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Post-doctoral Research Report

Extensions of finite irreducible modules for rank two Lie conformal algebras

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Fundamental Mathematics

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Contents

研究报告题目: 秩为二李共形代数有限不可约模的扩张研究

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摘　　要

本研究基于 Kac 提出的李共形代数公理化体系, 该理论从代数角度刻画了二维共形场论中手征场的算子积展开（OPE）的奇异部分. 李共形代数理论不仅与顶点代数、具有局部性的无限维李代数以及哈密顿形式体系等数学理论密切相关, 在非线性演化方程等物理问题中也有重要应用, 这使得对其结构理论和表示理论的研究具有深刻的数学与物理意义. Biswal 等人系统研究了秩为二的李共形代数, 将其分为半单李共形代数、可解李共形代数以及两类非半单非可解李共形代数, 并完整刻画了它们的代数结构. 在相应有限不可约模的分类工作完成的基础上, 本文进一步研究了秩为二李共形代数不可约模的扩张问题. 本报告共包含五章内容.

第一章介绍了研究背景以及本文的主要结果.

第二章回顾了一些必要的基本定义、相关符号和已有的重要结论.

第三章首先考虑了半单的秩为二李共形代数. 基于 Kac 等人有限半单李共形代数的分类结果, 结合半单李代数的性质, 可知秩为二的半单李共形代数同构于两个 Virasoro 共形代数的直和. 通过求解多元多项式方程组, 我们完整分类了这类代数的不可约模扩张. 随后, 我们采用类似方法研究了可解的秩为二李共形代数, 通过引入积分技巧并利用系数矩阵及斜对称矩阵的性质, 最终给出了相应的分类结果.

第四章集中讨论了第一类非半单非可解的秩为二李共形代数不可约模的扩张分类问题.

第五章重点研究了第二类非半单非可解的秩为二李共形代数. 特别地, 当 $Q=0$ 时, 这类代数即为 $\mathcal{W}(a,b)$ 代数, 其模扩张分类已由 Luo 等人完成. 当 $Q\neq 0$ 时, 该类代数结构特征可细分为五种子情形, 我们给出了各类情形的不可约模扩张的完整分类.

关键词: 共形代数; 李共形代数; 半单; 可解; 非半单非可解; 共形模; 不可约模; 扩张; 秩为二.

REPORT: Extensions of finite irreducible modules for rank two Lie conformal algebras

SPECIALIZATION: Fundamental Mathematics

POSTGRADUATE: Mengjun Wang

MENTOR: Professor Gongxiang Liu

Abstract

This study is based on the axiomatic system of Lie conformal algebras proposed by Kac, which provides an algebraic characterization of the singular part of the operator product expansion (OPE) for chiral fields in two-dimensional conformal field theory. The theory of Lie conformal algebras is not only closely related to mathematical theories such as vertex algebras, infinite-dimensional Lie algebras with locality, and Hamiltonian formal systems, but also has significant applications in physical problems like nonlinear evolution equations. This makes the study of their structural and representation theories profoundly meaningful both mathematically and physically. Biswal et al. systematically investigated rank-two Lie conformal algebras, classifying them into semisimple Lie conformal algebras, solvable Lie conformal algebras, and two types of non-semisimple and non-solvable Lie conformal algebras, while completely characterizing their algebraic structures. Building upon the completed classification of finite irreducible modules, this paper further studies the extension problems of irreducible modules for rank two Lie conformal algebras. This report consists of five chapters.

In Chapter 1, we introduce the research background and the main results of this paper.

In Chapter 2, we review necessary fundamental definitions, relevant notations, and existing important results.

In Chapter 3, we first consider semisimple rank two Lie conformal algebras. Based on the classification results of finite semisimple Lie conformal algebras by Kac et al. and the properties of semisimple Lie algebras, it is shown that semisimple rank two Lie conformal algebras are isomorphic to the direct sum of two Virasoro conformal algebras. By solving systems of multivariate polynomial equations, we provide a complete classification of the extensions of irreducible modules for this class of algebras. Subsequently, using similar methods, we study solvable rank two Lie conformal algebras.

By introducing integration techniques and utilizing properties of coefficient matrices and skew-symmetric matrices, we ultimately establish the corresponding classification results.

In Chapter 4, we focus on the classification of extensions of irreducible modules for non-semisimple and non-solvable rank two Lie conformal algebras of Type I.

In Chapter 5, we investigate non-semisimple and non-solvable rank two Lie conformal algebras of Type II. Notably, when $Q = 0$, this class of algebras corresponds precisely to the $\mathcal{W}(a, b)$ algebras, whose module extension classification has been completed by Luo et al. When $Q \neq 0$, the structures of these algebras can be further divided into five subcases, for each of which we provide a complete classification of the extensions of irreducible modules.

Keywords: Conformal algebra; Lie conformal algebra; Semisimple; Solvable; Non-semisimple and non-solvable; Conformal module; Irreducible module; Extension; Rank two.

Chapter 1 Introduction

§1.1 Background

Since vertex algebras were introduced by Borcherds [5] in 1986, they have found extensive applications in both mathematics and physics [3, 16]. In particular, the locality condition between fields in the definition of vertex algebras characterizes the independence of measurements at spacelike separated points, which carries significant physical meaning. However, the complex operational rules involved make the study of related theories relatively challenging.

Through formal Fourier transforms, the investigation of locality between two fields can be transformed into an examination of the singular part of their operator product expansion (OPE). In the 1990s, Kac [9, 22] introduced an axiomatic definition of the Lie conformal algebra, which gives an algebraic description of the singular part of the operator product extansion (OPE) of the chiral fields in 2-dimensional conformal field theory. In addition to being closely related to vertex algebra and conformal field theory, the theory of Lie conformal algebras is also closely associated with infinite-dimensional Lie algebras [1], Hamiltonian formal systems of nonlinear evolution equations [2], and quantum physics [10], and thus has received more attention in recent years.

A conformal algebra is called finite if it is a finitely generated $\mathbb{C}[\partial]$-module, and the rank of a finite conformal algebra is just its rank as a $\mathbb{C}[\partial]$-module. It was shown in [4] that a rank two conformal algebra is one of the following four types up to isomorphism:

(i) a semisimple conformal algebra;

(ii) a solvable conformal algebra;

(iii) the direct sum of a commutative Lie conformal algebra of rank one and the Virasoro conformal algebra Vir (called in this paper the Lie conformal algebra of Type I);

(iv) and what we called the Lie conformal algebra of Type II(see case (2ii) in Proposition 2.1.9).

The classification of their finitely irreducible conformal modules can be found in [6, 32]. Even for finite semisimple Lie conformal algebras, however, conformal modules of Lie conformal algebras are generally not completely reducible. Therefore, solving the extension problem plays an essential role in studying the representation theory of conformal algebras. For instance, extensions of finite irreducible conformal modules over the Virasoro, the current, the Neveu-Schwarz and the semi-direct sum of the Virasoro and the current conformal algebras were investigated by Cheng, Kac and Wakimoto in [7,8], that over finite Lie conformal algebras of planar Galilean type were studied in [14], that over supercurrent conformal algebras were classified by Lam in [17], that over Lie conformal algebras $\mathcal{W}(a,b,r)$ were discussed in [24], and that over the Schrödinger-Virasoro conformal algebras were considered by Yuan and Ling in [35]. In this study, extensions of finite irreducible modules over rank two conformal algebras are characterized by dealing with certain polynomial equations induced by corresponding module actions.

§1.2 Main results

It was proved in [9] that any finite semisimple Lie conformal algebra is the direct sum of the following algebras:

$$Cur(\mathfrak{g}), \quad Vir \ltimes Cur(\mathfrak{g}), \quad Vir \oplus Vir,$$

where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra. Let $\mathcal{R}$ be a conformal algebra of rank two. Since there exists no semisimple Lie algebra of dimension less than 3, we only need to focus on $Vir \oplus Vir$ for the semisimple case and the results are listed in Theorems 3.1.2, 3.1.4 and 3.1.7.

Theorem 3.1.2 *Let $\mathcal{R}$ be a direct sum of two Virasoro conformal algebras. Then nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\delta,\alpha,\beta} \longrightarrow 0 \tag{1.1}$$

exist only when $(\delta_1,\delta_2) = (1,0), \alpha_1 \in \{1,2\}, \beta_1 + \eta = 0$ or $(\delta_1,\delta_2) = (0,1), \alpha_2 \in \{1,2\}, \beta_2 + \eta = 0$. In these cases, there exists a unique (up to a scalar) nontrivial extension, i.e. $dim(Ext(V_{\delta,\alpha,\beta}, \mathbb{C}c_\eta)) = 1$. Moreover, they are given (up to equivalence) *by*

$$\mathcal{R}_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta,$$
$$A_\lambda v = \delta_1(\partial + \alpha_1\lambda + \beta_1)v + f(\lambda)c_\eta, \quad B_\lambda v = \delta_2(\partial + \alpha_2\lambda + \beta_2)v + g(\lambda)c_\eta. \tag{1.2}$$

The values of $\delta_i, \alpha_i, \beta_i, i = 1, 2$ and η, along with the corresponding polynomials $f(\lambda)$ and $g(\lambda)$ giving rise to nontrivial extensions, are listed as follows:

(i) If $\delta_1 = 1, \alpha_1 \in \{1, 2\}, \beta_1 + \eta = 0$, then $\delta_2 = 0, g(\lambda) = 0, 0 \neq \alpha_2, \beta_2 \in \mathbb{C}$ and

$$f(\lambda) = \begin{cases} s_1\lambda^2, & \alpha_1 = 1, \\ s_2\lambda^3, & \alpha_1 = 2, \end{cases}$$

with nonzero constants s_1, s_2.

(ii) If $\delta_2 = 1, \alpha_2 \in \{1, 2\}, \beta_2 + \eta = 0$, then $\delta_1 = 0, f(\lambda) = 0, 0 \neq \alpha_1, \beta_1 \in \mathbb{C}$ and

$$g(\lambda) = \begin{cases} t_1\lambda^2, & \alpha_2 = 1, \\ t_2\lambda^3, & \alpha_2 = 2, \end{cases}$$

with nonzero constants t_1, t_2.

Theorem 3.1.4 *Let $\mathcal{R}$ be a direct sum of two Virasoro conformal algebras. Then nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow V_{\delta,\alpha,\beta} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0 \tag{1.3}$$

exist only when $\delta_1 = 1, \alpha_1 = 1, \beta_1 + \eta = 0$ or $\delta_2 = 1, \alpha_2 = 1, \beta_2 + \eta = 0$. In these cases, there exists a unique (up to a scalar) nontrivial extension, i.e. $dim(Ext(\mathbb{C}c_\eta, V_{\delta,\alpha,\beta})) = 1$. Moreover, they are given (up to equivalence) by

$$A_\lambda v = \delta_1(\partial + \alpha_1\lambda + \beta_1)v, \quad B_\lambda v = \delta_2(\partial + \alpha_2\lambda + \beta_2)v,$$
$$A_\lambda c_\eta = f(\partial, \lambda)v, \quad B_\lambda c_\eta = g(\partial, \lambda)v, \quad \partial c_\eta = \eta c_\eta + h(\partial)v. \tag{1.4}$$

The values of $\delta_i, \alpha_i, \beta_i, i = 1, 2$ and η along with the corresponding polynomials $f(\partial, \lambda)$, $g(\partial, \lambda)$ and $h(\partial)$ giving rise to nontrivial extensions, are listed as follows:

(i) If $\delta_1 = 1, \alpha_1 = 1, \beta_1 + \eta = 0$, *then* $\delta_2 = 0, g(\partial, \lambda) = 0, \alpha_2, \beta_2 \in \mathbb{C}$ *and* $f(\partial, \lambda) = h(\partial) = s$ *with nonzero constant* s.

(ii) If $\delta_2 = 1, \alpha_2 = 1, \beta_2 + \eta = 0$, *then* $\delta_1 = 0, f(\partial, \lambda) = 0, \alpha_1, \beta_1 \in \mathbb{C}$ *and* $g(\partial, \lambda) = h(\partial) = t$ *with nonzero constant* t.

Theorem 3.1.7 *Let* $\mathcal{R}$ *be a direct sum of two Virasoro conformal algebras. Then nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow V_{\delta,\alpha,\beta} \longrightarrow E \longrightarrow V_{\bar{\delta},\bar{\alpha},\bar{\beta}} \longrightarrow 0 \tag{1.5}$$

only exist in the following cases. Moreover, they are given (up to equivalence) by

$$\begin{aligned} &A_\lambda v = \delta_1(\partial + \alpha_1\lambda + \beta_1)v, \quad B_\lambda v = \delta_2(\partial + \alpha_2\lambda + \beta_2)v, \\ &A_\lambda \bar{v} = \bar{\delta}_1(\partial + \bar{\alpha}_1\lambda + \bar{\beta}_1)\bar{v} + f(\partial, \lambda)v, \quad B_\lambda \bar{v} = \bar{\delta}_2(\partial + \bar{\alpha}_2\lambda + \bar{\beta}_2)\bar{v} + g(\partial, \lambda)v. \end{aligned} \tag{1.6}$$

The value of $\delta_i, \bar{\delta}_i, \alpha_i, \bar{\alpha}_i, \beta_i, \bar{\beta}_i, i = 1, 2$ *and the corresponding polynomials* $f(\partial, \lambda)$ *and* $g(\partial, \lambda)$ *giving rise to nontrivial extensions, are listed as follows:*

1. *In the case that* $\delta_1 = \bar{\delta}_1 = 1, \delta_2 = \bar{\delta}_2 = 0$, $\alpha_2, \beta_2, \bar{\alpha}_2, \bar{\beta}_2 \in \mathbb{C}, g(\partial, \lambda) = 0, \beta_1 = \bar{\beta}_1, \bar{\alpha}_1 - \alpha_1 \in \{0, 1, 2, 3, 4, 5, 6\}, \alpha_1, \bar{\alpha}_1 \neq 0$ *and*

   (i) $\bar{\alpha}_1 = \alpha_1$, $f(\partial, \lambda) = s_0 + s_1\lambda$, *where* $(s_0, s_1) \neq (0, 0)$.

   (ii) $\bar{\alpha}_1 - \alpha_1 = 2$, $f(\partial, \lambda) = s\lambda^2(2(\partial + \beta_1) + \lambda)$, *where* $s \neq 0$.

   (iii) $\bar{\alpha}_1 - \alpha_1 = 3$, $f(\partial, \lambda) = s(\partial + \beta_1)\lambda^2(\partial + \beta_1 + \lambda)$, *where* $s \neq 0$.

   (iv) $\bar{\alpha}_1 - \alpha_1 = 4$, $f(\partial, \lambda) = s\lambda^2(4(\partial + \beta_1)^3 + 6(\partial + \beta_1)^2\lambda - (\partial + \beta_1)\lambda^2 + \alpha_1\lambda^3)$, *where* $s \neq 0$.

   (v) $\bar{\alpha}_1 = 1$ *and* $\alpha_1 = -4$, $f(\partial, \lambda) = s((\partial+\beta_1)^4\lambda^2 - 10(\partial+\beta_1)^2\lambda^4 - 17(\partial+\beta_1)\lambda^5 - 8\lambda^6)$, *where* $s \neq 0$.

   (vi) $\bar{\alpha}_1 = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha_1 = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$. $f(\partial, \lambda) = s((\partial + \beta_1)^4\lambda^3 - (2\alpha_1 + 3)(\partial + \beta_1)^3\lambda^4 - 3\alpha_1(\partial + \beta_1)^2\lambda^5 - (3\alpha_1 + 1)(\partial + \beta_1)\lambda^6 - (\alpha_1 + \frac{9}{28})\lambda^7)$, *where* $s \neq 0$.

   The value of $dim(Ext(V_{\bar{\delta},\bar{\alpha},\bar{\beta}}, V_{\delta,\alpha,\beta}))$ *is 2 in subcase (i), and 1 in subcases (ii)-(vi).*

2. *In the case that* $\delta_1 = \bar{\delta}_1 = 0, \delta_2 = \bar{\delta}_2 = 1$, $\alpha_1, \beta_1, \bar{\alpha}_1, \bar{\beta}_1 \in \mathbb{C}, f(\partial, \lambda) = 0, \beta_2 = \bar{\beta}_2, \bar{\alpha}_2 - \alpha_2 \in \{0, 1, 2, 3, 4, 5, 6\}, \alpha_2, \bar{\alpha}_2 \neq 0$, *and*

(i) $\bar{\alpha}_2 = \alpha_2$, $g(\partial, \lambda) = t_0 + t_1\lambda$, *where* $(t_0, t_1) \neq (0, 0)$.

(ii) $\bar{\alpha}_2 - \alpha_2 = 2$, $g(\partial, \lambda) = t\lambda^2(2(\partial + \beta_2) + \lambda)$, *where* $t \neq 0$.

(iii) $\bar{\alpha}_2 - \alpha_2 = 3$, $g(\partial, \lambda) = t(\partial + \beta_2)\lambda^2(\partial + \beta_2 + \lambda)$, *where* $t \neq 0$.

(iv) $\bar{\alpha}_2 - \alpha_2 = 4$, $g(\partial, \lambda) = t\lambda^2(4(\partial + \beta_2)^3 + 6(\partial + \beta_2)^2\lambda - (\partial + \beta_2)\lambda^2 + \alpha_2\lambda^3)$, *where* $t \neq 0$.

(v) $\bar{\alpha}_2 = 1$ *and* $\alpha_2 = -4$, $g(\partial, \lambda) = t((\partial+\beta_2)^4\lambda^2 - 10(\partial+\beta_2)^2\lambda^4 - 17(\partial+\beta_2)\lambda^5 - 8\lambda^6)$, *where* $t \neq 0$.

(vi) $\bar{\alpha}_2 = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha_2 = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$, $g(\partial, \lambda) = t((\partial + \beta_2)^4\lambda^3 - (2\alpha_2 + 3)(\partial + \beta_2)^3\lambda^4 - 3\alpha_2(\partial + \beta_2)^2\lambda^5 - (3\alpha_2 + 1)(\partial + \beta_2)\lambda^6 - (\alpha_2 + \frac{9}{28})\lambda^7)$, *where* $t \neq 0$.

The value of $dim(Ext(V_{\bar{\delta},\bar{\alpha},\bar{\beta}}, V_{\delta,\alpha,\beta}))$ *is 2 in subcase (i), and 1 in subcases (ii)-(vi).*

The nontrivial extensions for solvable rank two conformal algebras can be seen in Theorems 3.2.3, 3.2.5 and 3.2.10, while those for non-semisimple and non-solvable Lie conformal algebras of Type I are described in Theorems 4.1.2, 4.2.2 and 4.3.2.

Theorem 3.2.3 *For a solvable rank two Lie conformal algebra* $\mathcal{R}$*, nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\phi_A,\phi_B} \longrightarrow 0 \tag{1.7}$$

exist only if $p(\lambda) \neq 0$ *and* $Q_1(\partial, \lambda) = 0$*. Moreover, they are given (up to equivalence) by*

$$\begin{aligned} &\mathcal{R}_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta, \\ &A_\lambda v = \phi_A(\lambda)v + f(\lambda)c_\eta, \quad B_\lambda v = \phi_B(\lambda)v + g(\lambda)c_\eta. \end{aligned} \tag{1.8}$$

The values of η *along with the corresponding polynomials* $\phi_A(\lambda), \phi_B(\lambda), f(\lambda)$ *and* $g(\lambda)$ *giving rise to nontrivial extensions, are listed as follows:* $\eta \in \mathbb{C}$, $\phi_A(\lambda) = -p(\lambda)$, $\phi_B(\lambda) = 0$, $f(\lambda) = 0$ *and* $g(\lambda)$ *is a nonzero constant. Thus* $dim(Ext(V_{\phi_A,\phi_B}, \mathbb{C}c_\eta)) = \delta_{\phi_A(\lambda)+p(\lambda),0}$.

Theorem 3.2.5 *For a solvable rank two Lie conformal algebra* $\mathcal{R}$*, nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow V_{\phi_A,\phi_B} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0 \tag{1.9}$$

do not exist, that is, $dim(Ext(\mathbb{C}c_\eta, V_{\phi_A,\phi_B})) = 0$.

Theorem 3.2.10 *For a solvable rank two Lie conformal algebra* $\mathcal{R}$*, nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow V_{\phi_A,\phi_B} \longrightarrow E \longrightarrow V_{\bar{\phi}_A,\bar{\phi}_B} \longrightarrow 0 \tag{1.10}$$

always exist. Moreover, they are given (up to equivalence) by

$$\begin{aligned} &A_\lambda v = \phi_A(\lambda)v, \quad B_\lambda v = \phi_B(\lambda)v, \\ &A_\lambda \bar{v} = \bar{\phi}_A(\lambda)\bar{v} + f(\partial,\lambda)v, \quad B_\lambda \bar{v} = \bar{\phi}_B(\lambda)\bar{v} + g(\partial,\lambda)v. \end{aligned} \tag{1.11}$$

The corresponding polynomials $\phi_A(\lambda), \phi_B(\lambda), \bar{\phi}_A(\lambda), \bar{\phi}_B(\lambda), f(\partial,\lambda)$ *and* $g(\partial,\lambda)$ *giving rise to nontrivial extensions, are listed as follows:*

1. *In the case that* $p(\lambda) = Q_1(\partial,\lambda) = 0$.

   (i) *If* $\phi_A(\lambda) = \bar{\phi}_A(\lambda) = 0, \phi_B(\lambda) = \bar{\phi}_B(\lambda) \neq 0$*, then* $f(\partial,\lambda) = s(\lambda), g(\partial,\lambda) = t(\lambda)$*, where* s, t *are polynomials, and either* $s \neq 0$ *or* $t(\lambda)$ *is not a scalar multiple of* $\lambda\phi_B(\lambda)$.

   (ii) *If* $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, \phi_B(\lambda) = \bar{\phi}_B(\lambda) = 0$*, then* $f(\partial,\lambda) = s(\lambda), g(\partial,\lambda) = t(\lambda)$*, where* s, t *are polynomials, and either* $t \neq 0$ *or* $s(\lambda)$ *is not a scalar multiple of* $\lambda\phi_A(\lambda)$.

   (iii) *If* $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, \phi_B(\lambda) = \bar{\phi}_B(\lambda) \neq 0$*, then* $f(\partial,\lambda) = s(\lambda), g(\partial,\lambda) = t(\lambda)$*, where* s, t *are polynomials, and* $s(\lambda), t(\lambda)$ *are not the same scalar multiple of* $\lambda\phi_A(\lambda), \lambda\phi_B(\lambda)$ *respectively.*

2. *In the case that* $p(\lambda) = 0, Q_1(\partial,\lambda) \neq 0$*, we always have* $\phi_B(\lambda) = \bar{\phi}_B(\lambda) = 0$.

   (i) *If* $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, g(\partial,\lambda) = 0$*, then* $f(\partial,\lambda) = s(\lambda)$*, where the polynomial* $s(\lambda)$ *is not a scalar multiple of* $\lambda\phi_A(\lambda)$.

   (ii) *If* $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, g(\partial,\lambda) = t(\lambda) \neq 0$ *such that the coefficient matrix* $M = \{q_{ij}\}$ *of* $Q_1(-\lambda-\mu,\lambda)t(\lambda+\mu)$ *is of rank 2 and for* $q_{i_0,j_0} \neq 0$, $\phi_A(\lambda) = \frac{a_{i0}}{f_{i_0,j_0}}(\sum_k f_{k,j_0}\lambda^k) - \frac{a_{j_0}}{f_{i_0,j_0}}(\sum_k f_{k,i_0}\lambda^k)$ *with the coefficients* a_{i_0-1}, a_{j_0-1}

of $\lambda^{i_0-1}, \lambda^{j_0-1}$ *in* ϕ_A *are not all 0, then*

$$f(\partial,\lambda)=\begin{cases}-\frac{1}{a_{i_0-1}}(\sum_k q_{k,j_0}\lambda^k)\partial+s(\lambda), & \text{if } a_{i_0-1}\neq 0,\\ -\frac{1}{a_{j_0-1}}(\sum_k q_{k,i_0}\lambda^k)\partial+s(\lambda), & \text{if } a_{i_0-1}=0,\end{cases}$$

where s *is a polynomial.*

3. *In the case that* $p(\lambda)\neq 0, Q_1(\partial,\lambda)=0$*, we have* $\phi_B(\lambda)=\bar{\phi}_B(\lambda)=0$.

   (i) *If* $\phi_A(\lambda)=\bar{\phi}_A(\lambda)\neq 0$*, then* $f(\partial,\lambda)=s(\lambda), g(\partial,\lambda)=0$*, where* s_1, s_2 *are polynomials, and* $s(\lambda)$ *is not a scalar multiple of* $\lambda\phi_A(\lambda)$.

   (ii) *If* $\phi_A(\lambda)\neq\bar{\phi}_A(\lambda)$*, then* $f(\partial,\lambda)=0, \phi_A(\lambda)-\bar{\phi}_A(\lambda)=p(\lambda)$*, and*

$$g(\partial,\lambda)=\begin{cases}k_1(\partial+\frac{1}{r}\lambda)+k_2, & p(\lambda)=r\phi_A(\lambda) \text{ and } r\neq 1,\\ k_1, & p(\lambda) \text{ is not a scalar multiple of } \phi_A(\lambda),\end{cases}$$

   where $k_1, k_2\in\mathbb{C}$ *and* $g(\partial,\lambda)\neq 0$.

The space of $Ext(V_{\bar{\phi}_A,\bar{\phi}_B}, V_{\phi_A,\phi_B})$ *is of infinite dimension in all of the above subcases but (3)-(ii).*

Theorem 4.1.2 *For a rank two Lie conformal algebra* $\mathcal{R}$ *that is of Type I, nontrivial extensions of finite irreducible conformal modules of the form*

$$0\longrightarrow\mathbb{C}c_\eta\longrightarrow E\longrightarrow V_{\alpha,\beta,\phi}\longrightarrow 0 \tag{1.12}$$

exist only when $(\delta_1,\delta_2)=(1,0), \alpha\in\{1,2\}, \beta+\eta=0$*. Moreover, they are given (up to equivalence) by*

$$\begin{aligned}&\mathcal{R}_\lambda c_\eta=0,\quad \partial c_\eta=\eta c_\eta\\ &A_\lambda v=\delta_1(\partial+\alpha\lambda+\beta)v+f(\lambda)c_\eta,\quad B_\lambda v=\delta_2\phi(\lambda)v+g(\lambda)c_\eta.\end{aligned} \tag{1.13}$$

The values of η*, along with the corresponding polynomials* $f(\lambda)$ *and* $g(\lambda)$ *giving rise to* *nontrivial extensions, are listed as follows:* $g(\lambda)=0$ *and*

$$f(\lambda)=\begin{cases} s_1\lambda^2, & \alpha=1,\\ s_2\lambda^3, & \alpha=2,\end{cases}$$

with nonzero constants s_1, s_2. *In these cases,* $dim(Ext(V_{\alpha,\beta,\phi},\mathbb{C}c_\eta))=1$.

Theorem 4.2.2 *For a rank two Lie conformal algebra* $\mathcal{R}$ *that is of Type I, nontrivial extensions of finite irreducible conformal modules of the form*

$$0\longrightarrow V_{\alpha,\beta,\phi}\longrightarrow E\longrightarrow \mathbb{C}c_\eta\longrightarrow 0 \tag{1.14}$$

exist only when $\delta_1=1,\alpha=1,\beta+\eta=0$. *Moreover, the space of* $Ext(\mathbb{C}c_\eta, V_{\alpha,\beta,\phi})$ *is 1-dimensional, and the unique nontrivial extension is given (up to equivalence) as follows:* $\delta_2=0, g(\partial,\lambda)=0$ *and* $f(\partial,\lambda)=h(\partial)=s$ *with nonzero constant* s.

Theorem 4.3.2 *For a rank two Lie conformal algebra* $\mathcal{R}$ *that is of Type I, nontrivial extensions of finite irreducible conformal modules of the form*

$$0\longrightarrow V_{\alpha,\beta,\phi}\longrightarrow E\longrightarrow V_{\bar\alpha,\bar\beta,\bar\phi}\longrightarrow 0 \tag{1.15}$$

exist only when $(\delta_1,\delta_2)=(\bar{\delta_1},\bar{\delta_2})$. *Moreover, they are given (up to equivalence) by*

$$\begin{aligned} &A_\lambda v=\delta_1(\partial+\alpha\lambda+\beta)v, \quad B_\lambda v=\delta_2\phi(\lambda)v,\\ &A_\lambda \bar v=\bar\delta_1(\partial+\bar\alpha\lambda+\bar\beta)\bar v+f(\partial,\lambda)v, \quad B_\lambda\bar v=\bar\delta_2\bar\phi(\lambda)\bar v+g(\partial,\lambda)v. \end{aligned} \tag{1.16}$$

The value of $\delta_i,\bar\delta_i, i=1,2,\alpha,\bar\alpha,\beta,\bar\beta$, *and the corresponding polynomials* $\phi(\lambda),\bar\phi(\lambda), f(\partial,\lambda)$ *and* $g(\partial,\lambda)$ *giving rise to nontrivial extensions, are listed as follows:*

1. *In the case that* $(\delta_1,\delta_2)=(\bar{\delta_1},\bar{\delta_2})=(1,0)$, $g=0,\beta=\bar\beta,\bar\alpha-\alpha\in\{0,1,2,3,4,5,6\},\alpha,\bar\alpha\neq 0$, *and*

   (i) $\bar\alpha=\alpha$, $f(\partial,\lambda)=s_0+s_1\lambda$, *where* $(s_0,s_1)\neq(0,0)$.

   (ii) $\bar\alpha-\alpha=2$, $f(\partial,\lambda)=s\lambda^2(2(\partial+\beta)+\lambda)$, *where* $s\neq 0$.

   (iii) $\bar\alpha-\alpha=3$, $f(\partial,\lambda)=s(\partial+\beta)\lambda^2((\partial+\beta)+\lambda)$, *where* $s\neq 0$.

   (iv) $\bar\alpha-\alpha=4$, $f(\partial,\lambda)=s\lambda^2(4(\partial+\beta)^3+6(\partial+\beta)^2\lambda-(\partial+\beta)\lambda^2+\alpha_1\lambda^3)$, *where* $s\neq 0$.

(v) $\bar{\alpha}=1$ *and* $\alpha=-4$, $f(\partial,\lambda)=s((\partial+\beta)^4\lambda^2-10(\partial+\beta)^2\lambda^4-17(\partial+\beta)\lambda^5-8\lambda^6)$, *where* $s\neq 0$.

(vi) $\bar{\alpha}=\frac{7}{2}\pm\frac{\sqrt{19}}{2}$ *and* $\alpha=-\frac{5}{2}\pm\frac{\sqrt{19}}{2}$, $f(\partial,\lambda)=s((\partial+\beta)^4\lambda^3-(2\alpha+3)(\partial+\beta)^3\lambda^4-3\alpha(\partial+\beta)^2\lambda^5-(3\alpha+1)(\partial+\beta)\lambda^6-(\alpha+\frac{9}{28})\lambda^7)$, *where* $s\neq 0$.

The value of $dim(Ext(V_{\bar{\alpha},\bar{\beta},\bar{\phi}},V_{\alpha,\beta,\phi}))$ *is 2 in subcase (i), and 1 in subcases (ii)-(vi).*

2. *In the case that* $(\delta_1,\delta_2)=(\bar{\delta_1},\bar{\delta_2})=(0,1)$, $\phi(\lambda)=\bar{\phi}(\lambda), f(\partial,\lambda)=0, g(\partial,\lambda)=t(\lambda)$ *with polynomials* t *and* $t(\lambda)$ *is not a scalar multiple of* $\lambda\phi(\lambda)$. *Then the space* $Ext(V_{\bar{\alpha},\bar{\beta},\bar{\phi}},V_{\alpha,\beta,\phi})$ *is infinite-dimensional.*

If $\mathcal{R}$ is a non-semisimple and non-solvable Lie conformal algebra of Type II, one can find a basis $\{A,B\}$ such that

$$[A_\lambda A]=(\partial+2\lambda)A+Q(\partial,\lambda)B,\quad [A_\lambda B]=(\partial+a\lambda+b)B,\quad [B_\lambda B]=0,$$

where $a,b\in\mathbb{C}$ and $Q(\partial,\lambda)$ is some skew-symmetric polynomial depending on a,b, i.e. $Q(\partial,\lambda)=-Q(\partial,-\partial-\lambda)$. If $Q(\partial,\lambda)=\beta(\partial+2\lambda), a=1, b=0$, $\mathcal{R}$ is the algebra called $\mathcal{L}(\beta)$ in [28]. If $Q(\partial,\lambda)=0$, $\mathcal{R}$ is just the Lie conformal algebra $\mathcal{W}(a,b)$ whose extension problem has been investigated in [20, 21]. Particularly, $\mathcal{W}(1-b,0)$ is the Lie conformal algebra $\mathcal{W}(b)$ in [19] and $\mathcal{W}(1,0)$ is just the Heisenberg-Virasoro Lie conformal algebra in [18]. So in this case we consider $\mathcal{R}$ under the condition that $Q(\partial,\lambda)\neq 0$ and the results can be found in Theorems 5.2.2, 5.2.4 and 5.2.6. Fixed $a=1,b=0,Q(\partial,\lambda)=\partial+2\lambda$, our results are consistent with those mentioned in [36].
Theorem 5.2.2 *For a rank two Lie conformal algebra* $\mathcal{R}$ *that is of Type II with* $Q\neq 0$, *nontrivial extensions of finite irreducible conformal modules of the form*

$$0\longrightarrow \mathbb{C}c_\eta\longrightarrow E\longrightarrow V_{\alpha,\beta}\longrightarrow 0 \tag{1.17}$$

exist only if $\beta+\eta=0$. *Moreover, they are given (up to equivalence) by*

$$\begin{aligned}&\mathcal{R}_\lambda c_\eta=0,\quad \partial c_\eta=\eta c_\eta,\\ &A_\lambda v=(\partial+\alpha\lambda+\beta)v+f(\lambda)c_\eta,\quad B_\lambda v=g(\lambda)c_\eta.\end{aligned} \tag{1.18}$$

The values of α *along with the corresponding polynomials* $f(\lambda)$ *and* $g(\lambda)$ *giving rise to* *nontrivial extensions, are listed as follows:* $g(\lambda) = 0$ *and*

$$f(\lambda) = \begin{cases} s_1\lambda^2, & \alpha = 1, \\ s_2\lambda^3, & \alpha = 2, \end{cases}$$

with nonzero constants s_1, s_2. *In these cases,* $dim(Ext(V_{\alpha,\beta}, \mathbb{C}c_\eta)) = 1$.

