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| 导 师 姓 名 | 刘公祥 教授 |

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王栓宏 教授

朱晓胜 教授

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导师签名：

# Finite GK-dimensional Nichols algebras over non-abelian groups


by

**Yongliang Zhang**

Supervised by

Professor Gongxiang Liu


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# 南京大学研究生毕业论文中文摘要首页用纸

毕业论文题目：非阿贝尔群上具有有限 GK-维数的 Nichols 代数

基础数学 专业 2019 级博士生姓名： 张永亮

指导教师（姓名、职称）： 刘公祥 教授

## 摘　要

我们通过研究无限二面体群 $\mathbb{D}_\infty$ 和四元数群 $\mathbb{Q}_8$ 上的 Nichols 代数，对具有有限 Gelfand-Kirillov 维数（简称 GK-维数）的 Hopf 代数的分类做出了一定贡献。我们找到了 $\mathbb{D}_\infty$ 和 $\mathbb{Q}_8$ 上的所有有限维不可约 Yetter-Drinfeld 模，并确定了哪些不可约 Yetter-Drinfeld 模对应的 Nichols 代数具有有限 GK-维数。进一步，我们考虑了半单 Yetter-Drinfeld 模对应的 Nichols 代数的 GK-维数。

**关键词：**Nichols 代数；Hopf 代数；Yetter-Drinfeld 模

# 南京大学研究生毕业论文英文摘要首页用纸

THESIS: Finite GK-dimensional Nichols algebras over non-abelian groups

SPECIALIZATION: Fundamental Mathematics

POSTGRADUATE: Yongliang Zhang

MENTOR: Professor Gongxiang Liu

## ABSTRACT

We contribute to the classification of Hopf algebras with finite Gelfand-Kirillov dimension, GK-dimension for short, through the study of Nichols algebras over the infinite dihedral group $\mathbb{D}_\infty$, and the quaternion group $\mathbb{Q}_8$. We find all the irreducible Yetter-Drinfeld modules $V$ over $\mathbb{D}_\infty$ and $\mathbb{Q}_8$, and determine which Nichols algebras $\mathcal{B}(V)$ of $V$ are finite GK-dimensional. Furthermore, we consider GK-dimensions of Nichols algebras of semisimple Yetter-Drinfeld modules.

KEYWORDS: Nichols algebras; Hopf algebras; Yetter-Drinfeld modules

# CONTENTS

# Chapter 1 Introduction

## 1.1 Background

The Gelfand-Kirillov dimension, was first introduced by I. M. Gelfand and A. A. Kirillov in [28-29]. In 1976, Borho and Kraft studied systematically the properties of the Gelfand-Kirillov dimension in [22]. This dimension now serves as one of the standard invariants in the study of noncommutative algebras. In the past decades, Hopf algebras with finite GK-dimensions were investigated, see [15,23-26,30-31,37-38,43-46,48-50].

Nichols algebras, appeared for the first time in the thesis of W. Nichols [39], and the small quantum group $u_q(\mathrm{sl}_3)$, was introduced, where $q$ is a primitive cubic root of one. Also, by Woronowicz [47], they were discovered independently as the invariant part of his non-commutative differential calculus. It was observed in [16-17] that Nichols algebras are basic invariants of pointed Hopf algebras.

As stated in [18], the classification problem of pointed Hopf algebras has three parts:

(1) Structure of the Nichols algebras $\mathcal{B}(V)$.

(2) The "lifting" problem: describe all pointed Hopf algebras $A$ with $G(A) = \Gamma$ such that

$$\mathrm{gr}A \cong \mathcal{B}(V)\#\Bbbk\Gamma,$$

where $\mathrm{gr}A$ is the graded coalgebra associated to the coradical filtration, and $G(A)$ is the group of the group-like elements of $A$.

(3) Generation in degree one: determine which Hopf algebras $A$ are generated by group-like and skew-primitive elements, that is $\mathrm{gr}A$ is generated in degree one.

For the first step of the “Lifting Method” stated by Nicolás Andruskiewitsch [14], the following problems arise naturally: Given a group $\Gamma$, and a Yetter-Drinfeld modules $V$ over $\Bbbk\Gamma$,

(1) when is $\dim \mathcal{B}(V)$ finite? or

(2) when is GKdim$\mathcal{B}(V)$ finite?

For the first problem, there are two main results:

(1) István Heckenberger gave the classification of all finite-dimensional Nichols algebras of diagonal type in [33], using the Weyl groupoid defined in [32];

(2) The defining relations of the finite-dimensional Nichols algebras of diagonal type were given by Iván Angiono in [19].

see the survey [3] for more details on the classification of $\mathcal{B}(V)$, when $\mathcal{B}(V)$ is of diagonal type. Let $G$ be a group, and $\mathbb{C}$ the field of all complex numbers. One problem is to find all the Nichols algebras $\mathcal{B}(V)$ with finite dimension for any $V \in {}^{G}_{G}\mathcal{YD}$, the Yetter-Drinfeld modules over the group algebra $\Bbbk G$. The cases when $G$ is a finite simple group were studied in [7-12,17,27]. Precisely, for the symmetric groups or alternating groups, it was proved in [17] and [27] that, except for some particular cases, $\mathcal{O}_\sigma^{\mathbb{S}_m}$ or $\mathcal{O}_\sigma^{\mathbb{A}_m}$ collapses ( a conjugacy class $\mathcal{O}$ collapses if $\dim \mathcal{B}(\mathcal{O}, \mathbf{q}) = \infty$ for any 2-cocycle $\mathbf{q}$). Similarly, it is shown [7-12] that, if $\mathcal{O}$ is a non-trivial unipotent conjugacy class in a Chevalley or Steinberg group, or a sporadic group different from the Moster $M$, $\mathcal{O}$ collapses except for some particular cases.

For the second problem, M. Rosso [41] pointed out that the finite Gelfand-Kirillov dimension is a crucial requirement for $U_q^+(g)$. Great progress was acheived when $G$ is an abelian group, see [4,6]. It was proved that, if $V(\varepsilon, \ell)$ is a block, then GKdim$\mathcal{B}(V(\varepsilon, \ell)) < \infty$ if and only $\ell = 2$ and $\varepsilon^2 = 1$. Also, the Gelfand-Kirillov dimension of Nichols algebras of direct sums of blocks and points are considered.

If $G$ is a non-abelian group, whether GKdim$\mathcal{B}(V) < \infty$ is largely unknown, where $V \in {}^{G}_{G}\mathcal{YD}$. In this thesis, we consider the infinite dihedral group $\mathbb{D}_\infty$ and the quaternion group $\mathbb{Q}_8$. We prove the main results:

**Theorem 1.1.1.** The only Nichols algebras of the finite dimensional irreducible Yetter-Drinfeld modules over $\Bbbk\mathbb{D}_\infty$ with finite GK-dimension, up to isomorphism, are those in the following list.

(1) $\mathcal{B}(\mathcal{O}_{h^n}, \rho_{\pm 1})$ for $n \in \mathbb{N}$.

(2) $\mathcal{B}(\mathcal{O}_1, S_0^+)$.

(3) $\mathcal{B}(\mathcal{O}_1, S_0^-)$.

(4) $\mathcal{B}(\mathcal{O}_1, S_\lambda^+)$.

(5) $\mathcal{B}(\mathcal{O}_1, S_\lambda^-)$.

**Theorem 1.1.2.** The only Nichols algebras of the finite dimensional irreducible Yetter-Drinfeld modules over $\Bbbk\mathbb{Q}_8$ with finite GK-dimension, up to isomorphism, are those in the following list.

(1) $\mathcal{B}(\mathcal{O}_1, \rho_i)$, with $1 \leqslant i \leqslant 5$.

(2) $\mathcal{B}(\mathcal{O}_x, \phi_0)$, $\mathcal{B}(\mathcal{O}_x, \phi_2)$

(3) $\mathcal{B}(\mathcal{O}_{x^2}, \rho_i)$, with $1 \leqslant i \leqslant 5$.

(4) $\mathcal{B}(\mathcal{O}_y, \phi_0)$, $\mathcal{B}(\mathcal{O}_y, \phi_2)$.

(5) $\mathcal{B}(\mathcal{O}_{xy}, \phi_0)$, $\mathcal{B}(\mathcal{O}_{xy}, \phi_2)$.

## 1.2 Organization

In Chapter 2, we recall some basic definitions, including braided vector spaces, Yetter-Drinfeld modules,racks,Nichols algebras, Gelfand-Kirillov dimension, and the groups we consider in this thesis.

In Chapter 3, we give the irreducible Yetter-Drinfeld modules $V \in {}^G_G\mathcal{YD}$ over the infinite dihedral group $G = \mathbb{D}_\infty$, and consider when $\mathrm{GKdim}\mathcal{B}(V) < \infty$. Also, we consider when Nichols algebras of semisimple Yetter-Drinfeld modules are finite GK-dimensional.

In Chapter 4, we consider the quaternion group $\mathbb{Q}_8$. We list all the irreducible Yetter-Drinfeld modules $V$ over $\mathbb{Q}_8$, and give the cases when GKdim$\mathcal{B}(V) < \infty$. In addition, we give a large numbers of semisimple Yetter-Drinfeld modules whose Nichols algebras are finite GK-dimensional.

# Chapter 2 Preliminaries

## 2.1 Notations

| | |
|---|---|
| $\Bbbk$ | algebraic closed field |
| $\mathbf{i}$ | the imaginary unit $\sqrt{-1}$ |
| $q$ | $e^{\pi \mathbf{i}/2} \in \mathbb{G}_4$ |
| $\mathbb{N}$ | the set of all natural numbers |
| $\mathbb{Z}$ | the set of all integers |
| $\mathbb{C}$ | the set of all complex numbers |
| $G$ | group |
| $H$ | Hopf algebra |
| ${}^H_H\mathcal{YD}$ | the category of Yetter-Drinfeld modules over the Hopf algebra $H$ |
| $\mathcal{O}_g$ | conjugacy class in a group $G$ corresponding to $g$. |
| $G^g$ | the set of centralizers of $g$ in $G$. |
| $Z(A)$ | the center of an algebra $A$ |
| GL(V) | the general linear group over $V$ |
| GKdim$A$ | the Gefand-Kirillov-dimension of an algebra $A$ |

## 2.2 Coalgebras and Hopf algebras

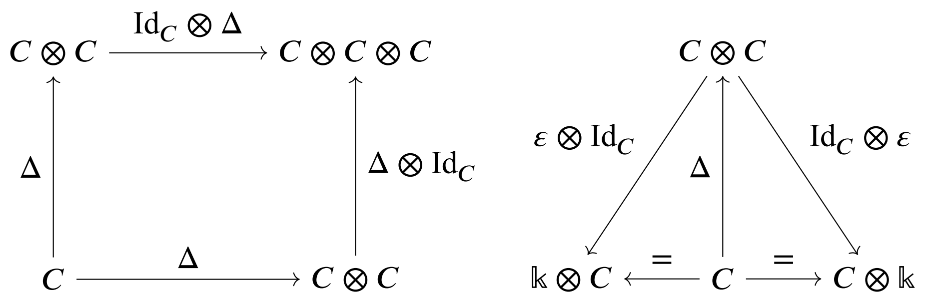
**Definition 2.2.1.** [40] A coalgebra over the field $\Bbbk$ is a triple $(C, \Delta, \varepsilon)$, where $C$ is a vector space over $\Bbbk$, and $\Delta : C \to C \otimes C$, $\varepsilon : C \to \Bbbk$ are linear maps such that

$$(\Delta \otimes \mathrm{Id}_C) \circ \Delta = (\mathrm{Id}_C \otimes \Delta) \circ \Delta,$$

and

$$(\varepsilon \otimes \mathrm{Id}_C) \circ \Delta = \mathrm{Id}_C = (\mathrm{Id}_C \otimes \varepsilon) \circ \Delta,$$

namely, the following diagrams




commute. ■

**Definition 2.2.2.** [40] A bialgebra over $\Bbbk$ is a tuple $(A, m, \eta, \Delta, \varepsilon)$, where $(A, m, \eta)$ is an algebra and $(A, \Delta, \varepsilon)$ is a coalgebra over $\Bbbk$, such that $\Delta$ and $\varepsilon$ are algebra maps. ■

**Definition 2.2.3.** [40] A Hopf algebra over $\Bbbk$ is a bialgebra $A$ over $\Bbbk$ such that the identity map $\mathrm{Id}_A$ has an inverse $S$ in the convolution algebra $\mathrm{End}(A)$, namely, there exist $S \in \mathrm{End}(A)$ such that

$$m \circ (S \otimes \mathrm{Id}_A) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{Id}_A \otimes S) \circ \Delta.$$ ■

## 2.3 Braided vector spaces

**Definition 2.3.1.** [1] Let $V$ be a vector space , $c \in \mathrm{GL}(V \otimes V)$. $(V, c)$ is said to be a braided vector space if $c$ is a solution of the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c),$$ ■

We call a braided space $(V, c)$ diagonal type if there exists a matrix $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}}$ with

$q_{ij} \in \Bbbk^{\times}$ and $q_{ii} \neq 1$ for any $i, j \in \mathbb{I}$ such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, i, j \in \mathbb{I}.$$

## 2.4 Yetter-Drinfeld modules

**Definition 2.4.1.** [1,Definition 3] Let $H$ be a Hopf algebra with bijective antipode $S$. A Yetter-Drinfeld module over $H$ is a vector space $V$ provided with

(1) a structure of left $H$-module $\mu : H \otimes V \to V$ and

(2) a structure of left $H$-comodule $\delta : V \to H \otimes V$, such that

for all $h \in H$ and $v \in V$, the following compatibility condition holds:

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}.$$

The category of left Yetter-Drinfeld modules is denoted by ${}^H_H\mathcal{YD}$. ■

In particular, if $H = \Bbbk G$ is the group algebra of the group $G$, then a Yetter-Drinfeld module over $H$ is a $G$-graded vector space $M = \oplus_{g \in G} M_g$ provided with a $G$-module structure such that $g \cdot M_h = M_{ghg^{-1}}$.

It can be shown that each Yetter-Drinfeld module $V \in {}^H_H\mathcal{YD}$ is a braided vector space with the braiding structure

$$c(x \otimes y) = x_{(-1)} \cdot y \otimes x_{(0)}, \quad x, y \in {}^H_H\mathcal{YD}.$$

The category of Yetter-Drinfeld modules over $\Bbbk G$ is denoted by ${}^G_G\mathcal{YD}$. Let $\mathcal{O} \subseteq G$ be a conjugacy class of $G$, then we denote by ${}^G_G\mathcal{YD}(\mathcal{O})$ the subcategory of ${}^G_G\mathcal{YD}$ consisting of all $M \in {}^G_G\mathcal{YD}$ with $M = \bigoplus_{s \in \mathcal{O}} M_s$.

