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Morita theories in tensor triangulated categories

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毕业论文题目: **张量三角范畴中的森田理论**

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摘　　要

本博士论文主要研究的对象是：有限张量范畴的稳定范畴和有限张量范畴的导出范畴，它们是张量三角范畴的两个主要例子。具体来说，我希望找到 Hopf 代数（或更一般地说有限张量范畴）中三种等价关系之间的联系。这三种等价关系分别是 gauge 等价（类似于张量范畴的森田等价）、稳定张量等价和导出张量等价。

本文的第一个主要结果表明,有限维非半单 Hopf 代数的稳定张量等价在某些条件下能诱导 gauge 等价。接着，我利用幺半 t-结构从导出张量范畴重构出有限维 Hopf 代数。即有限维 Hopf 代数的导出张量等价可以诱导 gauge 等价。由此我们还可以看出，有限维非半单 Hopf 代数的导出张量等价可以诱导它们之间的稳定张量等价。以上所有结果都可以借助 Frobenius-Perron 维数推广到有限张量范畴中。

关键词: 张量三角范畴; 稳定张量等价; Frobenius-Perron 维数; 导出张量等价; 幺半 t-结构 ; 重构定理.

THESIS: **Morita theories in tensor triangulated categories**

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Abstract

My dissertation focuses on the stable categories of finite tensor categories and the derived categories of finite tensor categories, which are tensor triangulated categories. Specifically I look for relations between three types of equivalences between Hopf algebras, or more general, between finite tensor categories. These three types of equivalences are gauge equivalences which are analogue of Morita equivalences for tensor categories, stable tensor equivalences and derived tensor equivalences.

My first two main results show that a stable tensor equivalence of finite-dimensional non-semisimple Hopf algebras induces a gauge equivalence under certain conditions. Later on, I have used monoidal t-structures to reconstruct a finite-dimensional Hopf algebra from its derived tensor category. Namely a derived tensor equivalence of Hopf algebras induces a gauge equivalence of these Hopf algebras. At this point we can also see that a derived tensor equivalence of finite-dimensional non-semisimple Hopf algebras can induce a stable tensor equivalence of these Hopf algebras. All the above results can be generalized into finite tensor categories with the help of Frobenius-Perron dimensions.

Keywords: Tensor triangulated categories; Stable tensor equivalences; Frobenius-Perron dimensions; Derived tensor equivalences; Monoidal t-structures; Construction theorem.

CONTENTS

Chapter 1 Introduction

Tensor triangulated categories combine the structural insights of triangulated categories with the operational framework of tensor categories. They are particularly prominent in modern homotopy theory and motivic homotopy theory, providing a way for the study of stable homotopy categories of spectra and derived categories of sheaves. In recent years, there has been tremendous interest in developing tensor triangulated categories. The spectrum of a tensor triangulated category was introduced by Paul Balmer [7], providing an algebra-geometric method to the study of tensor triangulated categories. It establishes an abstract framework to build bridges among different branches of mathematics, such as algebraic geometry, stable homotopy theory, modular representation theory, motivic theory, non-commutative topology, and symplectic geometry [6].

There are a number of approaches to do research on tensor triangulated categories. So far, limited work has been done in purely algebraic fields. Since tensor triangulated categories have both monoidal and triangulated structures, in terms of triangulated categories, there are two entry points: stable categories of Frobenius categories and derived categories. Much research on stable categories and derived categories has been done. However, not much attention has been paid to the tensor structures in these theories, such as derived Morita theory. Balmer established a classification of thick tensor ideal in tensor triangulated categories [7]. Schwede and Shipley showed in [49] that stable model categories with a single compact generator are equivalent to modules over a ring spectrum. Steen and Stevenson provided a detailed exposition of the conditions such that tensor triangulated categories do not contain thick tensor ideals admitting strong generators [50]. J. J. Zhang and J.-H. Zhou applied the Frobenius-Perron theory to tensor triangulated structures of quiver representations in [59]. They defined the concept of mtt-structure to recover a monoidal abelian equivalence from a derived tensor equivalence under some conditions. Shahram Biglari gave a Künneth formula in tensor triangulated categories [10].

In representation theory, derived Morita theory helps classify algebras up to derived equivalences. This classification is vital for understanding the structure of representation categories and their interactions. I am interested in developing the tensor triangulated equivalences of stable categories and derived categories, and exploring invariants under equivalences further.

Problems and results

All finite tensor categories are Frobenius categories [19, Chapter 6], hence their stable categories are tensor triangulated categories (see Lemma 2.3.2). An example of a finite tensor category is the category of finite-dimensional representations of a finite-dimensional Hopf algebra [18]. As it happens, Hopf algebras are self-injective algebras [36]. Moreover, the comultiplication and antipodes of Hopf algebras provide the tensor structure of H-mod. Let H and H' be finite-dimensional non-semisimple Hopf algebras.

***Problem** 1.* If H-$\underline{\text{mod}}$ is equivalent to H'-$\underline{\text{mod}}$ as a tensor triangulated category, then what are the relations between H and H'?

In order to compare two different equivalences between Hopf algebras: gauge equivalences (see Section 2.1) and stable equivalences, we should first present the relations between Morita equivalences (gauge equivalences are Morita equivalences) and stable equivalences. Ng and Schauenburg showed in [39] that H and H' are gauge equivalent if and only if H-mod and H'-mod are tensor equivalent. Even if the stable categories of two finite-dimensional algebras are equivalent, the corresponding algebraic structures may be quite different. For example, some direct sum of group algebras are not Morita equivalent, although they are stably equivalent. Therefore, Broué proposed the concept of stable equivalence of Morita type [12]. Linckelmann proved that a stable equivalence of Morita type between two self-injective algebras can be lifted to a Morita equivalence if and only if the equivalence maps any simple module to a simple module [32]. It is known that some stable equivalences can be induced by a functor between two categories of representations over self-injective k-algebras which maps projective modules to projective modules. If in addition, the functor is exact, then the two self-injective k-algebras are stably equivalent of Morita type [46].

In [18], the Frobenius-Perron dimension of a tensor category $\mathcal{C}$ has been defined, which is invariant under tensor equivalences. For a finite-dimensional Hopf algebra H, $\text{FPdim}(H\text{-}\mathsf{mod}) = \dim_{\Bbbk}(H)$. The invertibility of simple objects (see Section 3.2) and Frobenius-Perron dimension have been the main methods in [56] to establish a criterion to get a tensor equivalence from a stable equivalence. These results to be explained in Chapter 3 tell us that adding tensor structure attached to tensor triangulated categories may lead to a new version of Morita theory.

Proposition 1.0.1. (Proposition 3.2.3) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-semisimple finite tensor categories. Suppose $F : \mathcal{C} \to \mathcal{C}'$ is an exact $\Bbbk$-linear monoidal functor inducing a stable equivalence $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{C}'}$.*

If all simple objects in $\mathcal{C}$ and $\mathcal{C}'$ are invertible, then F is a tensor equivalence.

For Hopf algebras, the corresponding result is:

Corollary 1.0.2. (Corollary 3.2.4) *Let H and H' be finite-dimensional non-semisimple Hopf algebras. Suppose $F : H\text{-}\mathsf{mod} \to H'\text{-}\mathsf{mod}$ is an exact $\Bbbk$-linear monoidal functor inducing a stable equivalence $\underline{F} : H\text{-}\underline{\mathsf{mod}} \to H'\text{-}\underline{\mathsf{mod}}$.*

If H and H' are basic, then H and H' are gauge equivalent.

As an invariant of tensor equivalences, Frobenius-Perron dimension (say FPdim for short) can determine the equivalence between two tensor categories to some extend (see [18]). Thus by putting conditions on FPdim we can also recover the tensor equivalence from a stable tensor equivalence.

Theorem 1.0.1. (Theorem 3.2.12) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-semisimple finite tensor categories having no projective simple objects. Suppose $F : \mathcal{C} \to \mathcal{C}'$ is an exact $\Bbbk$-linear monoidal functor inducing a stable equivalence $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{C}'}$.*

If $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}')$, then F is a tensor equivalence.

For Hopf algebras, the corresponding result is:

Corollary 1.0.3. (Corollary 3.2.13) *Let H and H' be finite-dimensional non-semisimple Hopf algebras having no simple projective modules. Suppose $F : H\text{-}\mathsf{mod} \to H'\text{-}\mathsf{mod}$ is an exact $\Bbbk$-linear monoidal functor inducing a stable equivalence $\underline{F} : H\text{-}\underline{\mathsf{mod}} \to H'\text{-}\underline{\mathsf{mod}}$.*

If $\dim_{\Bbbk}(H) = \dim_{\Bbbk}(H')$, then H and H' are gauge equivalent.

