学校代码：10284
分 类 号：O154.2
密　　级：公开
U　D　C：510.2
学　　号：MG21210032

南京大學

硕 士 学 位 论 文

论 文 题 目	A generalized Hopf algebra structure constructed from arithmetic functions
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专 业 名 称	基础数学
研 究 方 向	Hopf 代数
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2024 年 5 月 9 日

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论文答辩日期　2024 年 5 月 22 日

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A generalized Hopf algebra structure constructed from arithmetic functions

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A dissertation submitted to

the graduate school of Nanjing University

in partial fulfilment of the requirements for the degree of

Master

in

Pure Mathematics

Department of Mathematics

Nanjing University

May 09, 2024

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毕业论文题目：A generalized Hopf algebra structure constructed from arithmetic functions

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摘 要

我们想将数论中重要的函数——完全乘性的数论函数一一对应到 Hopf 代数的元素。在发现 Hopf 代数中的群似元只能是完全乘性的数论函数的子集后，推广了 Hopf 代数的定义，主要是将其中张量积推广到了完全张量积 (complete tensor)，使其可以与完全乘性的数论函数一一对应。这样就可以用 Hopf 代数研究数论函数中一系列重要的对象，比如 L-函数等。

关键词：Hopf algebra；Arithmetic function；Completed tensor

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THESIS: A generalized Hopf algebra structure constructed from arithmetic functions

SPECIALIZATION: Pure Mathematics

POSTGRADUATE: Xu Wang

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ABSTRACT

We want to give a bijection between the completely multiplicative arithmetic functions and some elements of hopf algebras one by one. After finding that the group-like elements in Hopf algebra only can be a subset of the completely multiplicative arithmetic functions, we generalize the definition of Hopf algebra, mainly by extending the tensor product to the complete tensor, so that it can include all completely multiplicative arithmetic funtions. In this way, Hopf algebra can be used to study a series of important objects in number theory, such as L-functions.

KEYWORDS: Hopf algebra; Arithmetic function; Completed tensor

CONTENTS

Chapter 1 Introduction

In number theory, there are important functions, complete multiplicative arithmetic functions. It is algebra maps from $\mathbb{N}$ to $\mathbb{C}$ where the symbols $\mathbb{N}$, $\mathbb{C}$ denote the natural numbers $\{1, 2, 3, \cdots\}$ and complex numbers respectively. In the set of completely multiplicative arithmetic functions, there is an essential multiplication, Dirichlet multiplication[1]. Recalling the known knowledge of Hopf algebra[2], we guess that some set on $\mathbb{N}$ has a Hopf algebra structure.

We prove that a certain set V based on $\mathbb{N}$ does form a Hopf algebra according to the given comultiplication. It naturally occurs to us that the finite dual of this Hopf algebra must be a Hopf algebra[3]. We give a bijection between $Alg(V, \mathbb{C})$ and the completely multiplicative arithmetic function. Because we know the fact that a group-like element of a Hopf algebra is an algbra map, a group-like element of finite dual is an algebra map, which vanishes at infinite points.

So it is vital to find a larger set that can include all algebra maps. We think making infinite points non-zero is the key. We change the tensor product in a Hopf algebra to a complete tensor product[4], so that algebra maps are not necessary to be zero at infinite points. Similarly, we verify that all generalized group-like elements in the so-called generalized Hopf algebra is indeed $Alg(V, \mathbb{C})$.

Conclusively, we get a bijection between generalized group-like elements and complete multiplicative arithmetic functions. That means we can use Hopf algebra tools to study a series of important objects in number theory, such as L-functions.

Chapter 2 Preliminary

2.1 Some notations from Number theory

Here we recall some known notations and results from number theory for our convenience.

Definition 2.1 *An arithmetic function is any function* $f : \mathbb{N} \to \mathbb{C}$.

Definition 2.2 *An arithmetic function with* $f(1) \neq 0$ *and*

$$f(mn) = f(m)f(n),$$

whenever m and are coprime is called multiplicative. (Note that implies $f(1) = 1$.*)*

If f has this property not only for coprime m, n, *but for all* $m, n \in N$, *then f is called completely multiplicative arithmetic function.*

Definition 2.3 *The Dirichlet convolution product of arithmetic functions f and g is the function* $f * g$ *defined by*

$$f * g = \sum_{d|n} f(d)g(\frac{n}{d}).$$

Proposition 2.1 *The Dirichlet convolution product is commutative and associative. In other words,*

$$f * g = g * f \qquad and \qquad (f * g) * h = f * (g * h),$$

for any arithmetic functions f, g, and h.

Proof The sum in

$$\sum_{d|n} f(d)g(\frac{n}{d}),$$

runs over all pairs $d, e \in N$ with $de = n$, so it is equal to

$$\sum_{de=n} f(d)g(e),$$

and the latter expression is symmetric in f and g. To see that convolution is associative, notice that

$$((f * g) * h)(n) = \sum_{cde=n} f(c)g(d)h(e)(f * (g * h))(n), \forall n \in N.$$ ■

Definition 2.4 *Define the arithmetic function I by* $I(1) = 1$ *and* $I(n) = 0, \forall n > 1$.

Proposition 2.2 *For any arithmetic function f*

$$f * I = I * f = f.$$

Proof $(f * I)(n) = \sum_{d|n} f(d)I(\frac{n}{d}) = f(n)I(1) = f(n)$, since all the other summands are zero by the definition of I. ■

Proposition 2.3 *If f is an arithmetic function with* $f(1) \neq 0$*, then there is a unique arithmetic function g such that* $f * g = I$*. This function is denoted by* f^{-1}.

Proof The equation $f * g(1) = f(1)g(1)$ determines g(1). Then define g recursively as follows. Assuming that $g(1), \cdots, g(n-1)$ have been defined uniquely, the equation

$$(f * g)(n) = f(1)g(n) + \sum_{\substack{d|n \\ d>1}} f(d)g(\frac{n}{d}),$$

allows us to calculate $g(n)$ uniquely. ■

Definition 2.5 *Let G be a finite Abelian group. A character of G is a homomorphism*

$$\chi : G \to (\mathbb{C}^*, \cdot)$$

The multiplicative group C^* is $\mathbb{C}\backslash\{0\}$ equipped with the usual multiplication. For any group, the map

$$\chi_0 : G \to \mathbb{C}^*, \chi_0(g) = 1,$$

is a character by above, and we say it is the trival character.

Proposition 2.4 *Let G be a finite Abelian group. Then the characters of G form a group with respect to the multiplication*

$$(\chi \cdot \phi)(g) = \chi(g)\phi(g),$$

denoted $\widehat{G}$. The identity in $\widehat{G}$ is the trival character. The group is isomorphic to G.

In particular, any finite Abelian group G of order n has exactly n distinct character.

Definition 2.6 *Given $1 < q \in \mathbb{N}$, let $G = U(\mathbb{Z}/q\mathbb{Z})$ and fix a character χ in $\widehat{G}$. Extend χ to a function X on $\mathbb{N}$ by*

$$f(x) = \begin{cases} \chi(n), & n \in U(\mathbb{Z}/q\mathbb{Z}), \\ 0, & otherwise. \end{cases}$$

The function X is called a Dirchlet character modulo q.

Theorem 2.1 *A Dirichlet character is a completely multiplicative arithmetic function.*

Proof Let χ be a Dirchlet character modulo q. If two integers m, n are given, and at least one of them is not coprime to q, then neither is the product mn. Thus $\chi(mn) = \chi(m)\chi(n)$, otherwise, m and n both coprime to q, then

$$(m\, mod\, q) \cdot (n\, mod\, q) = (mn\, mod\, q).$$

χ is group character, that means χ is a group homomorphism, then $\chi(mn) = \chi(m)\chi(n)$.■

Definition 2.7 *Let χ be a Dirichlet character. Define a complex function by*

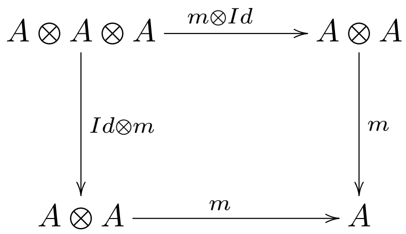
$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where $s \in \mathbb{C}$. Such functions are called L-funtions.

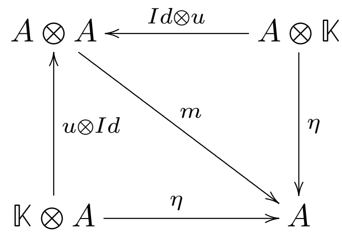
2.2 Some definitions of Hopf algebra

Definition 2.8 *Let $\mathbb{K}$ be a field. A $\mathbb{K}$-algebra (with unit) is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps, multiplication m: A⊗ A→A, and unit u: $\mathbb{K}$→A, such that the following diagrams are commutative:*

a) associativity



b) unit



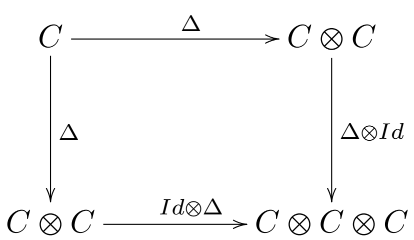
the Id denotes the identity mapping, and the η denotes the scalar multiplication.