**Definition 2.4.2.** [34,Definition 1.4.15] Let $g \in G$, and let $V$ be a left $\Bbbk G$-module. Define

$$M(g, V) = \Bbbk G \otimes_{\Bbbk G^g} V$$

as an object in ${}^{G}_{G}\mathcal{YD}(\mathcal{O}_g)$, where $M(g,V)$ is the induced $\Bbbk G$-module, the $G$-grading is given by

$$\deg(h \otimes v) = h \triangleright g, \quad \text{for all } h \in G, v \in V,$$

and the $\Bbbk G$-comodule structure is

$$\delta(h \otimes v) = (h \triangleright g) \otimes (h \otimes v). \qquad \blacksquare$$

Let $V \in {}^{G}_{G}\mathcal{YD}$. Let $I(V)$ be the largest coideal of $T(V)$ contained in $\oplus_{n\geqslant 2} T^n(V)$. The Nichols algebra of $V$ is defined by $\mathcal{B}(V) = T(V)/I(V)$. $\mathcal{B}(V)$ is called diagonal type if $(V,c)$ is of digonal type.

By the following lemmas and their proofs, we can find all the irreducible Yetter-Drinfeld modules $M(g,V)$ in ${}^{G}_{G}\mathcal{YD}$, once we have known the corresponding irreducible representations $(\rho, V)$ of $\Bbbk G^g$. The corresponding Nichols algebra of $M(\mathcal{O}_g, V)$ is denoted by $\mathcal{B}(\mathcal{O}_g, \rho)$ or $\mathcal{B}(\mathcal{O}_g, V)$.

**Lemma 2.4.1.** [34,Lemma 1.4.16] Let $g \in G$, $M \in {}^{G}_{G}\mathcal{YD}(\mathcal{O}_g)$. Then $M(g, M_g) \to M$ is an isomorphism of Yetter-Drinfeld modules in ${}^{G}_{G}\mathcal{YD}$.

**Lemma 2.4.2.** [34,Corollary 1.4.18] Let $\{\mathcal{O}_{g_l} | l \in L\}$ be the set of the conjugacy classes of $G$. There is a bijection between the disjoint union of the isomorphism classes of the simple left $\Bbbk G^{g_l}$-modules, $l \in L$, and the set of isomorphism classes of the simple Yetter-Drinfeld modules in ${}^{G}_{G}\mathcal{YD}$.

## 2.5 Racks

**Definition 2.5.1.** [13, Definition 1.1] Let $X$ be a non-empty set， $\triangleright : X \times X \to X$ be a function. $(X, \triangleright)$ is said to be a rack if

(1) for any $i \in X$, $\phi_i : X \to X$ is a bijection, where $\phi_i(j) = i \triangleright j$;

(2) $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k), \forall i, j, k \in X$. $\blacksquare$

**Example 2.5.1.** Let $G$ be a group，$X = \mathcal{O}$ is a conjugacy class in $G$, $\triangleright : X \times X \to X$ is the conjugacy action in $G$, that is, $i \triangleright j = iji^{-1}$. Then $(X, \triangleright)$ is a rack.

**Definition 2.5.2.** [1,Section 2.1.5] Let $W$ be a vector space, $(X, \triangleright)$ be a rack, and let $\mathbf{q} : X \times X \to \mathrm{GL}(W)$ be a 2-cocycle, that is, the following equation holds:

$$\mathbf{q}_{x,y\triangleright z}\mathbf{q}_{y,z} = \mathbf{q}_{x\triangleright y,x\triangleright z}\mathbf{q}_{x,z}.$$

Let $V = \Bbbk X \otimes W$, $e_x v : e_x \otimes v$ and let $c^{\mathbf{q}}$ be the braiding given by

$$c^q(e_x v \otimes e_y w) = e_{x\triangleright y}\mathbf{q}_{x,y}(w) \otimes e_x v, x, y \in X, v, w \in W.$$

Then $V$ said to be a braided vector space of rack type. ■

## 2.6 Nichols algebras

In this section, we recall the definition of Nichols algebras.

**Definition 2.6.1.** [34,Definition 1.6.16] Let $V \in {}^H_H\mathcal{YD}$. An $\mathbb{N}_0$-graded connected Hopf algebra $R$ in ${}^H_H\mathcal{YD}$ is a Nichols algebra of $V$, if

(1) $R(1) \cong V$ in ${}^H_H\mathcal{YD}$,

(2) $R$ is generated as an algebra by $R(1)$, and

(3) $R$ is strictly graded, that is, $P(R) = R(1)$. ■

Alternatively, the Nichols algebra can be constructed as a quotient of $T(V)$.

**Definition 2.6.2.** [34,Definition 1.6.18] Let $V \in {}^H_H\mathcal{YD}$. Let $I(V)$ be the largest coideal of $T(V)$ contained in $\bigoplus_{n\geqslant 2} T^n(V)$. The Nichols algebra of $V$ is defined by

$$\mathcal{B}(V) = T(V)/I(V).$$ ■

It is shown in [34,Theorem 1.6.18] that this $\mathcal{B}(V)$ is indeed a Nichols algebra of $V$ defined as Definition 2.6.1.

## 2.7 Gelfand-Kirillov dimension

The Gelfand-Kirillov dimension, GK-dimension for short, becomes a powerful tool to study noncommutative algebras, especially for those with infinite dimensions. For the definition and properties of the GK-dimension we refer to [36].

**Definition 2.7.1.** [36] The Gelfand-Kirillov dimension of a $\Bbbk$-algebra $A$ is

$$\mathrm{GKdim}(A) = \sup_V \overline{\lim} \log_n \dim(V^n),$$

where the supremum is taken over all finite dimensional subspaces $V$ of $A$. ■

As for finite GK-dimension, we need the following result

**Lemma 2.7.1.** [1,Theorem 6] If either its Weyl groupoid is infinite and $\dim V = 2$, or else $V$ is of affine Cartan type, then $\mathrm{GKdim}\mathcal{B}(V) = \infty$.

**Example 2.7.1.** [1] Let $A$ be a finitely generated $\Bbbk$-algebra. Then $\mathrm{GKdim}(A) = 0$ if and only if $\dim A < \infty$.

**Example 2.7.2.** [1] Let $A = \Bbbk[X_1, \cdots, X_d]$ be the polynomial algebra over $\Bbbk$ in $d$ indeterminates $X_1, \cdots, X_d$. Then

$$\mathrm{GKdim}(A) = d.$$

**Example 2.7.3.** [1] Let $\mathfrak{g}$ be a finite dimensional Lie algebra，$U(\mathfrak{g})$ 为 $\mathfrak{g}$ be its universal enveloping algebra. Then $\mathrm{GKdim}(U(\mathfrak{g})) = \dim \mathfrak{g}$.

For $\Bbbk$-algebras $A$ and $B$, by [36,Lemma 3.10]，we have $\mathrm{GKdim}(A \otimes_{\Bbbk} B) \leqslant \mathrm{GKdim}(A) + \mathrm{GKdim}(B)$. In particular，if $A$ is a left $H$-module algebra, then $\mathrm{GKdim}(A\#H) \leqslant \mathrm{GKdim}(A) + \mathrm{GKdim}(H)$, where $A\#H$ is the smash product of $A$ and $H$. For algebras with infinite GK-dimensions, the following result is useful:

**Lemma 2.7.2.** [2,Theorem 2.6] Let $G$ be a finitely generated group, $M \in {}^G_G\mathcal{YD}$ satisfies $\mathcal{O} = \mathrm{supp} M$ is a infinite conjugacy class. Then $\mathrm{GKdim}\mathcal{B}(M)\#\Bbbk G = \infty$.

This lemma can be used as a criteria for infinite GK-dimensions.

## 2.8 The infinite dihedral group $\mathbb{D}_\infty$

As we are familiar,

$$\mathbb{D}_\infty = \langle h, g | g^2 = 1, ghg = h^{-1} \rangle = \{1, g, h^n, gh^n, h^n g, gh^n g | n \in \mathbb{N}\},$$

where

$$g^{-1} = g, \quad (h^n)^{-1} = gh^n g, \quad (gh^n)^{-1} = gh^n, \quad (h^n g)^{-1} = h^n g, \quad (gh^n g)^{-1} = h^n.$$

Consider the conjugacy classes in $\mathbb{D}_\infty$.

$$\mathcal{O}_1 = \{1\}, \quad \mathcal{O}_{h^n} = \{h^n, gh^n g\} = \{h^n, h^{-n}\}, \forall n \in \mathbb{N},$$
$$\mathcal{O}_g = \{g, gh^{2n}, h^{2n} g | n \in \mathbb{N}\}, \quad \mathcal{O}_{gh} = \{h^{2k-1} g, gh^{2k-1} | k \in \mathbb{N}\}.$$

The centralizers of one element in each conjugacy class are as follows:

$$G^1 = \mathbb{D}_\infty, \quad G^g = \{x \in G \mid xg = gx\} = \{1, g\} \cong \mathbb{Z}_2,$$
$$G^{h^n} = \{1, h^k, gh^k g | k \in \mathbb{N}\} \cong \mathbb{Z}, \quad G^{gh} = \{1, gh\} \cong \mathbb{Z}_2.$$

Consider all the cosets of $G^g$ in $G$. For any $n \in \mathbb{N}$,

$$gG^g = G^g, \quad h^n g G^g = h^n G^g = \{h^n, h^n g\},$$
$$gh^n g G^g = \{gh^n, gh^n g\}, \quad gh^n G^g = \{gh^n, gh^n g\}.$$

Therefore, a representative of complete cosets of $G$ over $G^g$ is

$$\{1, h^n, gh^n | n \in \mathbb{N}\},$$

Similarly, consider all the cosets of $G^h$. Representatives of complete cosets of $G$

over $G^h$ and $G^{gh}$ are

$$\{1, g\}, \quad \{1, h^n, gh^n | n \in \mathbb{N}\},$$

respectively.

## 2.9 The quaternion group $\mathbb{Q}_8$

As we are familiar, $\mathbb{Q}_8 = \langle x, y | x^4 = 1, y^2 = x^2, xy = yx^{-1} \rangle$ is generated by two elements, precisely,

$$\mathbb{Q}_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$$

where $x^{-1} = x^3$, $x^{-2} = x^2$,$y^{-1} = x^2y$, and $(xy)^{-1} = x^3y$.

There are five conjugacy classes in $G = \mathbb{Q}_8$:

$$\mathcal{O}_1 = \{1\}, \mathcal{O}_x = \{x, x^3\}, \mathcal{O}_{x^2} = \{x^2\}, \mathcal{O}_y = \{y, x^2y\}, \mathcal{O}_{xy} = \{xy, x^3y\}.$$

Choose one element in each conjugacy class, and we compute the centralizers:

$$G^1 = G, \quad G^x = \langle x \rangle \cong \mathbb{Z}_4, \quad G^{x^2} = G, \quad G^y = \langle y \rangle \cong \mathbb{Z}_4, \quad G^{xy} = \langle xy \rangle \cong \mathbb{Z}_4.$$

By [35,Exercise 17.1],there are four 1-dimensional irreducible representations $(\rho_i, V_i)_{1\leqslant i\leqslant 4} \in$ Irr($\mathbb{Q}_8$), and one 2-dimensional irreducible representation, denoted by $(\rho_5, V_5)$. The character table of $(\rho_i, V_i)$ is as follows:

| | 1 | $x^2$ | $x$ | $y$ | $xy$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | -1 | -1 |
| $\chi_3$ | 1 | 1 | -1 | 1 | -1 |
| $\chi_4$ | 1 | 1 | -1 | -1 | 1 |
| $\chi_5$ | 2 | -2 | 0 | 0 | 0 |

# Chapter 3 Finite GK-dimensional Nichols Algebras over the Infinite Dihedral Group $\mathbb{D}_\infty$

## 3.1 The Nichols algebra $\mathcal{B}(\mathcal{O}_{h^n}, \rho)$

**Proposition 3.1.1.** For any $n \in \mathbb{N}$ and any irreducible representation $(\rho, V) \in \mathrm{Irr}(G^{h^n})$, $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{h^n}, \rho) < \infty$ if and only if $(\rho, V)$ is the trivial representation or the sign representation.

Proof. Consider an irreducible Yetter-Drinfeld-module $\Bbbk G \otimes_{\Bbbk G^{h^n}} V = (1 \otimes V) \oplus (g \otimes V)$, where $(\rho, V) \in \mathrm{Irr}(G^{h^n})$ is an irreducible representation of $\Bbbk G^{h^n}$. By the theory of representations of groups [42], $V$ one-dimensional, because $G^{h^n}$ is and abelian group. Let $V = \Bbbk x$. The module structure of $M(\mathcal{O}_{h^n}, \rho)$ is as follows:

$$
\begin{aligned}
&g \cdot (1 \otimes x) = g \otimes \rho(1)(x), && h^n \cdot (1 \otimes x) = 1 \otimes \rho(h^n)(x),\\
&gh^n \cdot (1 \otimes x) = g \otimes \rho(h^n)(x), && h^n g \cdot (1 \otimes x) = g \otimes \rho(h^{-n})(x),\\
&gh^n g \cdot (1 \otimes x) = 1 \otimes \rho(gh^n g)(x), && g \cdot (g \otimes x) = 1 \otimes \rho(1)(x),\\
&h^n \cdot (g \otimes x) = g \otimes \rho(h^{-n})(x), && gh^n \cdot (g \otimes x) = 1 \otimes \rho(gh^n g)(x),\\
&h^n g \cdot (g \otimes x) = 1 \otimes \rho(h^n)(x), && gh^n g \cdot (g \otimes x) = g \otimes \rho(h^n)(x).
\end{aligned}
$$

The comodule structure $\delta : M(\mathcal{O}_{h^n}, \rho) \to \Bbbk G \otimes M(\mathcal{O}_{h^n}, \rho)$ of $M(\mathcal{O}_{h^n}, \rho)$ is

$$
\begin{aligned}
\delta(1 \otimes x) &= (1 \triangleright h) \otimes (1 \otimes x) = h \otimes (1 \otimes x),\\
\delta(g \otimes x) &= (g \triangleright h) \otimes (g \otimes x) = h^{-1} \otimes (g \otimes x).
\end{aligned}
$$

Then $\Bbbk G \otimes_{\Bbbk G^{h^n}} V$ is a Yetter-Drinfeld module over $G$.

Now we will compute the GK-dimension of the Nichols algebra of $M(\mathcal{O}_{h^n}, \rho)$.

First consider the case of the trivial representation $(\epsilon, V)$ of $G^{h^n}$. That is, the module structure of $M(\mathcal{O}_{h^n}, \epsilon)$ is trivial. The braiding of $M(\mathcal{O}_{h^n}, \rho)$ is given as follows.

Write $x_1 = 1 \otimes x, x_2 = g \otimes x$. Then we have

$$c(x_1 \otimes x_1) = x_1 \otimes x_1, \quad c(x_1 \otimes x_2) = x_2 \otimes x_1,$$
$$c(x_2 \otimes x_1) = x_1 \otimes x_2, \quad c(x_2 \otimes x_2) = x_2 \otimes x_2.$$

The braiding matrix is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore, $\mathcal{B}(\mathcal{O}_{h^n}, \epsilon) \cong S(W)$, the symmetric algebra over $W$, by [1,Example 31], which has GK-dimension 2.

