have $\Delta^{op}(x)R = R\Delta(x)$. Multiply both sides of this equation by $e_{s_1} \otimes e_{s_2}$ where $s_1, s_2 \in S$ and we note that $e_{s_1} \otimes e_{s_2}$ is an element in the center, so we get $(e_{s_1} \otimes e_{s_2})\Delta^{op}(x)R = R\Delta(x)(e_{s_1} \otimes e_{s_2})$. On the one hand, we have the following equation

$$\begin{aligned}
(e_{s_1} \otimes e_{s_2})\Delta^{op}(x)R &= (e_{s_1} \otimes e_{s_2})[\sum_{g,h\in G} \tau(h, g)e_g \otimes e_h](x \otimes x)R \\
&= [\tau(s_2, s_1)e_{s_1} \otimes e_{s_2}](x \otimes x)R \\
&= (x \otimes x)[\tau(s_2, s_1)e_{s_1} \otimes e_{s_2}]R \\
&= (x \otimes x)[\tau(s_2, s_1)w^1(s_1, s_2)(e_{s_1} \otimes e_{s_2})] \\
&= \tau(s_2, s_1)w^1(s_1, s_2)e_{s_1}x \otimes e_{s_2}x
\end{aligned}$$

On the other hand, the following equation hold

$$\begin{aligned} R\Delta(x)(e_{s_1} \otimes e_{s_2}) &= R[\sum_{g,h\in G} \tau(g,h)e_g \otimes e_h](x \otimes x)(e_{s_1} \otimes e_{s_2}) \\ &= R[\sum_{g,h\in G} \tau(g,h)e_g \otimes e_h](e_{s_1} \otimes e_{s_2})(x \otimes x) \\ &= R[\tau(s_1,s_2)e_{s_1} \otimes e_{s_2}](x \otimes x) \\ &= [\tau(s_1,s_2)w^1(s_1,s_2)e_{s_1} \otimes e_{s_2}](x \otimes x) \\ &= \tau(s_1,s_2)w^1(s_1,s_2)e_{s_1}x \otimes e_{s_2}x. \end{aligned}$$

Therefore, $(e_{s_1} \otimes e_{s_2})\Delta^{op}(x)R = R\Delta(x)(e_{s_1} \otimes e_{s_2})$ holds if and only if $\tau(s_1,s_2) = \tau(s_2,s_1)$. $\square$

Corollary 3.2.2 *If there are $t_1, t_2 \in T$ such that $t_1^{-1}(t_1 \triangleleft x) \neq t_2^{-1}(t_2 \triangleleft x)$, then $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ has no non-trivial quasitriangular structure.*

Proof: If $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ has a non-trivial quasitriangular structure, then there is $b \in S$ such that $t \triangleleft x = tb$ for $t \in T$ by (ii) of Proposition 3.2.1. Therefore $t^{-1}(t \triangleleft x) \equiv b$ for $t \in T$ and we have completed the proof. $\square$

The following proposition determine all possible trivial quasitriangular structures.

Proposition 3.2.3 *The element R is a trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ if and only if*

(i) $R = \sum_{g,h\in G} w(g,h)e_g \otimes e_h$ *for some bicharacter w on G;*

(ii) $w(g \triangleleft x, h \triangleleft x) = w(g,h)\eta(g,h)$ *where $\eta(g,h) = \tau(g,h)\tau(h,g)^{-1}$ for $g,h \in G$.*

Proof: We can assume that $R = \sum_{g,h\in G} w(g,h)e_g \otimes e_h$ is a trivial quasitriangular structure on it. Owing to $(\Delta \otimes \mathrm{Id})(R) = R_{13}R_{23}$ and $(\mathrm{Id} \otimes \Delta)(R) = R_{13}R_{12}$, we know (i). Expanding $\Delta^{op}(x)R = R\Delta(x)$, one can get (ii). $\square$

Next, we give a simple criterion to the quasitriangularity of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Corollary 3.2.4 *If there are $s_1, s_2 \in S$ such that $\eta(s_1,s_2) \neq 1$, then there is no quasi-triangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.*

Proof: By Proposition 3.2.1, we know that there is no non-trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Assume that there is a trivial quasitriangular structure, then we can use (ii) of Proposition 3.2.3 to get $w(s_1, s_2) = w(s_1, s_2)\eta(s_1, s_2)$. But this equation can't happen because of our assumption, so there is no trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. $\square$

The following proposition is a direct application of the above Corollary 3.2.4.

Proposition 3.2.5 *Let A_{32n^2} be the Hopf algebras in Example 2.1.9, then there is no quasitriangular structure on A_{32n^2} for any $n \in \mathbb{N}$.*

Proof: It can be seen that $a^{2n}, b \in S$ and $\eta(a^{2n}, b) = -1$, thus there is no quasitriangular structure by Corollary 3.2.4. $\square$

The following proposition is an application of above results and we get all universal $\mathcal{R}$-matrices of $H_{2n^2}(n \geq 3)$.

Proposition 3.2.6 *All universal $\mathcal{R}$-matrices of $H_{2n^2}(n \geq 3)$ are given by*

$$R = \sum_{1\leq i,j,k,l\leq n} \alpha^{ik+jl}\beta^{il+jk} e_{a^ib^j} \otimes e_{a^kb^l}$$

for some $\alpha, \beta \in \Bbbk$ satisfying $\alpha^n = \beta^n = 1$.

Proof: Since $n \geq 3$, we know $a^{-1}(a\triangleleft x) \neq b^{-1}(b\triangleleft x)$. Therefore H_{2n^2} has no non-trivial quasitriangular structure by Corollary 3.2.2. Assume that $R = \sum_{g,h\in G} w(g, h)e_g \otimes e_h$ is a trivial quasitriangular structure on H_{2n^2}, then w is a bicharacter on G and it satisfies the following equations by Proposition 3.2.3

$$\begin{aligned} &w(a, a)^n = 1, \quad w(a, b)^n = 1, \\ &w(b, a) = w(a, b), \quad w(b, b) = w(a, a). \end{aligned} \tag{3.10}$$

Let $w(a, a) := \alpha, w(a, b) := \beta$ and using the above series of equations (3.10), we get what we want. $\square$

Remark 3.2.7

(i) If $n = 2$, then H_8 is the 8-dimensional Kac-Paljutkin algebra K_8. All possible quasitriangular structures on K_8 were given in [71]. Proposition 3.2.6 above does not consider quasitriangular structures on K_8. In fact, trivial quasitriangular structures on K_8 can be given by Proposition 3.2.6, which only needs to set the parameter $n = 2$ in Proposition 3.2.6. Non-trivial quasitriangular structures on K_8 can be completely determined by using the (ii) in Lemma 2.2.1 and the equation $\Delta^{op}(x)R = R\Delta(x)$;

(ii) Because of Proposition 3.2.1 above and our aim is to find all non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we agree that $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ satisfies the conditions (i)-(iii) in Proposition 3.2.1 in the following content.

(iii) If we let $\eta(g,h) = \tau(g,h)\tau(h,g)^{-1}$ for $g, h \in G$, then η is a bicharacter on G due to τ is a 2-cocycle on the abelian group G and so (iii) of the Proposition 3.1.5 is equivalent to $\eta(s_1, s_2) = 1$ for $s_1, s_2 \in S$. We will often use η without explaination in the following content.

Chapter 4 Symmetries of quasitriangular structures on Hopf algebras

We will define symmetries of quasitriangular structures on Hopf algebras and give some relevant propositions in this chapter. Then we apply these propositions to the special Hopf algebras $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$.

§4.1 Symmetries of quasitriangular structures and some results

We will define symmetries of quasitriangular structures on Hopf algebras and give some relevant propositions in this section.

Let $(H, m, \eta, \Delta, \epsilon)$ be a Hopf algebra and let $R \in H \otimes H$. If $\varphi : H \to H^{op}$ is a Hopf isomorphism, then we denote $(\varphi \otimes \varphi) \circ \tau(R)$ as R_φ for the sake of convenience, here τ is the flip map and $H^{op} = (H, m \circ \tau, \eta, \Delta, \epsilon)$. Now we can define φ-symmetry of quasitriangular structures on Hopf algebras as follows

Definition 4.1.1 *Let $\varphi : H \to H^{op}$ be a Hopf isomorphism and let $R \in H \otimes H$, then we call R is φ-symmetric if $R = R_\varphi$. Moreover if $R = R_\varphi$ and it is a quasitriangular structure on H then we call R is a φ-symmetric quasitriangular structure.*

The reason why we introduced the above definition is due to the following propositions

Proposition 4.1.2 *Let $\varphi : H \to H^{op}$ be a Hopf isomorphism and let $R \in H \otimes H$, then we have*

(i) *l_R is algebra map $\Leftrightarrow$ r_{R_φ} is anti-algebra map;*

(ii) *r_R is anti-algebra map $\Leftrightarrow$ l_{R_φ} is algebra map;*

(iii) *$\Delta^{op}(h)R = R\Delta(h)$ for $h \in H$ $\Leftrightarrow$ $\Delta^{op}(\varphi(h))R_\varphi = R_\varphi\Delta(\varphi(h))$ for $h \in H$;*

(iv) *R is invertible $\Leftrightarrow$ R_φ is invertible.*

Proof: Since $R = (R_\varphi)_{\varphi^{-1}}$, we only need to prove half of the proposition. Let φ^* be the dual map of φ, then $\varphi^* : H^* \to H^{cop}$ is a Hopf isomorphism, here $H^{cop} = (H, m, \eta, \tau\circ\Delta, \epsilon)$. If we denote $l_R(f) := (f\otimes \mathrm{Id})(R)$, $r_R(f) := (\mathrm{Id}\otimes f)(R)$ respectively, then we claim that the following equations hold

$$l_{R_\varphi} = \varphi\circ r_R\circ\varphi^*,\ r_{R_\varphi} = \varphi\circ l_R\circ\varphi^*. \tag{4.1}$$

Directly we have

$$\begin{aligned} r_{R_\varphi}(f) &= (\mathrm{Id}\otimes f)[(\varphi\otimes\varphi)\circ\tau(R)] = (\varphi\otimes f\circ\varphi)\circ\tau(R)\\ &= (\varphi\otimes\varphi^*(f))\circ\tau(R) = (\varphi^*(f)\otimes\varphi)(R)\\ &= \varphi[(\varphi^*(f)\otimes\mathrm{Id})(R)] = \varphi[l_R\circ\varphi^*(f)]\\ &= (\varphi\circ l_R\circ\varphi^*)(f), \end{aligned}$$

and

$$\begin{aligned} l_{R_\varphi}(f) &= (f\otimes\mathrm{Id})[(\varphi\otimes\varphi)\circ\tau(R)] = (f\circ\varphi\otimes\varphi)\circ\tau(R)\\ &= (\varphi^*(f)\otimes\varphi)\circ\tau(R) = (\varphi\otimes\varphi^*(f))(R)\\ &= \varphi[(\mathrm{Id}\otimes\varphi^*(f))(R)] = \varphi[r_R\circ\varphi^*(f)]\\ &= (\varphi\circ r_R\circ\varphi^*)(f), \end{aligned}$$

so (i),(ii) hold. Suppose that $R = \Sigma_{i=1}^n r_i\otimes r^i$ and $\Delta^{op}(h)R = R\Delta(h)$. Taking a $k\in H$, then we can write it by $k = \varphi(h),\ h\in H$ due to φ is bijective map. Using φ is Hopf isomorphism, we get

$$\begin{aligned} \Delta^{op}(\varphi(h))R_\varphi &= [\varphi(h_{(2)})\otimes\varphi(h_{(1)})]R_\varphi = [\varphi(h_{(2)})\otimes\varphi(h_{(1)})][\Sigma_{i=1}^n\varphi(r^i)\otimes\varphi(r_i)]\\ &= \Sigma_{i=1}^n\varphi(r^ih_{(2)})\otimes\varphi(r_ih_{(1)}) = (\varphi\otimes\varphi)[\Sigma_{i=1}^n r^ih_{(2)}\otimes r_ih_{(1)}] \end{aligned}$$

and

$$\begin{aligned} R_\varphi\Delta(\varphi(h)) &= R_\varphi[\varphi(h_{(1)})\otimes\varphi(h_{(2)})] = [\Sigma_{i=1}^n\varphi(r^i)\otimes\varphi(r_i)][\varphi(h_{(1)})\otimes\varphi(h_{(2)})]\\ &= \Sigma_{i=1}^n\varphi(h_{(1)}r^i)\otimes\varphi(h_{(2)}r_i) = (\varphi\otimes\varphi)[\Sigma_{i=1}^n h_{(1)}r^i\otimes h_{(2)}r_i]. \end{aligned}$$

Because $\Delta^{op}(h)R = R\Delta(h)$, we know $(h_{(2)} \otimes h_{(1)})(\Sigma_{i=1}^n r_i \otimes r^i) = (\Sigma_{i=1}^n r_i \otimes r^i)(h_{(1)} \otimes h_{(2)})$. If we use the flip map τ acting on both sides of this equation, then we get $\Sigma_{i=1}^n r^i h_{(2)} \otimes r_i h_{(1)} = \Sigma_{i=1}^n h_{(1)} r^i \otimes h_{(2)} r_i$ and hence $\Delta^{op}(\varphi(h))R_\varphi = R_\varphi \Delta(\varphi(h))$ for $h \in H$. That is to say (iii) holds. Assume R^{-1} is the inverse of R, then it can be seen that $R_\varphi (R^{-1})_\varphi = 1 \otimes 1$ and thus we have (iv). $\square$

Recall that we call $(H, *)$ is a $*$-Hopf algebra over $\mathbb{C}$ if $* : H \to H$ is an anti-multiplicative conjugate linear involution and comultiplicative, where H is a Hopf algebra. Similar to Proposition 4.1.2, we have the following proposition.

Proposition 4.1.3 *Let $(H, *)$ be a $*$-Hopf algebra and let $R \in H \otimes H$. If we denote $R_* := (* \otimes *) \circ \tau(R)$, then we have*

(i) l_R is algebra map $\Leftrightarrow$ r_{R_*} is anti-algebra map;

(ii) r_R is anti-algebra map $\Leftrightarrow$ l_{R_*} is algebra map;

(iii) $\Delta^{op}(h)R = R\Delta(h)$ for $h \in H$ $\Leftrightarrow$ $\Delta^{op}(h)R_* = R_*\Delta(h)$ for $h \in H$;

(iv) R is invertible $\Leftrightarrow$ R_* is invertible.

Proof: Similar to the proof of Proposition 4.1.2.

Remark 4.1.4 So far, all known semisimple Hopf algebras over $\mathbb{C}$ have involutions, i.e they are $*$-Hopf algebras. Therefore, the discussion on the symmetry of quasitriangular structures are applicable to the known semisimple Hopf algebras. Moreover, the following propositions in this section are also hold for $*$-Hopf algebras when we replace the φ with $*$ and we don't plan to list them.

Proposition 4.1.5 *Let $\varphi : H \to H^{op}$ be a Hopf isomorphism and let $R \in H \otimes H$. If R is φ-symmetric, then $l_R : H^* \to H$ is an algebra map if and only if $r_R : H^* \to H^{op}$ is an algebra map.*

Proof: Since $R = R_\varphi$ and we have proved $l_{R_\varphi} = \varphi \circ r_R \circ \varphi^*$, $r_{R_\varphi} = \varphi \circ l_R \circ \varphi^*$ in Proposition 4.1.2, we get $l_R = \varphi \circ r_R \circ \varphi^*$ and $r_R = \varphi \circ l_R \circ \varphi^*$. Using these equations we get what we want. $\square$

Corollary 4.1.6 *Let $R \in H \otimes H$. If R is φ-symmetric and it is invertible, then R is a quasitriangular structure if and only if $l_R : H^* \to H$ is an algebra map and $\Delta^{op}(h)R = R\Delta(h)$ for $h \in H$.*

Proof: By Proposition 4.1.5 and Lemma 2.2.1, we get what we want. □

§4.2 Applying to $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$

In order to apply the results of Section 4.1 to our case, we give the following conclusions.

Proposition 4.2.1 *Let $\varphi : \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2 \to (\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2)^{op}$ be a linear map which is determined by $\varphi(e_g) := e_{g\triangleleft x}$, $\varphi(e_g x) := e_g x$, then φ is a Hopf isomorphism.*

Proof: Obviously φ is bijective, thus we only need to show φ is a bialgebra map. To show φ is an algebra map, the only non-trivial thing is to check $\varphi(e_g)\varphi(e_h x) = \varphi[(e_h x)e_g]$. Directly we have $\varphi(e_g)\varphi(e_h x) = e_{g\triangleleft x}(e_h x) = \delta_{g\triangleleft x,h} e_h x$ and $\varphi[(e_h x)e_g] = \varphi(e_h e_{g\triangleleft x} x) = \delta_{g\triangleleft x,h} e_h x$, so $\varphi(e_g)\varphi(e_h x) = \varphi[(e_h x)e_g]$. To prove that φ is a coalgebra map, we consider the dual map φ^*. Denote the dual basis of $\{e_g, e_g x\}_{g\in G}$ by $\{E_g, X_g\}_{g\in G}$, then it can be seen that $\varphi^*(E_g) = E_{g\triangleleft x}$ and $\varphi^*(X_g) = X_g$. Therefore it is easy to see that φ^* is an algebra map and this implies that φ is a coalgebra map. □

Let R be the form (ii) in Proposition 3.1.5 and let φ be the Hopf isomorphism in Proposition 4.2.1 above, then R_φ is given by

$$R_\varphi = \sum_{s_1,s_2\in S} w^1(s_2,s_1) e_{s_1} \otimes e_{s_2} + \sum_{s\in S,t\in T} w^3(t\triangleleft x, s) e_s x \otimes e_t + \sum_{t\in T,s\in S} w^2(s, t\triangleleft x) e_t \otimes$$
$$e_s x + \sum_{t_1,t_2\in T} w^4(t_2,t_1) e_{t_1} x \otimes e_{t_2} x. \tag{4.2}$$

Corollary 4.2.2 *R is a quasitriangular structure on $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ if and only if R_φ is a quasitriangular structure on $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$.*

Proof: Owing to the Proposition 4.1.2 and Proposition 4.2.1, we get what we want. □

Corollary 4.2.3 *Let R be the form* (ii) *in Proposition 3.1.5, and if* $w^i (1 \leq i \leq 4)$ *satisfy the following conditions*

(i) $w^1(s_1, s_2) = w^1(s_2, s_1)$ *for* $s_1, s_2 \in S$;

(ii) $w^2(s, t) = w^3(t \triangleleft x, s)$ *for* $s \in S, t \in T$;

(iii) $w^4(t_1, t_2) = w^4(t_2, t_1)$ *for* $t_1, t_2 \in T$;

then R is a quasitriangular structure if and only if l_R is an algebra map and $\Delta^{op}(h)R = R\Delta(h)$ *for* $h \in \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$.

Proof: Since R_φ is given by the form (4.2) and (i)-(iii), it can be seen that R is φ-symmetric. Thanks to Corollary 4.1.6, we get what we want. $\square$

Remark 4.2.4 For our convenience, we agree that φ mentioned in the following content refers to the φ in Proposition 4.2.1.

Chapter 5 Quasitriangular functions on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

In this chapter, we prove that non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ are in one-one correspondence to some special functions on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, which we call quasitriangular functions on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. After that, we will focus on quasitriangular functions and give a criterion for when a function is a quasitriangular function.

§5.1 The one to one correspondence between quasitriangular functions and quasitriangular structures

In this section, we give the definition of quasitriangular functions on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and prove that there is a natural one-one correspondence between quasitriangular functions and quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Let R be the form (ii) in Proposition 3.1.5 and we will use this R without explanation in the following sections, then

Lemma 5.1.1 *The equations $\Delta^{op}(h)R = R\Delta(h)$ hold for $h \in \Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ if and only if the following equations hold*

$$w^2(s, t \triangleleft x) = w^2(s,t)\eta(s,t), \ s \in S, t \in T, \tag{5.1}$$

$$w^3(t \triangleleft x, s) = w^3(t,s)\eta(t,s), \ s \in S, t \in T, \tag{5.2}$$

$$\tau(t_2,t_1)w^4(t_1 \triangleleft x, t_2 \triangleleft x) = \tau(t_1 \triangleleft x, t_2 \triangleleft x)w^4(t_1,t_2), \ t_1,t_2 \in T. \tag{5.3}$$

Proof: Since R is invertible and $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is generated by $\{e_g, x|\ g \in G\}$ as algebra, $\Delta^{op}(h) = R\Delta(h)R^{-1}$ for $h \in \Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is equivalent to $\Delta^{op}(h) = R\Delta(h)R^{-1}$ for $h \in \{e_g, x|\ g \in G\}$. We first prove that $\Delta^{op}(e_g)R = R\Delta(e_g)$ for $g \in G$. Taking $s \in S, t \in T$, then directly we have

$$\Delta^{op}(e_s)R = [\sum_{\substack{s_1,s_2\in S\\ s_1s_2=s}} w^1(s_1,s_2)e_{s_1}\otimes e_{s_2}] + [\sum_{\substack{t_1,t_2\in T\\ t_1t_2=s}} w^4(t_1,t_2)e_{t_1}x\otimes e_{t_2}x]$$

and

$$R\Delta(e_s) = [\sum_{\substack{s_1,s_2\in S\\ s_1s_2=s}} w^1(s_1,s_2)e_{s_1}\otimes e_{s_2}] + [\sum_{\substack{t_1,t_2\in T\\ t_1t_2=s}} w^4(t_1\triangleleft x, t_2\triangleleft x)e_{t_1\triangleleft x}x\otimes e_{t_2\triangleleft x}x].$$

Owing to $t_1t_2 = (t_1\triangleleft x)(t_2\triangleleft x)$ by definition, thus $\Delta^{op}(s)R = R\Delta(s)$. Similarly, we have

$$\Delta^{op}(e_t)R = R\Delta(e_t) = [\sum_{\substack{s\in S,t'\in T\\ st'=s}} w^2(s,t')e_sx\otimes e_{t'}] + [\sum_{\substack{s\in S,t'\in T\\ st'=s}} w^3(t',s)e_{t'}\otimes e_sx],$$

but $G = S\cup T$ and so we have showed $\Delta^{op}(e_g)R = R\Delta(e_g)$ for $g\in G$. Next we prove that $\Delta^{op}(x)R = R\Delta(x)$ is equivalent to above equations (5.1)-(5.3). On the one hand, we have the following equation

$$\begin{aligned}\Delta^{op}(x)R &= [\sum_{g,h\in G}\tau(h,g)e_g\otimes e_h](x\otimes x)R\\ &= [\sum_{s_1,s_2\in S}\tau(s_2,s_1)w^1(s_1,s_2)e_{s_1}\otimes e_{s_2} + \sum_{s\in S,t\in T}\tau(t,s)w^2(s,t\triangleleft x)e_sx\otimes e_t+\\ &\sum_{t\in T,s\in S}\tau(s,t)w^3(t\triangleleft x,s)e_t\otimes e_sx+\\ &\sum_{t_1,t_2\in T}\tau(t_2,t_1)w^4(t_1\triangleleft x,t_2\triangleleft x)e_{t_1}x\otimes e_{t_2}x](x\otimes x),\end{aligned}$$

On the other hand, the following equations hold

$$\begin{aligned}R\Delta(x) &= R[\sum_{g,h\in G}\tau(g,h)e_g\otimes e_h](x\otimes x)\\ &= [\sum_{s_1,s_2\in S}\tau(s_1,s_2)w^1(s_1,s_2)e_{s_1}\otimes e_{s_2} + \sum_{s\in S,t\in T}\tau(s,t)w^2(s,t)e_sx\otimes e_t+\\ &\sum_{t\in T,s\in S}\tau(t,s)w^3(t,s)e_t\otimes e_sx+\\ &\sum_{t_1,t_2\in T}\tau(t_1\triangleleft x,t_2\triangleleft x)w^4(t_1,t_2)e_{t_1}x\otimes e_{t_2}x](x\otimes x).\end{aligned}$$

Therefore, $\Delta^{op}(x)R = R\Delta(x)$ holds if and only if equations (5.1)-(5.3) hold. □

If R is a quasitriangular structure, then R is completely determined by w^4. The following lemma states this fact. For simplify, we denote $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ by H_G and we will use this notation in this section without explanation.

Lemma 5.1.2 *If $l(f_1)l(f_2) = l(f_1f_2)$ and $r(f_1)r(f_2) = r(f_2f_1)$ for $f_1, f_2 \in (H_G)^*$, then $w^i(1 \leq i \leq 3)$ of R are completely determined by w^4 as follows*

(i) $w^1(s_1, s_2) = \frac{w^4(s_1t_1,s_2t_2)w^4(t_1,t_2)}{w^4(s_1t_1,t_2)w^4(t_1,s_2t_2)}$;

(ii) $w^2(s,t) = \tau(s,t_1)\frac{w^4(st_1,t)}{w^4(t_1,t)}$;

(iii) $w^3(t,s) = \tau(s,t_1)\frac{w^4(t\triangleleft x,st_1)}{w^4(t\triangleleft x,t_1)}$;

where $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$.