Theorem 5.2.4 *For a rank two Lie conformal algebra* $\mathcal{R}$ *that is of Type II, nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow V_{\alpha,\beta} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0 \tag{1.19}$$

exist only if $\beta + \eta = 0$ *and* $\alpha = 1$. *Moreover, they are given (up to equivalence) by*

$$\begin{aligned} &A_\lambda v = (\partial + \alpha\lambda + \beta)v, \quad B_\lambda v = 0, \\ &A_\lambda c_\eta = f(\partial, \lambda)v, \quad B_\lambda c_\eta = g(\partial, \lambda)v, \quad \partial c_\eta = \eta c_\eta + h(\partial)v, \end{aligned} \tag{1.20}$$

and $dim(Ext(\mathbb{C}c_\eta, V_{\alpha,\beta})) = 1$. *The corresponding polynomials* $f(\partial, \lambda), g(\partial, \lambda)$ *and* $h(\partial)$ *giving rise to nontrivial extensions, are listed as follows:* $g(\partial, \lambda) = 0$ *and* $f(\partial, \lambda) = h(\partial) = s$ *with nonzero constant* s.

Theorem 5.2.6 *For a rank two Lie conformal algebra* $\mathcal{R}$ *that is of Type II, nontrivial extensions of finite irreducible conformal modules of the form*

$$0 \longrightarrow V_{\alpha,\beta} \longrightarrow E \longrightarrow V_{\bar{\alpha},\bar{\beta}} \longrightarrow 0 \tag{1.21}$$

exist only if $\beta = \bar{\beta}$. *Moreover, they are given (up to equivalence) by*

$$\begin{aligned} &A_\lambda v = (\partial + \alpha\lambda + \beta)v, \quad B_\lambda v = 0, \\ &A_\lambda \bar{v} = (\partial + \bar{\alpha}\lambda + \bar{\beta})\bar{v} + f(\partial, \lambda)v, \quad B_\lambda \bar{v} = g(\partial, \lambda)v. \end{aligned} \tag{1.22}$$

The value of $\alpha, \bar{\alpha}$, *and the corresponding polynomials* $f(\partial, \lambda)$ *and* $g(\partial, \lambda)$ *giving rise to nontrivial extensions, are listed as follows (by replacing* ∂ *by* $\partial + \beta$*):*

1. *In the case when* $a = 1$, *where* $Q(\partial, \lambda) = c(\partial + 2\lambda)$ *for some nonzero constant* c, $\bar{\alpha} - \alpha \in \{0, 1, 2, 3, 4, 5, 6\}, \alpha, \bar{\alpha} \neq 0$ *and*

(i) $\bar{\alpha} = \alpha$, $f(\partial,\lambda) = s_0 + s_1\lambda, g(\partial,\lambda) = 0$, *where* $(s_0, s_1) \neq (0,0)$.

(ii) $\bar{\alpha} - \alpha = 1$, $f(\partial,\lambda) = \frac{ct}{\alpha}\partial, g(\partial,\lambda) = t\lambda$, *where* $t \neq 0$.

(iii) $\bar{\alpha} - \alpha = 2$ *with* $\alpha \neq -1$, $f(\partial,\lambda) = s\lambda^2(2\partial + \lambda), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(iv) $\bar{\alpha} = 1$ *and* $\alpha = -1$, $f(\partial,\lambda) = s\lambda^2(2\partial+\lambda) - ct(\partial^2 - \lambda^2), g(\partial,\lambda) = t(\partial\lambda + \lambda^2)$, *where* $(s,t) \neq (0,0)$.

(v) $\bar{\alpha} - \alpha = 3$, $f(\partial,\lambda) = s\partial\lambda^2(\partial + \lambda), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(vi) $\bar{\alpha} - \alpha = 4$, $f(\partial,\lambda) = s\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \alpha_1\lambda^3), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(vii) $\bar{\alpha} = 1$ *and* $\alpha = -4$, $f(\partial,\lambda) = s(\partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(viii) $\bar{\alpha} = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial,\lambda) = s(\partial^4\lambda^3 - (2\alpha+3)\partial^3\lambda^4 - 3\alpha\partial^2\lambda^5 - (3\alpha+1)\partial\lambda^6 - (\alpha + \frac{9}{28})\lambda^7), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

The value of dim$(Ext(V_{\bar{\alpha},\bar{\beta}}, V_{\alpha,\beta}))$ *is 2 in subcase (i) and (iv), and 1 in the other subcases.*

2. *In the case when* $a = 0$, *where* $Q(\partial,\lambda) = c(\partial + 2\lambda)(\partial+\lambda)\lambda + d(\partial+2\lambda)\partial$ *for some nonzero constants* c, d, $\bar{\alpha} - \alpha \in \{0,1,2,3,4,5,6\}, \alpha, \bar{\alpha} \neq 0$ *and*

(i) $\bar{\alpha} = \alpha$, $f(\partial,\lambda) = s_0 + s_1\lambda, g(\partial,\lambda) = 0$, *where* $(s_0, s_1) \neq (0,0)$.

(ii) $\bar{\alpha} - \alpha = 1$, $f(\partial,\lambda) = -\frac{ct}{\alpha}\partial\lambda - \frac{dt}{\alpha}\partial, g(\partial,\lambda) = t$, *where* $t \neq 0$.

(iii) $\bar{\alpha} - \alpha = 2$ *with* $\alpha \neq -1$, $f(\partial,\lambda) = s\lambda^2(2\partial + \lambda), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(iv) $\bar{\alpha} = 1$ *and* $\alpha = -1$, $f(\partial,\lambda) = s\lambda^2(2\partial + \lambda) + ct\partial^2\lambda + dt(\partial^2 - \lambda^2), g(\partial,\lambda) = t(\partial + \lambda)$, *where* $(s,t) \neq (0,0)$.

(v) $\bar{\alpha} - \alpha = 3$, $f(\partial,\lambda) = s\partial\lambda^2(\partial + \lambda), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(vi) $\bar{\alpha} - \alpha = 4$, $f(\partial,\lambda) = s\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \alpha_1\lambda^3), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(vii) $\bar{\alpha} = 1$ *and* $\alpha = -4$, $f(\partial,\lambda) = s(\partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

(viii) $\bar{\alpha} = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial,\lambda) = s(\partial^4\lambda^3 - (2\alpha+3)\partial^3\lambda^4 - 3\alpha\partial^2\lambda^5 - (3\alpha+1)\partial\lambda^6 - (\alpha + \frac{9}{28})\lambda^7), g(\partial,\lambda) = 0$, *where* $s \neq 0$.

The value of dim$(Ext(V_{\bar{\alpha},\bar{\beta}}, V_{\alpha,\beta}))$ *is 2 in subcase (i) and (iv), and 1 in the other subcases.*

3. *In the case when* $a=-1$, *where* $Q(\partial,\lambda)=c(\partial+2\lambda)\partial^2+d(\partial+2\lambda)(\partial+\lambda)\partial\lambda$ *for some nonzero constants* c,d, $\bar{\alpha}-\alpha\in\{0,1,2,3,4,5,6\},\alpha,\bar{\alpha}\neq 0$ *and*

   (i) $\bar{\alpha}=\alpha$, $f(\partial,\lambda)=s_0+s_1\lambda, g(\partial,\lambda)=0$, *where* $(s_0,s_1)\neq(0,0)$.

   (ii) $\bar{\alpha}-\alpha=2$ *with* $\alpha\neq-\frac{1}{2}$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   (iii) $\bar{\alpha}=\frac{3}{2}$ *and* $\alpha=-\frac{1}{2}$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda)-2dt\partial^2\lambda-ct(2\partial^2-\lambda^2), g(\partial,\lambda)=t$, *where* $(s,t)\neq(0,0)$.

   (iv) $\bar{\alpha}-\alpha=3$ *with* $\alpha\neq-1$, $f(\partial,\lambda)=s\partial\lambda^2(\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   (v) $\bar{\alpha}=2$ *and* $\alpha=-1$, $f(\partial,\lambda)=s\partial\lambda^2(\partial+\lambda)-\frac{dt}{4}(2\partial^3\lambda+\lambda^4)-\frac{ct}{2}(\partial^3-2\partial\lambda^2-2\lambda^3), g(\partial,\lambda)=t(\partial+\frac{1}{2}\lambda)$, *where* $(s,t)\neq(0,0)$.

   (vi) $\bar{\alpha}-\alpha=4$, $f(\partial,\lambda)=s\lambda^2(4\partial^3+6\partial^2\lambda-\partial\lambda^2+\alpha_1\lambda^3), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   (vii) $\bar{\alpha}=1$ *and* $\alpha=-4$, $f(\partial,\lambda)=s(\partial^4\lambda^2-10\partial^2\lambda^4-17\partial\lambda^5-8\lambda^6), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   (viii) $\bar{\alpha}=\frac{7}{2}\pm\frac{\sqrt{19}}{2}$ *and* $\alpha=-\frac{5}{2}\pm\frac{\sqrt{19}}{2}$, $f(\partial,\lambda)=s(\partial^4\lambda^3-(2\alpha+3)\partial^3\lambda^4-3\alpha\partial^2\lambda^5-(3\alpha+1)\partial\lambda^6-(\alpha+\frac{9}{28})\lambda^7), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   The value of dim$(Ext(V_{\bar{\alpha},\bar{\beta}},V_{\alpha,\beta}))$ *is 2 in subcase (i) and (v), and 1 in the other subcases.*

4. *In the case when* $a=-4$, *where* $Q(\partial,\lambda)=c(\partial+2\lambda)(\partial+\lambda)^3\lambda^3$ *for some nonzero constant* c, $\bar{\alpha}-\alpha\in\{0,1,2,3,4,5,6,7\},\alpha,\bar{\alpha}\neq 0$, *and*

   (i) $\bar{\alpha}=\alpha$, $f(\partial,\lambda)=s_0+s_1\lambda, g(\partial,\lambda)=0$, *where* $(s_0,s_1)\neq(0,0)$.

   (ii) $\bar{\alpha}-\alpha=2$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   (iii) $\bar{\alpha}-\alpha=3$, $f(\partial,\lambda)=s\partial\lambda^2(\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   (iv) $\bar{\alpha}-\alpha=4$, $f(\partial,\lambda)=s\lambda^2(4\partial^3+6\partial^2\lambda-\partial\lambda^2+\alpha_1\lambda^3), g(\partial,\lambda)=0$, *where* $s\neq 0$.

   (v) $\bar{\alpha}-\alpha=5$ *with* $\alpha\notin\{-2,-4\}$, $f(\partial,\lambda)=-\frac{3}{\alpha(\alpha+2)(\alpha+4)}ct\partial^3\lambda^3+\frac{9(\alpha+1)}{2\alpha(\alpha+2)(\alpha+4)}ct\partial^2\lambda^4-\frac{9(\alpha+1)(2\alpha+1)}{10\alpha(\alpha+2)(\alpha+4)}ct\partial\lambda^5+\frac{(\alpha+1)(2\alpha+1)}{10(\alpha+2)(\alpha+4)}ct\lambda^6, g(\partial,\lambda)=t$, *where* $t\neq 0$.

   (vi) $\bar{\alpha}=3$ *and* $\alpha=-2$, $f(\partial,\lambda)=\frac{3}{8}ct\partial^4\lambda^2-\frac{3}{2}ct\partial^2\lambda^4-\frac{57}{40}ct\partial\lambda^5-\frac{2}{5}ct\lambda^6, g(\partial,\lambda)=t$, *where* $t\neq 0$.

   (vii) $\bar{\alpha}=1$ *and* $\alpha=-4$, $f(\partial,\lambda)=s(\partial^4\lambda^2-10\partial^2\lambda^4-17\partial\lambda^5-8\lambda^6), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(viii) $\bar{\alpha}-\alpha=6$ *with* $\alpha\notin\{-\frac{5}{2},-\frac{5}{2}\pm\frac{\sqrt{19}}{2}\}$, $f(\partial,\lambda)=-\frac{3}{(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial^4\lambda^3+\frac{3(2\alpha+3)}{(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial^3\lambda^4-\frac{9(\alpha+1)(2\alpha+3)}{5(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial^2\lambda^5+\frac{(\alpha+1)(2\alpha+1)(2\alpha+3)}{5(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial\lambda^6-\frac{\alpha(\alpha+1)(2\alpha+1)(2\alpha+3)}{70(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\lambda^7, g(\partial,\lambda)=t(\partial-\frac{\alpha}{5}\lambda)$, *where* $t\neq 0$.

(ix) $\bar{\alpha}=\frac{7}{2}$ *and* $\alpha=-\frac{5}{2}$, $f(\partial,\lambda)=\frac{36}{665}ct\bar{\partial}^5\lambda^2-\frac{54}{113}ct\bar{\partial}^3\lambda^4-\frac{387}{665}ct\bar{\partial}^2\lambda^5-\frac{218}{665}ct\bar{\partial}\lambda^6+\frac{127}{1862}ct\lambda^7, g(\partial,\lambda)=t(\partial+\frac{1}{2}\lambda)$, *where* $t\neq 0$.

(x) $\bar{\alpha}=\frac{7}{2}\pm\frac{\sqrt{19}}{2}$ *and* $\alpha=-\frac{5}{2}\pm\frac{\sqrt{19}}{2}$, $f(\partial,\lambda)=s(\partial^4\lambda^3-(2\alpha+3)\partial^3\lambda^4-3\alpha\partial^2\lambda^5-(3\alpha+1)\partial\lambda^6-(\alpha+\frac{9}{28})\lambda^7), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(xi) $\bar{\alpha}=1$ *and* $\alpha=-6$, $f(\partial,\lambda)=\frac{1}{35}ct\partial^5\lambda^3+\frac{2}{7}ct\partial^4\lambda^4+\frac{36}{35}ct\partial^3\lambda^5+\frac{12}{7}ct\partial^2\lambda^6+\frac{66}{49}ct\partial\lambda^7+\frac{99}{245}ct\lambda^8, g(\partial,\lambda)=t(\partial^2+\frac{11}{5}\partial\lambda+\frac{6}{5}\lambda^2)$, *where* $t\neq 0$.

The value of $dim(Ext(V_{\bar{\alpha},\bar{\beta}},V_{\alpha,\beta}))$ *is 2 in subcase (i), and 1 in the other subcases.*

5. *In the case when* $a=-6$, *where* $Q(\partial,\lambda)=c(\partial+2\lambda)[11(\partial+\lambda)^4\lambda^4+2(\partial+\lambda)^3\partial^2\lambda^3]$ *for some nonzero constant* c, $\bar{\alpha}-\alpha\in\{0,1,2,3,4,5,6,7,8\},\alpha,\bar{\alpha}\neq 0$, *and*

(i) $\bar{\alpha}=\alpha$, $f(\partial,\lambda)=s_0+s_1\lambda, g(\partial,\lambda)=0$, *where* $(s_0,s_1)\neq(0,0)$.

(ii) $\bar{\alpha}-\alpha=2$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(iii) $\bar{\alpha}-\alpha=3$, $f(\partial,\lambda)=s\partial\lambda^2(\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(iv) $\bar{\alpha}-\alpha=4$, $f(\partial,\lambda)=s\lambda^2(4\partial^3+6\partial^2\lambda-\partial\lambda^2+\alpha_1\lambda^3), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(v) $\bar{\alpha}=1$ *and* $\alpha=-4$, $f(\partial,\lambda)=s(\partial^4\lambda^2-10\partial^2\lambda^4-17\partial\lambda^5-8\lambda^6), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(vi) $\bar{\alpha}=\frac{7}{2}\pm\frac{\sqrt{19}}{2}$ *and* $\alpha=-\frac{5}{2}\pm\frac{\sqrt{19}}{2}$, $f(\partial,\lambda)=s(\partial^4\lambda^3-(2\alpha+3)\partial^3\lambda^4-3\alpha\partial^2\lambda^5-(3\alpha+1)\partial\lambda^6-(\alpha+\frac{9}{28})\lambda^7), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(vii) $\bar{\alpha}=4\pm\frac{\sqrt{22}}{2}$ *and* $\alpha=-3\pm\frac{\sqrt{22}}{2}$, $f(\partial,\lambda)=-\frac{40}{7(\alpha+3)}ct\partial^5\lambda^3+\frac{100(\alpha+2)}{7(\alpha+3)}ct\partial^4\lambda^4+\frac{40(5\alpha+1)}{7(\alpha+3)}ct\partial^3\lambda^5+\frac{20(16\alpha+11)}{7(\alpha+3)}ct\partial^2\lambda^6+\frac{10(154\alpha+101)}{49(\alpha+3)}ct\partial\lambda^7+\frac{823\alpha+539}{98(\alpha+3)}ct\lambda^8, g(\partial,\lambda)=t$, *where* $t\neq 0$.

(viii) $\bar{\alpha}=7$ *and* $\alpha=-1$, $f(\partial,\lambda)=-\frac{2}{7}ct\partial^6\lambda^3+\frac{9}{7}ct\partial^5\lambda^4-\frac{9}{7}ct\partial^4\lambda^5+\frac{2}{7}ct\partial^3\lambda^6, g(\partial,\lambda)=t(\partial+\frac{1}{7}\lambda)$, *where* $t\neq 0$.

(ix) $\bar{\alpha}=2$ *and* $\alpha=-6$, $f(\partial,\lambda)=-\frac{2}{7}ct\partial^6\lambda^3-3ct\partial^5\lambda^4-12ct\partial^4\lambda^5-24ct\partial^3\lambda^6-\frac{180}{7}ct\partial^2\lambda^7-\frac{99}{7}ct\partial\lambda^8-\frac{22}{7}ct\lambda^9, g(\partial,\lambda)=t(\partial+\frac{6}{7}\lambda)$, *where* $t\neq 0$.

The value of $dim(Ext(V_{\bar{\alpha},\bar{\beta}},V_{\alpha,\beta}))$ *is 2 in subcase (i), and 1 in the other subcases.*

Chapter 2 Preliminaries

In this chapter, we recall the definition of Lie conformal algebras, conformal modules and their extensions, and some known results that are useful in this paper. For more details, one can refer to [4, 6, 7, 22, 32].

§2.1 Lie conformal algebras

First, we introduce the two equivalent definitions of Lie conformal algebras. The distribution notion has advantages in attaching to Lie algebras, while it is more convenient to compute with λ-bracket.

Definition 2.1.1 *A* ***Lie conformal algebra*** $\mathcal{R}$ *is a* $\mathbb{C}[\partial]$*-module equipped with a* $\mathbb{C}$*-linear map (called* ***λ-bracket****)* $\mathcal{R}\otimes\mathcal{R}\to\mathcal{R}[\lambda]$, $a\otimes b\mapsto[a_\lambda b]$, *satisfying the axioms (C1)-(C4) for all* $a,b,c\in\mathcal{R}$.

(C1) $[a_\lambda b]\in\mathbb{C}[\lambda]\otimes\mathcal{R}$,

(C2) $[\partial a_\lambda b]=-\lambda[a_\lambda b],\ [a_\lambda\partial b]=(\partial+\lambda)[a_\lambda b]$,

(C3) $[a_\lambda b]=-[b_{-\lambda-\partial}a]$,

(C4) $[a_\lambda[b_\mu c]]=[[a_\lambda b]_{\lambda+\mu}c]+[b_\mu[a_\lambda c]]$.

A Lie conformal algebra $\mathcal{R}$ is called **finite** if $\mathcal{R}$ is finitely generated as a $\mathbb{C}[\partial]$-module. The **rank** of a finite Lie conformal algebra is just its rank as a $\mathbb{C}[\partial]$-module.

Obviously, we can define subalgebras, ideals, quotients, simple algebras and homomorphisms of Lie conformal algebras.

For $a,b,c\in\mathcal{R}$, set

$$[a_\lambda b]=\sum_{k\in\mathbb{N}_+}\lambda^{(k)}a_{(k)}b, \tag{2.1}$$

where $\lambda^{(k)}:=\frac{\lambda^k}{k!}$. Then we can get a family of bilinear products $\{\bullet_{(k)}\bullet\}$ on $\mathcal{R}$ and equivalently rephrase the above Lie conformal axioms.

(C1′) $a_{(k)}b=0$ for $k\gg 0$,

(C2′) $\partial a_{(k)}b=-ka_{(k-1)}b,\ \ a_{(k)}\partial b=\partial(a_{(k)}b)+ka_{(k-1)}b$,

(C3′) $a_{(k)}b=-\sum_j(-1)^{j+k}\partial^{(j)}b_{(j+k)}a$, (C4′) $a_{(k)}(b_{(l)}c) - b_l(a_{(k)}c) = \sum_{j=0}^{k} \binom{k}{j}(a_{(j)}b)_{k+l-j}c.$

There is a Lie algebra structure contained in the distribution notion of a Lie conformal algebra.

Lemma 2.1.2 *Let $(\mathcal{R}, \{\bullet_{(k)}\bullet)$ be a Lie conformal algebra. Then $\bullet_{(0)}\bullet$ is a Lie bracket of $\mathcal{R}$, and with respect to the 0-th product $\partial\mathcal{R}$ is an ideal of $\mathcal{R}$ so that $\mathcal{R}/\partial\mathcal{R}$ is a Lie algebra.*

Below are some important examples of Lie conformal algebras.

Example 2.1.3 *Let $Vir = \mathbb{C}[\partial]\otimes L$ be the rank-one $\mathbb{C}[\partial]$-module generated by L. Then*

$$[L_\lambda L] = (\partial + 2\lambda)L$$

defines a Lie conformal algebra structure on Vir. The Lie conformal algebra Vir is called the ***Virasoro conformal algebra****.*

Example 2.1.4 *Given a Lie algebra $\mathfrak{g}$, let $Cur(\mathfrak{g}) = \mathbb{C}[\partial] \otimes \mathfrak{g}$. Then*

$$[a_\lambda b] = [a, b], \ \forall a, b \in \mathfrak{g}$$

defines a Lie conformal algebra structure on $Cur(\mathfrak{g})$. The Lie conformal algebra $Cur(\mathfrak{g})$ is called the ***current Lie conformal algebra*** *associated with $\mathfrak{g}$.*

Example 2.1.5 *Let $Cur(\mathfrak{g})$ be the current Lie conformal algebra associated to the finite-dimensional Lie algebra $\mathfrak{g}$. Then the $\mathbb{C}[\partial]$-module $Vir \oplus Cur(\mathfrak{g})$ can be given a conformal algebra structure by*

$$[L_\lambda L] = (\partial + 2\lambda)L, \ [g_\lambda h] = [g, h], \ [L_\lambda g] = (\partial + \lambda)g,$$

where L is the standard generator of Vir, $g, h \in \mathfrak{g}$. This Lie conformal algebra is called the ***semidirect sum of*** *Vir* ***and*** *$Cur(\mathfrak{g})$, denoted by $Vir \ltimes Cur(\mathfrak{g})$.*

Besides, by utilizing the correspondence between formal distribution Lie algebras and Lie conformal algebras, one can effectively construct a class of finite non-simple Lie conformal algebras. In [26], Su and Yuan investigated two non-simple Lie conformal algebras derived from the Schrödinger-Virasoro Lie algebra and the extended

Schrödinger-Virasoro Lie algebra, which were generalized in [27]. Similarly, a class of Lie conformal algebras $W(b)$ was obtained from the infinite-dimensional Lie algebra $\mathcal{W}(a,b)$, which is the semidirect sum of the centerless Virasoro algebra and the intermediate series module $A(a,b)$ [33]. Furthermore, from the twisted deformation of Schrödinger-Virasoro type Lie conformal algebras, the Schrödinger-Virasoro Lie algebra studied in [29] was obtained.

Let $\mathcal{R}$ be a Lie conformal algebra, I and J its ideals. The bracket $[I_\lambda J]$ is the subspace of $\mathcal{R}$ that is spanned by all products $i_{(n)}j$ with $i \in I, j \in J, n \in \mathbb{N}_+$.

Definition 2.1.6 *A Lie conformal algebra $\mathcal{R}$ is called* ***solvable*** *if its derived series terminates at zero, i.e., there exists $n \in \mathbb{N}$ such that:*

$$\mathcal{R}^{(0)} = \mathcal{R}, \quad \mathcal{R}^{(k+1)} = [\mathcal{R}^{(k)}{}_\lambda \mathcal{R}^{(k)}], \quad \textit{and} \quad \mathcal{R}^{(n)} = 0,$$

where $[\cdot_\lambda \cdot]$ denotes the λ-bracket defining the Lie conformal algebra structure.

A Lie conformal algebra $\mathcal{R}$ is called ***semisimple*** *if it contains no nonzero solvable ideals.*

The classification of rank one Lie conformal algebras and all finite semisimple ones was established by Kac et al. in [9], where Virasoro conformal algebra plays a central role.

Proposition 2.1.7 *[9, Proposition 3.3] If $\mathcal{R}$ is a non-commutative rank one Lie conformal algebra, then $\mathcal{R}$ must be isomorphic to Vir.*

Theorem 2.1.8 *[9, Theorem 7.1] Let $\mathcal{R}$ be a finite semi-simple conformal algebra. Then $\mathcal{R}$ can be uniquely decomposed in a finite direct sum of conformal algebras, where each summand is isomorphic to one of the following:*

(a) Vir,

(b) $Cur(\mathfrak{g})$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra,

(b) $Vir \ltimes Cur(\mathfrak{g})$, where $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra.

A super version of the above classification can be seen in [11].

Let $\mathcal{R}$ be a free rank two Lie conformal algebra. If $\mathcal{R}$ is semisimple, as shown in Theorem 2.1.8, $\mathcal{R}$ is the direct sum of Vir, $Cur(\mathfrak{g})$, $Vir \ltimes Cur(\mathfrak{g})$, where $Cur(\mathfrak{g})$

is the current conformal algebra associated with a finite-dimensional semisimple Lie algebra $\mathfrak{g}$. Since there exists no semisimple Lie algebra of dimension less than 3, then $\mathcal{R}$ is isomorphic to the direct sum of two Virasoro Lie conformal algebras. As for the non-semisimple case, we have the following proposition. For more details, one can refer to [4, 15, 23, 34].

Proposition 2.1.9 *[4, Theorem 3.21] Let $\mathcal{R}$ be a rank two Lie conformal algebra that is not semisimple.*

(1) If $\mathcal{R}$ is solvable, then there is a basis $\{A, B\}$ such that

$$[A_\lambda A] = Q_1(\partial, \lambda)B,\ [A_\lambda B] = p(\lambda)B,\ [B_\lambda B] = 0, \tag{2.2}$$

for some polynomial $p(\lambda)$ and some skew-symmetric polynomial $Q_1(\partial, \lambda)$ satisfying $p(\lambda)Q_1(\partial, \lambda) = 0$.

(2) If $\mathcal{R}$ is neither solvable nor semisimple, then there are two classes.

(2i) $\mathcal{R}$ is the direct sum of a rank one commutative Lie conformal algebra and the Virasoro Lie conformal algebra. That is, there is a basis $\{A, B\}$ of $\mathcal{R}$ satisfying

$$[A_\lambda A] = (\partial + 2\lambda)A,\ [A_\lambda B] = 0,\ [B_\lambda B] = 0. \tag{2.3}$$

(2ii) There is a basis $\{A, B\}$ of $\mathcal{R}$ such that

$$\begin{aligned} [A_\lambda A] &= (\partial + 2\lambda)A + Q(\partial, \lambda)B, \\ [A_\lambda B] &= (\partial + a\lambda + b)B,\ [B_\lambda B] = 0, \end{aligned} \tag{2.4}$$

where $a, b \in \mathbb{C}$ and $Q(\partial, \lambda)$ is some skew-symmetric polynomial depending on a, b. Moreover, $Q(\partial, \lambda) \neq 0$ only when $a \in \{1, 0, -1, -4, -6\}$ and $b = 0$, in which case we document the explicit formula for $Q(\partial, \lambda)$ in the following table.

a	$Q(\partial,\lambda),\ c,d\in\mathcal{C},$
1	$c(\partial+2\lambda)$
0	$c(\partial+2\lambda)(\partial+\lambda)\lambda+d(\partial+2\lambda)\partial$
-1	$c(\partial+2\lambda)\partial^2+d(\partial+2\lambda)(\partial+\lambda)\partial\lambda$
-4	$c(\partial+2\lambda)(\partial+\lambda)^3\lambda^3$
-6	$c(\partial+2\lambda)[11(\partial+\lambda)^4\lambda^4+2(\partial+\lambda)^3\partial^2\lambda^3]$

Remark 2.1.10 *A polynomial $Q(\partial,\lambda)$ is called* ***skew-symmetric*** *if $Q(\partial,\lambda)=-Q(\partial,-\lambda-\partial)$.*

In this study, we refer to the two classes of non-solvable and non-semisimple rank two conformal algebras mentioned above as **Lie conformal algebras of Type I and Type II**, respectively. A Lie conformal algebras of Type II with $Q(\partial,\lambda)=0$ is called $\mathcal{W}(a,b)$ **algebra**.

§2.2 Conformal modules of Lie conformal algebras and their extensions

Now, we can introduce the definition of conformal modules.

Definition 2.2.1 *Let $\mathcal{R}$ be a Lie conformal algebra. A* ***conformal $\mathcal{R}$-module*** *V is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $\mathcal{R}\otimes V\to V[\lambda]$, $a\otimes v\mapsto a_\lambda v$, satisfying the following axioms:*

$$(\partial a)_\lambda v=-\lambda(a_\lambda v),\quad a_\lambda(\partial v)=(\partial+\lambda)a_\lambda v,$$

$$a_\lambda(b_\mu v)=[a_\lambda b]_{\lambda+\mu}v+b_\mu(a_\lambda v),$$

for all a, $b\in\mathcal{R}$ and $v\in V$.

In the sequel, for convenience, a conformal $\mathcal{R}$-module is also called an $\mathcal{R}$-module. A conformal module V is called **irreducible** if there is no nonzero submodule W such that $W\neq V$, and V is said to be a trivial $\mathcal{R}$-module if $\mathcal{R}$ acts on V trivially. For any $\eta\in\mathbb{C}$, we can obtain a trivial $\mathcal{R}$-module $\mathbb{C}c_\eta=\mathbb{C}$, which is determined by η, via the action $\partial c_\eta=\eta c_\eta,\mathcal{R}_\lambda c_\eta=0$. It is easy to check that the modules $\mathbb{C}c_\eta$ with $\eta\in\mathbb{C}$ exhaust all trivial irreducible $\mathcal{R}$-modules.

The complete classification of the finite nontrivial irreducible modules of the Virasoro conformal algebra was provided in [6,9], that of $\mathcal{R}$ of rank two was described in [32], and that of other important Lie conformal algebras were investigated in [12,13,25,31] and so on.

Proposition 2.2.2 *[6, 9] Any non-trivial free rank one Vir-module has the form $M_{a,b} = \mathbb{C}[\partial]v$, such that*

$$L_\lambda v = (\partial + a\lambda + b)v,$$

for $a, b \in \mathbb{C}$. Moreover, if $a \neq 0$, then $M_{a,b}$ is irreducible and any non-trivial finite irreducible Vir-module is in such form. If $a = 0$, then $M_{0,b}$ has the unique finite irreducible proper Vir-submodule $(\partial + b)M_{0,b}$, which is isomorphic to $M_{1,b}$.