In general, for any irreducible representation of $G^{h^n}$, we have $h \cdot x = ax$, for some $a \in \mathbb{k}^\times$. Write $\rho_a$ for the representation. Therefore, the braiding of $M(\mathcal{O}_{h^n}, \rho_a)$ is

$$c(x_1 \otimes x_1) = ax_1 \otimes x_1, \qquad c(x_1 \otimes x_2) = a^{-1}x_2 \otimes x_1,$$
$$c(x_2 \otimes x_1) = a^{-1}x_1 \otimes x_2, \quad c(x_2 \otimes x_2) = ax_2 \otimes x_2.$$

The braiding matrix is

$$\begin{bmatrix} a & a^{-1} \\ a^{-1} & a \end{bmatrix}.$$

If $a = -1$, then we see that $M(\mathcal{O}_{h^n}, \rho_a)$ is of Cartan type with Cartan matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which is of finite type. Therefore $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{h^n}, \rho_a) < \infty$ by [32,Theorem 1]. If $a^2 \neq 1$, then the corresponding Dynkin diagram is $\overset{a}{\circ}\overset{a^{-2}}{\rule[3pt]{60pt}{0.4pt}}\overset{a}{\circ}$. By [5,Theorem 1.2] and [2,Remark 1.6] or by going through the list of [33,59-124], we have $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{h^n}, \rho_a) = \infty$ for all $a^2 \neq 1$. ■

## 3.2 The Nichols algebra $\mathcal{B}(\mathcal{O}_g, \rho)$

Since all the irreducible representations of $\mathbb{Z}_2$ are the unit representation and sign representation, we have the corresponding irreducible Yetter-Drinfeld modules $M(\mathcal{O}_g, \epsilon)$ and $M(\mathcal{O}_g, \text{sign})$.

Let $X = \{1, h^n, gh^n | n \in \mathbb{N}\}$. Then

$$M(\mathcal{O}_g, \rho) = \bigoplus_{y \in X} h_y \otimes \Bbbk x,$$

where the degree of each $h_y$ is given by

$$\deg(h_y) = h_y \triangleright g.$$

### 3.2.1 The Yetter-Drinfeld module $M(\mathcal{O}_g, \text{sign})$

The module structure of $M(\mathcal{O}_g, \text{sign})$ is

$$\begin{aligned}
&g \cdot (1 \otimes x) = -1 \otimes x, && h \cdot (1 \otimes x) = h \otimes x,\\
&g \cdot (gh \otimes x) = h \otimes x, && h \cdot (gh \otimes x) = -1 \otimes x,\\
&g \cdot (h^n \otimes x) = gh^n \otimes x, && h \cdot (h^n \otimes x) = h^{n+1} \otimes x,\\
&g \cdot (gh^n \otimes x) = h^n \otimes x, && h \cdot (gh^n \otimes x) = gh^{n-1} \otimes x, n \geqslant 2.
\end{aligned}$$

The comodule structure is

$$\delta(1 \otimes x) = g \otimes (1 \otimes x), \quad \delta(h^n \otimes x) = h^{2n} g \otimes (h^n \otimes x), \quad \delta(gh^n \otimes x) = gh^{2n} \otimes (gh^n \otimes x).$$

For any $n \geqslant 1$, let

$$a_n = h^n \otimes x, \quad b_n = gh^n \otimes x, \quad a_0 = 1 \otimes x.$$

Then the module structure is as follows:

$$g \cdot a_0 = -a_0, \qquad g \cdot a_n = b_n, \qquad g \cdot b_n = a_n,$$

$$h \cdot a_0 = a_1, \qquad h \cdot a_n = a_{n+1}, \qquad h \cdot b_n = \begin{cases} b_{n-1}, & n \geqslant 2 \\ -a_0, & n = 1 \end{cases}.$$

The comodule structure is

$$\delta(a_0) = g \otimes a_0, \quad \delta(a_n) = h^{2n} g \otimes a_n, \quad \delta(b_n) = g h^{2n} \otimes b_n.$$

The braiding of $M(\mathcal{O}_g, \text{sign})$ is

$$c(a_m \otimes a_n) = h^{2m} \cdot b_n \otimes a_m.$$

From the module structure we obtain that $h^{2m} \cdot b_n = b_{n-2m}$ if $2m < n$, and $h^{2m} \cdot b_n = -a_{2m-n}$ if $2m \geqslant n$. Therefore,

$$c(a_m \otimes a_n) = \begin{cases} b_{n-2m} \otimes a_m, & 2m < n \\ -a_{2m-n} \otimes a_m, & 2m \geqslant n \end{cases}.$$

For any $n \geqslant 1$ and $m \geqslant 0$,

$$c(a_m \otimes b_n) = a_{2m+n} \otimes a_m.$$

For $n \geqslant 2$ and $m \geqslant 1$, we have

$$c(b_m \otimes b_n) = g h^{2m} \cdot b_n \otimes b_m = \begin{cases} a_{n-2m} \otimes b_m, & 2m < n \\ -b_{2m-n} \otimes b_m, & 2m \geqslant n \end{cases},$$

$$c(b_m \otimes b_1) = g \cdot a_{2m-1} \otimes b_m = -b_{2m-1} \otimes b_m,$$

$$c(b_m \otimes a_n) = g \cdot a_{2m+n} \otimes b_m = b_{2m+n} \otimes b_m.$$

### 3.2.2 The Yetter-Drinfeld module $M(\mathcal{O}_g, \epsilon)$

The module structure of $M(\mathcal{O}_g, \epsilon)$ is

$$
\begin{aligned}
&g \cdot (1 \otimes x) = 1 \otimes \rho(g)(x) = 1 \otimes x,\\
&h \cdot (1 \otimes x) = h \otimes \rho(1)(x) = h \otimes x,\\
&g \cdot (h^n \otimes x) = gh^n \otimes \rho(1)(x) = gh^n \otimes x,\\
&h \cdot (h^n \otimes x) = h^{n+1} \otimes \rho(1)(x) = h^{n+1} \otimes x,\\
&g \cdot (gh^n \otimes x) = h^n \otimes \rho(1)(x) = h^n \otimes x,\\
&h \cdot (gh \otimes x) = 1 \otimes \rho(g)(x) = 1 \otimes x,\\
&h \cdot (gh^n \otimes x) = gh^{n-1} \otimes \rho(1)(x) = gh^{n-1} \otimes x, n \geqslant 2.
\end{aligned}
$$

The comodule structure is

$$\delta(1 \otimes x) = g \otimes (1 \otimes x), \quad \delta(h^n \otimes x) = h^{2n} g \otimes (h^n \otimes x), \quad \delta(gh^n \otimes x) = gh^{2n} \otimes (gh^n \otimes x).$$

For $n \geqslant 1$, let

$$a_n = h^n \otimes x, \quad b_n = gh^n \otimes x, \quad a_0 = 1 \otimes x.$$

Then the action and coaction are

$$
\begin{aligned}
&g \cdot a_0 = a_0, \quad g \cdot a_n = b_n, \quad g \cdot b_n = a_n,\\
&h \cdot a_0 = a_1, \quad h \cdot a_n = a_{n+1}, \quad h \cdot b_n = \begin{cases} b_{n-1}, & n \geqslant 2 \\ a_0, & n = 1 \end{cases},\\
&\delta(a_0) = g \otimes a_0, \quad \delta(a_n) = h^{2n} g \otimes a_n, \quad \delta(b_n) = gh^{2n} \otimes b_n.
\end{aligned}
$$

The braiding of $M(\mathcal{O}_g, \epsilon)$ is as follows:

$$c(a_m \otimes a_n) = h^{2m} g \cdot a_n \otimes a_m = h^{2m} \cdot b_n \otimes a_m.$$

If $2m < n$, then

$$h^{2m} \cdot b_n = b_{n-2m}.$$

If $2m \geqslant n$, then

$$h^{2m} \cdot b_n = a_{2m-n}.$$

Therefore, we have

$$c(a_m \otimes a_n) = \begin{cases} b_{n-2m} \otimes a_m, & 2m < n \\ a_{2m-n} \otimes a_m, & 2m \geqslant n \end{cases},$$

$$c(a_m \otimes b_n) = h^{2m} g \cdot b_n \otimes a_m = a_{2m+n} \otimes a_m.$$

hold for any $n, m \in \mathbb{N}$. For any $n \geqslant 2$, we have

$$c(b_m \otimes b_n) = gh^{2m} \cdot b_n \otimes b_m = \begin{cases} a_{n-2m} \otimes b_m, & 2m < n \\ b_{2m-n} \otimes b_m, & 2m \geqslant n \end{cases},$$

$$c(b_m \otimes b_1) = gh^{2m} \cdot b_1 \otimes b_m = g \cdot a_{2m-1} \otimes b_m = b_{2m-1} \otimes b_m,$$

$$c(b_m \otimes a_n) = gh^{2m} \cdot a_n \otimes b_m = g \cdot a_{2m+n} \otimes b_m = b_{2m+n} \otimes b_m.$$

Clearly, $\dim \mathcal{B}(\mathcal{O}_g, \rho) = \infty$, since the Yetter-Drinfeld modules are of infinite dimension. For the GK-dimension we have

**Proposition 3.2.1.** $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_g, \rho) = \infty$ for $\rho = \mathrm{sign}$ and $\rho = \epsilon$.

Proof. Using Lemma 2.7.2, let $M = M(\mathcal{O}_g, \rho)$. Then $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_g, \rho)\#\Bbbk\mathbb{D}_\infty = \infty$ since $\mathrm{supp}M(\mathcal{O}, \rho) = \mathcal{O}_g$ is an infinite conjugacy class. But $\mathrm{GKdim}\Bbbk\mathbb{D}_\infty < \infty$, this implies the $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_g, \rho) = \infty$, since $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_g, \rho)\#\Bbbk\mathbb{D}_\infty \leqslant \mathrm{GKdim}\mathcal{B}(\mathcal{O}_g, \rho) + \mathrm{GKdim}\Bbbk\mathbb{D}_\infty$. ■

## 3.3 The Nichols algebra $\mathcal{B}(\mathcal{O}_{gh}, \rho)$

Since $G^{gh} \cong \mathbb{Z}_2$ has only 2 irreducible representations, the unit representation and sign representation, we have the irreducible Yetter-Drinfeld modules $M(\mathcal{O}_{gh}, \epsilon)$ and $M(\mathcal{O}_{gh}, \text{sign})$.

Let $X = \{1, h^n, gh^n | n \in \mathbb{N}\}$. Then

$$M(\mathcal{O}_{gh}, \text{sign}) = \bigoplus_{y \in X} h_y \otimes \Bbbk x,$$

where $h_y$ is a renumeration of $X$, and

$$\deg(h_y) = h_y \triangleright gh.$$

The module structure is

$$\begin{aligned}
&g \cdot (1 \otimes x) = h \otimes \rho(gh)(x), && g \cdot (h^n \otimes x) = gh^n \otimes \rho(1)(x),\\
&g \cdot (gh^n \otimes x) = h^n \otimes \rho(1)(x), && h \cdot (1 \otimes x) = h \otimes \rho(1)(x),\\
&h \cdot (h^n \otimes x) = h^{n+1} \otimes \rho(1)(x), && h \cdot (gh^{n+1} \otimes x) = gh^n \otimes \rho(1)(x),\\
&h \cdot (gh \otimes x) = h \otimes \rho(gh)(x),
\end{aligned}$$

where $n \geqslant 1$.

The comodule structure is

$$\begin{aligned}
&\delta(1 \otimes x) = (1 \triangleright gh) \otimes (1 \otimes x) = gh \otimes (1 \otimes x),\\
&\delta(h^n \otimes x) = (h^n \triangleright gh) \otimes (h^n \otimes x) = h^{2n-1} g \otimes (h^n \otimes x),\\
&\delta(gh^n \otimes x) = (gh^n \triangleright gh) \otimes (gh^n \otimes x) = gh^{2n-1} \otimes (gh^n \otimes x).
\end{aligned}$$

### 3.3.1 The Yetter-Drinfeld module $M(\mathcal{O}_{gh}, \text{sign})$

Let

$$a_0 = 1 \otimes x, \quad a_n = h^n \otimes x, \quad b_n = gh^n \otimes x.$$

Then we obtain the module structure

$$g \cdot a_0 = -a_1, \qquad g \cdot a_n = b_n, \qquad g \cdot b_n = a_n,$$
$$h \cdot a_n = a_{n+1}, \qquad h \cdot b_{n+1} = b_n, \qquad h \cdot b_1 = -a_1.$$

The comodule structure is

$$\delta(a_0) = gh \otimes a_0, \quad \delta(a_n) = h^{2n-1}g \otimes a_n, \quad \delta(b_n) = gh^{2n-1} \otimes b_n.$$

The braiding structure is

$$c(a_m \otimes a_n) = \begin{cases} b_{n-2m+1} \otimes a_m & n > 2m-1 \\ -a_{2m-n} \otimes a_m & n \leqslant 2m-1 \end{cases}, \qquad c(a_m \otimes b_n) = a_{n+1} \otimes a_m,$$

$$c(b_m \otimes b_n) = \begin{cases} a_{n-2m+1} \otimes b_m & n > 2m-1 \\ -b_{2m-n+1} \otimes b_m & n \leqslant 2m-1 \end{cases}, \qquad c(a_m \otimes a_0) = -a_{2m} \otimes a_m,$$

$$c(a_m \otimes b_1) = a_{2m} \otimes a_m, \qquad c(b_m \otimes a_n) = b_{n+2m-1} \otimes b_m,$$
$$c(b_m \otimes a_0) = b_{2m-1} \otimes b_m, \qquad c(b_m \otimes b_1) = -b_{2m-1} \otimes b_m,$$
$$c(a_0 \otimes a_0) = b_1 \otimes a_0, \qquad c(a_0 \otimes a_n) = b_{n+1} \otimes a_0,$$
$$c(a_0 \otimes b_n) = a_{n-1} \otimes a_0, \qquad c(a_0 \otimes b_1) = -b_1 \otimes a_0,$$
$$c(b_1 \otimes a_0) = b_1 \otimes b_1, \qquad c(b_1 \otimes a_n) = b_{n+1} \otimes b_1,$$
$$c(b_1 \otimes b_n) = a_{n-1} \otimes b_1, \qquad c(b_1 \otimes b_1) = -b_1 \otimes b_1.$$

### 3.3.2 The Yetter-Drinfeld module $M(\mathcal{O}_{gh}, \epsilon)$

As in the case $M(\mathcal{O}_{gh}, \text{sign})$, we write

$$a_0 = 1 \otimes x, \quad a_n = h^n \otimes x, \quad b_n = gh^n \otimes x.$$

Then we have the action and coaction

$$\begin{aligned} &g \cdot a_0 = a_1, && g \cdot a_n = b_n, && g \cdot b_n = a_n, \\ &h \cdot a_n = a_{n+1}, && h \cdot b_{n+1} = b_n, && h \cdot b_1 = a_1, \\ &\delta(a_0) = gh \otimes a_0, && \delta(a_n) = h^{2n-1}g \otimes a_n, && \delta(b_n) = gh^{2n-1} \otimes b_n. \end{aligned}$$

The braiding structure is

$$c(a_m \otimes a_n) = \begin{cases} b_{n-2m+1} \otimes a_m, & n > 2m-1 \\ a_{2m-n} \otimes a_m, & n \leqslant 2m-1 \end{cases}, \qquad c(a_m \otimes b_n) = a_{n+1} \otimes a_m,$$

$$c(b_m \otimes b_n) = \begin{cases} a_{n-2m+1} \otimes b_m, & n > 2m-1 \\ -b_{2m-n+1} \otimes b_m, & n \leqslant 2m-1 \end{cases}, \qquad c(a_m \otimes a_0) = a_{2m} \otimes a_m,$$

$$\begin{aligned} &c(a_m \otimes b_1) = a_{2m} \otimes a_m, && c(b_m \otimes a_n) = b_{n+2m-1} \otimes b_m, \\ &c(b_m \otimes a_0) = b_{2m-1} \otimes b_m, && c(b_m \otimes b_1) = b_{2m-1} \otimes b_m, \\ &c(a_0 \otimes a_0) = b_1 \otimes a_0, && c(a_0 \otimes a_n) = b_{n+1} \otimes a_0, \\ &c(a_0 \otimes b_n) = a_{n-1} \otimes a_0, && c(a_0 \otimes b_1) = b_1 \otimes a_0, \\ &c(b_1 \otimes a_0) = b_1 \otimes b_1, && c(b_1 \otimes a_n) = b_{n+1} \otimes b_1, \\ &c(b_1 \otimes b_n) = a_{n-1} \otimes b_1, && c(b_1 \otimes b_1) = b_1 \otimes b_1. \end{aligned}$$

It is easy to see that $\dim M(\mathcal{O}_{gh}, \epsilon) = \dim M(\mathcal{O}_{gh}, \text{sign}) = \infty$. For the GK-dimension of $\mathcal{B}(\mathcal{O}_{gh}, \rho)$, we have

**Proposition 3.3.1.** $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{gh}, \rho) = \infty$ for $\rho = \text{sign}$ and $\rho = \epsilon$.