Definition 2.9 *For any $\mathbb{K}$-space V and W, the twist map $\tau : V \otimes W \to W \otimes V$, is given by $\tau(v \otimes w) = w \otimes v$.*

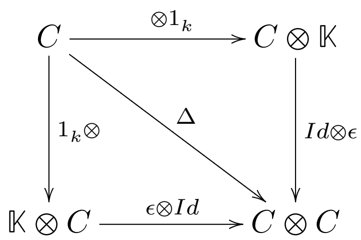
Obviously, A is commutative $\iff$ $m \circ \tau = m$ on $A \otimes A$.

Definition 2.10 *A $\mathbb{K}$-coalgebra (with counit) is a $\mathbb{K}$-vector space C together with two $\mathbb{K}$-linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to \mathbb{K}$, such that the following diagrams are commutative:*

a) coassociativity



b) counit



the two upper maps in b) are given by $c \mapsto 1 \otimes c$, and $c \mapsto c \otimes 1$, for any $c \in C$.

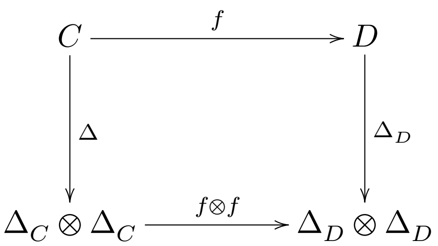
We say C is cocommutataive if $\tau \circ \Delta = \Delta$.

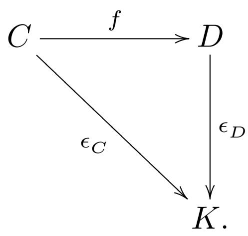
Definition 2.11 *Let C be any coalgebra,and $c \in C$.*

a) c is called group-like if $\Delta c = c \otimes c$ and $\epsilon(c) = 1$. The set of group-like elements in C is denoted by $G(C)$.

b) For $g, h \in G(C)$, c is called g,h-primitive if $\Delta c = c \otimes g + h \otimes c$. The set of all g,h-primitive elements is denoted by $P_{g,h}(C)$. $P_{1,1}(C)$ are simply called the primitive elements of C, denoted by P(C).

Definition 2.12 *Let C and D be coalgebras, with comultiplication Δ_C and Δ_D, and counits ϵ_C and ϵ_D, respectively. A coalgebra map $f : C \to D$ is a linear map, such that $\Delta_D \circ f = (f \otimes f)\Delta_c$ and $\epsilon_C = \epsilon_D \circ f$, that means the following diagrams are commutative*





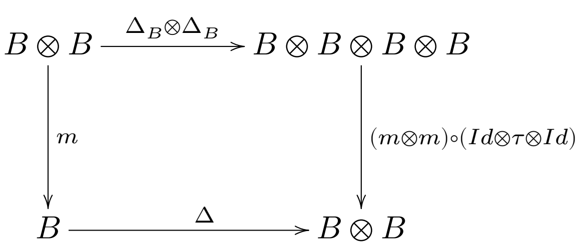
Similarly, we have the definition of algebra map.

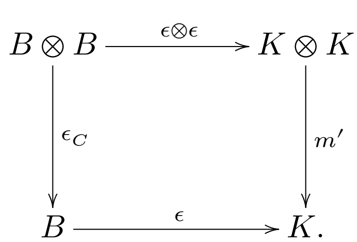
Definition 2.13 *A $\mathbb{K}$-space B is a bialgebra if (B, m, u) is an algebra, (B, Δ, ϵ) is a coalgebra, and either of the following (equivalent) conditions holds:*

a) Δ and ϵ are algebra morphisms

b) m and u are coalgebra morphisms.

For example, noticing the multiplication of $B \otimes B$, we get following commutative diagrams from a)





Theorem 2.2 *Let C be a coalgebra and A an algebra. Then* $Hom_{\mathbb{K}}(C, A)$ *becomes an algebra under the convolution product* $f * g(c) = m \circ (f \otimes g)(\Delta c), \forall f, g \in Hom_{\mathbb{K}}(C, A), c \in C$. *The unit element in* $Hom_{\mathbb{K}}(C, A)$ *is* $u\epsilon$.

Let C be any coalgebra with comultiplication $\Delta : C \to C \otimes C$. The sigma notation for Δ is given as follows: for any $c \in C$, we write

$$\Delta c = c_1 \otimes c_2.$$

The subscripts 1 and 2 are symbolic, and do not indicate particular elements of C, this notation is analogous to notation used in physics. In these notes we usually simplify the notation by omitting parentheses. In particular, the coassociativity diagram gives that

$$c_1 \otimes c_{2_1} \otimes c_{2_2} = c_{1_1} \otimes c_{1_2} \otimes c_2,$$

this element is written as $c_1 \otimes c_2 \otimes c_3 = \Delta_2(c)$.

Definition 2.14 *Let* $(H, m, u, \Delta, \epsilon)$ *be a bialgebra. Then* H *is a Hopf algebra if there exists an element* $S \in Hom_K(H, H)$*, which is an inverse to* Id_H *under the convolution* $*$. *S is called an antipode for H. Note that in sigma notation, S satisfies*

$$\sum (Sh_1)h_2 = \epsilon(h)1_H = \sum h_1(Sh_2), \forall h \in H.$$

Definition 2.15 *We also have the obvious definitions of morphisms and ideals: a map* $f : H \to K$ *of Hopf algebras is a Hopf morphism, if it is a bialgebra morphism and* $f(S_H h) = S_K f(h), \forall h \in H$. *A subspace I of H is a Hopf ideal if it is a biideal and* $S(I) \subset I$*; in this situation* H/I *is a Hopf algebra with structure induced from H.*

Proposition 2.5 *Let H be a Hopf algebra with antipode S.*

a) S is an anti-algebra morphism; that is

$$S(hk) = S(k)S(h), \forall h, k \in H, \text{ and } S(1) = 1.$$

b) S is an anti-coalgebra morphism; that is

$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta, \; and \; \epsilon \circ S = \epsilon.$$

By sigma notation, b) means $\sum (Sh)_1 \otimes (Sh)_2 = \sum S(h_2) \otimes S(h_1)$.

2.3 Filtered module

Definition 2.16 *Let k be a ring. A filtered module (A,F) is a k-module A equipped with a filtration*

$$A = F_0A \supset F_1A \supset F_2A \supset \cdots \supset F_kA \supset F_{k+1}A \supset \cdots$$

made up of submodules.

There are two examples: let I be an ideal of the ring k and M be a k-module. Then the submodules $F_nM := I^nM$ form a filtration of M; let $F_n(T) = 0, \forall n \geqslant 1$, then any k-module T can be a filtered module. We name the filtration by trivial filtration.

Definition 2.17 *A filtered map $f : (A, F) \to (B, G)$ between two filtered module is an element of $Hom_k(A, B)$, that is a linear map preserving the respective filtration: $f(F_nA) \subset G_nB$, for any $n \in N = \{0, 1, \cdots\}$.*

The induced filtration on the tensor product of two filtered modules A and B is given by

$$F_t(A \otimes B) = \sum_{n+m=t} Im(F_nA \otimes G_mB \to A \otimes B)$$

For any decreasing filtration $A = F_0A \supset \cdots$ induces a sequence of surjective maps:

$$0 = A/F_0A \overset{p_0}{\leftarrow} A/F_1A \overset{p_1}{\leftarrow} A/F_2A \overset{p_2}{\leftarrow} \cdots,$$

where p_k is the reduction modulo F_kA. Its limit,denoted by:

$$\hat{A} := \lim_{k \in N} A/F_kA,$$

is made up of elements of the following form:

$$\hat{A} = \{(x_0, x_1, \cdots)|x_k \in A/F_kA, \quad p_k(x_{k+1}) = x_k\}.$$

If we denote the structure maps by:

$$q_k : \hat{A} \longrightarrow A/F_kA$$
$$(x_0, x_1, \cdots) \mapsto x_k$$

then the limit module $\hat{A}$ is endowed with the following canonical filtration:

$$\hat{F}_k\hat{A} := ker\, q_k = (0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots)$$

.

Let $\pi_n : A \to A/F_nA$ be the canonical projections. The canonical map $\pi : A \to \hat{A}, x \mapsto (\pi_0(x), \pi_1(x), \cdots)$, associated to them, is filtered.

Definition 2.18 *A complete module is a filtered module (A,F) such that the canonical morphism*

$$\pi : A \to \hat{A} = \lim_{n \in N} A/F_nA$$

is an isomorphism.

Definition 2.19 *The complete tensor product of two complete modules* (A, F) *and* (B, G) *is defined by the completion of their filtered tensor product*

$$A\hat{\otimes}B := \widehat{A \otimes B}.$$

In addition, we wil show that when the two filtered modules are not necessarily complete, the completion of their tensor product $\widehat{A \otimes B}$ is equal to $\hat{A}\hat{\otimes}\hat{B}$ by the Theorem 4.2.