Morita theory for derived categories of algebras has developed well. A finite-dimensional algebra derived equivalent to a self-injective algebra is itself self-injective [55]. However, derived equivalence does not preserve the Hopf structure, that is a finite-dimensional algebra derived equivalent to a Hopf algebra need not be a Hopf algebra. For example, a field k (Hopf algebra) and the n×n matrix over k (not Hopf algebra since there is no counit) are derived equivalent. This means that tensor structures will give another version of derived Morita theory. It is known that a bounded derived category of a finite tensor category also inherits a tensor structure (see Lemma 2.3.9). In particular, $D^b(H\text{-}\mathsf{mod})$ becomes a tensor triangulated category. However, if H is not semisimple, then $K^b(\mathcal{P}_H)$ is not a monoidal category since it does not contain identity.

Problem *2.* If $D^b(H\text{-}\mathsf{mod})$ is equivalent to $D^b(H'\text{-}\mathsf{mod})$ as a tensor triangulated category, then what are the relations between H and H'?

Reconstruction results were already obtained by Bondal and Orlov in different context [11]. More precisely, when considering a smooth algebraic variety V with ample either canonical or anticanonical sheaf, the variety V is uniquely determined by its derived category of coherent sheaves. Furthermore, Balmer showed that the derived category of coherent sheaves on a smooth variety, when considered as a monoidal category in addition to its triangulated category structure (i.e. as a tensor triangulated category), completely determines the variety uniquely [5]. As a corollary, a tensor triangulated equivalence between the derived categories of the perfect complexes over two reduced noetherian schemes induces an isomorphism between these two schemes.

As I mentioned before, in [59] Zhang and Zhou defined mtt-structures (latter on, we will give a different version in Definition 4.1.6) on a tensor triangulated category. They observed that under certain strong assumptions, the heart of an mtt-structure is a tensor category. Then they solved a version of *Problem 2* under some conditions in the case of hereditary weak bialgebras. We redefined monoidal t-structures, and it happens that all the equivalent monoidal t-structures in a bounded derived category with 0 contained in the set of deviation (see Section 4.1) are equal. Based on our latest results, it is possible to reconstruct a finite-dimensional Hopf algebra from its derived tensor category by using the monoidal t-structure in Chapter 4.

Theorem 1.0.2. (Theorem 4.2.3) *Let $\mathcal{C}$ and $\mathcal{C}'$ be finite tensor categories. $D^b(\mathcal{C})$ is equivalent to $D^b(\mathcal{C}')$ as a tensor triangulated category if only if $\mathcal{C}$ and $\mathcal{C}'$ are tensor equivalent.*

For Hopf algebras, the corresponding result is:

Corollary 1.0.3. (Corollary 4.2.4) *Let H and H' be finite-dimensional Hopf algebras. $D^b(H\text{-}\mathsf{mod})$ is equivalent to $D^b(H'\text{-}\mathsf{mod})$ as a tensor triangulated category if only if H and H' are gauge equivalent.*

The stable categories and derived categories are not completely independent. For a finite-dimensional self-injective k-algebra A, there is a triangle equivalence [47]:

$$F : D^b(A\text{-mod})/K^b(\mathcal{P}_A) \to A\text{-}\underline{\mathsf{mod}}$$

where $K^b(\mathcal{P}_A)$ is the homotopy category of bounded complexes over $\mathcal{P}_A$ the full subcategory of finite generated projective A-modules. In addition, if two finite-dimensional self-injective algebras are derived equivalent then they are stably equivalent [47]. As a corollary of Theorem 1.0.2, we get the following result:

Theorem 1.0.4. (Theorem 4.2.10) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-semisimple finite tensor categories. If $D^b(\mathcal{C}) \simeq D^b(\mathcal{C}')$ as tensor triangulated categories, then*

$$\underline{\mathcal{C}} \simeq \underline{\mathcal{C}'}$$

as tensor triangulated categories.

For Hopf algebras, the corresponding result is:

Corollary 1.0.5. (Corollary 4.2.11) *Let H and H' are two non-semisimple Hopf algebras. If $D^b(H\text{-mod}) \simeq D^b(H'\text{-mod})$ as tensor triangulated categories, then*

$$H\text{-}\underline{\mathsf{mod}} \simeq H'\text{-}\underline{mod}$$

as tensor triangulated categories.

***Problem** 3.* How closely is *Problem 1* related to *Problem 2* ?

In [7], Balmer defines the concept of thick tensor ideal. Given a tensor triangulated category $\mathcal{C}$ and a thick tensor ideal $\mathcal{I}$, one can deduce that the Verdier quotient $\mathcal{C}/\mathcal{I}$ is still a tensor triangulated category. It is direct to check that the essential image of the natural embedding $K^b(\mathcal{P}_H) \to D^b(H\text{-mod})$ is a thick tensor ideal, which means the quotient category $D^b(H\text{-mod})/K^b(\mathcal{P}_H)$ is still a tensor triangulated category. Then a triangle equivalent functor F above is indeed a monoidal functor in the case of Hopf algebras, that is, $D^b(H\text{-mod})/K^b(\mathcal{P}_H)$ is equivalent to $H\text{-}\underline{\mathsf{mod}}$ as tensor triangulated categories. From this perspective we can also obtain Corollary 4.2.11.

In order to better understand the tensor triangulated equivalences, I am also concerned about the issue of invariants. The tensor structure of a tensor triangulated category equips Grothendieck

(Green) groups with ring structures. Derived (stable) Grothendieck (Green) rings are invariants of derived (stable) tensor equivalences between finite tensor categories. I have started to compute some examples, such as Taft algebras. Work on the following problem is ongoing but not included in this dissertation.

***Problem** 4.* Are there computable invariants of a tensor triangulated equivalence?

Organization of the dissertation

This dissertation is built up as follows:

In Chapter 1, motivations, main problems and results are stated.

In Chapter 2, the elementary knowledge about tensor triangulated categories will be introduced. The stable categories of finite tensor categories and the derived categories of finite tensor categories, as two main examples of tensor triangulated categories, come into our picture accompanied by some rich compatibility between tensor functors and translation functors.

Chapter 3 is devoted to the relations between finite tensor categories and the corresponding stable tensor categories. We will see a stable equivalence between two finite tensor categories can induce a correspondence between simple objects. The main results demonstrate that a stable equivalence induced by an exact k-linear monoidal functor can be lifted to a tensor equivalence by utilizing the invertibility of simple objects and the restriction of Frobenius-Perron dimensions.

Chapter 4 presents the definition of monoidal t-structure $\mathbb{t}$ and the deviation of $\mathbb{t}$. After that, we notice that all the equivalent monoidal t-structures on a tensor triangulated category are the same. This leads to our reconstruction theory, spelled out: Bounded derived categories of two finite tensor categories $\mathcal{C}$ and $\mathcal{C}'$ are equivalent as tensor triangulated categories if only if $\mathcal{C}$ and $\mathcal{C}'$ are tensor equivalent. As a corollary, stable tensor equivalences can be realized from derived tensor equivalences between finite tensor categories.

In Chapter 5, we will be concerned with a concrete example of Hopf algebras called Taft alegbras $H_n(q)$. Our purpose is to compute all the indecomposable complexes in $D^b(H_n(q))$, and then describe the derived Green ring in the case of Sweedler's 4-dimensional Hopf algebra.

Throughout this dissertation, $\mathbb{k}$ is assumed to be an algebraically closed field. All vector spaces, algebras, coalgebras, and Hopf algebras are over $\mathbb{k}$. For any $\mathbb{k}$-algebra, the category of finitely generated modules over A is denoted by $A\text{-}\mathsf{mod}$. All the categories we consider are $\Bbbk$-linear essentially small categories.

Chapter 2 On tensor triangulated categories

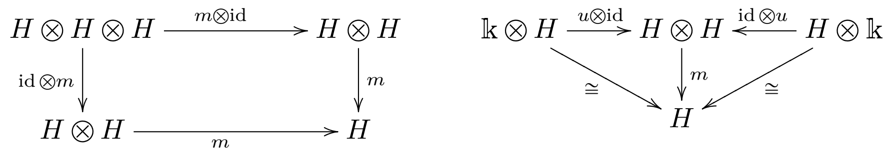
Triangulated categories were introduced by Verdier [57]. These become tensor triangulated categories by adding tensor structures. In literature, slightly different definitions of tensor triangulated categories are used. The authors considered symmetric tensor structures due to geometric reasons in [6, 8, 24, 29, 34, 40, 50]. May proposed further compatibility axioms between tensor and octahedra [34]. I emphasize that the definition of tensor triangulated categories in this dissertation will not include the symmetric assumption. However, we can still obtain the coherence between tensor structures and cosyzygy functors in the case of stable tensor categories, and natural isomorphisms providing compatibility between tensor structures and shift functors in the case of derived tensor categories.

In this chapter, I will firstly introduce two basic structures of tensor triangulated categories, tensor categories and triangulated categories. In Section 2.1, some preliminaries on Hopf algebras and tensor categories are presented (see [16, 18, 36] and the references given there). Section 2.2 is devoted to the study of triangulated categories. I refer the readers to [4, 22, 31, 58] for details. In Section 2.2, I also summarize without proofs the relevant material on an condition allowing to recover an equivalence between Frobenius categories from a stable equivalence. After that another main example called derived categories will be given. Then the definition of tensor triangulated categories will be introduced in Section 2.3, followed by giving concrete examples of tensor triangulated categories we are studying, and some compatibility conditions between tensor bifunctors and translation functors.