Proof: We first show (ii). Taking $s \in S, t_1 \in T$, then we have $l(X_s)l(X_{t_1}) = l(X_sX_{t_1})$ by our assumption. We expand this equation as follows

$$l(X_s)l(X_{t_1}) = [\sum_{t\in T} w^2(s,t)e_t][\sum_{t\in T} w^4(t_0,t_1)e_tx] = [\sum_{t\in T} w^2(s,t)w^4(t_0,t)e_tx]$$

and

$$l(X_sX_{t_1}) = \tau(s,t_1)l(X_{st_1}) = [\sum_{t\in T} \tau(s,t_1)w^4(st_1,t)e_tx],$$

so we have $w^2(s,t)w^4(t_1,t) = \tau(s,t_1)w^4(st_1,t)$ and this implies that (ii) holds. Then we will show (i). Let $s_1, s_2 \in S$ and let $t_1, t_2 \in T$. Owing to $r(E_{s_2})r(E_{t_2}) = r(E_{t_2}E_{s_2})$ by assumption, we can expand this equation as follows

$$r(E_{s_2})r(E_{t_2}) = [\sum_{s_1\in S} w^1(s_1,s_2)e_{s_1}][\sum_{s_1\in S} w^2(s_1,t_2)e_{s_1}x] = [\sum_{s_1\in S} w^1(s_1,s_2)w^2(s_1,t_2)e_{s_1}x]$$

and

$$r(E_{t_2}E_{s_2}) = r(E_{s_2t_2}) = [\sum_{s_1\in S} w^2(s_1,s_2t_2)e_{s_1}x],$$

thus we get $w^1(s_1,s_2)=\frac{w^2(s_1,s_2t_2)}{w^2(s_1,t_2)}$. But we have showed the following equations

$$w^2(s_1,s_2t_2)=\tau(s_1,t_1)\frac{w^4(s_1t_1,s_2t_2)}{w^4(t_1,s_2t_2)},\ w^2(s_1,t_2)=\tau(s_1,t_1)\frac{w^4(s_1t_1,t_2)}{w^4(t_1,t_2)},$$

therefore we know *(i)* holds. To show (iii), we consider R_φ(here φ is the Hopf isomorphism in Proposition 4.2.1). Since the proof of Proposition 4.1.2, we know R_φ satisfy $l_{R_\varphi}(f_1)l_{R_\varphi}(f_2)=l_{R_\varphi}(f_1f_2)$ and $r_{R_\varphi}(f_1)r_{R_\varphi}(f_2)=r_{R_\varphi}(f_2f_1)$ for $f_1,f_2\in(H_G)^*$, i.e R_φ such that the conditions of this Lemma. Denote the $w^i(1\le i\le 4)$ of R_φ by $w'^i(1\le i\le 4)$, then we have $w'^2(s,t)=w^3(t\triangleleft x,s)$ and $w'^4(t_1,t_2)=w^4(t_2,t_1)$ for $s\in S$ and $t_1,t_2\in T$ by (4.2). But we have proved that (ii) holds, we get that $w'^2(s,t)=\tau(s,t_1)\frac{w'^4(st_1,t)}{w'^4(t_1,t)}$. And hence we know $w^3(t\triangleleft x,s)=\tau(s,t_1)\frac{w^4(t,st_1)}{w^4(t,t_1)}$ and this implies (iii). □

The following lemma gives a criterion for when R is a non-trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Lemma 5.1.3 *The R is a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ if and only if the following equations hold*

$$l(E_g)l(E_h)=l(E_gE_h),\ l(X_g)l(X_h)=l(X_gX_h),\ g,h\in G, \tag{5.4}$$

$$r(E_g)r(E_h)=r(E_hE_g),\ r(X_g)r(X_h)=r(X_hX_g),\ g,h\in G, \tag{5.5}$$

$$\tau(t_2,t_1)w^4(t_1\triangleleft x,t_2\triangleleft x)=\tau(t_1\triangleleft x,t_2\triangleleft x)w^4(t_1,t_2),\ t_1,t_2\in T. \tag{5.6}$$

Proof: Since the Lemma 2.2.1, Lemma 5.1.1 and the definition of quasitriangular structures, we know that if R is a quasitriangular structure then it satisfies the above equations (5.4)-(5.6). Conversely, suppose R such that equations (5.4)-(5.6), we will first prove that $l(f_1)l(f_2)=l(f_1f_2)$ and $r(f_1)r(f_2)=r(f_2f_1)$ for $f_1,f_2\in(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$. For $s\in S,t\in T$, since

$$l(E_s)=\sum_{s'\in S}w^1(s,s')e_{s'},\ l(E_t)=\sum_{s'\in S}w^3(t,s')e_{s'}x$$

and

$$l(X_s) = \sum_{t' \in T} w^2(s,t')e_{t'},\ l(X_t) = \sum_{t' \in T} w^4(t,t')e_{t'}x,$$

it can be seen that $l(E_g)l(X_h) = l(E_gX_h) = 0$ and $l(X_g)l(E_h) = l(X_gE_h) = 0$ for $g,h \in G$. Because $(\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$ is linear spanned by $\{E_g, X_g |\ g \in G\}$, we get that $l(f_1)l(f_2) = l(f_1f_2)$ for $f_1, f_2 \in (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$. Using a similar discussion, we will know that $r(f_1)r(f_2) = r(f_2f_1)$ for $f_1, f_2 \in (\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2)^*$. Secondly, we will prove that $\Delta^{op}(h)R = R\Delta(h)$ for $h \in \Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Owing to the Lemma 5.1.1, we only need to prove that $w^2(s,t\triangleleft x) = w^2(s,t)\eta(s,t)$ and $w^3(t\triangleleft x,s) = w^3(t,s)\eta(t,s)$ for $s \in S, t \in T$. Let $t_0 \in T$, then we have the following equations by (ii) of Lemma 5.1.2

$$w^2(s,t) = \tau(s,t_0)\frac{w^4(st_0,t)}{w^4(t_0,t)},\ w^2(s,t\triangleleft x) = \tau(s,t_0\triangleleft x)\frac{w^4(st_0\triangleleft x,t\triangleleft x)}{w^4(t_0\triangleleft x,t\triangleleft x)}. \tag{5.7}$$

Due to the assumption, we have

$$w^4(st_0\triangleleft x,t\triangleleft x) = \frac{\tau(st_0\triangleleft x,t\triangleleft x)}{\tau(t,st_0)}w^4(st_0,t) \tag{5.8}$$

and

$$w^4(t_0\triangleleft x,t\triangleleft x) = \frac{\tau(t_0\triangleleft x,t\triangleleft x)}{\tau(t,t_0)}w^4(t_0,t). \tag{5.9}$$

So $w^2(s,t\triangleleft x) = w^2(s,t)\frac{\tau(s,t_0\triangleleft x)}{\tau(s,t_0)}\frac{\tau(st_0\triangleleft x,t\triangleleft x)}{\tau(t,st_0)}\frac{\tau(t,t_0)}{\tau(t_0\triangleleft x,t\triangleleft x)}$. Using τ is two cocycle, we get

$$\begin{aligned}
\frac{\tau(s,t_0\triangleleft x)}{\tau(s,t_0)}\frac{\tau(st_0\triangleleft x,t\triangleleft x)}{\tau(t,st_0)}\frac{\tau(t,t_0)}{\tau(t_0\triangleleft x,t\triangleleft x)} &= \frac{\tau(s,t_0t)\tau(t_0\triangleleft x,t\triangleleft x)}{\tau(s,t_0)\tau(t,st_0)}\frac{\tau(t,t_0)}{\tau(t_0\triangleleft x,t\triangleleft x)} \\
&= \frac{\tau(s,t_0t)\tau(t_0\triangleleft x,t\triangleleft x)}{\eta(s,t_0)\tau(t_0,s)\tau(t,st_0)}\frac{\tau(t,t_0)}{\tau(t_0\triangleleft x,t\triangleleft x)} \\
&= \frac{\tau(s,t_0t)\tau(t_0\triangleleft x,t\triangleleft x)}{\eta(s,t_0)\tau(tt_0,s)\tau(t,t_0)}\frac{\tau(t,t_0)}{\tau(t_0\triangleleft x,t\triangleleft x)} \\
&= \frac{\eta(s,tt_0)}{\eta(s,t_0)} = \eta(s,t).
\end{aligned}$$

Therefore $w^2(s,t\triangleleft x) = w^2(s,t)\eta(s,t)$. To show $w^3(t\triangleleft x,s) = w^3(t,s)\eta(t,s)$, we consider R_φ and denote the $w^i(1 \le i \le 4)$ of R_φ by $w'^i(1 \le i \le 4)$, then we have $w'^2(s,t) = w^3(t\triangleleft x,s)$ and $w'^4(t_1,t_2) = w^4(t_2,t_1)$ for $s \in S, t_1,t_2 \in T$ by (4.2). Owing to $\tau(t_1\triangleleft x,t_2\triangleleft$

$x)\tau(t_1,t_2) = \tau(t_2 \triangleleft x, t_1 \triangleleft x)\tau(t_2,t_1) = \sigma(t_1t_2)\sigma(t_1)^{-1}\sigma(t_2)^{-1}$, then we have $\frac{\tau(t_1 \triangleleft x, t_2 \triangleleft x)}{\tau(t_2,t_1)} = \frac{\tau(t_2 \triangleleft x, t_1 \triangleleft x)}{\tau(t_1,t_2)}$ and hence it is easy to see that R_φ also satisfies the conditions of this Lemma. But we have showed that $w'^2(s, t \triangleleft x) = w'^2(s,t)\eta(s,t)$, so $w^3(t,s) = w^3(t \triangleleft x, s)\eta(s,t)$. Since $\eta(s,t)^{-1} = \eta(t,s)$, we know $w^3(t \triangleleft x, s) = w^3(t,s)\eta(t,s)$ and therefore we have completed the proof. □

Since H_G is determined by the data $(G, \triangleleft, \sigma, \tau)$, naturally we can guess that all non-trivial quasitriagular structures on H_G can be expressed by using the data $(G, \triangleleft, \sigma, \tau)$. To confirm this conjecture, we use the following propositions.

Proposition 5.1.4 *If R is a universal $\mathcal{R}$-matrix of H_G, then*

(i) $\tau(s,t_1)\frac{w^4(st_1,t)}{w^4(t_1,t)} = \tau(s,t_2)\frac{w^4(st_2,t)}{w^4(t_2,t)}$;

(ii) $\tau(s,t_1)\frac{w^4(t,st_1)}{w^4(t,t_1)} = \tau(s,t_2)\frac{w^4(t,st_2)}{w^4(t,t_2)}$;

(iii) $w^4(t,t_1)w^4(t^{-1}, t_1 \triangleleft x)\sigma(t_1) = \tau(t,t^{-1})$;

(iv) $w^4(t_1,t)w^4(t_1 \triangleleft x, t^{-1})\sigma(t_1) = \tau(t,t^{-1})$;

(v) $w^4(t_1 \triangleleft x, t_2 \triangleleft x) = \frac{\tau(t_1 \triangleleft x, t_2 \triangleleft x)}{\tau(t_2,t_1)} w^4(t_1,t_2)$;

where $s \in S,\ t, t_1, t_2 \in T$.

Proof: Since (ii) of Lemma 5.1.2, we know $w^2(s,t) = \tau(s,t_1)\frac{w^4(st_1,t)}{w^4(t_1,t)} = \tau(s,t_2)\frac{w^4(st_2,t)}{w^4(t_2,t)}$ for $s \in S$ and $t, t_1 \in T$. Therefore (i) holds. Similarly, we get (ii) due to (iii) of Lemma 5.1.2. Owing to R is a universal $\mathcal{R}$-matrix, we have $l(X_t)l(X_{t^{-1}}) = l(X_tX_{t^{-1}})$. Then we expand the equation as follows

$$
\begin{aligned}
l(X_t)l(X_{t^{-1}}) &= [\sum_{t_1 \in T} w^4(t,t_1)e_{t_1}x][\sum_{t_1 \in T} w^4(t^{-1},t_1)e_{t_1}x] \\
&= [\sum_{t_1 \in T} w^4(t,t_1)w^4(t, t_1 \triangleleft x)\sigma(t_1)e_{t_1}]
\end{aligned}
$$

and

$$
l(X_tX_{t^{-1}}) = \tau(t,t^{-1})l(X_1) = \tau(t,t^{-1})[\sum_{t \in T} w^2(1,t_1)e_{t_1}x] = [\sum_{t \in T} \tau(t,t^{-1})e_{t_1}x],
$$

so we have (iii). Similarly, we get $r(X_t)r(X_{t^{-1}}) = r(X_{t^{-1}}X_t)$. And if we expand this equation then we get $w^4(t_1,t)w^4(t_1 \triangleleft x, t^{-1})\sigma(t_1) = \tau(t^{-1},t)$. But $\tau(t^{-1},t) = \tau(t,t^{-1})$ due to $\eta(t,t^{-1}) = 1$, therefore we know (iv) holds. (v) is a conclusion of Lemma 5.1.3 and so we have completed the proof. $\square$

In fact, given a function $w : T \times T \to \Bbbk^\times$ that satisfies the above conditions, we can find a unique quasitriangular structure R that satisfies $w^4 = w$. And we will show this in Theorem 5.1.11. Because of this reason, we introduce the concept of quasitriangular functions on H_G.

Definition 5.1.5 *A quasitriangular function on H_G is a function $w : T \times T \to \Bbbk^\times$ such that (i)-(v) in Proposition 5.1.4, i.e it satisfies the following condtions*

(i) $\tau(s,t_1)\frac{w(st_1,t)}{w(t_1,t)} = \tau(s,t_2)\frac{w(st_2,t)}{w(t_2,t)}$;

(ii) $\tau(s,t_1)\frac{w(t,st_1)}{w(t,t_1)} = \tau(s,t_2)\frac{w(t,st_2)}{w(t,t_2)}$;

(iii) $w(t,t_1)w(t^{-1},t_1 \triangleleft x)\sigma(t_1) = \tau(t,t^{-1})$;

(iv) $w(t_1,t)w(t_1 \triangleleft x, t^{-1})\sigma(t_1) = \tau(t,t^{-1})$;

(v) $w(t_1 \triangleleft x, t_2 \triangleleft x) = \frac{\tau(t_1 \triangleleft x, t_2 \triangleleft x)}{\tau(t_2,t_1)} w(t_1,t_2)$;

where $s \in S,\ t, t_1, t_2 \in T$.

It can be seen that the definition of quasitriangular functions is expressed by the data $(G, \triangleleft, \sigma, \tau)$. Furthermore, we will see that non-trivial quasitriangular structures on H_G are in one-one correspondence to quasitriangular functions on it in Corollary 5.1.12. Since we will often deal with the two maps l_R, r_R in the later sections, we give the following lemmas about them

Lemma 5.1.6 *Let R be the form (ii) in Proposition 3.1.5, then we have*

(i) $l(E_{s_1})l(E_{s_2}) = l(E_{s_1 s_2}) \Leftrightarrow w^1(s_1 s_2, s) = w^1(s_1,s)w^1(s_2,s),\ s \in S$;

(ii) $l(E_s)l(E_t) = l(E_{st}) \Leftrightarrow w^1(s,s')w^3(t,s') = w^3(st,s'),\ s' \in S$;

(iii) $l(X_{s_1})l(X_{s_2}) = l(X_{s_1}X_{s_2}) \Leftrightarrow w^2(s_1,t)w^2(s_2,t) = \tau(s_1,s_2)w^2(s_1 s_2,t),\ t \in T$;

(iv) $l(X_s)l(X_t) = l(X_sX_t) \Leftrightarrow w^2(s,t')w^4(t,t') = \tau(s,t)w^4(st,t'),\ t' \in T$;

where $s, s_1, s_2 \in S$ *and* $t, t_1, t_2 \in T$.

Proof: We only show (i) and the other things can be proved in a similar way. Since

$$l(E_{s_1})l(E_{s_2}) = [\sum_{s\in S} w^1(s_1, s)e_s][\sum_{s\in S} w^1(s_2, s)e_s] = \sum_{s\in S} w^1(s_1, s)w^1(s_2, s)e_s$$

and

$$l(E_{s_1s_2}) = \sum_{s\in S} w^1(s_1s_2, s)e_s,$$

we know (i) holds. □

Lemma 5.1.7 *Let* R *be the form (ii) in Proposition 3.1.5, then we have*

(i) $r(E_{s_1})r(E_{s_2}) = r(E_{s_1s_2}) \Leftrightarrow w^1(s, s_1s_2) = w^1(s, s_1)w^1(s, s_2),\ s \in S$;

(ii) $r(E_s)r(E_t) = r(E_{st}) \Leftrightarrow w^1(s', s)w^2(s', t) = w^2(s', st),\ s' \in S$;

(iii) $r(X_{s_1})r(X_{s_2}) = r(X_{s_2}X_{s_1}) \Leftrightarrow w^3(t, s_1)w^3(t, s_2) = \tau(s_2, s_1)w^3(t, s_1s_2),\ t \in T$;

(iv) $r(X_t)r(X_s) = r(X_sX_t) \Leftrightarrow w^3(t' \triangleleft x, s)w^4(t', t) = \tau(s, t)w^4(t', st),\ t' \in T$;

where $s, s_1, s_2 \in S$ *and* $t, t_1, t_2 \in T$.

Proof: Similar to the proof of Lemma 5.1.6 above. □

Lemma 5.1.8 *Let* R *be the form (ii) in Proposition 3.1.5, and if* w^4 *is a quasitriangular function on* H_G *and* $w^i(1 \le i \le 3)$ *are given in Lemma 5.1.2, then* $l(X_g)l(X_h) = l(X_gX_h),\ g, h \in G$.

Proof: Since $w^2(s, t) = \tau(s, t_1)\frac{w^4(st_1,t)}{w^4(t_1,t)}$ for $t_1 \in T$ by assumption and (iv) of the Lemma 5.1.6, we have $l(X_s)l(X_{t_1}) = l(X_sX_{t_1})$. Similarly, if we repeat part of the proof in Proposition 5.1.4, then we will get that $l(X_t)l(X_{t^{-1}}) = l(X_tX_{t^{-1}})$ is equivalent to $w^4(t, t_1)w^4(t^{-1}, t_1 \triangleleft x)\sigma(t_1) = \tau(t, t^{-1})$ for $t_1 \in T$. But we have assumed that $w^4(t, t_1)w^4(t^{-1}, t_1 \triangleleft x)\sigma(t_1) = \tau(t, t^{-1})$ for $t_1 \in T$, therefore we have $l(X_t)l(X_{t^{-1}}) =$

$l(X_tX_{t^{-1}})$. To show $l(X_g)l(X_h) = l(X_gX_h)$ for $g, h \in G$, we only need to show the following equations hold

$$l(X_{t_1})l(X_{t_2}) = l(X_{t_1}X_{t_2}),\ l(X_{t_1})l(X_s) = l(X_{t_1}X_s),\ l(X_{s_1})l(X_{s_2}) = l(X_{s_1}X_{s_2}),$$

where $s, s_1, s_2 \in S$ and $t_1, t_2 \in T$. Since $|S| = |T|$ and $T = T^{-1}$, where $T^{-1} := \{t^{-1}|\ t \in T\}$, we can assume $t_1 = st$ and $t_2 = t^{-1}$, then we have

$$\begin{aligned} l(X_{t_1})l(X_{t_2}) &= l(X_{st})l(X_{t^{-1}}) = [\tau(s,t)^{-1}l(X_s)l(X_t)]l(X_{t^{-1}}) \\ &= \tau(s,t)^{-1}l(X_s)[l(X_t)l(X_{t^{-1}})] = \tau(s,t)^{-1}l(X_s)l(X_tX_{t^{-1}}) \\ &= \tau(s,t)^{-1}l(X_s)[\tau(t,t^{-1})l(X_1)] = \tau(s,t)^{-1}\tau(t,t^{-1})l(X_s). \end{aligned}$$

It can be seen that $X_{st}X_{t^{-1}} = \tau(s,t)^{-1}\tau(t,t^{-1})X_s$ by using the τ is a 2-cocycle, and hence $l(X_{t_1})l(X_{t_2}) = l(X_{t_1}X_{t_2})$. For $s \in S$, we can find t, t' such that $s = tt'$ due to $|S| = |T|$. Because $t_1t \in S$ by definition, we have

$$\begin{aligned} l(X_{t_1})l(X_s) &= l(X_{t_1})l(X_{tt'}) = l(X_{t_1})[\tau(t,t')^{-1}l(X_t)l(X_{t'})] \\ &= \tau(t,t')^{-1}[l(X_{t_1})l(X_t)]l(X_{t'}) = \tau(t,t')^{-1}l(X_{t_1}X_t)l(X_{t'}) \\ &= \tau(t,t')^{-1}\tau(t_1,t)l(X_{t_1t})l(X_{t'}) = \tau(t,t')^{-1}\tau(t_1,t)l(X_{t_1t}X_{t'}). \end{aligned}$$

Similarly, one can show $X_{t_1}X_{tt'} = \tau(t,t')^{-1}\tau(t_1,t)X_{t_1t}X_{t'}$ and hence $l(X_{t_1})l(X_s) = l(X_{t_1}X_s)$. To show $l(X_{s_1})l(X_{s_2}) = l(X_{s_1}X_{s_2})$ for $s_1, s_2 \in S$, we assume that $s_2 = tt'$ for some $t, t' \in T$. Then we have

$$\begin{aligned} l(X_{s_1})l(X_{s_2}) &= l(X_{s_1})l(X_{tt'}) = l(X_{s_1})[\tau(t,t')^{-1}l(X_t)l(X_{t'})] \\ &= \tau(t,t')^{-1}[l(X_{s_1})l(X_t)]l(X_{t'}) = \tau(t,t')^{-1}l(X_{s_1}X_t)l(X_{t'}) \\ &= \tau(t,t')^{-1}\tau(s_1,t)l(X_{s_1t})l(X_{t'}) = \tau(t,t')^{-1}\tau(s_1,t)l(X_{s_1t}X_{t'}). \end{aligned}$$

One can check that $X_{s_1}X_{s_2} = \tau(t,t')^{-1}\tau(s_1,t)X_{s_1t}X_{t'}$, so $l(X_{s_1})l(X_{s_2}) = l(X_{s_1}X_{s_2})$. Therefore we have completed the proof. $\square$

Lemma 5.1.9 *Let R be in Lemma 5.1.8, then we have $l(E_g)l(E_h) = l(E_gE_h)$ for $g, h \in G$.*

Proof: We mimic the proof of above Lemma 5.1.8. Let $s \in S, t \in T$, then we have

$$l(E_s)l(E_t) = [\sum_{s' \in S} w^1(s,s')e_{s'}][\sum_{s' \in S} w^3(t,s')e_{s'}x] = [\sum_{s' \in S} w^1(s,s')w^3(t,s')e_{s'}x] \tag{5.10}$$

and

$$l(E_sE_t) = l(E_{st}) = [\sum_{s' \in S} w^3(st,s')e_{s'}x]. \tag{5.11}$$

therefore we need to show $w^1(s,s')w^3(t,s') = w^3(st,s')$ for $s' \in S$ if we want to prove $l(E_s)l(E_t) = l(E_sE_t)$. Let $t_1 := t \triangleleft x$ and taking $t_2 \in T$, since we have assumed w^3 such that (iii) of Lemma 5.1.2, we have

$$w^3(t,s') = \tau(s',t_2)\frac{w^4(t_1,s't_2)}{w^4(t_1,t_2)},\ w^3(st,s') = \tau(s',t_2)\frac{w^4(st_1,s't_2)}{w^4(st_1,t_2)}.$$

And hence $\frac{w^3(st,s')}{w^3(t,s')} = \frac{w^4(st_1,s't_2)w^4(t_1,t_2)}{w^4(st_1,t_2)w^4(t_1,s't_2)}$. Because w^1 satisfy the (i) of Lemma 5.1.2, we know $\frac{w^3(st,s')}{w^3(t,s')} = w^1(s,s')$ and thus we get $l(E_s)l(E_t) = l(E_sE_t)$. Then we prove that $l(E_t)l(E_{t^{-1}}) = l(E_1)$. Since

$$\begin{aligned} l(E_t)l(E_{t^{-1}}) &= [\sum_{s' \in S} w^3(t,s')e_{s'}x][\sum_{s' \in S} w^3(t^{-1},s')e_{s'}x] \\ &= [\sum_{s' \in S} w^3(t,s')w^3(t^{-1},s')\sigma(s')e_{s'}] \end{aligned}$$

and

$$\begin{aligned} w^3(t,s')w^3(t^{-1},s') &= [\tau(s',t_1)\frac{w^4(t \triangleleft x, s't_1)}{w^4(t \triangleleft x, t_1)}][\tau(s',t_1 \triangleleft x)\frac{w^4(t^{-1} \triangleleft x, s't_1 \triangleleft x)}{w^4(t^{-1} \triangleleft x, t_1 \triangleleft x)}] \\ &= \tau(s',t_1)\tau(s',t_1 \triangleleft x)\frac{w^4(t \triangleleft x, s't_1)w^4(t^{-1} \triangleleft x, s't_1 \triangleleft x)}{w^4(t \triangleleft x, t_1)w^4(t^{-1} \triangleleft x, t_1 \triangleleft x)} \\ &= \tau(s',t_1)\tau(s',t_1 \triangleleft x)\frac{\tau(t \triangleleft x, t^{-1} \triangleleft x)\sigma(t_1 \triangleleft x)}{\tau(t \triangleleft x, t^{-1} \triangleleft x)\sigma(s't_1 \triangleleft x)} \\ &= \tau(s',t_1)\tau(s',t_1 \triangleleft x)\sigma(t_1 \triangleleft x)\frac{1}{\sigma(s't_1 \triangleleft x)} = \sigma(s')^{-1}. \end{aligned}$$

The first equality follows from the assumption about w^3, and the third one follows from Relation (iii) in Proposition 5.1.4 and the last one follows from the compatibility of σ and τ. Thus we have showed $w^3(t,s')w^3(t^{-1},s')\sigma(s') = 1$, and this implies $l(E_t)l(E_{t^{-1}}) = l(E_1)$. Because we have proved that $l(E_s)l(E_t) = l(E_sE_t)$ and $l(E_t)l(E_{t^{-1}}) = l(E_1)$ for $s \in S, t \in T$, if we repeat the proof of Lemma 5.1.8 then we can obtain $l(E_g)l(E_h) = l(E_gE_h),\ g, h \in G$. $\square$

Lemma 5.1.10 *Let R be in Lemma 5.1.8, then r_R is an algebra anti-homomorphism.*

Proof: If we consider R_φ then it is easy to see that R_φ also satisfies the conditions of Lemma 5.1.8, so we can apply Lemma 5.1.8-5.1.9 to R_φ, i.e we know l_{R_φ} is an algebra map. Since we have showed $l_{R_\varphi} = \varphi \circ r_R \circ \varphi^*$, then l_{R_φ} is an algebra map implies r_R is antihomomorphism. $\square$

Now we prove the inverse of Proposition 5.1.4 above also holds.