Proposition 2.2.3 *[32, Theorem 3.2] Suppose that $\mathcal{R} = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]B$ is a Lie conformal algebra of rank two. Then any non-trivial finite irreducible $\mathcal{R}$-module is free of rank one. Moreover, if $V = \mathbb{C}[\partial]v$ is a non-trivial irreducible $\mathcal{R}$-module, then the action of $\mathcal{R}$ on V has to be one of the following cases:*

(i) If $\mathcal{R} = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]B$ is a direct sum of two Virasoro Lie conformal algebras with $[A_\lambda B] = 0$, then either

$$A_\lambda v = (\partial + \alpha_1\lambda + \beta_1)v,\ B_\lambda v = 0,\ \textit{for some } \beta_1,\ 0 \neq \alpha_1 \in \mathbb{C},$$

or

$$A_\lambda v = 0,\ B_\lambda v = (\partial + \alpha_2\lambda + \beta_2)v,\ \textit{for some } \beta_2,\ 0 \neq \alpha_2 \in \mathbb{C}.$$

(ii) If $\mathcal{R}$ is solvable with the relations (2.2), then we have $A_\lambda v = \phi_A(\lambda)v$, $B_\lambda v = \phi_B(\lambda)v$, where $\phi_A(\lambda)$, $\phi_B(\lambda)$ are not zero simultaneously. Moreover, $\phi_B(\lambda) \neq 0$ only if $p(\lambda) = Q_1(\partial, \lambda) = 0$.

(iii) Suppose that $\mathcal{R}$ is the Lie conformal algebra defined in (2.3), then either

$$A_\lambda v = (\partial + \alpha\lambda + \beta)v,\ B_\lambda v = 0,\ \textit{for some } \beta,\ 0 \neq \alpha \in \mathbb{C},$$

or

$$A_\lambda v = 0,\ B_\lambda v = \phi(\lambda)v,\ \textit{for some nonzero } \phi(\lambda) \in \mathbb{C}[\lambda].$$

(iv) Suppose that $\mathcal{R}$ is the Lie conformal algebra defined in (2.4). Then

$$A_\lambda v = (\partial + \alpha\lambda + \beta),\ \ B_\lambda v = \gamma v,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ only if $a = 1$, $b = 0$ and $Q(\partial, \lambda) = 0$. Further, if $\gamma = 0$, then $\alpha \neq 0$.

Definition 2.2.4 *Let V and W be two modules over a Lie conformal algebra $\mathcal{R}$. An* ***extension*** *of W by V is an exact sequence of $\mathcal{R}$-modules of the form*

$$0 \longrightarrow V \xrightarrow{i} E \xrightarrow{p} W \longrightarrow 0, \tag{2.5}$$

where E is isomorphic to $V \oplus W$ as a vector space. Two extensions $0 \longrightarrow V \xrightarrow{i} E \xrightarrow{p} W \longrightarrow 0$ and $0 \longrightarrow V \xrightarrow{i'} E' \xrightarrow{p'} W \longrightarrow 0$ are said to be ***equivalent*** *if there exists a homomorphism of modules such that the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & E & \xrightarrow{p} & W & \longrightarrow & 0 \\ & & {\scriptstyle 1_V}\big\downarrow & & {\scriptstyle \Psi}\big\downarrow & & {\scriptstyle 1_W}\big\downarrow & & \\ 0 & \longrightarrow & V & \xrightarrow{i'} & E' & \xrightarrow{p'} & W & \longrightarrow & 0. \end{array} \tag{2.6}$$

Obviously, the direct sum of modules $V \oplus W$ gives rise to an extension $0 \to V \to V \oplus W \to W \to 0$. Any extension $0 \to V \to E \to W \to 0$, which is equivalent to $0 \to V \to V \oplus W \to W \to 0$, is called **trivial extension**.

In general, an extension can be thought of as the direct sum of vector spaces $E = V \oplus W$, where V is a submodule of E, while for $w \in W$ we have

$$a_\lambda \cdot w = a_\lambda w + f_{a_\lambda}(w),\ \ a \in \mathcal{R},$$

where $f_{a_\lambda} : W \to \mathbb{C}[\lambda] \otimes V$ is a linear map satisfying the **cocycle** condition:

$$f_{[a_\lambda b]_{\lambda+\mu}}(w) = f_{a_\lambda}(b_\mu w) + a_\lambda f_{b_\mu}(w) - f_{b_\mu}(a_\lambda w) - b_\mu f_{a_\lambda}(w),\ \ b \in \mathcal{R}.$$

The set of all cocycles forms a vector space $\mathcal{E}xt(W, V)$ over $\mathbb{C}$. Cocycles equivalent to trivial extensions are called **coboundaries**. They form a subspace $\mathcal{E}xt^c(W, V)$ and the quotient space $\mathcal{E}xt(W, V)/\mathcal{E}xt^c(W, V)$ is denoted by $Ext(W, V)$. The dimension of the quotient space is called the **dimension** of the space of extensions of W by V, denoted by $dim(Ext(W,V))$. From now on, we will say two extensions are **equivalent** if they belong to the same equivalent class unless confusion is possible.

Example 2.2.5 *Let $\mathcal{R}$ be an arbitrary conformal algebra, and we can consider extensions of trivial irreducible $\mathcal{R}$-modules of the form*

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow \mathbb{C}c_{\bar{\eta}} \longrightarrow 0. \tag{2.7}$$

In this case, E as a vector space is isomorphic to $\mathbb{C}c_\eta \oplus \mathbb{C}c_{\bar{\eta}}$, where $\mathbb{C}c_\eta$ is an $\mathcal{R}$-submodule, and the following identities hold in E:

$$R_\lambda c_{\bar{\eta}} = f_R(\lambda)c_\eta, \quad \partial c_{\bar{\eta}} = \bar{\eta}c_{\bar{\eta}} + tc_\eta, \tag{2.8}$$

for any $R \in \mathcal{R}$, where $\eta, \bar{\eta}, t \in \mathbb{C}$ and $f_R(\lambda)$ is some polynomial depending on R. Since E is an $\mathcal{R}$-module, it follows from $R_\lambda(\partial c_{\bar{\eta}}) = (\partial + \lambda)R_\lambda c_{\bar{\eta}}$ that

$$\bar{\eta}f_R(\lambda) = (\eta + \lambda)f_R(\lambda)$$

which implies $f_R(\lambda) = 0$ for any $R \in \mathcal{R}$. Assume that (2.7) is a trivial extension, that is, there exists $c_\sigma = kc_\eta + lc_{\bar{\eta}}$, where $k, l \in \mathbb{C}$ and $l \neq 0$, such that

$$\partial c_\sigma = \bar{\eta}c_\sigma = \bar{\eta}kc_\eta + \bar{\eta}lc_{\bar{\eta}}, \quad \mathcal{R}_\lambda c_\sigma = 0.$$

On the other hand, it follows from (2.8) that

$$\partial c_\sigma = k\partial c_\eta + l\partial c_{\bar{\eta}} = (\eta k + tl)c_\eta + \bar{\eta}lc_{\bar{\eta}}.$$

So we have $(\bar{\eta} - \eta)k = tl$.

If $\bar{\eta} \neq \eta$, for arbitrary t, we can find such $k, l \in \mathbb{C}$, and E is always a trivial extension. If $\bar{\eta} = \eta$, E is trivial only when $t = 0$. Therefore, $dim(Ext(\mathbb{C}c_\eta, \mathbb{C}c_{\bar{\eta}}))=\delta_{\eta,\bar{\eta}}$ and when $\eta = \bar{\eta}$, the nontrivial extensions are given (up to equivalence) by

$$\mathcal{R}_\lambda c_{\bar{\eta}} = 0, \quad \partial c_{\bar{\eta}} = \bar{\eta}c_{\bar{\eta}} + kc_\eta$$

with $k \neq 0$.

In [7], extensions over the Virasoro conformal modules of the following types have been classified:

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow M_{\alpha,\beta} \longrightarrow 0 \tag{2.9}$$

$$0 \longrightarrow M_{\alpha,\beta} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0 \tag{2.10}$$

$$0 \longrightarrow M_{\bar{\alpha},\bar{\beta}} \longrightarrow E \longrightarrow M_{\alpha,\beta} \longrightarrow 0. \tag{2.11}$$

The corresponding results are listed as follows.

Theorem 2.2.6 *[7, Proposition 2.1] Nontrivial extensions of Virasoro conformal modules of the form (2.9) exist if and only if* $\beta + \eta = 0$ *and* $\alpha = 1$ *or* 2. *In these cases, they are given (up to equivalence) by*

$$L_\lambda v_\alpha = (\partial + \alpha\lambda + \beta)v_\alpha + f(\lambda)c_\eta,$$

where

(i) $f(\lambda) = c_2\lambda^2$, *for* $\alpha = 1$ *and* $c_2 \neq 0$.

(ii) $f(\lambda) = c_3\lambda^3$, *for* $\alpha = 2$ *and* $c_3 \neq 0$.

Furthermore, all trivial cocycles are given by scalar multiples of the polynomial $f(\lambda) = \alpha\lambda + \beta + \eta$.

Theorem 2.2.7 *[7, Proposition 2.2] Nontrivial extensions of Virasoro conformal modules of the form (2.10) exist if and only if* $\beta + \eta = 0$ *and* $\alpha = 1$. *In these cases, they are given (up to equivalence) by*

$$L_\lambda c_\eta = f(\partial, \lambda)v_\alpha, \quad \partial c_\eta = \eta c_\eta + p(\partial)v_\alpha,$$

where $f(\partial, \lambda) = p(\partial) = k$ *for some nonzero* $k \in \mathbb{C}$.

Furthermore, all trivial cocycles are given by the same scalar multiples of the polynomial $f(\partial, \lambda) = (\partial + \alpha\lambda + \beta)\phi(\partial + \lambda)$ *and* $p(\partial) = (\partial - \eta)\phi(\partial)$, *where* ϕ *is a polynomial.*

Theorem 2.2.8 *[7, Theorem 3.1] Nontrivial extensions of Virasoro conformal modules of the form (2.11) exist if and only if* $\beta = \bar{\beta}$ *and* $\alpha - \bar{\alpha} = 0, 1, 2, 3, 4, 5, 6$. *In these cases, they are given (up to equivalence) by*

$$L_\lambda v_\alpha = (\partial + \alpha\lambda + \beta)v_\alpha + f(\partial, \lambda)v_{\bar{\alpha}}.$$

The complete list of values of α *and* $\bar{\alpha}$ *along with the corresponding polynomials* $f(\partial, \lambda)$, *is given as follows, whose nonzero scalar multiples give rise to nontrivial extensions (by replacing* ∂ *by* $\partial + \beta$*):*

(i) $\alpha = \bar{\alpha}$ *with* $\alpha \in \mathbb{C}$. $f(\partial, \lambda) = a_0 + a_1\lambda$, *where* $(a_0, a_1) \neq (0, 0)$.

(ii) $\alpha = 1$ *and* $\bar{\alpha} = 0$. $f(\partial, \lambda) = a_0\partial + b_0\partial\lambda + b_1\lambda^2$, *where* $(a_0, b_0, b_1) \neq (0, 0, 0)$.

(iii) $\alpha - \bar{\alpha} = 2$ *with* $\alpha \in \mathbb{C}$. $f(\partial, \lambda) = \lambda^2(2\partial + \lambda)$.

(iv) $\alpha - \bar{\alpha} = 3$ *with* $\alpha \in \mathbb{C}$. $f(\partial, \lambda) = \partial\lambda^2(\partial + \lambda)$.

(v) $\alpha - \bar{\alpha} = 4$ *with* $\alpha \in \mathbb{C}$. $f(\partial, \lambda) = \lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \bar{\alpha}\lambda^3)$.

(vi) $\alpha = 5$ *and* $\bar{\alpha} = 0$. $f(\partial, \lambda) = 5\partial^4\lambda^2 + 10\partial^2\lambda^4 - \partial\lambda^5$.

(vii) $\alpha = 1$ *and* $\bar{\alpha} = -4$. $f(\partial, \lambda) = \partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6$.

(viii) $\alpha = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\bar{\alpha} = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$. $f(\partial, \lambda) = \partial^4\lambda^3 - (2\bar{\alpha} + 3)\partial^3\lambda^4 - 3\bar{\alpha}\partial^2\lambda^5 - (3\bar{\alpha} + 1)\partial\lambda^6 - (\bar{\alpha} + \frac{9}{28})\lambda^7$.

Furthermore, all trivial cocycles are given by scalar multiples of the polynomial $f(\partial, \lambda) = (\partial + \alpha\lambda + \beta)\phi(\partial) - (\partial + \bar{\alpha}\lambda + \bar{\beta})\phi(\partial + \lambda)$, *where* ϕ *is a polynomial.*

Chapter 3 Extensions of finite irreducible modules of semisimple or solvable rank two Lie conformal algebras

§3.1 Extensions of finite irreducible modules of semisimple rank two Lie conformal algebras

In this section, we consider $\mathcal{R}$ as a semisimple rank two Lie conformal algebra. Then $\mathcal{R}$ is the direct sum of two Virasoro Lie conformal algebras. We can assume $\mathcal{R} = \mathbb{C}[\partial]A \oplus \mathbb{C}[\partial]B$ with

$$[A_\lambda A] = (\partial + 2\lambda)A,\ [A_\lambda B] = 0,\ [B_\lambda B] = (\partial + 2\lambda)B. \tag{3.1}$$

Let V be a non-trivial finite irreducible $\mathcal{R}$-module. According to case (i) in Proposition 2.2.3,

$$V \cong V_{\delta,\alpha,\beta} = \mathbb{C}[\partial]v, \quad A_\lambda v = \delta_1(\partial + \alpha_1\lambda + \beta_1)v, \quad B_\lambda v = \delta_2(\partial + \alpha_2\lambda + \beta_2)v, \tag{3.2}$$

where $\delta_i \in \{0,1\}, \beta_i, 0 \neq \alpha_i \in \mathbb{C}$ for $i = 1, 2$, and $\delta_1^2 + \delta_2^2 = 1$.

By Definition 2.2.1, an $\mathcal{R}$-module structure on V is given by $A_\lambda, B_\lambda \in End_{\mathbb{C}}(V)[\lambda]$ such that

$$[A_\lambda, A_\mu] = (\lambda - \mu)A_{\lambda+\mu}, \tag{3.3}$$

$$[A_\lambda, B_\mu] = 0, \tag{3.4}$$

$$[B_\lambda, B_\mu] = (\lambda - \mu)B_{\lambda+\mu}, \tag{3.5}$$

$$[\partial, A_\lambda] = -\lambda A_\lambda, \tag{3.6}$$

$$[\partial, B_\lambda] = -\lambda B_\lambda. \tag{3.7}$$

§3.1.1 $0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\delta,\alpha,\beta} \longrightarrow 0$

First, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\delta,\alpha,\beta} \longrightarrow 0. \tag{3.8}$$

Then E is isomorphic to $\mathbb{C}c_\eta \oplus V_{\delta,\alpha,\beta} = \mathbb{C}c_\eta \oplus \mathbb{C}[\partial]v$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned}
&\mathcal{R}_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta, \\
&A_\lambda v = \delta_1(\partial + \alpha_1\lambda + \beta_1)v + f(\lambda)c_\eta, \quad B_\lambda v = \delta_2(\partial + \alpha_2\lambda + \beta_2)v + g(\lambda)c_\eta,
\end{aligned} \tag{3.9}$$

where $f(\lambda), g(\lambda) \in \mathbb{C}[\lambda]$.

Lemma 3.1.1 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (3.8) are given by (3.9), where $f(\lambda)$ and $g(\lambda)$ are the same scalar multiples of $\delta_1(\alpha_1\lambda+\eta+\beta_1)$ and $\delta_2(\alpha_2\lambda+\eta+\beta_2)$, respectively.*

Proof: Assume that (3.8) is a trivial extension, that is, there exists $v' = kc_\eta + l(\partial)v \in E$, where $k \in \mathbb{C}$ and $0 \neq l(\partial) \in \mathbb{C}[\partial]$, such that

$$\begin{aligned}
A_\lambda v' = \delta_1(\partial + \alpha_1\lambda + \beta_1)v' = k\delta_1(\eta + \alpha_1\lambda + \beta_1)c_\eta + \delta_1 l(\partial)(\partial + \alpha_1\lambda + \beta_1)v, \\
B_\lambda v' = \delta_2(\partial + \alpha_2\lambda + \beta_2)v' = k\delta_2(\eta + \alpha_2\lambda + \beta_2)c_\eta + \delta_2 l(\partial)(\partial + \alpha_2\lambda + \beta_2)v.
\end{aligned}$$

On the other hand, it follows from (3.9) that

$$\begin{aligned}
A_\lambda v' = f(\lambda)l(\eta + \lambda)c_\eta + \delta_1 l(\partial + \lambda)(\partial + \alpha_1\lambda + \beta_1)v, \\
B_\lambda v' = g(\lambda)l(\eta + \lambda)c_\eta + \delta_2 l(\partial + \lambda)(\partial + \alpha_2\lambda + \beta_2)v.
\end{aligned}$$

We can obtain that $l(\partial)$ is a nonzero constant by comparing both expressions for $A_\lambda v'$ and $B_\lambda v'$. Thus $f(\lambda)$ and $g(\lambda)$ are the same scalar multiple of $\delta_1(\alpha_1\lambda + \eta + \beta_1)$ and $\delta_2(\alpha_2\lambda + \eta + \beta_2)$, respectively.

Conversely, if $f(\lambda) = k\delta_1(\alpha_1\lambda + \eta + \beta_1)$ and $g(\lambda) = k\delta_2(\alpha_2\lambda + \eta + \beta_2)$ for some $k \in \mathbb{C}$, setting $v' = kc_\eta + v$ we can deduce that (3.8) is a trivial extension.

Theorem 3.1.2 *Let $\mathcal{R}$ be a direct sum of two Virasoro conformal algebras. Then nontrivial extensions of finite irreducible conformal modules of the form (3.8) exist only* *when* $(\delta_1,\delta_2)=(1,0),\alpha_1\in\{1,2\},\beta_1+\eta=0$ *or* $(\delta_1,\delta_2)=(0,1),\alpha_2\in\{1,2\},\beta_2+\eta=0$. *In these cases, there exists a unique (up to a scalar) nontrivial extension, i.e.* $dim(Ext(V_{\delta,\alpha,\beta},\mathbb{C}c_\eta))=1$. *Moreover, they are given (up to equivalence) by (3.9). The values of* $\delta_i,\alpha_i,\beta_i,i=1,2$ *and* η*, along with the corresponding polynomials* $f(\lambda)$ *and* $g(\lambda)$ *giving rise to nontrivial extensions, are listed as follows:*

(i) If $\delta_1=1,\alpha_1\in\{1,2\},\beta_1+\eta=0$*, then* $\delta_2=0,g(\lambda)=0,0\neq\alpha_2,\beta_2\in\mathbb{C}$ *and*

$$f(\lambda)=\begin{cases}s_1\lambda^2, & \alpha_1=1,\\ s_2\lambda^3, & \alpha_1=2,\end{cases}$$

with nonzero constants s_1,s_2.

(ii) If $\delta_2=1,\alpha_2\in\{1,2\},\beta_2+\eta=0$*, then* $\delta_1=0,f(\lambda)=0,0\neq\alpha_1,\beta_1\in\mathbb{C}$ *and*

$$g(\lambda)=\begin{cases}t_1\lambda^2, & \alpha_2=1,\\ t_2\lambda^3, & \alpha_2=2,\end{cases}$$

with nonzero constants t_1,t_2.

Proof: Applying both sides of (3.4) to v and comparing the corresponding coefficients, we obtain

$$\delta_2(\eta+\lambda+\alpha_2\mu+\beta_2)f(\lambda)-\delta_1(\eta+\mu+\alpha_1\lambda+\beta_1)g(\mu)=0. \tag{3.10}$$

If $(\delta_1,\delta_2)=(1,0)$, (3.10) implies $g(\mu)=0$ and it reduces to the case of Virasoro conformal algebra. We can deduce the result by Proposition 2.1 in [7]. A similar discussion can be made with $(\delta_1,\delta_2)=(0,1)$.

§3.1.2 $0\longrightarrow V_{\delta,\alpha,\beta}\longrightarrow E\longrightarrow\mathbb{C}c_\eta\longrightarrow 0$

Next, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0\longrightarrow V_{\delta,\alpha,\beta}\longrightarrow E\longrightarrow\mathbb{C}c_\eta\longrightarrow 0. \tag{3.11}$$

Then E is isomorphic to $V_{\delta,\alpha,\beta} \oplus \mathbb{C}c_\eta = \mathbb{C}[\partial]v \oplus \mathbb{C}c_\eta$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v = \delta_1(\partial + \alpha_1\lambda + \beta_1)v, \quad B_\lambda v = \delta_2(\partial + \alpha_2\lambda + \beta_2)v, \\ &A_\lambda c_\eta = f(\partial, \lambda)v, \quad B_\lambda c_\eta = g(\partial, \lambda)v, \quad \partial c_\eta = \eta c_\eta + h(\partial)v, \end{aligned} \tag{3.12}$$

where $f(\partial, \lambda), g(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$ and $h(\partial) \in \mathbb{C}[\partial]$.

Lemma 3.1.3 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (3.11) are given by (3.12), and $f(\partial, \lambda) = \delta_1\varphi(\partial + \lambda)(\partial + \alpha_1\lambda + \beta_1), g(\partial, \lambda) = \delta_2\varphi(\partial + \lambda)(\partial + \alpha_2\lambda + \beta_2)$ and $h(\partial) = (\partial - \eta)\varphi(\partial)$, where φ is a polynomial.*

Proof: Assume that (3.11) is a trivial extension, that is, there exists $c'_\eta = kc_\eta + l(\partial)v \in E$, where $0 \neq k \in \mathbb{C}$ and $l(\partial) \in \mathbb{C}[\partial]$, such that $A_\lambda c'_\eta = B_\lambda c'_\eta = 0$ and $\partial c'_\eta = \eta c'_\eta$.

On the other hand, it follows from (3.12) that

$$\begin{aligned} A_\lambda c'_\eta &= (kf(\partial, \lambda) + \delta_1 l(\partial + \lambda)(\partial + \alpha_1\lambda + \beta_1))v, \\ B_\lambda c'_\eta &= (kg(\partial, \lambda) + \delta_2 l(\partial + \lambda)(\partial + \alpha_2\lambda + \beta_2))v, \\ \partial c'_\eta &= k\eta c_\eta + (kh(\partial) + \partial l(\partial))v. \end{aligned}$$

We can obtain the result by comparing both expressions for $A_\lambda c'_\eta, B_\lambda c'_\eta$ and $\partial c'_\eta$.

Conversely, if $f(\partial, \lambda) = \delta_1\varphi(\partial+\lambda)(\partial+\alpha_1\lambda+\beta_1), g(\partial, \lambda) = \delta_2\varphi(\partial+\lambda)(\partial+\alpha_2\lambda+\beta_2)$ and $h(\partial) = (\partial - \eta)\varphi(\partial)$ for some polynomial φ, setting $c'_\eta = c_\eta - \varphi(\partial)v$, we can deduce that (3.11) is a trivial extension.

Theorem 3.1.4 *Let $\mathcal{R}$ be a direct sum of two Virasoro conformal algebras. Then nontrivial extensions of finite irreducible conformal modules of the form (3.11) exist only when $\delta_1 = 1, \alpha_1 = 1, \beta_1 + \eta = 0$ or $\delta_2 = 1, \alpha_2 = 1, \beta_2 + \eta = 0$. In these cases, there exists a unique (up to a scalar) nontrivial extension, i.e. $dim(Ext(\mathbb{C}c_\eta, V_{\delta,\alpha,\beta})) = 1$. Moreover, they are given (up to equivalence) by (3.12). The values of $\delta_i, \alpha_i, \beta_i, i = 1, 2$ and η along with the corresponding polynomials $f(\partial, \lambda), g(\partial, \lambda)$ and $h(\partial)$ giving rise to nontrivial extensions, are listed as follows:*

(i) If $\delta_1 = 1, \alpha_1 = 1, \beta_1 + \eta = 0$, then $\delta_2 = 0, g(\partial, \lambda) = 0, \alpha_2, \beta_2 \in \mathbb{C}$ and $f(\partial, \lambda) = h(\partial) = s$ with nonzero constant s.

(ii) If $\delta_2 = 1, \alpha_2 = 1, \beta_2 + \eta = 0$, then $\delta_1 = 0, f(\partial, \lambda) = 0, \alpha_1, \beta_1 \in \mathbb{C}$ and $g(\partial, \lambda) = h(\partial) = t$ with nonzero constant t.

Proof: Applying both sides of (3.6) and (3.7) to c_η and comparing the corresponding coefficients gives the following equations

$$(\partial + \lambda - \eta) f(\partial, \lambda) = \delta_1 h(\partial + \lambda)(\partial + \alpha_1 \lambda + \beta_1), \tag{3.13}$$

$$(\partial + \lambda - \eta) g(\partial, \lambda) = \delta_2 h(\partial + \lambda)(\partial + \alpha_2 \lambda + \beta_2). \tag{3.14}$$

We only need to consider the case that $(\delta_1, \delta_2) = (1, 0)$. Then $g(\partial, \lambda) = 0$ by (3.14) and the result can be deduced by Proposition 2.2 in [7].

§3.1.3 $0 \longrightarrow V_{\delta,\alpha,\beta} \longrightarrow E \longrightarrow V_{\bar{\delta},\bar{\alpha},\bar{\beta}} \longrightarrow 0$

Finally, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow V_{\delta,\alpha,\beta} \longrightarrow E \longrightarrow V_{\bar{\delta},\bar{\alpha},\bar{\beta}} \longrightarrow 0. \tag{3.15}$$

Then E is isomorphic to $V_{\delta,\alpha,\beta} \oplus V_{\bar{\delta},\bar{\alpha},\bar{\beta}} = \mathbb{C}[\partial]v \oplus \mathbb{C}[\partial]\bar{v}$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v = \delta_1(\partial + \alpha_1 \lambda + \beta_1)v, \quad B_\lambda v = \delta_2(\partial + \alpha_2 \lambda + \beta_2)v, \\ &A_\lambda \bar{v} = \bar{\delta}_1(\partial + \bar{\alpha}_1 \lambda + \bar{\beta}_1)\bar{v} + f(\partial, \lambda)v, \quad B_\lambda \bar{v} = \bar{\delta}_2(\partial + \bar{\alpha}_2 \lambda + \bar{\beta}_2)\bar{v} + g(\partial, \lambda)v, \end{aligned} \tag{3.16}$$

where $f(\partial, \lambda), g(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$.

Lemma 3.1.5 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (3.15) are given by (3.16), and $f(\partial, \lambda) = \delta_1 \varphi(\partial + \lambda)(\partial + \alpha_1 \lambda + \beta_1) - \bar{\delta}_1 \varphi(\partial)(\partial + \bar{\alpha}_1 \lambda + \bar{\beta}_1)$ and $g(\partial, \lambda) = \delta_2 \varphi(\partial + \lambda)(\partial + \alpha_2 \lambda + \beta_2) - \bar{\delta}_2 \varphi(\partial)(\partial + \bar{\alpha}_2 \lambda + \bar{\beta}_2)$ for some polynomial φ.*

Proof: Assume that (3.15) is a trivial extension, that is, there exists $\bar{v}' = k(\partial)v + l(\partial)\bar{v} \in E$, where $k(\partial), l(\partial) \in \mathbb{C}[\partial]$ and $l(\partial) \neq 0$, such that

$$A_\lambda \bar{v}' = \bar{\delta}_1(\partial + \bar{\alpha}_1 \lambda + \bar{\beta}_1)\bar{v}' = \bar{\delta}_1 k(\partial)(\partial + \bar{\alpha}_1 \lambda + \bar{\beta}_1)v + \bar{\delta}_1 l(\partial)(\partial + \bar{\alpha}_1 \lambda + \bar{\beta}_1)\bar{v},$$

$$B_\lambda \bar{v}' = \bar{\delta}_2(\partial + \bar{\alpha}_2\lambda + \bar{\beta}_2)\bar{v}' = \bar{\delta}_2 k(\partial)(\partial + \bar{\alpha}_2\lambda + \bar{\beta}_2)v + \bar{\delta}_2 l(\partial)(\partial + \bar{\alpha}_2\lambda + \bar{\beta}_2)\bar{v}.$$

On the other hand, it follows from (3.16) that

$$A_\lambda \bar{v}' = (\delta_1 k(\partial+\lambda)(\partial + \alpha_1\lambda + \beta_1) + l(\partial+\lambda)f(\partial,\lambda))v + \bar{\delta}_1 l(\partial+\lambda)(\partial + \bar{\alpha}_1\lambda + \bar{\beta}_1)\bar{v},$$
$$B_\lambda \bar{v}' = (\delta_2 k(\partial+\lambda)(\partial + \alpha_2\lambda + \beta_2) + l(\partial+\lambda)g(\partial,\lambda))v + \bar{\delta}_2 l(\partial+\lambda)(\partial + \bar{\alpha}_2\lambda + \bar{\beta}_2)\bar{v}.$$

Comparing both expressions for $A_\lambda \bar{v}'$ and $B_\lambda \bar{v}'$, we can obtain that $l(\partial)$ is a nonzero constant. Then we can give the expressions of $f(\partial,\lambda)$ and $g(\partial,\lambda)$.

Conversely, if $f(\partial,\lambda) = \delta_1\varphi(\partial+\lambda)(\partial+\alpha_1\lambda+\beta_1) - \bar{\delta}_1\varphi(\partial)(\partial+\bar{\alpha}_1\lambda+\bar{\beta}_1)$ and $g(\partial,\lambda) = \delta_2\varphi(\partial+\lambda)(\partial+\alpha_2\lambda+\beta_2) - \bar{\delta}_2\varphi(\partial)(\partial+\bar{\alpha}_2\lambda+\bar{\beta}_2)$ for some polynomial φ, setting $\bar{v}' = -\varphi(\partial)v + \bar{v}$ we can deduce that (3.15) is a trivial extension.

Before classifying all nontrivial extensions of the form (3.15), we give the following lemma for later use.

Lemma 3.1.6 *The equation*

$$c(\partial+\lambda,\mu)(\partial+a\lambda+b) - c(\partial,\mu)(\partial+\mu+\bar{a}\lambda+\bar{b}) = 0 \tag{3.17}$$

for unknown polynomials $c(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$ with $a, \bar{a}, b, \bar{b} \in \mathbb{C}$ has only zero solution.

Proof: Putting $\lambda = 0$ in (3.17), we get $c(\partial,\mu)(b-\mu-\bar{b}) = 0$. So $c = 0$.

Theorem 3.1.7 *Let $\mathcal{R}$ be a direct sum of two Virasoro conformal algebras. Then nontrivial extensions of finite irreducible conformal modules of the form (3.15) only exist in the following cases. Moreover, they are given (up to equivalence) by (3.16). The value of $\delta_i, \bar{\delta}_i, \alpha_i, \bar{\alpha}_i, \beta_i, \bar{\beta}_i, i = 1,2$ and the corresponding polynomials $f(\partial,\lambda)$ and $g(\partial,\lambda)$ giving rise to nontrivial extensions, are listed as follows:*

1. *In the case that $\delta_1 = \bar{\delta}_1 = 1, \delta_2 = \bar{\delta}_2 = 0$, $\alpha_2, \beta_2, \bar{\alpha}_2, \bar{\beta}_2 \in \mathbb{C}, g(\partial,\lambda) = 0, \beta_1 = \bar{\beta}_1, \bar{\alpha}_1 - \alpha_1 \in \{0,1,2,3,4,5,6\}, \alpha_1, \bar{\alpha}_1 \neq 0$ and*
   - *(i) $\bar{\alpha}_1 = \alpha_1$, $f(\partial,\lambda) = s_0 + s_1\lambda$, where $(s_0,s_1) \neq (0,0)$.*
   - *(ii) $\bar{\alpha}_1 - \alpha_1 = 2$, $f(\partial,\lambda) = s\lambda^2(2(\partial+\beta_1)+\lambda)$, where $s \neq 0$.*
   - *(iii) $\bar{\alpha}_1 - \alpha_1 = 3$, $f(\partial,\lambda) = s(\partial+\beta_1)\lambda^2(\partial+\beta_1+\lambda)$, where $s \neq 0$.*

(iv) $\bar{\alpha}_1 - \alpha_1 = 4$, $f(\partial,\lambda) = s\lambda^2(4(\partial+\beta_1)^3 + 6(\partial+\beta_1)^2\lambda - (\partial+\beta_1)\lambda^2 + \alpha_1\lambda^3)$, *where* $s \neq 0$.

(v) $\bar{\alpha}_1 = 1$ *and* $\alpha_1 = -4$, $f(\partial,\lambda) = s((\partial+\beta_1)^4\lambda^2 - 10(\partial+\beta_1)^2\lambda^4 - 17(\partial+\beta_1)\lambda^5 - 8\lambda^6)$, *where* $s \neq 0$.

(vi) $\bar{\alpha}_1 = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha_1 = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$. $f(\partial,\lambda) = s((\partial+\beta_1)^4\lambda^3 - (2\alpha_1+3)(\partial+\beta_1)^3\lambda^4 - 3\alpha_1(\partial+\beta_1)^2\lambda^5 - (3\alpha_1+1)(\partial+\beta_1)\lambda^6 - (\alpha_1+\frac{9}{28})\lambda^7)$, *where* $s \neq 0$.