§2.1 Tensor categories

This section is a mild touching on tensor categories. The main focus will be on the categories of finitely generated modules over Hopf algebras, as an important example of tensor categories. Hence some basic knowledge about Hopf algebra will firstly come into our picture. Next, the definition of a tensor category is introduced, along with some important properties and results that we will use later.

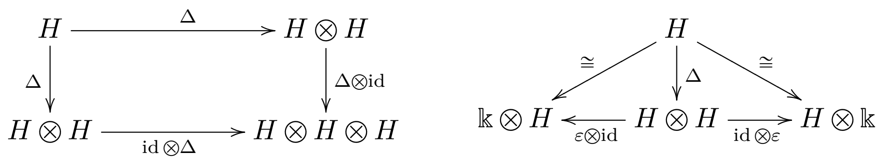
- **Algebra.** An algebra H is a triple (H, m, u), where H is a $\Bbbk$-vector space, and $m : H \otimes H \to H$, $u : \Bbbk \to H$ are linear maps, such that following diagrams both commute:



m and u are called the *multiplication* and the *unit*, respectively.

The definition of coalgebra is "dual" to the definition of algebra:

- **Coalgebra.** A *coalgebra* H is a triple (H, Δ, ε), where H is a $\mathbb{k}$-vector space, and $\Delta : H \to H \otimes H$, $\varepsilon : H \to \mathbb{k}$ are linear maps, such that the following diagrams both commute:



  Δ and ε are called the *comultiplication* and the *counit*, respectively.

- **Hopf algebra.** Suppose that (H, m, u) is a $\mathbb{k}$-algebra, and (H, Δ, ε) is a $\mathbb{k}$-coalgebra. H is said to be a *Hopf algebra* over $\mathbb{k}$, if

  (1) Δ and ε are both algebra maps (H is called a *bialgebra*);

  (2) There is a linear map $S : H \to H$ called the *antipode*, such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

  hold on H.

Some definitions in the case of Hopf algebras are also required. We use the symbols Δ, ε and S respectively, for the comultiplication, counit and antipode of a Hopf algebra H, and write $J = \sum_{i=1}^{r} J_i \otimes J^i$ which is expressed $J = J_i \otimes J^i$ using the Einstein summation convention for any element in $H \otimes H$.

A gauge transformation of a Hopf algebra $H = (H, m, u, \Delta, \varepsilon, S)$ is an invertible element $J = J_i \otimes J^i$ of $H \otimes H$ such that:

$$(J \otimes 1)((\Delta \otimes \mathrm{id})(J)) = (1 \otimes J)((\mathrm{id} \otimes \Delta)(J))$$

Define an algebra map $\Delta^J : H \to H \otimes H$ by

$$\Delta^J(h) = J\Delta(h)J^{-1}$$

and

$$S^J(h) = J_i S(J^i) S(h) S((J^{-1})_j)(J^{-1})^j$$

for each $h \in H$. Then $H^J = (H, m, u, \Delta^J, \varepsilon, S^J)$ is also a Hopf algebra [16]. Two Hopf algebras H and H' are said to be guage equivalent, if there is a gauge transformation J of H, such that H' and H^J are isomorphic as Hopf algebras [16].

As we can see, Hopf algebras carry coalgebraic structures which gives their module categories richer structures than abelian categories. More precisely, they admit tensor structures.

Definition 2.1.1. ([18, Definition 2.2.8]) A *monoidal category* is a sextuple $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ where

- $\mathcal{C}$ is a category with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product bifunctor*;
- $a_{-,-,-} : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ is a natural isomorphism called *associativity constraint*;
- $\mathbf{1}$ is an object of $\mathcal{C}$ with two natural isomorphisms:

  $$l_- : \mathbf{1} \otimes - \to - \quad \text{and} \quad r_- : - \otimes \mathbf{1} \to -$$

  called *left and right unit constraints* respectively;

subject to the following two axioms:

(1) pentagon axiom

$$\begin{array}{ccccc} & & ((W \otimes X) \otimes Y) \otimes Z & & \\ & \swarrow^{a_{W,X,Y} \otimes \mathrm{id}_Z} & & \searrow^{a_{W \otimes X, Y, Z}} & \\ (W \otimes (X \otimes Y)) \otimes Z & & & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow{\scriptstyle a_{W, X \otimes Y, Z}} & & & & \downarrow{\scriptstyle a_{W,X,Y \otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & & \xrightarrow{\mathrm{id}_W \otimes a_{X,Y,Z}} & & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

(2) the triangle axiom

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

is commutative for all $X, Y \in \mathcal{C}$.

Definition 2.1.2. ([18, Definition 2.4.1]) Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ and $(\mathcal{C}', \otimes', \mathbf{1}', a', l', r')$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J), where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, and

$$J_{-,-} : F(-) \otimes' F(-) \xrightarrow{\sim} F(- \otimes -)$$

is a natural isomorphism, such that $F(\mathbf{1}) \cong \mathbf{1}'$ and the following diagram is commutative for all $X, Y, Z \in \mathcal{C}$ (the monoidal structure axiom)

$$\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\downarrow{\scriptstyle J_{X,Y} \otimes' \mathrm{id}_{F(Z)}} & & \downarrow{\scriptstyle \mathrm{id}_{F(X)} \otimes' J_{Y,Z}} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
\downarrow{\scriptstyle J_{X \otimes Y, Z}} & & \downarrow{\scriptstyle J_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(a_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z))
\end{array}$$

A monoidal functor is said to be an *equivalence of monoidal categories* if it is an equivalence of ordinary categories.

Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category, and let X be an object of C. An object X^* in $\mathcal{C}$ is said to be a *left dual* of X if there exist morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$, called the *evaluation* and *coevaluation*, such that the following compositions are the identity morphisms.

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$
$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

Likewise, an object ${}^*X$ in $\mathcal{C}$ is said to be a *right dual* of X if there are morphisms $\mathrm{ev}'_X : X \otimes {}^*X \to \mathbf{1}$ and $\mathrm{coev}'_X : \mathbf{1} \to {}^*X \otimes X$ such that the following compositions are the identity morphisms.

$$X \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{id}_X} X$$
$${}^*X \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{id}_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathrm{id}_{{}^*X} \otimes \mathrm{ev}'_X} {}^*X$$

An object in a monoidal category is called *rigid* if it has left and right dual. A monoidal category is called *rigid* if every object of $\mathcal{C}$ is rigid.

Remark 2.1.3. (1) When a left (resp. right) dual of $X \in \mathcal{C}$ exists, then the functor $X^* \otimes -$ is the left adjoint of $X \otimes -$ (resp. ${}^*X \otimes -$ is the right adjoint of $X \otimes -$) ([18, Proposition 2.10.8]).

(2) If $X \in \mathcal{C}$ has a left (resp. right) dual object, then it is unique up to isomorphism ([18, Proposition 2.10.5]).

Definition 2.1.4. ([18, Definition 1.8.5 and Definition 1.8.6]) A $\Bbbk$-linear abelian category $\mathcal{A}$ is said to be *finite* if the following two conditions are satisfied:

(1) the $\Bbbk$-linear space $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite dimensional for any two objects X, Y in $\mathcal{A}$;

(2) every object in $\mathcal{A}$ has finite length;

(3) $\mathcal{A}$ has enough projectives; and

(4) there are only finitely many isomorphism classes of simple objects.

If $\mathcal{A}$ only satisfies the first two conditions, we call it *locally finite.*

Equivalently, a $\Bbbk$-linear abelian category $\mathcal{A}$ is said to be *finite* if it is equivalent to A-mod over a finite dimensional $\Bbbk$-algebra.

Definition 2.1.5. ([18, Definition 4.1.1]) Let $\mathcal{C}$ be a locally finite $\Bbbk$-linear abelian rigid monoidal category. We call $\mathcal{C}$ a *tensor category* over $\Bbbk$ if the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear on morphisms and $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \cong \Bbbk$.

Lemma 2.1.6. ([18, Propositin 4.2.1]) *The bifunctor in a tensor category is biexact.*

Example 2.1.7. Considering a finite-dimensional Hopf algebra H over $\Bbbk$, H-mod is a monoidal categoy with $\otimes_{\Bbbk}$ being the tensor product of H-modules over $\Bbbk$ and the unit object $\Bbbk$. Moreover, for any H-module X, antipode S and S^{-1} define two different actions of H on the $\Bbbk$-linear dual space X^* making H-modules X^* into the left and right dual of X respectively. To sum up, $(H\text{-}\mathsf{mod}, \otimes_{\Bbbk}, \Bbbk)$ is a finite tensor category.