Proof. Similar to the proof of Proposition 3.2.1. ■

## 3.4 The Nichols algebra $\mathcal{B}(\mathcal{O}_1, \rho)$

To determine the Nichols algebras associated to the conjugacy class $\mathcal{O}_1$, we need to find all the left simple $\mathbb{D}_\infty$-modules. Let $S$ be any left simple $\mathbb{D}_\infty$-module. Then $\mathrm{End}_{\Bbbk\mathbb{D}_\infty}(S) = \Bbbk \mathrm{id}_S$ by Schur's lemma. For any $a \in Z(\Bbbk\mathbb{D}_\infty)$, the map $f_a : S \to$

$S, s \mapsto a \cdot s$, is a module map. So $(h + h^{-1}) \cdot s = \lambda s$ for some $\lambda \in \Bbbk$, and hence the representation

$$\rho : \Bbbk\mathbb{D}_\infty \longrightarrow \mathrm{End}(S)$$

induces a representation

$$\bar{\rho} : \Bbbk\mathbb{D}_\infty/\langle h + h^{-1} - \lambda\rangle \longrightarrow \mathrm{End}(S).$$

In other words, every simple left $\Bbbk\mathbb{D}_\infty$-module is a simple left $\Bbbk\mathbb{D}_\infty/\langle h + h^{-1} - \lambda\rangle$-module.

### 3.4.1 Representations of $\Bbbk\mathbb{D}_\infty/\langle h + h^{-1} - \lambda\rangle$

**Lemma 3.4.1.** The center of $\Bbbk\mathbb{D}_\infty$ is $\Bbbk[h + h^{-1}]$, and for $\lambda \in \Bbbk$,

$$\dim \Bbbk\mathbb{D}_\infty/\langle h + h^{-1} - \lambda\rangle < \infty.$$

Proof. Let $A_\lambda = \Bbbk\mathbb{D}_\infty/\langle h + h^{-1} - \lambda\rangle$. In $A_\lambda$, $h + h^{-1} = \lambda$, by direct computation, the following relations hold:

$$hg = \lambda g - gh, \quad h^2 = \lambda h - 1, \quad h^3 = \lambda h^2 - h, \quad \cdots, \quad h^{n+1} = \lambda h^n - h^{n-1} = 0.$$

Therefore, $h^n$ can be spaned by 1 and $h$ in $R$. Hence $gh^n$, $h^n g$, and $gh^n g$ can be spaned by $1, g, h, gh$. We see that $\dim A_\lambda \leqslant 4$. ■

By the following lemma, we need to find all primitive othogonal idempotents of $A_\lambda$.

**Lemma 3.4.2.** [21,Corollary 5.17] Suppose that $A_A = e_1 A \bigoplus \cdots \bigoplus e_n A$ is a decomposition of $A$ into indecomposable submodules. Every simple right $A$-module is isomorphic to one of the modules

$$S(1) = \mathrm{top} e_1 A, \quad \cdots, \quad S(n) = \mathrm{top} e_n A.$$

Now compute the idempotents of $A_\lambda$. Let

$$(x_1 + x_2 g + x_3 h + x_4 gh)^2 = x_1 + x_2 g + x_3 h + x_4 gh.$$

By the following lemma, we will find the primitive idempotents of an algebra.

**Lemma 3.4.3.** [21,Corollary 4.7 ] An idempotent $e \in A$ is primitive if and only if the algebra $eAe \cong \mathrm{End} eA$ has only two idempotents 0 and $e$, that is, the algebra $eAe$ is local.

Taking $x_3 = 0$, $x_2 = \pm\frac{1}{2}, x_4 = 0$, we obtain

$$e_1 = \frac{1}{2}(1+g), \quad e_2 = \frac{1}{2}(1-g), \quad 1 = e_1 + e_2,$$

which is a decomposition of 1. By direct computation, the following equalities hold:

$$e_1 A_\lambda = k(g+1) + k(gh+h), \quad e_2 A_\lambda = k(g-1) + k(gh-h).$$

Since $\dim A_\lambda < \infty$, the Jacobson radical $\mathrm{rad}\, e_1 A_\lambda = \mathrm{Nilrad}\, e_1 A_\lambda$ is the nil ideal, we need to find all nilpotent elements of $e_1 A_\lambda$ and $e_2 A_\lambda$.
Let $a = 1 + g, b = h + gh$. Then we have

$$a^2 = 2a, \quad ab = 2b,$$
$$ba = \lambda a, \quad b^2 = \lambda b.$$

**Lemma 3.4.4.** $(x_1 a + x_2 b)^n = (2x_1 + \lambda x_2)^{n-1}(x_1 a + x_2 b)$.

Proof. If $n = 2$, then

$$\begin{aligned}(x_1 a + x_2 b)^2 &= x_1^2 a^2 + x_1 x_2 ab + x_1 x_2 ba + x_2^2 b^2 = 2x_1^2 a + 2x_1 x_2 b + \lambda x_1 x_2 a + \lambda x_2^2 b \\ &= (2x_1^2 + \lambda x_1 x_2)a + (\lambda x_2^2 + 2x_1 x_2)b = (2x_1 + \lambda x_2)x_1 a + (2x_1 + \lambda x_2)x_2 b.\end{aligned}$$

By induction on *n*,

$$(x_1a + x_2b)^{n+1} = (x_1a + x_2b)^n(x_1a + x_2b) = (2x_1 + \lambda x_2)^n(x_1a + x_2b). \qquad \blacksquare$$

Let $[x_1(g+1) + x_2(gh+h)]^n = 0$. We have $x_1 = -\frac{\lambda}{2}x_2$. Therefore, the set of all nilpotent elements is

$$\text{Nilrad } e_1A_\lambda = \mathbb{k}(-\frac{\lambda}{2}a + b).$$

Therefore, rad $e_1A_\lambda = \mathbb{k}(-\frac{\lambda}{2}a + b)$.

Let $c = 1 - g, d = h - gh$. Then

$$c^2 = 2c, \quad d^2 = \lambda d, \quad cd = 2d, \quad dc = \lambda c.$$

**Lemma 3.4.5.** $(x_1c + x_2d)^n = (2x_1 + \lambda x_2)^{n-1}(x_1c + x_2d)$

Let $(x_1c + x_2d)^n = 0$. We have $x_1 = -\frac{\lambda}{2}x_2$. Therefore, the set of all nilpotent elements is

$$\text{Nilrad } e_2A_\lambda = \mathbb{k}(-\frac{\lambda}{2}c + d),$$

and rad $e_2A_\lambda = \mathbb{k}(-\frac{\lambda}{2}c + d)$.

Consequently,

**Lemma 3.4.6.** Let $e_1 = \frac{1}{2}(1+g)$ and $e_2 = \frac{1}{2}(1-g)$. Then the simple right modules of $A_\lambda$ are

$$e_1A_\lambda/\mathbb{k}(-\frac{\lambda}{2}a + b) \quad \text{and} \quad e_2A_\lambda/\mathbb{k}(-\frac{\lambda}{2}c + d),$$

where $\lambda \in \mathbb{k}, a = 1 + g, b = h + gh, c = 1 - g, d = h - gh$.

In particular, the simple modules of $A_0$ are $e_1A_0$ and $e_2A_0$.

$\Bbbk\mathbb{D}_\infty$-modules We can consider the left simple modules of $\Bbbk\mathbb{D}_\infty$. Let

$$e_1 = \frac{1}{2}(1+g), \quad e_2 = \frac{1}{2}(1-g).$$

Then

$$A_\lambda e_1 = \Bbbk(1+g) + \Bbbk(h^{-1} + gh), \qquad A_\lambda e_2 = \Bbbk(1-g) + \Bbbk(h^{-1} - gh).$$

Let $a = 1+g, b = h^{-1} + gh, c = 1-g, d = h^{-1} - gh$. Then we have

$$\begin{aligned} a^2 &= 2a, & ab &= \lambda a, & ba &= 2b, & b^2 &= \lambda b, \\ c^2 &= 2c, & cd &= \lambda c, & dc &= 2d, & d^2 &= \lambda d. \end{aligned}$$

By the first paragraph of this section and Lemma 3.4.6 we have

**Lemma 3.4.7.** The simple left $\Bbbk\mathbb{D}_\infty$-modules are

$$A_\lambda e_1/\Bbbk(-\frac{\lambda}{2}a + b) \quad \text{and} \quad A_\lambda e_2/\Bbbk(-\frac{\lambda}{2}c + d).$$

Precisely, if $\lambda \neq 0$, then the corresponding simple modules are

$$S_\lambda^+ = \Bbbk a, \quad S_\lambda^- = \Bbbk c,$$

where the module structures are

$$g \cdot a = a, \quad h \cdot a = \frac{\lambda}{2}a, \quad g \cdot c = -c, \quad h \cdot c = \frac{\lambda}{2}c.$$

If $\lambda = 0$, then

$$\begin{aligned} S_0^+ &= A_0 e_1 = \Bbbk(1+g) + \Bbbk(h^{-1} + gh), \\ S_0^- &= A_0 e_2 = \Bbbk(1-g) + \Bbbk(h^{-1} - gh). \end{aligned}$$

The module structures are

$$g \cdot a = a, \qquad h \cdot a = -b, \qquad g \cdot b = -b, \qquad h \cdot b = a,$$
$$g \cdot c = -c, \qquad g \cdot d = d, \qquad h \cdot c = -d, \qquad h \cdot d = c.$$

Proof. If $\lambda \neq 0$, then the module structures of $S_0^+$ and $S_0^-$ are

$$\begin{aligned}
g \cdot a &= g(1+g) = g + 1 = a,\\
h \cdot a &= h(1+g) = h + hg = \lambda - h^{-1} + (\lambda g - gh)\\
&= \lambda(1+g) - (h^{-1} - gh) = \lambda a - b = \frac{\lambda}{2}a,\\
g \cdot c &= g(1-g) = g - 1 = -c,\\
h \cdot c &= h(1-g) = h - hg = (\lambda - h^{-1}) - (\lambda - gh)\\
&= \lambda(1-g) - (h^{-1} - gh) = \lambda c - d = \frac{\lambda}{2}c.
\end{aligned}$$

If $\lambda = 0$, then the module structures of $S_0^+$ and $S_0^-$ are

$$\begin{aligned}
g \cdot a &= g(1+g) = g + 1 = a,\\
h \cdot a &= h(1+g) = h + hg = (-h^{-1}) - gh = -b,\\
g \cdot b &= g(h^{-1} + gh) = gh^{-1} + h = -gh - h^{-1} = -b,\\
h \cdot b &= h(h^{-1} + gh) = 1 + hgh = a,\\
g \cdot c &= g(1-g) = g - 1 = -c\\
g \cdot d &= g(h^{-1} - gh) = gh^{-1} - h = hg - (-h^{-1}) = h^{-1} - gh = d,\\
h \cdot c &= h(1-g) = h - hg = -h^{-1} - (-gh) = -d,\\
h \cdot d &= h(h^{-1} - gh) = 1 - hgh = 1 - g = c.
\end{aligned}$$

■

### 3.4.2 The irreducible Yetter-Drinfeld modules

Let $G = \mathbb{D}_\infty$. From Lemma 2.4.1 and Lemma 2.4.2 we obtain all the irreducible Yetter-Drinfeld modules in ${}^G_G\mathcal{YD}(\mathcal{O}_1)$:

$$M(1, S_0^+) = \Bbbk G \otimes_{\Bbbk G} S_0^+ = 1 \otimes \Bbbk a + 1 \otimes \Bbbk b,$$
$$M(1, S_0^-) = \Bbbk G \otimes_{\Bbbk G} S_0^- = 1 \otimes \Bbbk c + 1 \otimes \Bbbk d,$$
$$M(1, S_\lambda^+) = \Bbbk G \otimes_{\Bbbk G} S_\lambda^+ = 1 \otimes \Bbbk a,$$
$$M(1, S_\lambda^-) = \Bbbk G \otimes_{\Bbbk G} S_\lambda^- = 1 \otimes \Bbbk c.$$

Let $v_1 = 1 \otimes a$ and $v_2 = 1 \otimes b$. Then the $A_\lambda$-module structure is

$$g \cdot v_1 = g \cdot (1 \otimes a) = 1 \otimes g \cdot a = 1 \otimes a = v_1,$$
$$g \cdot v_2 = g \cdot (1 \otimes b) = 1 \otimes g \cdot b = 1 \otimes (-b) = -v_2,$$
$$h \cdot v_1 = h \cdot (1 \otimes a) = 1 \otimes h \cdot a = 1 \otimes (-b) = -v_2,$$
$$h \cdot v_2 = h \cdot (1 \otimes b) = 1 \otimes h \cdot b = 1 \otimes a = v_1.$$

The comodule structure is

$$\delta(v_1) = \delta(1 \otimes a) = 1 \otimes (1 \otimes a) = 1 \otimes v_1, \quad \delta(v_2) = \delta(1 \otimes b) = 1 \otimes (1 \otimes b) = 1 \otimes v_2.$$

The braiding structure is

$$c(v_1 \otimes v_1) = v_1 \otimes v_1, \quad c(v_1 \otimes v_2) = v_2 \otimes v_1,$$
$$c(v_2 \otimes v_1) = v_1 \otimes v_2, \quad c(v_2 \otimes v_2) = v_2 \otimes v_2.$$

Therefore, the braiding matrix is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Hence GKdim$B(V) = 2$ by [1,Example 31].