Chapter 3 Hopf structure of V

3.1 M- coefficient

We find a function is essential through explainning the Hopf algebra structure of some set.

Definition 3.1 *Define a function M*

$$M : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$(i, j) \mapsto \begin{cases} 1 & i = 1 \text{ or } j = 1 \\ \prod_{i=1}^{s} \binom{a_i + b_i}{a_i} & i \geqslant 2, j \geqslant 2, \end{cases}$$

where $i = p_1^{a_1} \cdots p_s^{a_s}, j = p_1^{b_1} \cdots p_s^{b_s}, i \geqslant 2, j \geqslant 2.$

The image of (i, j) is called the M-coefficient of (i, j).

Lemma 3.1 *M is a symmetric binary function.*

Proof Just note that $\binom{a_i + b_i}{a_i} = \binom{b_i + a_i}{a_i}$. ■

Lemma 3.2 *The function M has the following equation:*

$$M(i, j) = \sum_{\substack{d|i, n|j \\ dn = m}} M(d, n) M(\frac{i}{d}, \frac{j}{n}), \forall m \in \mathbb{N}, m|ij.$$

Proof Let $i = p_1^{a_1} \cdots p_s^{a_s}, j = p_1^{b_1} \cdots p_s^{b_s}, i \geqslant 2, j \geqslant 2$. For m is a divisor of ij, we can write $m = p_1^{c_1} \cdots p_s^{c_s}$,$0 \leqslant c_i \leqslant a_i + b_i$, then $d = p_1^{g_1} \cdots p_s^{g_s}, n = p_1^{h_1} \cdots p_s^{h_s}$. We have

$$\sum_{\substack{d|i, n|j \\ dn = m}} M(d, n) M(\frac{i}{d}, \frac{j}{n})$$

$$= \sum_{g_1+h_1=c_1} \cdots \sum_{g_s+h_s=c_s} \binom{g_1+h_1}{g_1} \cdots \binom{g_s+h_s}{g_s} \binom{a_1-g_1+b_1-h_1}{a_1-g_1} \cdots \binom{a_s-g_s+b_s-h_s}{a_s-g_s}$$

$$= \sum_{max\{0,c_t-b_t\} \leqslant g_t \leqslant min\{c_t,a_t\}} \binom{g_1+h_1}{g_1} \cdots \binom{a_1-g_1+b_1-h_1}{a_1-g_1} \cdots \binom{a_s-g_s+b_s-h_s}{a_s-g_s}$$

$$= \sum_{max\{0,c_1-b_1\} \leqslant g_1 \leqslant min\{c_1,a_1\}} \cdots \sum_{max\{0,c_s-b_s\} \leqslant g_s \leqslant min\{c_s,a_s\}} \prod_{k=1}^{s} \binom{c_k}{g_k} \binom{a_k+b_k-c_k}{a_k-g_k}$$

$$= \prod_{k=1}^{s} \left(\sum_{max\{0,c_k-b_k\} \leqslant g_k \leqslant min\{c_k,a_k\}} \binom{c_k}{g_k} \binom{a_k+b_k-c_k}{a_k-g_k} \right)$$

$$= \prod_{k=1}^{s} \binom{a_k+b_k}{a_k}$$

For $i = 1$ or $j = 1$, since $M(i, j)$ is a symmetric binary function, we only claim $M(1, j)$ holds

$$M(1, j) = \sum_{\substack{d|1,n|j \\ dn=m}} M(d, n) M(\frac{1}{d}, \frac{j}{n}) = M(1, m) M(1, \frac{j}{m}). \qquad \blacksquare$$

In that proof, we notice that

a) From

$$g_i + h_i = c_i$$
$$0 \leqslant c_i \leqslant a_i + b_i,$$

we have $max\{0, c_i - b_i\} \leqslant g_i \leqslant min\{c_i, a_i\}$.

b) From $(x + 1)^{c_k}(x + 1)^{a_k+b_k-c_k} = (x + 1)^{a_k+b_k}$, comparing x^{a_k} coefficients, we have

$$\sum_{max\{0,c_k-b_k\} \leqslant g_k \leqslant min\{c_k,a_k\}} \binom{c_k}{g_k} \binom{a_k+b_k-c_k}{a_k-g_k} = \binom{a_k+b_k}{a_k},$$

then

$$0 \leqslant g_k \leqslant c_k$$
$$0 \leqslant g_k{}' \leqslant a_k + b_k - c_k$$
$$g_k + g_k{}' = a_k.$$

Lemma 3.3 $M(i,j)M(ij,k) = M(i,jk)M(j,k), \forall i,j,k \in \mathbb{N}.$

Proof For some special cases

$$\begin{aligned} i = 1, \quad & M(1,j)M(j,k) = M(1,jk)M(j,k) \\ j = 1, \quad & M(i,1)M(i,k) = M(i,k)M(1,k) \\ k = 1, \quad & M(i,j)M(ij,1) = M(1,j)M(j,1). \end{aligned}$$

In another case，assume

$$i = p_1^{a_1} \cdots p_s^{a_s}, j = p_1^{b_1} \cdots p_s^{b_s}, k = p_1^{c_1} \cdots p_s^{c_s}, i,j,k \geqslant 2,$$

then

$$M(i,j) = \prod_{k=1}^{s} \binom{a_k + b_k}{a_k}, \qquad M(ij,k) = \prod_{k=1}^{s} \binom{a_k + b_k + c_k}{a_k + b_k}$$
$$M(i,jk) = \prod_{k=1}^{s} \binom{a_k + b_k + c_k}{a_k}, \qquad M(j,k) = \prod_{k=1}^{s} \binom{b_k + c_k}{b_k}.$$

Notice that

$$\begin{aligned} M(i,j)M(ij,k) = & \binom{a_k + b_k}{a_k} \binom{a_k + b_k + c_k}{a_k + b_k} \\ = & \frac{(a_k + b_k)!}{a_k! b_k!} \frac{(a_k + b_k + c_k)!}{(a_k + b_k)! c_k!} \end{aligned}$$

$$
\begin{aligned}
&=\frac{(a_k+b_k+c_k)!}{a_k!b_k!c_k!}\\
&=\frac{(b_k+c_k)!}{b_k!c_k!}\frac{(a_k+b_k+c_k)!}{(b_k+c_k)!a_k!}\\
&=\binom{b_k+c_k}{b_k}\binom{a_k+b_k+c_k}{a_k}\\
&=M(i,jk)M(j,k).
\end{aligned}
$$

So the lemma holds. ■

3.2 V is a Hopf algebra

Theorem 3.1 *Let V be a vector space on complex field, spanned by $\{e_n|n\in\mathbb{N}\}$, and define $m(e_i\otimes e_j)=M(i,j)e_{ij}$, $u(k)=ke_1$,*

$$
\Delta(e_n)=\sum_{d|n}e_d\otimes e_{\frac{n}{d}},\epsilon(e_n)=\begin{cases}1,\ n=1\\0,\ n\neq 1.\end{cases}
$$

Then V is a Hopf algebra under a given antipode S.

Proof Now we want to prove the theorem by the definition of Hopf algebra. We claim V is an algebra, a coalgebra and give the definition of the antipode.

Firstly, V is an algebra, that can be checked by following steps. Notice the diagram of Definition 2.7 $a)$ $associativity$, and recall the lemma of M- coefficient, then

$$
\begin{aligned}
m(m\otimes Id)(e_i\otimes e_j\otimes e_k)&=m(M(i,j)e_{ij}\otimes e_k)\\
&=M(i,j)m(e_{ij}\otimes e_k)\\
&=M(i,j)M(ij,k)e_{ijk}\\
&=M(i,jk)M(j,k)e_{ijk}\\
&=M(j,k)m(e_i\otimes e_{jk})
\end{aligned}
$$

$$= M(i, jk)m(e_i \otimes m(e_j \otimes e_k))$$
$$= m(Id \otimes m)(e_i \otimes e_j \otimes e_k).$$

Similarly, check $b)$ $unit$ of the same diagram. Because the commutativity of multiplication, we only prove the following equation:

$$\begin{aligned}
&m(Id \otimes u)(e_i \otimes k) \forall i \in N, k \in C \\
=&m(e_i \otimes ke_1) \\
=&km(e_i \otimes e_1) \\
=&kM(i, 1)e_i \\
=&ke_i \\
=&\eta(e_i \otimes k)
\end{aligned}$$

Next, we will show V is a coalgebra. Notice that the diagram of Definition 2.9,and check the $a)$ $coassociativity$

$$\begin{aligned}
(\Delta \otimes Id)\Delta(e_n) &= (\Delta \otimes Id)(\sum_{d|n} e_d \otimes e_{\frac{n}{d}}) \\
&= \sum_{d'|d} \sum_{d|n} e_{d'} \otimes e_{\frac{d}{d'}} \otimes e_{\frac{n}{d}} \\
&= \sum_{pqr=n} e_p \otimes e_q \otimes e_r \\
&= (Id \otimes \Delta)(\sum_{d|n} e_d \otimes e_{\frac{n}{d}}) \\
&= (Id \otimes \Delta)\Delta(e_n).
\end{aligned}$$