Theorem 5.1.11 *Assume w is a quasitriangular function on H_G, then there is a u-nique R such that it is a non-trivial quasitriangular structure on H_G and the w^4 of it is equal to the w.*

Proof: Uniqueness can be obtained directly from Lemma 5.1.2. To show the existence, we will use the w to construct a non-trivial quasitriangular structure. We define $w^i(1 \leq i \leq 4)$ of R through letting $w^4 := w$ and let $w^i(1 \leq i \leq 3)$ be given by (i)-(iii) of Lemma 5.1.2. Since w is a quasitriangular function, we know w^2 and w^3 are well defined. By direct calculation we can get $w^1(s_1, s_2) = \frac{w^2(s_1,s_2t_2)}{w^2(s_1,t_2)}$ for $t_2 \in T$, so w^1 is also well-defined. Owing to Lemma 5.1.8-5.1.10, we know R such that the equations (5.4), (5.5) in Lemma 5.1.3. Furthermore, the R satisfies the equation (5.6) of Lemma 5.1.3 by the definition of quasitriangular function, so R is a non-trivial quasitriangular structure on H_G due to Lemma 5.1.3. $\square$

Corollary 5.1.12 *There is a bijective map between the set of non-trivial quasitriangular structures on H_G and the set of quasitriangular functions on H_G.*

Proof: Denote the set of non-trivial quasitriangular structures on H_G as N and we write the set of quasitriangular functions on H_G as F, then we can define a map $\phi : N \to F$ by $\phi(R) := w^4$. Since Proposition 5.1.4, we know ϕ is well defined. Owing to Theorem 5.1.11, we get ϕ is bijective. $\square$

§5.2 The criterion of a quasitriangular function

In this section, we mainly discuss how to know a function is a quasitriangular function, and give the criterion of a quasitriangular function.

Given a function $w : T \times T \to \Bbbk^\times$ and taking $t_0 \in T$, then we can define functions $w^2 : S \times T \to \Bbbk^\times$ and $w^3 : T \times S \to \Bbbk^\times$ as follows

$$w^2(s,t) := \tau(s,t_0)\frac{w^4(st_0,t)}{w^4(t_0,t)},\ w^3(t,s) := \tau(s,t_0)\frac{w^4(t \triangleleft x, st_0)}{w^4(t \triangleleft x, t_0)},\ s \in S, t \in T. \quad (5.12)$$

Let V_G be the subspace of $(H_G)^*$ which is linear spanned by $\{X_g |\ g \in G\}$, then we can define $l_w : V_G \to H_G$ and $r_w : V_G \to H_G$ through letting

$$l_w(X_s) := \sum_{t' \in T} w^2(s,t')e_{t'},\ l_w(X_t) := \sum_{t' \in T} w(t,t')e_{t'}x, \quad (5.13)$$

$$r_w(X_s) := \sum_{t' \in T} w^3(t',s)e_{t'},\ r_w(X_t) := \sum_{t' \in T} w(t',t)e_{t'}x. \quad (5.14)$$

It can be seen that V_G is a subalgebra of $(H_G)^*$. In order to determine when the function w is a quasitriangular function on $(H_G)^*$, we give the following propositions

Proposition 5.2.1 *The function w satisfies (i)-(iv) in Definition 5.1.5 if and only if l_w is an algebra homomorphism and r_w is an algebra anti-homomorphism.*

Proof: If w such that (i)-(iv) in Definition 5.1.5 then we can repeat the proof of Lemma 5.1.8, and hence we know l_w is an algebra homomorphism and r_w is an algebra antihomomorphism. On the contrary, if l_w is an algebra homomorphism and r_w is an algebra antihomomorphism then we have the following equations

$$l(X_s)l(X_{t_1}) = l(X_s X_{t_1}),\ l(X_t)l(X_{t^{-1}}) = l(X_t X_{t^{-1}}),$$

$$r(X_s)r(X_{t_1}) = r(X_{t_1} X_s),\ r(X_t)r(X_{t^{-1}}) = r(X_{t^{-1}} X_t).$$

Expand these equations above, then we know that w such that (i)-(iv) in Definition 5.1.5. $\square$

Proposition 5.2.1 above will often be used to solve quasitriangular functions on H_G in the following sections. Given a function $w : T \times T \to \Bbbk^\times$ and taking $t_0 \in T$, we have defined w^2, w^3 through the equalities (5.12). Furthermore, we can define another function $w^1 : S \times S \to \Bbbk^\times$ as follows

$$w^1(s_1, s_2) = \frac{w^4(s_1t_0, s_2t_0)w^4(t_0, t_0)}{w^4(s_1t_0, t_0)w^4(t_0, s_2t_0)},\ s_1, s_2 \in S. \tag{5.15}$$

Then we have the following proposition

Proposition 5.2.2 *Taking $t_0 \in T$ and if w such that (i)-(iv) in Definition 5.1.5, then w is a quasitriangular function on H_G if and only if the following equations hold*

(i) $w^1(s, b) = w^1(b, s) = \eta(t_0, s)$, *here w^1 is given by the* (5.15) *above;*

(ii) $w(t_0 \triangleleft x, t_0 \triangleleft x) = \frac{\tau(t_0 \triangleleft x, t_0 \triangleleft x)}{\tau(t_0, t_0)} w(t_0, t_0)$*;*

Proof: If w is a quasitriangular function on H_G, then we only need to show (i). Since w is a quasitriangular function on H_G, we can find a unique $R \in H_G \otimes H_G$ such that R is a non-trivial quasitriangular structure on H_G and the w^4 of it is equal to the w by Theorem 5.1.11. Since Lemma 5.1.2, we know the $w^i (1 \leq i \leq 3)$ of R are given by the equations (5.12), (5.15). Owing to Lemma 5.1.1, we have

$$w^2(s, t \triangleleft x) = w^2(s, t)\eta(s, t),\ w^3(t \triangleleft x, s) = w^3(t, s)\eta(t, s), \tag{5.16}$$

where $s \in S, t \in T$. Owing to (ii) of Lemma 5.1.6, we get $w^3(bt, s) = w^1(b, s)w^3(t, s)$. But $bt = t \triangleleft x$ because of the Remark 3.2.7, we know $w^3(t \triangleleft x, s) = w^1(b, s)w^3(t, s)$. Since $w^3(t \triangleleft x, s) = w^3(t, s)\eta(t, s)$ by (5.16), we get $w^1(b, s) = \eta(t, s)$. Due to η is a bicharacter and the Remark 3.2.7, we know $\eta(t, s) = \eta(t_0, s)$ and hence $w^1(b, s) = \eta(t_0, s)$. Similarly, we can show $w^1(s, b) = \eta(t_0, s)$ and thus we have shown (i). Conversely, if w satisfies (i), (ii), then we can construct a $R \in H_G \otimes H_G$ such that $w^4 = w$ and the $w^i (1 \leq i \leq 3)$ of it are given by the equations (5.12), (5.15). To show w is a quasitriangular function, we need only to prove that $w(t_1 \triangleleft x, t_2 \triangleleft x) = \frac{\tau(t_1 \triangleleft x, t_2 \triangleleft x)}{\tau(t_2, t_1)} w(t_1, t_2)$ for $t_1, t_2 \in T$. Repeating the proofs of Lemmas 5.1.8-5.1.10, then we know l_R is an algebra homomorphism and r_R is an algebra anti-homomorphism. So we have $l_R(E_b)l_R(E_t) = l_R(E_{bt})$ and $r_R(E_b)r_R(E_t) = r_R(E_{bt})$ for $t \in T$. But we have already seen that these two equalities implies that $w^3(t \triangleleft x, s) = w^1(b, s)w^3(t, s)$ and $w^2(s, t \triangleleft x) = w^1(s, b)w^2(s, t)$. Because of (i), we get

$$w^2(s, t \triangleleft x) = w^2(s, t)\eta(s, t), \ w^3(t \triangleleft x, s) = w^3(t, s)\eta(t, s). \tag{5.17}$$

Since w^2, w^3 are given by the equations (5.12), one can get

$$w^4(s_1t_0, s_2t_0) = \tau(s_1, t_0)^{-1}w^2(s_1, s_2t_0)w^4(t_0, s_2t_0) \tag{5.18}$$

and

$$w^4(t_0, s_2t_0) = \tau(s_2, t_0)^{-1}w^4(t_0, t_0)w^3(t_0 \triangleleft x, s_2), \tag{5.19}$$

where $s_1, s_2 \in S$. Using equations (5.18) and (5.19) together, then we get

$$w^4(s_1t_0, s_2t_0) = \frac{w^2(s_1, s_2t_0)w^3(t_0 \triangleleft x, s_2)w^4(t_0, t_0)}{\tau(s_1, t_0)\tau(s_2, t_0)}. \tag{5.20}$$

Similarly, one can get

$$w^4(s_1t_0 \triangleleft x, s_2t_0 \triangleleft x) = \frac{w^2(s_1, s_2t_0 \triangleleft x)w^3(t_0, s_2)w^4(t_0 \triangleleft x, t_0 \triangleleft x)}{\tau(s_1, t_0 \triangleleft x)\tau(s_2, t_0 \triangleleft x)}. \tag{5.21}$$

Combining the equations (5.17), (5.20), (5.21), we obtain

$$\begin{aligned}\frac{w^4(s_1t_0 \triangleleft x, s_2t_0 \triangleleft x)}{w^4(s_1t_0, s_2t_0)} &= \eta(s_1, s_2t_0)\frac{1}{\eta(t_0, s_2)}\frac{w^4(t_0 \triangleleft x, t_0 \triangleleft x)}{w^4(t_0, t_0)}\frac{\tau(s_1, t_0)\tau(s_2, t_0)}{\tau(s_1, t_0 \triangleleft x)\tau(s_2, t_0 \triangleleft x)}\\ &= \eta(s_1, s_2t_0)\frac{1}{\eta(t_0, s_2)}\frac{\tau(t_0 \triangleleft x, t_0 \triangleleft x)}{\tau(t_0, t_0)}\frac{\tau(s_1, t_0)\tau(s_2, t_0)}{\tau(s_1, t_0 \triangleleft x)\tau(s_2, t_0 \triangleleft x)}.\end{aligned}$$

Using the following Lemma 5.2.3, we obtain $\frac{w^4(s_1t_0 \triangleleft x, s_2t_0 \triangleleft x)}{w^4(s_1t_0, s_2t_0)} = \frac{\tau(s_1t_0 \triangleleft x, s_2t_0 \triangleleft x)}{\tau(s_2t_0, s_1t_0)}$. Since $T = t_0S$, we know $w(t_1 \triangleleft x, t_2 \triangleleft x) = \frac{\tau(t_1 \triangleleft x, t_2 \triangleleft x)}{\tau(t_2, t_1)}w(t_1, t_2)$ for $t_1, t_2 \in T$. □

Proposition 5.2.2 above simplifies the test for the condition (v) in Definition 5.1.5, so it will be used frequently in next sections. The following lemma is used in the proof of Proposition 5.2.2 above.

Lemma 5.2.3 $\eta(s_1, s_2t_0)\frac{1}{\eta(t_0, s_2)}\frac{\tau(t_0 \triangleleft x, t_0 \triangleleft x)}{\tau(t_0, t_0)}\frac{\tau(s_1, t_0)\tau(s_2, t_0)}{\tau(s_1, t_0 \triangleleft x)\tau(s_2, t_0 \triangleleft x)} = \frac{\tau(s_1t_0 \triangleleft x, s_2t_0 \triangleleft x)}{\tau(s_2t_0, s_1t_0)}$.

Proof: Directly we have

$$
\begin{aligned}
X_{s_1}X_{s_2}X_{t_0}X_{t_0} &= X_{s_1}(X_{s_2}X_{t_0})X_{t_0} \\
&= \eta(s_2,t_0)X_{s_1}(X_{t_0}X_{s_2})X_{t_0} \\
&= \eta(s_2,t_0)(X_{s_1}X_{t_0})(X_{s_2}X_{t_0}) \\
&= \eta(s_2,t_0)[\tau(s_1,t_0)X_{s_1t_0}][\tau(s_2,t_0)X_{s_2t_0}] \\
&= \eta(s_2,t_0)\tau(s_1,t_0)\tau(s_2,t_0)X_{s_1t_0}X_{s_2t_0} \\
&= \eta(s_2,t_0)\tau(s_1,t_0)\tau(s_2,t_0)\tau(s_1t_0,s_2t_0)X_{s_1s_2t_0^2}
\end{aligned}
$$

and

$$
\begin{aligned}
X_{s_1}X_{s_2}X_{t_0}X_{t_0} &= X_{s_1}X_{s_2}(X_{t_0}X_{t_0}) \\
&= X_{s_1}X_{s_2}[\frac{\tau(t_0,t_0)}{\tau(t_0\lhd x,t_0\lhd x)}X_{t_0\lhd x}X_{t_0\lhd x}] \\
&= \frac{\tau(t_0,t_0)}{\tau(t_0\lhd x,t_0\lhd x)}X_{s_1}(X_{s_2}X_{t_0\lhd x})X_{t_0\lhd x} \\
&= \frac{\tau(t_0,t_0)}{\tau(t_0\lhd x,t_0\lhd x)}X_{s_1}[\eta(s_2,t_0\lhd x)X_{t_0\lhd x}X_{s_2}]X_{t_0\lhd x} \\
&= \eta(s_2,t_0\lhd x)\frac{\tau(t_0,t_0)}{\tau(t_0\lhd x,t_0\lhd x)}(X_{s_1}X_{t_0\lhd x})(X_{s_2}X_{t_0\lhd x}) \\
&= \eta(s_2,t_0\lhd x)\frac{\tau(t_0,t_0)}{\tau(t_0\lhd x,t_0\lhd x)}\tau(s_1,t_0\lhd x)\tau(s_2,t_0\lhd x)X_{s_1t_0\lhd x}X_{s_2t_0\lhd x}.
\end{aligned}
$$

Because $X_{s_1t_0\lhd x}X_{s_2t_0\lhd x}=\tau(s_1t_0\lhd x,s_2t_0\lhd x)X_{s_1s_2t_0^2}$, we know

$$
\frac{\eta(s_1t_0,s_2t_0)}{\eta(s_2,t_0\lhd x)}\eta(s_2,t_0)\frac{\tau(t_0\lhd x,t_0\lhd x)}{\tau(t_0,t_0)}\frac{\tau(s_1,t_0)\tau(s_2,t_0)}{\tau(s_1,t_0\lhd x)\tau(s_2,t_0\lhd x)}=\frac{\tau(s_1t_0\lhd x,s_2t_0\lhd x)}{\tau(s_2t_0,s_1t_0)}.
$$

To complete the proof, we only need to show $\eta(s_2,t_0)=\eta(t_0,s_2)^{-1}$ and $\frac{\eta(s_1t_0,s_2t_0)}{\eta(s_2,t_0\lhd x)}=\eta(s_1,s_2t_0)$. By the definition of η, we have $\eta(s_2,t_0)=\eta(t_0,s_2)^{-1}$. Since

$$
\begin{aligned}
\frac{\eta(s_1t_0,s_2t_0)}{\eta(s_2,t_0\lhd x)} &= \frac{\eta(s_1,s_2t_0)\eta(t_0,s_2t_0)}{\eta(s_2,t_0\lhd x)}=\frac{\eta(s_1,s_2t_0)\eta(t_0,s_2)}{\eta(s_2,t_0\lhd x)} \\
&= \eta(s_1,s_2t_0)\eta(t_0,s_2)\eta(t_0\lhd x,s_2)=\eta(s_1,s_2t_0)\eta(t_0t_0\lhd x,s_2) \\
&= \eta(s_1,s_2t_0),
\end{aligned}
$$

the last equality follows from the assumption about H_G in Remark 3.2.7, we know $\frac{\eta(s_1t_0,s_2t_0)}{\eta(s_2,t_0\triangleleft x)} = \eta(s_1, s_2t_0)$. $\square$

Chapter 6 Solutions of quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

In this chapter, we analogize the problem of solving a system of linear equations and naturally reduce the problem of solving quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ to the problem of solving all general solutions as well as giving a special solution for it. After that we give all general solutions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and get a necessary and sufficient condition for the existence of a special solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

§6.1 General solutions for quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

In this section, we introduce the concepts of general solution and special solution of non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Then we give all general solutions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Let R, R' be non-trivial quastriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and assume that the four maps associated with R (resp. R') are $w^i(1 \leq i \leq 4)$ (resp. $w'^i(1 \leq i \leq 4)$), then we can use these maps to define four other maps $v^i(1 \leq i \leq 4)$ as follows

$$v^1(s_1, s_2) := \frac{w^1(s_1, s_2)}{w'^1(s_1, s_2)},\ v^2(s, t) := \frac{w^2(s, t)}{w'^2(s, t)}$$
$$v^3(t, s) := \frac{w^3(t, s)}{w'^3(t, s)},\ v^4(t_1, t_2) := \frac{w^4(t_1, t_2)}{w'^4(t_1, t_2)},$$

where $s, s_1, s_2 \in S$ and $t, t_1, t_2 \in T$. Using the data $(G, \triangleleft, \sigma, \tau)$ of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ we can induce another data $(G', \triangleleft', \sigma', \tau')$ by making $G' := G$, $\triangleleft' := \triangleleft$ and $\sigma'(g) := 1$, $\tau'(g, h) := 1$ for $g, h \in G$. Then the data $(G', \triangleleft', \sigma', \tau')$ determines a Hopf algebra by Definition 2.1.2 and we simply denote it as $\Bbbk^G\#\Bbbk\mathbb{Z}_2$. Then we have

Proposition 6.1.1 *Let R'' be the form (ii) in Proposition 3.1.5 and the $(w'')^i(1 \leq i \leq 4)$ of it are the $v^i(1 \leq i \leq 4)$ above, then R'' is a quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$.*

Proof: Since R, R' are non-trivial quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we know w^4, w'^4 are quasitriangular functions on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Then we can easily check that v^4

is a quasitriangular function on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$. And furthermore, $v^i(1\leq i\leq 3)$ are given as follows because of Lemma 5.1.2

$$v^1(s_1,s_2)=\frac{v^4(s_1t_1,s_2t_2)v^4(t_1,t_2)}{v^4(s_1t_1,t_2)v^4(t_1,s_2t_2)},\ v^2(s,t)=\frac{v^4(st_1,t)}{v^4(t_1,t)},\ v^3(t,s)=\frac{v^4(t\triangleleft x,st_1)}{v^4(t\triangleleft x,t_1)}.$$

Therefore we know R'' is a quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$ due to the proof of Theorem 5.1.11. $\square$

We can view R'' as $\frac{R}{R'}$ and then we can analogize the solutions of a system of linear equations and give the following definition

Definition 6.1.2 *We call a quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$ as a* general solution *for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Naturally, we call a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ as a* special solution *for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.*

Then the problem of solving all quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ can be reduced to solving all general solutions and finding a special solution. And we will give all general solutions in this subsection. To do this, we first give the following lemma

Lemma 6.1.3 *Assume that H is a finite abelian group and $\phi: H\to H$ is a group isomorphism. Let $S_H:=\{h\in H|\ \phi(h)=h\}$ and let $T_H:=\{h\in H|\ \phi(h)\neq h\}$. If $|S_H|=|T_H|$ and there is $c\in H$ such that $c^2=1$ and $\phi(h)=hc$ for $h\in T_H$, then there are $s_1,...,s_n\in S_H$ and $a\in T_H$ such that $H=\langle s_i,a|\ s_i^{k_i}=1,a^2=s_1^{m_1}...s_n^{m_n},s_is_j=s_js_i,as_i=s_ia\rangle_{1\leq i,j\leq n}$ as group for some natural numbers n,k_i,m_j.*

Proof: Since S_H is a subgroup of H, we can find $s_1...s_m\in S_H$ such that $S_H=\langle s_i|\ s_i^{k_i}=1,s_is_j=s_js_i\rangle_{1\leq i,j\leq n}$ for some natural numbers $k_i(1\leq i\leq n)$. Because T_H is not empty, we can find $a\in T_H$. But $a^2\in S_H$ due to $\phi(a^2)=(ac)^2=a^2$, so we can assume $a^2=s_1^{m_1}...s_n^{m_n}$ for some natural numbers $m_i(1\leq i\leq n)$. Let H' be a group such that $H'=\langle S_i,A|\ S_i^{k_i}=1,A^2=S_1^{m_1}...S_n^{m_n},S_iS_j=S_jS_i,AS_i=S_iA\rangle_{1\leq i,j\leq n}$ as group, then we will prove that $H\cong H'$ as group and hence we completed the proof. We define a group homomorphism $f:H'\to H$ through letting $f(S_i):=s_i,f(A):=a$ for $1\leq i\leq n$, then f is well defined by the definition of H'. Owing to $aS_H\subseteq T_H$ and $|S_H|=|T_H|$, we obtain $T_H=aS_H$. Thus we can see that f is surjective. To show f is injective, we only need to show $|H'|\leq|H|$. Let $S_{H'}:=\langle S_i\rangle_{1\leq i\leq n}$, then

$f|_{S_{H'}} : S_{H'} \to S_H$ is onto by definition and hence $|S_{H'}| \geq |S_H|$. But $|S_{H'}| \leq |S_H|$ by definition of H', so we know $|S_{H'}| = |S_H|$. Directly, we have $H' = S_{H'} \cup AS_{H'}$ and therefore $|H'| \leq 2|S_{H'}|$. Since $|H| = 2|S_H|$ and $|S_{H'}| = |S_H|$, we have $|H'| \leq |H|$. $\square$

Corollary 6.1.4 *For the data $(G, \triangleleft, \sigma, \tau)$, there are $s_1, ..., s_n \in S$ and $a \in T$ such that $G \cong \langle s_i, a| \ s_i^{k_i} = 1, a^2 = s_1^{m_1}...s_n^{m_n}, s_is_j = s_js_i, as_i = s_ia\rangle_{1\leq i,j\leq n}$ as group for some natural numbers n, k_i, m_j.*

Proof: Since the assumption about $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ in Remark 3.2.7 and the Lemma 6.1.3 above, we get what we want. $\square$

Since Corollary 6.1.4 above, we agree that $G = \langle s_i, a| \ s_i^{k_i} = 1, a^2 = s_1^{m_1}...s_n^{m_n}, s_is_j = s_js_i, as_i = s_ia\rangle_{1\leq i,j\leq n}$ as group for some natural numbers n, k_i, m_j in the following content, where $s_i \in S, a \in T$ for $1 \leq i \leq n$. Let $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ as before, we associate a free object with it as follows. We define F_G as a free $\Bbbk$ algebra generated by set $\{x_{s_i}, x_a, x_1, e_{s_i}, e_a, e_1\}_{1\leq i\leq n}$, and let I_G be the ideal generated by $\{x_{s_i}x_1 - x_{s_i}, x_1x_{s_i} - x_{s_i}, x_{s_i}x_{s_j} - x_{s_j}x_{s_i}, x_{s_i}x_a - \eta(s_i, a)x_ax_{s_i}, x_{s_i}^{k_i} - P_{s_i^{k_i}}x_1, x_a^2 - \tau(a,a)P^{-1}_{s_1^{m_1}...s_n^{m_n}}x_{s_1}^{m_1}...x_{s_n}^{m_n}\}$ $\cup\{x_1e_1, e_1x_1, e_{s_i}e_1 - e_{s_i}, e_1e_{s_i} - e_{s_i}, e_{s_i}e_{s_j} - e_{s_j}e_{s_i}, e_{s_i}e_a - e_ae_{s_i}, e_{s_i}^{k_i} - e_1, e_a^2 - e_{s_1}^{m_1}...e_{s_n}^{m_n}\}$, where $1 \leq i, j \leq n$ and $P_{g_1^{j_1}...g_n^{j_n}} \in \Bbbk$ is defined by the following equation

$$X_{g_1}^{j_1}...X_{g_n}^{j_n} = P_{g_1^{j_1}...g_n^{j_n}}X_{g_1^{j_1}...g_n^{j_n}}, \ g_1, ..., g_n \in G, j_1, ..., j_n \in \mathbb{N}. \tag{6.1}$$

For convenience, we agree that $P_{g_1^{j_1}...g_n^{j_n}}$ in the following content refers to the $P_{g_1^{j_1}...g_n^{j_n}}$ in equation (6.1) above. Then we have the following lemma.

Lemma 6.1.5 *Denote the dual Hopf algebra of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ by H^*, then $H^* \cong A_G/I_G$ as an algebra.*

Proof: We define an algebra map $\pi : A_G/I_G \to H^*$ by setting

$$\pi(x_{s_i}) = X_{s_i}, \ \pi(x_a) = X_a, \ \pi(x_1) = X_1, \ \pi(e_{s_i}) = E_{s_i}, \ \pi(e_a) = E_a, \ \pi(e_1) = E_1.$$

Then we will show that π is well defined and it is bijective. Since Lemma 3.1.1 and the definition of I_G, we know π is well defined. Next we show π is bijective. Because $\{X_{s_i}, X_a, E_{s_i}, E_a\} \subseteq \mathrm{Im}\pi$, we know π is surjective. Owing to the definition of I_G, we obtain $\dim(A_G/I_G) \leq \dim(H^*)$. But we have shown π is surjective, so $\dim(A_G/I_G) = \dim(H^*)$ and hence π is bijective. $\square$

Let R be the form (ii) on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ in Proposition 3.1.5. For our purposes, we assume that $w^i(1 \leq i \leq 4)$ of R satisfy $w^1(1,s) = w^1(s,1) = 1$ and $w^2(1,t) = w^3(t,1)$ for $s \in S, t \in T$ in the following content. Then we have

Lemma 6.1.6 *The map l_R is an algebra homomorphism if and only if the following conditions hold*

(i) $l(E_{s_1})^{i_1}...l(E_{s_n})^{i_n} = l(E_{s_1^{i_1}...s_n^{i_n}}),\ l(X_{s_1})^{i_1}...l(X_{s_n})^{i_n} = P_{s_1^{i_1}...s_n^{i_n}} l(X_{s_1^{i_1}...s_n^{i_n}})$;

(ii) $l(E_s)l(E_a) = l(E_{sa}),\ l(X_s)l(X_a) = \tau(s,a)l(X_{sa})$;

(iii) $l(E_{s_i})^{k_i} = l(E_1),\ l(X_{s_i})^{k_i} = P_{{s_i}^{k_i}} l(X_1)$;

(v) $l(E_a)^2 = l(E_{s_1^{m_1}...s_1^{m_n}}),\ l(X_a)^2 = P_{a^2}\ l(X_{s_1^{m_1}...s_1^{m_n}})$;

(vi) $w^2(s, t \triangleleft x) = \eta(s,t)w^2(s,t)$;

where $s \in S,\ 1 \leq i \leq n$.