An exact and faithful $\Bbbk$-linear functor between two tensor categories over $\Bbbk$ is called a *tensor functor* if it is a monoidal functor. Recall that a *tensor equivalence* is a $\Bbbk$-linear monoidal equivalence. Gauge equivalences are closely related to tensor equivalences, which can be seen by the following lemma which is the dual form of [48, Corollary 5.9]. Here I state [39, Theorem 2.2] in the case of Hopf algebras.

Lemma 2.1.8. ([39, Theorem 2.2]) *Let H and H' be finite-dimensional Hopf algebras over $\Bbbk$. If H-mod and H'-mod are tensor equivalent, then H is gauge equivalent to H as Hopf algebras.*

Remark 2.1.9. Assume (F, ξ) is a tensor equivalence from H-mod to H'-mod in Lemma 2.1.8, where $\xi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ is a natural isomorphism for all X, $Y \in H$-mod. One can verify

$J := \xi_{H,H}^{-1}(\Bbbk \otimes \Bbbk)$ is invertible in $H \otimes H$, and then J is a gauge transformation of H. Hence H' is isomorphic to H^J as Hopf algebras.

The following basic properties about tensor categories are used in Section 2.3 to help us understand some concrete examples of tensor triangulated categories.

Lemma 2.1.10. ([19, Proposition 2.3]) *Any projective object in a tensor category is also injective, and vice versa.*

Lemma 2.1.11. ([27, Corollary 2, p.441]) *Let P be a projective object in a tensor category $\mathcal{C}$. Then $P \otimes X$ and $X \otimes P$ are both projective for any object $X \in \mathcal{C}$.*

Now Grothendieck (Green) rings are brought into the picture.

Let $\mathcal{C}$ be an abelian category over $\Bbbk$. We denote by $[X]$ the isomorphism classes of any object X in $\mathcal{C}$. Let K be the free abelian group generated by the isomorphism classes of objects in $\mathcal{C}$ and K_0 the subgroup generated by $[X] - [Y] + [Z]$ for all short exact sequences

$$0 \to X \to Y \to Z \to 0.$$

Then the *Grothendieck group* $\mathrm{Gr}(\mathcal{C})$ is defined to be the factor group K/K_0. Additionally, if we only consider K_0 generated by elements for all split short exact sequences, then the factor group K/K_0 is called *Green group* denoted by $\mathrm{G}_0(\mathcal{C})$.

Moreover, if $(\mathcal{C}, \otimes, \mathbf{1})$ is a tensor category. Then the tensor product on $\mathcal{C}$ induces a natural multiplication on $\mathrm{Gr}(\mathcal{C})$ (or $\mathrm{G}_0(\mathcal{C})$) defined by $[X][Y] := [X \otimes Y]$ which is associative ([18, Lemma 4.5.1]). Hence $\mathrm{Gr}(\mathcal{C})$ (or $\mathrm{G}_0(\mathcal{C})$) becomes a ring with unit $[\mathbf{1}]$ called a *Grothendieck ring* (*Green ring*).

Remark 2.1.12. For a tensor category $\mathcal{C}$, the Grothendieck (Green) ring $\mathrm{Gr}(\mathcal{C})$ ($\mathrm{G}_0(\mathcal{C})$) possesses a $\mathbb{Z}$-basis given by isomorphism classes of simple (indecomposable) objects in $\mathcal{C}$. Actually, $\mathrm{G}_0(\mathcal{C})$ is a quotient ring of $\mathrm{Gr}(\mathcal{C})$.

Remark 2.1.13. ([18, Remark 4.5.6]) Let $\mathcal{C}$ and $\mathcal{C}'$ be tensor categories and let $F : \mathcal{C} \to \mathcal{C}'$ be a tensor functor. Then F defines a homomorphism of rings $[F] : \mathrm{Gr}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}')$ ($[F] : \mathrm{G}_0(\mathcal{C}) \to \mathrm{G}_0(\mathcal{C}')$)). If moreover, F is an equivalence, then $[F]$ is an isomorphism.

§2.2 Triangulated categories

§2.2.1 Definitions

Let $\mathcal{C}$ be an additive category and $T : \mathcal{C} \to \mathcal{C}$ an automorphism called *translation functor*. A sextuple (X, Y, Z, u, v, w) in $\mathcal{C}$ is $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ where $X, Y, Z \in \mathcal{C}$ and u, v, w are morphisms in $\mathcal{C}$.

A *morphism of sextuples* from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', u') is a triple (f, g, h) of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle Tf}\downarrow \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX' \end{array}$$

We call this morphism an isomorphism if f, g, h are isomorphisms in $\mathcal{C}$.

A set $\mathcal{E}$ of sextuples in $\mathcal{C}$ is call *triangulation* of $\mathcal{C}$ if the following conditions are satisfied. The element of $\mathcal{E}$ are then called *triangles*.

- (TR1)
  - (i) Every sextuple isomorphic to a triangle is a triangle.
  - (ii) Every morphism $u : X \to Y$ in $\mathcal{C}$ can be embedded into a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$.
  - (iii) For any object $X \in \mathcal{C}$, $X \xrightarrow{\mathrm{id}_X} X \to 0 \to TX$ is a triangle.
- (TR2) If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is a triangle, then so is $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{-Tu} TY$.
- (TR3) Given two triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') and the following commutative diagram:

  $$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ X' & \xrightarrow{u'} & Y' \end{array}$$

  there exists a morphism (f, g, h) from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w').
- (TR4) (The octahedral axiom) Suppose that $X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{i'} TX$, $Y \xrightarrow{v} Z \xrightarrow{j} X' \xrightarrow{j'} TY$ and $X \xrightarrow{vu} Z \xrightarrow{k} Y' \xrightarrow{k'} TX$ are triangles in $\mathcal{E}$. Then there exists $f : Z' \to Y'$ and $g : Y' \to X'$ such

that $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{(Ti)j'} TZ'$ is also a triangle in $\mathcal{E}$ and the following diagram is commutative.

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & TX \\
\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f} & & \| \\
X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & TX \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Tu} \\
 & & X' & = & X' & \xrightarrow{j'} & TY \\
 & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle (Ti)j'} & & \\
 & & TY & \xrightarrow{Ti} & TZ' & &
\end{array}$$

Definition 2.2.1. ([22, Chapter I]) An additive category $\mathcal{C}$ together with a translation functor T and a triangulation $\mathcal{E}$ is called a *triangulated category.* We then call every triangle in $\mathcal{E}$ *a distinguished triangle.*

Definition 2.2.2. ([58, Definition 1.4.3]) A full subcategory $\mathcal{D}$ of a triangulated category $\mathcal{C} = (\mathcal{C}, T, \mathcal{E})$ is called a *triangulated subcategory* if the following conditions are satisfied:

(1) $\mathcal{D}$ is closed under isomorphisms;

(2) T restricts to an automorphism of $\mathcal{D}$;

(3) $\mathcal{D}$ is closed under extension. (i.e. if $X \to Y \to Z \to TX \in \mathcal{E}$ and $X, Z \in \mathcal{D}$, then $Y \in \mathcal{D}$.)

Let $(\mathcal{C}, T, \mathcal{E})$ and $(\mathcal{C}', T', \mathcal{E}')$ be triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called *exact* if there exists a natural isomorphism $\varphi : FT \to T'F$ such that

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\varphi_{F(X)} \circ F(w)} T'F(X)$$

is in $\mathcal{E}$ whenever $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is in $\mathcal{E}$.

If an exact functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories, we call F a *triangle equivalence.* $\mathcal{C}$ and $\mathcal{C}'$ are then called *triangle equivalent.*

§2.2.2 Examples of triangulated categories

In this subsection, two commonly used triangulated categories will be introduced: stable category and derived category. Readers can find relevant definitions and results in [4, 22, 31, 58].

- **Stable category**

The first example of trianguated categories is coming from the stable categories of special categories called Frobenius categories. Before working with this kind of triangulated category, I should first present some of the definitions involved.

Let $\mathcal{B}$ be an additive category embedded as a full and extension closed subcategory in some abelian category $\mathcal{A}$. Following Quillen [42, Chapter 2] the pair $(\mathcal{B}, \mathcal{S})$ is called an *exact category* where $\mathcal{S}$ is the set of exact sequences in $\mathcal{A}$ with terms in $\mathcal{B}$. We call a morphism $u : X \to Y$ in $\mathcal{B}$ a *proper monomorphism* if there is an exact sequence $0 \xrightarrow{u} Y \to Z \to 0$ in $\mathcal{S}$. Similarly, a morphism $v : Y \to Z$ in $\mathcal{B}$ is *proper epimorphism* if there is an exact sequence $0 \to X \to Y \xrightarrow{v} Z \to 0$ in $\mathcal{S}$.

An object $P \in \mathcal{B}$ is called *$\mathcal{S}$-projective* if for any proper epimorphism $v : Y \to Z$ and morphism $f : P \to Z$ in $\mathcal{B}$ there exists $g : P \to Y$ such that $f = v \circ g$. Likewise, an object $I \in \mathcal{B}$ is called *$\mathcal{S}$-injective* if for any proper monomorphism $u : X \to Y$ and morphism $f : X \to I$ in $\mathcal{B}$ there exists $g : Y \to I$ such that $f = g \circ u$.