Then check the $b)$ $counit$. Because of the cocommutativity of V, we only prove the following equation:

$$
\begin{aligned}
(Id \otimes \epsilon)\Delta(e_n) &= (Id \otimes \epsilon)(\sum_{d|n} e_d \otimes e_{\frac{n}{d}}) \\
&= \sum_{d|n} e_d \otimes \epsilon(e_{\frac{n}{d}}) \\
&= e_n \otimes 1.
\end{aligned}
$$

Secondly we need to prove V is a bialgebra by two conditions in Definiton 2.12. It can be showed by following calculation. Δ is an algebra map,

$$
\begin{aligned}
&(m \otimes m)(Id \otimes \tau \otimes m)(\Delta \otimes \Delta)(e_i \otimes e_j) \\
=&(m \otimes m)(\sum_{d|i} \sum_{n|j} e_d \otimes e_n \otimes e_{\frac{i}{d}} \otimes e_{\frac{j}{n}}) \\
=& \sum_{\substack{d|i,n|j \\ dn=m'}} M(d,n)M(\frac{i}{d}, \frac{j}{n}) e_{dn} \otimes e_{\frac{ij}{dn}} \\
=&M(i,j) \sum_{m'|ij} e'_m \otimes e_{\frac{ij}{m'}} \\
=&\Delta m(e_i \otimes e_j).
\end{aligned}
$$

And ϵ is also an algebra map. We know $ij = 1 \iff i = 1, j = 1, i, j \in \mathbb{N}$,

$$
\begin{aligned}
&\epsilon m(e_i \otimes e_j) \\
=&M(i,j)\epsilon(e_i j) \\
=&\delta_{ij,1} \\
=&\epsilon(e_i)\epsilon(e_j) \\
=&m(\epsilon \otimes \epsilon)(e_i \otimes e_j).
\end{aligned}
$$

Finally, we claim the existence of the inverse S of Id. We give the construction of the antipode S recursively. For $(Id * S)(e_1) = u\epsilon(e_1)$, then $S(e_1) = e_1$, and

$(Id * S)(e_2) = u\epsilon(e_2)$, then $S(e_2) = -e_2, \cdots$, then $S(e_n) = -\sum_{\substack{d|n \\ d>1}} e_d S(e_{\frac{n}{d}})$.

Because V is cocommutative, the antipode exists. Now, we give the expression of S explicitly. Since any prime p only has a trivial decomposition, $S(e_p) = -e_p$. Because S is an algebra map,

$$\begin{aligned} S(e_p^r) &= M(p, p^{r-1})^{-1} S(e_p e_p^{r-1}) \forall r \in \mathbb{N} \\ &= -\frac{1}{r} e_p S(e_p^{r-1}) \\ &= (-\frac{1}{r} e_p)(-\frac{1}{r-1} e_p S(e_p^{r-2})) \\ &= (-\frac{1}{r} e_p)(-\frac{1}{r-1} e_p) \cdots (-e_p) \\ &= (-1)^r e_p^r. \end{aligned}$$

When $i \geqslant 2, i \in N$， write $i = p_1^{a_1} \cdots p_s^{a_s}$, if p, q is coprime， then $M(p, q) = 1$,

$$\begin{aligned} S(e_i) &= S(e_{p_1^{a_1} \cdots p_s^{a_s}}) \\ &= S(e_{p_1^{a_1}}) \cdots S(e_{p_s^{a_s}}) \\ &= (-1)^{a_1 + \cdots + a_s} e_i. \end{aligned}$$

If we denote 1 as zero power of some prime numbers, we have

$$S(e_i) = (-1)^{a_1 + \cdots + a_s} e_i, i = p_1^{a_1} \cdots p_s^{a_s}, i \in \mathbb{N}.$$

Now we have proved that V is a Hopf algebra by definition. ■

Naturally, we want to discuss some properties of V.

3.3 Properties of V

Proposition 3.1 *V is a domain.*

Proof Assuming v_1, v_2 are nonzero vectors of V, we claim v_1v_2 is a nonzero vector of V. Let $v_1 = \sum_{i=1}^{m} k_i e_i, k_m \neq 0$, and $v_2 = \sum_{i=1}^{n} h_i e_i, h_n \neq 0$. If $v_1 v_2 = 0$. comparing the the highest degree, we have $M(m,n)k_m h_n = 0$, so $M(m,n) = 0$. That is a contradiction, then V is a domain. ■

Proposition 3.2 *V is not a PID.*

Proof Assume $< e_2, e_3 >=< c >$, where c is a nonzero vector of V, then

$$e_2 = k_2 c, \quad k_2 \in V$$
$$e_3 = k_3 c, \quad k_3 \in V.$$

Set

$$k_2 = \sum_{j=1}^{m_2} r_i e_i, \quad r_i \in \mathbb{C}$$
$$k_3 = \sum_{j=1}^{m_3} s_i e_i, \quad s_i \in \mathbb{C}$$
$$c = \sum_{i=1}^{n} c_i e_i, \quad c_i \in \mathbb{C}.$$

For $e_2 = k_2 c$, comparing the the highest degree, we have $M(r_{m_2}, c_n) e_{m_2 n} = e_2$, then $m_2 n = 2$, so $n = 1$ or $n = 2$. When $n = 1$, we have $< c >= V$, but e_1 is not generated by e_2 and e_3; when $n = 2$, we have $m_2 = 1$, then $k_2 = r_1 e_1$. Since $e_2 = k_2 c$, then $k_2 = r_1 c$, and $e_3 = k_3 c$, then $e_3 \in < e_2 >$. It is a contradict.

Conclusively, V is not a PID. ■

Proposition 3.3 *V is a connected Hopf algebra.*

Proof Let $v = \sum_{i=1}^{n} k_i e_i \in V, v \neq 0$, then

$$\Delta(v) = \sum_{i=1}^{n} k_i \sum_{d|i} e_d \otimes e_{\frac{i}{d}}.$$

We know that any coalgebra generated by a nonzero vector must contain <e_1>， after all 1 is a divisor of any positive integer. Because <e_1> has been a simple subcoalgebra, any subcoalgebra strictly containing <e_1> is not simple, as the subcoalgebra contains three distinct subcoalgebras at least.

That implies <e_1> is the unique simple subcoalgebra of V. By definiton, V is pointed. ■

Group-like elements and primitive elements of V

We give all the group-like elements and primitive elements of V.

a) group-like elements

Let $c = \sum_{i=1}^{n} k_i e_i, c \in V, k_n \neq 0$, then

$$\begin{aligned} \Delta(c) &= \sum_{i=1}^{n} k_i \Delta(e_i) \\ &= k_i \sum_{d|i} e_d \otimes e_{\frac{i}{d}} \end{aligned}$$

$$c \otimes c = (\sum_{i=1}^{n} k_i e_i) \otimes (\sum_{i=1}^{n} k_i e_i).$$

Because $\Delta(c) = c \otimes c$, comparing coefficents of both sides, we have $c = k_1 e_1$. Then e_1 is the unique group-like element of V.

b) primitive elements

Let c be a g, h-primitive element, where g, h are group-like elements of V. By the definition ,we have $g = h = e_1$, so $\Delta(c) = c \otimes e_1 + e_1 \otimes c$. Let $c = \sum_{i=1}^{n} k_i e_i, c \in V, k_n \neq 0$. From the definition

$$\sum_{i=1}^{n} k_i \Delta(e_i) = \sum_{i=1}^{n} k_i \Delta(e_i \otimes e_1) + \sum_{i=1}^{n} k_i \Delta(e_1 \otimes e_i).$$

Moreover,

$$\sum_{i=1}^{n} k_i \sum_{d|i} e_d \otimes e_{\frac{i}{d}} = \sum_{i=1}^{n} k_i \Delta(e_i \otimes e_1) + \sum_{i=1}^{n} k_i \Delta(e_1 \otimes e_i).$$

Noticing the right side of the equation is the trivial decomposition of i, we have the fact that the left side of the equation is also a trivial decomposition. That means i is a prime number. Trivial calculation show $i = 1$ does not holds.

So primitive elements of the form $\sum_{p \in P} k_p e_p$, where P is the set of all prime numbers.

Chapter 4 Dual of V

4.1 Bijection between algebra maps and completely multiplicative arithmetic functions

There is a bijection between algebra maps of $V \to \mathbb{C}$ and completely multiplicative arithmetic functions. Given an algebra morphism f, such that $f(e_i e_j) = f(e_{ij})$, we define

$$\bar{f}(n) = f(e_{p_1})^{t_1} \cdots f(e_{p_r})^{t_r}, \quad n = p_1^{t_1} \cdots p_r^{t_r}.$$

Then $\bar{f}$ is a completely multipicative arithmetic function, coinciding with f at every points of prime numbers.