Proof: We define an algebra map $\pi : H^* \to H$ by setting

$$\pi(X_{s_i}) = l(X_{s_i}),\ \pi(X_a) = l(X_a),\ \pi(X_1) = l(X_1),$$
$$\pi(E_{s_i}) = l(E_{s_i}),\ \pi(E_a) = l(E_a),\ \pi(E_1) = l(E_1).$$

Then we will show that π is well defined and $\pi = l_R$. To show π is well defined, the only non-trivial case is to prove that $\pi(X_s)\pi(X_a) = \eta(s,a)l(X_a)l(X_s)$. Directly we have $\pi(X_s)\pi(X_a) = l(X_s)l(X_a)$ and

$$l(X_s)l(X_a) = [\sum_{t\in T} w^2(s,t)e_t][\sum_{t\in T} w^4(a,t)e_t x] = [\sum_{t\in T} w^2(s,t)w^4(a,t)e_t x].$$

Similarly, we get $\pi(X_a)\pi(X_s) = l(X_a)l(X_s)$ and

$$l(X_a)l(X_s) = [\sum_{t\in T} w^4(a,t)e_t x][\sum_{t\in T} w^2(s,t)e_t] = [\sum_{t\in T} w^2(s, t\triangleleft x)w^4(a,t)e_t x].$$

Owing to (v), we obtain $\pi(X_s)\pi(X_a) = \eta(s,a)l(X_a)l(X_s)$. Due to (i),(ii), we get $\pi = l_R$ and thus we have completed the proof. $\square$

Similar to Lemma 6.1.6, we have

Lemma 6.1.7 *The map r_R is an algebra anti-homomorphism if and only if the following conditions hold*

(i) $r(E_{s_1})^{i_1}...r(E_{s_n})^{i_n} = r(E_{s_1^{i_1}...s_n^{i_n}}),\ r(X_{s_1})^{i_1}...r(X_{s_n})^{i_n} = P_{s_1^{i_1}...s_n^{i_n}} r(X_{s_1^{i_1}...s_n^{i_n}})$;

(ii) $r(E_s)r(E_a) = r(E_{sa}),\ r(X_a)r(X_s) = \tau(s,a)r(X_{sa})$;

(iii) $r(E_{s_i})^{k_i} = r(E_1),\ r(X_{s_i})^{k_i} = P_{{s_i}^{k_i}}\ r(X_1),\ r(X_a)^2 = P_{a^2}\ r(X_{s_1^{m_1}...s_1^{m_n}})$;

(iv) $r(E_a)^2 = r(E_{s_1^{m_1}...s_1^{m_n}}),\ r(X_a)^2 = P_{a^2}\ r(X_{s_1^{m_1}...s_1^{m_n}})$;

(v) $w^3(t \triangleleft x, s) = \eta(t,s)w^3(t,s)$;

where $s \in S,\ 1 \leq i \leq n$.

Proof: Consider the R_φ, then it can be seen that R_φ such that the conditions of Lemma 6.1.6 and so l_{R_φ} is an algebra map. Since the proof of Proposition 4.1.2, we know l_{R_φ} is an algebra map if and only if r_R is an algebra anti-homomorphism and hence we have completed the proof. $\square$

The following proposition can be used to determine when R is a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Proposition 6.1.8 *Let R be the form (ii) on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ in Proposition 3.1.5, then R is a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ if and only if R such that the conditions of Lemma 6.1.6, Lemma 6.1.7 and $w^4(ab,ab) = \frac{\tau(ab,ab)}{\tau(a,a)}w^4(a,a)$.*

Proof: If R is a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, Since l_R is an algebra homomorphism and r_R is an algebra anti-homomorphism and thus R such that the conditions of Lemma 6.1.6, Lemma 6.1.7. Owing to Lemma 5.1.1, we get $w^4(ab,ab) = \frac{\tau(ab,ab)}{\tau(a,a)}w^4(a,a)$. Conversely, if R such that the conditions of Lemma 6.1.6, Lemma 6.1.7 and $w^4(ab,ab) = \frac{\tau(ab,ab)}{\tau(a,a)}w^4(a,a)$, then l_R is an algebra homomorphism and r_R is an algebra anti-homomorphism. Therefore w^4 satisfies (i)-(iv) in Definition 5.1.5 due to

Proposition 5.2.1. Then we will use Proposition 5.2.2 to get what we want. To do this, we only need to show $w^1(s,b) = w^1(b,s) = \eta(a,s)$. Because l_R is an algebra map and (ii) of Lemma 5.1.7, we get $w^2(s,ab) = w^2(s,a)w^1(s,b)$. Owing to Lemma 6.1.6, we obtain $w^2(s, a \triangleleft x) = w^2(s,a)\eta(s,a)$. But $a \triangleleft x = ab$ by assumption, so we have $w^1(s,b) = \eta(s,a)$. Due to $\frac{\eta(s,a)}{\eta(a,s)} = \eta(s,a^2)$ and $a^2 \in S$, we know $\eta(s,a) = \eta(a,s)$ and hence $w^1(s,b) = \eta(a,s)$. Similarly, one can show $w^1(b,s) = \eta(a,s)$ and so we have completed the proof. $\square$

In practice, we usually use the following corollary to determine when R is a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ because it's easier to be checked.

Corollary 6.1.9 *Let R be the form (ii) on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ in Proposition 3.1.5, then R is a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ if and only if R such that the following conditions*

(i) *R satisfies (i)-(iv) of Lemma 6.1.6;*

(ii) *R satisfies (i)-(iv) of Lemma 6.1.7;*

(iii) $w^1(b,s) = w^1(s,b) = \eta(a,s),\ s \in S$;

(iv) $w^4(ab,ab) = \frac{\tau(ab,ab)}{\tau(a,a)} w^4(a,a)$;

Proof: Since the proof of Proposition 6.1.8, we know that necessity holds. Conversely, owing to Proposition 6.1.8 above, we only need to show $w^2(s, t \triangleleft x) = \eta(s,t)w^2(s,t)$ and $w^3(t \triangleleft x, s) = \eta(t,s)w^3(t,s)$ for $s \in S, t \in T$. Since (i),(ii) of Lemma 6.1.7, we know R satisfies (ii) of Lemma 5.1.7. Then we get $w^2(s,tb) = w^2(s,t)w^1(s,b)$. But we have $w^1(s,b) = \eta(a,s)$, so $w^2(s,tb) = w^2(s,t)\eta(a,s)$ by Lemma 5.1.7. Due to $\frac{\eta(a,s)}{\eta(s,t)} = \eta(at,s)$ and $at \in S$, we get $\frac{\eta(a,s)}{\eta(s,t)} = 1$ and hence $w^2(s, t \triangleleft x) = \eta(s,t)w^2(s,t)$. Similarly, one can prove that $w^3(t \triangleleft x, s) = \eta(t,s)w^3(t,s)$ and thus we have completed the proof. $\square$

For the convenience of calculation, we assume $b = s_1^{p_1}...s_n^{p_n}$ for some natural numbers $p_1, ..., p_n$ in the following sections. Then all general solutions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ will be given by the following Theorems 6.1.13-6.1.14.

Proposition 6.1.10 *Let R be a general solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, that is to say R is a non-trivial quasitriangular structure on $\Bbbk^G\#\Bbbk\mathbb{Z}_2$, and if we denote*

$$\alpha_{ij} := w^1(s_i, s_j),\ \beta_i := w^2(s_i, a),\ \gamma_i := w^3(a, s_i),\ \delta := w^4(a, a),$$

then the following equations hold

(i) $\alpha_{ij}^{k_i} = \alpha_{ij}^{k_j} = 1,\ 1 \le i, j \le n;$

(ii) $\beta_i^{k_i} = 1,\ \beta_i^2 = \alpha_{i1}^{m_1}...\alpha_{in}^{m_n},\ 1 \le i \le n;$

(iii) $\gamma_i^{k_i} = 1,\ \gamma_i^2 = \alpha_{1i}^{m_1}...\alpha_{ni}^{m_n},\ 1 \le i \le n;$

(iv) $\delta^2 = \beta_1^{m_1+p_1}...\beta_n^{m_n+p_n} = \gamma_1^{m_1+p_1}...\gamma_n^{m_n+p_n};$

(v) $\alpha_{1i}^{p_1}...\alpha_{ni}^{p_n} = \alpha_{i1}^{p_1}...\alpha_{in}^{p_n} = 1,\ \beta_1^{p_1}...\beta_n^{p_n} = \gamma_1^{p_1}...\gamma_n^{p_n};$

Proof: Using (i) of Lemma 5.1.6 and (i) of Lemma 5.1.7, we know that w^1 is a bicharacter on S. Then we have $w^1(s_i, s_j)^{k_i} = w^1(s_i, s_j)^{k_j} = 1$ and so (i) holds. To show (ii), we note that $l(X_{s_i})^{k_i} = l(X_1)$ and $r(E_a)^2 = r(E_{s_1^{m_1}...s_n^{m_n}})$ and if we use (iii) of Lemma 5.1.6, we get $w^2(s_i, a)^{k_i} = 1$ through letting $t = a$ and so we have $\beta_i^{k_i} = 1$. Similarly, since

$$r(E_a)^2 = \sum_{s\in S} w^2(s, a)^2 e_s,\ r(E_{s_1^{m_1}...s_n^{m_n}}) = \sum_{s\in S} w^1(s, s_1^{m_1}...s_n^{m_n})e_s,$$

so we have $w^2(s_i, a)^2 = w^1(s_i, s_1^{m_1}...s_n^{m_n})$ through letting $s = s_i$. Because we have shown w^1 is a bicharacter on S, we obtain $w^1(s_i, s_1^{m_1}...s_n^{m_n}) = \alpha_{i1}^{m_1}...\alpha_{in}^{m_n}$ and hence (ii) holds. If we consider R_φ, then we know R_φ is also a general solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and so R_φ such that (ii). Due to the $w'^i(1 \le i \le 4)$ of R_φ such that $w'^2(s_i, a) = w^3(ab, s_i)$ by definition of R_φ and we have $w^3(ab, s_i) = w^3(a, s_i)w^1(b, s_i)$ by (ii) of Lemma 5.1.6, we obtain $w'^2(s_i, a) = w^3(a, s_i)w^1(b, s_i)$. But $w^1(b, s_i) = 1$ because of (iii) of Corollary 6.1.9, so $w^3(ab, s_i) = w^3(a, s_i) = \gamma_i$ and hence (iii) holds. To show (iv), we first show (v). Since (iii) of Corollary 6.1.9, we know $w^1(s_i, b) = w^1(b, s_i) = 1$. But we have shown w^1 is a bicharacter on S and because $b = s_1^{p_1}...s_n^{p_n}$ by the assumption, we know $\alpha_{1i}^{p_1}...\alpha_{ni}^{p_n} = \alpha_{i1}^{p_1}...\alpha_{in}^{p_n} = 1$. Because of (iv) in Corollary 6.1.9, we have $w^4(ab, ab) = w^4(a, a)$. Using (iv) of Lemma 5.1.6, we get $w^4(ab, ab) = w^2(b, ab)w^4(a, ab)$. With the help of the (iv) of Lemma 5.1.7, we obtain $w^4(a,ab) = w^4(a,a)w^3(a,b)$ and so we have $w^2(b,ab)w^3(a,b) = 1$. Due to $w^2(b,ab) = w^2(b,a)w^1(b,b) = w^2(b,a)$ by (ii) of Lemma 5.1.6, we know $w^2(b,a)w^3(a,b) = 1$. Since $w^2(b,a) = \beta_1^{p_1}...\beta_n^{p_n}$ and $w^3(a,b) = \gamma_1^{p_1}...\gamma_n^{p_n}$, we get $\beta_1^{p_1}...\beta_n^{p_n} = \gamma_1^{-p_1}...\gamma_n^{-p_n}$. But $w^3(a,b)^2 = w^3(a,1) = 1$ by (iii) of Lemma 5.1.7, we get $\gamma_1^{2p_1}...\gamma_n^{2p_n} = 1$ and thus $\beta_1^{p_1}...\beta_n^{p_n} = \gamma_1^{p_1}...\gamma_n^{p_n}$. Therefore (v) holds. To show (iv), we only need to show $\delta^2 = \beta_1^{m_1+p_1}...\beta_n^{m_n+p_n}$ due to the same reason with the proof of (iii). Since $l(X_a)^2 = l(X_{a^2}) = l(X_{s_1^{m_1}...s_n^{m_n}})$ and the following equations hold

$$l(X_a)^2 = \sum_{s\in S} w^4(a,t)w^4(a,t \triangleleft x)e_t,\ l(X_{s_1^{m_1}...s_n^{m_n}}) = \sum_{t\in T} w^2(s_1^{m_1}...s_n^{m_n},t)e_t,$$

we have

$$w^4(a,a)w^4(a,ab) = w^2(s_1^{m_1}...s_n^{m_n},a) \tag{6.2}$$

through letting $t = a$. Since (iii) of Lemma 5.1.6, we get $w^2(s,t)w^2(s',t) = w^2(ss',t)$ for $s \in S, t \in T$. So we have

$$w^2(s_1^{m_1}...s_n^{m_n},a) = \beta_1^{m_1}...\beta_n^{m_n}. \tag{6.3}$$

Since the (iii) in Lemma 5.1.2, we have $w^3(ab,b) = \frac{w^4(a,ab)}{w^4(a,a)}$. Owing to (ii) of Lemma 5.1.6, we can get $w^3(ab,b) = w^3(a,b)w^1(b,b)$. But we have known $w^1(b,b) = 1$, so $w^3(ab,b) = w^3(a,b)$. Furthermore, owing to $w^4(a,ab) = w^4(a,a)w^3(a,b)$ and $w^3(a,b) = \gamma_1^{-p_1}...\gamma_n^{-p_n}$, we have

$$w^4(a,ab) = w^4(a,a)\gamma_1^{-p_1}...\gamma_n^{-p_n}. \tag{6.4}$$

Combining equations (6.2)-(6.4) and (v), we obtain (iv). □

In fact, the four tuple $(\alpha_{ij},\beta_i,\gamma_i,\delta)_{1\le i,j\le n}$ in the above proposition completely determine R.

Proposition 6.1.11 *Let R be in Proposition 6.1.10 and let $(\alpha_{ij},\beta_i,\gamma_i,\delta)_{1\le i,j\le n}$ be in Proposition 6.1.10, then the following equations hold*

(i) $w^1(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}) = \prod_{k=1}^{n}\prod_{l=1}^{n}\alpha_{kl}^{i_k j_l}$;

(ii) $w^2(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}a) = (\prod_{k=1}^{n} \beta_k^{i_k}) \prod_{k=1}^{n}\prod_{l=1}^{n} \alpha_{kl}^{i_k j_l}$;

(iii) $w^3(s_1^{i_1}...s_n^{i_n}a, s_1^{j_1}...s_n^{j_n}) = (\prod_{k=1}^{n} \gamma_k^{j_k}) \prod_{k=1}^{n}\prod_{l=1}^{n} \alpha_{kl}^{i_k j_l}$;

(iv) $w^4(s_1^{i_1}...s_n^{i_n}a, s_1^{j_1}...s_n^{j_n}a) = (\prod_{k=1}^{n} \beta_k^{i_k})(\prod_{k=1}^{n} \gamma_k^{j_k}) \prod_{k=1}^{n}\prod_{l=1}^{n} \alpha_{kl}^{i_k j_l}\delta$;

where $0 \leq i_1, ..., i_n \leq n-1$ *and* $0 \leq j_1, ..., j_n \leq n-1$.

Proof: Since w^1 is a bicharacter on S, we get (i). Owing to (ii) of Lemma 5.1.7, we know $w^2(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}a) = w^1(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n})w^2(s_1^{i_1}...s_n^{i_n}, a)$. Due to (iii) of Lemma 5.1.6, we obtain $w^2(s_1^{i_1}...s_n^{i_n}, a) = \prod_{k=1}^{n} \beta_k^{i_k}$ and so we have (ii). Similarly, we can show (iii). Thanks to (iv) of Lemma 5.1.6, we get

$$w^4(s_1^{i_1}...s_n^{i_n}a, s_1^{j_1}...s_n^{j_n}a) = w^2(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}a)w^4(a, s_1^{j_1}...s_n^{j_n}a). \tag{6.5}$$

Using (iv) of Lemma 5.1.7, we have

$$w^4(a, s_1^{j_1}...s_n^{j_n}a) = w^3(a \triangleleft x, s_1^{j_1}...s_n^{j_n})w^4(a, a). \tag{6.6}$$

Because of (5.2) in Lemma 5.1.1, we get

$$w^3(a \triangleleft x, s_1^{j_1}...s_n^{j_n}) = w^3(a, s_1^{j_1}...s_n^{j_n}). \tag{6.7}$$

Since the equations (6.5)-(6.7) and (ii),(iii), we know (iv). □

Conversely, given a four tuple $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq n}$ satisfies conditions (i)-(v) of Proposition 6.1.10, then we have

Proposition 6.1.12 *Given a four tuple* $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq n}$ *satisfying conditions (i)-(v) of Proposition 6.1.10 and let* R *be the form (ii) on* $\Bbbk^G \# \Bbbk\mathbb{Z}_2$ *in Proposition 3.1.5. If* $w^i (1 \leq i \leq 4)$ *of* R *are given by (i)-(iv) in Proposition 6.1.11 by using the four tuple, then* R *is a general solution for* $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$.

Proof: Since Corollary 6.1.9, we only need to show R such that the conditions of Corollary 6.1.9. Because the definition of w^1, we know w^1 is a bicharacter on S and hence we get $l(E_{s_1})^{i_1}...l(E_{s_n})^{i_n} = l(E_{s_1^{i_1}...s_n^{i_n}})$. To show $l(X_{s_1})^{i_1}...l(X_{s_n})^{i_n} = P_{s_1^{i_1}...s_n^{i_n}} l(X_{s_1^{i_1}...s_n^{i_n}})$, we need only to prove that $w^2(-,t)$ is a character on S for $t \in T$ by (iii) of Lemma 5.1.6. By definition of w^2, we get $w^2(-, s_1^{j_1}...s_n^{j_n}a)$ is a character on S. Owing to $aS = T$, we obtain $w^2(-,t)$ is a character on S for $t \in T$ and so (i) of Lemma 6.1.6 holds. To show (ii) of Lemma 6.1.6, we only need to prove that $w^1(s,s')w^3(a,s) = w^3(as,s')$ and $w^2(s,t)w^4(a,t) = w^4(as,t)$ for $s,s' \in S,\ t \in T$ because of (iii), (iv) of Lemma 5.1.6. And these equalities are not difficult to check and so (ii) of Lemma 6.1.6 hold. To show (iii) of Lemma 6.1.6, note that w^1 is a bicharacter on S and $\alpha_{ij}^{k_i} = 1$ by assumption and hence $l(E_{s_i})^{k_i} = l(E_1)$. Similarly, because $w^2(-,t)$ is a character on S for $t \in T$ and $\beta_i^{k_i} = 1$ by our conditions, therefore we get $l(X_{s_i})^{k_i} = l(X_1)$ and so we know (iii) of Lemma 6.1.6 hold. To show (iv), it can be seen that $w^3(t,-)$ is a bicharacter on S and $\gamma_i^2 = \alpha_{1i}^{m_1}...\alpha_{ni}^{m_n}$ by assumption and so we have $l(E_a)^2 = l(E_{s_1^{m_1}...s_1^{m_n}})$. By definition, we have

$$l(X_a)^2 = [\sum_{t\in T} w^4(a,t)e_t x][\sum_{t\in T} w^4(a,t)e_t x] = \sum_{t\in T} w^4(a,t)w^4(a,t \triangleleft x)e_t$$

and $l(X_{s_1^{m_1}...s_1^{m_n}}) = \sum_{t\in T} w^2(s_1^{m_1}...s_1^{m_n},t)e_t x$. To show $l(X_a)^2 = l(X_{s_1^{m_1}...s_1^{m_n}})$, we need only to show $w^4(a,t)w^4(a,t\triangleleft x) = w^2(s_1^{m_1}...s_1^{m_n},t)$ for $t \in T$. For the simplest case $t = a$, we have $w^4(a,a)w^4(a,ab) = \delta^2 \prod_{k=1}^n \gamma_k^{p_k}$ and $w^2(s_1^{m_1}...s_1^{m_n},a) = \prod_{k=1}^n \beta_k^{m_k}$ by definition. Since (v) of Proposition 6.1.10, we have $\beta_1^{p_1}...\beta_n^{p_n} = \gamma_1^{p_1}...\gamma_n^{p_n}$. And because $b^2 = 1$ and $b = s_1^{p_1}...s_n^{p_n}$, we get $\gamma_1^{2p_1}...\gamma_n^{2p_n} = 1$. Therefore we get $w^4(a,t)w^4(a,a\triangleleft x) = w^2(s_1^{m_1}...s_1^{m_n},a)$. For the case $t = s_1^{j_1}...s_1^{j_n}a$, if we use the equalities $w^4(a,a)w^4(a,a\triangleleft x) = w^2(s_1^{m_1}...s_1^{m_n},a)$, $\gamma_i^2 = \alpha_{1_i}^{m_1}...\alpha_{n_i}^{m_n}$, then we can also prove that $w^4(a,s_1^{j_1}...s_1^{j_n}a)w^4(a,s_1^{j_1}...s_1^{j_n}ab) = w^2(s_1^{m_1}...s_1^{m_n}, s_1^{j_1}...s_1^{j_n}a)$ and hence we have show (iv) of Lemma 6.1.6. Therefore we have prove that (i) of Corollary 6.1.9 holds. To prove that R such that (ii) of Corollary 6.1.9, we consider the R_φ. If we denote the four maps associated with R_φ as $w'^i (1 \le i \le 4)$, then it can be seen that the following equations hold

$$w'^1(s_i,s_j) = \alpha_{ji},\ w'^2(s_i,a) = \gamma_i,\ w'^3(a,s_i) = \beta_i,\ w'^4(a,a) = \delta.$$

Furthermore, one can get that the four tuple $(\alpha_{ji},\gamma_i,\beta_i,\delta)_{1\le i,j\le n}$ satisfies conditions (i)-(v) of Proposition 6.1.10 and hence R_φ such that the conditions of this Proposition and hence R_φ such that (i) of Corollary 6.1.9. And this implies that R satisfies (ii) of Corollary 6.1.9. The (iii) of Corollary 6.1.9 holds by (v) of Proposition 6.1.10. Finally we prove that $w^4(ab,ab) = w^4(a,a)$. Due to $\beta_1^{p_1}...\beta_n^{p_n} = \gamma_1^{p_1}...\gamma_n^{p_n}$, $\gamma_1^{2p_1}...\gamma_n^{2p_n} = 1$ and the definition of w^4, we know that $w^4(ab,ab) = w^4(a,a)$ and thus we have completed the proof. □

Combining Propositions 6.1.10-6.1.12, we obtain the following theorems

Theorem 6.1.13 *Given a four tuple* $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq n}$ *satisfying conditions* (i)-(v) *of Proposition 6.1.10, then there exists a unique general solution* R *for* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ *such that the following equations*

$$w^1(s_i,s_j) = \alpha_{ij},\ w^2(s_i,a) = \beta_i,\ w^3(a,s_i) = \gamma_i,\ w^4(a,a) = \delta.$$

Proof: By Propositions 6.1.10-6.1.12, we get what we want. □

Theorem 6.1.14 *Let* R *be a general solution for* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$*, then we can find a unique four tuple* $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq n}$ *satisfying the conditions* (i)-(v) *of Proposition 6.1.10 and the following equalities*

$$\alpha_{ij} := w^1(s_i,s_j),\ \beta_i := w^2(s_i,a),\ \gamma_i := w^3(a,s_i),\ \delta := w^4(a,a).$$

Proof: By Proposition 6.1.10, we get what we want. □

Let $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq n} := (1,1,1,1)_{1\leq i,j\leq n}$, then we can see that the four tuple such that the conditions of Proposition 6.1.10 and so we get a general solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. And hence we know that the general solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ always exists. Since we are interested in the number of quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, we will give the following theorem. Let N_Q be the set of all non-trivial quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and let T_Q be the set of trivial quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. For simple, we denote A_Q as all quasitriangular structures of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Then we have

Theorem 6.1.15 *If* $N_Q \neq \emptyset$ *and* $T_Q \neq \emptyset$*, then we have* $|N_Q| = |T_Q|$*. Moreover, if* $G = \mathbb{Z}_{n_1}\times ... \times \mathbb{Z}_{n_r}$*, then the cardinality of* A_Q *is a factor of* $2|G|^r$*.*

Proof: Let $W := \{\omega$ is bicharacter on G such that $\omega(g,h) = \omega(g\triangleleft x, h\triangleleft x)$ for $g,h \in G\}$. By Proposition 3.2.3, we know $|T_Q| = |W|$. Owing to Theorem 6.1.13-6.1.14 and using a little observation, one can obtain that the number of general solutions for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is also $|W|$. If $N_Q \neq \emptyset$, then we get that $|N_Q|$ is equal to the cardinality of the set of all general solutions by definition. Hence we have $|N_Q| = |T_Q|$. It is easy to see that $|W|$ is a factor of $|G|^r$, so we know $|A_Q|$ is a factor of $2|G|^r$. □

Remark 6.1.16 In fact, the above result implies that the number of quasitriangular structures of $\Bbbk^G\#\Bbbk\mathbb{Z}_2$ is $2|T_Q|$. If we put this observation and the Propositions 3.2.5-3.2.6 together, we know that A_Q can be taken any number in $\{0, |T_Q|, 2|T_Q|\}$. Furthermore, if $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ has a non-trivial quasitriangular structure, then the number of quasitriangular structures only depends on part of data $(G,\triangleleft)$. And this result seems to be very interesting. For example, we can quickly know that the number of quastriangular structures of D_4(the dihedral group with degree 2) is equal to the number of quastriangular structures of K_8 or have the double relationship! Trivial quasitriangular structures are easy to give, thus the number of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ can be evaluated very quickly!