An exact category $(\mathcal{B}, \mathcal{S})$ is called a *Frobenius category* if $(\mathcal{B}, \mathcal{S})$ has enough $\mathcal{S}$-projective objects and enough $\mathcal{S}$-injective objects, and an object is $\mathcal{S}$-projective if and only if it is $\mathcal{S}$-injective [23, p.386]. Note that here we say that $(\mathcal{B}, \mathcal{S})$ has *enough $\mathcal{S}$-projective objects* means for any $Y \in \mathcal{B}$ there exists a proper epimorphism $v : P \to Y$ with P an $\mathcal{S}$-projective in $\mathcal{B}$. In parallel, we say that $(\mathcal{B}, \mathcal{S})$ has *enough $\mathcal{S}$-injective objects* if for any $X \in \mathcal{B}$ there exists a proper monomorphism $u : X \to I$ with I an $\mathcal{S}$-injective in $\mathcal{B}$.

Example 2.2.3. A finite tensor category is a Frobenius category by Lemma 2.1.10. Thus for a finite-dimensional Hopf algebra H, H-mod is also a Frobenius category.

Recall that an artin algebra A is said to be *self-injective* if it is injective as an A-module. A-mod is a Frobenius category where A is a self-injective algebra.

Let $\mathcal{B}$ be a Frobenius category. The *stable category* of $\mathcal{B}$ written as $\underline{\mathcal{B}}$ is defined as follows: The objects of $\underline{\mathcal{B}}$ are the same as those of $\mathcal{B}$; For any objects $X, Y \in \underline{\mathcal{B}}$, the morphisms from X to Y are given by the quotient space

$$\underline{\mathrm{Hom}}_{\mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{B}}(X, Y)/\mathcal{I}(X, Y),$$

where $\mathcal{I}(X, Y)$ is the subspace of $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ consisting of homomorphisms which factor through an injective object. We say two Frobenius categories $\mathcal{B}$ and $\mathcal{B}'$ are *stably equivalent*, if $\underline{\mathcal{B}}$ and $\underline{\mathcal{B}'}$ are $\Bbbk$-linear equivalent.

For simplicity of presentations, we stipulate the following notations.

Notation 2.2.1. We use A-$\underline{\mathsf{mod}}$ to denote the stable category of A-mod where A is any self-injective algebra. Talking about any stable categories $\underline{\mathcal{B}}$ of a Frobenius category $\mathcal{B}$, the following notations are always used:

- For $X, Y \in \underline{\mathcal{B}}$, let $\underline{f}$ denote the morphism in the quotient space $\underline{\mathrm{Hom}}_{\mathcal{B}}(X, Y)$ represented by $f \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$. We use the diagram below to indicate $\underline{f} = 0$:

$$f : X \xrightarrow{i} I \xrightarrow{j} Y,$$

where $f = j \circ i$ in $\mathrm{Hom}_{\mathcal{B}}(X, Y)$ and I is an injective object in $\mathcal{B}$.

- Given a $\Bbbk$-linear functor $F : \mathcal{B} \to \mathcal{B}'$, if F transforms injective objects to injective objects, then it induces a functor from $\underline{\mathcal{B}}$ to $\underline{\mathcal{B}'}$:

$$\underline{F} : \underline{\mathcal{B}} \to \underline{\mathcal{B}'}, \quad X \mapsto F(X), \quad \underline{f} \mapsto \underline{F(f)},$$

where $X \in \underline{\mathcal{B}}$ and f is a morphism in $\underline{\mathcal{B}}$.

Lemma 2.2.2. ([22, p.11]) *Assume that $(\mathcal{B}, \mathcal{S})$ is a Frobenius category and $X \in \mathcal{B}$. If*

$$0 \to X \xrightarrow{m_X} I \xrightarrow{p_X} \mathrm{Coker}(m_X) \to 0 \quad \textit{and} \quad 0 \to X \xrightarrow{m'_X} I' \xrightarrow{p'_X} \mathrm{Coker}(m) \to 0$$

are in $\mathcal{S}$ such that I, I' are $\mathcal{S}$-injective, then $\mathrm{Coker}(m_X) \cong \mathrm{Coker}(m'_X)$ *in $\underline{\mathcal{B}}$.*

According to [4, p.125], we can define a *cosyzygy functor* $\Omega^{-1} : \underline{\mathcal{B}} \to \underline{\mathcal{B}}$ as follows. For any objects $X \in \underline{\mathcal{B}}$ choose a fixed exact sequence in $\mathcal{S}$: $0 \to X \xrightarrow{m_X} I(X) \xrightarrow{p_X} \mathrm{Coker}(m_X) \to 0$ and define $\Omega^{-1}(X) := \mathrm{Coker}(m_X)$. Based on Lemma 2.2.2, Ω^{-1} is well-defined on objects. For any $u : X \to Y$ in $\mathcal{B}$, there is a commutative diagram in $\mathcal{B}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{p_X} & \Omega^{-1}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle I(u)} & & \downarrow{\scriptstyle \Omega^{-1}(u)} & & \\
0 & \longrightarrow & Y & \xrightarrow{m_Y} & I(Y) & \xrightarrow{p_Y} & \Omega^{-1}(Y) & \longrightarrow & 0
\end{array}$$

where the existence of $I(u)$ is based on the definition of $\mathcal{S}$-injective $I(X)$, and $\Omega^{-1}(u)$ is given by the universal property of cokernel. It is not difficult to see $\underline{\Omega^{-1}(u)}$ is independent of the choice of $I(u)$ in

$\underline{\mathcal{B}}$. So we can define $\Omega^{-1}(\underline{u}) := \underline{\Omega^{-1}(u)}$. A result in [22, p.13] tells us that the cosyzygy functor Ω^{-1} is an automorphism of $\underline{\mathcal{B}}$.

Let $(\mathcal{B}, \mathcal{S})$ as a subcategory in some abelian category $\mathcal{A}$ be a Frobenius category and $\underline{\mathcal{B}}$ the stable category. Suppose that $X, Y \in \mathcal{B}$ and $u \in \mathrm{Hom}_{\mathcal{B}}(X, Y)$. We consider the following commutative diagram in $\mathcal{B}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{m_X} & I(X) & \xrightarrow{p_X} & \Omega^{-1}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle i_u} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{v} & C_u & \xrightarrow{w} & \Omega^{-1}(X) & \longrightarrow & 0 \end{array}$$

where $I(X)$ is $\mathcal{S}$-injective, and C_u is the pushout of u and m_X. Since $\mathcal{B}$ is closed under extension in $\mathcal{A}$, the pushout $C_u \in \mathcal{B}$ coincides with the pushout in $\mathcal{A}$.

We call the sextuple $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} \Omega^{-1}(X)$ *a standard triangle* in $\underline{\mathcal{B}}$. Let $\mathcal{E}$ be the class of all the sextuples which are isomorphic to standard triangles in $\underline{\mathcal{B}}$.

Theorem 2.2.4. ([22, Chapter 2, Theorem 2.6]) *Let $(\mathcal{B}, \mathcal{S})$ be a Frobenius category. Then $(\underline{\mathcal{B}}, \Omega^{-1}, \mathcal{E})$ is a triangulated category where Ω^{-1} is the translation functor and $\mathcal{E}$ is the triangulation.*

A great deal of mathematical effort in the representation theory of algebras has been devoted to the study of self-injective algebras. The following proposition tells us when a stable equivalence can be lifted to a Morita equivalence.

Lemma 2.2.5. ([32, Proposition 2.5]) *Let A and A' be self-injective $\Bbbk$-algebras having no projective simple modules and $F : A\text{-}\mathsf{mod} \to A'\text{-}\mathsf{mod}$ be an exact functor. Suppose F induces a stable equivalence $\underline{F} : A\text{-}\underline{\mathsf{mod}} \to A'\text{-}\underline{\mathsf{mod}}$. Then F is an equivalence if and only if F maps any simple A-module to a simple A'-module.*

This result will prove extremely useful in Section 3.2 to help us get first two main theorems. It is direct to give the category version of Lemma 2.2.5.

Lemma 2.2.6. *Let $\mathcal{B}$ and $\mathcal{B}'$ be Frobenius categories having no projective simple objects and $F : \mathcal{B} \to \mathcal{B}'$ be an exact functor. Suppose F induces a stable equivalence $\underline{F} : \underline{\mathcal{B}} \to \underline{\mathcal{B}'}$. Then F is an equivalence if and only if F maps any simple object to a simple object.*

- **Homotopy category**

Suppose $\mathcal{A}$ is an abelian category. A *chain complex* of objects in $\mathcal{A}$ is a diagram

$$X = \cdots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \longrightarrow \cdots$$

where $X^n \in \mathcal{A}$ with maps $d_X^n : X^n \to X^{n+1}$ in $\mathcal{A}$ such that $d_X^{n+1} \circ d_X^n = 0$ for any $n \in \mathbb{Z}$. The map d_X^n is called a *differential* of X. The kernel of d_X^n is called a n-*cycle* of X, denoted by $Z^n = Z^n(X)$. The image of d_X^{n-1} is called a n-*boundary* of X, denoted by $B^n = B^n(X)$. The n-th *cohomology* of X is the subquotient $H^n(X) = Z^n/B^n$ of X^n, $n \in \mathbb{Z}$.