Consider $\Bbbk G \otimes_{\Bbbk G} S_0^- = 1 \otimes \Bbbk c + 1 \otimes \Bbbk d$. Let $w_1 = 1 \otimes c$ and $w_2 = 1 \otimes d$. Then the $A_\lambda$-module structure is

$$g \cdot w_1 = g(1 \otimes c) = 1 \otimes g \cdot c = 1 \otimes (-c) = -w_1,$$
$$g \cdot w_2 = g(1 \otimes d) = 1 \otimes g \cdot d = 1 \otimes d = w_2,$$
$$h \cdot w_1 = h(1 \otimes c) = 1 \otimes h \cdot c = 1 \otimes (-d) = -w_2,$$
$$h \cdot w_2 = h(1 \otimes d) = 1 \otimes h \cdot d = 1 \otimes c = w_1.$$

The comodule structure is

$$\delta(w_1) = \delta(1 \otimes c) = 1 \otimes (1 \otimes c) = 1 \otimes w_1, \quad \delta(w_2) = \delta(1 \otimes d) = 1 \otimes (1 \otimes d) = 1 \otimes w_2.$$

By direct computation, the braiding structure is

$$c(w_1 \otimes w_1) = w_1 \otimes w_1, \quad c(w_1 \otimes w_2) = w_2 \otimes w_1,$$
$$c(w_2 \otimes w_1) = w_1 \otimes w_2, \quad c(w_2 \otimes w_2) = w_2 \otimes w_2.$$

Therefore, the braiding matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Hence GKdim$\mathcal{B}(\mathcal{O}_1, S_0^-) = 1$ by [1,Example 31].

Consider $\Bbbk G \otimes_{\Bbbk G} S_\lambda^+ = 1 \otimes \Bbbk a$. Let $w = 1 \otimes a$. Then the $A_\lambda$-module structure is

$$g \cdot w = g \cdot (1 \otimes a) = 1 \otimes g \cdot a = 1 \otimes a = w,$$
$$h \cdot w = h \cdot (1 \otimes a) = 1 \otimes h \cdot a = 1 \otimes \frac{\lambda}{2} a = \frac{\lambda}{2} w.$$

The comodule structure is

$$\delta(w) = \delta(1 \otimes a) = 1 \otimes (1 \otimes a) = 1 \otimes w.$$

The braiding structure is

$$c(w \otimes w) = w_{(-1)} \cdot w \otimes w_{(0)} = w \otimes w.$$

Therefore, $\mathrm{GKdim} B(\mathcal{O}_1, S_\lambda^+) = 1$.

Now we consider $\Bbbk G \otimes_{\Bbbk G} S_\lambda^- = 1 \otimes \Bbbk c$. Let $v = 1 \otimes c$. Then the $A_\lambda$-module structure is

$$g \cdot v = g \cdot (1 \otimes c) = 1 \otimes g \cdot c = 1 \otimes (-c) = -v,$$
$$h \cdot v = h \cdot (1 \otimes c) = 1 \otimes h \cdot c = 1 \otimes \frac{\lambda}{2} c = \frac{\lambda}{2} v.$$

The comodule structure is

$$\delta(v) = \delta(1 \otimes c) = 1 \otimes (1 \otimes c) = 1 \otimes v.$$

The braiding structure is

$$c(v \otimes v) = v_{(-1)} \cdot v \otimes v_{(0)} = v \otimes v.$$

Therefore, $\mathrm{GKdim} B(\mathcal{O}_1, S_\lambda^-) = 1$.

## 3.5 Finite GK-dimensional Nichols algebras of semisimple Yetter-Drinfeld modules over $\mathbb{D}_\infty$

Through the discusion above, we have the following finite dimensional irreducible Yetter-Drinfeld modules $V$ with $\mathrm{GKdim}\mathcal{B}(V) < \infty$.

$$M(\mathcal{O}_{h^n}, \rho_{\pm 1}) = \Bbbk x_1 + \Bbbk x_2,$$
$$M(\mathcal{O}_1, S_0^+) = \Bbbk G \otimes_{\Bbbk G} S_0^+ = 1 \otimes \Bbbk a + 1 \otimes \Bbbk b,$$
$$M(\mathcal{O}_1, S_0^-) = \Bbbk G \otimes_{\Bbbk G} S_0^- = 1 \otimes \Bbbk c + 1 \otimes \Bbbk d,$$
$$M(\mathcal{O}_1, S_\lambda^+) = \Bbbk G \otimes_{\Bbbk G} S_\lambda^+ = 1 \otimes \Bbbk a,$$

$$M(\mathcal{O}_1, S_\lambda^-) = \Bbbk G \otimes_{\Bbbk G} S_\lambda^- = 1 \otimes \Bbbk c.$$

where $\lambda \neq 0$. In this section we consider when the Nichols algebra of a semisimple Yetter-Drinfeld module has a finite Gelfand-Kirillov dimension.

**Proposition 3.5.1.** Let $M = M(\mathcal{O}_{h^m}, \rho_a) \oplus M(\mathcal{O}_{h^n}, \rho_b)$ with $a, b \in \{\pm 1\}$. Then $\mathrm{GKdim}\mathcal{B}(M) < \infty$ if and only if $a = b$.

Proof. The braiding matrix is given as follows:

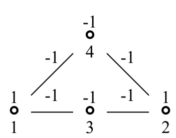
$$q = \begin{bmatrix} a & a^{-1} & b & b^{-1} \\ a^{-1} & a & b^{-1} & b \\ a & a^{-1} & b & b^{-1} \\ a^{-1} & a & b^{-1} & b \end{bmatrix}.$$

If $a = b = 1$, then $\mathcal{B}(M(\mathcal{O}_{h^m}, \rho_a) \oplus M(\mathcal{O}_{h^n}, \rho_b) \cong S(M(\mathcal{O}_{h^m}, \rho_a) \oplus M(\mathcal{O}_{h^n}, \rho_b))$, whose GK-dimension is 4 [1,Example 27]. If $a = b = -1$, then the braiding matrix is

$$q = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

In this case, $\mathrm{GKdim}(M(\mathcal{O}_{h^m}, \rho_a) \oplus M(\mathcal{O}_{h^n}, \rho_b)) < \infty$ by [1,Example 27].

In the last, if $a = 1, b = -1$, then the corresponding Dynkin diagram is



By [2,Lemma 1.4] and [20,Theorem 1.2] we have $\mathrm{GKdim}(M(\mathcal{O}_{h^m}, \rho_a) \oplus M(\mathcal{O}_{h^n}, \rho_b)) = \infty$. ■

**Proposition 3.5.2.** Let $M = M(\mathcal{O}_1, S_\lambda^\alpha) \oplus M(\mathcal{O}_1, S_\mu^\beta)$ with $\lambda, \mu \in \Bbbk, \alpha, \beta \in \{+, -\}$. Then $\mathrm{GKdim}\mathcal{B}(M) < \infty$.

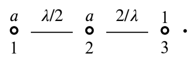
Proof. By computation of the braiding matrices of $M$ in all cases, we have the braiding matrices are all of the form $q = (q_{ij})$, where $q_{ij} = 1$ for all $i, j$. Therefore GKdim$\mathcal{B}(M) < \infty$ by [1,Example 31]. ■

**Proposition 3.5.3.** Let $M = M(\mathcal{O}_{h^n}, \rho_a) \oplus M(\mathcal{O}_1, S^{\alpha}_{\lambda})$ with $\lambda \in \Bbbk^{\times}$, $\alpha \in \{+, -\}$ and $a \in \{1, -1\}$. Then GKdim$\mathcal{B}(M) < \infty$ if and only if $\lambda = 2$.

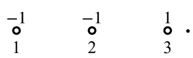
Proof. The braiding matrix is

$$q = \begin{bmatrix} a & 1/a & \lambda/2 \\ 1/a & a & 2/\lambda \\ 1 & 1 & 1 \end{bmatrix}.$$

If $a = 1, \lambda = 2$, then Therefore GKdim$\mathcal{B}(M) < \infty$ by [1,Example 31].
If $\lambda \neq 2$, then the Dynkin diagram is as follows:




Therefore GKdim$\mathcal{B}(M) = \infty$ by going through the list of [33,59-124].
If $a = -1, \lambda = 2$, then the Dynkin diagram is




Thus GKdim$\mathcal{B}(M) < \infty$ by [1,Example 27]. ■

# Chapter 4 Finite GK-dimensional Nichols Algebras over the quaternion group $\mathbb{Q}_8$

## 4.1 The Nichols algebras $\mathcal{B}(\mathcal{O}_1, \rho)$

Consider the conjugacy $\mathcal{O}_1 = \{1\}$. $G^1 = G = \mathbb{Q}_8$. For $(\rho_1, V_1)$, let $V_1 = \Bbbk v_1$. The module structure is

$$x \cdot v_1 = v_1, \quad y \cdot v_1 = v_1.$$

The corresponding Yetter-Drinfeld module $W_1 = 1 \otimes V_1$ The module structure is

$$x \cdot (1 \otimes v_1) = 1 \otimes \rho_1(x)(v_1) = 1 \otimes v_1$$

The comodule structure is

$$\delta(1 \otimes v_1) = 1 \otimes (1 \otimes v_1)$$

Let $w_1 = 1 \otimes v_1$ The braiding structure is

$$c(w_1 \otimes w_1) = w_1 \otimes w_1$$

Therefore $\dim \mathcal{B}(W_1) = \infty$, and $\mathrm{GKdim}\mathcal{B}(W_1) = 1$.

For $(\rho_2, V_2)$, let $V_1 = \Bbbk v_2$. The module structure is

$$x \cdot v_2 = v_2, \quad y \cdot v_2 = -v_2.$$

The corresponding Yetter-Drinfeld module is $W_2 = 1 \otimes V_2$. The $\Bbbk G$-module structure of $W_2$ is

$$x \cdot (1 \otimes v_2) = 1 \otimes \rho_2(x)(v_2) = 1 \otimes v_2$$
$$y \cdot (1 \otimes v_2) = 1 \otimes \rho_2(y)(v_2) = -1 \otimes v_2$$

The comodule structure is

$$\delta(1 \otimes v_2) = 1 \otimes (1 \otimes v_2)$$

Let $w_2 = 1 \otimes v_2$ The braiding structure is

$$c(w_2 \otimes w_2) = w_2 \otimes w_2$$

Therefore $\dim \mathcal{B}(W_2) = \infty$, and $\mathrm{GKdim}\mathcal{B}(W_2) = 1$.

For $(\rho_3, V_3)$, let $V_3 = \Bbbk v_3$.The module structure is

$$x \cdot v_3 = -v_3, \quad y \cdot v_3 = v_3.$$

The corresponding Yetter-Drinfeld module $W_3 = 1 \otimes V_3$ The module structure is

$$x \cdot (1 \otimes v_3) = 1 \otimes \rho_3(x)(v_3) = -1 \otimes v_3$$
$$y \cdot (1 \otimes v_3) = 1 \otimes \rho_3(y)(v_3) = 1 \otimes v_3$$

The comodule structure is

$$\delta(1 \otimes v_3) = 1 \otimes (1 \otimes v_3)$$

Let $w_3 = 1 \otimes v_3$ The braiding structure is

$$c(w_3 \otimes w_3) = w_3 \otimes w_3$$

Therefore $\dim \mathcal{B}(W_3) = \infty$, and $\mathrm{GKdim}\mathcal{B}(W_3) = 1$.

For $(\rho_4, V_4)$, let $V_4 = \Bbbk v_4$. The module structure is

$$x \cdot v_4 = -v_4, \quad y \cdot v_4 = -v_4.$$

The corresponding Yetter-Drinfeld module $W_4 = 1 \otimes V_4$ The module structure is

$$x \cdot (1 \otimes v_4) = 1 \otimes \rho_4(x)(v_4) = -1 \otimes v_4$$
$$y \cdot (1 \otimes v_4) = 1 \otimes \rho_4(y)(v_4) = -1 \otimes v_4$$

The comodule structure is

$$\delta(1 \otimes v_4) = 1 \otimes (1 \otimes v_4)$$

Let $w_4 = 1 \otimes v_4$ The braiding structure is

$$c(w_4 \otimes w_4) = w_4 \otimes w_4$$

Therefore $\dim \mathcal{B}(W_4) = \infty$, and $\mathrm{GKdim}\mathcal{B}(W_4) = 1$.

For $(\rho_5, V_5)$, let $\{v_1, v_2\}$ be a basis of $V_5$. The module structure is

$$x \cdot v_1 = \mathbf{i}v_1, \quad y \cdot v_1 = v_2, \quad x \cdot v_2 = -\mathbf{i}v_2, \quad y \cdot v_2 = -v_1.$$

The corresponding Yetter-Drinfeld module $W_5 = 1 \otimes V_5$ The module structure is

$$x \cdot (1 \otimes v_1) = 1 \otimes \rho_5(x)(v_1) = \mathbf{i} \otimes v_1,$$
$$y \cdot (1 \otimes v_1) = 1 \otimes \rho_5(y)(v_1) = 1 \otimes v_2,$$
$$x \cdot (1 \otimes v_2) = 1 \otimes \rho_5(x)(v_2) = -\mathbf{i}(1 \otimes v_2),$$
$$y \cdot (1 \otimes v_2) = 1 \otimes \rho_5(y)(v_2) = -1 \otimes v_1.$$

The comodule structure is

$$\delta(1 \otimes v_1) = 1 \otimes (1 \otimes v_1), \quad \delta(1 \otimes v_2) = 1 \otimes (1 \otimes v_2).$$

Let $u_1 = 1 \otimes v_1, u_2 = 1 \otimes v_2$. The braiding structure is

$$c(u_1 \otimes u_1) = u_1 \otimes u_1, \quad c(u_1 \otimes u_2) = u_2 \otimes u_1,$$
$$c(u_2 \otimes u_1) = u_1 \otimes u_2, \quad c(u_2 \otimes u_2) = u_2 \otimes u_2.$$

Therefore $\dim \mathcal{B}(W_5) = \infty$, and $\mathrm{GKdim}\mathcal{B}(W_5) = 2$ by [1,Example 31].

Therefore, we have proved

**Proposition 4.1.1.** For any irreducible representation $(\rho, V) \in \mathrm{Irr}(G^1)$, $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_1, \rho) < \infty$, and $\dim \mathcal{B}(\mathcal{O}_1, \rho) = \infty$.

## 4.2 The Nichols algebras $\mathcal{B}(\mathcal{O}_x, \phi)$

As shown above, $\mathcal{O}_x = \{x, x^3\}$. $G^x = \{1, x, x^2, x^3\} \cong \mathbb{Z}_4$ is the cyclic group with order 4. Therefore, all the irrreducible representations are $(\phi_t, U_t), 0 \leqslant t \leqslant 3$, which are all 1-dimensional. Precisely,

$$x^k \cdot u_t = \phi_t(x^k)(u_t) = e^{t\pi \mathbf{i} k/2} u_t.$$

where $\mathbf{i} = \sqrt{-1}$.

Clearly, $(\phi_0, U_0)$ is the trivial representation of $G^x$. Since

$$G = G^x \cup yG^x,$$

the corresponding Yetter-Drinfeld module is $M(\mathcal{O}_x, \phi_0) = \Bbbk G \otimes_{\Bbbk G^x} U_0 = 1 \otimes U_0 \oplus y \otimes U_0$.