Next, we use Theorems 6.1.13-6.1.14 to give all the general solutions on the Hopf algebra $H_{b:y}$ as an application.

Example 6.1.17 Recall that the 16 dimensional semisimple Hopf algebra $H^1_{b:y}$ in Example 2.1.8, the group $G = \langle a, b | a^4 = b^2 = 1, ab = ba\rangle$ and $a \triangleleft x = a^3, b \triangleleft x = b$. It can be seen that $S = \{1, a^2, b, a^2b\}$ and $T = \{a, a^3, ab, a^3b\}$. Thus we can assume $s_1 = a^2, s_2 = b$ by Corollary 6.1.4. Since the definition of $H_{b:y}$, we get $m_1 = p_1 = 1$ and $m_2 = p_2 = 0$. Let $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq 2}$ be a four tuple satisfying conditions (i)-(v) of Proposition 6.1.10, then we get the following equations

$$\alpha_{1i} = \alpha_{i1} = 1,\ \alpha_{22}^2 = 1,\ \beta_i^2 = \gamma_i^2 = 1,\ \beta_1 = \gamma_1,\ \delta^2 = 1. \tag{6.8}$$

For simple, we denote α_{22} as α. Now if we use Proposition 6.1.12, then we get a general solution R for $H_{b:y}$. In order to see the R more clearly, we list the $w^i(1 \leq i \leq 4)$ of R as follows

w^1	1	a^2	b	a^2b
1	1	1	1	1
a^2	1	1	1	1
b	1	1	α	α
a^2b	1	1	α	α

w^2	a	a^3	ab	a^3b
1	1	1	1	1
a^2	β_1	β_1	β_1	β_1
b	β_2	β_2	$\alpha\beta_2$	$\alpha\beta_2$
a^2b	$\beta_1\beta_2$	$\beta_1\beta_2$	$\alpha\beta_1\beta_2$	$\alpha\beta_1\beta_2$

w^3	1	a^2	b	a^2b
a	1	γ_1	γ_2	$\gamma_1\gamma_2$
a^3	1	γ_1	γ_2	$\gamma_1\gamma_2$
ab	1	γ_1	$\alpha\gamma_2$	$\alpha\gamma_1\gamma_2$
a^3b	1	γ_1	$\alpha\gamma_2$	$\alpha\gamma_1\gamma_2$

w^4	a	a^3	ab	a^3b
a	δ	$\gamma_1\delta$	$\gamma_2\delta$	$\gamma_1\gamma_2\delta$
a^3	$\beta_1\delta$	δ	$\beta_1\gamma_2\delta$	$\gamma_2\delta$
ab	$\beta_2\delta$	$\beta_2\gamma_1\delta$	$\alpha\beta_2\gamma_2\delta$	$\alpha\beta_2\gamma_1\gamma_2\delta$
a^3b	$\beta_1\beta_2\delta$	$\beta_2\delta$	$\alpha\beta_1\beta_2\gamma_2\delta$	$\alpha\beta_2\gamma_2\delta$

It can be known from Theorems 6.1.13-6.1.14 that the above table gives all general solutions for $H_{b:y}$ when $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq 2}$ takes all the four tuples that satisfy equation 6.8.

§6.2 Special solutions for quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

In this section, we will imitate the method used in the section 5.1 to give a necessary and sufficient condition for the existence of a special solution on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Proposition 6.2.1 *Let R be a special solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, and if we denote*

$$\alpha_{ij} := w^1(s_i, s_j),\ \beta_i := w^2(s_i, a),\ \gamma_i := w^3(a, s_i),\ \delta := w^4(a, a),$$

then the following equations hold

(i) $\alpha_{ij}^{k_i} = \alpha_{ij}^{k_j} = 1,\ 1 \leq i, j \leq n;$

(ii) $\beta_i^{k_i} = P_{s_i^{k_i}},\ \beta_i^2\sigma(s_i) = \alpha_{i1}^{m_1}...\alpha_{in}^{m_n},\ 1 \leq i \leq n;$

(iii) $\gamma_i^{k_i} = P_{s_i^{k_i}},\ \gamma_i^2\sigma(s_i) = \alpha_{1i}^{m_1}...\alpha_{ni}^{m_n},\ 1 \leq i \leq n;$

(iv) $\delta^2 = [\tau(a,a)\tau(b,a)\tau(b,b)^{-1}\sigma(a)^{-1}P^{-1}_{s_1^{m_1}...s_n^{m_n}}P^{-1}_{s_1^{p_1}...s_n^{p_n}}]\beta_1^{m_1+p_1}...\beta_n^{m_n+p_n};$

(v) $\delta^2 = [\tau(a,a)\tau(a,b)\tau(b,b)^{-1}\sigma(a)^{-1}P^{-1}_{s_1^{m_1}...s_n^{m_n}}P^{-1}_{s_1^{p_1}...s_n^{p_n}}]\gamma_1^{m_1+p_1}...\gamma_n^{m_n+p_n};$

(vi) $\alpha_{1i}^{p_1}...\alpha_{ni}^{p_n} = \alpha_{i1}^{p_1}...\alpha_{in}^{p_n} = \eta(a, s_i)$, $\beta_1^{p_1}...\beta_n^{p_n} = \gamma_1^{p_1}...\gamma_n^{p_n}\eta(a, b)$;

Proof: We mimic the proof of Proposition 6.1.10 as follows. Since w^1 is a bicharacter on S, we have $w^1(s_i, s_j)^{k_i} = w^1(s_i, s_j)^{k_j} = 1$ and so (i) holds. To show (ii), we note that $l(X_{s_i})^{k_i} = P_{s_i^{k_i}} l(X_1)$ and $r(E_a)^2 = r(E_{s_1^{m_1}...s_n^{m_n}})$, if we use (iii) of Lemma 5.1.6, we get $w^2(s_i, a)^{k_i} = P_{s_i^{k_i}}$ and so we have $\beta_i^{k_i} = P_{s_i^{k_i}}$. Similarly, since

$$r(E_a)^2 = \sum_{s\in S} w^2(s, a)^2\sigma(s)e_s,\ r(E_{s_1^{m_1}...s_n^{m_n}}) = \sum_{s\in S} w^1(s, s_1^{m_1}...s_n^{m_n})e_s,$$

we have $w^2(s_i, a)^2\sigma(s_i) = w^1(s_i, s_1^{m_1}...s_n^{m_n})$ through letting $s = s_i$. Because we have shown w^1 is a bicharacter on S, we obtain $w^1(s_i, s_1^{m_1}...s_n^{m_n}) = \alpha_{i1}^{m_1}...\alpha_{in}^{m_n}$ and hence (ii) holds. If we consider R_φ, then we know R_φ is also a special solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ and so R_φ such that (ii). Due to the $w'^i (1 \le i \le 4)$ of R_φ such that $w'^2(s_i, a) = w^3(ab, s_i)$ by definition of R_φ and $w^3(ab, s_i) = w^3(a, s_i)w^1(b, s_i)$ by (ii) of Lemma 5.1.6, we obtain $w'^2(s_i, a) = w^3(a, s_i)w^1(b, s_i)$. But $w^1(b, s_i) = \eta(a, s_i)$ because of (iii) in Corollary 6.1.9, so $w^3(ab, s_i) = w^3(a, s_i)\eta(a, s_i) = \gamma_i\eta(a, s_i)$. Because $\eta(a, s_i)^{k_i} = \eta(a, s_i)^2 = 1$ and (ii) holds for R_φ, we know (iii) holds. To show (iv), we first show (vi). Since (iii) in Corollary 6.1.9, we know $w^1(s_i, b) = w^1(b, s_i) = \eta(a, s_i)$. But we have known w^1 is a bicharacter on S and $b = s_1^{p_1}...s_n^{p_n}$ by the assumption, we know $\alpha_{1i}^{p_1}...\alpha_{ni}^{p_n} = \alpha_{i1}^{p_1}...\alpha_{in}^{p_n} = \eta(a, s_i)$. Because of (iv) in Corollary 6.1.9, we have $w^4(ab, ab) = \frac{\tau(ab,ab)}{\tau(a,a)}w^4(a, a)$. Using (iv) of Lemma 5.1.6, we get $w^4(ab, ab) = \tau(b, a)^{-1}w^2(b, ab)w^4(a, ab)$. With the help of the (iv) of Lemma 5.1.7, we obtain $w^4(a, ab) = \tau(a, b)^{-1}w^4(a, a)w^3(a, b)$ and so we have $\tau(b, a)^{-1}\tau(a, b)^{-1}w^2(b, ab)w^3(a, b) = \frac{\tau(ab,ab)}{\tau(a,a)}$. Due to $w^2(b, ab) = \eta(b, a)w^2(b, a)$ by (5.1) of Lemma 5.1.1, we know $\tau(b, a)^{-1}\tau(a, b)^{-1}\eta(b, a)w^2(b, a)w^3(a, b) = \frac{\tau(ab,ab)}{\tau(a,a)}$. It can be seen that $\frac{\tau(b,a)\tau(a,b)\tau(ab,ab)}{\tau(a,a)\eta(b,a)} = \tau(b, b)\eta(a, b)$. Moreover, since $w^2(b, a) = P^{-1}_{s_1^{p_1}...s_n^{p_n}}\beta_1^{p_1}...\beta_n^{p_n}$ and $w^3(a, b) = P^{-1}_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{p_1}...\gamma_n^{p_n}$, we get $(\beta_1^{p_1}...\beta_n^{p_n})(\gamma_1^{p_1}...\gamma_n^{p_n}) = \tau(b, b)\eta(a, b)P^2_{s_1^{p_1}...s_n^{p_n}}$. Due to (iii) of Lemma 5.1.6, we obtain $w^3(a, b)^2 = \tau(b, b)$ and hence $w^3(a, b) = \tau(b, b)w^3(a, b)^{-1} = \tau(b, b)P_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{-p_1}...\gamma_n^{-p_n}$. Therefore (vi) holds. To show (iv), since $l(X_a)^2 = \tau(a, a)l(X_{a^2}) = \tau(a, a)l(X_{s_1^{m_1}...s_n^{m_n}})$ and the following equations hold

$$l(X_a)^2 = \sum_{s\in S} w^4(a, t)w^4(a, t \triangleleft x)\sigma(t)e_t,\ l(X_{s_1^{m_1}...s_n^{m_n}}) = \sum_{t\in T} w^2(s_1^{m_1}...s_n^{m_n}, t)e_t,$$

we have

$$w^4(a,a)w^4(a,ab)\sigma(a) = \tau(a,a)w^2(s_1^{m_1}...s_n^{m_n},a) \tag{6.9}$$

through letting $t = a$. Since Lemma 5.1.6, we get $w^2(s,t)w^2(s',t) = \tau(s,s')w^2(ss',t)$ for $s \in S, t \in T$, so we have

$$w^2(s_1^{m_1}...s_n^{m_n},a) = P^{-1}_{s_1^{m_1}...s_n^{m_n}}\beta_1^{m_1}...\beta_n^{m_n}. \tag{6.10}$$

Since the (iii) in Lemma 5.1.2, we have $w^3(ab,b) = \tau(b,a)\frac{w^4(a,ab)}{w^4(a,a)}$. Owing to (5.2) in Lemma 5.1.1, we can get $w^3(ab,b) = w^3(a,b)\eta(a,b)$. Because we have known $w^3(a,b) = \tau(b,b)P_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{-p_1}...\gamma_n^{-p_n}$ and $w^3(ab,b) = \tau(b,a)\frac{w^4(a,ab)}{w^4(a,a)}$, we have

$$w^4(a,ab) = w^4(a,a)\tau(b,a)^{-1}\eta(a,b)\tau(b,b)P_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{-p_1}...\gamma_n^{-p_n}. \tag{6.11}$$

Combining equations (6.9)-(6.11) and (v), we obtain (iv). Finally, if we consider R_φ, then we have (iv) holds for R_φ. Therefore we get (v) holds for R. $\square$

Similar to Proposition 6.1.11, we have

Proposition 6.2.2 *Let R be in Proposition 6.2.1 and let $(\alpha_{ij},\beta_i,\gamma_i,\delta)_{1\le i,j\le n}$ be in Proposition 6.2.1, then the following equations hold*

(i) $w^1(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}) = \prod_{k=1}^n\prod_{l=1}^n \alpha_{kl}^{i_kj_l}$;

(ii) $w^2(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}a) = P^{-1}_{s_1^{i_1}...s_n^{i_n}}(\prod_{k=1}^n \beta_k^{i_k})\prod_{k=1}^n\prod_{l=1}^n \alpha_{kl}^{i_kj_l}$;

(iii) $w^3(s_1^{i_1}...s_n^{i_n}a, s_1^{j_1}...s_n^{j_n}) = P^{-1}_{s_1^{j_1}...s_n^{j_n}}(\prod_{k=1}^n \gamma_k^{j_k})\prod_{k=1}^n\prod_{l=1}^n \alpha_{kl}^{i_kj_l}$;

(iv) $w^4(s_1^{i_1}...s_n^{i_n}a, s_1^{j_1}...s_n^{j_n}a) = \lambda(i_1,...,i_n,j_1,...,j_n)(\prod_{k=1}^n \beta_k^{i_k})(\prod_{k=1}^n \gamma_k^{j_k})\prod_{k=1}^n\prod_{l=1}^n \alpha_{kl}^{i_kj_l}\delta$;

where $0 \le i_1,...,i_n \le (n-1)$, $0 \le j_1,...,j_n \le (n-1)$ *and* $\lambda(i_1,...,i_n,j_1,...,j_n) := P^{-1}_{s_1^{i_1}...s_n^{i_n}}P^{-1}_{s_1^{j_1}...s_n^{j_n}}\tau(s_1^{i_1}...s_n^{i_n},a)^{-1}\tau(a,s_1^{j_1}...s_n^{j_n})^{-1}$.

Proof: Since w^1 is a bicharacter on S, we get (i). Owing to (ii) of Lemma 5.1.7, we know $w^2(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}a) = w^1(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n})w^2(s_1^{i_1}...s_n^{i_n},a)$. Due to (iii) of

Lemma 5.1.6, we obtain $w^2(s_1^{i_1}...s_n^{i_n}, a) = P^{-1}_{s_1^{p_1}...s_n^{p_n}} \prod\limits_{k=1}^{n} \beta_k^{i_k}$ and so we have (ii). Similarly, we can show (iii). Thanks to (iv) of Lemma 5.1.6, we get

$$w^4(s_1^{i_1}...s_n^{i_n}a, s_1^{j_1}...s_n^{j_n}a) = \tau(s_1^{i_1}...s_n^{i_n}, a)^{-1}w^2(s_1^{i_1}...s_n^{i_n}, s_1^{j_1}...s_n^{j_n}a)w^4(a, s_1^{j_1}...s_n^{j_n}a). \quad (6.12)$$

Using (iv) of Lemma 5.1.7, we have

$$w^4(a, s_1^{j_1}...s_n^{j_n}a) = \tau(s_1^{j_1}...s_n^{j_n}, a)^{-1}w^3(a \triangleleft x, s_1^{j_1}...s_n^{j_n})w^4(a, a). \quad (6.13)$$

Because of (5.2) in Lemma 5.1.1, we get

$$w^3(a \triangleleft x, s_1^{j_1}...s_n^{j_n}) = \eta(a, s_1^{j_1}...s_n^{j_n})w^3(a, s_1^{j_1}...s_n^{j_n}). \quad (6.14)$$

Since the equations (6.12)-(6.14) and (ii),(iii), we know (iv) holds. $\square$

Conversely, given a four tuple $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\le i,j\le n}$ satisfies conditions (i)-(vi) of Proposition 6.2.1, then we have

Proposition 6.2.3 *Let R be the form (ii) on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ in Proposition 3.1.5, and if $w^i(1 \le i \le 4)$ of R are given by (i)-(iv) in Proposition 6.2.2 by using the four tuple above, then R is a special solution for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.*

Proof: Since Corollary 6.1.9, we only need to show R such that the conditions of Corollary 6.1.9. Because the definition of w^1, we know w^1 is a bicharacter on S and hence we get $l(E_{s_1})^{i_1}...l(E_{s_n})^{i_n} = l(E_{s_1^{i_1}...s_n^{i_n}})$. To show $l(X_{s_1})^{i_1}...l(X_{s_n})^{i_n} = P_{s_1^{i_1}...s_n^{i_n}}l(X_{s_1^{i_1}...s_n^{i_n}})$, we only need to show $\prod_{k=1}^{n} w^2(s_k, t)^{i_k} = P_{s_1^{i_1}...s_n^{i_n}}w^2(s_1^{i_1}...s_n^{i_n}, t)$ for $t \in T$. Owing to $aS = T$, we can assume $t = s_1^{j_1}...s_n^{j_n}a$. Since $w^2(s_k, s_1^{j_1}...s_n^{j_n}a) = \beta_k \prod_{l=1}^{n} \alpha_{kl}^{j_l}$, we obtain

$$\prod_{k=1}^{n} w^2(s_k, s_1^{j_1}...s_n^{j_n}a)^{i_k} = \prod_{k=1}^{n} \beta_k^{i_k} \prod_{k,l=1}^{n} \alpha_{kl}^{i_k j_l}. \quad (6.15)$$

Because the definition of w^2 and the equation (6.15) above, we get $\prod_{k=1}^{n} w^2(s_k, t)^{i_k} = P_{s_1^{i_1}...s_n^{i_n}}w^2(s_1^{i_1}...s_n^{i_n}, t)$ for $t \in T$ and so (i) of Lemma 6.1.6 holds. To show (ii) of Lemma 6.1.6, we only need to prove that $w^1(s, s')w^3(a, s) = w^3(as, s')$ and $w^2(s, t)w^4(a, t) = \tau(s, a)w^4(as, t)$ for $s, s' \in S,\ t \in T$ due to Lemma 5.1.6. And these equalities are not difficult to check and so (ii) of Lemma 6.1.6 hold. To show (iii) of Lemma 6.1.6, note that w^1 is a bicharacter on S and $\alpha_{ij}^{k_i} = 1$ by assumption and hence $l(E_{s_i})^{k_i} = l(E_1)$. To show $l(X_{s_i})^{k_i} = P_{s_i^{k_i}} l(X_1)$, we only need to prove that $w^2(s_i,t)^{k_i} = P_{s_i^{k_i}}$ for $t \in T$. Since the definition of w^2 and $\beta_i^{k_i} = P_{s_i^{k_i}}$ by assumption, we get $w^2(s_i,t)^{k_i} = P_{s_i^{k_i}}$ for $t \in T$ and so we know (iii) of Lemma 6.1.6 hold. To show (iv), it can be seen that $l(E_a)^2 = l(E_{s_1^{m_1}...s_1^{m_n}})$ is equivalent to $w^3(a,s)^2\sigma(s) = w^1(a^2,s)$ for $s \in S$. Since (iii) in Proposition 6.2.1 and w^1 is a bicharacter, we know $w^3(a,s_i)^2\sigma(s_i) = w^1(a^2,s_i)$. By definition, we have $w^3(a, s_1^{j_1}...s_n^{j_n}) = P^{-1}_{s_1^{j_1}...s_n^{j_n}} \prod_{k=1}^n w^3(a,s_k)^{j_k}$ and so we get

$$\begin{aligned} w^3(a, s_1^{j_1}...s_n^{j_n})^2 &= P^{-2}_{s_1^{j_1}...s_n^{j_n}} \prod_{k=1}^n w^3(a,s_k)^{2j_k} \\ &= P^{-2}_{s_1^{j_1}...s_n^{j_n}} \prod_{k=1}^n w^1(a^2,s_k)^{j_k}\sigma(s_k)^{-j_k} \\ &= P^{-2}_{s_1^{j_1}...s_n^{j_n}} (\prod_{k=1}^n \sigma(s_k)^{-j_k}) w^1(a^2, s_1^{j_1}...s_n^{j_n}) \end{aligned}$$

To show $w^3(a, s_1^{j_1}...s_n^{j_n})^2\sigma(s_1^{j_1}...s_n^{j_n}) = w^1(a^2, s_1^{j_1}...s_n^{j_n})$, we only need to show

$$P^{-2}_{s_1^{j_1}...s_n^{j_n}} \prod_{k=1}^n \sigma(s_k)^{-j_k}\sigma(s_1^{j_1}...s_n^{j_n}) = 1.$$

But the equation above follows from the following Lemma 6.2.4 and so we have $l(E_a)^2 = l(E_{s_1^{m_1}...s_1^{m_n}})$. By definition, we have

$$l(X_a)^2 = [\sum_{t\in T} w^4(a,t)e_t x][\sum_{t\in T} w^4(a,t)e_t x] = \sum_{t\in T} w^4(a,t)w^4(a,t\triangleleft x)\sigma(t)e_t$$

and $l(X_{s_1^{m_1}...s_1^{m_n}}) = \sum_{t\in T} w^2(s_1^{m_1}...s_1^{m_n},t)e_t x$. To show $l(X_a)^2 = \tau(a,a)l(X_{s_1^{m_1}...s_1^{m_n}})$, we only need to show $w^4(a,t)w^4(a,t\triangleleft x)\sigma(t) = \tau(a,a)w^2(s_1^{m_1}...s_1^{m_n},t)$ for $t \in T$. For the simplest case $t = a$, we have $w^4(a,a)w^4(a,ab) = \tau(a,b)^{-1}P^{-1}_{s_1^{p_1}...s_n^{p_n}} \prod_{k=1}^n \gamma_k^{p_k}\delta^2$ and $w^2(s_1^{m_1}...s_1^{m_n},a) = P^{-1}_{s_1^{m_1}...s_n^{m_n}} \prod_{k=1}^n \beta_k^{m_k}$ by definition. Since the proof of Proposition 6.2.1, we have $w^3(a,b) = P^{-1}_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{p_1}...\gamma_n^{p_n} = \tau(b,b)P_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{-p_1}...\gamma_n^{-p_n}$ and $w^4(a,a)w^4(a,ab) = \tau(a,b)^{-1}\tau(b,b)P_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{-p_1}...\gamma_n^{-p_n}\delta^2$. Owing to (iv), (vi) of Proposition 6.2.1, we get $w^4(a,a)w^4(a,ab)\sigma(a) = \tau(a,a)w^2(s_1^{m_1}...s_1^{m_n},a)$. For the case $t =$

$s_1^{j_1}...s_1^{j_n}a$, using the following equalities

$$w^4(a,a)w^4(a,ab)\sigma(a) = \tau(a,a)w^2(s_1^{m_1}...s_1^{m_n},a),\ \gamma_i^2\sigma(s_i) = \alpha_{1_i}^{m_1}...\alpha_{n_i}^{m_n}$$

and the following Lemma 6.2.4, we can prove that $w^4(a,t)w^4(a,t\triangleleft x)\sigma(t) = \tau(a,a)w^2(s_1^{m_1}...s_1^{m_n},t)$ and hence we have show (iv) of Lemma 6.1.6. To prove that R such that (i)-(iv) of Lemma 6.1.7, we consider the R_φ. If we denote the four maps associated with R_φ as $w'^i(1 \le i \le 4)$, then it can be seen that the following equations hold

$$w'^1(s_i,s_j) = \alpha_{ji},\ w'^2(s_i,a) = \eta(a,s_i)\gamma_i,\ w'^3(a,s_i) = \eta(s_i,a)\beta_i,\ w'^4(a,a) = \delta.$$

Furthermore, one can check that the four tuple $(\alpha_{ji}, \eta(a,s_i)\gamma_i, \eta(s_i,a)\beta_i, \delta)_{1\le i,j\le n}$ satisfies conditions (i)-(vi) of Proposition 6.2.1 and hence R_φ such that the conditions of this Proposition. Therefore R_φ such that (i)-(iv) of Lemma 6.1.6. And this implies that R such that (i)-(iv) of Lemma 6.1.7. Since w^1 is a bicharacter on S and (vi) of Proposition 6.2.1, we know (iii) of Corollary 6.1.9 holds. Finally we prove that $w^4(ab,ab) = \frac{\tau(ab,ab)}{\tau(a,a)}w^4(a,a)$. Since the definition of w^4, we have

$$w^4(ab,ab) = P^{-2}_{s_1^{p_1}...s_n^{p_n}}\tau(b,a)^{-1}\tau(a,b)^{-1}\prod_{k=1}^{n}\beta^{p_k}\prod_{k=1}^{n}\gamma^{p_k}\eta(a,b)\delta$$

.