A *morphism of chain complexes* $f : X \to Y$ is a family of maps $f^n : X^n \to Y^n$ in $\mathcal{A}$ such that the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n & \longrightarrow & Y^{n+1} & \longrightarrow & \cdots
\end{array}$$

We denote by $C(\mathcal{A})$ *the category of chain complexes* in $\mathcal{A}$. The objects are chain complexes and morphisms are morphisms of chain complexes. The category of chain complexes $C(\mathcal{A})$ is again an abelian category. ([54, Theorem 1.2.3])

A complex X is *bounded* if there are only finitely many n such that $X^n \neq 0$; *upper-bounded* (resp. *lower-bounded*) if there is α (resp. β) such that $X^n = 0$ for all $n > \alpha$ (resp. $n < \beta$). The notations $C^b(\mathcal{A})$, $C^-(\mathcal{A})$, $C^+(\mathcal{A})$ are used to represent the full subcategories of $C(\mathcal{A})$ consisting of all bounded complexes, upper-bounded and lower-bounded complexes respectively.

Let f, $g : X \to Y$ be two morphisms of chain complexes in $C(\mathcal{A})$. A *homotopy* between f and g is a sequence $(s^n)_{n \in \mathbb{Z}}$ of morphisms $s^n : X^n \to Y^{n-1}$ such that for each n, there exists an equality:

$$f^n - g^n = d_Y^{n-1} \circ s^n + s^{n+1} \circ d_X^n.$$

In case there is such a homotopy, we call f and g are *homotopic* denoted by

$$s : f \sim g.$$

If $g = 0$, then f is called *null-homotopy* (i.e. homotopic to zero) denoted $f \sim 0$.

Let

$$\mathrm{Htp}(X, Y) := \{f : X \to Y \text{ is a morphism of complexes and } f \sim 0\}.$$

It is straightforward to see $\mathrm{Htp}(X, Y)$ is an additive subgroup of $\mathrm{Hom}_{C(\mathcal{A})}(X, Y)$.

Now we are ready to define a *homotopy category* $K(\mathcal{A})$, it is a category having the same objects as $C(\mathcal{A})$ with morphisms

$$\mathrm{Hom}_{K(\mathcal{A})}(X,Y) = \mathrm{Hom}_{C(\mathcal{A})}(X,Y)/\mathrm{Htp(X, Y)}.$$

Similar to category of complexes, we can also define a ull subcategory $K^+(\mathcal{A})$ of $K(\mathcal{A})$ called the *upper-bounded homotopy category* of $\mathcal{A}$, the *lower-bounded homotopy category* $K^-(\mathcal{A})$, and the *bounded homotopy category* $K^b(\mathcal{A})$.

Let us fix an abelian category $\mathcal{A}$, its category of complexes $C(\mathcal{A})$ and the homotopy category $K(\mathcal{A})$. The homotopy category $K(\mathcal{A})$ is an additive category, but in general not abelian any more. There is a fact saying that an abelian triangulated category is semisimple [21]. In the following, we will know $K(\mathcal{A})$ carries a triangular structure.

Let [1] be the *shift funtor* of $C(\mathcal{A})$, that is for $X \in C(\mathcal{A})$, $(X[1])^n = X^{n+1}$, $d^n_{X[1]} = -d^{n+1}_X$, $\forall n \in \mathbb{Z}$.

$$X = \cdots \longrightarrow X^n \xrightarrow{d^n_X} X^{n+1} \xrightarrow{d^{n+1}_X} X^{n+2} \longrightarrow \cdots$$

$$X[1] = \cdots \longrightarrow X^{n+1} \xrightarrow{-d^{n+1}_X} X^{n+2} \xrightarrow{-d^{n+2}_X} X^{n+3} \longrightarrow \cdots$$

For morphism $f : X \to Y$, $f[1] : X[1] \to Y[1]$ with $(f[1])^n = f^{n+1}$, $\forall n \in \mathbb{Z}$. In the same way, one can define shift functor $[-1]$ shifted right by one step.

Let X, Y be objects in $C(\mathcal{A})$ and $f : X \to Y$ a morphism. The *mapping cone* $\mathrm{Cone}(f)$ of f is the following complex in $C(\mathcal{A})$:

$$(\mathrm{Cone}(f))^n := X^{n+1} \bigoplus Y^n, \quad \forall n \in \mathbb{Z}$$

$$d^n_{\mathrm{Cone}(f)} := \begin{pmatrix} -d^{n+1}_X & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} : X^{n+1} \bigoplus Y^n \to X^{n+2} \bigoplus Y^{n+1}$$

Every mapping cone determines a sextuple which is a triangle:

$$X \xrightarrow{f} Y \xrightarrow{\binom{0}{1}} \mathrm{Cone}(f) \xrightarrow{(1\ 0)} X[1]$$

Let $\mathcal{E}$ be the class of all the sextuples which are isomorphic to the triangles constructed by mapping cone in $K(\mathcal{A})$.

Theorem 2.2.7. ([31, Theorem 2.3.1]) *Let $\mathcal{A}$ be an abelian category. Then $(K(\mathcal{A}), [1], \mathcal{E})$ is a triangulated category where* [1] *is the shift functor. Moreover* $[1] : K(\mathcal{A}) \to K(\mathcal{A})$ *is an automorphism with inverse* $[-1]$.

Remark 2.2.8. $K^b(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ are triangulated subcategories.

The following definition is introduced to define derived categories.

Definition 2.2.9. ([31, Definition 2.5.1]) Suppose $f : X \to Y$ is a morphism in $K(\mathcal{A})$ and f induces an isomorphism on cohomology, that is, $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism in $\mathcal{A}$ for any integer n. Then we call f a *quasi-isomorphism*. The objects X and Y are then called *quasi-isomorphic*.

- **Derived category**

Definition 2.2.10. ([58, Definition 3.1.1]) Let $\mathcal{A}$ be an additive category. A class of morphisms S in $\mathcal{A}$ is called a *multiplicative system* if it satisfies the following conditios: (the notation "$\Rightarrow$" means the morphism in S)

(S1) S is closed under composition and $\mathrm{id}_X \in S$ for any $X \in \mathcal{A}$.

(S2) For any diagram in $\mathcal{A}$ with $s \in S$,

$$\begin{array}{ccc} & & Z \\ & & \Downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array}$$

there is $t : W \to X$ in S and $g : W \to Z$ such that the following diagram is commutative.

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle t}\Downarrow & & \Downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array}$$

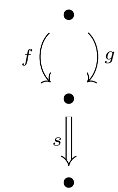
Dually, for any diagram in $\mathcal{A}$ with $s \in S$,

$$\begin{array}{ccc} & & Z \\ & & \Uparrow{\scriptstyle s} \\ X & \xleftarrow{f} & Y \end{array}$$

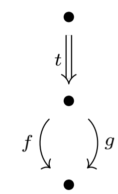
there is a commutative diagram with $t \in S$.

$$\begin{array}{ccc} W & \xleftarrow{g} & Z \\ {\scriptstyle t}\Uparrow & & \Uparrow{\scriptstyle s} \\ X & \xleftarrow{f} & Y \end{array}$$

(S3) These is $s \in S$ such that the following diagram is commutative,



that means $s \circ f = s \circ g$, if and only if there is $t \in S$ such that the following diagram commute,

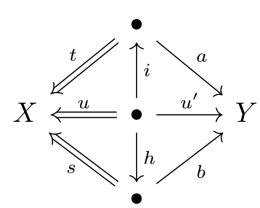


that is $f \circ t = g \circ t$.

Given an additive category $\mathcal{A}$ and a multiplicative system S. Let $X, Y \in \mathcal{A}$. A *right roof* (b, s) from X to Y is a diagram of morphisms below:

$$X \overset{s}{\Longleftarrow} \bullet \overset{b}{\longrightarrow} Y$$

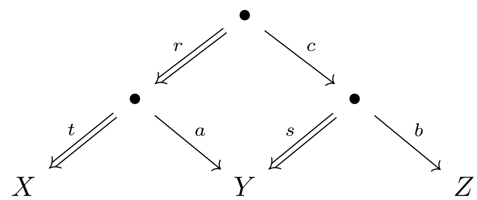
Two right roofs are called *equivalent* denoted by $(a, t) \sim (b, s)$ if there exists the following commutative diagram



where $u \in S$.