Let $v_1 = 1 \otimes u_0$, $v_2 = y \otimes u_0$. The $G$-module structure is

$$x \cdot v_1 = x \cdot (1 \otimes u_0) = 1 \otimes \phi_0(x)(u_0) = 1 \otimes u_0 = v_1,$$

$$y \cdot v_1 = y \cdot (1 \otimes u_0) = y \otimes \phi_0(1)(u_0) = y \otimes u_0 = v_2.$$

$$x \cdot v_2 = x \cdot (y \otimes u_0) = y \otimes \phi_0(x^3)(u_0) = y \otimes u_0 = v_2,$$

$$y \cdot v_2 = y \cdot (y \otimes u_0) = 1 \otimes \phi_0(x^2)(u_0) = 1 \otimes u_0 = v_1.$$

The $G$-comdule is

$$\delta(v_1) = \delta(1 \otimes u_0) = (1 \triangleright x) \otimes v_1 = x \otimes v_1,$$

$$\delta(v_2) = \delta(y \otimes u_0) = (y \triangleright x) \otimes v_2 = x^3 \otimes v_2.$$

The braiding structure is

$$c(v_1 \otimes v_1) = v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = x \cdot v_1 \otimes v_1 = v_1 \otimes v_1,$$

$$c(v_1 \otimes v_2) = v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = x \cdot v_2 \otimes v_1 = v_2 \otimes v_1,$$

$$c(v_2 \otimes v_1) = v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = x^3 \cdot v_1 \otimes v_2 = v_1 \otimes v_2,$$

$$c(v_2 \otimes v_2) = v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = x^3 \cdot v_2 \otimes v_2 = v_2 \otimes v_2,$$

which is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore, $\dim \mathcal{B}(\mathcal{O}_x, \phi_0) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_x, \phi_0) = 2$ by [1,Example 31].

For $(\phi_1, U_1)$, we compute the Yetter-Drinfeld module

$$M(\mathcal{O}_x, \phi_1) = 1 \otimes U_1 \oplus y \otimes U_1.$$

Let $v_1 = 1 \otimes u_1$, $v_2 = y \otimes u_1$. $q = e^{\pi \mathbf{i}/2} \in \mathbb{G}_4'$.The $G$-module structure is

$$x \cdot v_1 = x \cdot (1 \otimes u_1) = 1 \otimes \phi_1(x)(u_1) = e^{\pi \mathbf{i}/2} v_1,$$

$$y \cdot v_1 = y \cdot (1 \otimes u_1) = y \otimes \phi_1(1)(u_1) = v_2.$$

The $G$-comodule is

$$\delta(v_1) = \delta(1 \otimes u_1) = (1 \triangleright x) \otimes v_1 = x \otimes v_1,$$
$$\delta(v_2) = \delta(y \otimes u_1) = (y \triangleright x) \otimes v_2 = x^3 \otimes v_2.$$

The braiding structure is

$$c(v_1 \otimes v_1) = v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = x \cdot v_1 \otimes v_1 = qv_1 \otimes v_1,$$
$$c(v_1 \otimes v_2) = v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = x \cdot v_2 \otimes v_1 = q^3 v_2 \otimes v_1,$$
$$c(v_2 \otimes v_1) = v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = x^3 \cdot v_1 \otimes v_2 = q^3 v_1 \otimes v_2,$$
$$c(v_2 \otimes v_2) = v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = x^3 \cdot v_2 \otimes v_2 = qv_2 \otimes v_2.$$

This is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} q & q^3 \\ q^3 & q \end{bmatrix}$$

which is of affine Cartan type. Therefore $\dim \mathcal{B}(\mathcal{O}_x, \phi_1) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_x, \phi_1) = \infty$ by going through the list of Heckenberger's classification [34] and Lemma 2.7.1.

For $(\phi_2, U_2)$, we compute the Yetter-Drinfeld module

$$M(\mathcal{O}_x, \phi_2) = 1 \otimes U_2 \oplus y \otimes U_2.$$

Let $v_1 = 1 \otimes u_2$, $v_2 = y \otimes u_2$, $q = e^{\pi \mathbf{i}/2} \in \mathbb{G}_4'$. Then $\phi_2(u_2) = q^2 u_2$. The $G$-module structure is

$$x \cdot v_1 = x \cdot (1 \otimes u_2) = 1 \otimes \phi_2(x)(u_2) = e^{2\pi \mathbf{i}/2}(1 \otimes u_2) = q^2 v_1,$$
$$y \cdot v_1 = y \cdot (1 \otimes u_2) = y \otimes \phi_2(1)(u_2) = v_2,$$
$$x \cdot v_2 = x \cdot (y \otimes u_2) = y \otimes \phi_2(x^3)(u_2) = q^2 v_2,$$
$$y \cdot v_2 = y \cdot (y \otimes u_2) = 1 \otimes \phi_2(x^2)(u_2) = v_1.$$

The $\Bbbk G$-comodule is

$$\delta(v_1) = \delta(1 \otimes u_2) = (1 \triangleright x) \otimes v_1 = x \otimes v_1,$$
$$\delta(v_2) = \delta(y \otimes u_2) = (y \triangleright x) \otimes v_2 = x^3 \otimes v_2.$$

The braiding structure is

$$c(v_1 \otimes v_1) = v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = x \cdot v_1 \otimes v_1 = q^2 v_1 \otimes v_1,$$
$$c(v_1 \otimes v_2) = v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = x \cdot v_2 \otimes v_1 = q^2 v_2 \otimes v_1,$$
$$c(v_2 \otimes v_1) = v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = x^3 \cdot v_1 \otimes v_2 = q^2 v_1 \otimes v_2,$$
$$c(v_2 \otimes v_2) = v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = x^3 \cdot v_2 \otimes v_2 = q^2 v_2 \otimes v_2,$$

This is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

By [1,Example 27], $\dim \mathcal{B}(\mathcal{O}_x, \phi_2) = 4$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_x, \phi_2) = 0$.

For the $(\phi_3, U_3)$, we compute the Yetter-Drinfeld module

$$M(\mathcal{O}_x, \phi_3) = 1 \otimes U_3 \oplus y \otimes U_3.$$

Let $v_1 = 1 \otimes u_3$, $v_2 = y \otimes u_3$. $q = e^{\pi \mathbf{i}/2} \in \mathbb{G}_4'$. The $G$-module structure is

$$x \cdot v_1 = x \cdot (1 \otimes u_3) = 1 \otimes \phi_1(x)(u_3) = e^{3\pi \mathbf{i}/2}(1 \otimes u_3) = e^{3\pi \mathbf{i}/2} v_1,$$
$$y \cdot v_1 = y \cdot (1 \otimes u_3) = y \otimes \phi_1(1)(u_3) = v_2.$$

The $G$-comodule is

$$\delta(v_1) = \delta(1 \otimes u_3) = (1 \triangleright x) \otimes v_1 = x \otimes v_1,$$

$$\delta(v_2) = \delta(y \otimes u_3) = (y \triangleright x) \otimes v_2 = x^3 \otimes v_2.$$

The braiding structure is

$$c(v_1 \otimes v_1) = v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = x \cdot v_1 \otimes v_1 = e^{3\pi \mathbf{i}/2} v_1 \otimes v_1,$$
$$c(v_1 \otimes v_2) = v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = x \cdot v_2 \otimes v_1 = e^{3\pi \mathbf{i}/2} v_2 \otimes v_1,$$
$$c(v_2 \otimes v_1) = v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = x^3 \cdot v_1 \otimes v_2 = e^{\pi \mathbf{i}/2} v_1 \otimes v_2,$$
$$c(v_2 \otimes v_2) = v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = x^3 \cdot v_2 \otimes v_2 = e^{3\pi \mathbf{i}/2} v_2 \otimes v_2,$$

This is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} e^{3\pi \mathbf{i}/2} & e^{3\pi \mathbf{i}/2} \\ e^{\pi \mathbf{i}/2} & e^{\pi \mathbf{i}/2} \end{bmatrix}$$

By [1,Example 27], $\dim \mathcal{B}(\mathcal{O}_x, \phi_3) = 4$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_x, \phi_3) = 0$.

**Proposition 4.2.1.** For any irreducible representation $(\phi, V) \in \mathrm{Irr}(G^x)$, $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_x, \phi) < \infty$ if and only if $\phi = \phi_0$ or $\phi = \phi_2$; $\dim \mathcal{B}(\mathcal{O}_x, \phi) < \infty$ if and only if $\phi = \phi_2$.

## 4.3 The Nichols algebras $\mathcal{B}(\mathcal{O}_{x^2}, \rho)$

It is shown that $\mathcal{O}_{x^2} = \{x^2\}$, and $G^{x^2} = G$. It is shown that there are four 1-dim irreducible representations $(\rho_1, V_1), (\rho_2, V_2), (\rho_3, V_3), (\rho_4, V_4)$ and one 2-dim irreducible representation, denoted by $(\rho_5, V_5)$.

For $(\rho_1, V_1)$ The corresponding Yetter-Drinfeld module $M(\mathcal{O}_{x^2}, \rho_1) = 1 \otimes V_1$. Let $w_1 = 1 \otimes v_1$. The module structure is

$$x \cdot w_1 = x \cdot (1 \otimes v_1) = 1 \otimes \rho_1(x)(v_1) = 1 \otimes v_1 = w_1,$$
$$y \cdot w_1 = y \cdot (1 \otimes v_1) = 1 \otimes \rho_1(y)(v_1) = 1 \otimes v_1 = w_1.$$

The comodule structure is

$$\delta(w_1) = \rho(1 \otimes v_1) = x^2 \otimes (1 \otimes v_1) = x^2 \otimes w_1.$$

The braiding structure is

$$c(w_1 \otimes w_1) = w_1^{(-1)} \cdot w_1 \otimes w_1^{(0)} = x^2 \cdot w_1 \otimes w_1 = w_1 \otimes w_1.$$

Therefore, $\dim \mathcal{B}(\mathcal{O}_{x^2}, \rho_1) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{x^2}, \rho_1) = 1$.

For $(\rho_2, V_2)$ The corresponding Yetter-Drinfeld module $M(\mathcal{O}_{x^2}, \rho_2) = 1 \otimes V_2$. Let $w_2 = 1 \otimes v_2$. The module structure is

$$x \cdot w_2 = x \cdot (1 \otimes v_2) = 1 \otimes \rho_2(x)(v_2) = 1 \otimes v_2 = w_2,$$
$$y \cdot w_2 = y \cdot (1 \otimes v_2) = 1 \otimes \rho_2(y)(v_2) = -1 \otimes v_2 = -w_2.$$

The comodule structure is

$$\delta(1 \otimes v_2) = x^2 \otimes (1 \otimes v_2) = x^2 \otimes w_2.$$

The braiding structure is

$$c(w_2 \otimes w_2) = w_2^{(-1)} \cdot w_2 \otimes w_2^{(0)} = x^2 \cdot w_2 \otimes w_2 = w_2 \otimes w_2.$$

Therefore, $\dim \mathcal{B}(\mathcal{O}_{x^2}, \rho_2) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{x^2}, \rho_2) = 1$.

For $(\rho_3, V_3)$ The corresponding Yetter-Drinfeld module $M(\mathcal{O}_{x^2}, \rho_3) = 1 \otimes V_3$. Let $w_3 = 1 \otimes v_3$. The module structure is

$$x \cdot w_3 = x \cdot (1 \otimes v_3) = 1 \otimes \rho_3(x)(v_3) = -1 \otimes v_3 = -w_3,$$
$$y \cdot w_3 = y \cdot (1 \otimes v_3) = 1 \otimes \rho_3(y)(v_3) = 1 \otimes v_3 = w_3.$$

The comodule structure is

$$\delta(1 \otimes v_3) = x^2 \otimes (1 \otimes v_3) = x^2 \otimes w_3.$$

The braiding structure is

$$c(w_3 \otimes w_3) = w_3^{(-1)} \cdot w_3 \otimes w_3^{(0)} = x^2 \cdot w_3 \otimes w_3 = w_3 \otimes w_3.$$

Therefore, $\dim \mathcal{B}(\mathcal{O}_{x^2}, \rho_3) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{x^2}, \rho_3) = 1$.

For $(\rho_4, V_4)$ The corresponding Yetter-Drinfeld module $M(\mathcal{O}_{x^2}, \rho_4) = 1 \otimes V_4$. Let $w_4 = 1 \otimes v_4$. The module structure is

$$x \cdot w_4 = x \cdot (1 \otimes v_4) = 1 \otimes \rho_4(x)(v_4) = -1 \otimes v_4 = -w_4,$$
$$y \cdot w_4 = y \cdot (1 \otimes v_4) = 1 \otimes \rho_4(y)(v_4) = -1 \otimes v_4 = -w_4.$$

The comodule structure is

$$\delta(1 \otimes v_4) = x^2 \otimes (1 \otimes v_4) = x^2 \otimes w_4.$$

The braiding structure is

$$c(w_4 \otimes w_4) = w_4^{(-1)} \cdot w_4 \otimes w_4^{(0)} = x^2 \cdot w_4 \otimes w_4 = w_4 \otimes w_4.$$

Therefore, $\dim \mathcal{B}(\mathcal{O}_{x^2}, \rho_4) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{x^2}, \rho_4) = 1$.

For $(\rho_5, V_5)$, the corresponding Yetter-Drinfeld module $M(\mathcal{O}_{x^2}, \rho_5) = 1 \otimes V_5$. Let $w_5 = 1 \otimes v_5$, $w_6 = 1 \otimes v_6$. The $G$-module structure is

$$x \cdot w_5 = x \cdot (1 \otimes v_5) = 1 \otimes \rho_5(x)(v_5) = \mathbf{i}(1 \otimes v_5) = \mathbf{i}w_5,$$
$$y \cdot w_5 = y \cdot (1 \otimes v_5) = 1 \otimes \rho_5(y)(v_5) = 1 \otimes v_6 = w_6,$$
$$x \cdot w_6 = x \cdot (1 \otimes v_6) = 1 \otimes \rho_5(x)(v_6) = -\mathbf{i}(1 \otimes v_6) = -\mathbf{i}w_6,$$
$$y \cdot w_6 = y \cdot (1 \otimes v_6) = 1 \otimes \rho_5(y)(v_6) = -1 \otimes v_5 = -w_5.$$

The $G$-comodule structure is

$$\delta(w_5) = x^2 \otimes w_5, \quad \delta(w_6) = x^2 \otimes w_6.$$

The braiding structure is

$$\begin{aligned}
c(w_5 \otimes w_5) &= w_5^{(-1)} \cdot w_5 \otimes w_5^{(0)} = x^2 \cdot w_5 \otimes w_5 = -w_5 \otimes w_5, \\
c(w_5 \otimes w_6) &= w_5^{(-1)} \cdot w_6 \otimes w_5^{(0)} = x^2 \cdot w_6 \otimes w_5 = -w_6 \otimes w_5, \\
c(w_6 \otimes w_5) &= w_6^{(-1)} \cdot w_5 \otimes w_6^{(0)} = x^2 \cdot w_6 \otimes w_5 = -w_6 \otimes w_6, \\
c(w_6 \otimes w_6) &= w_6^{(-1)} \cdot w_6 \otimes w_6^{(0)} = x^2 \cdot w_6 \otimes w_6 = -w_6 \otimes w_6.
\end{aligned}$$

which is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Therefore, $\mathcal{B}(\mathcal{O}_{x^2}, \rho_5) \cong \Lambda M(\mathcal{O}_{x^2}, \rho_5)$, the exterior algebra over $M(\mathcal{O}_{x^2}, \rho_5)$, which implies $\dim \mathcal{B}(\mathcal{O}_{x^2}, \rho_5) = 4$ and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{x^2}, \rho_5) = 0$.