Due to $P^{-1}_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{p_1}...\gamma_n^{p_n} = \tau(b,b)P_{s_1^{p_1}...s_n^{p_n}}\gamma_1^{-p_1}...\gamma_n^{-p_n}$ and $\beta_1^{p_1}...\beta_n^{p_n} = \eta(a,b)\gamma_1^{p_1}...\gamma_n^{p_n}$, we know that $w^4(ab,ab) = \frac{\tau(b,b)}{\tau(a,b)\tau(b,a)}w^4(a,a)$. Using the fact τ is a 2-cocycle, we can see that $\frac{\tau(b,b)}{\tau(a,b)\tau(b,a)} = \frac{\tau(ab,ab)}{\tau(a,a)}$ and hence we get $w^4(ab,ab) = \frac{\tau(ab,ab)}{\tau(a,a)}w^4(a,a)$. Therefore we have completed the proof. □

We have the following equality

Lemma 6.2.4 *$P^{-2}_{s_1^{j_1}...s_n^{j_n}}\prod_{k=1}^{n}\sigma(s_k)^{-j_k}\sigma(s_1^{j_1}...s_n^{j_n}) = 1$, where $j_1,...,j_n \in \mathbb{N}$.*

Proof: On the one hand, it can be seen that $\Delta(E_s) = E_s \otimes E_s + \sigma(s)X_s \otimes X_s$ for $s \in S$ and so we have $\Delta(E_{s_1^{j_1}...s_n^{j_n}}) = E_{s_1^{j_1}...s_n^{j_n}} \otimes E_{s_1^{j_1}...s_n^{j_n}} + \sigma(s_1^{j_1}...s_n^{j_n})X_{s_1^{j_1}...s_n^{j_n}} \otimes X_{s_1^{j_1}...s_n^{j_n}}$. On the other hand, we have $\Delta(E_{s_1})^{j_1}...\Delta(E_{s_n})^{j_n} = [E_{s_1} \otimes E_{s_1} + \sigma(s_1)X_{s_1} \otimes X_{s_1}]^{j_1}...[E_{s_n} \otimes E_{s_n} + \sigma(s_n)X_{s_n} \otimes X_{s_n}]^{j_n} = E_{s_1^{j_1}...s_n^{j_n}} \otimes E_{s_1^{j_1}...s_n^{j_n}} + \sigma(s_1)^{j_1}...\sigma(s_n)^{j_n}P^2_{s_1^{j_1}...s_n^{j_n}}X_{s_1^{j_1}...s_n^{j_n}} \otimes X_{s_1^{j_1}...s_n^{j_n}}$. Since $E_{s_1}^{j_1}...E_{s_n}^{j_n} = E_{s_1^{j_1}...s_n^{j_n}}$, we get $P^2_{s_1^{j_1}...s_n^{j_n}}\prod_{k=1}^{n}\sigma(s_k)^{j_k} = \sigma(s_1^{j_1}...s_n^{j_n})$. □

By Propositions 6.2.1-6.2.3, we get the following theorem

Theorem 6.2.5 *There exists a quasitriangular structure for* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ *if and only if there exists a four tuple* $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\le i,j\le n}$ *satisfies conditions (i)-(vi) of Proposition 6.2.1.*

Proof: By Propositions 6.2.1-6.2.3, we get what we want. □

To use the above Theorem 6.2.5 more convenient, we give the following corollary.

Corollary 6.2.6 *There exists a quasitriangular structure for* $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ *if and only if there exists a bicharacter* w^1 *on* S *and a pairing* $(\beta_i, \gamma_i)_{1\le i\le n}$ *satisfying the following conditions*

(i) $\beta_i^{k_i} = P_{s_i^{k_i}},\ \gamma_i^{k_i} = P_{s_i^{k_i}};$

(ii) $\beta_1^{p_1}...\beta_n^{p_n} = \gamma_1^{p_1}...\gamma_n^{p_n}\eta(a,b),\ \beta_1^{m_1}...\beta_n^{m_n} = \gamma_1^{m_1}...\gamma_n^{m_n};$

(iii) $w^1(s_i,b) = w^1(b,s_i) = \eta(a,s_i),\ w^1(s_i,a^2) = \beta_i^2\sigma(s_i),\ w^1(a^2,s_i) = \gamma_i^2\sigma(s_i);$

where $1 \le i \le n$.

Proof: If there exists a quasitriangular structure, then we know w^1, $(\beta_i,\gamma_i)_{1\le i\le n}$ of Proposition 6.2.1 such that the conditions (i)-(iii). Conversely, let $\alpha_{ij} := w^1(s_i, s_j)$ and let δ be given by (iv) of Proposition 6.2.1, then we know that $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\le i,j\le n}$ satisfies conditions (i)-(vi) of Proposition 6.2.1 by our conditions (i)-(iii). And hence there exists a quasitriangular structure by Theorem 6.2.5. □

Next we give a sufficient condition for the existence of a special solution on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.

Corollary 6.2.7 *If there is a bicharacter* w^1 *on* S *and a set* $\{\beta_i \in \Bbbk \mid 1 \le i \le n\}$ *satisfying the following conditions*

(i) $\beta_i^{k_i} = P_{s_i^{k_i}};$

(ii) $w^1(s_i,a^2) = w^1(a^2,s_i) = \beta_i^2\sigma(s_i);$

(iii) $w^1(s_i,b) = w^1(b,s_i) = \eta(a,s_i);$

where $1 \le i \le n$, *then there exists a quasitriangular structure on* $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$.

Proof: Let $\gamma_i := \beta_i \eta(a, s_i)$, then we can see that w^1 and $(\beta_i, \gamma_i)_{1 \le i \le n}$ satisfy the conditions of Corollary 6.2.6. And so we get what we want. $\square$

We give the following examples to illustrate our results in this section.

Example 6.2.8 Let $K(8n, \sigma, \tau)$ be in Example 2.1.4, then we can assume $s_1 = a^2,\ s_2 = b$. It can be seen that we can give a bicharacter on S satisfying the conditions of Corollary 6.2.7 through the following equations

$$w^1(s_1, s_1) := \beta^2 \sigma(s_1),\ w^1(s_1, s_2) = w^1(s_2, s_1) := \eta(a, s_1),\ w^1(s_2, s_2) := \eta(a, s_2),$$

where $\beta \in \Bbbk$ such that $\beta^n = P_{s_1^n}$. That is to say there is a special solution for $K(8n, \sigma, \tau)$.

Example 6.2.9 We have given all general solutions for $H^1_{b:y}$ in Example 6.1.17, now let's use Theorem 6.2.5 to give a special solution R_0 for $H^1_{b:y}$. To do this, we first give a four tuple $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1 \le i,j \le 2}$ as follows

$$w^1(s_1, s_1) = w^1(s_2, s_2) := 1,\ w^1(s_1, s_2) = w^1(s_2, s_1) := -1,\ \beta_i = \gamma_i := 1,\ \delta := 1,$$

where $1 \le i \le 2$. Then it can be seen that the four tuple satisfies the conditions (i)-(vi) of Proposition 6.2.1 and hence we can use Theorem 6.1.13 to get a special solution for $H_{b:y}$. Using the Proposition 6.2.2, we know the $w^i (1 \le i \le 4)$ of R_0 are given as follows

w^1	1	a^2	b	a^2b
1	1	1	1	1
a^2	1	1	-1	-1
b	1	-1	1	-1
a^2b	1	-1	-1	1

w^2	a	a^3	ab	a^3b
1	1	1	1	1
a^2	1	1	-1	-1
b	1	-1	1	-1
a^2b	1	-1	-1	1

w^3	1	a^2	b	a^2b
a	1	1	1	1
a^3	1	1	-1	-1
ab	1	-1	1	-1
a^3b	1	-1	-1	1

w^4	a	a^3	ab	a^3b
a	1	1	1	1
a^3	1	1	-1	-1
ab	-1	1	-1	1
a^3b	-1	1	1	-1

Chapter 7 φ-symmetric quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

In this chapter, we apply the conclusions in Chapter 5 to give all φ-symmetric quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. Then we prove that all quasitriangular structures on $K(8n,\sigma,\tau),\ A(8n,\sigma,\tau)$ are φ-symmetric and give all quasitriangular structures on $K(8n,\sigma,\tau),\ A(8n,\sigma,\tau)$.

§7.1 φ-symmetric quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$

Let φ be the Hopf isomorphism in Proposition 4.2.1. By Corollary 4.2.3, we know the most simple quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ are φ-symmetric quasitriangular structures. We will give a necessary and sufficient condition for the existence of φ-symmetric quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ in this section. Before that, we give the following definition.

Definition 7.1.1 *A quasitriangular function w on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is called a φ-symmetric quasitriangular function if it satisfies $w(t_1,t_2)=w(t_2,t_1)$ for $t_1,t_2\in T$.*

The following proposition is the reason why we give the above definition.

Proposition 7.1.2 *Let R be the form (ii) in Proposition 3.1.5, then R is a φ-symmetric quasitriangular structure if and only if w^4 is a φ-symmetric quasitriangular function and $w^i(1\le i\le 3)$ are given by (i)-(iii) in Lemma 5.1.2.*

Proof: If R is a φ-symmetric quasitriangular structure, then we know w^4 is a quasitriangular function due to Proposition 5.1.4. By definition of φ-symmetric quasitriangular structure, we get $w(t_1,t_2)=w(t_2,t_1)$ for $t_1,t_2\in T$. Moreover, since Lemma 5.1.2, we obtain that $w^i(1\le i\le 3)$ are given by (i)-(iii) in Lemma 5.1.2 and so we have proved the necessity. Conversely, if w^4 is a φ-symmetric quasitriangular function and $w^i(1\le i\le 3)$ are given by (i)-(iii) in Lemma 5.1.2, then we get R is a quasitriangular structure by Theorem 5.1.11. To show R is a φ-symmetric quasitriangular structure, we only need to prove that $w^2(s,t)=w^3(t\triangleleft x,s)$ and $w^1(s_1,s_2)=w^1(s_2,s_1)$ for $s,s_1,s_2\in S,\ t\in T$

by definition. Since (i)-(iii) of Lemma 5.1.2 and $w^4(t_1, t_2) = w^4(t_2, t_1)$ for $t_1, t_2 \in T$, we get $w^2(s, t) = w^3(t \triangleleft x, s)$ and $w^1(s_1, s_2) = w^1(s_2, s_1)$. $\square$

Corollary 7.1.3 *Let R be a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$, then R is a φ-symmetric quasitriangular structure if and only if $w^1(s_i, s_j) = w^1(s_j, s_i)$, $w^2(s_i, a) = w^3(ab, s_i)$ for $1 \leq i, j \leq n$.*

Proof: The necessity is obvious. In order to prove the sufficiency, we only need to prove that $w(t_1, t_2) = w(t_2, t_1)$ for $t_1, t_2 \in T$ because of Proposition 7.1.2 above. Since $aS = T$, we can assume that $t_1 = as$ and $t_2 = as'$ for some $s, s' \in S$. Then we have $w^4(as, as') = \tau(s, a)^{-1}w^2(s, as')w^4(a, as')$ by (ii) of Lemma 5.1.2. Because (ii) of Lemma 5.1.7, we have $w^2(s, as') = w^2(s, a)w^1(s, s')$. Owing to $w^4(a, as') = \tau(s', a)^{-1}w^4(a, a)w^3(ab, s')$ by (iii) of Lemma 5.1.2, we get

$$w^4(as, as') = \tau(s, a)^{-1}\tau(s', a)^{-1}w^4(a, a)w^1(s, s')w^2(s, a)w^3(ab, s').$$

Since w^1 is a bicharacter on S, we get $w^1(s, s') = w^1(s', s)$. Due to (iii) of Lemma 5.1.6 and (iii) of Lemma 5.1.7, we know $w^2(s, a) = w^3(ab, s)$ and $w^3(ab, s') = w^2(s', a)$. Therefore we have $w^4(as, as') = w^4(as', as)$. $\square$

As an application of the results of Section 6.2, we give the following proposition.

Proposition 7.1.4 *There exists a φ-symmetric quasitriangular structure for $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ if and only if there exists a bicharacter w^1 on S and a set $\{\beta_i \in \Bbbk |\ 1 \leq i \leq n\}$ satisfies the following conditions*

(i) $w^1(s_i, s_j) = w^1(s_j, s_i)$;

(ii) $\beta_i^{k_i} = P_{s_i^{k_i}}$, $w^1(s_i, a^2) = \beta_i^2\sigma(s_i)$;

(iii) $w^1(s_i, b) = \eta(a, s_i)$;

where $n = |S|$ and $1 \leq i, j \leq n$.

Proof: If R is a φ-symmetric quasitriangular structure, then we can find a bicharacter w^1 on S and a pairing $(\beta_i, \gamma_i)_{1\leq i\leq n}$ satisfy (ii), (iii) by Corollary 6.2.6. Since Corollary 7.1.3, we know w^1 satisfies (i). Conversely, it can be seen that w^1 and $\{\beta_i \in \Bbbk |\ 1 \leq i \leq$

$n\}$ such that conditions of Corollary 6.2.7, so we can find a quasitriangular structure R on $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ satisfies w^1 of R is exactly the w^1 and $w^2(s_i, a) = \beta_i$. Then we will show that R is φ-symmetric and hence we complete the proof. Since Corollary 7.1.3, we only need to prove that $w^1(s_i, s_j) = w^1(s_j, s_i)$, $w^2(s_i, a) = w^3(ab, s_i)$ for $1 \leq i, j \leq n$. Owing to (i), we know $w^1(s_i, s_j) = w^1(s_j, s_i)$. Because of the proof of Corollary 6.2.7, we get $w^3(a, s_i) = \eta(a, s_i)w^2(s_i, a)$. Due to (ii) of Lemma 5.1.1, we obtain $w^3(ab, s_i) = w^3(a, s_i)\eta(a, s_i)$. Therefore $w^3(ab, s_i) = \eta(a, s_i)^2 w^2(s_i, a)$. But $\eta(a, s_i)^2 = \eta(a^2, s_i) = 1$, so $w^3(ab, s_i) = w^2(s_i, a)$. $\square$

Remark 7.1.5 In fact, not all non-trivial quasitriangular structures on Hopf algebras $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ are φ-symmetric. For example, the R_0 in Example 6.2.9 is not φ-symmetric. This can be seen from the fact that w^4 of R_0 is not a φ-symmetric quasitriangular function. Therefore we know that symmetry of a quasitriangular structure is a special property.

§7.2 All quasitriangular structures on $K(8n, \sigma, \tau)$, $A(8n, \sigma, \tau)$

We regard $K(8n, \sigma, \tau)$ and $A(8n, \sigma, \tau)$ as Hopf algebras which are easy to deal with due to the numbers of generators of G are very small. Moreover, we will see that $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ has a quotient, either $K(8n, \sigma, \tau)$ or $A(8n, \sigma, \tau)$. For these reasons, we will study these Hopf algebras and give all the quasitriangular structures on $K(8n, \sigma, \tau)$, $A(8n, \sigma, \tau)$ in this section. Let $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ be in Definition 2.1.2, and if there is a subgroup H of G such that $H \triangleleft x = H$, then we have another data $(H, \triangleleft|_H, \sigma|_H, \tau|_{H\times H})$. For our convenience, we denote the data $(H, \triangleleft|_H, \sigma|_H, \tau|_{H\times H})$ as $(H, \triangleleft, \sigma, \tau)$.

Proposition 7.2.1 *The Hopf algebra $\Bbbk^H \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ is a quotient of $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$.*

Proof: We define a linear map $\psi : \Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2 \to \Bbbk^H \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$ by letting

$$\psi(e_h) := e_h,\ \psi(e_g) := 0,\ \psi(e_h x) := e_h x,\ \psi(e_g x) := 0,$$

where $h \in H$, $g \notin H$. Then it can be seen that ψ is a morphism of Hopf algebras and ψ is surjective. So we have completed the proof. $\square$

Corollary 7.2.2 *The Hopf algebra $K(8n,\sigma,\tau)$ is a quotient of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ or the Hopf algebra $A(8n,\sigma,\tau)$ is a quotient of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$.*

Proof: Owing to (ii) of Proposition 3.1.5, there is $b \in S$ such that $b^2 = 1$ and $t\triangleleft x = tb$ for $t \in T$. Taking $a \in T$ and let $H := \langle a, b\rangle$ as subgroup of G, then we know that $\Bbbk^H\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is a quotient of $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ by Proposition 7.2.1. Next we will show that $H = \langle a, b|\ a^{2n} = 1, b^2 = 1, ab = ba\rangle$ or $H = \langle a, b|\ a^{4n} = 1, b = a^{2n}\rangle$ as group for some $n \in \mathbb{N}$ and thus we complete the proof. If $b \in \langle a\rangle$, then we can assume $b = a^m$ for some $m \in \mathbb{N}$. Since $b^2 = 1$, then we have $a^{2m} = 1$. we claim that m is an even number in this case. Otherwise, m is odd and then we have $a^m \in S$. Because $a^2 \in S$ by definition and $(2, m) = 1$, so we get $a \in S$. But this is a contradiction, and hence we can assume that $m = 2n$. Then it can be seen that $H = \langle a, b|\ a^{4n} = 1, b = a^{2n}\rangle$ as group. If $b \notin \langle a\rangle$, then we will show that $H = \langle a, b|\ a^{2n} = 1, b^2 = 1, ab = ba\rangle$. Since $a^2 \in S$ and $a \notin S$, we can assume that the order of a is $2n$ for some $n \in \mathbb{N}$. Let $i, j \in \mathbb{N}$ and if $a^ib^j = 1$, then we have $2|j$ due to $b \notin \langle a\rangle$. Then we know $a^i = 1$ and hence $(2n)|i$. Therefore we get $H = \langle a, b|\ a^{2n} = 1, b^2 = 1, ab = ba\rangle$. □

Not only $K(8n,\sigma,\tau)$, $A(8n,\sigma,\tau)$ have the simple form, but also the quasitriangular structures on them are very simple.

Proposition 7.2.3 *All quasitriangular structures on $K(8n,\sigma,\tau)$, $A(8n,\sigma,\tau)$ are φ-symmetric.*

Proof: Let R be a non-trivial quasitriangular structure on $K(8n,\sigma,\tau)$, then we will show that R is φ-symmetric. Owing to the definition of $K(8n,\sigma,\tau)$, we can assume that $s_1 = a^2$, $s_2 = b$. Since Corollary 7.1.3, we only need to show that the following equations hold

$$w^1(s_1, s_2) = w^1(s_2, s_1),\ w^2(s_1, a) = w^3(ab, s_1),\ w^2(s_2, a) = w^3(ab, s_2).$$

Because (iii) of Corollary 6.2.6, we get $w^1(s_1, b) = w^1(b, s_1) = \eta(a, s_1)$. But $s_2 = b$ and so we obtain $w^1(s_1, s_2) = w^1(s_2, s_1)$. Due to (5.3) of Lemma 5.1.1, we have $w^4(ab, a) = w^4(a, ab)$. Since $l(X_a)^2 = \tau(a, a)l(X_{a^2})$, we get $w^4(a, t)w^4(a, t\triangleleft x)\sigma(t) = \tau(a, a)w^2(a^2, t)$

by expanding the equation. Let $t = a$, then we have

$$w^4(a,a)w^4(a,ab)\sigma(a) = \tau(a,a)w^2(a^2,a). \tag{7.1}$$

Similarly, we obtain that $w^4(t,a)w^4(t \triangleleft x, a)\sigma(t) = \tau(a,a)w^3(t,a^2)$ by expanding $r(X_a)^2 = \tau(a,a)r(X_{a^2})$. Let $t = a$, then we have

$$w^4(a,a)w^4(ab,a)\sigma(a) = \tau(a,a)w^3(a,a^2). \tag{7.2}$$

Since $w^4(a,ab) = w^4(ab,a)$ and the equations (7.1), (7.2), we get $w^2(a^2,a) = w^3(a,a^2)$. Because of (5.2) in Lemma 5.1.1, the know $w^3(ab,a^2) = w^3(a,a^2)$ and so the equation $w^2(s_1,a) = w^3(ab,s_1)$ holds. To show $w^2(s_2,a) = w^3(ab,s_2)$, we use (ii) of Lemma 5.1.2 and we get $w^2(b,a) = \tau(b,a)\frac{w^4(ab,a)}{w^4(a,a)}$. Similarly, we get $w^3(ab,b) = \tau(b,a)\frac{w^4(a,ab)}{w^4(a,a)}$ by (iii) of Lemma 5.1.2. Because we have known $w^4(a,ab) = w^4(ab,a)$, we get $w^2(b,a) = w^3(ab,b)$ and so $w^2(s_2,a) = w^3(ab,s_2)$. Therefore R is φ-symmetric. Similarly, one can prove that all quasitriangular structures on $A(8n,\sigma,\tau)$ are φ-symmetric. □

Let $Q_K :=$ {non-trivial quasitriangular structures on $K(8n,\sigma,\tau)$ }, then we have

Theorem 7.2.4 $Q_K \overset{1-1}{\longleftrightarrow} \{(\beta_1,\beta_2,\delta)|\ \beta_1^n = P_{s_1^n},\ \beta_2^2 = P_{s_2^2},\ \delta^2 = \frac{\tau(a,a)\tau(b,a)}{\tau(b,b)\sigma(a)}\beta_1\beta_2\}$, *where* $s_1 = a^2,\ s_2 = b$.

Proof: Given a non-trivial quasitriangular structure R on $K(8n,\sigma,\tau)$, we can define a triple (β_1,β_2,δ) through letting $\beta_1 := w^2(s_1,a),\ \beta_2 := w^2(s_2,a),\ \delta := w^4(a,a)$. Since (ii), (iv) of Proposition 6.2.1, we know $\beta_1^n = P_{s_1^n},\ \beta_2^2 = P_{s_2^2},\ \delta^2 = \frac{\tau(a,a)\tau(b,a)}{\tau(b,b)\sigma(a)}\beta_1\beta_2$. Conversely, let (β_1,β_2,δ) be a triple satisfying $\beta_1^n = P_{s_1^n},\ \beta_2^2 = P_{s_2^2},\ \delta^2 = \frac{\tau(a,a)\tau(b,a)}{\tau(b,b)\sigma(a)}\beta_1\beta_2$, then we claim that there is a unique quasitriangular structure R such that $w^2(s_1,a) = \beta_1,\ w^2(s_2,a) = \beta_2,\ w^4(a,a) = \delta$. To do this, let w^1 be a bicharacter on S which is determined as follows

$$w^1(s_1,s_1) := \beta_1^2\sigma(s_1),\ w^1(s_1,s_2) = w^1(s_2,s_1) := 1,\ w^1(s_2,s_2) := \eta(a,s_2). \tag{7.3}$$

then we will use Proposition 7.1.4 to get a quasitriangular structure R such that $w^2(s_1,a) = \beta_1,\ w^2(s_2,a) = \beta_2,\ w^4(a,a) = \delta$. We first show w^1 is well defined. To show this, the only non-trivial thing is to prove that $[\beta_1^2\sigma(s_1)]^n = 1$. Since Lemma 6.2.4, we get $P_{s_1^n}^2\sigma(s_1)^n = 1$ and so $[\beta_1^2\sigma(s_1)]^n = 1$. Then we prove that w^1 and the set $\{\beta_1, \beta_2\}$ such that the conditions of Proposition 7.1.4. To prove this, the only non-trivial thing is to show $\beta_2^2\sigma(s_2) = 1$. Owing to $\tau(a,b)\tau(ab,b) = \sigma(ab)\sigma(a)^{-1}\sigma(b)^{-1}$ and $\sigma(ab) = \sigma(a)$, we know $\tau(b,b)\sigma(b) = 1$. Due to $\beta_2^2\sigma(s_2) = \tau(b,b)\sigma(b)$, we have $\beta_2^2\sigma(s_2) = 1$. Now we can use Proposition 7.1.4 and Proposition 6.2.2 to get a φ-symmetric quasitriangular structure R satisfying $w^2(s_1,a) = \beta_1,\ w^2(s_2,a) = \beta_2,\ w^4(a,a) = \delta$. Since Lemma 5.1.2, we know R is unique if it is a φ-symmetric quasitriangular structure and it satifies that $w^2(s_1,a) = \beta_1,\ w^2(s_2,a) = \beta_2,\ w^4(a,a) = \delta$. Finally, by Proposition 7.2.3, we know that this correspondence we have discussed is one-one. □

Remark 7.2.5 In fact, from the proof of the above theorem, we know that all non-trivial quasitriangular structures on $K(8n,\sigma,\tau)$ are given by (i)-(iv) of Proposition 6.2.2, where (β_1,β_2,δ) are in Theorem 7.2.4 and w^1 is defined by (7.3) above and $\alpha_{ij} = w^1(s_i,s_j),\ \gamma_i = \beta_i\eta(a,s_i)$ for $1 \le i \le 2$. Suppose $\eta(a,b) = 1$ for $K(8n,\sigma,\tau)$. Given a non-trivial quasitriangular structure on $K(8n,\sigma,\tau)$, if we compare the coefficient of $e_a \otimes e_{s_2}x$ for R_{21} and $R^{-1} = (S\otimes \mathrm{Id})(R)$, then we can get that $R_{21} \neq R^{-1}$. Furthermore, one can obtain that there is no trivial triangular structure such that it is triangular. That is to say $K(8n,\sigma,\tau)$ is not triangular when $\eta(a,b) = 1$. And these Hopf algebras exist in large numbers, such as $K(8n,\zeta)$ in Example 2.1.5.

Similar to above Theorem 7.2.4, let $Q_A := \{$non-trivial quasitriangular structures on $A(8n,\sigma,\tau)$ $\}$, then we have

Theorem 7.2.6 $Q_A \xleftrightarrow{1-1} \{(\beta,\delta) |\ \beta^{2n} = P_{s^{2n}},\ \delta^2 = \frac{\tau(a,a)\tau(b,a)}{\tau(b,b)\sigma(a)}P_{s^n}\beta^{1+n}\}$, *where* $s = a^2$.

Proof: Let R be a non-trivial quasitriangular structure and let $w^1(s,s) = \beta, w^4(a,a) = \delta$, then it can be seen that (β,δ) such that the following conditions

$$\beta^{2n} = P_{s^{2n}},\ \delta^2 = \frac{\tau(a,a)\tau(b,a)}{\tau(b,b)\sigma(a)}P_{s^n}\beta^{1+n}. \tag{7.4}$$

due to (ii),(iv) of Proposition 6.2.1. Conversely, if (β,δ) satisfies the equation (7.4), then we can use (i)-(iv) of Proposition 6.2.2 to define a R as follows

$$w^1(s,s) := \beta^2\sigma(s),\ w^2(s,a) = w^3(a,s) := \beta,\ w^4(a,a) := \delta. \tag{7.5}$$

Then we can see that the four tuple $(\beta^2\sigma(s), \beta, \beta, \delta)$ satisfies conditions (i)-(vi) of Proposition 6.2.1 and thus R is a non-trivial quasitriangular structure. Moreover, since Lemma 5.1.2, we know R is unique if it is a φ-symmetric quasitriangular structure and satisfies $w^2(s,a) = w^3(s,a) = \beta,\ w^4(a,a) = \delta$. Finally, since Proposition 7.2.3, we know that this correspondence we have discussed is one-to-one. $\square$

Remark 7.2.7 From the proof of the above theorem, we know that all non-trivial quasitriangular structures on $A(8n,\sigma,\tau)$ are given by (i)-(iv) of Proposition 6.2.2, where (β,δ) are in Theorem 7.2.6 and w^1 is defined by (7.5) above and $\alpha_{11} = w^1(s,s),\ \gamma = \beta$.