Lemma 2.2.11. ([58, Lemma 3.2.1]) *The relation between right roofs from X to Y is an equivalence relation.*

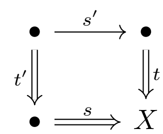
The equivalence class of a right roof (a, t) from X to Y is denoted $a \circ t^{-1}$. If $a \circ t^{-1}$ is an equivalence class of a right roof from X to Y and $b \circ s^{-1}$ is an equivalence class of a right roof from Y to Z, then we have the following diagram by (S2) in definition 2.2.10:



The composition is defined to be $(b \circ s^{-1}) \circ (a \circ t^{-1}) := (b \circ c) \circ (t \circ r)^{-1}$.

Definition 2.2.12. ([58, Definition 3.2.2]) Let S be a multiplicative system in an additive category $\mathcal{A}$. The localization of $\mathcal{A}$ at S is the category $S^{-1}\mathcal{A}$, whose objects are the same as in $\mathcal{A}$; for objects $X, Y \in S^{-1}\mathcal{A}$, $\mathrm{Hom}_{S^{-1}\mathcal{A}}(X, Y)$ are the set of all equivalence classes of right roofs from X to Y.

Moreover, given two morphisms $b \circ s^{-1}$ and $a \circ t^{-1}$ in $\mathrm{Hom}_{S^{-1}\mathcal{A}}(X, Y)$, we also the following commutative diagram,



which means we can find $r \in S$ such that $a \circ t^{-1} = a' \circ r^{-1}$, $b \circ s^{-1} = b' \circ r^{-1}$. Namely, let $r = t \circ s' = s \circ t'$, $a' = a \circ s'$, and $b' = b \circ t'$.

Define the addition of two equivalence classes to be $a \circ t^{-1} + b \circ s^{-1} := (a' + b') \circ r^{-1}$ which is well-defined by using (S2) and (S3) [58].

Fact 2.2.3. Let $\alpha = a \circ s^{-1} \in \mathrm{Hom}_{S^{-1}\mathcal{A}}(X, Y)$ where $s \in S$. Then α is a zero morphism if and only if there is $t \in S$ such that $s \circ t \in S$ and $a \circ t$ is a zero morphism in $\mathcal{A}$.

A *localization functor* $F : \mathcal{A} \to S^{-1}\mathcal{A}$ is defined as follows: for any object $X \in \mathcal{A}$, $F(X) = X$; for any $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$, $F(f) = f \circ (\mathrm{id}_X)^{-1} \in \mathrm{Hom}_{S^{-1}\mathcal{A}}(X, Y)$.

Lemma 2.2.13. ([58, Lemma 3.2.6]) *Let S be a multiplicative system in an additive category $\mathcal{A}$. Then the quotient category $S^{-1}\mathcal{A}$ is an additive category and the localization functor F is an additive functor.*

Here comes a question: when does $S^{-1}\mathcal{A}$ become a triangulated category? Next we will see that for a special class of multiplicative systems, the quotient category $S^{-1}\mathcal{A}$ becomes a triangulated category.

Definition 2.2.14. ([58, Definition 3.5.1]) A triangulated subcategory $\mathcal{D}$ of a triangulated category $(\mathcal{C}, T, \mathcal{E})$ is called a *thick subcategory* if it satisfies the following condition: If $X \xrightarrow{f} Y \to Z \to TX \in \mathcal{E}$ with $Z \in \mathcal{D}$ and f can factor through an object $W \in \mathcal{D}$, then $X \in \mathcal{D}$, $Y \in \mathcal{D}$.

Let $\mathcal{D}$ be a thick subcategory of a triangulated category $(\mathcal{C}, T, \mathcal{E})$. Then

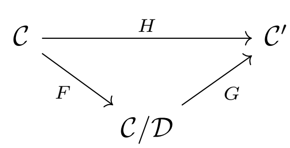
$$S := \phi(\mathcal{D}) = \{f : X \to Y \mid \text{there is } \ X \xrightarrow{f} Y \to Z \to TX \in \mathcal{E} \text{ such that } \ Z \in \mathcal{D}\}$$

is a multiplicative system and $\mathcal{D} = \ker F$ where $F : \mathcal{C} \to S^{-1}\mathcal{C}$ is the localization functor (see [58, Lemma 3.5.5]). Under this situation, the *Verdier quotient* of a thick subcategory $\mathcal{D}$ with respect to $\mathcal{C}$ is defined to be $\mathcal{C}/\mathcal{D} := S^{-1}\mathcal{C}$ and the localization functor $F : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ is called *Verdier functor.*

Theorem 2.2.15. ([58, Theorem 3.4.2, Corollary 3.5.7]) *Let $\mathcal{D}$ be a thick subcategory of a triangulated category $(\mathcal{C}, T, \mathcal{E})$ with $S := \phi(\mathcal{D})$. Then*

(1) *$(\mathcal{C}/\mathcal{D}, T)$ is also a triangulated category.*

(2) *Suppose that $H : \mathcal{C} \to \mathcal{C}'$ is an exact functor between triangulated categories, and H maps any object in $\mathcal{D}$ into the zero object in $\mathcal{C}'$. Then there is an unique exact functor $G : \mathcal{C}/\mathcal{D} \to \mathcal{C}'$ such that the follwoing diagram commute.*



(3) *The Verdier functor $F : \mathcal{C} \to \mathcal{C}/\mathcal{D}$ is an exact functor and $\mathcal{D} = \ker F$. Moreover, $F(f)$ is an isomorphism if and only if $f \in S$, and any morphism $a \circ s^{-1} \in \mathcal{C}/\mathcal{D}$ is an isorphism if and only if $a \in S$.*

Last, by applying the above localization theory to homotopy categories, we will see that the subcategory $\mathcal{D}$ of objects with zero n^{th}-cohomology for $\forall n \in \mathbb{Z}$ in the homotopy category is exactly a thick subcategory. Then its corresponding $\phi(\mathcal{D})$ is a multiplicative system that satisfies Theorem 2.2.15.

Proposition 2.2.16. ([58, Proposition 5.1.1]) *Let $K^*(\mathcal{A})$ be a homotopy category of an abelian category $\mathcal{A}$ where $* \in \{\varnothing, b, -, +\}$ and $\mathcal{D} := \{X \in K^*(\mathcal{A}) \mid H^n(X) = 0,\ \forall n \in \mathbb{Z}\}$. Then $\mathcal{D}$ is a thick subcategory and $\phi(\mathcal{D})$ is the class of all quasi-isomorphism in $K^*(\mathcal{A})$.*

Let $K^*(\mathcal{A})$ be a homotopy category of an abelian category $\mathcal{A}$ where $* \in \{\varnothing, b, -, +\}$ and $\mathcal{D} = \{X \in K^*(\mathcal{A}) \mid H^n(X) = 0,\ \forall n \in \mathbb{Z}\}$. Define the (unbounded, bounded, upper-bounded, lower-bounded) *derived category* $D^*(\mathcal{A})$ to be the Verdier quotient $D^*(\mathcal{A}) := K^*(\mathcal{A})/\mathcal{D}$ where $* \in \{\varnothing, b, -, +\}$. Theorem 2.2.15 tells us that they are triangulated categories.

Let $\mathcal{P}$ be the full subcategory consisting of projrctive objects. A full subcategory $K^{-,b}(\mathcal{P})$ of $K(\mathcal{P})$ includes all the complexes over $\mathcal{P}$, which is upper-bounded with finitely many non-zero cohomologies. Additionally, if $\mathcal{A}$ has enough projective objects, then $D^b(\mathcal{A}) \cong K^{-,b}(\mathcal{P})$ as triangulated categories [58].

Lemma 2.2.17. ([26, Exercise 4.2, p.129]) *Let $f : P \to Q$ be a chain map of projective complexes in $K^{-,b}(\mathcal{P})$. Then f is a homotopy equivalence if and only if $H^n(f) : H^n(P) \to H^n(Q)$ is isomorphic for each $n \in \mathbb{Z}$.*

Example 2.2.18. Consider A-mod which is an abelian category for a finite-dimensional algebra A. The categories of chain complexes $C^*(A\text{-mod})$ where $* \in \{\varnothing, b, -, +\}$ contain all the chain complexes of A-modules such that all the differentials are homomorphisms of A- modules. Then we can get the corresponding homotopy categories $K^*(A\text{-mod})$ and derived categories $D^*(A\text{-mod})$ where $* \in \{\varnothing, b, -, +\}$.

§2.3 Tensor triangulated categories

According to [37], a *tensor (monoidal) triangulated category* is a triangulated category $\mathcal{C}$ having a monoidal structure [33, Chapter VII]

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and a unit object $\mathbf{1} \in \mathcal{C}$, such that the bifunctor $- \otimes -$ is exact in each variable.

Two tensor triangulated categories $\mathcal{C}$ and $\mathcal{C}'$ are said to be *tensor triangulated equivalent* if there is a monoidal functor making $\mathcal{C}$ and $\mathcal{C}'$ be triangulated equivalent.