The value of $dim(Ext(V_{\bar{\delta},\bar{\alpha},\bar{\beta}}, V_{\delta,\alpha,\beta}))$ *is 2 in subcase (i), and 1 in subcases (ii)-(vi).*

2. *In the case that* $\delta_1 = \bar{\delta}_1 = 0, \delta_2 = \bar{\delta}_2 = 1$, $\alpha_1, \beta_1, \bar{\alpha}_1, \bar{\beta}_1 \in \mathbb{C}, f(\partial,\lambda) = 0, \beta_2 = \bar{\beta}_2, \bar{\alpha}_2 - \alpha_2 \in \{0,1,2,3,4,5,6\}, \alpha_2, \bar{\alpha}_2 \neq 0$, *and*

   (i) $\bar{\alpha}_2 = \alpha_2$, $g(\partial,\lambda) = t_0 + t_1\lambda$, *where* $(t_0,t_1) \neq (0,0)$.

   (ii) $\bar{\alpha}_2 - \alpha_2 = 2$, $g(\partial,\lambda) = t\lambda^2(2(\partial+\beta_2)+\lambda)$, *where* $t \neq 0$.

   (iii) $\bar{\alpha}_2 - \alpha_2 = 3$, $g(\partial,\lambda) = t(\partial+\beta_2)\lambda^2(\partial+\beta_2+\lambda)$, *where* $t \neq 0$.

   (iv) $\bar{\alpha}_2 - \alpha_2 = 4$, $g(\partial,\lambda) = t\lambda^2(4(\partial+\beta_2)^3 + 6(\partial+\beta_2)^2\lambda - (\partial+\beta_2)\lambda^2 + \alpha_2\lambda^3)$, *where* $t \neq 0$.

   (v) $\bar{\alpha}_2 = 1$ *and* $\alpha_2 = -4$, $g(\partial,\lambda) = t((\partial+\beta_2)^4\lambda^2 - 10(\partial+\beta_2)^2\lambda^4 - 17(\partial+\beta_2)\lambda^5 - 8\lambda^6)$, *where* $t \neq 0$.

   (vi) $\bar{\alpha}_2 = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha_2 = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$, $g(\partial,\lambda) = t((\partial+\beta_2)^4\lambda^3 - (2\alpha_2+3)(\partial+\beta_2)^3\lambda^4 - 3\alpha_2(\partial+\beta_2)^2\lambda^5 - (3\alpha_2+1)(\partial+\beta_2)\lambda^6 - (\alpha_2+\frac{9}{28})\lambda^7)$, *where* $t \neq 0$.

   The value of $dim(Ext(V_{\bar{\delta},\bar{\alpha},\bar{\beta}}, V_{\delta,\alpha,\beta}))$ *is 2 in subcase (i), and 1 in subcases (ii)-(vi).*

Proof: Applying both sides of (3.3), (3.4) and (3.5) to $\bar{v}$ and comparing the corresponding coefficients of v, we obtain

$$\bar{\delta}_1 f(\partial,\lambda)(\partial+\lambda+\bar{\alpha}_1\mu+\bar{\beta}_1) + \delta_1 f(\partial+\lambda,\mu)(\partial+\alpha_1\lambda+\beta_1) - \bar{\delta}_1 f(\partial,\mu)(\partial+\mu+\bar{\alpha}_1\lambda+\bar{\beta}_1) - \delta_1 f(\partial+\mu,\lambda)(\partial+\alpha_1\mu+\beta_1) = (\lambda-\mu)f(\partial,\lambda+\mu), \tag{3.18}$$

$$\bar{\delta}_2 f(\partial,\lambda)(\partial+\lambda+\bar{\alpha}_2\mu+\bar{\beta}_2) + \delta_1 g(\partial+\lambda,\mu)(\partial+\alpha_1\lambda+\beta_1) - \bar{\delta}_1 g(\partial,\mu)(\partial+\mu+\bar{\alpha}_1\lambda+\bar{\beta}_1) - \delta_2 f(\partial+\mu,\lambda)(\partial+\alpha_2\mu+\beta_2) = 0, \tag{3.19}$$

$$\bar{\delta}_2 g(\partial,\lambda)(\partial+\lambda+\bar{\alpha}_2\mu+\bar{\beta}_2) + \delta_2 g(\partial+\lambda,\mu)(\partial+\alpha_2\lambda+\beta_2)$$

$$-\bar{\delta}_2 g(\partial,\mu)(\partial+\mu+\bar{\alpha}_2\lambda+\bar{\beta}_2)-\delta_2 g(\partial+\mu,\lambda)(\partial+\alpha_2\mu+\beta_2)=(\lambda-\mu)g(\partial,\lambda+\mu). \tag{3.20}$$

If $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(1,1,0,0)$ or $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(0,0,1,1)$, $g(\partial,\lambda)=0$ or $f(\partial,\lambda)=0$ follows from (3.19) and Lemma 3.1.6. Substituting these results back into the original equations and simplifying, we obtain precisely the same equation solved in Theorem 3.2 of [7] (or equivalently, Theorem 2.7 in [21]).

If $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(1,0,0,1)$, then putting $\mu=0$ in (3.18), we can obtain

$$f(\partial+\lambda,0)(\partial+\alpha_1\lambda+\beta_1)=f(\partial,\lambda)(\partial+\lambda+\beta_1).$$

So when $\alpha_1=1$, we have $f(\partial,\lambda)=f(\partial+\lambda,0)=s(\partial+\lambda)$ for some polynomial s. If $\alpha_1\neq 1$, we can denote $f(\partial,\lambda)=h(\partial,\lambda)(\partial+\alpha_1\lambda+\beta_1)$, and thus $f(\partial+\lambda,0)=h(\partial+\lambda,0)(\partial+\lambda+\beta_1)$. Then one can deduce that $h(\partial,\lambda)=h(\partial+\lambda,0)$. It is not difficult to check that $f(\partial,\lambda)=s(\partial+\lambda)(\partial+\alpha_1\lambda+\beta_1)$ for some polynomial s. On the other hand, dealing with (3.20) in a similar way, we have $g(\partial,\lambda)=t(\partial)(\partial+\bar{\alpha}_2\lambda+\bar{\beta}_2)$ for $\bar{\alpha}_2\neq 0$, where t is a polynomial. Putting these results in (3.19), we can obtain

$$\begin{cases} s(\partial+\lambda)(\partial+\lambda+\bar{\alpha}_2\mu+\bar{\beta}_2)+t(\partial+\lambda)(\partial+\lambda+\bar{\alpha}_2\mu+\bar{\beta}_2)(\partial+\lambda+\beta_1)=0, & \alpha_1=1,\\ s(\partial+\lambda)(\partial+\alpha_1\lambda+\beta_1)(\partial+\lambda+\bar{\alpha}_2\mu+\bar{\beta}_2) & \\ \quad +t(\partial+\lambda)(\partial+\lambda+\bar{\alpha}_2\mu+\bar{\beta}_2)(\partial+\alpha_1\lambda+\beta_1)=0, & \alpha_1\neq 1. \end{cases} \tag{3.21}$$

The solutions are concluded as follows.

(i) If $\alpha_1=1$, then $f(\partial,\lambda)=-t(\partial+\lambda)(\partial+\lambda+\beta_1), g(\partial,\lambda)=t(\partial)(\partial+\bar{\alpha}_2\lambda+\bar{\beta}_2)$ for some polynomial t. The extension is trivial.

(ii) If $\alpha_1\neq 1$, then $f(\partial,\lambda)=s(\partial+\lambda)(\partial+\alpha_1\lambda+\beta_1), g(\partial,\lambda)=-s(\partial)(\partial+\bar{\alpha}_2\lambda+\bar{\beta}_2)$ for some polynomial t. The extension is trivial.

If $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(0,1,1,0)$, one can deduce the result similarly.

§3.2 Extensions of finite irreducible modules of solvable rank two Lie conformal algebras

In this section, we classify the extension of irreducible modules over a solvable rank two Lie conformal algebra $\mathcal{R}$. Then there is a basis $\{A, B\}$ of $\mathcal{R}$ such that

$$[A_\lambda A] = Q_1(\partial, \lambda)B,\ [A_\lambda B] = p(\lambda)B,\ [B_\lambda B] = 0, \tag{3.22}$$

for some polynomial $p(\lambda)$ and some skew-symmetric polynomial $Q_1(\partial, \lambda)$ satisfying $p(\lambda)Q_1(\partial, \lambda) = 0$ [4]. If V is a non-trivial finite irreducible $\mathcal{R}$-module, it was shown in [32] that

$$V \cong V_{\phi_A,\phi_B} = \mathbb{C}[\partial]v, \quad A_\lambda v = \phi_A(\lambda)v, \quad B_\lambda v = \phi_B(\lambda)v, \tag{3.23}$$

where $\phi_A(\lambda)$, $\phi_B(\lambda)$ are not zero simultaneously. Moreover, $\phi_B(\lambda) \neq 0$ only if $p(\lambda) = Q_1(\partial, \lambda) = 0$.

By definition 2.2.1, an $\mathcal{R}$-module structure on V is given by $A_\lambda, B_\lambda \in End_{\mathbb{C}}(V)[\lambda]$ such that

$$[A_\lambda, A_\mu] = Q_1(-\lambda-\mu, \lambda)B_{\lambda+\mu}, \tag{3.24}$$

$$[A_\lambda, B_\mu] = p(\lambda)B_{\lambda+\mu}, \tag{3.25}$$

$$[B_\lambda, B_\mu] = 0, \tag{3.26}$$

$$[\partial, A_\lambda] = -\lambda A_\lambda, \tag{3.27}$$

$$[\partial, B_\lambda] = -\lambda B_\lambda. \tag{3.28}$$

§3.2.1 $0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\phi_A,\phi_B} \longrightarrow 0$

First, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\phi_A,\phi_B} \longrightarrow 0. \tag{3.29}$$

Then E is isomorphic to $\mathbb{C}c_\eta \oplus V_{\phi_A,\phi_B} = \mathbb{C}c_\eta \oplus \mathbb{C}[\partial]v$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &\mathcal{R}_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta, \\ &A_\lambda v = \phi_A(\lambda)v + f(\lambda)c_\eta, \quad B_\lambda v = \phi_B(\lambda)v + g(\lambda)c_\eta, \end{aligned} \tag{3.30}$$

where $f(\lambda), g(\lambda) \in \mathbb{C}[\lambda]$.

Lemma 3.2.1 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (3.29) are given by (3.30), and $f(\lambda), g(\lambda)$ are the same scalar multiples of $\phi_A(\lambda), \phi_B(\lambda)$, respectively.*

Proof: Assume that (3.29) is a trivial extension, that is, there exists $v' = kc_\eta + l(\partial)v \in E$, where $k \in \mathbb{C}$ and $0 \neq l(\partial) \in \mathbb{C}[\partial]$, such that

$$\begin{aligned} A_\lambda v' &= \phi_A(\lambda)v' = k\phi_A(\lambda)c_\eta + \phi_A(\lambda)l(\partial)v, \\ B_\lambda v' &= \phi_B(\lambda)v' = k\phi_B(\lambda)c_\eta + \phi_B(\lambda)l(\partial)v. \end{aligned}$$

On the other hand, it follows from (3.30) that

$$\begin{aligned} A_\lambda v' &= f(\lambda)l(\eta+\lambda)c_\eta + \phi_A(\lambda)l(\partial+\lambda)v, \\ B_\lambda v' &= g(\lambda)l(\eta+\lambda)c_\eta + \phi_B(\lambda)l(\partial+\lambda)v. \end{aligned}$$

We can obtain that $l(\partial)$ is a nonzero constant by comparing both expressions for $A_\lambda v'$ and $B_\lambda v'$. Thus $f(\lambda)$ and $g(\lambda)$ are the same scalar multiple of $\phi_A(\lambda)$ and $\phi_B(\lambda)$ respectively.

Conversely, if $f(\lambda) = k\phi_A(\lambda)$ and $g(\lambda) = k\phi_B(\lambda)$ for some $k \in \mathbb{C}$, setting $v' = kc_\eta + v$ we can deduce that (3.29) is a trivial extension.

The following key lemma plays a crucial role in simplifying the calculations in our classification of nontrivial extensions.

Lemma 3.2.2 *Let $a(\lambda), b(\mu), c(\lambda), d(\mu)$ be four polynomials in $\mathbb{C}[\lambda, \mu]$. If $a(\lambda)$ and $b(\mu)$ are not 0 simultaneously, the equation*

$$a(\lambda)d(\mu) - b(\mu)c(\lambda) = 0$$

implies that $c(\lambda)$ *and* $d(\mu)$ *are the same multiples of* $a(\lambda)$ *and* $b(\mu)$ *respectively. Particularly, if both* $a(\lambda)$ *and* $b(\mu)$ *are not 0, the multiple is a scalar multiple.*

Proof: Without loss of generality, we assume that $a(\lambda) \neq 0$. If $b(\mu) = 0$, then $d(\mu) = 0$. In this case, the conclusion is founded whatever $c(\lambda)$ is. If $b(\mu) \neq 0$, we have $\frac{c(\lambda)}{a(\lambda)} = \frac{d(\mu)}{b(\mu)} := e(\lambda, \mu)$. It is easy to see that $e(\lambda, \mu)$ is a constant, and the proof is done.

Theorem 3.2.3 *For a solvable rank two Lie conformal algebra* $\mathcal{R}$*, nontrivial extensions of finite irreducible conformal modules of the form (3.29) exist only if* $p(\lambda) \neq 0$ *and* $Q_1(\partial, \lambda) = 0$*. Moreover, they are given (up to equivalence) by (3.30). The values of* η *along with the corresponding polynomials* $\phi_A(\lambda), \phi_B(\lambda), f(\lambda)$ *and* $g(\lambda)$ *giving rise to nontrivial extensions, are listed as follows:* $\eta \in \mathbb{C}$, $\phi_A(\lambda) = -p(\lambda)$, $\phi_B(\lambda) = 0$, $f(\lambda) = 0$ *and* $g(\lambda)$ *is a nonzero constant. Thus* $dim(Ext(V_{\phi_A,\phi_B}, \mathbb{C}c_\eta)) = \delta_{\phi_A(\lambda)+p(\lambda),0}$.

Proof: Applying both sides of (3.24), (3.25) and (3.26) to v and comparing the corresponding coefficients, we obtain

$$Q_1(-\lambda - \mu, \lambda)\phi_B(\lambda + \mu) = p(\lambda)\phi_B(\lambda + \mu) = 0, \tag{3.31}$$

$$f(\lambda)\phi_A(\mu) - f(\mu)\phi_A(\lambda) = Q_1(-\lambda - \mu, \lambda)g(\lambda + \mu), \tag{3.32}$$

$$f(\lambda)\phi_B(\mu) - g(\mu)\phi_A(\lambda) = p(\lambda)g(\lambda + \mu), \tag{3.33}$$

$$g(\lambda)\phi_B(\mu) - g(\mu)\phi_B(\lambda) = 0. \tag{3.34}$$

We first consider the case that $p(\lambda) = Q_1(\partial, \lambda) = 0$ and take it in (3.32), (3.33) and (3.34). Since $\phi_A(\lambda)$, $\phi_B(\lambda)$ are not zero simultaneously, we can assume that $\phi_A(\lambda) \neq 0$. By Lemma 3.2.2 and (3.32), we have $f(\lambda) = k\phi_A(\lambda)$ for some $k \in \mathbb{C}$. It can be deduced from (3.33) that $g(\mu) = k\phi_B(\mu)$ for the same k. According to Lemma 3.2.1, the extension is trivial in this case.

Now we assume that $p(\lambda) = 0$ and $Q_1(\partial, \lambda) \neq 0$, which implies that $\phi_B(\lambda) = 0$. Then $\phi_A(\lambda)$ must not be zero and $g(\lambda)$ is forced to be 0 by (3.33). Taking these results in (3.32) and applying Lemma 3.2.2, we can see that $f(\lambda)$ is a scalar multiple of $\phi_A(\lambda)$ and thus the extension is trivial.

Lastly, we discuss the case that $p(\lambda) \neq 0$ but $Q_1(\partial, \lambda) = 0$. Then $\phi_B(\lambda) = 0$ and $\phi_A(\lambda) \neq 0$. In this case, (3.32) means that $f(\lambda) = k\phi_A(\lambda)$ for some $k \in \mathbb{C}$ by

Lemma 3.2.2. So if $g(\lambda) = 0$, the extension is trivial. If $g(\lambda) \neq 0$, we can obtain that $\phi_A(\lambda) = -p(\lambda)$ by comparing the coefficients of the highest term of μ in (3.33), which implies $g(\lambda)$ is a nonzero constant. By Lemma 3.2.1, the corresponding extension is nontrivial.

Since $p(\lambda)Q_1(\partial, \lambda) = 0$, we have completed the proof.

§3.2.2 $0 \longrightarrow V_{\phi_A,\phi_B} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0$

Next, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow V_{\phi_A,\phi_B} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0. \tag{3.35}$$

Then E is isomorphic to $V_{\phi_A,\phi_B} \oplus \mathbb{C}c_\eta = \mathbb{C}[\partial]v \oplus \mathbb{C}c_\eta$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v = \phi_A(\lambda)v, \quad B_\lambda v = \phi_B(\lambda)v, \\ &A_\lambda c_\eta = f(\partial, \lambda)v, \quad B_\lambda c_\eta = g(\partial, \lambda)v, \quad \partial c_\eta = \eta c_\eta + h(\partial)v, \end{aligned} \tag{3.36}$$

where $f(\partial, \lambda), g(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$ and $h(\partial) \in \mathbb{C}[\partial]$.

Lemma 3.2.4 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (3.35) are given by (3.36), and $f(\partial, \lambda) = \varphi(\partial + \lambda)\phi_A(\lambda), g(\partial, \lambda) = \varphi(\partial + \lambda)\phi_B(\lambda)$ and $h(\partial) = (\partial - \eta)\varphi(\partial)$, where φ is a polynomial.*

Proof: Assume that (3.35) is a trivial extension, that is, there exists $c'_\eta = kc_\eta + l(\partial)v \in E$, where $0 \neq k \in \mathbb{C}$ and $l(\partial) \in \mathbb{C}[\partial]$, such that $A_\lambda c'_\eta = B_\lambda c'_\eta = 0$ and $\partial c'_\eta = \eta c'_\eta = k\eta c_\eta + \eta l(\partial)v$.

On the other hand, it follows from (3.36) that

$$\begin{aligned} A_\lambda c'_\eta &= (kf(\partial, \lambda) + l(\partial + \lambda)\phi_A(\lambda))v, \\ B_\lambda c'_\eta &= (kg(\partial, \lambda) + l(\partial + \lambda)\phi_B(\lambda))v, \\ \partial c'_\eta &= k\eta c_\eta + (kh(\partial) + \partial l(\partial))v. \end{aligned}$$

We can obtain the result by comparing both expressions for $A_\lambda c'_\eta, B_\lambda c'_\eta$ and $\partial c'_\eta$.

Conversely, if $f(\partial,\lambda) = \varphi(\partial+\lambda)\phi_A(\lambda), g(\partial,\lambda) = \varphi(\partial+\lambda)\phi_B(\lambda)$ and $h(\partial) = (\partial-\eta)\varphi(\partial)$ for some polynomial φ, setting $c'_\eta = c_\eta - \varphi(\partial)v$ we can deduce that (3.35) is a trivial extension.

Theorem 3.2.5 *For a solvable rank two Lie conformal algebra $\mathcal{R}$, nontrivial extensions of finite irreducible conformal modules of the form (3.35) do not exist, that is, $dim(Ext(\mathbb{C}c_\eta, V_{\phi_A,\phi_B})) = 0$.*

Proof: Applying both sides of (3.27) and (3.28) to c_η and comparing the corresponding coefficients gives the following equations

$$(\partial+\lambda-\eta)f(\partial,\lambda) = h(\partial+\lambda)\phi_A(\lambda), \tag{3.37}$$

$$(\partial+\lambda-\eta)g(\partial,\lambda) = h(\partial+\lambda)\phi_B(\lambda). \tag{3.38}$$

Since $\phi_A(\lambda)$ and $\phi_B(\lambda)$ are not all equal to zero, the above equations imply that there exists a polynomial φ such that $h(\partial) = (\partial-\eta)\varphi(\partial)$. Then we have $f(\partial,\lambda) = \varphi(\partial+\lambda)\phi_A(\lambda)$ and $g(\partial,\lambda) = \varphi(\partial+\lambda)\phi_B(\lambda)$. By Lemma 3.2.4, extensions of finite irreducible $\mathcal{R}$-modules of the form (3.35) are always trivial.

§3.2.3 $0 \longrightarrow V_{\phi_A,\phi_B} \longrightarrow E \longrightarrow V_{\bar\phi_A,\bar\phi_B} \longrightarrow 0$

Finally, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow V_{\phi_A,\phi_B} \longrightarrow E \longrightarrow V_{\bar\phi_A,\bar\phi_B} \longrightarrow 0. \tag{3.39}$$

Then E is isomorphic to $V_{\phi_A,\phi_B} \oplus V_{\bar\phi_A,\bar\phi_B} = \mathbb{C}[\partial]v \oplus \mathbb{C}[\partial]\bar v$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v = \phi_A(\lambda)v, \quad B_\lambda v = \phi_B(\lambda)v, \\ &A_\lambda \bar v = \bar\phi_A(\lambda)\bar v + f(\partial,\lambda)v, \quad B_\lambda \bar v = \bar\phi_B(\lambda)\bar v + g(\partial,\lambda)v, \end{aligned} \tag{3.40}$$

where $f(\partial,\lambda), g(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$.

Lemma 3.2.6 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (3.39) are given by (3.40), and* $f(\partial,\lambda) = \varphi(\partial+\lambda)\phi_A(\lambda) - \varphi(\partial)\bar{\phi}_A(\lambda)$ *and* $g(\partial,\lambda) = \varphi(\partial+\lambda)\phi_B(\lambda) - \varphi(\partial)\bar{\phi}_B(\lambda)$ *for some polynomial* φ.

Proof: Assume that (3.39) is a trivial extension, that is, there exists $\bar{v}' = k(\partial)v + l(\partial)\bar{v} \in E$, where $k(\partial), l(\partial) \in \mathbb{C}[\partial]$ and $l(\partial) \neq 0$, such that

$$A_\lambda \bar{v}' = \bar{\phi}_A(\lambda)\bar{v}' = k(\partial)\bar{\phi}_A(\lambda)v + l(\partial)\bar{\phi}_A(\lambda)\bar{v},$$
$$B_\lambda \bar{v}' = \bar{\phi}_B(\lambda)\bar{v}' = k(\partial)\bar{\phi}_B(\lambda)v + l(\partial)\bar{\phi}_B(\lambda)\bar{v}.$$

On the other hand, it follows from (3.40) that

$$A_\lambda \bar{v}' = (k(\partial+\lambda)\phi_A(\lambda) + l(\partial+\lambda)f(\partial,\lambda))v + l(\partial+\lambda)\bar{\phi}_A(\lambda)\bar{v},$$
$$B_\lambda \bar{v}' = (k(\partial+\lambda)\phi_B(\lambda) + l(\partial+\lambda)g(\partial,\lambda))v + l(\partial+\lambda)\bar{\phi}_B(\lambda)\bar{v}.$$

Comparing both expressions for $A_\lambda \bar{v}'$ and $B_\lambda \bar{v}'$, we can obtain that $l(\partial)$ is a nonzero constant. Then we can give the expressions of $f(\partial,\lambda)$ and $g(\partial,\lambda)$.

Conversely, if $f(\partial,\lambda) = \varphi(\partial+\lambda)\phi_A(\lambda) - \varphi(\partial)\bar{\phi}_A(\lambda)$ and $g(\partial,\lambda) = \varphi(\partial+\lambda)\phi_B(\lambda) - \varphi(\partial)\bar{\phi}_B(\lambda)$ for some polynomial φ, setting $\bar{v}' = -\varphi(\partial)v + \bar{v}$ we can deduce that (3.39) is a trivial extension.

To better characterize the classification procedure of nontrivial extensions, we advance part of the computation in the following lemmas.

Lemma 3.2.7 *The solutions of the equation*

$$c(\partial,\lambda)b(\mu) + c(\partial+\lambda,\mu)a(\lambda) - c(\partial,\mu)b(\lambda) - c(\partial+\mu,\lambda)a(\mu) = 0 \tag{3.41}$$

for unknown polynomial $c(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$ *are given as follows.*

- *If* $a(\lambda) = b(\lambda) = 0$, *the equation holds for any polynomial in* $\mathbb{C}[\partial,\lambda]$.
- *If* $a(\lambda) = b(\lambda) \neq 0$, *then* $c(\partial,\lambda) = a(\lambda)(\varphi_1(\partial+\lambda) - \varphi_1(\partial)) + \varphi_2(\lambda)$, *where* φ_1, φ_2 *are polynomials.*
- *If* $a(\lambda) \neq b(\lambda)$, *then* $c(\partial,\lambda) = a(\lambda)\varphi(\partial+\lambda) - b(\lambda)\varphi(\partial)$ *for some polynomial* φ.

Proof: If $a(\lambda) = b(\lambda) = 0$ or $c(\partial, \lambda) = 0$, the result is obvious.

Now we assume $a(\lambda) = b(\lambda) \neq 0$, $c(\partial, \lambda) \neq 0$, and $c(\partial, \lambda) = \sum_{i=0}^{m} c_i(\lambda)\partial^i$ with $c_m(\lambda) \neq 0$. The result can be obtained by induction on m. When $m = 0$, the variation of (3.41)

$$(c(\partial + \lambda, \mu) - c(\partial, \mu))a(\lambda) = (c(\partial + \mu, \lambda) - c(\partial, \lambda))a(\mu) \tag{3.42}$$

implies the original equation is established. Assume the conclusion holds for $m = n$ ($n \geq 0$) and consider the case that $m = n + 1$. Comparing the coefficients of ∂^n of (3.42), we have

$$\lambda c_{n+1}(\mu)a(\lambda) = \mu c_{n+1}(\lambda)a(\mu).$$

So $c_{n+1}(\lambda) = k\lambda a(\lambda)$ for some nonzero constant k by Lemma 3.2.2. Let $k_{n+1} = \frac{k}{n+2}$, $d(\partial, \lambda) = k_{n+1}a(\lambda)((\partial + \lambda)^{n+2} - \partial^{n+2})$ and $e(\partial, \lambda) = c(\partial, \lambda) - d(\partial, \lambda)$. Since both c and d satisfy (3.42), by induction, $e(\partial, \lambda) = a(\lambda)(\varphi_0(\partial + \lambda) - \varphi_0(\partial)) + \varphi_2(\lambda)$ with polynomials φ_0, φ_2. Set $\varphi_1(\partial) = \varphi_0(\partial) + k_{n+1}\partial^{n+2}$, and the expression of $c(\partial, \lambda)$ follows.

For $a(\lambda) \neq b(\lambda)$, we set $c(\partial, \lambda) = \sum_{i=0}^{m} c_i(\lambda)\partial^i$ with $c_m(\lambda) \neq 0$. If $m = 0$, the equation (3.41) can be rewritten as

$$c_0(\lambda)(a(\mu) - b(\mu)) - c_0(\mu)(a(\lambda) - b(\lambda)) = 0,$$

which means $c(\partial, \lambda) = k(a(\lambda) - b(\lambda))$ for some nonzero constant k. Thus the conclusion holds for $m = 0$. Assume the conclusion holds for $m = n$ ($n \geq 0$) and consider the case that $m = n + 1$. The coefficients of ∂^{n+1} of the two sides of (3.41) imply $c_{n+1}(\lambda) = t(a(\lambda) - b(\lambda))$ with $t \neq 0$. Let $\varphi_1(\partial) = t\partial^{n+1}$, $d(\partial, \lambda) = a(\lambda)\varphi_1(\partial + \lambda) - b(\lambda)\varphi_1(\partial)$ and $e(\partial, \lambda) = c(\partial, \lambda) - d(\partial, \lambda)$. Then $d(\partial, \lambda)$ satisfies (3.41), and so does $e(\partial, \lambda)$. By the assumption, we have $e(\partial, \lambda) = a(\lambda)\varphi_2(\partial + \lambda) - b(\lambda)\varphi_2(\partial)$ for some polynomial φ_2. Setting $\varphi(\partial) = \varphi_1(\partial) + \varphi_2(\partial)$, one can find that the conclusion also holds for $m = n+1$.

Lemma 3.2.8 *Let $f(\lambda, \mu)$ be a nonzero polynomial in $\mathbb{C}[\lambda, \mu]$ satisfying $f(\lambda, \mu) = -f(\mu, \lambda)$. Denote the coefficient of $\lambda^i\mu^j$ in f by f_{ij} and the antisymmetric matrix consisting of f_{ij}'s by M. Assume $f_{i_0, j_0} \neq 0$ ($i_0 < j_0$). Then the following statements are equivalent.*

(i) There exist polynomials P_1, P_2 such that

$$P_1(\lambda)P_2(\mu) - P_1(\mu)P_2(\lambda) = f(\lambda, \mu). \tag{3.43}$$

(ii) For any i, j, k, l, $f_{ij}f_{kl} - f_{ik}f_{jl} + f_{jk}f_{il} = 0$.

(iii) $rank(M) = 2$.

Proof: (i)⇒(ii). Assume that (3.43) has been established and the expressions of P_1, P_2 are given by $P_1(\lambda) = \sum_{i=0}^{n} p_{i1}\lambda^i$, $P_2(\lambda) = \sum_{i=0}^{n} p_{i2}\lambda^i$. Let $\mathbf{P}_1 = (p_{01}, p_{11}, \cdots, p_{n1})^T$ and $\mathbf{P}_2 = (p_{02}, p_{12}, \cdots, p_{n2})^T$. Then (ii) follows from $f_{ij} = p_{i1}p_{j2} - p_{j1}p_{i2}, \forall i, j$.

(ii)⇒(iii). Assume (ii) and perform the following elementary row and column transformations on M as follows.

$$M \xrightarrow[\mathbf{c}_k - \frac{f_{k-1,j_0}}{f_{i_0,j_0}}\mathbf{c}_{i_0+1} + \frac{f_{k-1,i_0}}{f_{i_0,j_0}}\mathbf{c}_{j_0+1}, \forall k \neq i_0+1, j_0+1]{\mathbf{r}_k - \frac{f_{k-1,j_0}}{f_{i_0,j_0}}\mathbf{r}_{i_0+1} + \frac{f_{k-1,i_0}}{f_{i_0,j_0}}\mathbf{r}_{j_0+1}, \forall k \neq i_0+1, j_0+1} \xrightarrow[\mathbf{c}_{i_0+1} \leftrightarrow \mathbf{c}_1]{\mathbf{r}_{i_0+1} \leftrightarrow \mathbf{r}_1} \xrightarrow[\mathbf{c}_{j_0+1} \leftrightarrow \mathbf{c}_2]{\mathbf{r}_{j_0+1} \leftrightarrow \mathbf{r}_2}$$
$$diag\left\{\begin{pmatrix} 0 & f_{i_0,j_0} \\ -f_{i_0,j_0} & 0 \end{pmatrix}, 0, \cdots, 0\right\}.$$

Thus $rank(M) = 2$.

(iii)⇒(i). Assume that $rank(M) = 2$ and denote the order of M by m ($m \geq 2$). Then there exists an invertible matrix $P = \{q_{ij}\}$ of order m such that

$$M = Pdiag\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0, \cdots, 0\right\}P^T.$$

Define $P_1(\lambda) = \sum_{i=0}^{m-1} q_{i+1,1}\lambda^i$, $P_2(\lambda) = \sum_{i=0}^{m-1} q_{i+1,2}\lambda^i$. It is easy to check that P_1, P_2 satisfy 3.43.

For $f(\lambda, \mu)$ meeting the condition in Lemma 3.2.8, based on the proof of (ii)⇒(iii) and (iii)⇒(i), we can write down a pair of polynomials P_1, P_2 satisfying (3.43) as follows:

$$P_1(\lambda) = \sum_k f_{k,j_0}\lambda^k, \; P_2(\lambda) = -\frac{1}{f_{i_0,j_0}}\sum_k f_{k,i_0}\lambda^k. \tag{3.44}$$

With the above discussion, we can solve the following equation.

Lemma 3.2.9 *Let $f(\lambda,\mu)$ be a polynomial satisfying the condition in Lemma 3.2.8 with P_1, P_2 defined in (3.44). The equation*

$$a(\lambda)b(\mu) - a(\mu)b(\lambda) = f(\lambda,\mu) \tag{3.45}$$

for unknown polynomial $b(\partial) \in \mathbb{C}[\partial]$ has solutions only when $a(\lambda) = \frac{a_{i_0}}{f_{i_0,j_0}}P_1(\lambda) + a_{j_0}P_2(\lambda)$ and not all a_{i_0}, a_{j_0} equal 0. If solutions do exist, they are given as follows.