Conversely, if F is a completely multiplicative arithmetic function, which means $F(mn) = F(m)F(n)$. We define

$$f(e_n) = \frac{1}{(t_1)! \cdots (t_r)!} F(p_1)^{t_1} \cdots F(p_r)^{t_r}, \quad n = (p_1)^{t_1} \cdots (p_r)^{t_r},$$

By the following calculation we can show f is an algebra morphism from V to $\mathbb{C}$. For

$$n = (p_1)^{t_1} \cdots (p_r)^{t_r},$$
$$m = (p_1)^{s_1} \cdots (p_r)^{s_r},$$

we have

$$f(e_n e_m) = M(n,m) f(e_{nm})$$
$$= \binom{t_1+s_1}{t_1} \cdots \binom{t_r+s_r}{t_r} \frac{1}{(t_1+s_1)! \cdots (t_r+s_r)!} F(p_1)^{t_1+s_1} \cdots F(p_r)^{t_r+s_r}$$

$$= \frac{(t_1+s_1)!}{t_1!s_1!}\cdots\frac{(t_r+s_r)!}{t_r!s_r!}\frac{1}{(t_1+s_1)!\cdots(t_r+s_r)!}F(p_1)^{t_1+s_1}\cdots F(p_r)^{t_r+s_r}$$

$$= \frac{1}{(t_1)!\cdots(t_r)!}F(p_1)^{t_1}\cdots F(p_r)^{t_r}\frac{1}{(s_1)!\cdots(s_r)!}F(p_1)^{s_1}\cdots F(p_r)^{s_r}$$

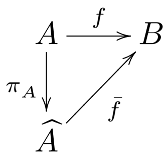
$$= f(e_n)f(e_m).$$

Notice that the Dirichlet character function is a completely multipicative arithmetic function, then we get a corresponding algebra map from V to $\mathbb{C}$.

Now we just need consider a set containing all algebra maps V to $\mathbb{C}$. Luckly the finite dula of V is nearly to what we want, though the elements of it will vanish at infinite point. So we try extending $V^* \otimes V^*$ to make sure the operation Δ^* is closed, then the extended set will contain all algebra maps. In the process, we need some useful properties of complete tensor.

4.2 Universal property of complete module

Theorem 4.1 *The universal property of the limit $\hat{A}$: any filtered map $A \xrightarrow{f} B$ where B is a complete module , factor uniquely through the canonical map as following:*



Proof Note that B is complete $\Leftrightarrow$ (B, G) is a filtered module and $B \overset{\pi_B}{\longrightarrow} \widehat{B}$ is an isomorphism.
Define

$$\bar{f} : \hat{A} \to B$$

$$(a_0 + F_0A, a_1 + F_1A_1 \cdots) \mapsto \pi_B^{-1}(f_1(a_0) + G_0B, f(a_1) + G_1B_2 \cdots).$$

a) The definition of $\bar{f}$ is well defined. If

$$(a_0 + F_0A, a_1 + F_1A, \cdots) = (a'_0 + F_0A, a'_1 + F_1A, \cdots),$$

we have $a_i - a'_i \in F_iA$. Recall the definition of the filtered map, we have $f(F_iA) \subset G_iB$, so $f(a_i) - f(a_i) \in G_iB$, then $f(a_i) - f(a_i) \in G_iB$. Thus

$$(f(a_0) + G_0B, f(a_1) + G_1B \cdots) = (f(a'_0) + G_0B, f(a'_1) + G_1B \cdots).$$

b) We claim it is a commutative diagram. $\bar{f}\pi_A(a) = \bar{f}(a + F_0A, a + F_1A, \cdots) = \pi_B^{-1}(f(a) + G_0B, f(a) + G_1B, \cdots) = f(a), \forall a \in A$ then $\bar{f}\pi_A = f$.
c) $\bar{f}$ is a filtered map. For any $(0, \cdots, 0, a_{k+1} + F_{k+1}A, a_{k+2} + F_{k+2}A, \cdots) \in \hat{f}_k\hat{A}$,

$$\begin{aligned}&\bar{f}(0, \cdots, 0, a_{k+1} + F_{k+1}A, a_{k+2} + F_{k+2}A, \cdots)\\=&\pi_B^{-1}(0, \cdots, 0, f(a_{k+1}) + G_{k+1}B, f(a_{k+2}) + G_{k+2}B, \cdots)\\\in&\pi_B^{-1}\hat{G}_k\hat{B}\\\subset&G_kB.\end{aligned}$$

d) We will show it is the unique map satifying the conditions. Assume there is another filtered map $\bar{g}$ satisfying $\bar{g}\pi_A = f$, and $\bar{g} \neq \bar{f}$. Then there exists $(a_0 + F_0A, a_1 + F_1A, \cdots) \in \hat{A}$, such that

$$\pi_B\bar{g}(a_0 + F_0A, a_1 + F_1A, \cdots) \neq \pi_B\bar{f}(a_0 + F_0A, a_1 + F_1A \cdots).$$

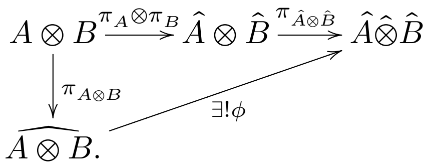
Note that π_B is an isomorphism. Denote $(b_0 + G_0B, \cdots)$ and $(b'_0 + G_0B, \cdots)$ for both side respectively, there must be some k, such that $b_k + G_kB \neq b'_k + G_kB$. Then

$$\begin{aligned}&(a_0 + F_0A, a_1 + F_1A, \cdots, a_k + F_kA, a_{k+1} + F_{k+1}A, \cdots)\\=&(a_0 + F_0A, a_1 + F_1A, \cdots, a_k + F_kA, a_k + F_{k+1}A, \cdots) + (0, \cdots, 0, a_{k+1} - a_k + F_{k+1}A)\\=&(a_k + F_0A, a_k + F_1A, \cdots, a_k + F_kA, a_k + F_{k+1}A, \cdots) + (0, \cdots, 0, a_{k+1} - a_k + F_{k+1}A)\\=&\pi_A(a_k) + (0, \cdots, 0, a_{k+1} - a_k + F_{k+1}A)\end{aligned}$$

We also have $(0, \cdots, 0, a_{k+1} - a_k + F_{k+1}A) \in \hat{F}_k\hat{A}$, and $\pi_B\bar{f}$, $\pi_B\bar{g}$ are filtered maps, so the value of $b_k + G_kB$ and $b'_k + G_kB$ depends on $\pi_B\bar{f}\pi_A(a_k)$ and $\pi_B\bar{g}\pi_A(a_k)$ respectively. But we know $\pi_B\bar{f}\pi_A = \pi_B\bar{g}\pi_A$. so $b_k + G_kB = b'_k + G_kB$, thus it contradict with our assumption. Then $\bar{f}$ is unique in this way. ■

Theorem 4.2 $\widehat{A \otimes B} \cong \hat{A}\hat{\otimes}\hat{B}$.

Proof For the complete module $\hat{A}\hat{\otimes}\hat{B}$, we have a filtered map $\pi_{\hat{A}\otimes\hat{B}} \circ \pi_A \otimes \pi_B$. By the universal property of the completion of $A \otimes B$, we get ϕ as following:



Since the commutative map is unique, we just define ϕ so that ϕ satifies the commucative diagram.

$$
\begin{aligned}
&\phi(\sum a_0^i \otimes b_0^i + F_0\widehat{A \otimes B}, \sum a_1^i \otimes b_1^i + F_1\widehat{A \otimes B}, \cdots) \\
=&(\sum (a_0^i + F_0A, a_0^i + F_1A, \cdots) \otimes (b_0^i + G_0B, b_0^i + G_1B, \cdots) + F_0(\hat{A} \otimes \hat{B}), \\
&\sum (a_1^i + F_0A, a_1^i + F_1A, \cdots) \otimes (b_1^i + G_0B, b_1^i + G_1B, \cdots) + F_1(\hat{A} \otimes \hat{B}), \cdots).
\end{aligned} \tag{4-1}
$$

In particular, we get

$$
\begin{aligned}
&\phi((a_0 \otimes b_0 + F_0\widehat{A \otimes B}), \sum (a_1 \otimes b_1 + F_1\widehat{A \otimes B}), \cdots) \\
=&((a_0 + F_0A, a_0 + F_1A, \cdots) \otimes (b_0 + G_0B, b_0 + G_1B, \cdots) + F_0(\hat{A} \otimes \hat{B}), \\
&(a_1 + F_0A, a_1 + F_1A, \cdots) \otimes (b_1 + G_0B, b_1 + G_1B, \cdots) + F_1(\hat{A} \otimes \hat{B}), \cdots).
\end{aligned}
$$

a) We claim the definition is well-defined. When

$$
\begin{aligned}
&(\sum a_0^i \otimes b_0^i + F_0\widehat{A \otimes B}, \sum a_1^i \otimes b_1^i + F_1\widehat{A \otimes B}, \cdots) \\
=&(\sum c_0^i \otimes c_0^i + F_0\widehat{A \otimes B}, \sum d_1^i \otimes d_1^i + F_1\widehat{A \otimes B}, \cdots),
\end{aligned}
$$

we get

$$\sum a_k^i \otimes b_k^i - \sum b_k^i \otimes c_k^i \in F_{,}(A \otimes B) = \sum_{m+n=k} Im(F_m A \otimes G_n B \to A \otimes B).$$

From the following equations, the definiton is well-defined.