**Proposition 4.3.1.** Let $(\rho, V) \in \mathrm{Irr}(G^{x^2})$ be any irreducible representation.
Then $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{x^2}, \rho) < \infty$. Moreover, $\dim \mathcal{B}(\mathcal{O}_{x^2}, \rho) < \infty$ if and only if $\rho = \rho_5$.

## 4.4 The Nichols algebras $\mathcal{B}(\mathcal{O}_y, \phi)$

For the conjugacy class $\mathcal{O}_y$, $G^y = \{1, y, x^2y, x^2\} = \langle y \rangle \cong \mathbb{Z}_4$. As we known, Therefore, all the irrreducible representations are $(\phi_t, U_t), 0 \leqslant t \leqslant 3$, which are all 1-dimensional. Precisely,

$$y^k \cdot u_t = \phi_t(y^k)(u_t) = e^{t\pi \mathbf{i} k/2} u_t.$$

where $\mathbf{i} = \sqrt{-1}$.

Clearly, $(\phi_0, U_0)$ is the trivial representation of $G^y$. Since

$$G = G^y \cup xG^y,$$

the corresponding Yetter-Drinfeld module is $M(\mathcal{O}_y, \phi_0) = \Bbbk G \otimes_{\Bbbk G^y} U_0 = 1 \otimes U_0 \oplus x \otimes U_0$.

Let $v_1 = 1 \otimes u_0$, $v_2 = x \otimes u_0$. The $G$-module structure is

$$\begin{aligned}
x \cdot v_1 &= x \cdot (1 \otimes u_0) = x \otimes \phi_0(1)(u_0) = x \otimes u_0 = v_2, \\
y \cdot v_1 &= y \cdot (1 \otimes u_0) = 1 \otimes \phi_0(y)(u_0) = 1 \otimes u_0 = v_1, \\
x \cdot v_2 &= x \cdot (x \otimes u_0) = 1 \otimes \phi_0(y^2)(u_0) = 1 \otimes u_0 = v_1, \\
y \cdot v_2 &= y \cdot (x \otimes u_0) = x \otimes \phi_0(y^3)(u_0) = x \otimes u_0 = v_2.
\end{aligned}$$

The $G$-comdule is

$$\begin{aligned}
\delta(v_1) &= \delta(1 \otimes u_0) = (1 \triangleright y) \otimes v_1 = y \otimes v_1, \\
\delta(v_2) &= \delta(x \otimes u_0) = (x \triangleright y) \otimes v_2 = y^3 \otimes v_2.
\end{aligned}$$

The braiding structure is

$$\begin{aligned}
c(v_1 \otimes v_1) &= v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = y \cdot v_1 \otimes v_1 = v_1 \otimes v_1, \\
c(v_1 \otimes v_2) &= v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = y \cdot v_2 \otimes v_1 = v_2 \otimes v_1, \\
c(v_2 \otimes v_1) &= v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = y^3 \cdot v_1 \otimes v_2 = v_1 \otimes v_2, \\
c(v_2 \otimes v_2) &= v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = y^3 \cdot v_2 \otimes v_2 = v_2 \otimes v_2,
\end{aligned}$$

which is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore, $\dim \mathcal{B}(\mathcal{O}_y, \phi_0) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_y, \phi_0) = 2$ by [1,Example 31].

For $(\phi_1, U_1)$, the corresponding Yetter-Drinfeld module is $M(\mathcal{O}_y, \phi_1) = \Bbbk G \otimes_{\Bbbk G^y} U_1 = 1 \otimes U_1 \oplus x \otimes U_1$.

Let $v_1 = 1 \otimes u_1$, $v_2 = x \otimes u_1$. The $G$-module structure is

$$
\begin{aligned}
x \cdot v_1 &= x \cdot (1 \otimes u_1) = x \otimes \phi_1(1)(u_1) = x \otimes u_1 = v_2, \\
y \cdot v_1 &= y \cdot (1 \otimes u_1) = 1 \otimes \phi_1(y)(u_1) = q(1 \otimes u_1) = qv_1, \\
x \cdot v_2 &= x \cdot (x \otimes u_1) = 1 \otimes \phi_1(y^2)(u_1) = q^2(1 \otimes u_1) = q^2 v_1, \\
y \cdot v_2 &= y \cdot (x \otimes u_0) = x \otimes \phi_1(y^3)(u_1) = q^3(x \otimes u_1) = q^3 v_2.
\end{aligned}
$$

The $G$-comdule is

$$
\begin{aligned}
\delta(v_1) &= \delta(1 \otimes u_1) = (1 \triangleright y) \otimes v_1 = y \otimes v_1, \\
\delta(v_2) &= \delta(x \otimes u_1) = (x \triangleright y) \otimes v_2 = y^3 \otimes v_2.
\end{aligned}
$$

The braiding structure is

$$
\begin{aligned}
c(v_1 \otimes v_1) &= v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = y \cdot v_1 \otimes v_1 = qv_1 \otimes v_1, \\
c(v_1 \otimes v_2) &= v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = y \cdot v_2 \otimes v_1 = q^3 v_2 \otimes v_1, \\
c(v_2 \otimes v_1) &= v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = y^3 \cdot v_1 \otimes v_2 = q^3 v_1 \otimes v_2, \\
c(v_2 \otimes v_2) &= v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = y^3 \cdot v_2 \otimes v_2 = qv_2 \otimes v_2,
\end{aligned}
$$

which is of diagonal type with braiding matrix

$$
\mathbf{q} = \begin{bmatrix} q & q^3 \\ q^3 & q \end{bmatrix}
$$

Therefore, by going through the list of Heckenberger's classification [34] and Lemma 2.7.1, $\dim \mathcal{B}(\mathcal{O}_y, \phi_1) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_y, \phi_1) = \infty$.

For $(\phi_2, U_2)$, the corresponding Yetter-Drinfeld module is

$$
M(\mathcal{O}_y, \phi_2) = \Bbbk G \otimes_{\Bbbk G^y} U_2 = 1 \otimes U_2 \oplus x \otimes U_2.
$$

Let $v_1 = 1 \otimes u_2, v_2 = x \otimes u_2$. The $\boldsymbol{G}$-module structure is

$$x \cdot v_1 = x \cdot (1 \otimes u_2) = x \otimes \phi_1(1)(u_2) = x \otimes u_2 = v_2,$$
$$y \cdot v_1 = y \cdot (1 \otimes u_2) = 1 \otimes \phi_1(y)(u_2) = q^2(1 \otimes u_2) = q^2 v_1,$$
$$x \cdot v_2 = x \cdot (x \otimes u_2) = 1 \otimes \phi_1(y^2)(u_2) = 1 \otimes u_2 = v_1,$$
$$y \cdot v_2 = y \cdot (x \otimes u_2) = x \otimes \phi_1(y^3)(u_2) = q^6(x \otimes u_2) = q^2 v_2.$$

The $\boldsymbol{G}$-comdule is

$$\delta(v_1) = \delta(1 \otimes u_2) = (1 \rhd y) \otimes v_1 = y \otimes v_1,$$
$$\delta(v_2) = \delta(x \otimes u_2) = (x \rhd y) \otimes v_2 = y^3 \otimes v_2.$$

The braiding structure is

$$c(v_1 \otimes v_1) = v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = y \cdot v_1 \otimes v_1 = q^2 v_1 \otimes v_1,$$
$$c(v_1 \otimes v_2) = v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = y \cdot v_2 \otimes v_1 = q^2 v_2 \otimes v_1,$$
$$c(v_2 \otimes v_1) = v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = y^3 \cdot v_1 \otimes v_2 = q^2 v_1 \otimes v_2,$$
$$c(v_2 \otimes v_2) = v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = y^3 \cdot v_2 \otimes v_2 = q^2 v_2 \otimes v_2,$$

which is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Therefore, by [1,Example 31], $\dim \mathcal{B}(\mathcal{O}_y, \phi_2) = 4$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_y, \phi_2) = 0$.

For $(\phi_3, U_3)$, the corresponding Yetter-Drinfeld module is

$$M(\mathcal{O}_y, \phi_3) = \Bbbk G \otimes_{\Bbbk G^y} U_3 = 1 \otimes U_3 \oplus x \otimes U_3.$$

Let $v_1 = 1 \otimes u_3$, $v_2 = x \otimes u_3$. The $G$-module structure is

$$\begin{aligned}
x \cdot v_1 &= x \cdot (1 \otimes u_3) = x \otimes \phi_3(1)(u_3) = x \otimes u_3 = v_2,\\
y \cdot v_1 &= y \cdot (1 \otimes u_2) = 1 \otimes \phi_3(y)(u_3) = q^3(1 \otimes u_3) = q^3 v_1,\\
x \cdot v_2 &= x \cdot (x \otimes u_3) = 1 \otimes \phi_3(y^2)(u_3) = q^2(1 \otimes u_3) = q^2 v_1,\\
y \cdot v_2 &= y \cdot (x \otimes u_3) = x \otimes \phi_3(y^3)(u_3) = q(x \otimes u_3) = q v_2.
\end{aligned}$$

The $\Bbbk G$-comdule structure is

$$\begin{aligned}
\delta(v_1) &= \delta(1 \otimes u_3) = (1 \triangleright y) \otimes v_1 = y \otimes v_1,\\
\delta(v_2) &= \delta(x \otimes u_3) = (x \triangleright y) \otimes v_2 = y^3 \otimes v_2.
\end{aligned}$$

The braiding structure is

$$\begin{aligned}
c(v_1 \otimes v_1) &= v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = y \cdot v_1 \otimes v_1 = q^3 v_1 \otimes v_1,\\
c(v_1 \otimes v_2) &= v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = y \cdot v_2 \otimes v_1 = q v_2 \otimes v_1,\\
c(v_2 \otimes v_1) &= v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = y^3 \cdot v_1 \otimes v_2 = q v_1 \otimes v_2,\\
c(v_2 \otimes v_2) &= v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = y^3 \cdot v_2 \otimes v_2 = q^3 v_2 \otimes v_2,
\end{aligned}$$

which is of diagonal type with braiding matrix

$$\mathbf{q} = \begin{bmatrix} q^3 & q \\ q & q^3 \end{bmatrix}$$

Therefore, by [1,Theorem 6] and going through the list of Heckenberger's classification, $\dim \mathcal{B}(\mathcal{O}_y, \phi_3) = \infty$, and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_y, \phi_3) = \infty$.

**Proposition 4.4.1.** For any irreducible representation $(\phi, V) \in \mathrm{Irr}(G^y)$, $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_y, \phi) < \infty$ if and only if $\phi = \phi_0$ or $\phi = \phi_2$; and $\dim \mathcal{B}(\mathcal{O}_y, \phi) < \infty$ if and only if $\phi = \phi_2$;

## 4.5 The Nichols algebras $\mathcal{B}(\mathcal{O}_{xy}, \phi)$

For the conjugacy class $\mathcal{O}_{xy} = \{xy, x^3y\}$, $G^{xy} = \{1, xy, x^2, x^3y\} = \langle xy \rangle \cong \mathbb{Z}_4$. As we known, Therefore, all the irrreducible representations are $(\phi_t, U_t), 0 \leqslant t \leqslant 3$, which are all 1-dimensional. Precisely,

$$(xy)^k \cdot u_t = \phi_t((xy)^k)(u_t) = e^{t\pi \mathbf{i}k/2}u_t.$$

where $\mathbf{i} = \sqrt{-1}$.

Clearly, $(\phi_0, U_0)$ is the trivial representation of $G^{xy}$. Since

$$G = G^{xy} \cup yG^{xy},$$

the corresponding Yetter-Drinfeld module is

$$M(\mathcal{O}_{xy}, \phi_0) = \Bbbk G \otimes_{\Bbbk G^{xy}} U_0 = 1 \otimes U_0 \oplus y \otimes U_0.$$

Let $v_1 = 1 \otimes u_0$, $v_2 = y \otimes u_0$. The $G$-module structure is

$$\begin{aligned}
x \cdot v_1 &= x \cdot (1 \otimes u_0) = y \otimes \phi_0(xy)(u_0) = y \otimes u_0 = v_2, \\
y \cdot v_1 &= y \cdot (1 \otimes u_0) = y \otimes \phi_0(1)(u_0) = y \otimes u_0 = v_2, \\
x \cdot v_2 &= x \cdot (y \otimes u_0) = 1 \otimes \phi_0(xy)(u_0) = 1 \otimes u_0 = v_1, \\
y \cdot v_2 &= y \cdot (y \otimes u_0) = 1 \otimes \phi_0((xy)^2)(u_0) = 1 \otimes u_0 = v_1.
\end{aligned}$$

and we have $(xy) \cdot v_1 = v_1, (xy) \cdot v_2 = v_2$.

The $\Bbbk G$-comdule is

$$\begin{aligned}
\delta(v_1) &= \delta(1 \otimes u_0) = (1 \triangleright xy) \otimes v_1 = xy \otimes v_1, \\
\delta(v_2) &= \delta(x \otimes u_0) = (x \triangleright xy) \otimes v_2 = x^3y \otimes v_2 = (xy)^3 \otimes v_2.
\end{aligned}$$

The braiding structure is

$$
\begin{aligned}
c(v_1 \otimes v_1) &= v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = xy \cdot v_1 \otimes v_1 = v_1 \otimes v_1, \\
c(v_1 \otimes v_2) &= v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = xy \cdot v_2 \otimes v_1 = v_2 \otimes v_1, \\
c(v_2 \otimes v_1) &= v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = (xy)^3 \cdot v_1 \otimes v_2 = v_1 \otimes v_2, \\
c(v_2 \otimes v_2) &= v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = (xy)^3 \cdot v_2 \otimes v_2 = v_2 \otimes v_2,
\end{aligned}
$$

which is of diagonal type with braiding matrix

$$
\mathbf{q} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

Therefore $\dim \mathcal{B}(\mathcal{O}_{xy}, \phi_0) = \infty$ and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{xy}, \phi_0) = 2$ by [1,Example 31].