- *If $a_{i_0} \neq 0$, then $b(\lambda) = ka(\lambda) + \frac{f_{i_0,j_0}}{a_{i_0}}P_2(\lambda)$, where $k \in \mathbb{C}$.*
- *If $a_{i_0} = 0, a_{j_0} \neq 0$, then $b(\lambda) = ka(\lambda) - \frac{1}{a_{j_0}}P_1(\lambda)$, where $k \in \mathbb{C}$.*

Proof: First we consider the special case that the coefficient matrix M of f is

$$diag\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0, \cdots, 0\right\}.$$

In this case, $P_1(\lambda) = 1, P_2(\lambda) = \lambda$. Denote the order of M by $n+1$ and write $a(\lambda) = \sum_{i=0}^{n} a_i\lambda^i, b(\lambda) = \sum_{i=0}^{n} b_i\lambda^i$. Taking them in (3.45), we have

$$\begin{cases} a_0b_1 - a_1b_0 = 1, \\ a_ib_j - a_jb_i = 0, \forall i < j, (i,j) \neq (0,1). \end{cases} \tag{3.46}$$

So the equation has solutions only when not all a_0, a_1 are equal to 0. Assume the solutions do exist. If $a_0 \neq 0$, (3.46) implies $b_1 = \frac{1+a_1b_0}{a_0}, b_j = \frac{a_jb_0}{a_0}, \forall j > 1$ and then $a_j = 0, \forall j > 1$. Thus, $a(\lambda) = a_0 + a_1\lambda$ and $b(\lambda) = b_0 + \frac{1+a_1b_0}{a_0}\lambda$. If $a_0 = 0$, then (3.46) implies $a_1 \neq 0, b_0 = -\frac{1}{a_1}$ and $a_j, b_j = 0, \forall j > 1$. Therefore, $a(\lambda) = a_1\lambda$ and $b(\lambda) = -\frac{1}{a_1} + b_1\lambda$.

For the general case, we use $\mathbf{a} = (a_0, a_1, \cdots, a_n)^T, \mathbf{b} = (b_0, b_1, \cdots, b_n)^T$ to mean the coefficient matrices of $a(\lambda), b(\lambda)$ and P to mean the invertible matrix corresponding to $P_1(\lambda), P_2(\lambda)$ as mentioned in the proof of (iii)$\Rightarrow$(i). Denote $\mathbf{a}' = P^{-1}\mathbf{a}, \mathbf{b}' = P^{-1}\mathbf{b}$.

Since $M = \mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T$, we have

$$diag\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0, \cdots, 0\right\} = \mathbf{a}'\mathbf{b}'^T - \mathbf{b}'\mathbf{a}'^T.$$

By the above discussion, $\mathbf{b}'$ exists only when $\mathbf{a}' = ({a'}_0, {a'}_1, 0, \cdots, 0)$ with ${a'}_0, {a'}_1$ not being 0 simultaneously, which is equivalent to $a(\lambda) = {a'}_0 P_1(\lambda) + {a'}_1 P_2(\lambda)$ with $({a'}_0, {a'}_1) \neq (0, 0)$. Focusing on the coefficient of $\lambda^{i_0}, \lambda^{j_0}$, we have ${a'}_0 = \frac{a_{i0}}{f_{i_0,j_0}}, {a'}_1 = a_{j_0}$. Similarly, we can obtain the general expression of $b(\lambda)$ in different cases.

Theorem 3.2.10 *For a solvable rank two Lie conformal algebra $\mathcal{R}$, nontrivial extensions of finite irreducible conformal modules of the form (3.39) always exist. Moreover, they are given (up to equivalence) by (3.40). The corresponding polynomials $\phi_A(\lambda), \phi_B(\lambda), \bar{\phi}_A(\lambda), \bar{\phi}_B(\lambda), f(\partial, \lambda)$ and $g(\partial, \lambda)$ giving rise to nontrivial extensions, are listed as follows:*

1. *In the case that $p(\lambda) = Q_1(\partial, \lambda) = 0$.*

   (i) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) = 0, \phi_B(\lambda) = \bar{\phi}_B(\lambda) \neq 0$, then $f(\partial, \lambda) = s(\lambda), g(\partial, \lambda) = t(\lambda)$, where s, t are polynomials, and either $s \neq 0$ or $t(\lambda)$ is not a scalar multiple of $\lambda\phi_B(\lambda)$.

   (ii) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, \phi_B(\lambda) = \bar{\phi}_B(\lambda) = 0$, then $f(\partial, \lambda) = s(\lambda), g(\partial, \lambda) = t(\lambda)$, where s, t are polynomials, and either $t \neq 0$ or $s(\lambda)$ is not a scalar multiple of $\lambda\phi_A(\lambda)$.

   (iii) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, \phi_B(\lambda) = \bar{\phi}_B(\lambda) \neq 0$, then $f(\partial, \lambda) = s(\lambda), g(\partial, \lambda) = t(\lambda)$, where s, t are polynomials, and $s(\lambda), t(\lambda)$ are not the same scalar multiple of $\lambda\phi_A(\lambda), \lambda\phi_B(\lambda)$ respectively.

2. *In the case that $p(\lambda) = 0, Q_1(\partial, \lambda) \neq 0$, we always have $\phi_B(\lambda) = \bar{\phi}_B(\lambda) = 0$.*

   (i) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, g(\partial, \lambda) = 0$, then $f(\partial, \lambda) = s(\lambda)$, where the polynomial $s(\lambda)$ is not a scalar multiple of $\lambda\phi_A(\lambda)$.

   (ii) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, g(\partial, \lambda) = t(\lambda) \neq 0$ such that the coefficient matrix $M = \{q_{ij}\}$ of $Q_1(-\lambda - \mu, \lambda)t(\lambda + \mu)$ is of rank 2 and for $q_{i_0,j_0} \neq 0$, $\phi_A(\lambda) = \frac{a_{i0}}{f_{i_0,j_0}}(\sum_k f_{k,j_0}\lambda^k) - \frac{a_{j_0}}{f_{i_0,j_0}}(\sum_k f_{k,i_0}\lambda^k)$ with the coefficients a_{i_0-1}, a_{j_0-1}

of $\lambda^{i_0-1}, \lambda^{j_0-1}$ in ϕ_A are not all 0, then

$$f(\partial,\lambda)=\begin{cases}-\frac{1}{a_{i_0-1}}(\sum_k q_{k,j_0}\lambda^k)\partial+s(\lambda), & \text{if } a_{i_0-1}\neq 0,\\ -\frac{1}{a_{j_0-1}}(\sum_k q_{k,i_0}\lambda^k)\partial+s(\lambda), & \text{if } a_{i_0-1}=0,\end{cases}$$

where s is a polynomial.

3. *In the case that $p(\lambda)\neq 0, Q_1(\partial,\lambda)=0$, we have $\phi_B(\lambda)=\bar{\phi}_B(\lambda)=0$.*

   (i) *If $\phi_A(\lambda)=\bar{\phi}_A(\lambda)\neq 0$, then $f(\partial,\lambda)=s(\lambda), g(\partial,\lambda)=0$, where s_1, s_2 are polynomials, and $s(\lambda)$ is not a scalar multiple of $\lambda\phi_A(\lambda)$.*

   (ii) *If $\phi_A(\lambda)\neq\bar{\phi}_A(\lambda)$, then $f(\partial,\lambda)=0, \phi_A(\lambda)-\bar{\phi}_A(\lambda)=p(\lambda)$, and*

$$g(\partial,\lambda)=\begin{cases}k_1(\partial+\frac{1}{r}\lambda)+k_2, & p(\lambda)=r\phi_A(\lambda) \text{ and } r\neq 1,\\ k_1, & p(\lambda) \text{ is not a scalar multiple of } \phi_A(\lambda),\end{cases}$$

where $k_1, k_2\in\mathbb{C}$ and $g(\partial,\lambda)\neq 0$.

The space of $Ext(V_{\bar{\phi}_A,\bar{\phi}_B}, V_{\phi_A,\phi_B})$ is of infinite dimension in all of the above subcases but (3)-(ii).

Proof: Applying both sides of (3.24), (3.25) and (3.26) to $\bar{v}$ and comparing the corresponding coefficients, we obtain

$$Q_1(-\lambda-\mu,\lambda)\bar{\phi}_B(\lambda+\mu)=p(\lambda)\bar{\phi}_B(\lambda+\mu)=0, \tag{3.47}$$

$$f(\partial,\lambda)\bar{\phi}_A(\mu)+f(\partial+\lambda,\mu)\phi_A(\lambda)-f(\partial,\mu)\bar{\phi}_A(\lambda)-f(\partial+\mu,\lambda)\phi_A(\mu)$$
$$=Q_1(-\lambda-\mu,\lambda)g(\partial,\lambda+\mu), \tag{3.48}$$

$$f(\partial,\lambda)\bar{\phi}_B(\mu)+g(\partial+\lambda,\mu)\phi_A(\lambda)-g(\partial,\mu)\bar{\phi}_A(\lambda)-f(\partial+\mu,\lambda)\phi_B(\mu)=p(\lambda)g(\partial,\lambda+\mu), \tag{3.49}$$

$$g(\partial,\lambda)\bar{\phi}_B(\mu)+g(\partial+\lambda,\mu)\phi_B(\lambda)-g(\partial,\mu)\bar{\phi}_B(\lambda)-g(\partial+\mu,\lambda)\phi_B(\mu)=0. \tag{3.50}$$

Case 1. $p(\lambda)=Q_1(\partial,\lambda)=0$.

(i) If $\phi_A(\lambda)=\bar{\phi}_A(\lambda)=0, \phi_B(\lambda)=\bar{\phi}_B(\lambda)\neq 0$, we can obtain $f(\partial,\lambda)\in\mathbb{C}[\lambda]$ from (3.49). And by (3.50) and Lemma 3.2.7, we have $g(\partial,\lambda)=\phi_B(\lambda)(t_1(\partial+\lambda)-$

$t_1(\partial)) + t_2(\lambda)$ for some polynomials t_1, t_2. According to Lemma 3.2.6, this extension is equivalent to the extension with the same $f(\partial, \lambda)$ and $g(\partial, \lambda) = t_2(\lambda)$, and is nontrivial only if $f(\partial, \lambda) \neq 0$ or $t_2(\lambda)$ is not a scalar multiple of $\lambda\phi_B(\lambda)$.

(ii) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) = 0, \phi_B(\lambda) \neq \bar{\phi}_B(\lambda)$, (3.49) implies $f(\partial, \lambda) = 0$ by comparing the coefficients of the highest order term with respect to ∂. Meanwhile, (3.50) means $g(\partial, \lambda) = \phi_B(\lambda)t(\partial + \lambda) - \bar{\phi}_B(\lambda)t(\partial)$ for some polynomial t by Lemma 3.2.7. In this subcase, the extension is trivial.

(iii) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, \phi_B(\lambda) = \bar{\phi}_B(\lambda) \neq 0$, we can deduce that $f(\partial, \lambda) = \phi_A(\lambda)(s_1(\partial + \lambda) - s_1(\partial)) + s_2(\lambda), g(\partial, \lambda) = \phi_B(\lambda)(t_1(\partial + \lambda) - t_1(\partial)) + t_2(\lambda)$ for some polynomials s_1, s_2, t_1, t_2 from (3.48) and (3.50). Taking them in (3.49) and setting $r(\lambda) = t_1(\lambda) - s_1(\lambda)$, we can obtain

$$r(\partial + \lambda + \mu) - r(\partial + \lambda) - r(\partial + \mu) + r(\partial) = 0,$$

which implies $r(\lambda) = r_1\lambda + r_0$ for some $r_0, r_1 \in \mathbb{C}$. This extension is equivalent to the extension with $f(\partial, \lambda) = s_2(\lambda)$ and $g(\partial, \lambda) = t'(\lambda)$ where $t'(\lambda) = r_1\lambda\phi_B(\lambda) + t_2(\lambda)$. By Lemma 3.2.6, the extension is nontrivial if and only if $s_2(\lambda), t'(\lambda)$ are not the same scalar multiple of $\lambda\phi_A(\lambda), \lambda\phi_B(\lambda)$ respectively.

(iv) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0, \phi_B(\lambda) \neq \bar{\phi}_B(\lambda)$, then $f(\partial, \lambda) = \phi_A(\lambda)(s_1(\partial + \lambda) - s_1(\partial)) + s_2(\lambda), g(\partial, \lambda) = \phi_B(\lambda)t(\partial + \lambda) - \bar{\phi}_B(\lambda)t(\partial)$ for some polynomials s_1, s_2, t from (3.48) and (3.50). Taking them in (3.49) and setting $r(\lambda) = t(\lambda) - s_1(\lambda)$, we can obtain

$$\phi_A(\lambda)\phi_B(\mu)(r(\partial+\lambda+\mu)-r(\partial+\mu))-\phi_A(\lambda)\bar{\phi}_B(\mu)(r(\partial+\lambda)-r(\partial)) = s_2(\lambda)(\phi_B(\mu)-\bar{\phi}_B(\mu)).$$

Denote the degree of $r(\lambda)$ by m. If $m \geq 2$, comparing the coefficients of ∂^{m-1} on each side of the above equation, we can get a contradiction. Let $r(\lambda) = r_1\lambda + r_0$ with $r_1, r_2 \in \mathbb{C}$. Then we have $s_2(\lambda) = r_1\lambda\phi_A(\lambda)$. And the extension is always trivial in this subcase because $f(\partial, \lambda) = \phi_A(\lambda)(s_1(\partial + \lambda) - s_1(\partial)) = \phi_A(\lambda)(t(\partial + \lambda) - t(\partial))$.

(v) If $\phi_A(\lambda) \neq \bar{\phi}_A(\lambda), \phi_B(\lambda) \neq \bar{\phi}_B(\lambda)$, then $f(\partial, \lambda) = \phi_A(\lambda)s(\partial + \lambda) - \bar{\phi}_A(\lambda)s(\partial)$, $g(\partial, \lambda) = \phi_B(\lambda)t(\partial+\lambda)-\bar{\phi}_B(\lambda)t(\partial)$ for some polynomials s, t. If $r(\partial) = s(\partial)-t(\partial) \neq 0$, then $r(\partial)$ can be written as $r(\partial) = \sum_{i=0}^{m} r_i\partial^i$ with $r_m \neq 0$. Taking them in (3.49) and considering the coefficients of ∂^m give

$$r_m(\phi_A(\lambda)-\bar{\phi}_A(\lambda))(\bar{\phi}_B(\mu)-\phi_B(\mu))=0,$$

which contradicts the assumption. So we can deduce that $s(\partial)=t(\partial)$ and thus the extension is trivial.

The other subcases can be learned from the symmetry of $(\phi_A,\bar{\phi}_A)$ and $(\phi_B,\bar{\phi}_B)$.

Case 2. $p(\lambda)=0, Q_1(\partial,\lambda)\neq 0$. In this case, $\phi_B(\lambda)=\bar{\phi}_B(\lambda)=0$ by (3.47) and then $\phi_A(\lambda)$ and $\bar{\phi}_A(\lambda)$ are nonzero polynomials.

(i) If $\phi_A(\lambda)=\bar{\phi}_A(\lambda)\neq 0$, then (3.49) implies $g(\partial,\lambda)=t(\lambda)\in\mathbb{C}[\lambda]$. Put it in (3.48) and take the partial derivative of both sides of the equation with respect to ∂, and we can obtain

$$(f_\partial(\partial+\lambda,\mu)-f_\partial(\partial,\mu))\phi_A(\lambda)-(f_\partial(\partial+\mu,\lambda)-f_\partial(\partial,\lambda))\phi_A(\mu)=0. \tag{3.51}$$

By Lemma 3.2.7, $f_\partial(\partial,\lambda)=\phi_A(\lambda)(v_1(\partial+\lambda)-v_1(\partial))+v_2(\lambda)$, where v_1,v_2 are polynomials. Let $s_1(\partial)=\int v_1(\partial)d\partial$, and then

$$f(\partial,\lambda)=\int f_\partial(\partial,\lambda)d\partial=\phi_A(\lambda)(s_1(\partial+\lambda)-s_1(\partial))+v_2(\lambda)\partial+v_3(\lambda),$$

where v_3 is a polynomial. Taking this result in (3.48) again, one can get

$$\lambda\phi_A(\lambda)v_2(\mu)-\mu\phi_A(\mu)v_2(\lambda)=Q_1(-\lambda-\mu,\lambda)t(\lambda+\mu). \tag{3.52}$$

With this equation, we have the following two subceses.

If $t=0$, $v_2(\lambda)=k\lambda a(\lambda)$ for some constant k. Let ${s'}_1(\partial)=s_1(\partial)+\frac{k}{2}\partial^2$. Then $f(\partial,\lambda)=\phi_A(\lambda)({s'}_1(\partial+\lambda)-{s'}_1(\partial))+s_2(\lambda)$ for some polynomials ${s'}_1,s_2$. In this case, the extension is nontrivial only if s_2 is not a scalar multiple of $\lambda\phi_A(\lambda)$.

If $t\neq 0$, then there exists $v_2(\lambda)$ satisfying (3.52) only when $Q(\lambda,\mu)=Q_1(-\lambda-\mu,\lambda)t(\lambda+\mu),\lambda\phi_A(\lambda)$ meet the condition in Lemma 3.2.8 and 3.2.9. Under these conditions, we can give the expression of $v_2(\lambda)$ and then that of $f(\partial,\lambda)$. In this case, the extension is nontrivial.

(ii) If $\phi_A(\lambda)\neq\bar{\phi}_A(\lambda)$, then (3.49) implies $g(\partial,\lambda)=0$. So by (3.48) and Lemma 3.2.7, we have $f(\partial,\lambda)=\phi_A(\lambda)s(\partial+\lambda)-\bar{\phi}_A(\lambda)s(\partial)$ for some polynomial s. Thus the extension is trivial under the condition.

Case 3. $p(\lambda) \neq 0, Q_1(\partial, \lambda) = 0$. In this case, $\phi_B(\lambda) = \bar{\phi}_B(\lambda) = 0$ by (3.47) and then $\phi_A(\lambda)$ and $\bar{\phi}_A(\lambda)$ are nonzero polynomials.

(i) If $\phi_A(\lambda) = \bar{\phi}_A(\lambda) \neq 0$, then $f(\partial, \lambda) = \phi_A(\lambda)(s_1(\partial + \lambda) - s_1(\partial)) + s_2(\lambda)$ with polynomials s_1, s_2 by (3.48). Assume $g(\partial, \lambda) \neq 0$. Comparing the coefficients of the highest item with respect to ∂ in (3.49), we get $p(\lambda) = 0$, which contracts the given condition. So $g(\partial, \lambda) = 0$ and the extension is nontrivial only when $s_2(\lambda)$ is not a scalar multiple of $\lambda\phi_A(\lambda)$.

(ii) If $\phi_A(\lambda) \neq \bar{\phi}_A(\lambda)$, then $f(\partial, \lambda) = \phi_A(\lambda)s(\partial+\lambda) - \bar{\phi}_A(\lambda)s(\partial)$ with polynomial s by (3.48). Assume $g(\partial, \lambda) \neq 0$. Let $g(\partial, \lambda) = \sum_{i=0}^{m} g_i(\lambda)\partial^i$. Comparing the coefficients of ∂^m in (3.49), we have $p(\lambda) = \phi_A(\lambda) - \bar{\phi}_A(\lambda)$ and $g_m(\lambda) = k_1 \in \mathbb{C}^\times$. If $m \geq 1$, one can obtain $p(\lambda) = r\phi_A(\lambda)$ for some nonzero constant r and $g_{m-1}(\lambda) = \frac{mk_1}{r}\lambda + k_2$ with $k_2 \in \mathbb{C}$ by comparing the coefficients of ∂^{m-1} in (3.48). If $m \geq 2$, the coefficients of ∂^{m-2} imply $r = 1$ and then $\bar{\phi}_A(\lambda) = 0$. Thus we get a contradiction. The extension is nontrivial if and only if $g(\partial, \lambda) \neq 0$.

Chapter 4 Extensions of finite irreducible modules of rank two Lie conformal algebras that are of Type I

Let $\mathcal{R}$ be the conformal algebra defined in (2.3). Then there is a basis $\{A, B\}$ such that

$$[A_\lambda A] = (\partial + 2\lambda)A,\ [A_\lambda B] = 0,\ [B_\lambda B] = 0. \tag{4.1}$$

In this chapter, we deal with the extension problem over $\mathcal{R}$. If V is a non-trivial finite irreducible $\mathcal{R}$-module, then either

$$V \cong V_{\alpha,\beta,\phi} = \mathbb{C}[\partial]v, \quad A_\lambda v = \delta_1(\partial + \alpha\lambda + \beta)v, \quad B_\lambda v = \delta_2\phi(\lambda)v, \tag{4.2}$$

where $\delta_1, \delta_2 \in \{0, 1\}, \delta_1^2 + \delta_2^2 = 1, \beta, 0 \neq \alpha \in \mathbb{C}$, and ϕ is a nonzero polynomial.

By definition 2.2.1, the $\mathcal{R}$-module structure on V given by $A_\lambda, B_\lambda \in End_{\mathbb{C}}(V)[\lambda]$ satisfies

$$[A_\lambda, A_\mu] = (\lambda - \mu)A_{\lambda+\mu}, \tag{4.3}$$

$$[A_\lambda, B_\mu] = 0, \tag{4.4}$$

$$[B_\lambda, B_\mu] = 0, \tag{4.5}$$

$$[\partial, A_\lambda] = -\lambda A_\lambda, \tag{4.6}$$

$$[\partial, B_\lambda] = -\lambda B_\lambda. \tag{4.7}$$

§4.1 $0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\alpha,\beta,\phi} \longrightarrow 0$

First, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\alpha,\beta,\phi} \longrightarrow 0. \tag{4.8}$$

Then E is isomorphic to $\mathbb{C}c_\eta \oplus V_{\alpha,\beta,\phi} = \mathbb{C}c_\eta \oplus \mathbb{C}[\partial]v$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\mathcal{R}_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta$$
$$A_\lambda v = \delta_1(\partial + \alpha\lambda + \beta)v + f(\lambda)c_\eta, \quad B_\lambda v = \delta_2\phi(\lambda)v + g(\lambda)c_\eta, \tag{4.9}$$

where $f(\lambda), g(\lambda) \in \mathbb{C}[\lambda]$.

Lemma 4.1.1 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (4.8) are given by (4.9), and $f(\lambda)$ and $g(\lambda)$ are the same scalar multiple of $\delta_1(\alpha\lambda + \eta + \beta)$ and $\delta_2\phi(\lambda)$ respectively.*

Proof: Assume that (4.8) is a trivial extension, that is, there exists $v' = kc_\eta + l(\partial)v \in E$, where $k \in \mathbb{C}$ and $0 \neq l(\partial) \in \mathbb{C}[\partial]$, such that

$$A_\lambda v' = \delta_1(\partial + \alpha\lambda + \beta)v' = \delta_1 k(\eta + \alpha\lambda + \beta)c_\eta + \delta_1 l(\partial)(\partial + \alpha\lambda + \beta)v,$$
$$B_\lambda v' = \delta_2\phi(\lambda)v' = \delta_2 k\phi(\lambda)c_\eta + \delta_2 l(\partial)\phi(\lambda)v.$$

On the other hand, it follows from (4.9) that

$$A_\lambda v' = f(\lambda)l(\eta + \lambda)c_\eta + \delta_1 l(\partial + \lambda)(\partial + \alpha\lambda + \beta)v,$$
$$B_\lambda v' = g(\lambda)l(\eta + \lambda)c_\eta + \delta_2 l(\partial + \lambda)\phi(\lambda)v.$$

We can obtain that $l(\partial)$ is a nonzero constant by comparing both expressions for $A_\lambda v'$ and $B_\lambda v'$. Thus $f(\lambda)$ and $g(\lambda)$ are the same scalar multiple of $\delta_1(\alpha\lambda + \eta + \beta)$ and $\delta_2\phi(\lambda)$ respectively.

Conversely, if $f(\lambda) = \delta_1 k(\alpha\lambda + \eta + \beta)$ and $g(\lambda) = \delta_2 k\phi(\lambda)$ for some $k \in \mathbb{C}$, setting $v' = kc_\eta + v$ we can deduce that (4.8) is a trivial extension.

Theorem 4.1.2 *For a rank two Lie conformal algebra $\mathcal{R}$ that is of Type I, nontrivial extensions of finite irreducible conformal modules of the form (4.8) exist only when $(\delta_1, \delta_2) = (1, 0), \alpha \in \{1, 2\}, \beta + \eta = 0$. Moreover, they are given (up to equivalence) by (4.9). The values of η, along with the corresponding polynomials $f(\lambda)$ and $g(\lambda)$ giving* *rise to nontrivial extensions, are listed as follows:* $g(\lambda)=0$ *and*

$$f(\lambda)=\begin{cases} s_1\lambda^2, & \alpha=1,\\ s_2\lambda^3, & \alpha=2,\end{cases}$$

with nonzero constants s_1, s_2*. In these cases,* $dim(Ext(V_{\alpha,\beta,\phi},\mathbb{C}c_\eta))=1$*.*

Proof: Applying both sides of (4.3), (4.4) and (4.5) to v and comparing the corresponding coefficients, we obtain

$$\delta_1(\eta+\lambda+\alpha\mu+\beta)f(\lambda)-\delta_1(\eta+\mu+\alpha\lambda+\beta)f(\mu)=(\lambda-\mu)f(\lambda+\mu), \tag{4.10}$$

$$\delta_2\phi(\mu)f(\lambda)-\delta_1(\eta+\mu+\alpha\lambda+\beta)g(\mu)=0, \tag{4.11}$$

$$\delta_2\phi(\mu)g(\lambda)-\delta_2\phi(\lambda)g(\mu)=0. \tag{4.12}$$

If $(\delta_1,\delta_2)=(1,0)$, (4.11) implies $g(\mu)=0$ and it reduces to the case of Virasoro conformal algebra. We can deduce the result by Proposition 2.1 in [7]. If $(\delta_1,\delta_2)=(0,1)$, then $f(\lambda)=0$ by (4.11). Applying Lemma 3.2.2 to (4.12), we have $g(\lambda)=k\phi(\lambda)$ for some constant k and then the extension is trivial.

§4.2 $\quad 0\longrightarrow V_{\alpha,\beta,\phi}\longrightarrow E\longrightarrow \mathbb{C}c_\eta\longrightarrow 0$

Next, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0\longrightarrow V_{\alpha,\beta,\phi}\longrightarrow E\longrightarrow \mathbb{C}c_\eta\longrightarrow 0. \tag{4.13}$$

Then E is isomorphic to $V_{\alpha,\beta,\phi}\oplus\mathbb{C}c_\eta=\mathbb{C}[\partial]v\oplus\mathbb{C}c_\eta$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v=\delta_1(\partial+\alpha\lambda+\beta)v,\quad B_\lambda v=\delta_2\phi(\lambda)v,\\ &A_\lambda c_\eta=f(\partial,\lambda)v,\quad B_\lambda c_\eta=g(\partial,\lambda)v,\quad \partial c_\eta=\eta c_\eta+h(\partial)v, \end{aligned} \tag{4.14}$$

where $f(\partial,\lambda), g(\partial,\lambda)\in\mathbb{C}[\partial,\lambda]$ and $h(\partial)\in\mathbb{C}[\partial]$.

Lemma 4.2.1 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (4.13) are given by (4.14), and* $f(\partial,\lambda)=\delta_1\varphi(\partial+\lambda)(\partial+\alpha\lambda+\beta), g(\partial,\lambda)=\delta_2\varphi(\partial+\lambda)\phi(\lambda)$ *and* $h(\partial)=(\partial-\eta)\varphi(\partial)$*, where φ is a polynomial.*

Proof: Assume that (4.13) is a trivial extension, that is, there exists $c'_\eta=kc_\eta+l(\partial)v\in E$, where $0\neq k\in\mathbb{C}$ and $l(\partial)\in\mathbb{C}[\partial]$, such that $A_\lambda c'_\eta=B_\lambda c'_\eta=0$ and $\partial c'_\eta=\eta c'_\eta=k\eta c_\eta+\eta l(\partial)v$.

On the other hand, it follows from (4.14) that

$$\begin{aligned}A_\lambda c'_\eta&=(kf(\partial,\lambda)+\delta_1 l(\partial+\lambda)(\partial+\alpha\lambda+\beta))v,\\ B_\lambda c'_\eta&=(kg(\partial,\lambda)+\delta_2 l(\partial+\lambda)\phi(\lambda))v,\\ \partial c'_\eta&=k\eta c_\eta+(kh(\partial)+\partial l(\partial))v.\end{aligned}$$

We can obtain the result by comparing both expressions for $A_\lambda c'_\eta, B_\lambda c'_\eta$ and $\partial c'_\eta$.

Conversely, if $f(\partial,\lambda)=\delta_1\varphi(\partial+\lambda)(\partial+\alpha\lambda+\beta), g(\partial,\lambda)=\delta_2\varphi(\partial+\lambda)\phi(\lambda)$ and $h(\partial)=(\partial-\eta)\varphi(\partial)$ for some polynomial φ, setting $c'_\eta=c_\eta-\varphi(\partial)v$, we can deduce that (4.13) is a trivial extension.

Theorem 4.2.2 *For a rank two Lie conformal algebra $\mathcal{R}$ that is of Type I, nontrivial extensions of finite irreducible conformal modules of the form (4.13) exist only when* $\delta_1=1,\alpha=1,\beta+\eta=0$*. Moreover, the space of* $Ext(\mathbb{C}c_\eta,V_{\alpha,\beta,\phi})$ *is 1-dimensional, and the unique nontrivial extension is given (up to equivalence) as follows:* $\delta_2=0, g(\partial,\lambda)=0$ *and* $f(\partial,\lambda)=h(\partial)=s$ *with nonzero constant s.*

Proof: Applying both sides of (4.6) and (4.7) to c_η and comparing the corresponding coefficients gives the following equations

$$(\partial+\lambda-\eta)f(\partial,\lambda)=\delta_1 h(\partial+\lambda)(\partial+\alpha\lambda+\beta),\tag{4.15}$$

$$(\partial+\lambda-\eta)g(\partial,\lambda)=\delta_2 h(\partial+\lambda)\phi(\lambda).\tag{4.16}$$

If $(\delta_1,\delta_2)=(1,0)$, then $g=0$ by (4.16) and the result can be deduced by Proposition 2.2 in [7]. If $(\delta_1,\delta_2)=(0,1)$, then $f=0$. (4.16) and Lemma 4.2.1 imply that the extension is trivial in this case.

§4.3 $0 \longrightarrow V_{\alpha,\beta,\phi} \longrightarrow E \longrightarrow V_{\bar{\alpha},\bar{\beta},\bar{\phi}} \longrightarrow 0$

Finally, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow V_{\alpha,\beta,\phi} \longrightarrow E \longrightarrow V_{\bar{\alpha},\bar{\beta},\bar{\phi}} \longrightarrow 0. \tag{4.17}$$

Then E is isomorphic to $V_{\alpha,\beta,\phi} \oplus V_{\bar{\alpha},\bar{\beta},\bar{\phi}} = \mathbb{C}[\partial]v \oplus \mathbb{C}[\partial]\bar{v}$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v = \delta_1(\partial + \alpha\lambda + \beta)v, \quad B_\lambda v = \delta_2\phi(\lambda)v, \\ &A_\lambda \bar{v} = \bar{\delta}_1(\partial + \bar{\alpha}\lambda + \bar{\beta})\bar{v} + f(\partial,\lambda)v, \quad B_\lambda \bar{v} = \bar{\delta}_2\bar{\phi}(\lambda)\bar{v} + g(\partial,\lambda)v, \end{aligned} \tag{4.18}$$

where $f(\partial,\lambda), g(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$.

Lemma 4.3.1 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (4.17) are given by (4.18), and $f(\partial,\lambda) = \delta_1\varphi(\partial+\lambda)(\partial+\alpha\lambda+\beta) - \bar{\delta}_1\varphi(\partial)(\partial+\bar{\alpha}\lambda+\bar{\beta})$ and $g(\partial,\lambda) = \delta_2\varphi(\partial+\lambda)\phi(\lambda) - \bar{\delta}_2\varphi(\partial)\bar{\phi}(\lambda)$ for some polynomial φ.*

Proof: Assume that (4.17) is a trivial extension, that is, there exists $\bar{v}' = k(\partial)v + l(\partial)\bar{v} \in E$, where $k(\partial), l(\partial) \in \mathbb{C}[\partial]$ and $l(\partial) \neq 0$, such that

$$\begin{aligned} &A_\lambda \bar{v}' = \bar{\delta}_1(\partial + \bar{\alpha}\lambda + \bar{\beta})\bar{v}' = \bar{\delta}_1 k(\partial)(\partial + \bar{\alpha}\lambda + \bar{\beta})v + \bar{\delta}_1 l(\partial)(\partial + \bar{\alpha}\lambda + \bar{\beta})\bar{v}, \\ &B_\lambda \bar{v}' = \bar{\delta}_2\bar{\phi}(\lambda)\bar{v}' = \bar{\delta}_2 k(\partial)\bar{\phi}(\lambda)v + \bar{\delta}_2 l(\partial)\bar{\phi}(\lambda)\bar{v}. \end{aligned}$$

On the other hand, it follows from (4.18) that

$$\begin{aligned} &A_\lambda \bar{v}' = (\delta_1 k(\partial+\lambda)(\partial+\alpha\lambda+\beta) + l(\partial+\lambda)f(\partial,\lambda))v + \bar{\delta}_1 l(\partial+\lambda)(\partial+\bar{\alpha}\lambda+\bar{\beta})\bar{v}, \\ &B_\lambda \bar{v}' = (\delta_2 k(\partial+\lambda)\phi(\lambda) + l(\partial+\lambda)g(\partial,\lambda))v + \bar{\delta}_2 l(\partial+\lambda)\bar{\phi}(\lambda)\bar{v}. \end{aligned}$$

Comparing both expressions for $A_\lambda \bar{v}'$ and $B_\lambda \bar{v}'$, we can obtain that l is a nonzero constant. And then we can give the expressions of $f(\partial,\lambda)$ and $g(\partial,\lambda)$.