$$\begin{aligned}
&\sum (a_k^i + F_0 A, a_k^i + F_1 A, \cdots) \otimes (b_k^i + G_0 B, b_k^i + G_0 B, \cdots) \\
-&\sum (b_k^i + F_0 A, b_k^i + F_1 A, \cdots) \otimes (d_k^i + G_0 B, d_k^i + G_0 B, \cdots) \\
=&\pi_A \otimes \pi_B (\sum a_k^i \otimes b_k^i - \sum c_k^i \otimes d_k^i) \\
\in&F_k(\hat{A} \otimes \hat{B}).
\end{aligned}$$

b) Now, we say ϕ is a filtered map.

$$\begin{aligned}
&\phi(0, \cdots, 0, \sum a_{k+1}^i \otimes b_{k+1}^i + F_{k+1}(A \otimes B), \cdots) \\
=&(0, \cdots, 0, \sum a_{k+1}^i + F_0 A, a_{k+1}^i + F_1 A, \cdots) \otimes (b_{k+1}^i + G_0 B, b_{k+1}^i + G_1 B, \cdots) + \\
&F_{k+1}(\hat{A} \otimes \hat{B}, \cdots) \\
\in&F_k(\hat{A} \otimes \hat{B}).
\end{aligned}$$

So ϕ is a filtered map, and obviously $\phi \pi_{A \otimes B} = \pi_{\hat{A} \otimes \hat{B}} (\pi_A \otimes \pi_B)$.

c) We want to show ϕ is a inverse map by giving the inverse map explicitly. We will use the universal property again, after all the $\hat{A} \hat{\otimes} \hat{B}$ is a completion of $\hat{A} \otimes \hat{B}$.

$$\begin{array}{ccc}
\hat{A} \otimes \hat{B} & \xrightarrow{f} & \widehat{A \otimes B} \\
\downarrow \pi_{\hat{A} \hat{\otimes} \hat{B}} & \nearrow \exists ! \phi & \\
\hat{A} \hat{\otimes} \hat{B}. & &
\end{array}$$

We just need to define f, then get a map from $\hat{A} \hat{\otimes} \hat{B}$ to $\hat{A} \otimes \hat{B}$. Define f as

$$\begin{aligned}
&f(a_0 + F_0 A, a_1 + F_1 A, \cdots) \otimes (b_0 + G_0 B, b_1 + G_1 B, \cdots) \\
=&(a_0 \otimes b_0 + F_0(A \otimes B), a_1 \otimes b_1 + F_1(A \otimes B), \cdots).
\end{aligned}$$

Notice that

$$\begin{aligned}&a_{k+1}\otimes b_{k+1}-a_k\otimes b_k\\=&a_{k+1}\otimes b_{k+1}-a_k\otimes b_{k+1}+a_k\otimes b_{k+1}-a_k\otimes b_k\\\in&F_kA\otimes G_0B+F_0A\otimes G_Kb\\\subset&F_k(A\otimes B),\end{aligned}$$

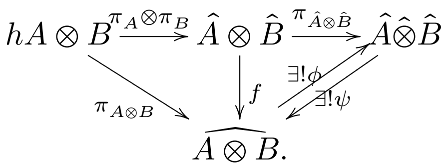
so f is well defined. Now we claim f is a filtered map.

When $m+n=k$, for any$(0,\cdots,0,a_{m+1}+F_{m+1}A,a_{m+2}+F_{m+2}A,\cdots)\otimes(0,\cdots,0,a_{m+1}+F_{m+1}A,a_{m+2}+F_{m+2}A,\cdots)\in\sum_{m+n=k}F_mA\otimes G_nB$. By the definition of f,

$$\begin{aligned}&f((0,\cdots,0,a_{m+1}+F_{m+1}A,a_{m+2}+F_{m+2}A,\cdots)\otimes\\&(0,\cdots,0,a_{m+1}+F_{m+1}A,a_{m+2}+F_{m+2}A,\cdots))\\=&(c_0+F_0(A\otimes B),c_1+F_1(A\otimes B),\cdots).\end{aligned}$$

When $t\leqslant max\{m,n\}$, $c_t=0$; when $t>max\{m,n\}$,$c_t=a_t\otimes b_t\in F_mA\otimes G_nB\subset F_k(A\otimes B)$. That means $(c_0+F_0(A\otimes B),c_1+F_1(A\otimes B),\cdots)\in F_k(\widehat{A\otimes B})$, thus f is a filtered map. Thus we get ψ and $\psi\pi_{\hat{A}\hat{\otimes}\hat{B}}=\hat{f}$.

Finally we get the diagram



Notice that

$$\begin{aligned}(\psi\phi)\pi_{A\otimes B}=&\psi(\phi\pi_{A\otimes B})\\=&\psi(\pi_{\hat{A}\hat{\otimes}\hat{B}}(\pi_A\otimes\pi_B))\\=&(\psi\pi_{\hat{A}\hat{\otimes}\hat{B}})(\pi_A\otimes\pi_B)\\=&f(\pi_A\otimes\pi_B)\end{aligned}$$

$$=\pi_{A\otimes B}.$$

Using the universal property of $\widehat{A\otimes B}$, we get $\psi\phi = Id$.

Besides,

$$\begin{aligned}
&\phi f(a_0 + F_0A, a_1 + F_1A, \cdots) \otimes (b_0 + G_0B, b_1 + G_1B, \cdots)\\
=&\phi(a_0 \otimes b_0 + F_0A \otimes B, a_1 \otimes b_1 + F_1(A \otimes B), \cdots)\\
=&((a_0 + F_0A, a_0 + F_1A, \cdots) \otimes (b_0 + G_0B, b_0 + G_1B, \cdots) + F_0(\hat{A} \otimes \hat{B}), \cdots),
\end{aligned}$$

and

$$\begin{aligned}
&\pi_{\hat{A}\otimes\hat{B}}(a_0 + F_0A, a_1 + F_1A, \cdots) \otimes (b_0 + G_0B, b_1 + G_1B, \cdots)\\
=&((a_0 + F_0A, a_1 + F_1A, \cdots) \otimes (b_0 + G_0B, b_1 + G_1B, \cdots) + F_0(\hat{A} \otimes \hat{B}), \cdots).
\end{aligned}$$

Since

$$\begin{aligned}
&(a_0 + F_0A, a + F_1A, \cdots) - (a_k + F_0A, a_k + F_1A, \cdots)\\
=&(0, \cdots, 0, a_{k+1} - a_k + F_{k+1}A, \cdots) \in \hat{F}_k\hat{A}\\
&(b_0 + G_0B, b + G_1B, \cdots) - (b_k + G_0B, b_k + G_1B, \cdots)\\
=&(0, \cdots, 0, b_{k+1} - b_k + G_{k+1}B, \cdots) \in \hat{G}_k\hat{B},
\end{aligned}$$

we have

$$\begin{aligned}
&(a_0 + F_0A, a_1 + F_1A, \cdots) \otimes (b_0 + G_0B, b_1 + G_1B, \cdots) + F_k(\hat{A} \otimes \hat{B}), \forall k\\
-&(a_k + F_0A, a_k + F_1A, \cdots) \otimes (b_k + G_0B, b_k + G_1B, \cdots) + F_k(\hat{A} \otimes \hat{B})\\
\in&\hat{F}_k\hat{A} \otimes \hat{G}_0\hat{B} + \hat{F}_0\hat{A} \otimes \hat{G}_k\hat{B}\\
\subset&F_k(\hat{A} \otimes \hat{B}).
\end{aligned}$$

So $\phi f = \pi_{\hat{A}\otimes\hat{B}}$, and then

$$\begin{aligned}\phi\psi\pi_{\hat{A}\otimes\hat{B}} =&\phi(\psi\pi_{\hat{A}\otimes\hat{B}})\\ =&\phi f\\ =&\pi_{\hat{A}\otimes\hat{B}}.\end{aligned}$$

By uniqueness, we have $\phi f = Id$. ■

4.3 Completion of U and $U \otimes U$

Lemma 4.1 *Let* $U =< e_1^*, e_2^*, \cdots >$, *and* $F_n(U) =< e_i^* | i > n >$. *Then* $\hat{U} \cong V^*$.

Proof When $f \in V^*$, we have

$$f = \sum_{i=1}^{\infty} f(e_i)e_i^*.$$

The left expression is well defined, for every element of V is a finite sum.