Chapter 8 Construction of minimal quasitriangular Hopf algebras

In this chapter, we will study two classes of minimal quasitriangular Hopf algebras which are full rank minimal quasitriangular Hopf algebras and minimal triangular Hopf algebras. We mainly consider how to construct these two classes of Hopf algebras through the conclusions we have discussed.

§8.1 Full rank minimal quasitriangular Hopf algebras

Since we also feel interested in minimal quasitriangular Hopf algebras, we want to identify all minimal quasitriangular Hopf algebras among $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. We observed that this problem is complex for the general case, so we choose to study a special class of minimal quasitriangular Hopf algebras which we call full rank minimal quasitriangular Hopf algebras in this section. This kind of Hopf algebras have a remarkable feature, that is, the Hopf algebra structures on them are determined by their quasitriangular structures. We will see that it is easy to know when a Hopf algebra is a full rank minimal quasitriangular Hopf algebra and we can find a large number of full rank minimal quasitriangular Hopf algebras form Hopf algebras $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. For a quasitriangular Hopf algebra (H, R), we will denote $H_l := \{(f \otimes \mathrm{Id})|\ f \in H\}$, $H_r := \{(\mathrm{Id} \otimes f)|\ f \in H\}$ respectively in this section. Next we give the definition of full rank minimal quasitriangular Hopf algebras.

Definition 8.1.1 *A quasitriangular Hopf algebra* (H, R) *is called by a* full rank minimal quasitriangular Hopf algebra *if* $H = H_l$.

It can be seen that $H = H_l$ is equivalent to $H = H_r$, thus the condition $H = H_l$ in Definition 8.1.1 can be replaced by $H = H_r$. By definition, we know a full rank minimal quasitriangular Hopf algebra is a special class of minimal quasitriangular Hopf algebras. From the result in [21, Theorem 2.2], we know minimal triangular Hopf algebras belong to full rank minimal quasitriangular Hopf algebras. The following lemma is obviously.

Lemma 8.1.2 *A quasitriangular Hopf algebra (H, R) is a full rank minimal quasitriangular Hopf algebra if and only* $\mathrm{Rank}(R) = \dim(H)$.

The following lemma shows that full rank minimal quasitriangular Hopf algebras and minimal quasitriangular Hopf algebras are exactly coincident in some special cases.

Lemma 8.1.3 *If (H, R) is a quasitriangular Hopf algebra such that $H_l = H_r$, then (H, R) is a full rank minimal quasitriangular Hopf algebra if and only (H, R) is a minimal quasitriangular Hopf algebra.*

Proof: By definition, we only need to prove sufficiency. If (H, R) is a minimal quasitriangular Hopf algebra, then we have $H = H_l H_r$. Since $H_l = H_r$ and H_l is subalgebra of H, we get $H = H_l$. $\square$

Before we continue to study full rank minimal quasitriangular Hopf algebras, we give the following examples of them.

Example 8.1.4 Let H be a finite abelian group, then we can find a Hopf isomorphism $\phi : \Bbbk[H] \to \Bbbk^H$ due to H is commutative. Let $R = \sum_{h,k \in H} \phi(h)(k) e_h \otimes e_k$, then we can get that $(\Bbbk^H, R)$ is a full rank minimal quasitriangular Hopf algebra. Therefore we know that finite abelian group belongs to full rank minimal quasitriangular Hopf algebras.

In fact all minimal triangualar Hopf algebras are full rank minimal quasitriangular Hopf algebras and this fact has been figured in [21, Theorem 2.2]. Another example of full rank minimal quasitriangular Hopf algebras is the 8-dimension Kac algebra.

Example 8.1.5 *The 8-dimension Kac algebra K_8* [1, Section 2.3.1] belongs to $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$. By definition, the data $(G, \triangleleft, \sigma, \tau)$ of K_8 is given by the following way

(i) $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = b^2 = 1, ab = ba \rangle$ and $a \triangleleft x = b, b \triangleleft x = a$.

(ii) $\sigma(a^i b^j) = (-1)^{ij}$ for $1 \leq i, j \leq 2$.

(iii) $\tau(a^i b^j, a^k b^l) = (-1)^{jk}$ for $1 \leq i, j, k, l \leq 2$.

All possible quasitriangular structures on K_8 were given in [71] and we will choose non-trivial quasitriangular structures of K_8 to get full rank minimal quasitriangular

Hopf algebras. Let $\gamma \in \mathbb{k}$ such that $\gamma^4 = -1$, define R_γ as below

$$\begin{aligned} R_\gamma \colon &= [e_1 \otimes e_1 + e_1 \otimes e_{ab} + e_{ab} \otimes e_1 - e_{ab} \otimes e_{ab}] + \\ &[e_1 x \otimes e_a + e_1 x \otimes e_b - \gamma^2 e_{ab} x \otimes e_a + \gamma^2 e_{ab} x \otimes e_b] + \\ &[e_a \otimes e_1 x + e_b \otimes e_1 x + \gamma^2 e_a \otimes e_{ab} x - \gamma^2 e_b \otimes e_{ab} x] + \\ &[\gamma^{-1} e_a x \otimes e_a x + \gamma e_a x \otimes e_b x + \gamma e_b x \otimes e_a x + \gamma^{-1} e_b x \otimes e_b x]. \end{aligned}$$

From [71], we know all non-trivial quasitriangular structures on K_8 have been gotten when R_γ run over $\gamma^4 = -1$. Moreover, one can see that $\mathrm{Rank}(R_\gamma) = 8$ and hence (K_8, R_γ) is a full rank minimal quasitriangular Hopf algebra. Moreover, one can check that R_γ is not triangular and trivial universal $\mathcal{R}$-matrices are also not triangular, that is to say the set of full rank minimal Hopf algebras is larger than the set of minimal triangular Hopf algebras.

Proposition 8.1.6 *If (H, R) is a full rank minimal quasitriangular Hopf algebra, then the set of group like elements $\mathrm{G}(H)$ is an abelian group.*

Proof: Owing to (H, R) is a full rank, we know $l_R : H^{*cop} \to H$ is Hopf isomorphism. Thus we only need to show $\mathrm{G}(H^{*cop})$ is an abelian group. By Lemma 3.1.2, we know $\mathrm{C}(H^*)$ is a commutative algebra. It can be seen that $S^*(\mathrm{C}(H^*)) = \mathrm{C}(H^{*cop})$, so we obtain $\mathrm{C}(H^{*cop})$ is a commutative algebra. Since $\mathrm{G}(H^{*cop}) \subseteq \mathrm{C}(H^{*cop})$, we obtain that $\mathrm{G}(H^{*cop})$ is an abelian group. $\square$

Using this result, we can easily get that the quantum double $D(\mathbb{k}S_3)$ is not full rank minimal quasitriangular Hopf algebra. And this example shows that minimal quasitriangular Hopf algebras are larger than full rank minimal quasitriangular Hopf algebras. Naturally, we can ask when $\mathbb{k}^G \#_{\sigma,\tau} \mathbb{k}\mathbb{Z}_2$ is full rank minimal quasitriangular Hopf algebra. Below we give sufficient conditions for $\mathbb{k}^G \#_{\sigma,\tau} \mathbb{k}\mathbb{Z}_2$ to be a full rank minimal quasitriangular Hopf algebra. Then we use our results to get a series of full rank minimal quasitriangular Hopf algebras. For simplicity, we still denote $\mathbb{k}^G \#_{\sigma,\tau} \mathbb{k}\mathbb{Z}_2$ as H_G. By Proposition 3.2.1, if there is a non-trivial quasitriangular structure R on H_G, then we have $|S| = |T|$. Observe that if we let $S = \{s_1, \cdots, s_m\}$ and $T = \{t_1, \cdots, t_m\}$, then the functions $w^i (1 \leq i \leq 4)$ of R can be viewed as 4 matrices, which are $(w^1(s_i, s_j))_{1 \leq i,j \leq m}$, $(w^2(t_i, s_j))_{1 \leq i,j \leq m}$, $(w^3(s_i, t_j))_{1 \leq i,j \leq m}$, $(w^4(t_i, t_j))_{1 \leq i,j \leq m}$.

So when we say $w^i(1 \leq i \leq 4)$ we mean that they are matrices in the following content.

Proposition 8.1.7 *If R is a non-trivial quasitriangular structure on H_G, then (H_G, R) is a full rank minimal quasitriangular Hopf algebra if only if one of $w^i(1 \leq i \leq 4)$ is non-degenerated matrix.*

Proof: Assume that (H_G, R) is a full rank minimal quasitriangular Hopf algebra, then we have $\mathrm{Rank}(R) = \dim(H)$ by Lemma 8.1.2. Thus we know $w^i(1 \leq i \leq 4)$ are non-degenerated matrices and this implies the necessity holds. Conversely, let $1 \leq j \leq 4$ and suppose w^j is non-degenerated matrix, then we will show $w^i(1 \leq i \leq 4)$ are non-degenerated matrices. Owing to Proposition 6.2.2 and w^j is non-degenerated matrix, we know w^1 is a non-degenerated matrix. Using Proposition 6.2.2 again, we get $w^i(1 \leq i \leq 4)$ are non-degenerated matrices. And this implies $\mathrm{Rank}(R) = \dim(H)$ and hence we know (H_G, R) is a full rank minimal quasitriangular Hopf algebra. $\square$

Recall the Example 6.2.9, then one can see that $H_{b:y}$ is a full rank minimal quasitriangular Hopf algebra by using the above Proposition 8.1.7. For non-trivial φ-symmetric quasitriangular structures on H_G, we have the following proposition.

Proposition 8.1.8 *If R is a non-trivial φ-symmetric quasitriangular structure on H_G, then (H_G, R) is a full rank minimal quasitriangular Hopf algebra if only if (H_G, R) is a minimal quasitriangular Hopf algebra.*

Proof: By Lemma 8.1.3, we only need to prove that $(H_G)_l = (H_G)_r$. Since l_R is an algebra map and $TT = S$, it can be seen that $(H_G)_l = \langle l(X_t), l(E_t) \mid t \in T\rangle$ as algebra. Similarly, one can get that $(H_G)_r = \langle r(X_t), r(E_t) \mid t \in T\rangle$ as algebra. Because R is φ-symmetric and R_φ has the form (4.2), we know $l(X_t) = r(X_t)$ and $l(E_t) = r(E_{t \triangleleft x})$ for $t \in T$. Therefore we obtain $(H_G)_l = (H_G)_r$. $\square$

To get a series of full rank minimal quasitriangular Hopf algebras, we use the following propositions. For the Hopf algebra $K(8n, \sigma, \tau)$, we keep using the notation $P_{s_1^n}$ in Theorem 7.2.4. Then we have

Theorem 8.1.9 *$K(8n, \sigma, \tau)$ is a full rank minimal quasitriangular Hopf algebra if and only if $\eta(a, b) = -1$ and there is $\beta \in \Bbbk$ such that $\beta^n = P_{(a^2)^n}$ and $\beta^2\sigma(a^2)$ is a primitive nth root of 1.*

Proof: Assume that $(K(8n,\sigma,\tau),R)$ is a full rank minimal quasitriangular Hopf algebra, then it can be seen that R is a non-trivial quasitriangular structure. By Proposition 8.1.7, we know w^1 is non-degenerated matrix. To calculate the matrix w^1, we denote $s_j := a^{2j-2}$, $s_{n+j} := a^{2j-2}b$ for $1 \leq j \leq n$. Owing to (vi) of Proposition 6.2.1, we know $w^1(s,b) = \eta(a,s)$ for $s \in S$. In particular, we get $w^1(s_i,b) = \eta(a,s_i) = \eta(a,a^{2i-2}) = 1$ for $1 \leq i \leq n$. Since w^1 is a bicharacter on S, we have the following equalities for $1 \leq i,j \leq n$

$$w^1(s_i,s_{n+j}) = w^1(s_i,s_jb) = w^1(s_i,s_j)w^1(s_i,b) = w^1(s_i,s_j). \tag{8.1}$$

Similarly, one can get the following equations

$$w^1(s_i,s_{n+j}) = w^1(s_i,s_j),\ w^1(s_{i+n},s_{n+j}) = w^1(s_i,s_j)\eta(a,b), \tag{8.2}$$

where $1 \leq i,j \leq n$. Let A be a matrix defined by $A := (w^1(s_i,s_j))_{1\leq i,j\leq n}$. Due to equations (8.1) and (8.2), we have

$$(w^1(s_i,s_j))_{1\leq i,j\leq 2n} = \begin{pmatrix} A & A \\ A & \eta(a,b)A \end{pmatrix} \sim \begin{pmatrix} A & 0 \\ 0 & (\eta(a,b)-1)A \end{pmatrix}.$$

Here "$\sim$" means that two matrices can be gotten each other through elementary operations. Therefore we know w^1 is non-degenerated if and only if $\eta(a,b) = -1$ and A is non-degenerated matrix. Let $\beta := w^2(a^2,a)$, then we have $\beta^n = P_{(a^2)^n}$ and $w^1(a^2,a^2) = \beta^2\sigma(a^2)$ by (ii) of Proposition 6.2.1. Because w^1 is a bicharacter on S, we obtain $w^1(s_i,s_j) = (\beta^2\sigma(a^2))^{(i-1)(j-1)}$ for $1 \leq i,j \leq n$. Since A is non-degenerated matrix, we know $\beta^2\sigma(a^2)$ is a primitive nth root of 1. Therefore we have proved the necessity. To show the sufficiency, we suppose that $\eta(a,b) = -1$, $\beta^n = P_{(a^2)^n}$ and $\beta^2\sigma(a^2)$ is a primitive nth root of 1, then we will construct a non-trivial quasitriangular structure on $K(8n,\sigma,\tau)$ such that $(K(8n,\sigma,\tau),R)$ is a full rank minimal quasitriangular Hopf algebra. Let $\beta',\delta \in \Bbbk$ such that $\beta'^2 = P_{b^2}$, $\delta^2 = \frac{\tau(a,a)\tau(b,a)}{\tau(b,b)\sigma(a)}\beta\beta'$, then we can get a non-trivial quasitriangular structure R such that $w^2(a^2,a) = \beta$ by Theorem 7.2.4. Since we have showed w^1 is non-degenerated in this case, we get $(K(8n,\sigma,\tau),R)$ is a full rank minimal quasitriangular Hopf algebra. $\square$

To use Theorem 8.1.9 more conveniently, we give the following corollary.

Corollary 8.1.10 *Let $K(8n, \sigma, \tau)$ as before. If $\tau(a, a^i) = 1$ for $i \in \mathbb{N}$ and $\eta(a, b) = -1$, then $K(8n, \sigma, \tau)$ is a full rank minimal quasitriangular Hopf algebra if and only if there is $\omega \in \Bbbk$ such that $\omega^n = 1$ and $\omega^2\sigma(a^2)$ is a primitive nth root of 1.*

Proof: By Theorem 8.1.9, we get what we want. $\square$

Using Corollary 8.1.10, we can give a series of full rank minimal quasitriangular Hopf algebras as follows

Corollary 8.1.11 *Let $K(8n, \zeta)$ be the Hopf algebras given in Example 2.1.5, then we have the following conclusions:*

(i) if n is even and $n \geq 4$, then $K(8n, \zeta)$ is full rank minimal quasitriangular Hopf algebra.

(ii) if n is odd or $n = 2$, then $K(8n, \zeta)$ is not full rank minimal quasitriangular Hopf algebra.

Proof: Firstly, we show (i). By the definition of $K(8n, \zeta)$, $\sigma(a^2) = -\zeta^2$. If n is even and bigger than 4, we can find a $\omega \in \Bbbk$ such that $\omega^n = 1$ and $\omega^2 = -1$. Then we have $\omega^2\sigma(a^2) = \zeta^2$ and thus $\omega^2\sigma(a^2)$ is a primitive nth root of 1. By Corollary 8.1.10, we know that $K(8n, \zeta)$ is full rank minimal quasitriangular Hopf algebra.

Secondly, we show (ii). If n is odd, then we have $(\omega^2\sigma(a^2))^n = [-(\omega\zeta)^2]^n = -1$ for arbitrary $\omega \in \Bbbk$ such that $\omega^n = 1$. Hence $\omega^2\sigma(a^2)$ is not a primitive nth root of 1. As a result $K(8n, \zeta)$ is not full rank minimal quasitriangular Hopf algebra. If $n = 2$, then $\sigma(a^2) = 1$. Let $\omega \in \Bbbk$ such that $\omega^2 = 1$. Thus $\omega^2\sigma(a^2) = 1$ which is not a primitive 2th root of 1. Therefore $K(16, \zeta)$ is not full rank minimal quasitriangular Hopf algebra. $\square$

So we have found a series of full rank minimal quasitriangular Hopf algebras which belong to $\Bbbk^G \#_{\sigma,\tau} \Bbbk\mathbb{Z}_2$.

§8.2 Minimal and triangular semisimple Hopf algebras

S. Gelaki raised the question of whether there is a minimal and triangular semisimple Hopf algebra in [21], and then he and P. Etingof constructed a series of minimal triangular semisimple Hopf algebra in [12]. Their method is to construct twists iteratively and give some minimal triangular structures on some semisimple Hopf algebras. Inspired by [21] and [12], we can naturally ask whether there are other ways to give minimal and triangular semisimple Hopf algebras? and if so, whether we can give all minimal triangular structures on them? To answer this question, we choose Hopf algebras $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ to explore this problem. We first give a criterion when a quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ is a triangular structure. Then we use this criterion and some results in Subsection 8.1 to construct all minimal triangular structures on $H^n_{b:y}$ for $n \in \mathbb{N}$ and these Hopf algebras are different from the minimal triangular semisimple Hopf algebras $((\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p), \overline{J}_{21}^{-1}\overline{J})$ and $\mathbb{C}[\widetilde{G}]^{\overline{J}}$ constructed in [12], where $\widetilde{G} = S_3 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_3)^*$. Our method is based on the result that all quasitriangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ have been determined. Therefore one can get all minimal triangular structures on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ by using our way. By the way, we prove that the semisimple Hopf algebra $H^1_{b:y}$ is the smallest Hopf algebra among the non-trivial semisimple minimal triangular Hopf algebras. For convenience, we assume that R in this subsection means that a non-trivial quasitriangular structure on $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$. We first give the following lemma.

Lemma 8.2.1 *Let R be a non-trivial quasitriangular structure. If $w^4(t_1,t_2)w^4(t_2 \triangleleft x, t_1 \triangleleft x)\sigma(t_1)\sigma(t_2) = 1,\ t_1, t_2 \in T$, then $w^1(s_1,s_2)w^1(s_2,s_1) = 1, w^2(s,t)w^3(t,s)\sigma(s) = 1$ for $s, s_1, s_2 \in S,\ t \in T$.*

Proof: By Lemma 5.1.2, we get the following equalities

$$w^2(s,t) = \tau(s,t_1)\frac{w^4(st_1,t)}{w^4(t_1,t)},\ w^3(t,s) = \tau(s,t_1 \triangleleft x)\frac{w^4(t \triangleleft x, st_1 \triangleleft x)}{w^4(t \triangleleft x, t_1 \triangleleft x)}.$$

Therefore we obtain

$$\begin{aligned} w^2(s,t)w^3(t,s)\sigma(s) &= \tau(s,t_1)\tau(s,t_1 \triangleleft x)\frac{w^4(st_1,t)w^4(t \triangleleft x, st_1 \triangleleft x)}{w^4(t_1,t)w^4(t \triangleleft x, t_1 \triangleleft x)}\sigma(s) \\ &= \tau(s,t_1)\tau(s,t_1 \triangleleft x)\frac{[\sigma(st_1)\sigma(t)]^{-1}}{[\sigma(t_1)\sigma(t)]^{-1}}\sigma(s) \\ &= \frac{\tau(s,t_1)\tau(s,t_1 \triangleleft x)\sigma(t_1)\sigma(s)}{\sigma(st_1)} = 1 \end{aligned}$$

Then we will show $w^1(s_1,s_2)w^1(s_2,s_1)=1$. Due to (ii) of Lemma 5.1.6, we obtain $w^1(s_1,s_2)=\frac{w^3(s_1t,s_2)}{w^3(t,s_2)}$. Similarly, one can get $w^1(s_2,s_1)=\frac{w^2(s_2,s_1t)}{w^2(s_2,t)}$ by (ii) of Lemma 5.1.7. Thus we have

$$w^1(s_1,s_2)w^1(s_2,s_1)=\frac{w^3(s_1t,s_2)}{w^3(t,s_2)}\frac{w^2(s_2,s_1t)}{w^2(s_2,t)}=\frac{\sigma(s_2)^{-1}}{\sigma(s_2)^{-1}}=1.$$

□

Using Lemma 8.2.1, we can easily obtain the following proposition.

Proposition 8.2.2 *The R is a triangular structure if and only if $w^4(t_1,t_2)w^4(t_2\triangleleft x,t_1\triangleleft x)\sigma(t_1)\sigma(t_2)=1$ for $t_1,t_2\in T$.*

Proof: Directly we have

$$\begin{aligned}R_{21}=&\sum_{s_1,s_2\in S}w^1(s_2,s_1)e_{s_1}\otimes e_{s_2}+\sum_{s\in S,t\in T}w^3(t,s)e_sx\otimes e_t+\\&\sum_{t\in T,s\in S}w^2(s,t)e_t\otimes e_sx+\sum_{t_1,t_2\in T}w^4(t_2,t_1)e_{t_1}x\otimes e_{t_2}x.\end{aligned}$$

So the following equation holds

$$\begin{aligned}RR_{21}=&\sum_{s_1,s_2\in S}w^1(s_1,s_2)w^1(s_2,s_1)e_{s_1}\otimes e_{s_2}+\sum_{s\in S,t\in T}w^2(s,t)w^3(t,s)\sigma(s)e_s\otimes e_t+\\&\sum_{t\in T,s\in S}w^2(s,t)w^3(t,s)\sigma(s)e_t\otimes e_s+\sum_{t_1,t_2\in T}w^4(t_1,t_2)w^4(t_2\triangleleft x,t_1\triangleleft x)\sigma(t_1)\sigma(t_2)e_{t_1}\otimes e_{t_2}.\end{aligned}$$

By Lemma 8.2.1, we get what we want. □

In order to use Proposition 8.2.2 more conveniently, we give the following corollary.

Corollary 8.2.3 *If R is φ-symmetric and $w^4(t_1,t_2)^2=1$ for $t_1,t_2\in T$, then R is triangular if and only if $\frac{\tau(t_1\triangleleft x,t_2\triangleleft x)}{\tau(t_2,t_1)}\sigma(t_1)\sigma(t_2)=1$ for $t_1,t_2\in T$.*

Proof: Since R is φ-symmetric, we get $w^4(t_2\triangleleft x,t_1\triangleleft x)=w^4(t_1\triangleleft x,t_2\triangleleft x)$. Owing to Lemma 5.1.1, we obtain $w^4(t_1\triangleleft x,t_2\triangleleft x)=w^4(t_1,t_2)\frac{\tau(t_1\triangleleft x,t_2\triangleleft x)}{\tau(t_2,t_1)}$. Therefore we can see that R is triangular if and only if $w^4(t_1,t_2)^2\frac{\tau(t_1\triangleleft x,t_2\triangleleft x)}{\tau(t_2,t_1)}\sigma(t_1)\sigma(t_2)=1$ by Proposition 8.2.2. Because $w^4(t_1,t_2)^2=1$, we get what we want. □

We give the application of Corollary 8.2.3 by the following example.

Example 8.2.4 Let $H_{b:y}^n$ be in Example 2.1.8. We will use Theorem 6.2.5 and Corollary 8.2.3 to give a minimal triangular structure on $H_{b:y}^1$. By definition of $H_{b:y}^1$, we know $S = \{1, a^2, b, a^2b\}$. Thus we can assume that $s_1 = a^2, s_2 = b$. To give a special solution for $H_{b:y}^1$, we first give a four tuple $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq 2}$ as follows

$$\alpha_{11} = \alpha_{22} := 1,\ \alpha_{12} = \alpha_{21} := -1,\ \beta_1 = \gamma_1 := 1,\ \beta_2 = -\gamma_2 := 1,\ \delta = 1.$$

Then it can be seen that the four tuple satisfies the conditions (i)-(vi) of Proposition 6.2.1 and hence we can use Theorem 6.1.13 to get a special solution R_0 for $H_{b:y}^1$. Using the Proposition 6.2.2, we know the $w^i (1 \leq i \leq 4)$ of R_0 are given as follows

w^1	1	a^2	b	a^2b
1	1	1	1	1
a^2	1	1	-1	-1
b	1	-1	1	-1
a^2b	1	-1	-1	1

w^2	a	a^3	ab	a^3b
1	1	1	1	1
a^2	1	1	-1	-1
b	1	-1	1	-1
a^2b	1	-1	-1	1

w^3	1	a^2	b	a^2b
a	1	1	-1	-1
a^3	1	1	1	1
ab	1	-1	-1	1
a^3b	1	-1	1	-1

w^4	a	a^3	ab	a^3b
a	1	1	-1	-1
a^3	1	1	1	1
ab	-1	1	1	-1
a^3b	-1	1	-1	1

By definition, we have $T = \{a^{2i+1}b^j |\ 1 \leq i, j \leq 2\}$. Then it can be checked that $\frac{\tau(t_1 \triangleleft x, t_2 \triangleleft x)}{\tau(t_2, t_1)}\sigma(t_1)\sigma(t_2) = 1$ for $t_1, t_2 \in T$. From the above tables, one can see that R_0 satisfies the conditions of Corollary 8.2.3. Moreover, we can get that w^1 is non-degenerated matrix and hence R_0 is full rank minimal quasitriangular structure by Proposition 8.1.7. Therefore we have given a full rank minimal and triangular structure on $H_{b:y}^1$.