Conversely, if $f(\partial,\lambda) = \delta_1\varphi(\partial+\lambda)(\partial+\alpha\lambda+\beta) - \bar{\delta}_1\varphi(\partial)(\partial+\bar{\alpha}\lambda+\bar{\beta})$ and $g(\partial,\lambda) = \delta_2\varphi(\partial+\lambda)\phi(\lambda) - \bar{\delta}_2\varphi(\partial)\bar{\phi}(\lambda)$ for some polynomial φ, setting $\bar{v}' = -\varphi(\partial)v + \bar{v}$ we can deduce that (4.17) is a trivial extension.

Theorem 4.3.2 *For a rank two Lie conformal algebra $\mathcal{R}$ that is of Type I, nontrivial extensions of finite irreducible conformal modules of the form (4.17) exist only when $(\delta_1, \delta_2) = (\bar{\delta_1}, \bar{\delta_2})$. Moreover, they are given (up to equivalence) by (4.18). The value of $\delta_i, \bar{\delta}_i, i = 1, 2, \alpha, \bar{\alpha}, \beta, \bar{\beta}$, and the corresponding polynomials $\phi(\lambda), \bar{\phi}(\lambda), f(\partial, \lambda)$ and $g(\partial, \lambda)$ giving rise to nontrivial extensions, are listed as follows:*

1. *In the case that $(\delta_1, \delta_2) = (\bar{\delta_1}, \bar{\delta_2}) = (1, 0)$, $g = 0, \beta = \bar{\beta}, \bar{\alpha} - \alpha \in \{0, 1, 2, 3, 4, 5, 6\}, \alpha, \bar{\alpha} \neq 0$, and*
   - *(i)* $\bar{\alpha} = \alpha$, $f(\partial, \lambda) = s_0 + s_1\lambda$, *where* $(s_0, s_1) \neq (0, 0)$.
   - *(ii)* $\bar{\alpha} - \alpha = 2$, $f(\partial, \lambda) = s\lambda^2(2(\partial + \beta) + \lambda)$, *where* $s \neq 0$.
   - *(iii)* $\bar{\alpha} - \alpha = 3$, $f(\partial, \lambda) = s(\partial + \beta)\lambda^2((\partial + \beta) + \lambda)$, *where* $s \neq 0$.
   - *(iv)* $\bar{\alpha} - \alpha = 4$, $f(\partial, \lambda) = s\lambda^2(4(\partial + \beta)^3 + 6(\partial + \beta)^2\lambda - (\partial + \beta)\lambda^2 + \alpha_1\lambda^3)$, *where* $s \neq 0$.
   - *(v)* $\bar{\alpha} = 1$ *and* $\alpha = -4$, $f(\partial, \lambda) = s((\partial+\beta)^4\lambda^2 - 10(\partial+\beta)^2\lambda^4 - 17(\partial+\beta)\lambda^5 - 8\lambda^6)$, *where* $s \neq 0$.
   - *(vi)* $\bar{\alpha} = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial, \lambda) = s((\partial+\beta)^4\lambda^3 - (2\alpha+3)(\partial+\beta)^3\lambda^4 - 3\alpha(\partial + \beta)^2\lambda^5 - (3\alpha + 1)(\partial + \beta)\lambda^6 - (\alpha + \frac{9}{28})\lambda^7)$, *where* $s \neq 0$.

   The value of dim$(Ext(V_{\bar{\alpha},\bar{\beta},\bar{\phi}}, V_{\alpha,\beta,\phi}))$ is 2 in subcase (i), and 1 in subcases (ii)-(vi).

2. *In the case that $(\delta_1, \delta_2) = (\bar{\delta_1}, \bar{\delta_2}) = (0, 1)$, $\phi(\lambda) = \bar{\phi}(\lambda), f(\partial, \lambda) = 0, g(\partial, \lambda) = t(\lambda)$ with polynomials t and $t(\lambda)$ is not a scalar multiple of $\lambda\phi(\lambda)$. Then the space $Ext(V_{\bar{\alpha},\bar{\beta},\bar{\phi}}, V_{\alpha,\beta,\phi})$ is infinite-dimensional.*

Proof: Applying both sides of (4.3), (4.4) and (4.5) to $\bar{v}$ and comparing the corresponding coefficients, we obtain

$$\bar{\delta}_1 f(\partial, \lambda)(\partial + \lambda + \bar{\alpha}\mu + \bar{\beta}) + \delta_1 f(\partial + \lambda, \mu)(\partial + \alpha\lambda + \beta) \\ - \bar{\delta}_1 f(\partial, \mu)(\partial + \mu + \bar{\alpha}\lambda + \bar{\beta}) - \delta_1 f(\partial + \mu, \lambda)(\partial + \alpha\mu + \beta) = (\lambda - \mu) f(\partial, \lambda + \mu), \tag{4.19}$$

$$\bar{\delta}_2 f(\partial, \lambda)\bar{\phi}(\mu) + \delta_1 g(\partial + \lambda, \mu)(\partial + \alpha\lambda + \beta)$$

$$-\bar{\delta}_1 g(\partial,\mu)(\partial+\mu+\bar{\alpha}\lambda+\bar{\beta})-\delta_2 f(\partial+\mu,\lambda)\phi(\mu)=0, \tag{4.20}$$

$$\bar{\delta}_2 g(\partial,\lambda)\bar{\phi}(\mu)+\delta_2 g(\partial+\lambda,\mu)\phi(\lambda)-\bar{\delta}_2 g(\partial,\mu)\bar{\phi}(\lambda)-\delta_2 g(\partial+\mu,\lambda)\phi(\mu)=0. \tag{4.21}$$

If $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(1,1,0,0)$, the result can be deduced from Lemma 3.1.6, Theorem 3.2 in [7] (or Theorem 2.7 in [21]) and Lemma 4.3.1.

If $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(0,0,1,1)$, then (4.19) implies $f(\partial,\lambda)=0$. Applying Lemma 3.2.7 to (4.21), we have $g(\partial,\lambda)=t(\partial+\lambda)\phi(\lambda)-t(\partial)\bar{\phi}(\lambda)$ for some polynomial t if $\phi(\lambda)\neq\bar{\phi}(\lambda)$ and $g(\partial,\lambda)=(t_1(\partial+\lambda)-t_1(\partial))\phi(\lambda)+t_2(\lambda)$ for some polynomial t_1,t_2 if $\phi(\lambda)=\bar{\phi}(\lambda)$. By Lemma 4.3.1, the extension is nontrivial only when $\phi=\bar{\phi}$ and $t_2(\lambda)$ is not a scalar multiple of $\lambda\phi(\lambda)$.

If $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(1,0,0,1)$, then putting $\mu=0$ in (4.19), we can obtain

$$f(\partial+\lambda,0)(\partial+\alpha\lambda+\beta)=f(\partial,\lambda)(\partial+\lambda+\beta).$$

So when $\alpha=1$, we have $f(\partial,\lambda)=f(\partial+\lambda,0)=s(\partial+\lambda)$ for some polynomial s. If $\alpha\neq1$, then one can deduce that $f(\partial,\lambda)=s(\partial+\lambda)(\partial+\alpha\lambda+\beta)$ for some polynomial s. Applying Lemma 3.2.7 to (4.21), we have $g(\partial,\lambda)=t(\partial)\bar{\phi}(\lambda)$, where t is a polynomial. Putting these results in (4.20), we can obtain

$$\begin{cases} s(\partial+\lambda)+t(\partial+\lambda)(\partial+\lambda+\beta)=0, & \alpha=1,\\ s(\partial+\lambda)+t(\partial+\lambda)=0, & \alpha\neq1, \end{cases} \tag{4.22}$$

The solutions are concluded as follows.

(i) If $\alpha=1$, then $f(\partial,\lambda)=-t(\partial+\lambda)(\partial+\lambda+\beta), g(\partial,\lambda)=t(\partial)\bar{\phi}(\lambda)$ for some polynomial t. The extension is trivial.

(ii) If $\alpha\neq1$, then $f(\partial,\lambda)=s(\partial+\lambda)(\partial+\alpha\lambda+\beta), g(\partial,\lambda)=-s(\partial)\bar{\phi}(\lambda)$ for some polynomial s. The extension is trivial.

If $(\delta_1,\bar{\delta}_1,\delta_2,\bar{\delta}_2)=(0,1,1,0)$, one can deduce the result similarly.

Chapter 5 Extensions of finite irreducible modules of rank two Lie conformal algebras that are of Type II

In this chapter, we investigate the extension problems under the condition that $\mathcal{R}$ is the conformal algebra defined in (2.4). Then there is a basis $\{A, B\}$ such that

$$[A_\lambda A] = (\partial + 2\lambda)A + Q(\partial, \lambda)B, \ [A_\lambda B] = (\partial + a\lambda + b)B, \ [B_\lambda B] = 0. \tag{5.1}$$

§5.1 Extensions of finite irreducible modules of $\mathcal{W}(a, b)$ algebras

When $Q(\partial, \lambda) = 0$, $\mathcal{R}$ is a $\mathcal{W}(a, b)$ algebra, which had been discussed in [21]. Recall the classification of all finite nontrivial $\mathcal{W}(a, b)$-module in [20].

Theorem 5.1.1 *[20, Theorem 3.10] Any finite nontrivial irreducible $\mathcal{W}(a, b)$-module M is free of rank one over $\mathbb{C}[\partial]$. Moreover,*

1. *If $(a, b) \neq (1, 0)$,*

$$M \cong M_{\alpha,\beta} = \mathbb{C}[\partial]v, \ A_\lambda v = (\partial + \alpha\lambda + \beta)v, \ B_\lambda v = 0,$$

   with $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$.

2. *If $(a, b) = (1, 0)$,*

$$M \cong M_{\alpha,\beta,\gamma} = \mathbb{C}[\partial]v, \ A_\lambda v = (\partial + \alpha\lambda + \beta)v, \ B_\lambda v = \gamma v,$$

   with $\alpha, \beta, \gamma \in \mathbb{C}$ and $(\alpha, \gamma) \neq (0, 0)$.

Then the corresponding results of extensions over the $\mathcal{W}(a, b)$-modules are listed as follows.

Theorem 5.1.2 *[21, Theorem 3.4] (1) If $(a, b) \neq (1, 0)$, nontrivial extensions of finite irreducible $\mathcal{W}(a, b)$-modules of the form*

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow M_{\alpha,\beta} \longrightarrow 0 \tag{5.2}$$

exist. Moreover, they are given (up to equivalence) by

$$A_\lambda v_\alpha = (\partial + \alpha\lambda + \beta)v_\alpha + f(\lambda)c_\eta, \quad B_\lambda v_\alpha = g(\lambda)c_\eta; \tag{5.3}$$

. *The values of β and η along with the pairs of polynomials $g(\lambda)$ and $f(\lambda)$, whose nonzero scalar multiples give rise to nontrivial extensions, are listed as follows:*

(i) if $g(\lambda) = 0$, then $\alpha = 1, 2$, $\beta + \eta = 0$ and $f(\lambda)$ is from the nonzero polynomials of Theorem 2.2.6;

(ii) if $a \neq 1$, $b = 0$ and $\beta + \eta = 0$, then $g(\lambda) = k$ for some nonzero complex number k, $\alpha = 1 - a$, and

$$f(\lambda) = \begin{cases} c_2\lambda^2, & \alpha = 1, \\ c_3\lambda^3, & \alpha = 2, \\ 0, & \text{otherwise}, \end{cases}$$

with $c_2, c_3 \in \mathbb{C}$;

(iii) if $a \neq 1$, $b + \beta + \eta = 0$ and $\beta + \eta \neq 0$, then $g(\lambda) = k$ for some nonzero complex number k, $\alpha = 1 - a$, and $f(\lambda) = 0$;

(iv) if $a = 1, b \neq 0$ and $b + \beta + \eta = 0$, then $g(\lambda) = k(1 - \frac{1}{b}\lambda)$ for some nonzero complex number k, $\alpha = 1$, and $f(\lambda) = 0$.

(2) If $(a, b) = (1, 0)$, nontrivial extensions of finite irreducible $\mathcal{W}(1, 0)$-modules of the form

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow M_{\alpha,\beta,\gamma} \longrightarrow 0 \tag{5.4}$$

exist if and only if $\beta + \eta = 0$ and $\gamma = 0$. Moreover, they are given (up to equivalence) by

$$L_\lambda v_\alpha = (\partial + \alpha\lambda + \beta)v_\alpha + f(\lambda)c_\eta, \quad W_\lambda v_\alpha = \gamma v_\alpha + g(\lambda)c_\eta; \tag{5.5}$$

, where, if $g(\lambda) = 0$, then $\alpha = 1, 2$ and $f(\lambda)$ is from the nonzero polynomials of Theorem 2.2.6, or else $g(\lambda) = k\lambda$ for some nonzero complex number k, $\alpha = 1$ and $f(\lambda) = c_2\lambda^2$ with $c_2 \in \mathbb{C}$.

Theorem 5.1.3 *[21, Theorem 3.6] (1) If $(a,b) \neq (1,0)$, nontrivial extensions of finite irreducible $\mathcal{W}(a,b)$-modules of the form*

$$0 \longrightarrow M_{\alpha,\beta} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0. \tag{5.6}$$

exist if and only if $\beta + \eta = 0$ and $\alpha = 1$. In this case, dimExt$(\mathbb{C}c_{-\beta}, M_{1,\beta}) = 1$, and the unique (up to equivalence) nontrivial extension is given by

$$A_\lambda c_\eta = kv_\alpha, \quad B_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta + kv_\alpha,$$

where k is a nonzero complex number.
(2) If $(a,b) = (1,0)$, nontrivial extensions of finite irreducible $\mathcal{W}(1,0)$-modules of the form

$$0 \longrightarrow M_{\alpha,\beta,\gamma} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0. \tag{5.7}$$

exist if and only if $\beta+\eta = 0$ and $(\alpha,\gamma) = (1,0)$. In this case, dimExt$(\mathbb{C}c_{-\beta}, M_{1,\beta,0}) = 1$, and the unique (up to equivalence) nontrivial extension is given by

$$A_\lambda c_\eta = kv_\alpha, \quad B_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta + kv_\alpha,$$

where k is a nonzero complex number.

Theorem 5.1.4 *[19, Theorem 3.7] Nontrivial extensions of finite irreducible $\mathcal{W}(a,0)$-modules of the form*

$$0 \longrightarrow M_{\bar{\alpha},\bar{\beta}} \longrightarrow E \longrightarrow M_{\alpha,\beta} \longrightarrow 0 \tag{5.8}$$

with $a \neq 1$ exist if and only if $\beta = \bar{\beta}$. For each $\beta \in \mathbb{C}$, these extensions are given (up to equivalence) by

$$A_\lambda v_\alpha = (\partial + \alpha\lambda + \beta)v_\alpha + f(\partial,\lambda)v_{\bar{\alpha}}, \quad B_\lambda v_\alpha = g(\partial,\lambda)v_{\bar{\alpha}}, \tag{5.9}$$

where $g(\partial,\lambda) = 0$ and $f(\partial,\lambda)$ is from the nonzero polynomials of Theorem 2.2.8, with $\alpha, \bar{\alpha} \neq 0$, or the values of α and $\bar{\alpha}$ along with the pairs of polynomials $g(\partial,\lambda)$ and

$f(\partial,\lambda)$, whose nonzero scalar multiples give rise to nontrivial extensions, are listed as follows (by replacing ∂ by $\partial+\beta$):

(1) When $a=3$, we have $\alpha=\bar{\alpha}=1$, $f(\partial,\lambda)=a_0+a_1\lambda$ and $g(\partial,\lambda)=\partial^2+\frac{3}{2}\partial\lambda+\frac{1}{2}\lambda^2$, where $a_0,a_1\in\mathbb{C}$.

(2) When $a=2$, we have $\alpha-\bar{\alpha}=-1$ or 0. Moreover,

- *(i) In the case $\alpha-\bar{\alpha}=-1$, $f(\partial,\lambda)=0$ and $g(\partial,\lambda)=1$.*
- *(ii) In the case $\alpha-\bar{\alpha}=0$, $f(\partial,\lambda)=a_0+a_1\lambda$ and $g(\partial,\lambda)=\partial+\bar{\alpha}\lambda$, where $a_0,a_1\in\mathbb{C}$.*

(3) When $a=0$, we have $\alpha-\bar{\alpha}=1,2$ or $\alpha=1,\bar{\alpha}=-2$. Moreover,

- *(i) In the case $\alpha-\bar{\alpha}=1$, $f(\partial,\lambda)=0$ and $g(\partial,\lambda)=1$.*
- *(ii) In the case $\alpha-\bar{\alpha}=2$, $f(\partial,\lambda)=a_0\lambda^2(2\partial+\lambda)$ and $g(\partial,\lambda)=\partial-\bar{\alpha}\lambda$, where $a_0\in\mathbb{C}$.*
- *(iii) In the case $\alpha=1,\bar{\alpha}=-2$, $f(\partial,\lambda)=a_0\partial\lambda^2(\partial+\lambda)$ and $g(\partial,\lambda)=\partial^2+3\partial\lambda+2\lambda^2$, where $a_0\in\mathbb{C}$.*

(4) When $a=-1$, we have $\alpha-\bar{\alpha}=2,3$ or $\alpha=1,\bar{\alpha}=-3$. Moreover,

- *(i) In the case $\alpha-\bar{\alpha}=2$, $f(\partial,\lambda)=a_0\lambda^2(2\partial+\lambda)$ and $g(\partial,\lambda)=1$, where $a_0\in\mathbb{C}$.*
- *(ii) In the case $\alpha-\bar{\alpha}=3$, $f(\partial,\lambda)=a_0\partial\lambda^2(\partial+\lambda)$ and $g(\partial,\lambda)=\partial-\frac{1}{2}\bar{\alpha}\lambda$, where $a_0\in\mathbb{C}$.*
- *(iii) In the case $\alpha=1,\bar{\alpha}=-3$, $f(\partial,\lambda)=a_0\lambda^2(4\partial^3+6\partial^2\lambda-\partial\lambda^2-3\lambda^3)$ and $g(\partial,\lambda)=\partial^2+\frac{5}{2}\partial\lambda+\frac{3}{2}\lambda^2$, where $a_0\in\mathbb{C}$.*

(5) When $a=-2$, we have $\alpha-\bar{\alpha}=3,4$ or $\alpha=1,\bar{\alpha}=-4$. Moreover,

- *(i) In the case $\alpha-\bar{\alpha}=3$, $f(\partial,\lambda)=a_0\partial\lambda^2(\partial+\lambda)$ and $g(\partial,\lambda)=1$, where $a_0\in\mathbb{C}$.*
- *(ii) In the case $\alpha-\bar{\alpha}=4$, $f(\partial,\lambda)=a_0\lambda^2(4\partial^3+6\partial^2\lambda-\partial\lambda^2+\bar{\alpha}\lambda^3)$ and $g(\partial,\lambda)=\partial-\frac{1}{3}\bar{\alpha}\lambda$, where $a_0\in\mathbb{C}$.*
- *(iii) In the case $\alpha=1,\bar{\alpha}=-4$, $f(\partial,\lambda)=a_0(\partial^4\lambda^2-10\partial^2\lambda^4-17\partial\lambda^5-8\lambda^6)$ and $g(\partial,\lambda)=\partial^2+\frac{7}{3}\partial\lambda+\frac{4}{3}\lambda^2$, where $a_0\in\mathbb{C}$.*

(6) When $a=-3$, we have $\alpha-\bar{\alpha}=4,5$ or $\alpha=1,\bar{\alpha}=-5$. Moreover,

(i) *In the case* $\alpha - \bar{\alpha} = 4$, $f(\partial, \lambda) = a_0\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \bar{\alpha}\lambda^3)$ *and* $g(\partial, \lambda) = 1$, *where* $a_0 \in \mathbb{C}$.

(ii) *In the case* $\alpha - \bar{\alpha} = 5, \alpha \neq 1$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = \partial - \frac{1}{4}\bar{\alpha}\lambda$.

(iii) *In the case* $\alpha = 1, \bar{\alpha} = -4$, $f(\partial, \lambda) = a_0(\partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6)$ *and* $g(\partial, \lambda) = \partial + \lambda$, *where* $a_0 \in \mathbb{C}$.

(iv) *In the case* $\alpha = 1, \bar{\alpha} = -5$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = \partial^2 + \frac{9}{4}\partial\lambda + \frac{5}{4}\lambda^2$.

(7) *When* $a = -4$, *we have* $\alpha - \bar{\alpha} = 5, 6$ *or* $\alpha = 1, \bar{\alpha} = -6$. *Moreover,*

(i) *In the case* $\alpha - \bar{\alpha} = 5, \alpha \neq 1$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = 1$.

(ii) *In the case* $\alpha = 1, \bar{\alpha} = -4$, $f(\partial, \lambda) = a_0(\partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6)$ *and* $g(\partial, \lambda) = 1$, *where* $a_0 \in \mathbb{C}$.

(iii) *In the case* $\alpha - \bar{\alpha} = 6, \alpha \neq \frac{7}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = \partial - \frac{1}{5}\bar{\alpha}\lambda$.

(iv) *In the case* $\alpha - \bar{\alpha} = 6, \alpha = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial, \lambda) = a_0(\partial^4\lambda^3 - (2\bar{\alpha} + 3)\partial^3\lambda^4 - 3\bar{\alpha}\partial^2\lambda^5 - (3\bar{\alpha} + 1)\partial\lambda^6 - (\bar{\alpha} + \frac{9}{28})\lambda^7)$ *and* $g(\partial, \lambda) = \partial - \frac{1}{5}\bar{\alpha}\lambda$, *where* $a_0 \in \mathbb{C}$.

(v) *In the case* $\alpha = 1, \bar{\alpha} = -6$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = \partial^2 + \frac{11}{5}\partial\lambda + \frac{6}{5}\lambda^2$.

(8) *When* $a = -5$, *we have* $\alpha - \bar{\alpha} = 6, 7$ *or* $\alpha = 1, \bar{\alpha} = -7$. *Moreover,*

(i) *In the case* $\alpha - \bar{\alpha} = 6, \alpha \neq \frac{7}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = 1$.

(ii) *In the case* $\alpha - \bar{\alpha} = 6, \alpha = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial, \lambda) = a_0(\partial^4\lambda^3 - (2\bar{\alpha} + 3)\partial^3\lambda^4 - 3\bar{\alpha}\partial^2\lambda^5 - (3\bar{\alpha} + 1)\partial\lambda^6 - (\bar{\alpha} + \frac{9}{28})\lambda^7)$ *and* $g(\partial, \lambda) = 1$, *where* $a_0 \in \mathbb{C}$.

(iii) *In the case* $\alpha - \bar{\alpha} = 7$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = \partial - \frac{1}{6}\bar{\alpha}\lambda$.

(iv) *In the case* $\alpha = 1, \bar{\alpha} = -7$, $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda) = \partial^2 + \frac{13}{6}\partial\lambda + \frac{7}{6}\lambda^2$.

(9) *When* $a = \frac{5}{3}$, *we have* $f(\partial, \lambda) = 0$. *Moreover,*

(i) *In the case* $\alpha - \bar{\alpha} = -\frac{2}{3}$, $g(\partial, \lambda) = 1$.

(ii) *In the case* $\alpha - \bar{\alpha} = \frac{1}{3}$, $g(\partial, \lambda) = \partial + \frac{3}{2}\bar{\alpha}\lambda$.

(iii) *In the case* $\alpha = 1, \bar{\alpha} = -\frac{1}{3}$, $g(\partial, \lambda) = \partial^2 + \frac{1}{2}\partial\lambda - \frac{1}{2}\lambda^2$;

(iv) In the case $\alpha = \frac{5}{3}, \bar{\alpha} = -\frac{2}{3}$. $g(\partial, \lambda) = \partial^3 + \frac{3}{2}\partial^2\lambda - \frac{3}{2}\partial\lambda^2 - \lambda^3$.

(10) When $a \neq 3, 2, 0, -1, -2, -3, -4, -5$ *or* $\frac{5}{3}$*, we have* $f(\partial, \lambda) = 0$*. Moreover,*

(i) In the case $\alpha - \bar{\alpha} = 1 - a$, $g(\partial, \lambda) = 1$.

(ii) In the case $\alpha - \bar{\alpha} = 2 - a$, $g(\partial, \lambda) = \partial - \frac{1}{1-a}\bar{\alpha}\lambda$.

(iii) In the case $\alpha = 1, \bar{\alpha} = a - 2$, $g(\partial, \lambda) = \partial^2 - \frac{1}{1-a}(1 + 2\bar{\alpha})\partial\lambda - \frac{1}{1-a}\bar{\alpha}\lambda^2$.

Theorem 5.1.5 *[21, Theorem 3.10] (A) If* $(a, b) \neq (1, 0)$*, nontrivial extensions of finite irreducible* $\mathcal{W}(a, b)$*-modules of the form*

$$0 \longrightarrow M_{\bar{\alpha},\bar{\beta}} \longrightarrow E \longrightarrow M_{\alpha,\beta} \longrightarrow 0 \tag{5.10}$$

exist. Moreover, they are given (up to equivalence) by

$$A_\lambda v_\alpha = (\partial + \alpha\lambda + \beta)v_\alpha + f(\partial, \lambda)v_{\bar{\alpha}}, \quad B_\lambda v_\alpha = g(\partial, \lambda)v_{\bar{\alpha}}. \tag{5.11}$$

The values of α *and* $\bar{\alpha}$*,* β *and* $\bar{\beta}$ *along with the pairs of polynomials* $g(\partial, \lambda)$ *and* $f(\partial, \lambda)$*, whose nonzero scalar multiples give rise to nontrivial extensions, are listed as follows (by replacing* ∂ *by* $\partial + \beta$ *only in (1) and (4)):*

(1) If $\beta - \bar{\beta} = 0$, $b \neq 0$*, then* $g(\partial, \lambda) = 0$, $f(\partial, \lambda)$ *is from the nonzero polynomials of Theorem 2.2.8 with* $\alpha, \bar{\alpha} \neq 0$.

(2) If $\beta - \bar{\beta} \neq 0$, $\beta - \bar{\beta} + b = 0$, $a \neq 1$*, then* $f(\partial, \lambda) = 0$ *and* $g(\partial, \lambda)$ *is as follows (where* m *is the highest degree of* $g(\partial, \lambda)$*):*

(i) If $m = 0$*, then* $\alpha - \bar{\alpha} = 1 - a$ *and* $g(\partial, \lambda) = 1$.

(ii) If $m = 1$*, then* $\alpha - \bar{\alpha} = 2 - a$ *and* $g(\partial, \lambda) = \partial - \frac{1}{1-a}\bar{\alpha}\lambda + \frac{1}{1-a}\bar{\alpha}b + \bar{\beta}$.

(iii) If $m = 2$*, then* $\alpha = 1, \bar{\alpha} = a - 2$ *and* $g(\partial, \lambda) = \partial^2 - \frac{1}{1-a}(1 + 2\bar{\alpha})\partial\lambda - \frac{1}{1-a}\bar{\alpha}\lambda^2 + a_{10}\partial + a_{11}\lambda + a_{00}$*, where* $a_{10} = 2\bar{\beta} + \frac{1}{1-a}(1 + 2\bar{\alpha})b$, $a_{11} = \frac{2b}{1-a}\bar{\alpha} - \frac{1}{1-a}(1 + 2\bar{\alpha})\bar{\beta}$*, and* $a_{00} = \bar{\beta}^2 + b\bar{\beta}\frac{1}{1-a}(1 + 2\bar{\alpha}) - b^2\frac{1}{1-a}\bar{\alpha}$.

(iv) If $m = 3$*, then* $\alpha = a = \frac{5}{3}, \bar{\alpha} = -\frac{2}{3}$ *and* $g(\partial, \lambda) = \partial^3 + \frac{3}{2}\partial^2\lambda - \frac{3}{2}\partial\lambda^2 - \lambda^3 + a_{20}\partial^2 + a_{21}\partial\lambda + a_{22}\lambda^2 + a_{10}\partial + a_{11}\lambda + a_{00}$*, where* $a_{20} = 3\bar{\beta} - \frac{3}{2}b$, $a_{21} = 3\bar{\beta} + 3b$, $a_{22} = -\frac{3}{2}\bar{\beta} + 3b$, $a_{10} = 3\bar{\beta}^2 - 3b\bar{\beta} - \frac{3}{2}b^2$, $a_{11} = \frac{3}{2}\bar{\beta}^2 + 3b\bar{\beta} - 3b^2$, $a_{00} = \bar{\beta}^3 - \frac{3}{2}b\bar{\beta}^2 - \frac{3}{2}b^2\bar{\beta} + b^3$.

(3) If $\beta-\bar\beta\neq 0$, $\beta-\bar\beta+b=0$, $a=1$, then $f(\partial,\lambda)=0$ and $g(\partial,\lambda)$ is as follows (where m is the highest degree of $g(\partial,\lambda)$):

(i) If $m=0$, then $\alpha-\bar\alpha=0$ and $g(\lambda)=1$.

(ii) If $m=1$, then $\alpha-\bar\alpha=1$ and $g(\lambda)=\lambda-b$.

(iii) If $m=2$, then $\alpha-\bar\alpha=2$ and $g(\partial,\lambda)=\partial\lambda-\bar\alpha\lambda^2-b\partial+(\bar\beta+2b\bar\alpha)\lambda-(b\bar\beta+b^2\bar\alpha)$.

(iv) If $m=3$, then $\alpha=1$, $\bar\alpha=-2$ and $g(\partial,\lambda)=\partial^2\lambda+3\partial\lambda^2+2\lambda^3-b\partial^2+(2\bar\beta-6b)\partial\lambda+(3\bar\beta-6b)\lambda^2+(-2\bar\beta b+3b^2)\partial+(\bar\beta^2-6b\bar\beta+6b^2)\lambda-\bar\beta^2 b+3b^2\bar\beta-2b^3$.

(4) If $\beta-\bar\beta=0$, $b=0$, then $f(\partial,\lambda)$ and $g(\partial,\lambda)$ satisfy the conclusions given in Theorem 5.1.4.

(B) If $(a,b)=(1,0)$, nontrivial extensions of finite irreducible $\mathcal{W}(1,0)$-modules of the form

$$0\longrightarrow M_{\bar\alpha,\bar\beta,\bar\gamma}\longrightarrow E\longrightarrow M_{\alpha,\beta,\gamma}\longrightarrow 0 \tag{5.12}$$

exist if and only if $\gamma=\bar\gamma$, $\beta=\bar\beta$. Moreover, they are given (up to equivalence) by

$$A_\lambda v_\alpha=(\partial+\alpha\lambda+\beta)v_\alpha+f(\partial,\lambda)v_{\bar\alpha},\quad B_\lambda v_\alpha=\gamma v_\alpha+g(\partial,\lambda)v_{\bar\alpha}. \tag{5.13}$$

The values of α and $\bar\alpha$, β and $\bar\beta$, γ and $\bar\gamma$ along with the pairs of polynomials $g(\partial,\lambda)$ and $f(\partial,\lambda)$, whose nonzero scalar multiples give rise to nontrivial extensions, are listed as follows (by replacing ∂ by $\partial+\beta$):
(1) If $\gamma=\bar\gamma=0$, then $f(\partial,\lambda)$ and $g(\partial,\lambda)$ are as follows:

(i) If $\alpha-\bar\alpha=0$, then $f(\partial,\lambda)=a_0+a_1\lambda$ and $g(\partial,\lambda)=b_0$ with $(a_0,a_1,b_0)\neq(0,0,0)$.

(ii) If $\alpha-\bar\alpha=1$, then $f(\partial,\lambda)=0$ and $g(\partial,\lambda)=b_1\lambda$ with $b_1\neq 0$.

(iii) If $\alpha-\bar\alpha=2$, then $f(\partial,\lambda)=a_3\lambda^2(2\partial+\lambda)$ and $g(\partial,\lambda)=b_2\lambda(\partial-\bar\alpha\lambda)$ with $(a_3,b_2)\neq(0,0)$.

(iv) If $(\alpha,\bar\alpha)=(1,-2)$, then $f(\partial,\lambda)=a_4\partial\lambda^2(\partial+\lambda)$ and $g(\partial,\lambda)=b_3\lambda(\partial^2+3\partial\lambda+2\lambda^2)$ with $(a_4,b_3)\neq(0,0)$.

(v) If $\alpha - \bar{\alpha} = 3$ and $\bar{\alpha} \neq -2$, then $f(\partial, \lambda) = a_4 \partial \lambda^2 (\partial + \lambda)$ and $g(\partial, \lambda) = 0$ with $a_4 \neq 0$.

(vi) If $\alpha - \bar{\alpha} = 4$, then $f(\partial, \lambda) = a_5 \lambda^2 (4\partial^3 + 6\partial^2 \lambda - \partial \lambda^2 + \bar{\alpha} \lambda^3)$ and $g(\partial, \lambda) = 0$ with $a_5 \neq 0$.