We define an isomorphism explicityly as

$$\begin{aligned}V^* &\to \widehat{U}\\ \sum_{i=1}^{\infty} f(e_i)e_i^* &\mapsto (0, f(e_1)e_1^* + U/F_1(U), ...),\end{aligned}$$

with its inverse is defined as

$$\begin{aligned}&\hat{U} \to V^*\\ &(a_0 + U/F_0(U), a_1 + U/F_1(U), a_2 + U/F_2(U), \cdots,) \mapsto \sum a_i.\end{aligned}$$

When

$$(a_0 + U/F_0(U), a_1 + U/F_1(U), a_2 + U/F_2(U), \cdots,)$$

$$=(b_0 + U/F_0(U), b_1 + U/F_1(U), b_2 + U/F_2(U), \cdots,),$$

we have

$$a_i - b_i \in F_i(U), \quad \forall i,$$

which implies $a_r(e_r) = b_r(e_r)$, then

$$\sum a_i(e_r) = a_r(e_r) = b_r(e_r) = \sum b_i(e_r),$$

so the map is well-defined. Now, we can prove the lemma. ■

From Theorem4.2, U, V not necessarily complete module. Let $\hat{U} = X$, and $\hat{V} = Y$, then

$$\widehat{U \otimes V} = \widehat{X \otimes Y}.$$

Recall the definition of induced filtration of $U \otimes U$ is

$$F_k(U \otimes U) = \sum_{n+m=k} Im(F_n A \otimes F_m A \hookrightarrow U \otimes U)$$

Similarly, we claim that $\widehat{U \otimes U} = (V \otimes V)^*$ by

$$\sigma : (V \otimes V)^* \to \widehat{U \otimes U} \tag{4-2}$$

$$\sum_{\substack{i=1 \\ j=1}}^{\infty} f(e_i \otimes e_j) e_i^* \otimes e_j^* \mapsto (0, 0, f(e_1 \otimes e_1) e_1^* \otimes e_1^*, ...),$$

is an isomorphism.

Corollary 4.1 $V^* \widehat{\otimes} V^* \cong (V \otimes V)^*$

Proof

$$V^* \hat{\otimes} V^* \overset{\hat{U}=V^*}{\cong} \hat{U} \hat{\otimes} \hat{U} \overset{Thm4.2}{\cong} \widehat{U \otimes U} \cong (V \otimes V)^*$$ ■

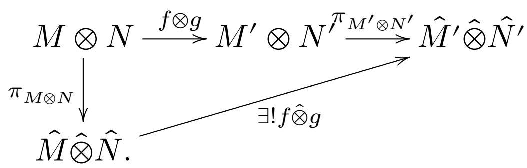
Conclusively, we find a set is closed under Δ^*. In the next section, we will give the definiton of generalized Hopf algebra.

Chapter 5 Generalized Hopf algbra

5.1 Definition of generalized Hopf algebra

Now we can formally give the definition to generalized Hopf algebra, after the following lemmas.

Lemma 5.1 *Given filtered maps $f : M \to M'$ and $g : N \to N'$, there is a map called $f\hat{\otimes}g$, making the following diagram commutates.*



Lemma 5.2 *If (A,F) is complete, then π^{-1} is also a filtered map, thus we have*

$$\pi_A(F_n A) = \hat{F}\hat{A}.$$

Proof For any $(0, \cdots, 0, a_{k+1} + F_{k+1}A, \cdots) \in \hat{F}\hat{A}$,

$$\begin{aligned} &\pi^{-1}(0, \cdots, 0, a_{k+1} + F_{k+1}A, \cdots) \\ &= \pi^{-1}(\pi(a_{k+1})) \\ &= a_{k+1}. \end{aligned}$$

Notice $a_{k+1} + F_k = 0$, thus $\pi^{-1}(\hat{F}_k\hat{A}) \subset F_k A$, then $\pi_A(F_n A) = \hat{F}\hat{A}$. ■

Definition 5.1 *We call a map which is a k-module isomorphism and keeps filtrations, a filtered isomorphism.*

Corollary 5.1 *$\hat{A} \cong \hat{B}$, if there exist a filtered isomorphism between A and B.*

Lemma 5.3 *$A\hat{\otimes}B \cong C\hat{\otimes}D$, if there exist filtered isomorphisms f, g between A and B, C and D.*

Proof $f\hat{\otimes}g$ works. ■

Lemma 5.4 *Let (A,F) be a filtered module then $\widehat{\widehat{A\otimes A}\otimes A} \cong \widehat{A\otimes \widehat{A\otimes A}}$.*

Proof Using lemma 5.2, we have $\hat{\hat{A}} \cong \hat{A}$ is filtered isomorphism.

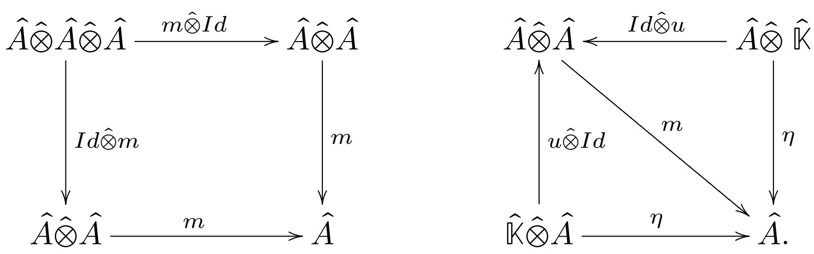
$$
\begin{aligned}
\widehat{\widehat{A\otimes A}\otimes A} &\overset{\hat{A}}{\cong} \widehat{\widehat{A\otimes A}}\hat{\otimes}\hat{\hat{A}} \\
&\cong \widehat{A\otimes A}\hat{\otimes}\hat{A} (Lemma\ 5.3) \\
&\cong (A\otimes \widehat{A)}\otimes A \\
&\cong A\otimes \widehat{(A\otimes} A) (Corollary\ 5.1) \\
&\cong A\otimes \widehat{\widehat{A\otimes A}}
\end{aligned}
$$

■

Because of the Lemma 5.4, we can identify $A\otimes \widehat{A\otimes A} = \hat{A}\hat{\otimes}\hat{A}\hat{\otimes}\hat{A}$ with $\widehat{\widehat{A\otimes A}\otimes A}$ and $\widehat{A\otimes \widehat{A\otimes A}}$. Now we can definite generalized $\mathbb{K}$-algebra almost word for word by the concepts of $\mathbb{K}$-algebra as following:

Definition 5.2 *A generalized $\mathbb{K}$ algebra (with unit) is a $\mathbb{K}$-vector space (A,F) together with two $\mathbb{K}$-filtered maps, multiplication m: $\hat{A}\hat{\otimes}\hat{A} \to \hat{A}$ (where $\hat{A}\hat{\otimes}\hat{A} = \widehat{X\otimes X}, X = \hat{A}$) and unit $u : \hat{\mathbb{K}} \to \hat{A}$, such that the following diagrams are commutative:*

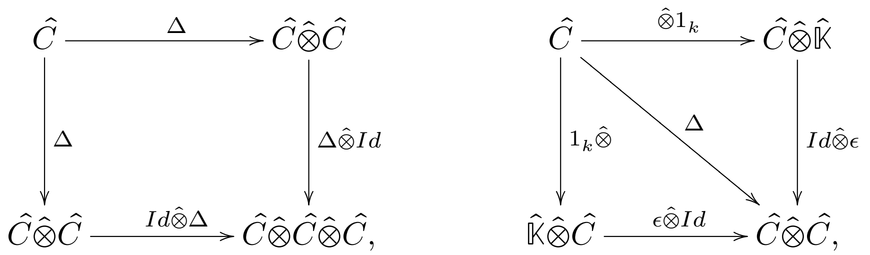
a) associativity *b) unit*



The Id denotes the identity mapping from $\hat{A}$ to $\hat{A}$, and the η dentoes the filtered map induced by scalar multiplication from $\mathbb{K}$ to A, where we endow $\mathbb{K}$ with the trivial filtration.

Definition 5.3 *A generalized $\mathbb{K}$-coalgebra (with counit) is a $\mathbb{K}$-vector space (C,F) together with two $\mathbb{K}$-linear maps, comultiplication$\Delta : \hat{C} \to \hat{C}\hat{\otimes}\hat{C}$ and counit$\epsilon : \hat{C} \to \hat{\mathbb{K}}$, such that the following diagrams are commutative:*

a) coassociativity *b) counit*



where the two upper maps in b) are given by $c \mapsto 1\widehat{\otimes}c$ and $c \mapsto c\widehat{\otimes}1$, for any $c \in C$.

Similarly, we can give the definiton of generalized algebra map and generalized bialgebra, and generalized convolution product.

Then we have the definition of generalize Hopf algebra:

Definition 5.4 *Let $(H, F, m, u, \Delta, \epsilon)$ be a generalized bialgebra. Then H is a generalized Hopf algebra if there exists an element $S \in Hom_{\mathbb{K}}(\hat{H}, \hat{H})$, which is an inverse to $Id_{\hat{H}}$ under the generalized convolution product.*

5.1.1 V^* is a generalized Hopf algebra

Recall the definition of V, which is a Hopf algebra with $(m_0, u_0, \Delta_0, \epsilon_0, S_0)$ and U. We say the (U, F_n) is a generalized Hopf algebra with $F_nU =< e_i^*|i > n >$.