For $(\phi_1, U_1)$, the corresponding Yetter-Drinfeld module is

$$
M(\mathcal{O}_{xy}, \phi_1) = \Bbbk G \otimes_{\Bbbk G^{xy}} U_1 = 1 \otimes U_1 \oplus x \otimes U_1.
$$

Let $v_1 = 1 \otimes u_1$, $v_2 = x \otimes u_1$. The $\Bbbk G$-module structure is

$$
\begin{aligned}
x \cdot v_1 &= x \cdot (1 \otimes u_1) = y \otimes \phi_1(xy)(u_1) = qy \otimes u_1 = qv_2, \\
y \cdot v_1 &= y \cdot (1 \otimes u_1) = y \otimes \phi_1(1)(u_1) = y \otimes u_1 = v_2, \\
x \cdot v_2 &= x \cdot (y \otimes u_1) = 1 \otimes \phi_1(xy)(u_1) = q(1 \otimes u_1) = qv_1, \\
y \cdot v_2 &= y \cdot (y \otimes u_1) = 1 \otimes \phi_1((xy)^2)(u_1) = q^2(1 \otimes u_1) = q^2 v_1
\end{aligned}
$$

and we have $(xy) \cdot v_1 = qv_1, (xy) \cdot v_2 = q^3 v_2$. The $\Bbbk G$-comdule is

$$
\begin{aligned}
\delta(v_1) &= \delta(1 \otimes u_1) = (1 \triangleright xy) \otimes v_1 = xy \otimes v_1, \\
\delta(v_2) &= \delta(y \otimes u_1) = (y \triangleright xy) \otimes v_2 = x^3 y \otimes v_2 = (xy)^3 \otimes v_2.
\end{aligned}
$$

The braiding structure is

$$c(v_1 \otimes v_1) = v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = xy \cdot v_1 \otimes v_1 = qv_1 \otimes v_1,$$
$$c(v_1 \otimes v_2) = v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = xy \cdot v_2 \otimes v_1 = q^3 v_2 \otimes v_1,$$
$$c(v_2 \otimes v_1) = v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = (xy)^3 \cdot v_1 \otimes v_2 = q^3 v_1 \otimes v_2,$$
$$c(v_2 \otimes v_2) = v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = (xy)^3 \cdot v_2 \otimes v_2 = qv_2 \otimes v_2,$$

Therefore $M(\mathcal{O}_{xy}, \phi_1)$ is of diagonal type with braiding matrix

$$\begin{bmatrix} q & q^3 \\ q^3 & q \end{bmatrix}$$

Thus, $\dim \mathcal{B}(\mathcal{O}_{xy}, \phi_1) = \infty$ since it is of Cartan type. Moreover, $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{xy}, \phi_1) = \infty$ by comparing the table by Heckenberger.

For $(\phi_2, U_2)$, the corresponding Yetter-Drinfeld module is

$$M(\mathcal{O}_{xy}, \phi_2) = \Bbbk G \otimes_{\Bbbk G^{xy}} U_2 = 1 \otimes U_2 \oplus x \otimes U_2.$$

Let $v_1 = 1 \otimes u_2$, $v_2 = x \otimes u_2$. The $\Bbbk G$-module structure is

$$x \cdot v_1 = x \cdot (1 \otimes u_1) = y \otimes \phi_1(xy)(u_2) = q^2 y \otimes u_1 = q^2 v_2,$$
$$y \cdot v_1 = y \cdot (1 \otimes u_2) = y \otimes \phi_1(1)(u_2) = y \otimes u_2 = v_2,$$
$$x \cdot v_2 = x \cdot (y \otimes u_2) = 1 \otimes \phi_1(xy)(u_2) = q^2(1 \otimes u_2) = q^2 v_1,$$
$$y \cdot v_2 = y \cdot (y \otimes u_2) = 1 \otimes \phi_1((xy)^2)(u_2) = (1 \otimes u_2) = v_1$$

and we have $(xy) \cdot v_1 = q^2 v_1, (xy) \cdot v_2 = q^2 v_2$. The $\Bbbk G$-comdule is

$$\delta(v_1) = \delta(1 \otimes u_2) = (1 \triangleright xy) \otimes v_1 = xy \otimes v_1,$$
$$\delta(v_2) = \delta(y \otimes u_2) = (y \triangleright xy) \otimes v_2 = x^3 y \otimes v_2 = (xy)^3 \otimes v_2.$$

The braiding structure is

$$\begin{aligned} c(v_1 \otimes v_1) &= v_1^{(-1)} \cdot v_1 \otimes v_1^{(0)} = xy \cdot v_1 \otimes v_1 = q^2 v_1 \otimes v_1, \\ c(v_1 \otimes v_2) &= v_1^{(-1)} \cdot v_2 \otimes v_1^{(0)} = xy \cdot v_2 \otimes v_1 = q^2 v_2 \otimes v_1, \\ c(v_2 \otimes v_1) &= v_2^{(-1)} \cdot v_1 \otimes v_2^{(0)} = (xy)^3 \cdot v_1 \otimes v_2 = q^2 v_1 \otimes v_2, \\ c(v_2 \otimes v_2) &= v_2^{(-1)} \cdot v_2 \otimes v_2^{(0)} = (xy)^3 \cdot v_2 \otimes v_2 = q^2 v_2 \otimes v_2, \end{aligned}$$

Therefore $M(\mathcal{O}_{xy}, \phi_2)$ is of diagonal type with braiding matrix

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Thus, $\dim \mathcal{B}(\mathcal{O}_{xy}, \phi_2) = 4$ and $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{xy}, \phi_2) = 0$ by [1,Example 31]. Therefore, we have proved:

**Proposition 4.5.1.** For any irreducible representation $(\phi, V) \in \mathrm{Irr}(G^{xy})$, $\mathrm{GKdim}\mathcal{B}(\mathcal{O}_{xy}, \phi) < \infty$ if and only if $\phi = \phi_0$ or $\phi = \phi_2$; and $\dim \mathcal{B}(\mathcal{O}_{xy}, \phi) < \infty$ if and only if $\phi = \phi_2$.

## 4.6 Finite GK-dimensional Nichols algebras of semisimple Yetter-Drinfeld modules over $\mathbb{Q}_8$

By the argument above, there are 16 mutually nonisomorphic irreducible Yetter-Drinfeld modules $V$ with $\mathrm{GKdim}\mathcal{B}(V) < \infty$. These are

$$\begin{aligned} &M(\mathcal{O}_1, \rho_i), \quad M(\mathcal{O}_{x^2}, \rho_i), \quad M(\mathcal{O}_x, \phi_0), \quad M(\mathcal{O}_x, \phi_2), \\ &M(\mathcal{O}_y, \phi_0), \quad M(\mathcal{O}_y, \phi_2), \quad M(\mathcal{O}_{xy}, \phi_0), \quad M(\mathcal{O}_{xy}, \phi_2), \end{aligned}$$

where $1 \leqslant i \leqslant 5$.

Now we consider the Nichols algebras of semisimple Yetter-Drinfeld modules over $\mathbb{Q}_8$.

**Proposition 4.6.1.** The following hold:

(1) $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_1, \rho_j)) < \infty$ for all $1 \leqslant i, j \leqslant 5$.

(2) Let $c \in \{x, y, xy\}$.Then $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_c, \phi_i) \oplus M(\mathcal{O}_c, \phi_j)) < \infty$ if and only if $i = j \in \{0, 2\}$.

(3) $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_{x^2}, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_j)) < \infty$ if and only if $1 \leqslant i, j \leqslant 4$, or $i = j = 5$.

(4) Let $c \in \{1, x^2\}$ and $j \in \{0, 2\}$. Then $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_c, \rho_i) \oplus M(\mathcal{O}_x, \phi_j)) < \infty$ if and only if $i \in \{1, 2\}$.

(5) $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_j)) < \infty$ if and only if $1 \leqslant i \leqslant 4$, and $1 \leqslant j \leqslant 5$.

(6) Let $c \in \{1, x^2\}$ and $j \in \{0, 2\}$. $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_c, \rho_i) \oplus M(\mathcal{O}_y, \phi_j)) < \infty$ if $i \in \{1, 3\}$.

(7) Let $c \in \{1, x^2\}$ and $j \in \{0, 2\}$.$\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_c, \rho_i) \oplus M(\mathcal{O}_{xy}, \phi_0)) < \infty$ if $i \in \{1, 4\}$.

PROOF. (1) For $1 \leqslant i, j \leqslant 4$ The braiding matrix of $M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_1, \rho_j)$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore, $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_1, \rho_j)) = 2$. For $1 \leqslant i \leqslant 4$,

The braiding matrix of $M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_1, \rho_5)$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus, $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_1, \rho_5)) = 3$ by [1,Example 27]. In a similar way, we see that the braiding matrix of $M(\mathcal{O}_1, \rho_5) \oplus M(\mathcal{O}_1, \rho_5)$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus, GKdim$\mathcal{B}(M(\mathcal{O}_1, \rho_5) \oplus M(\mathcal{O}_1, \rho_5)) = 4$ by [1,Example 27].

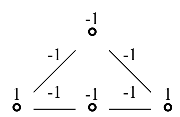
(2) The braiding matrix of $M(\mathcal{O}_x, \phi_0) \oplus M(\mathcal{O}_x, \phi_0)$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Therefore, GKdim$\mathcal{B}(M(\mathcal{O}_x, \phi_0) \oplus M(\mathcal{O}_x, \phi_0)) = 4$. The braiding matrix of $M(\mathcal{O}_x, \phi_2) \oplus M(\mathcal{O}_x, \phi_2)$ is

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

Therefore, GKdim$\mathcal{B}(M(\mathcal{O}_x, \phi_0) \oplus M(\mathcal{O}_x, \phi_0)) = 0$ by [1,Example 27]. For $M(\mathcal{O}_x, \phi_0) \oplus M(\mathcal{O}_x, \phi_2)$, we see the generalized Dynkin diagram is




Therefore, $\mathcal{B}(M(\mathcal{O}_x, \phi_0) \oplus M(\mathcal{O}_x, \phi_2)) = \infty$ by going through the tables of Heckenberger.

The cases when $c \in \{y, xy\}$ are proved in the same way.

(3) Let $1 \leqslant i, j \leqslant 4$. The braiding matrix is

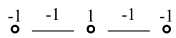
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore, $\mathcal{B}(M(\mathcal{O}_{x^2}, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_j)) = 2$. The braiding matrix of $M(\mathcal{O}_{x^2}, \rho_5) \oplus M(\mathcal{O}_{x^2}, \rho_5)$

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

Therefore, $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_{x^2}, \rho_5) \oplus M(\mathcal{O}_{x^2}, \rho_5)) = 0$ by [1,Example 27].

Let $1 \leqslant i \leqslant 4$. The generalized Dynkin diagram of $M(\mathcal{O}_{x^2}, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_5)$ is




Therefore, $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_{x^2}, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_5)) = \infty$.

(4) Let $c \in \{1, x^2\}$ and $i \in \{1, 2\}$. The braiding matrix of $M(\mathcal{O}_c, \rho_i) \oplus M(\mathcal{O}_x, \phi_0)$ is

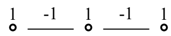
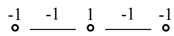
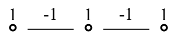
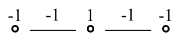
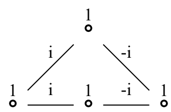
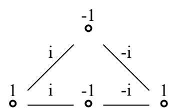
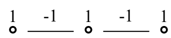
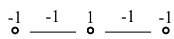
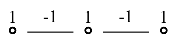
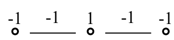
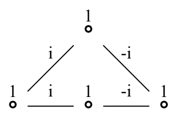
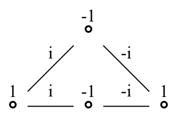
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore, $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_c, \rho_i) \oplus M(\mathcal{O}_x, \phi_0)) = 3$ by [1,Example 27].

The braiding matrix of $M(\mathcal{O}_c, \rho_i) \oplus M(\mathcal{O}_x, \phi_2)$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Therefore, $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_c, \rho_i) \oplus M(\mathcal{O}_x, \phi_0)) < \infty$ [1,Example 27]. For the remaining cases, see the table of generalized Dynkin diagrams.

| Cases | Gener. Dynkin diagrams | GKdim$\mathcal{B}$ |
|---|---|---|
| $M(\mathcal{O}_1, \rho_3) \oplus M(\mathcal{O}_x, \phi_0)$ |  | $\infty$ |
| $M(\mathcal{O}_1, \rho_3) \oplus M(\mathcal{O}_x, \phi_2)$ |  | $\infty$ |
| $M(\mathcal{O}_1, \rho_4) \oplus M(\mathcal{O}_x, \phi_0)$ |  | $\infty$ |
| $M(\mathcal{O}_1, \rho_4) \oplus M(\mathcal{O}_x, \phi_2)$ |  | $\infty$ |
| $M(\mathcal{O}_1, \rho_5) \oplus M(\mathcal{O}_x, \phi_0)$ |  | $\infty$ |
| $M(\mathcal{O}_1, \rho_5) \oplus M(\mathcal{O}_x, \phi_2)$ |  | $\infty$ |
| $M(\mathcal{O}_{x^2}, \rho_3) \oplus M(\mathcal{O}_x, \phi_0)$ |  | $\infty$ |
| $M(\mathcal{O}_{x^2}, \rho_3) \oplus M(\mathcal{O}_x, \phi_2)$ |  | $\infty$ |
| $M(\mathcal{O}_{x^2}, \rho_4) \oplus M(\mathcal{O}_x, \phi_0)$ |  | $\infty$ |
| $M(\mathcal{O}_{x^2}, \rho_4) \oplus M(\mathcal{O}_x, \phi_2)$ |  | $\infty$ |
| $M(\mathcal{O}_{x^2}, \rho_5) \oplus M(\mathcal{O}_x, \phi_0)$ |  | $\infty$ |
| $M(\mathcal{O}_{x^2}, \rho_5) \oplus M(\mathcal{O}_x, \phi_2)$ |  | $\infty$ |

The infiniteness of Gelfand-Kirillov dimension of the corresponding Nichols algebras are achieved by looking through the list in [32,Lemma 9] and by [20,Theorem 1.2]

(5) Let $1 \leqslant i, j \leqslant 4$. Then the braiding matrix of $M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_j)$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

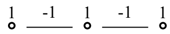
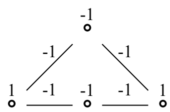
Therefore, $\mathrm{GKdim}\mathcal{B}(M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_j)) = 2$, by [1,Example 27] or [1,Example

31].

For $1 \leqslant i \leqslant 4$, the braiding matrix of $M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_5)$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Therefore, GKdim$\mathcal{B}(M(\mathcal{O}_1, \rho_i) \oplus M(\mathcal{O}_{x^2}, \rho_5)) < \infty$, by [1,Example 27]. For the remaining cases, see the following table.

| Cases | Gener. Dynkin diagrams | GKdim$\mathcal{B}$ |
|---|---|---|
| $M(\mathcal{O}_1, \rho_5) \oplus M(\mathcal{O}_{x^2}, \rho_i), 1 \leqslant i \leqslant 4$ |  | $\infty$ |
| $M(\mathcal{O}_1, \rho_5) \oplus M(\mathcal{O}_{x^2}, \rho_5)$ |  | $\infty$ |

(6) The braiding matrix of $M(\mathcal{O}_1, \rho_1) \oplus M(\mathcal{O}_y, \phi_0)$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Therefore, GKdim$\mathcal{B}(M(\mathcal{O}_1, \rho_1) \oplus M(\mathcal{O}_y, \phi_0)) = 3$, by [1,Example 27].

The braiding matrix of $M(\mathcal{O}_1, \rho_1) \oplus M(\mathcal{O}_y, \phi_2)$ is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Therefore, GKdim$\mathcal{B}(M(\mathcal{O}_1, \rho_1) \oplus M(\mathcal{O}_y, \phi_2)) < \infty$, by [1,Example 27].

The cases for GKdim$\mathcal{B}(M(\mathcal{O}_{x^2}, \rho_i) \oplus M(\mathcal{O}_y, \phi_j))$ are proved in a similar way.

(7) Similar to (6). ■

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# 攻读博士学位期间研究成果

## 攻读博士学位期间发表的学术论文

[1] Zhang, Y. L. Finite GK-dimensional Nichols Algebras over the Infinite Dihedral Group[J]. Algebras and Representation Theory, **2023**. DOI: https://doi.org/10.1007/s10468-023-10213-1.

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