(vii) If $(\alpha, \bar{\alpha}) = (1, -4)$, then $f(\partial, \lambda) = a_6(\partial^4 \lambda^2 - 10\partial^2 \lambda^4 - 17\partial \lambda^5 - 8\lambda^6)$ and $g(\partial, \lambda) = 0$ with $a_6 \neq 0$.

(viii) If $\alpha - \bar{\alpha} = 6, \alpha = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$, then $f(\partial, \lambda) = a_7(\partial^4 \lambda^3 - (2\bar{\alpha} + 3)\partial^3 \lambda^4 - 3\bar{\alpha}\partial^2 \lambda^5 - (3\bar{\alpha} + 1)\partial \lambda^6 - (\bar{\alpha} + \frac{9}{28})\lambda^7)$ and $g(\partial, \lambda) = 0$ with $a_7 \neq 0$.

(2) If $\gamma = \bar{\gamma} \neq 0$, then $f(\partial, \lambda)$ and $g(\partial, \lambda)$ are as follows:

(i) If $\alpha = \bar{\alpha}$, then $f(\partial, \lambda) = a_0 + a_1 \lambda$ and $g(\partial, \lambda) = b_0$ with $(a_0, a_1, b_0) \neq (0, 0, 0)$.

(ii) If $\alpha - \bar{\alpha} = 1$, then $f(\partial, \lambda) = a_2 \lambda^2$ and $g(\partial, \lambda) = b_1 \lambda$ with $(a_2, b_1) \neq (0, 0)$.

(iii) If $\alpha - \bar{\alpha} = 2$, then $f(\partial, \lambda) = \frac{b_2}{\beta} \partial \lambda^2 + a_3 \lambda^3$ and $g(\partial, \lambda) = b_2 \lambda^2$ with $(b_2, a_3) \neq (0, 0)$.

§5.2 Extensions of finite irreducible modules of rank two Lie conformal algebras that are of Type II with $Q(\partial, \lambda) \neq 0$

Now we consider the case that $Q(\partial, \lambda) \neq 0$, which means $b = 0$ and $a \in \{1, 0, -1, -4, -6\}$. If V is a non-trivial finite irreducible $\mathcal{R}$-module, then

$$V \cong V_{\alpha,\beta} = \mathbb{C}[\partial]v, \quad A_\lambda v = (\partial + \alpha\lambda + \beta)v, \quad B_\lambda v = 0, \tag{5.14}$$

where $\beta, 0 \neq \alpha \in \mathbb{C}$.

By definition 2.2.1, the $\mathcal{R}$-module structure on $V_{\alpha,\beta}$ given by $A_\lambda, B_\lambda \in End_{\mathbb{C}}(V)[\lambda]$ satisfies

$$[A_\lambda, A_\mu] = (\lambda - \mu)A_{\lambda+\mu} + Q(-\lambda - \mu, \lambda)B_{\lambda+\mu}, \tag{5.15}$$

$$[A_\lambda, B_\mu] = ((a - 1)\lambda - \mu)B_{\lambda+\mu}, \tag{5.16}$$

$$[B_\lambda, B_\mu] = 0, \tag{5.17}$$

$$[\partial, A_\lambda] = -\lambda A_\lambda, \tag{5.18}$$

$$[\partial, B_\lambda] = -\lambda B_\lambda. \tag{5.19}$$

§5.2.1 $\quad 0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\alpha,\beta} \longrightarrow 0$

First, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow \mathbb{C}c_\eta \longrightarrow E \longrightarrow V_{\alpha,\beta} \longrightarrow 0. \tag{5.20}$$

Then E is isomorphic to $\mathbb{C}c_\eta \oplus V_{\alpha,\beta} = \mathbb{C}c_\eta \oplus \mathbb{C}[\partial]v$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned}
&\mathcal{R}_\lambda c_\eta = 0, \quad \partial c_\eta = \eta c_\eta, \\
&A_\lambda v = (\partial + \alpha\lambda + \beta)v + f(\lambda)c_\eta, \quad B_\lambda v = g(\lambda)c_\eta,
\end{aligned} \tag{5.21}$$

where $f(\lambda), g(\lambda) \in \mathbb{C}[\lambda]$.

Lemma 5.2.1 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (5.20) are given by (5.21), and $f(\lambda)$ is a scalar multiple of $\alpha\lambda + \eta + \beta, g(\lambda) = 0$.*

Proof: Assume that (5.20) is a trivial extension, that is, there exists $v' = kc_\eta + l(\partial)v \in E$, where $k \in \mathbb{C}$ and $0 \neq l(\partial) \in \mathbb{C}[\partial]$, such that

$$A_\lambda v' = (\partial + \alpha\lambda + \beta)v' = k(\eta + \alpha\lambda + \beta)c_\eta + l(\partial)(\partial + \alpha\lambda + \beta)v, \quad B_\lambda v' = 0.$$

On the other hand, it follows from (5.21) that

$$\begin{aligned}
&A_\lambda v' = f(\lambda)l(\eta + \lambda)c_\eta + l(\partial + \lambda)(\partial + \alpha\lambda + \beta)v, \\
&B_\lambda v' = g(\lambda)l(\eta + \lambda)c_\eta.
\end{aligned}$$

We can obtain that $l(\partial)$ is a nonzero constant and $g = 0$ by comparing both expressions for $A_\lambda v'$ and $B_\lambda v'$. Thus $f(\lambda)$ is a scalar multiple of $\alpha\lambda + \eta + \beta$.

Conversely, if $f(\lambda) = k(\alpha\lambda + \eta + \beta)$ and $g(\lambda) = 0$ for some $k \in \mathbb{C}$, setting $v' = kc_\eta + v$ we can deduce that (5.20) is a trivial extension.

Theorem 5.2.2 *For a rank two Lie conformal algebra $\mathcal{R}$ that is of Type II with $Q \neq 0$, nontrivial extensions of finite irreducible conformal modules of the form (5.20) exist* *only if $\beta+\eta=0$. Moreover, they are given (up to equivalence) by (5.21). The values of α along with the corresponding polynomials $f(\lambda)$ and $g(\lambda)$ giving rise to nontrivial extensions, are listed as follows: $g(\lambda)=0$ and*

$$f(\lambda)=\begin{cases} s_1\lambda^2, & \alpha=1,\\ s_2\lambda^3, & \alpha=2,\end{cases}$$

with nonzero constants s_1, s_2. In these cases, $dim(Ext(V_{\alpha,\beta},\mathbb{C}c_\eta))=1$.

Proof: Applying both sides of (5.15) and (5.16) to v and comparing the corresponding coefficients, we obtain

$$\begin{aligned}&(\eta+\lambda+\alpha\mu+\beta)f(\lambda)-(\eta+\mu+\alpha\lambda+\beta)f(\mu)\\ &\qquad\qquad=(\lambda-\mu)f(\lambda+\mu)+Q(-\lambda-\mu,\lambda)g(\lambda+\mu), \qquad (5.22)\\ &-(\eta+\mu+\alpha\lambda+\beta)g(\mu)=((a-1)\lambda-\mu)g(\lambda+\mu). \qquad (5.23)\end{aligned}$$

Setting $\lambda=0$ in (5.23) gives

$$(\eta+\beta)g(\mu)=0.$$

If $\beta+\eta\neq 0$, then $g=0$. Putting $\mu=0$ in (5.22) and combining Lemma 5.2.1, one can deduce the extension is trivial.

Assume $\beta+\eta=0$. If $g=0$, then one can obtain the result by Proposition 2.1 in [7]. Now we consider that $g\neq 0$. If $a=1$, then (5.23) turns into

$$(\alpha\lambda+\mu)g(\mu)=\mu g(\lambda+\mu),$$

which implies $\alpha\in\{0,1\}$ and $g(\lambda)=\begin{cases} t, & \alpha=0,\\ t\lambda, & \alpha=1,\end{cases}$ for some nonzero constant t. On the other hand, under the condition that $a=1$, we have $Q(\partial,\lambda)=c(\partial+2\lambda), c\neq 0$. So (5.22) is equivalent to the equation

$$(\lambda+\alpha\mu)f(\lambda)-(\mu+\alpha\lambda)f(\mu)=(\lambda-\mu)f(\lambda+\mu)+c(\lambda-\mu)g(\lambda+\mu).$$

Taking the value of $\alpha, g(\lambda)$ and $\mu=0$ in the variant, one can get a contradiction. So

$g(\lambda) = 0$ if $a = 1$.

If $\beta + \eta = 0, g(\lambda) \neq 0, a \neq 1$, then (5.23) implies $\alpha = 1 - a$ and $g(\lambda) = t \in \mathbb{C}^{\times}$. So if $a = 0$, then $\alpha = 1$ and $Q(\partial, \lambda) = c\lambda(\partial + \lambda)(\partial + 2\lambda) + d\partial(\partial + 2\lambda)$. Putting $\mu = 0$ in (5.22), we have $-\lambda f(0) = -dt\lambda^2$. Thus $d = 0$ and $f(0) = 0$. Then putting $\mu = -\lambda$ in (5.22), we can see that $0 = 2ct\lambda^3$, that is, $c = 0$, which contracts with $Q(\partial, \lambda) \neq 0$. Hence $a \neq 0$. Similarly, one can check that $a \neq -1$ by putting $\mu = 0, -\lambda, -2\lambda$ in (5.22) one after another. For $a = -4$, we can assume $f(\lambda) = s\lambda^6$. By comparing the coefficient of $\lambda^5\mu^2$ in (5.22), we can deduce that $s = c = 0$. Thus, $Q(\partial, \lambda) = 0$. For $a = -6$, we can assume $f(\lambda) = s\lambda^8$ and we can get a contradiction by comparing the coefficient of $\lambda^6\mu^3$ in (5.22).

§5.2.2 $0 \longrightarrow V_{\alpha,\beta} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0$

Next, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow V_{\alpha,\beta} \longrightarrow E \longrightarrow \mathbb{C}c_\eta \longrightarrow 0. \tag{5.24}$$

Then E is isomorphic to $V_{\alpha,\beta} \oplus \mathbb{C}c_\eta = \mathbb{C}[\partial]v \oplus \mathbb{C}c_\eta$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v = (\partial + \alpha\lambda + \beta)v, \quad B_\lambda v = 0, \\ &A_\lambda c_\eta = f(\partial, \lambda)v, \quad B_\lambda c_\eta = g(\partial, \lambda)v, \quad \partial c_\eta = \eta c_\eta + h(\partial)v, \end{aligned} \tag{5.25}$$

where $f(\partial, \lambda), g(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$ and $h(\partial) \in \mathbb{C}[\partial]$.

Lemma 5.2.3 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (5.24) are given by (5.25), and $f(\partial, \lambda) = \varphi(\partial + \lambda)(\partial + \alpha\lambda + \beta), g(\partial, \lambda) = 0$ and $h(\partial) = (\partial - \eta)\varphi(\partial)$, where φ is a polynomial.*

Proof: Assume that (5.24) is a trivial extension, that is, there exists $c'_\eta = kc_\eta + l(\partial)v \in E$, where $0 \neq k \in \mathbb{C}$ and $l(\partial) \in \mathbb{C}[\partial]$, such that $A_\lambda c'_\eta = B_\lambda c'_\eta = 0$ and $\partial c'_\eta = \eta c'_\eta = k\eta c_\eta + \eta l(\partial)v$.

On the other hand, it follows from (5.25) that

$$A_\lambda c'_\eta = (kf(\partial,\lambda) + l(\partial+\lambda)(\partial+\alpha\lambda+\beta))v, \quad B_\lambda c'_\eta = kg(\partial,\lambda)v,$$
$$\partial c'_\eta = k\eta c_\eta + (kh(\partial) + \partial l(\partial))v.$$

We can obtain the result by comparing both expressions for $A_\lambda c'_\eta, B_\lambda c'_\eta$ and $\partial c'_\eta$.

Conversely, if $f(\partial,\lambda) = \varphi(\partial+\lambda)(\partial+\alpha\lambda+\beta), g(\partial,\lambda) = 0$ and $h(\partial) = (\partial-\eta)\varphi(\partial)$ for some polynomial φ, setting $c'_\eta = c_\eta - \varphi(\partial)v$, we can deduce that (5.24) is a trivial extension.

Theorem 5.2.4 *For a rank two Lie conformal algebra $\mathcal{R}$ that is of Type II, nontrivial extensions of finite irreducible conformal modules of the form (5.24) exist only if $\beta+\eta = 0$ and $\alpha = 1$. Moreover, they are given (up to equivalence) by (5.25) and $dim(Ext(\mathbb{C}c_\eta, V_{\alpha,\beta})) = 1$. The corresponding polynomials $f(\partial,\lambda), g(\partial,\lambda)$ and $h(\partial)$ giving rise to nontrivial extensions, are listed as follows: $g(\partial,\lambda) = 0$ and $f(\partial,\lambda) = h(\partial) = s$ with nonzero constant s.*

Proof: Applying both sides of (5.18) and (5.19) to c_η and comparing the corresponding coefficients gives the following equations

$$(\partial+\lambda-\eta)f(\partial,\lambda) = h(\partial+\lambda)(\partial+\alpha\lambda+\beta), \tag{5.26}$$
$$(\partial+\lambda-\eta)g(\partial,\lambda) = 0. \tag{5.27}$$

Then $g(\partial,\lambda) = 0$ by (5.27), and the result can be deduced by Proposition 2.2 in [7].

§5.2.3 $\quad 0 \longrightarrow V_{\alpha,\beta} \longrightarrow E \longrightarrow V_{\bar\alpha,\bar\beta} \longrightarrow 0$

Finally, we consider extensions of finite irreducible $\mathcal{R}$-modules of the form

$$0 \longrightarrow V_{\alpha,\beta} \longrightarrow E \longrightarrow V_{\bar\alpha,\bar\beta} \longrightarrow 0. \tag{5.28}$$

Then E is isomorphic to $V_{\alpha,\beta}\oplus V_{\bar{\alpha},\bar{\beta}} = \mathbb{C}[\partial]v\oplus\mathbb{C}[\partial]\bar{v}$ as a $\mathbb{C}[\partial]$-module, and the following identities hold in E:

$$\begin{aligned} &A_\lambda v = (\partial + \alpha\lambda + \beta)v, \quad B_\lambda v = 0, \\ &A_\lambda \bar{v} = (\partial + \bar{\alpha}\lambda + \bar{\beta})\bar{v} + f(\partial,\lambda)v, \quad B_\lambda \bar{v} = g(\partial,\lambda)v, \end{aligned} \tag{5.29}$$

where $f(\partial,\lambda), g(\partial,\lambda) \in \mathbb{C}[\partial,\lambda]$.

Lemma 5.2.5 *All trivial extensions of finite irreducible $\mathcal{R}$-modules of the form (5.28) are given by (5.29), and $f(\partial,\lambda) = \varphi(\partial+\lambda)(\partial+\alpha\lambda+\beta) - \varphi(\partial)(\partial+\bar{\alpha}\lambda+\bar{\beta})$ and $g(\partial,\lambda) = 0$ for some polynomial φ.*

Proof: Assume that (5.28) is a trivial extension, that is, there exists $\bar{v}' = k(\partial)v + l(\partial)\bar{v} \in E$, where $k(\partial), l(\partial) \in \mathbb{C}[\partial]$ and $l(\partial) \neq 0$, such that

$$A_\lambda \bar{v}' = (\partial+\bar{\alpha}\lambda+\bar{\beta})\bar{v}' = k(\partial)(\partial+\bar{\alpha}\lambda+\bar{\beta})v + l(\partial)(\partial+\bar{\alpha}\lambda+\bar{\beta})\bar{v},\ B_\lambda \bar{v}' = 0.$$

On the other hand, it follows from (5.29) that

$$\begin{aligned} &A_\lambda \bar{v}' = (k(\partial+\lambda)(\partial+\alpha\lambda+\beta) + l(\partial+\lambda)f(\partial,\lambda))v + l(\partial+\lambda)(\partial+\bar{\alpha}\lambda+\bar{\beta})\bar{v}, \\ &B_\lambda \bar{v}' = l(\partial+\lambda)g(\partial,\lambda)v. \end{aligned}$$

Comparing both expressions for $A_\lambda \bar{v}'$ and $B_\lambda \bar{v}'$, we can obtain that l is a nonzero constant. And then we can give the expressions of $f(\partial,\lambda)$ and $g(\partial,\lambda)$.

Conversely, if $f(\partial,\lambda) = \varphi(\partial+\lambda)(\partial+\alpha\lambda+\beta) - \varphi(\partial)(\partial+\bar{\alpha}\lambda+\bar{\beta})$ and $g(\partial,\lambda) = 0$ for some polynomial φ, setting $\bar{v}' = -\varphi(\partial)v + \bar{v}$, we can deduce that (5.28) is a trivial extension.

Theorem 5.2.6 *For a rank two Lie conformal algebra $\mathcal{R}$ that is of Type II, nontrivial extensions of finite irreducible conformal modules of the form (5.28) exist only if $\beta = \bar{\beta}$. Moreover, they are given (up to equivalence) by (5.29). The value of $\alpha, \bar{\alpha}$, and the corresponding polynomials $f(\partial,\lambda)$ and $g(\partial,\lambda)$ giving rise to nontrivial extensions, are listed as follows (by replacing ∂ by $\partial+\beta$):*

1. *In the case when $a = 1$, where $Q(\partial,\lambda) = c(\partial+2\lambda)$ for some nonzero constant c,*

$\bar{\alpha}-\alpha \in \{0,1,2,3,4,5,6\}, \alpha, \bar{\alpha} \neq 0$ *and*

(i) $\bar{\alpha}=\alpha$, $f(\partial,\lambda)=s_0+s_1\lambda, g(\partial,\lambda)=0$, *where* $(s_0,s_1)\neq(0,0)$.

(ii) $\bar{\alpha}-\alpha=1$, $f(\partial,\lambda)=\frac{ct}{\alpha}\partial, g(\partial,\lambda)=t\lambda$, *where* $t\neq 0$.

(iii) $\bar{\alpha}-\alpha=2$ *with* $\alpha\neq-1$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(iv) $\bar{\alpha}=1$ *and* $\alpha=-1$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda)-ct(\partial^2-\lambda^2), g(\partial,\lambda)=t(\partial\lambda+\lambda^2)$, *where* $(s,t)\neq(0,0)$.

(v) $\bar{\alpha}-\alpha=3$, $f(\partial,\lambda)=s\partial\lambda^2(\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(vi) $\bar{\alpha}-\alpha=4$, $f(\partial,\lambda)=s\lambda^2(4\partial^3+6\partial^2\lambda-\partial\lambda^2+\alpha_1\lambda^3), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(vii) $\bar{\alpha}=1$ *and* $\alpha=-4$, $f(\partial,\lambda)=s(\partial^4\lambda^2-10\partial^2\lambda^4-17\partial\lambda^5-8\lambda^6), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(viii) $\bar{\alpha}=\frac{7}{2}\pm\frac{\sqrt{19}}{2}$ *and* $\alpha=-\frac{5}{2}\pm\frac{\sqrt{19}}{2}$, $f(\partial,\lambda)=s(\partial^4\lambda^3-(2\alpha+3)\partial^3\lambda^4-3\alpha\partial^2\lambda^5-(3\alpha+1)\partial\lambda^6-(\alpha+\frac{9}{28})\lambda^7), g(\partial,\lambda)=0$, *where* $s\neq 0$.

The value of $dim(Ext(V_{\bar{\alpha},\bar{\beta}},V_{\alpha,\beta}))$ *is 2 in subcase (i) and (iv), and 1 in the other subcases.*

2. *In the case when* $a=0$, *where* $Q(\partial,\lambda)=c(\partial+2\lambda)(\partial+\lambda)\lambda+d(\partial+2\lambda)\partial$ *for some nonzero constants* c,d, $\bar{\alpha}-\alpha\in\{0,1,2,3,4,5,6\}, \alpha,\bar{\alpha}\neq 0$ *and*

(i) $\bar{\alpha}=\alpha$, $f(\partial,\lambda)=s_0+s_1\lambda, g(\partial,\lambda)=0$, *where* $(s_0,s_1)\neq(0,0)$.

(ii) $\bar{\alpha}-\alpha=1$, $f(\partial,\lambda)=-\frac{ct}{\alpha}\partial\lambda-\frac{dt}{\alpha}\partial, g(\partial,\lambda)=t$, *where* $t\neq 0$.

(iii) $\bar{\alpha}-\alpha=2$ *with* $\alpha\neq-1$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(iv) $\bar{\alpha}=1$ *and* $\alpha=-1$, $f(\partial,\lambda)=s\lambda^2(2\partial+\lambda)+ct\partial^2\lambda+dt(\partial^2-\lambda^2), g(\partial,\lambda)=t(\partial+\lambda)$, *where* $(s,t)\neq(0,0)$.

(v) $\bar{\alpha}-\alpha=3$, $f(\partial,\lambda)=s\partial\lambda^2(\partial+\lambda), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(vi) $\bar{\alpha}-\alpha=4$, $f(\partial,\lambda)=s\lambda^2(4\partial^3+6\partial^2\lambda-\partial\lambda^2+\alpha_1\lambda^3), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(vii) $\bar{\alpha}=1$ *and* $\alpha=-4$, $f(\partial,\lambda)=s(\partial^4\lambda^2-10\partial^2\lambda^4-17\partial\lambda^5-8\lambda^6), g(\partial,\lambda)=0$, *where* $s\neq 0$.

(viii) $\bar{\alpha}=\frac{7}{2}\pm\frac{\sqrt{19}}{2}$ *and* $\alpha=-\frac{5}{2}\pm\frac{\sqrt{19}}{2}$, $f(\partial,\lambda)=s(\partial^4\lambda^3-(2\alpha+3)\partial^3\lambda^4-3\alpha\partial^2\lambda^5-(3\alpha+1)\partial\lambda^6-(\alpha+\frac{9}{28})\lambda^7), g(\partial,\lambda)=0$, *where* $s\neq 0$.

The value of $dim(Ext(V_{\bar{\alpha},\bar{\beta}}, V_{\alpha,\beta}))$ *is 2 in subcase (i) and (iv), and 1 in the other subcases.*

3. *In the case when* $a = -1$*, where* $Q(\partial, \lambda) = c(\partial + 2\lambda)\partial^2 + d(\partial + 2\lambda)(\partial + \lambda)\partial\lambda$ *for some nonzero constants* c, d*,* $\bar{\alpha} - \alpha \in \{0, 1, 2, 3, 4, 5, 6\}, \alpha, \bar{\alpha} \neq 0$ *and*

   (i) $\bar{\alpha} = \alpha$*,* $f(\partial, \lambda) = s_0 + s_1\lambda, g(\partial, \lambda) = 0$*, where* $(s_0, s_1) \neq (0, 0)$*.*

   (ii) $\bar{\alpha} - \alpha = 2$ *with* $\alpha \neq -\frac{1}{2}$*,* $f(\partial, \lambda) = s\lambda^2(2\partial + \lambda), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   (iii) $\bar{\alpha} = \frac{3}{2}$ *and* $\alpha = -\frac{1}{2}$*,* $f(\partial, \lambda) = s\lambda^2(2\partial+\lambda) - 2dt\partial^2\lambda - ct(2\partial^2 - \lambda^2), g(\partial, \lambda) = t$*, where* $(s, t) \neq (0, 0)$*.*

   (iv) $\bar{\alpha} - \alpha = 3$ *with* $\alpha \neq -1$*,* $f(\partial, \lambda) = s\partial\lambda^2(\partial + \lambda), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   (v) $\bar{\alpha} = 2$ *and* $\alpha = -1$*,* $f(\partial, \lambda) = s\partial\lambda^2(\partial + \lambda) - \frac{dt}{4}(2\partial^3\lambda + \lambda^4) - \frac{ct}{2}(\partial^3 - 2\partial\lambda^2 - 2\lambda^3), g(\partial, \lambda) = t(\partial + \frac{1}{2}\lambda)$*, where* $(s, t) \neq (0, 0)$*.*

   (vi) $\bar{\alpha} - \alpha = 4$*,* $f(\partial, \lambda) = s\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \alpha_1\lambda^3), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   (vii) $\bar{\alpha} = 1$ *and* $\alpha = -4$*,* $f(\partial, \lambda) = s(\partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   (viii) $\bar{\alpha} = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$*,* $f(\partial, \lambda) = s(\partial^4\lambda^3 - (2\alpha + 3)\partial^3\lambda^4 - 3\alpha\partial^2\lambda^5 - (3\alpha + 1)\partial\lambda^6 - (\alpha + \frac{9}{28})\lambda^7), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   The value of $dim(Ext(V_{\bar{\alpha},\bar{\beta}}, V_{\alpha,\beta}))$ *is 2 in subcase (i) and (v), and 1 in the other subcases.*

4. *In the case when* $a = -4$*, where* $Q(\partial, \lambda) = c(\partial + 2\lambda)(\partial + \lambda)^3\lambda^3$ *for some nonzero constant* c*,* $\bar{\alpha} - \alpha \in \{0, 1, 2, 3, 4, 5, 6, 7\}, \alpha, \bar{\alpha} \neq 0$*, and*

   (i) $\bar{\alpha} = \alpha$*,* $f(\partial, \lambda) = s_0 + s_1\lambda, g(\partial, \lambda) = 0$*, where* $(s_0, s_1) \neq (0, 0)$*.*

   (ii) $\bar{\alpha} - \alpha = 2$*,* $f(\partial, \lambda) = s\lambda^2(2\partial + \lambda), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   (iii) $\bar{\alpha} - \alpha = 3$*,* $f(\partial, \lambda) = s\partial\lambda^2(\partial + \lambda), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   (iv) $\bar{\alpha} - \alpha = 4$*,* $f(\partial, \lambda) = s\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \alpha_1\lambda^3), g(\partial, \lambda) = 0$*, where* $s \neq 0$*.*

   (v) $\bar{\alpha} - \alpha = 5$ *with* $\alpha \notin \{-2, -4\}$*,* $f(\partial, \lambda) = -\frac{3}{\alpha(\alpha+2)(\alpha+4)}ct\partial^3\lambda^3 + \frac{9(\alpha+1)}{2\alpha(\alpha+2)(\alpha+4)}ct\partial^2\lambda^4 - \frac{9(\alpha+1)(2\alpha+1)}{10\alpha(\alpha+2)(\alpha+4)}ct\partial\lambda^5 + \frac{(\alpha+1)(2\alpha+1)}{10(\alpha+2)(\alpha+4)}ct\lambda^6, g(\partial, \lambda) = t$*, where* $t \neq 0$*.*

   (vi) $\bar{\alpha} = 3$ *and* $\alpha = -2$*,* $f(\partial, \lambda) = \frac{3}{8}ct\partial^4\lambda^2 - \frac{3}{2}ct\partial^2\lambda^4 - \frac{57}{40}ct\partial\lambda^5 - \frac{2}{5}ct\lambda^6, g(\partial, \lambda) = t$*, where* $t \neq 0$*.*

(vii) $\bar{\alpha} = 1$ *and* $\alpha = -4$, $f(\partial, \lambda) = s(\partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6), g(\partial, \lambda) = 0$, *where* $s \neq 0$.

(viii) $\bar{\alpha} - \alpha = 6$ *with* $\alpha \notin \{-\frac{5}{2}, -\frac{5}{2} \pm \frac{\sqrt{19}}{2}\}$, $f(\partial, \lambda) = -\frac{3}{(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial^4\lambda^3 + \frac{3(2\alpha+3)}{(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial^3\lambda^4 - \frac{9(\alpha+1)(2\alpha+3)}{5(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial^2\lambda^5 + \frac{(\alpha+1)(2\alpha+1)(2\alpha+3)}{5(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\partial\lambda^6 - \frac{\alpha(\alpha+1)(2\alpha+1)(2\alpha+3)}{70(2\alpha+5)(2\alpha^2+10\alpha+3)}ct\lambda^7, g(\partial, \lambda) = t(\partial - \frac{\alpha}{5}\lambda)$, *where* $t \neq 0$.

(ix) $\bar{\alpha} = \frac{7}{2}$ *and* $\alpha = -\frac{5}{2}$, $f(\partial, \lambda) = \frac{36}{665}ct\bar{\partial}^5\lambda^2 - \frac{54}{113}ct\bar{\partial}^3\lambda^4 - \frac{387}{665}ct\bar{\partial}^2\lambda^5 - \frac{218}{665}ct\bar{\partial}\lambda^6 + \frac{127}{1862}ct\lambda^7, g(\partial, \lambda) = t(\partial + \frac{1}{2}\lambda)$, *where* $t \neq 0$.

(x) $\bar{\alpha} = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial, \lambda) = s(\partial^4\lambda^3 - (2\alpha+3)\partial^3\lambda^4 - 3\alpha\partial^2\lambda^5 - (3\alpha+1)\partial\lambda^6 - (\alpha + \frac{9}{28})\lambda^7), g(\partial, \lambda) = 0$, *where* $s \neq 0$.

(xi) $\bar{\alpha} = 1$ *and* $\alpha = -6$, $f(\partial, \lambda) = \frac{1}{35}ct\partial^5\lambda^3 + \frac{2}{7}ct\partial^4\lambda^4 + \frac{36}{35}ct\partial^3\lambda^5 + \frac{12}{7}ct\partial^2\lambda^6 + \frac{66}{49}ct\partial\lambda^7 + \frac{99}{245}ct\lambda^8, g(\partial, \lambda) = t(\partial^2 + \frac{11}{5}\partial\lambda + \frac{6}{5}\lambda^2)$, *where* $t \neq 0$.

The value of $dim(Ext(V_{\bar{\alpha},\bar{\beta}}, V_{\alpha,\beta}))$ *is 2 in subcase (i), and 1 in the other subcases.*

5. *In the case when* $a = -6$, *where* $Q(\partial, \lambda) = c(\partial + 2\lambda)[11(\partial + \lambda)^4\lambda^4 + 2(\partial + \lambda)^3\partial^2\lambda^3]$ *for some nonzero constant* c, $\bar{\alpha} - \alpha \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}, \alpha, \bar{\alpha} \neq 0$, *and*

(i) $\bar{\alpha} = \alpha$, $f(\partial, \lambda) = s_0 + s_1\lambda, g(\partial, \lambda) = 0$, *where* $(s_0, s_1) \neq (0, 0)$.

(ii) $\bar{\alpha} - \alpha = 2$, $f(\partial, \lambda) = s\lambda^2(2\partial + \lambda), g(\partial, \lambda) = 0$, *where* $s \neq 0$.

(iii) $\bar{\alpha} - \alpha = 3$, $f(\partial, \lambda) = s\partial\lambda^2(\partial + \lambda), g(\partial, \lambda) = 0$, *where* $s \neq 0$.

(iv) $\bar{\alpha} - \alpha = 4$, $f(\partial, \lambda) = s\lambda^2(4\partial^3 + 6\partial^2\lambda - \partial\lambda^2 + \alpha_1\lambda^3), g(\partial, \lambda) = 0$, *where* $s \neq 0$.

(v) $\bar{\alpha} = 1$ *and* $\alpha = -4$, $f(\partial, \lambda) = s(\partial^4\lambda^2 - 10\partial^2\lambda^4 - 17\partial\lambda^5 - 8\lambda^6), g(\partial, \lambda) = 0$, *where* $s \neq 0$.

(vi) $\bar{\alpha} = \frac{7}{2} \pm \frac{\sqrt{19}}{2}$ *and* $\alpha = -\frac{5}{2} \pm \frac{\sqrt{19}}{2}$, $f(\partial, \lambda) = s(\partial^4\lambda^3 - (2\alpha+3)\partial^3\lambda^4 - 3\alpha\partial^2\lambda^5 - (3\alpha+1)\partial\lambda^6 - (\alpha + \frac{9}{28})\lambda^7), g(\partial, \lambda) = 0$, *where* $s \neq 0$.

(vii) $\bar{\alpha} = 4 \pm \frac{\sqrt{22}}{2}$ *and* $\alpha = -3 \pm \frac{\sqrt{22}}{2}$, $f(\partial, \lambda) = -\frac{40}{7(\alpha+3)}ct\partial^5\lambda^3 + \frac{100(\alpha+2)}{7(\alpha+3)}ct\partial^4\lambda^4 + \frac{40(5\alpha+1)}{7(\alpha+3)}ct\partial^3\lambda^5 + \frac{20(16\alpha+11)}{7(\alpha+3)}ct\partial^2\lambda^6 + \frac{10(154\alpha+101)}{49(\alpha+3)}ct\partial\lambda^7 + \frac{823\alpha+539}{98(\alpha+3)}ct\lambda^8, g(\partial, \lambda) = t$, *where* $t \neq 0$.

(viii) $\bar{\alpha} = 7$ *and* $\alpha = -1$, $f(\partial, \lambda) = -\frac{2}{7}ct\partial^6\lambda^3 + \frac{9}{7}ct\partial^5\lambda^4 - \frac{9}{7}ct\partial^4\lambda^5 + \frac{2}{7}ct\partial^3\lambda^6, g(\partial, \lambda) = t(\partial + \frac{1}{7}\lambda)$, *where* $t \neq 0$.