To further simplify Proposition 8.2.2, we introduce the following lemmas.

Lemma 8.2.5 *Let R be a non-trivial quasitriangular structure. Then $w^4(t_1, t_2)w^4(t_2 \triangleleft x, t_1 \triangleleft x)\sigma(t_1)\sigma(t_2) = 1$ for $t_1, t_2 \in T$ if and only if the following equations hold*

$$w^1(s_1, s_2)w^1(s_2, s_1) = 1,\ w^2(s, t)w^3(t, s)\sigma(s) = 1,\ w^4(a, a)w^4(ab, ab)\sigma(a)^2 = 1$$

for $s, s_1, s_2 \in S,\ t \in T$.

Proof: By Lemma 8.2.1, we know the necessity holds. To show the sufficiency, we will use Lemmas 5.1.6,5.1.7. Since $T = aS$, we need only to show the following equations

$$w^4(s_1a, s_2a)w^4(s_2ab, s_1ab)\sigma(s_1a)\sigma(s_2a) = 1,\ s_1, s_2 \in S.$$

By (iv) of Lemma 5.1.6, we get $w^4(s_1a, s_2a) = \tau(s_1, a)^{-1}w^2(s_1, s_2a)w^4(a, s_2a)$. Further, we can use (iv) of Lemma 5.1.7 to obtain $w^4(a, s_2a) = \tau(s_2, a)^{-1}w^3(a \triangleleft x, s_2)w^4(a, a)$. Therefore we have the following equation

$$w^4(s_1a, s_2a) = \tau(s_1, a)^{-1}\tau(s_2, a)^{-1}w^2(s_1, s_2a)w^3(a \triangleleft x, s_2)w^4(a, a). \tag{8.3}$$

Similarly, one can get the following equality

$$w^4(s_2ab, s_1ab) = \tau(s_2, ab)^{-1}\tau(s_1, ab)^{-1}w^2(s_2, s_1ab)w^3(a, s_1)w^4(ab, ab). \tag{8.4}$$

Note that $w^2(s_1, s_2a) = w^1(s_1, s_2)w^2(s_1, a)$ and $w^2(s_2, s_1ab) = w^1(s_2, s_1)w^2(s_2, a\triangleleft x)$ by Lemma 5.1.7, and if we use the assumption about w^2, w^3, we get

$$[w^2(s_1, s_2a)w^3(a \triangleleft x, s_2)][w^2(s_2, s_1ab)w^3(a, s_1)] = \sigma(s_1)^{-1}\sigma(s_2)^{-1}.$$

Since $w^4(a, a)w^4(ab, ab)\sigma(a)^2 = 1$ and the compatibility between σ and τ, one can obtain that $w^4(s_1a, s_2a)w^4(s_2ab, s_1ab)\sigma(s_1a)\sigma(s_2a) = 1$ and hence we have completed the proof. □

In fact, some conditions in Lemma 8.2.5 can be further simplified. Recall that we have assumed that $G = \langle s_i, a |\ s_i^{k_i} = 1, a^2 = s_1^{m_1}...s_n^{m_n}, s_is_j = s_js_i, as_i = s_ia\rangle_{1\le i,j\le n}$ as group for some natural numbers n, k_i, m_j, then we have

Lemma 8.2.6 *Let R be a non-trivial quasitriangular structure. Then the following statements are equivalent*

(i) $w^1(s, s')w^1(s', s) = 1,\ w^2(s, t)w^3(t, s)\sigma(s) = 1,\ s, s' \in S,\ t \in T;$

(ii) $w^1(s_i, s_j)w^1(s_j, s_i) = 1,\ w^2(s_i, a)w^3(a, s_i)\sigma(s_i) = 1,\ 1 \le i, j \le n;$

Proof: By definition, we only need to prove that (ii) implies (i). Now we assume that (ii) holds. Let $s, s' \in S$, we first show $w^1(s, s')w^1(s', s) = 1$. Owing to Lemmas 5.1.6,5.1.7, we obtain that w^1 is a bicharacter on S. Thus we have $w^1(s, s')w^1(s', s) = 1$. Then we will prove that $w^2(s, s'a)w^3(s'a, s)\sigma(s) = 1$. To do this, we use Lemmas 5.1.6,5.1.7 again. Then we get $w^2(s, s'a) = w^2(s, a)w^1(s, s')$ and $w^3(s'a, s) = w^3(a, s)w^1(s', s)$. Because $w^1(s, s')w^1(s', s) = 1$, we get $w^2(s, s'a)w^3(s'a, s)\sigma(s) = w^2(s, a)w^3(a, s)\sigma(s)$. Thus we only need to show $w^2(s, a)w^3(a, s)\sigma(s) = 1$. Let $s = s_1^{i_1}...s_n^{i_n}$ and keep the notation in Proposition 6.2.2, we can obtain $w^2(s, a)w^3(a, s) = P^{-2}_{s_1^{i_1}...s_n^{i_n}} \prod_{k=1}^n (\beta_k\gamma_k)^{i_k}$ by using (ii) and (iii) of Proposition 6.2.2. By assumption, we have $\beta_k\gamma_k = \sigma(s_k)^{-1}$. Therefore we get $w^2(s, a)w^3(a, s) = P^{-2}_{s_1^{i_1}...s_n^{i_n}} \prod_{k=1}^n \sigma(s_k)^{-i_k}$. Now we can use Lemma 6.2.4 to get $w^2(s, a)w^3(a, s) = \sigma(s_1^{i_1}...s_n^{i_n})^{-1}$. By definition, we have $s = s_1^{i_1}...s_n^{i_n}$ and hence we obtain $w^2(s, a)w^3(a, s)\sigma(s) = 1$. □

If we use Lemmas 8.2.5-8.2.6 and Proposition 8.2.2 together, then we obtain the following proposition which gives a simple criterion for R to be a triangular structure.

Proposition 8.2.7 *The R is a triangular structure if and only if the following conditions hold*

$$w^1(s_i, s_j)w^1(s_j, s_i) = 1,\ w^2(s_i, a)w^3(a, s_i)\sigma(s_i) = 1,\ w^4(a, a)w^4(ab, ab)\sigma(a)^2 = 1,$$

where $1 \leq i, j \leq n$.

Proof: By Lemmas 8.2.5-8.2.6 and Proposition 8.2.2, we get what we want. □

Now we're going to use Proposition 8.1.8 and Theorem 6.2.5 to answer the question at the beginning of this subsection. We first construct all the triangular structures on $H^n_{b:y}$, then we will find all minimal triangular structures among these triangular structures. To achieve our goal, we give the following lemma.

Lemma 8.2.8 *If n is even number, then there is no non-trivial quasitriangular structure on $H^n_{b:y}$.*

Proof: By definition of $H^n_{b:y}$, we know $S = \{a^{2i}b^j \mid i, j \geq 0\}$. Thus we can arrange that $s_1 = a^2, s_2 = b$. Assume that R is a non-trivial quasitriangular structure on $H^n_{b:y}$. For convenience, we keep the notation in Proposition 6.2.1. Owing to (ii) of

Proposition 6.2.1, we obtain $\beta_2^{2n} = 1$ and $\alpha_{21} = -\beta_2^2$. So we have $\alpha_{21}^n = (-1)^n$. Because $a \triangleleft x = a^{2n+1}$, we know $p_1 = n, p_2 = 0$. Due to (vi) of Proposition 6.2.1, one can get that $\alpha_{21}^n = -1$. But we already obtain $\alpha_{21}^n = (-1)^n$, so $(-1)^n = -1$ and this implies n is an odd number. $\square$

Due to Lemma 8.2.8, we only consider $H_{b:y}^n$ when n is an odd number. Given an odd number n, let $T_n :=$ {non-trivial triangular structures on $H_{b:y}^n$ }. Then we have

Theorem 8.2.9 $T_n \xleftrightarrow{1-1} \{(\alpha, \beta_1, \beta_2, \delta) \in \Bbbk^4 | \ \alpha^2 = \beta_1^2 = \beta_2^{2n} = \delta^2 = 1\}$.

Proof: Since $S = \{a^{2i}b^j | \ i, j \geq 0\}$, we can arrange that $s_1 = a^2, s_2 = b$. Let R be a non-trivial triangular structure on $H_{b:y}^n$. For simple, we keep the notation in Proposition 6.2.1. Then we define a map $\phi : T_n \to \{(\alpha, \beta_1, \beta_2, \delta) \in \Bbbk^4 | \ \alpha^2 = \beta_1^2 = \beta_2^{2n} = \delta^2 = 1\}$ by letting $\phi(R) := (\alpha_{22}, \beta_1, \beta_2, \delta)$. Due to Proposition 8.2.7, we know $w^1(s_2, s_2)^2 = 1$. Since $w^1(s_2, s_2) = \alpha_{22}$ by definition, we obtain $\alpha_{22}^2 = 1$. Using (iv),(v) of Proposition 6.2.1, one can get that $\delta^2 = \beta_1^{n+1} = \gamma_1^{n+1}$. Thanks to (vi) of Proposition 6.2.1, we know $\beta_1^n = \gamma_1^n$. So we have $\beta_1 = \gamma_1$. By Proposition 8.2.7, we know $w^2(s_1, a)w^3(a, s_1) = 1$. Because $w^2(s_1, a) = \beta_1$ and $w^3(a, s_1) = \gamma_1$ by definition, we obtain $\beta_1\gamma_1 = 1$ and so $\beta_1^2 = 1$. Owing to n is an odd number and $\delta^2 = \beta_1^{n+1}$, we get $\delta^2 = 1$. Note that the equation $\beta_2^{2n} = 1$ follows from (ii) of Proposition 6.2.1 directly and hence we have proved that ϕ is well defined. Then we will show ϕ is bijective. Let $(\alpha, \beta_1, \beta_2, \delta) \in \Bbbk^4$ such that $\alpha^2 = \beta_1^2 = \beta_2^{2n} = \delta^2 = 1$. If we define $\alpha_{ij}, \gamma_i \in \Bbbk$ for $1 \leq i, j \leq 2$ as follows

$$\alpha_{11} = 1, \ \alpha_{12} = -\beta_2^2, \ \alpha_{21} = -\beta_2^{-2}, \ \alpha_{22} = \alpha, \ \gamma_1 = \beta_1, \ \gamma_2 = -\beta_2^{-1}.$$

Then one can check that the four tuple $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1 \leq i,j \leq 2}$ satisfies conditions (i)-(vi) of Proposition 6.2.1, therefore we can obtain a non-trivial quasitriangular structure R such that $w^1(s_i, s_j) = \alpha_{ij}$ and $w^2(s_i, a) = \beta_i, w^3(a, s_i) = \gamma_i$ by Theorem 6.2.5. Now one can see that R satisfies the conditions of Proposition 8.2.7, so we know R is triangular structure. Moreover, it can be seen that $\phi(R) = (\alpha, \beta_1, \beta_2, \delta)$. Therefore we have prove that ϕ is surjective. Finally we will show ϕ is injective. Let R be a non-trivial triangular structure on $H_{b:y}^n$. For simple, we continue to use the notation in Proposition 6.2.1. Thanks to (ii) of Proposition 6.2.1, we get $\alpha_{11} = \beta_1^2$ and $\alpha_{21} = -\beta_2^2$. Similarly, one can obtain $\alpha_{12} = -\gamma_2^2$ by using (iii) of Proposition 6.2.1. Due to Proposition 8.2.7, we know $\gamma_1 = \beta_1^{-1}$ and $\gamma_2 = -\beta_2^{-1}$. Therefore we have seen that $(\alpha_{ij}, \beta_i, \gamma_i, \delta)_{1\leq i,j\leq 2}$ is completely determined by $(\alpha_{22}, \beta_1, \beta_2, \delta)$ and this implies that ϕ is injective. $\square$

Since we have obtained all triangular structures on $H^n_{b:y}$, we only need to identify all minimal structures from T_n. To this end, we first introduce the following lemmas. Recall that if (H, R) is a quasitriangular Hopf algebra, then the H_l, H_r have been defined by $H_l := \{(f \otimes \mathrm{Id})|\ f \in H\}$, $H_r := \{(\mathrm{Id} \otimes f)|\ f \in H\}$ respectively. For simple, we will denote $\Bbbk^G\#_{\sigma,\tau}\Bbbk\mathbb{Z}_2$ as H_G in this subsection. The following lemma is a part of the proof in [21, Theorem 2.2].

Lemma 8.2.10 *If R is a non-trivial triangular structure on H_G, then $(H_G)_l = (H_G)_r$.*

Proof: Using $R_{21} = R^{-1}$ and $R^{-1} = (S \otimes \mathrm{Id})(R)$, one can complete the proof. $\square$

Now we can give a necessary and sufficient condition for determining when a triangular structure R is minimal.

Corollary 8.2.11 *If R is a non-trivial triangular structure on H_G, then R is minimal if and only if w^1 is a non-degenerated matrix.*

Proof: By Lemmas 8.2.10,8.1.3 and Proposition 8.1.7, we get what we want. $\square$

Next we will give a criterion for when the w^1 in Corollary 8.2.11 is non-degenerated matrix. Assume that $H = \langle g_i |\ g_i^{n_i} = 1, g_ig_j = g_jg_i\rangle_{1\leq i,j\leq m}$ as group and let w be a bicharacter on H, then we will give a criterion for when $(w(g,h))_{g,h\in H}$ is non-degenerated matrix. Let $\alpha_{ij} := w(g_i, g_j)$ and let ω_i be a primitive n_ith root of 1 in $\Bbbk$. Since w is bicharacter, we can assume that $\alpha_{ij} = \omega_j^{m_{ij}}$. Then we can get a matrix $(m_{ij})_{1\leq i,j\leq m}$ and we denote it as M. Let $(i_1, ..., i_m), (j_1, ..., j_m) \in \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_m}$ such that $(i_1, ..., i_m) = (j_1, ..., j_m)$, then we will write it as $(i_1, ..., i_m) \equiv (j_1, ..., j_m)$.

Lemma 8.2.12 *The matrix $(w(g,h))_{g,h\in H}$ is non-degenerated if and only if the following equation has a unique solution $(0, ..., 0)$*

$$(i_1, ..., i_m)M \equiv (0, ..., 0),\ (i_1, ..., i_m) \in \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_m}.$$

Proof: Let $\chi_{g_1^{i_1}...g_m^{i_m}} : H \to \Bbbk$ be a character which is determined by $\chi_{g_1^{i_1}...g_m^{i_m}}(g_j) := \omega_j^{i_j}$ for $1 \leq i_k \leq n_k, 1 \leq j, k \leq m$. Then it is well known that $\{e_{g_1^{i_1}...g_m^{i_m}}|\ 1 \leq i_k \leq$

$n_k, 1 \leq k \leq m\}$ is a basis of orthogonal idempotents of $\Bbbk[H]$, where $e_{g_1^{i_1}...g_m^{i_m}} = \sum_{h\in H} \chi_{g_1^{i_1}...g_m^{i_m}}(h)$. Define a linear map $f : \Bbbk^H \to \Bbbk[H]$ by $f(e_g) := \sum_{h\in H} w(g,h)h$ for $g, h \in H$, then we can see that $(w(g,h))_{g,h\in H}$ is non-degenerated if and only if f is bijective. To study when f is bijective, we define anther linear map $\phi : \Bbbk[H] \to \Bbbk^H$ by $\phi(g_1^{i_1}...g_m^{i_m}) := \sum_{h\in H} \chi_{g_1^{i_1}...g_m^{i_m}}(h)^{-1}e_h$ where $1 \leq i_k \leq n_k, 1 \leq k \leq m$. Using the orthogonal relationship between the characters of H, we know that ϕ is bijective. Therefore we know f is bijective if and only if $\phi \circ f$ is bijective. By direct calculation we have $\phi \circ f(e_{g_1^{i_1}...g_m^{i_m}}) = n_1...n_m e_{g_1^{i'_1}...g_m^{i'_m}}$ where $(i'_1, ..., i'_m) = (i_1, ..., i_m)M$. Therefore we know that $\phi \circ f$ is bijective if and only if $(i_1, ..., i_m)M \neq (j_1, ..., j_m)M$ for $(i_1, ..., i_m) \neq (j_1, ..., j_m)$. However, one can see that $(i_1, ..., i_m)M \neq (j_1, ..., j_m)M$ for $(i_1, ..., i_m) \neq (j_1, ..., j_m)$ if and only if $(i_1, ..., i_m)M \equiv (0, ..., 0),\ (i_1, ..., i_m) \in \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_m}$ has a unique solution. Hence we have completed the proof. □

For the group H, if $m = 2$ and $n_1 = n_2 = n$ then we have a more simple way to determine when the matrix $(w(g,h))_{g,h\in H}$ is non-degenerated. For convenience, we denote $|M|$ as the determinant of M and write $(|M|, n)$ as the largest common factor of $|M|$ and n.

Corollary 8.2.13 *If $H = \langle g_i |\ g_i^n = 1, g_ig_j = g_jg_i\rangle_{1\leq i,j\leq 2}$ as group, then the matrix $(w(g,h))_{g,h\in H}$ is non-degenerated if and only if $(|M|, n)|m_{ij}$ for $1 \leq i, j \leq 2$.*

Proof: By Lemma 8.2.12, we only need to show that $(i_1, i_2)M \equiv (0, 0),\ (i_1, i_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ has a unique solution if and only if $(|M|, n)|m_{ij}$ for $1 \leq i, j \leq 2$. Assume that $(i_1, i_2)M \equiv (0, 0),\ (i_1, i_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ has a unique solution. Let $d := \frac{n}{(|M|,n)}$ and let M^* be the adjoint matrix of M, then we have $(dM^*)M = \mathrm{diag}(d|M|, d|M|)$. Therefore we have $\mathrm{diag}(d|M|, d|M|) \equiv (0, 0) \in \mathbb{Z}_n \times \mathbb{Z}_n$. Since $(i_1, i_2)M \equiv (0, 0),\ (i_1, i_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ has only one solution, we obtain $d(m_{22}, -m_{12}) \equiv (0, 0)$ and $d(-m_{21}, -m_{11}) \equiv (0, 0)$. And this implies that $(|M|, n)|m_{ij}$ for $1 \leq i, j \leq 2$. Conversely, we suppose that $(|M|, n)|m_{ij}$ for $1 \leq i, j \leq 2$, then we can find $m'_{ij} \in \mathbb{N}$ such that $m_{ij} = (|M|, n)m'_{ij}$ for $1 \leq i, j \leq 2$. Let $M' := (m'_{ij})_{1\leq i,j\leq 2}$, then it can be seen that $M = (|M|, n)M'$ and $(|M'|, d) = 1$. By definition, one can get that $(i_1, i_2)M \equiv (0, 0),\ (i_1, i_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ has a unique solution if and only if $(i_1, i_2)M \equiv (0, 0),\ (i_1, i_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$ has a unique solution. Due to $(|M'|, d) = 1$, we know M' is the inverse of M in $\mathrm{M}_2(\mathbb{Z}_d)$. Thus we obtain that $(i_1, i_2)M \equiv (0, 0),\ (i_1, i_2) \in \mathbb{Z}_d \times \mathbb{Z}_d$ has a unique solution. □

Next, we give all minimal triangular structures on $H^n_{b:y}$ by using the above conclusions. Let ω be a $2n$th primitive root of 1 and let $T'_n :=$ {minimal triangular structures on $H^n_{b:y}$ }, then we have

Theorem 8.2.14 $T'_n \stackrel{1-1}{\longleftrightarrow} \{(\alpha, \beta, \omega^k, \delta) \in \Bbbk^4 |\ \alpha^2 = \beta^2 = \delta^2 = 1,\ k \in \mathbb{N}$ *and* $(k^2, n)|k\}$.

Proof: We will use Theorem 8.2.9 and Corollary 8.2.11 to get what we want. Let R be a non-trivial triangular structure. Since the proof of Theorem 8.2.9, we know that the map $\phi : T_n \to \{(\alpha, \beta_1, \beta_2, \delta) \in \Bbbk^4 |\ \alpha^2 = \beta_1^2 = \beta_2^{2n} = \delta^2 = 1\}$ which is defined by $\phi(R) := (\alpha_{22}, \beta_1, \beta_2, \delta)$ is bijective. To complete the proof, we only need to show $\mathrm{Im}(\phi|_{T'_n}) = \{(\alpha, \beta, \omega^k, \delta) \in \Bbbk^4 |\ \alpha^2 = \beta^2 = \delta^2 = 1,\ k \in \mathbb{N}$ and $(k^2, n)|k\}$. Because we have proved that $\beta_2^{2n} = 1$ in Theorem 8.2.9, we can assume that $\beta_2 = \omega^k$ for some $k \in \mathbb{N}$. Then we claim that R is minimal if and only if $(k^2, n)|k$. By Corollary 8.2.11, we obtain that R is minimal if and only if the matrix w^1 is non-degenerated. Since $S = \langle s_i |\ s_i^{2n} = 1, s_i s_j = s_j s_i\rangle_{1\le i,j\le 2}$ as group, then we can use Corollary 8.2.13 to get that w^1 is non-degenerated if and only if $(|M|, n)|m_{ij}$ for $1 \le i, j \le 2$. Owing to the proof of Theorem 8.2.9, we know the following equations hold

$$\alpha_{11} = 1,\ \alpha_{12} = \alpha_{21}^{-1} = -\beta_2^2,\ \alpha_{22}^2 = 1.$$

By definition of the matrix M, we obtain $m_{11} = 0,\ m_{12} = 2k+n,\ m_{21} = -2k+n,\ m_{22} = ln$, where $l \in \mathbb{N}$. Thus we have $|M| = n^2 - 4k^2$. Due to n is an odd number, we get $(|M|, n) = (k^2, n)$. Therefore we know R is minimal triangular structure if and only if the corresponding four tuple $(\alpha_{22}, \beta_1, \omega^k, \delta)$ satisfies that $(k^2, n)|k$. $\square$

Finally, we will show that $H^1_{b:y}$ is the smallest Hopf algebra among non-trivial semisimple minimal triangular Hopf algebras. To do this, we first recall the only two non-trivial (self-dual) semisimple Hopf algebras $A_\pm$ of dimension 12(See [18] for details), where $A_\pm$ are the form $\Bbbk^{S_3}\#_{\sigma_\pm,\tau}\Bbbk\mathbb{Z}_2$.

Lemma 8.2.15 *The Hopf algebras* $A_\pm$ *are not minimal triangular Hopf algebras.*

Proof: We only show that A_+ is not minimal triangular and the other part can be proved in a similar way. Assume that A_+ is minimal triangular. Since A_+ is self-dual, we can find a braided structure $\langle , \rangle : A_+ \times A_+ \to \Bbbk$ such that $\langle , \rangle$ is non-degenerated and $\langle a,b\rangle = \langle s(b),a\rangle$ for $a,b \in A_+$. Next, we will show that such braided structure does not exist. Consider the $\langle,\rangle$ restrict to $\Bbbk^{S_3} \times \Bbbk^{S_3}$ and we denote it as $\overline{\langle,\rangle}$ for convenience. Then we get that $\overline{\langle,\rangle}$ is a braided structure on $\Bbbk^{S_3}$ such that $\langle a,b\rangle = \langle s(b),a\rangle$ for $a,b \in \Bbbk^{S_3}$. Using the viewpoint of dual, we get a triangular structure on $\Bbbk[S_3]$. But $\Bbbk[S_3] = \Bbbk^{\mathbb{Z}_3}\#\Bbbk\mathbb{Z}_2$ and hence we can use Proposition 3.1.5 to obtain that $\Bbbk[S_3]$ has only trivial quasitriangular structures. Further, one can obtain that the only triangular structure on $\Bbbk[S_3]$ is $R = 1 \otimes 1$. Thus we have $\overline{\langle e_g, e_h\rangle} = \epsilon(e_g)\epsilon(e_h)$ for $g,h \in S_3$. To determined the braided structure $\langle,\rangle$, we assume $\langle e_g, x\rangle = \beta(g)$, where $\beta(g) \in \Bbbk$. Directly we have $\langle e_g, e_h x\rangle = \sum_{kl=g}\langle e_k, x\rangle\langle e_l, e_h\rangle = \epsilon(e_h)\beta(g)$. Since $\langle e_g e_h, x\rangle = \sum_{k,l\in S_3}\tau(k,l)\langle e_g, e_k x\rangle\langle e_h, e_l x\rangle$, we obtain $\delta_{g,h}\beta(g) = \beta(g)\beta(h)$ for $g,h \in S_3$. Since $\langle,\rangle$ is non-degenerated, we can assume that $g_0 \in S_3$ satisfying $\beta(g_0) = 1$. Then one can see that $\beta(g) = 0$ when $g \neq g_0$. Let $g_1 \in S_3$ such that $g_1 \neq 1$ and $g_1 \neq g_0$, then we have $\langle e_{g_1}, b\rangle \equiv 0$ for $b \in A_+$. But this fact contradicts the assumption about the non-degeneracy of $\langle,\rangle$. Therefore we get A_+ is not minimal triangular. $\square$

Theorem 8.2.16 *The Hopf algebra $H^1_{b:y}$ is the smallest Hopf algebra among non-trivial semisimple minimal triangular Hopf algebras.*

Proof: By [1, Section 2.3], we know the non-trivial semisimple Hopf algebras with dimension<16 are the 8-dimension Kac algebra K_8 and the two 12-dimension semisimple Hopf algebras $A_\pm$. Since all quasitriangular structures on K_8 have been gotten in [71], one can easily check that K_8 is not minimal triangular. By Lemma 8.2.15, we know $A_\pm$ are not minimal triangular. Therefore we have completed the proof. $\square$

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E-mail address: xzkdh4712@163.com

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