Firstly, we claim it is a generalized algebra. We give a homomorphism

$$V^* \widehat{\otimes} V^* \to (V \otimes V)^*$$
$$(a_0, a_1, \cdots) \mapsto f,$$

in that equation

$$f : (V \otimes V) \to C$$
$$\sum k_{ij} e_i \otimes e_j \mapsto \sum_n \sum_{i+j=n} k_{ij} a_n (e_i \otimes e_j).$$

Then we have

$$m : V^* \widehat{\otimes} V^* \to V^*$$
$$(a_0, a_1, \cdots) \mapsto f \circ \Delta$$
$$u : \hat{C} \to V^*$$
$$(c_0, c_1, \cdots) \mapsto \sum c_i \epsilon_0.$$

We claim the V^* is a generalized algebra.

a) We prove the associativity of generalized algebra by its definition. If $f = (0, \sum f_i^{(1)} \otimes g_i^{(1)}, \cdots)$, then

$$\begin{aligned}(m \hat{\otimes}) Id(f)(e_s \otimes e_t) &= (0, \sum f_i^{(1)} \Delta_0 \otimes g_i^{(1)}, \cdots)(e_s \otimes e_t) \\ &= \sum f_i^{(i_0+j_0)} \Delta_0 \otimes g_i^{(n)} (e_s \otimes e_t) \\ &= f(\Delta_0 \otimes Id)(e_s \otimes e_t).\end{aligned}$$

By this equation, we can calculate

$$
\begin{aligned}
m(m\hat{\otimes}Id)(f)(e_n) &= \sum_{d|n}((m\hat{\otimes}Id)f)(e_d \otimes e_{\frac{d}{n}}), \forall f \in \hat{V}^*\hat{\otimes}\hat{V}^*\hat{\otimes}\hat{V}^* \\
&= \sum_{d|n} f(\Delta_0 \otimes Id)(e_d \otimes e_{\frac{d}{n}}) \\
&= f(\Delta_0 \otimes Id)\Delta_0(e_n) \\
&= f(Id \otimes \Delta_0)\Delta_0(e_n) \\
&= m(Id\hat{\otimes}m)(f)(e_n).
\end{aligned}
$$

b) Using the definition, we show its unit. Let $g = (0, \sum c_i^{(1)} \otimes a_i^{(1)}, \cdots)$

$$
\begin{aligned}
&(u\hat{\otimes}Id)g(e_s \otimes e_t) \\
=&(0, \sum c_i^{(1)}\epsilon_0 \otimes a_i^{(1)}, \cdots)(e_s \otimes e_t) \\
=&\sum c_i^{(s+t)}\epsilon_0(e_s) \otimes a_i^{(s+t)}(e_t) \\
=&g(\epsilon_0 \otimes Id)(e_s \otimes e_t).
\end{aligned}
$$

Note that we have an isomorphism,

$$
\begin{aligned}
C\hat{\otimes}V^* &\to (C \otimes V)^* \\
(0, \sum c_i^{(1)} \otimes a_i^{(1)}, \cdots) &\mapsto f
\end{aligned}
$$

where

$$
\begin{aligned}
f : C \otimes V &\to C \\
\sum k_n \otimes e_n &\mapsto \sum c_i^{(n)} k_n a_i^{(n)}(e_n),
\end{aligned}
$$

thus

$$
\begin{aligned}
m(u\hat{\otimes}Id)(g)(e_n) &= (u\hat{\otimes}Id)g\Delta(e_n)\\
&= g(\epsilon_0\otimes Id)\Delta(e_n)\\
&= g(1\otimes e_n)\\
&= \sum c_i^{(n)}a_i^{(n)}(e_n)\\
&= \eta(g)(e_n).
\end{aligned}
$$

By our definition, it is a generalized algebra.

Similarly, we can prove it is a coalgebra and the antipode is S_0^*, thus it is a generalized Hopf algebra.

Generally, a dual of a Hopf algebra which has countable basis is a generalized Hopf algebra by similar way.

5.1.2 A Hopf algebra is a generalized Hopf algebra

Another important example is Hopf algebra. For a Hopf algebra $H(m_0, u_0, \Delta_0, \epsilon_0, S_0)$, we give a trivial filtration $F_nH = 0,\ n \geqslant 1$, then it become a generalized Hopf algebra.

Taking m_0 as an example, we will see the m by the universal property in the following diagram

$$
\begin{array}{ccc}
H\otimes H & \xrightarrow{m_0} & H\\
\downarrow & & \downarrow\\
\widehat{H\otimes H} & \xrightarrow{m} & \hat{H}.
\end{array}
$$

Similarly, we can get u, Δ, and ϵ, and S, thus we can verify it is a generalized Hopf algebra.

5.2 Group-like elements of a generalized Hopf algebra V^*

In the final section of this article, we will show the generalized group-like elements in the generalized Hopf algebra is exactly the algebra maps from V to $\mathbb{C}$, which completely meet our goal.

Definition 5.5 *Let $(H, F)(m, u, \Delta, \epsilon)$ be a generalized bialgebra. c is called group-like if $\Delta c = c\hat{\otimes}c$ and $\epsilon(c) = 1$.*

Now, we recall maps from equations (4-1) and (4-2), then we have

$$(V \otimes V)^* \xrightarrow{\sigma} \widehat{U \otimes U} \xrightarrow{\phi} V^*\hat{\otimes}V^*.$$

The Δ in V^* is the transport of m defined in V.

$$\begin{aligned}
&\Delta(x)\\
=&\phi\sigma(xm)\\
=&\phi(0, xm(e_1 \otimes e_1)e_1^* \otimes e_1^* + F_1(U_1 \otimes U_1),\\
&xm(e_1 \otimes e_1)e_1^* \otimes e_1^* +\\
&xm(e_1 \otimes e_2)e_1^* \otimes e_2^* + xm(e_2 \otimes e_1)e_2^* \otimes e_1^* + F_2(U_1 \otimes U_1), \cdots)\\
=&((0, \cdots, 0) \otimes (0, \cdots, 0) + F_0(V^* \otimes V^*),\\
&(xm(e_1 \otimes e_1)e_1^*, 0, \cdots, 0) \otimes (e_1^*, 0, \cdots, 0) + F_1(V^* \otimes V^*),\\
&(xm(e_1 \otimes e_2)e_1*, 0, \cdots, 0) \otimes (0, e_2^*, 0, \cdots, 0)+\\
&(0, xm(e_2 \otimes e_1)e_2^*, 0, \cdots, 0) \otimes (e_1^*, 0, \cdots, 0), \cdots),
\end{aligned}$$

and by the definition of filtration of $V^* \otimes V^*$, $F_n(V^* \otimes V^*) = \sum_{i+j=n} F_i(V^*)F_j(V^*)$, then

$$\begin{aligned}
&x\hat{\otimes}x\\
=&((x(e_1), x(e_2), \cdots) \otimes (x(e_1), x(e_2), \cdots) + F_0(V^* \otimes V^*),\\
&(x(e_1), x(e_2), \cdots) \otimes (x(e_1), x(e_2), \cdots) + F_1(V^* \otimes V^*),\\
&(x(e_1), x(e_2), \cdots) \otimes (x(e_1), x(e_2), \cdots) + F_2(V^* \otimes V^*) \cdots)\\
=&((0, \cdots, 0) \otimes (0, \cdots, 0) + F_0(V^* \otimes V^*),\\
&(x(e_1)e_1^*, 0, \cdots) \otimes (x(e_1)e_1^*, 0, \cdots) + F_1(V^* \otimes V^*),
\end{aligned}$$

$(x(e_1)e_1^*, 0, \cdots) \otimes (x(e_1)e_1^*, 0, \cdots) + (x(e_1)e_1^*, 0, \cdots) \otimes (0, x(e_2)e_2^*, 0, \cdots) +$

$(0, x(e_2)e_2^*, 0, \cdots) \otimes (x(e_1)e_1^*, 0, \cdots) + F_2(V^* \otimes V^*).$

$\cdots).$

If x is a generalized group like element, then $x(e_ie_j) = x(e_i)x(e_j)$. Considering the basis of $V^* \otimes V^*$, we get x is a algebra homomorphism. Surely when x is an algebra homomorphism, $\Delta(x) = x\hat{\otimes}x$. Since $x(e_1) \neq 0$, $x(e_1) = 1$, thus $\epsilon(x) = x(e_1) = 1$, where ϵ is the transpose of u in V. So

$$G(V^*) = Alg(V, C).$$

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致　谢

我要特别感谢我的导师刘公祥教授对我的帮助。不仅是本文的选题到结束的各个关键环节的指导，还有三年来学习上的点拨、生活上的关心。

我要感谢我的各个授课老师：丁南庆、黄兆泳、杨俊峰、张翀、宗润弘等老师，他们和蔼可亲地传授道理，细致耐心地教授知识、严谨认真地回答疑问。惭愧地说许多知识我都没学会，即使扪心自问我勉强算是尽了力的，然而我终究不可能有再来一次的机会，虽然有些后悔，总是要带着些遗憾结束我的硕士生活了。

感谢王梦君博士后给我的许多建议和帮助，让我少走了很多弯路。大约有一些个性过强，对别人的建议思考的少，总是使得做事多有曲折。然而我大抵还是非常相信她的很多判断，终使得自己曲折的人生在写论文方面居然有些意外的顺利了。

感谢同门的师兄弟姐妹，虽然显然我没读过别的地方的硕士，但我总疑心我们是气氛最好的课题组，大家聚在一起既严肃又活泼。感谢我的室友们，魏寅、苏航、王得民、王振宇三年的陪伴，一起玩、学习、写论文，有过很多快活的时刻。

各位，再见！