(ix) $\bar{\alpha} = 2$ *and* $\alpha = -6$, $f(\partial, \lambda) = -\frac{2}{7}ct\partial^6\lambda^3 - 3ct\partial^5\lambda^4 - 12ct\partial^4\lambda^5 - 24ct\partial^3\lambda^6 - \frac{180}{7}ct\partial^2\lambda^7 - \frac{99}{7}ct\partial\lambda^8 - \frac{22}{7}ct\lambda^9, g(\partial, \lambda) = t(\partial + \frac{6}{7}\lambda)$, *where* $t \neq 0$.

The value of dim(Ext($V_{\bar{\alpha},\bar{\beta}}, V_{\alpha,\beta}$)) is 2 in subcase (i), and 1 in the other subcases.

Proof: Applying both sides of (5.15) and (5.16) to $\bar{v}$ and comparing the corresponding coefficients, we obtain

$$f(\partial,\lambda)(\partial+\lambda+\bar{\alpha}\mu+\bar{\beta})+f(\partial+\lambda,\mu)(\partial+\alpha\lambda+\beta)-f(\partial,\mu)(\partial+\mu+\bar{\alpha}\lambda+\bar{\beta}) \\ -f(\partial+\mu,\lambda)(\partial+\alpha\mu+\beta)=(\lambda-\mu)f(\partial,\lambda+\mu)+Q(-\lambda-\mu,\lambda)g(\partial,\lambda+\mu), \tag{5.30}$$

$$g(\partial+\lambda,\mu)(\partial+\alpha\lambda+\beta)-g(\partial,\mu)(\partial+\mu+\bar{\alpha}\lambda+\bar{\beta})=((a-1)\lambda-\mu)g(\partial,\lambda+\mu). \tag{5.31}$$

Setting $\lambda=0$ in (5.31) gives

$$g(\partial,\mu)(\beta-\bar{\beta})=0.$$

If $g(\partial,\mu)=0$, then the result follows from Theorem 3.2 in [7] (or Theorem 2.7 in [21]). Now we assume $g(\partial,\lambda)\neq 0$ so that $\beta=\bar{\beta}$. If no confusion is possible, we replace $\partial+\beta$ by ∂ in the sequel. By Proposition 3.8 and Corollary 3.10 in [21], for $a\in\{0,-1,-4,-6\}$, the nonzero solutions (up to a nonzero scalar t) of (5.31) are given by

$$g(\partial,\lambda)=\begin{cases}1, & \alpha-\bar{\alpha}=a-1,\\ \partial-\frac{1}{1-a}\alpha\lambda, & \alpha-\bar{\alpha}=a-2,\\ \partial^2-\frac{1}{1-a}(1+2\alpha)\partial\lambda-\frac{1}{1-a}\alpha\lambda^2, & \alpha=a-2,\bar{\alpha}=1,\end{cases} \tag{5.32}$$

and for $a=1$,

$$g(\partial,\lambda)=\begin{cases}1, & \alpha-\bar{\alpha}=0,\\ \lambda, & \alpha-\bar{\alpha}=-1,\\ \partial\lambda-\alpha\lambda^2, & \alpha-\bar{\alpha}=-2,\\ \partial^2\lambda+3\partial\lambda^2+2\lambda^3, & \alpha=-2,\bar{\alpha}=1.\end{cases} \tag{5.33}$$

Putting these results in (5.30), we can obtain the expression of $f(\partial,\lambda)$ as follows.

(1) If $a = 1, \alpha = \bar{\alpha}$, (5.30) is writen as

$$\begin{aligned} & f(\partial,\lambda)(\partial+\lambda+\alpha\mu) + f(\partial+\lambda,\mu)(\partial+\alpha\lambda) \\ & \quad - f(\partial,\mu)(\partial+\mu+\alpha\lambda) - f(\partial+\mu,\lambda)(\partial+\alpha\mu) \\ & \quad = (\lambda-\mu) f(\partial,\lambda+\mu) + ct(\lambda-\mu). \end{aligned} \tag{5.34}$$

By the nature of (5.34), we may assume that a solution to (5.34) is a homogeneous polynomial in ∂ and λ of degree 0, that is, $f(\partial,\lambda) = s$ for some constant s. Taking it in (5.34), we have $ct = 0$ which means $g(\partial,\lambda) = 0$ or $Q(\partial,\lambda) = 0$. This contradiction illustrates that the equation has no solution in this case.

(2) If $a = 1, \alpha - \bar{\alpha} = -1$, (5.30) is writen as

$$\begin{aligned} & f(\partial,\lambda)(\partial+\lambda+\mu+\alpha\mu) + f(\partial+\lambda,\mu)(\partial+\alpha\lambda) - f(\partial,\mu)(\partial+\lambda+\mu+\alpha\lambda) \\ & \quad - f(\partial+\mu,\lambda)(\partial+\alpha\mu) = (\lambda-\mu) f(\partial,\lambda+\mu) + ct(\lambda+\mu)(\lambda-\mu). \end{aligned} \tag{5.35}$$

By the nature of (5.35), we may assume that a solution to (5.35) is a homogeneous polynomial in ∂ and λ of degree 1, that is, $f(\partial,\lambda) = s_1\partial + s_2\lambda$ for some constant s_1, s_2. Taking it in (5.35), we have $s_1 = \frac{ct}{\alpha}$ with $\alpha \neq 0$ which means $f(\partial,\lambda) = \frac{ct}{\alpha}\partial + s_2\lambda$. For other homogeneous parts, one can refer to the case that $g(\partial,\lambda) = 0$.

(3) If $a = 1, \alpha - \bar{\alpha} = -2$, set $\bar{\partial} = \partial + \beta, F(\bar{\partial},\lambda) = f(\bar{\partial} - \beta, \lambda)$, and (5.30) is writen as

$$\begin{aligned} & F(\bar{\partial},\lambda)(\bar{\partial}+\lambda+2\mu+\alpha\mu) + F(\bar{\partial}+\lambda,\mu)(\bar{\partial}+\alpha\lambda) - F(\bar{\partial},\mu)(\bar{\partial}+2\lambda+\mu+\alpha\lambda) \\ & \quad - F(\bar{\partial}+\mu,\lambda)(\bar{\partial}+\alpha\mu) = (\lambda-\mu)F(\bar{\partial},\lambda+\mu) + ct(\lambda-\mu)(\bar{\partial}(\lambda+\mu) - \alpha(\lambda+\mu)^2). \end{aligned} \tag{5.36}$$

By the nature of (5.36), we may assume that a solution to (5.36) is a homogeneous polynomial in $\bar{\partial}$ and λ of degree 2, that is, $F(\bar{\partial},\lambda) = s_1\bar{\partial}^2 + s_2\bar{\partial}\lambda + s_3\lambda^2$ for some constant s_1, s_2, s_3. Taking it in (5.36), we have $\alpha = 0, F(\bar{\partial},\lambda) = ct\bar{\partial}^2 + s_2\bar{\partial}\lambda$ and $\alpha = -1, F(\bar{\partial},\lambda) = -ct(\bar{\partial}^2 - \lambda^2) + s_2(\bar{\partial}\lambda + \lambda^2)$.

(4) If $a = 1, \alpha = -2, \bar{\alpha} = 1$, set $\bar{\partial} = \partial + \beta, F(\bar{\partial},\lambda) = f(\bar{\partial} - \beta, \lambda)$, and (5.30) is

writen as

$$
\begin{aligned}
&F(\bar\partial,\lambda)(\bar\partial+\lambda+\mu)+F(\bar\partial+\lambda,\mu)(\bar\partial-2\lambda)\\
&\quad-F(\bar\partial,\mu)(\bar\partial+\mu+\lambda)-F(\bar\partial+\mu,\lambda)(\bar\partial-2\mu)\\
&\qquad=(\lambda-\mu)F(\bar\partial,\lambda+\mu)+ct(\lambda-\mu)(\bar\partial^2(\lambda+\mu)+3\bar\partial(\lambda+\mu)^2+2(\lambda+\mu)^3). \quad (5.37)
\end{aligned}
$$

By the nature of (5.37), we may assume that a solution to (5.37) is a homogeneous polynomial in $\bar\partial$ and λ of degree 3, that is, $F(\bar\partial,\lambda)=s_1\bar\partial^3+s_2\bar\partial^2\lambda+s_3\bar\partial\lambda^2+s_4\lambda^3$ for some constant s_1,s_2,s_3,s_4. Taking it in (5.37), we have $ct=0$ and get a contradiction.

(5) If $a=0,\alpha-\bar\alpha=-1$, set $\bar\partial=\partial+\beta, F(\bar\partial,\lambda)=f(\bar\partial-\beta,\lambda)$, and (5.30) is writen as

$$
\begin{aligned}
&F(\bar\partial,\lambda)(\bar\partial+\lambda+\mu+\alpha\mu)+F(\bar\partial+\lambda,\mu)(\bar\partial+\alpha\lambda)-F(\bar\partial,\mu)(\bar\partial+\lambda+\mu+\alpha\lambda)\\
&\quad-F(\bar\partial+\mu,\lambda)(\bar\partial+\alpha\mu)=(\lambda-\mu)F(\bar\partial,\lambda+\mu)-ct\lambda\mu(\lambda-\mu)-dt(\lambda+\mu)(\lambda-\mu).
\end{aligned} \quad (5.38)
$$

By the nature of (5.38), we may assume that a solution to (5.38) is a sum of homogeneous polynomials in $\bar\partial,\lambda$ of degree 1 and degree 2, that is, $F(\bar\partial,\lambda)=s_1\bar\partial^2+s_2\bar\partial\lambda+s_3\lambda^2+s_4\bar\partial+s_5\lambda$ for some constant s_1,s_2,s_3,s_4,s_5. Taking it in (5.38), we have $\alpha\neq 0$ and $F(\bar\partial,\lambda)=-\frac{ct}{\alpha}\bar\partial\lambda+s_3\lambda^2-\frac{dt}{\alpha}\bar\partial+s_5\lambda$.

(6) If $a=0,\alpha-\bar\alpha=-2$, set $\bar\partial=\partial+\beta, F(\bar\partial,\lambda)=f(\bar\partial-\beta,\lambda)$, and (5.30) is writen as

$$
\begin{aligned}
&F(\bar\partial,\lambda)(\bar\partial+\lambda+2\mu+\alpha\mu)+F(\bar\partial+\lambda,\mu)(\bar\partial+\alpha\lambda)-F(\bar\partial,\mu)(\bar\partial+2\lambda+\mu+\alpha\lambda)\\
&\quad-F(\bar\partial+\mu,\lambda)(\bar\partial+\alpha\mu)\\
&\qquad=(\lambda-\mu)F(\bar\partial,\lambda+\mu)-t(\lambda-\mu)(c\lambda\mu+d\lambda+d\mu)(\bar\partial-\alpha\lambda-\alpha\mu). \quad (5.39)
\end{aligned}
$$

By the nature of (5.39), we may assume that a solution to (5.39) is a sum of homogeneous polynomials in $\bar\partial,\lambda$ of degree 2 and degree 3, that is, $F(\bar\partial,\lambda)=s_1\bar\partial^3+s_2\bar\partial^2\lambda+s_3\bar\partial\lambda^2+s_4\lambda^3+s_5\bar\partial^2+s_6\bar\partial\lambda+s_7\lambda^2$ for some constant $s_1,s_2,s_3,s_4,s_5,s_6,s_7$. Taking it in (5.39), we have $\alpha=0, F(\bar\partial,\lambda)=-ct\bar\partial^2\lambda+s_3\bar\partial\lambda^2+s_4\lambda^3-dt\bar\partial^2+s_6\bar\partial\lambda$ and $\alpha=-1, F(\bar\partial,\lambda)=ct\bar\partial^2\lambda+s_3\bar\partial\lambda^2+s_4\lambda^3+dt\bar\partial^2+s_6\bar\partial\lambda+(s_6-dt)\lambda^2$.

(7) If $a=0,\alpha=-2,\bar\alpha=1$, set $\bar\partial=\partial+\beta, F(\bar\partial,\lambda)=f(\bar\partial-\beta,\lambda)$, and (5.30) is

writen as

$$F(\bar{\partial},\lambda)(\bar{\partial}+\lambda+\mu)+F(\bar{\partial}+\lambda,\mu)(\bar{\partial}-2\lambda)-F(\bar{\partial},\mu)(\bar{\partial}+\lambda+\mu)-F(\bar{\partial}+\mu,\lambda)(\bar{\partial}-2\mu)$$
$$=(\lambda-\mu)F(\bar{\partial},\lambda+\mu)-t(\lambda-\mu)(c\lambda\mu+d\lambda+d\mu)(\bar{\partial}^2+3\bar{\partial}(\lambda+\mu)+2(\lambda+\mu)^2). \tag{5.40}$$

By the nature of (5.40), we may assume that a solution to (5.40) is a sum of homogeneous polynomials in $\bar{\partial},\lambda$ of degree 3 and degree 4, that is, $F(\bar{\partial},\lambda)=\sum_{i=1}^{5}s_i\bar{\partial}^{5-i}\lambda^{i-1}+\sum_{i=6}^{9}s_i\bar{\partial}^{9-i}\lambda^{i-6}$ for some constant $s_i, i=1,2,\cdots,9$. Taking it in (5.40), we have $ct=dt=0$ and get a contradiction.

(8) If $a=-1,\alpha-\bar{\alpha}=-2$, set $\bar{\partial}=\partial+\beta, F(\bar{\partial},\lambda)=f(\bar{\partial}-\beta,\lambda)$, and (5.30) is writen as

$$F(\bar{\partial},\lambda)(\bar{\partial}+\lambda+2\mu+\alpha\mu)+F(\bar{\partial}+\lambda,\mu)(\bar{\partial}+\alpha\lambda)-F(\bar{\partial},\mu)(\bar{\partial}+2\lambda+\mu+\alpha\lambda)$$
$$-F(\bar{\partial}+\mu,\lambda)(\bar{\partial}+\alpha\mu)$$
$$=(\lambda-\mu)F(\bar{\partial},\lambda+\mu)+ct(\lambda-\mu)(\lambda+\mu)^2+dt\lambda\mu(\lambda-\mu)(\lambda+\mu). \tag{5.41}$$

By the nature of (5.41), we may assume that a solution to (5.41) is a sum of homogeneous polynomials in $\bar{\partial},\lambda$ of degree 2 and degree 3, that is, $F(\bar{\partial},\lambda)=s_1\bar{\partial}^3+s_2\bar{\partial}^2\lambda+s_3\bar{\partial}\lambda^2+s_4\lambda^3+s_5\bar{\partial}^2+s_6\bar{\partial}\lambda+s_7\lambda^2$ for some constant $s_1,s_2,s_3,s_4,s_5,s_6,s_7$. Taking it in (5.41), we have $\alpha=-\frac{1}{2}$ and $F(\bar{\partial},\lambda)=-2dt\bar{\partial}^2\lambda+s_3\bar{\partial}\lambda^2+s_4\lambda^3-2ct\bar{\partial}^2+s_6\bar{\partial}\lambda+(\frac{1}{2}s_6+ct)\lambda^2$.

(9) If $a=-1,\alpha-\bar{\alpha}=-3$, set $\bar{\partial}=\partial+\beta, F(\bar{\partial},\lambda)=f(\bar{\partial}-\beta,\lambda)$, and (5.30) is writen as

$$F(\bar{\partial},\lambda)(\bar{\partial}+\lambda+3\mu+\alpha\mu)+F(\bar{\partial}+\lambda,\mu)(\bar{\partial}+\alpha\lambda)-F(\bar{\partial},\mu)(\bar{\partial}+3\lambda+\mu+\alpha\lambda)$$
$$-F(\bar{\partial}+\mu,\lambda)(\bar{\partial}+\alpha\mu)$$
$$=(\lambda-\mu)F(\bar{\partial},\lambda+\mu)+t(\lambda^2-\mu^2)(d\lambda\mu+c\lambda+c\mu)(\bar{\partial}-\frac{\alpha}{2}\lambda-\frac{\alpha}{2}\mu). \tag{5.42}$$

By the nature of (5.42), we may assume that a solution to (5.42) is a sum of homogeneous polynomials in $\bar{\partial},\lambda$ of degree 3 and degree 4, that is, $F(\bar{\partial},\lambda)=\sum_{i=1}^{5}s_i\bar{\partial}^{5-i}\lambda^{i-1}+\sum_{i=6}^{9}s_i\bar{\partial}^{9-i}\lambda^{i-6}$ for some constant $s_i, i=1,2,\cdots,9$. Taking it in (5.42), we have $\alpha=-1, F(\bar{\partial},\lambda)=-\frac{dt}{2}\bar{\partial}^3\lambda+s_3\bar{\partial}^2\lambda^2+s_4\bar{\partial}\lambda^3-\frac{1}{2}(s_3-s_4+\frac{dt}{2})\lambda^4-\frac{ct}{2}\bar{\partial}^3+s_7\bar{\partial}^2\lambda+(s_7+$

$ct)\bar\partial\lambda^2+(s_7+ct)\lambda^3$.

(10) If $a=-1,\alpha=-3,\bar\alpha=1$, set $\bar\partial=\partial+\beta, F(\bar\partial,\lambda)=f(\bar\partial-\beta,\lambda)$, and (5.30) is writen as

$$\begin{aligned}F(\bar\partial,\lambda)(\bar\partial+\lambda+\mu)+F(\bar\partial+\lambda,\mu)(\bar\partial-3\lambda)-F(\bar\partial,\mu)(\bar\partial+\lambda+\mu)-F(\bar\partial+\mu,\lambda)(\bar\partial-3\mu)\\=(\lambda-\mu)F(\bar\partial,\lambda+\mu)+t(\lambda^2-\mu^2)(c\lambda+c\mu+d\lambda\mu)(\bar\partial^2+\frac{5}{2}\bar\partial(\lambda+\mu)+\frac{3}{2}(\lambda+\mu)^2).\end{aligned}\tag{5.43}$$

By the nature of (5.43), we may assume that a solution to (5.43) is a sum of homogeneous polynomials in $\bar\partial,\lambda$ of degree 4 and degree 5, that is, $F(\bar\partial,\lambda)=\sum_{i=1}^{6}s_i\bar\partial^{6-i}\lambda^{i-1}+\sum_{i=7}^{11}s_i\bar\partial^{11-i}\lambda^{i-7}$ for some constant $s_i, i=1,2,\cdots,11$. Taking it in (5.43), we have $ct=dt=0$, a contradiction.

(11) If $a=-4,\alpha-\bar\alpha=-5$, set $\bar\partial=\partial+\beta, F(\bar\partial,\lambda)=f(\bar\partial-\beta,\lambda)$, and (5.30) is writen as

$$\begin{aligned}F(\bar\partial,\lambda)(\bar\partial+\lambda+5\mu+\alpha\mu)+F(\bar\partial+\lambda,\mu)(\bar\partial+\alpha\lambda)-F(\bar\partial,\mu)(\bar\partial+5\lambda+\mu+\alpha\lambda)\\-F(\bar\partial+\mu,\lambda)(\bar\partial+\alpha\mu)=(\lambda-\mu)F(\bar\partial,\lambda+\mu)-ct\lambda^3\mu^3(\lambda-\mu).\end{aligned}\tag{5.44}$$

By the nature of (5.44), we may assume that a solution to (5.44) is a sum of homogeneous polynomials in $\bar\partial,\lambda$ of degree 6, that is, $F(\bar\partial,\lambda)=s_1\bar\partial^6+s_2\bar\partial^5\lambda+s_3\bar\partial^4\lambda^2+s_4\bar\partial^3\lambda^3+s_5\bar\partial^2\lambda^4+s_6\bar\partial\lambda^5+s_7\lambda^6$ for some constant $s_1,\cdots,s_7$. Taking it in (5.44), we have $\alpha=-2, F(\bar\partial,\lambda)=\frac{3}{8}ct\bar\partial^4\lambda^2+s_4\bar\partial^3\lambda^3+\frac{3}{2}(s_4-ct)\bar\partial^2\lambda^4+\frac{3}{40}(12s_4-19ct)\bar\partial\lambda^5+\frac{1}{5}(s_4-2ct)\lambda^6$ and $\alpha\notin\{0,-2,-4\}, F(\bar\partial,\lambda)=s_3\bar\partial^4\lambda^2+(\frac{2(\alpha+1)}{\alpha+2}s_3-\frac{3}{\alpha(\alpha+2)(\alpha+4)}ct)\bar\partial^3\lambda^3+(\frac{2\alpha+1}{\alpha+2}s_3+\frac{9(\alpha+1)}{2\alpha(\alpha+2)(\alpha+4)}ct)\bar\partial^2\lambda^4+(\frac{5\alpha+1}{5(\alpha+2)}s_3-\frac{9(\alpha+1)(2\alpha+1)}{10\alpha(\alpha+2)(\alpha+4)}ct)\bar\partial\lambda^5+(\frac{\alpha}{5(\alpha+2)}s_3+\frac{(\alpha+1)(2\alpha+1)}{10(\alpha+2)(\alpha+4)}ct)\lambda^6$.

(12) If $a=-4,\alpha-\bar\alpha=-6$, set $\bar\partial=\partial+\beta, F(\bar\partial,\lambda)=f(\bar\partial-\beta,\lambda)$, and (5.30) is writen as

$$\begin{aligned}F(\bar\partial,\lambda)(\bar\partial+\lambda+6\mu+\alpha\mu)+F(\bar\partial+\lambda,\mu)(\bar\partial+\alpha\lambda)-F(\bar\partial,\mu)(\bar\partial+6\lambda+\mu+\alpha\lambda)\\-F(\bar\partial+\mu,\lambda)(\bar\partial+\alpha\mu)=(\lambda-\mu)F(\bar\partial,\lambda+\mu)-ct\lambda^3\mu^3(\lambda-\mu)(\bar\partial-\frac{\alpha}{5}\lambda-\frac{\alpha}{5}\mu).\end{aligned}\tag{5.45}$$

By the nature of (5.45), we may assume that a solution to (5.45) is a sum of homogeneous polynomials in $\bar\partial,\lambda$ of degree 7, that is, $F(\bar\partial,\lambda)=\sum_{i=1}^{8}s_i\bar\partial^{8-i}\lambda^{i-1}$ for some constant $s_1, \cdots, s_8$. Taking it in (5.45), we have $\alpha = -\frac{5}{2}, F(\bar{\partial}, \lambda) = \frac{36}{665}ct\bar{\partial}^5\lambda^2 + s_4\bar{\partial}^4\lambda^3 + (2s_4 - \frac{54}{113}ct)\bar{\partial}^3\lambda^4 + (\frac{9}{5}s_4 - \frac{387}{665}ct)\bar{\partial}^2\lambda^5 + (\frac{4}{5}s_4 - \frac{218}{665}ct)\bar{\partial}\lambda^6 + (\frac{1}{7}s_4 + \frac{127}{1862}ct)\lambda^7$ and $\alpha \notin \{-\frac{5}{2}, -\frac{5}{2} \pm \frac{\sqrt{19}}{2}\}, F(\bar{\partial}, \lambda) = s_3\bar{\partial}^5\lambda^2 + (\frac{5(3\alpha+4)}{3(2\alpha+5)}s_3 - \frac{3}{(2\alpha+5)(2\alpha^2+10\alpha+3)}ct)\bar{\partial}^4\lambda^3 + (\frac{5(4\alpha+3)}{3(2\alpha+5)}s_3 + \frac{3(2\alpha+3)}{(2\alpha+5)(2\alpha^2+10\alpha+3)}ct)\bar{\partial}^3\lambda^4 + (\frac{5\alpha+2}{2\alpha+5}s_3 - \frac{9(\alpha+1)(2\alpha+3)}{5(2\alpha+5)(2\alpha^2+10\alpha+3)}ct)\bar{\partial}^2\lambda^5 + (\frac{6\alpha+1}{3(2\alpha+5)}s_3 + \frac{(\alpha+1)(2\alpha+1)(2\alpha+3)}{5(2\alpha+5)(2\alpha^2+10\alpha+3)}ct)\bar{\partial}\lambda^6 + (\frac{\alpha}{3(2\alpha+5)}s_3 - \frac{\alpha(\alpha+1)(2\alpha+1)(2\alpha+3)}{70(2\alpha+5)(2\alpha^2+10\alpha+3)}ct)\lambda^7$.

(13) If $a = -4, \alpha = -6, \bar{\alpha} = 1$, set $\bar{\partial} = \partial + \beta, F(\bar{\partial}, \lambda) = f(\bar{\partial} - \beta, \lambda)$, and (5.30) is writen as

$$\begin{aligned} &F(\bar{\partial}, \lambda)(\bar{\partial} + \lambda + \mu) + F(\bar{\partial} + \lambda, \mu)(\bar{\partial} - 6\lambda) - F(\bar{\partial}, \mu)(\bar{\partial} + \lambda + \mu) - F(\bar{\partial} + \mu, \lambda)(\bar{\partial} - 6\mu) \\ &\quad = (\lambda - \mu)F(\bar{\partial}, \lambda + \mu) - ct\lambda^3\mu^3(\lambda - \mu)(\bar{\partial}^2 + \frac{11}{5}\bar{\partial}(\lambda + \mu) + \frac{6}{5}(\lambda + \mu)^2). \end{aligned} \tag{5.46}$$

By the nature of (5.46), we may assume that a solution to (5.46) is a sum of homogeneous polynomials in $\bar{\partial}, \lambda$ of degree 8, that is, $F(\bar{\partial}, \lambda) = \sum_{i=1}^{9} s_i\bar{\partial}^{9-i}\lambda^{i-1}$ for some constant $s_i, i = 1, 2, \cdots, 9$. Taking it in (5.46), we have $F(\bar{\partial}, \lambda) = s_3\bar{\partial}^6\lambda^2 + (\frac{13}{3}s_3 + \frac{1}{35}ct)\bar{\partial}^5\lambda^3 + (\frac{25}{3}s_3 + \frac{2}{7}ct)\bar{\partial}^4\lambda^4 + (9s_3 + \frac{36}{35}ct)\bar{\partial}^3\lambda^5 + (\frac{17}{3}s_3 + \frac{12}{7}ct)\bar{\partial}^2\lambda^6 + (\frac{41}{21}s_3 + \frac{66}{49}ct)\bar{\partial}\lambda^7 + (\frac{2}{7}s_3 + \frac{99}{245}ct)\lambda^8$.

(14) If $a = -6, \alpha - \bar{\alpha} = -7$, set $\bar{\partial} = \partial + \beta, F(\bar{\partial}, \lambda) = f(\bar{\partial} - \beta, \lambda)$, and (5.30) is writen as

$$\begin{aligned} &F(\bar{\partial}, \lambda)(\bar{\partial} + \lambda + 7\mu + \alpha\mu) + F(\bar{\partial} + \lambda, \mu)(\bar{\partial} + \alpha\lambda) - F(\bar{\partial}, \mu)(\bar{\partial} + 7\lambda + \mu + \alpha\lambda) \\ &\quad - F(\bar{\partial} + \mu, \lambda)(\bar{\partial} + \alpha\mu) = (\lambda - \mu)F(\bar{\partial}, \lambda + \mu) + ct(\lambda - \mu)(11\lambda^4\mu^4 - 2\lambda^3\mu^3(\lambda + \mu)^2). \end{aligned} \tag{5.47}$$

By the nature of (5.47), we may assume that a solution to (5.47) is a sum of homogeneous polynomials in $\bar{\partial}, \lambda$ of degree 8, that is, $F(\bar{\partial}, \lambda) = \sum_{i=1}^{9} s_i\bar{\partial}^{9-i}\lambda^{i-1}$ for some constant $s_i, i = 1, 2, \cdots, 9$. Taking it in (5.47), we have $\alpha = -3 \pm \frac{\sqrt{22}}{2}, F(\bar{\partial}, \lambda) = s_3\bar{\partial}^6\lambda^2 + (\frac{3\alpha+5}{\alpha+3}s_3 - \frac{40}{7(\alpha+3)}ct)\bar{\partial}^5\lambda^3 + (\frac{5(\alpha+1)}{\alpha+3}s_3 + \frac{100(\alpha+2)}{7(\alpha+3)}ct)\bar{\partial}^4\lambda^4 + (\frac{5\alpha+3}{\alpha+3}s_3 + \frac{40(5\alpha+1)}{7(\alpha+3)}ct)\bar{\partial}^3\lambda^5 + (\frac{3\alpha+1}{\alpha+3}s_3 + \frac{20(16\alpha+11)}{7(\alpha+3)}ct)\bar{\partial}^2\lambda^6 + (\frac{7\alpha+1}{7(\alpha+3)}s_3 + \frac{10(154\alpha+101)}{49(\alpha+3)}ct)\bar{\partial}\lambda^7 + (\frac{\alpha}{7(\alpha+3)}s_3 + \frac{823\alpha+539}{98(\alpha+3)}ct)\lambda^8$.

(15) If $a = -6, \alpha - \bar{\alpha} = -8$, set $\bar{\partial} = \partial + \beta, F(\bar{\partial}, \lambda) = f(\bar{\partial} - \beta, \lambda)$, and (5.30) is writen as

$$\begin{aligned} &F(\bar{\partial}, \lambda)(\bar{\partial} + \lambda + 8\mu + \alpha\mu) + F(\bar{\partial} + \lambda, \mu)(\bar{\partial} + \alpha\lambda) \\ &\quad - F(\bar{\partial}, \mu)(\bar{\partial} + 8\lambda + \mu + \alpha\lambda) - F(\bar{\partial} + \mu, \lambda)(\bar{\partial} + \alpha\mu) \end{aligned}$$

$$= (\lambda-\mu)F(\bar{\partial},\lambda+\mu)+ct(\lambda-\mu)(11\lambda^4\mu^4-2\lambda^3\mu^3(\lambda+\mu)^2)(\bar{\partial}-\frac{\alpha}{7}\lambda-\frac{\alpha}{7}\mu). \tag{5.48}$$

By the nature of (5.48), we may assume that a solution to (5.48) is a sum of homogeneous polynomials in $\bar{\partial},\lambda$ of degree 9, that is, $F(\bar{\partial},\lambda)=\sum_{i=1}^{10}s_i\bar{\partial}^{10-i}\lambda^{i-1}$ for some constant $s_i, i=1,2,\cdots,10$. Taking it in (5.48), we have $\alpha=-1, F(\bar{\partial},\lambda)=s_3\bar{\partial}^7\lambda^2+(\frac{7}{5}s_3-\frac{2}{7}ct)\bar{\partial}^6\lambda^3+(\frac{7}{10}s_3+\frac{9}{7}ct)\bar{\partial}^5\lambda^4-(\frac{7}{10}s_3+\frac{9}{7}ct)\bar{\partial}^4\lambda^5-(\frac{7}{5}s_3-\frac{2}{7}ct)\bar{\partial}^3\lambda^6-s_3\bar{\partial}^2\lambda^7-\frac{7}{20}s_3\bar{\partial}\lambda^8-\frac{1}{20}s_3\lambda^9$ and $\alpha=-6, F(\bar{\partial},\lambda)=s_3\bar{\partial}^7\lambda^2+(\frac{28}{5}s_3-\frac{2}{7}ct)\bar{\partial}^6\lambda^3+(\frac{133}{10}s_3-3ct)\bar{\partial}^5\lambda^4+(\frac{91}{5}s_3-12ct)\bar{\partial}^4\lambda^5+(\frac{77}{5}s_3-24ct)\bar{\partial}^3\lambda^6+(8s_3-\frac{180}{7}ct)\bar{\partial}^2\lambda^7+(\frac{47}{20}s_3-\frac{99}{7}ct)\bar{\partial}\lambda^8+(\frac{3}{10}s_3-\frac{22}{7}ct)\lambda^9$.

(16) If $a=-6,\alpha=-8,\bar{\alpha}=1$, set $\bar{\partial}=\partial+\beta, F(\bar{\partial},\lambda)=f(\bar{\partial}-\beta,\lambda)$, and (5.30) is written as

$$\begin{aligned}&F(\bar{\partial},\lambda)(\bar{\partial}+\lambda+\mu)+F(\bar{\partial}+\lambda,\mu)(\bar{\partial}-8\lambda)-F(\bar{\partial},\mu)(\bar{\partial}+\lambda+\mu)-F(\bar{\partial}+\mu,\lambda)(\bar{\partial}-8\mu)\\&=(\lambda-\mu)F(\bar{\partial},\lambda+\mu)\\&\qquad+ct(\lambda-\mu)(11\lambda^4\mu^4-2\lambda^3\mu^3(\lambda+\mu)^2)(\bar{\partial}^2+\frac{15}{7}\bar{\partial}(\lambda+\mu)+\frac{8}{7}(\lambda+\mu)^2).\end{aligned} \tag{5.49}$$

By the nature of (5.49), we may assume that a solution to (5.49) is a sum of homogeneous polynomials in $\bar{\partial},\lambda$ of degree 10, that is, $F(\bar{\partial},\lambda)=\sum_{i=1}^{11}s_i\bar{\partial}^{11-i}\lambda^{i-1}$ for some constant $s_i, i=1,2,\cdots,11$. Taking it in (5.49), we have $ct=0$ and get a contradiction.

The final results follow from Lemma 5.2.5 and Theorem 3.2 in [7] (or Theorem 2.7 in [21]).

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