It is also available to define the Grothendieck group in a triangulated category $(\mathcal{C}, T, \mathcal{E})$. We denote by $[X]$ the isomorphism classes of any object X in $\mathcal{C}$. Let K be the free abelian group generated by the isomorphism classes of objects in $\mathcal{C}$ and K_0 the subgroup generated by $[X]-[Y]+[Z]$ for all distinguished triangles

$$X \to Y \to Z \to TX \in \mathcal{E}.$$

Then the *Grothendieck group* $\mathrm{Gr}(\mathcal{C})$ is defined to be the factor group K/K_0. Additionally, if we only consider K_0 generated by elements for all split distinguished triangles, then the factor group K/K_0 is called *Green group* denoted by $\mathrm{G}_0(\mathcal{C})$.

Furthermore, if now $(\mathcal{C}, \otimes, T)$ is a tensor triangulated category. Then the tensor product on $\mathcal{C}$ induces a natural multiplication on K defined by $[X][Y] := [X \otimes Y]$ and K_0 is an ideal as $- \otimes -$ is biexact in $\mathcal{C}$. Thus $\mathrm{Gr}(\mathcal{C})$ (or $\mathrm{G}_0(\mathcal{C})$) turns out to be a ring called a *Grothendieck ring* (*Green ring*) of a tensor triangulated category.

§2.3.1 Stable tensor categories

In retrospect, all finite tensor categories are Frobenius categories by Lemma 2.1.10. Meanwhile the stable categories of Frobenius categories are triangulated categories by Theorem 2.2.4. Our first destination is to show the following basic fact.

Lemma 2.3.1. *Let $\mathcal{C}$ be a finite tensor category. Then there are a natural isomorphism in $\underline{\mathcal{C}}$:*

$$\underline{e}_{X,Y} : \Omega^{-1}(X \otimes Y) \to \Omega^{-1}(X) \otimes Y, \quad (X, Y \in \underline{\mathcal{C}})$$

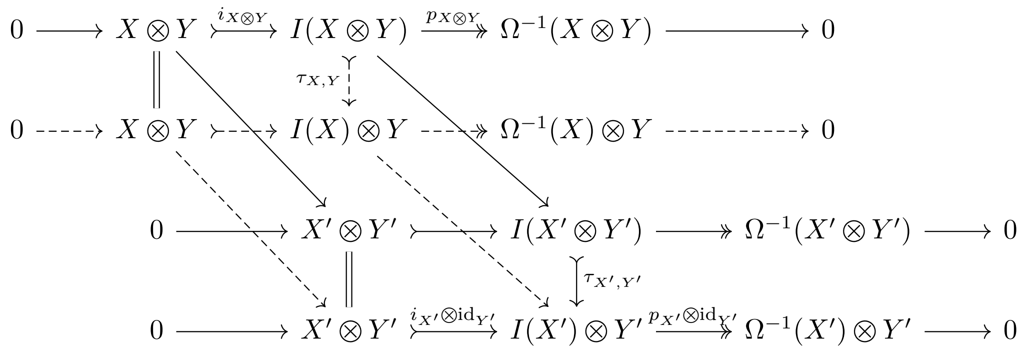
$$\underline{\theta}_{X,Y} : \Omega^{-1}(X \otimes Y) \to X \otimes \Omega^{-1}(Y), \quad (X, Y \in \underline{\mathcal{C}})$$

where Ω^{-1} is the cosyzygy functor.

Proof. We only prove the first natural isomorphism and will divide our proof in two steps. The first step is to establish a natural transformation:

$$e_{X,Y} : \Omega^{-1}(X \otimes Y) \to \Omega^{-1}(X) \otimes Y.$$

For any morphisms $f : X \to X'$ and $g : Y \to Y'$ in $\mathcal{C}$, we have the following commutative diagram:



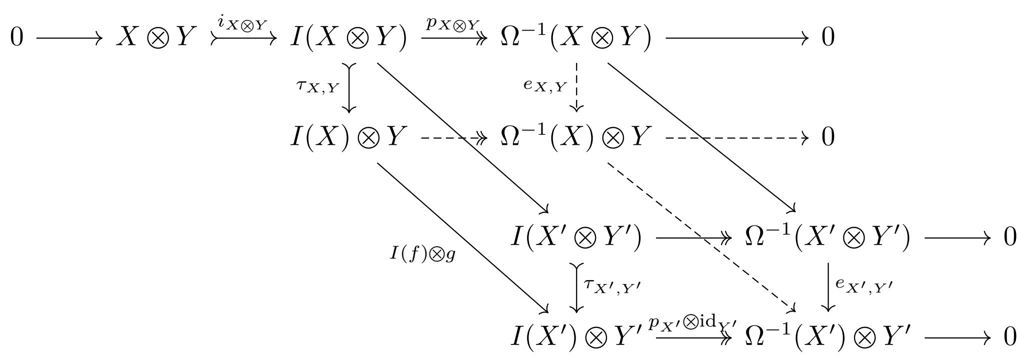
where $I(X)$ stands for the injective hull of any object X and all the four rows are exact sequences. The monomorphisms $\tau_{X',Y'}, \tau_{X,Y}$ are given by the universal property of injective hull such that the front and back squares commute. Furthermore, note that $I(X' \otimes Y')$, $I(X') \otimes Y'$ are injective, there exist morphisms

$$I(f \otimes g) : I(X \otimes Y) \to I(X' \otimes Y'),$$

$$I(f) \otimes g : I(X) \otimes Y \to I(X') \otimes Y'$$

making the top and right squares commute. The commutativity of the bottom square is routine to verify.

Next, we complete the above diagram as follows:



By the universal property of cokernel, the morphisms

$$e_{X,Y} : \Omega^{-1}(X \otimes Y) \to \Omega^{-1}(X) \otimes Y,$$

$$e_{X',Y'} : \Omega^{-1}(X' \otimes Y') \to \Omega^{-1}(X') \otimes Y',$$

$$\Omega^{-1}(f \otimes g) : \Omega^{-1}(X \otimes Y) \to \Omega^{-1}(X' \otimes Y'),$$

$$\Omega^{-1}(f)\otimes g:\Omega^{-1}(X)\otimes Y\to\Omega^{-1}(X')\otimes Y'$$

are given to make the back, front, top and bottom squares commute. Indeed, they are well-defined and unique in $\underline{\mathcal{C}}$. The right square is commutative since $e_{X',Y'}\circ\Omega^{-1}(f\otimes g)$ and $(\Omega^{-1}(f)\otimes g)\circ e_{X,Y}$ both satisfy the universal property of $\Omega^{-1}(X\otimes Y)$ as the cokernel of $i_{X\otimes Y}$. That is, $e_{X,Y}$ is a natural transformation in both variable.

The second step is to verify $e_{X,Y}$ is in fact an isomorphism in $\underline{\mathcal{C}}$. By the construction of $e_{X,Y}$, there is the following pullback diagram:

$$\begin{array}{ccc} I(X\otimes Y) & \xrightarrow{p_{X\otimes Y}} & \Omega^{-1}(X\otimes Y) \\ \downarrow{\scriptstyle \tau_{X,Y}} & & \downarrow{\scriptstyle e_{X,Y}} \\ I(X)\otimes Y & \xrightarrow{p_X\otimes \mathrm{id}} & \Omega^{-1}(X)\otimes Y \end{array}$$

which induces the following split exact sequence:

$$0\to I(X\otimes Y)\to\Omega^{-1}(X\otimes Y)\oplus(I(X)\otimes Y)\to\Omega^{-1}(X)\otimes Y\to 0.$$

Hence we have

$$\Omega^{-1}(X\otimes Y)\oplus(I(X)\otimes Y)\cong I(X\otimes Y)\oplus(\Omega^{-1}(X)\otimes Y)$$

in $\mathcal{C}$. Moreover by Lemma 2.1.11, we have

$$\Omega^{-1}(X\otimes Y)\cong\Omega^{-1}(X)\otimes Y$$

in $\underline{\mathcal{C}}$. Consequently, there is a natural isomorphism $\underline{e}_{X,Y}$ in $\underline{\mathcal{C}}$:

$$\underline{e}_{X,Y}:\Omega^{-1}(X\otimes Y)\cong\Omega^{-1}(X)\otimes Y$$

which completes the proof. □

Using the above Lemma 2.3.1, we can obtain that:

Lemma 2.3.2. *Let $\mathcal{C}$ be a finite tensor category, then $\underline{\mathcal{C}}$ is a tensor triangulated catgegory.*

Proof. Our problem reduces to prove $\mathcal{C}$ has a monoidal structure.

Firstly, there is a monodidal quotient functor $F:\mathcal{C}\longrightarrow\underline{\mathcal{C}}$ being identity on objects and sending every morphism f to $\underline{f}$. Next, the tensor product on $\underline{\mathcal{C}}$ written as $\underline{\otimes}$ can be defined as follows:

$$\underline{\otimes}:\underline{\mathcal{C}}\times\underline{\mathcal{C}}\to\underline{\mathcal{C}}$$

such that $X \underline{\otimes} Y = F(X \otimes Y)$ for any objects X, Y and $\underline{f} \underline{\otimes} \underline{g} = F(f \otimes g)$ for morphisms $\underline{f}, \underline{g}$. By Lemma 2.1.11, it is straightforward to see the tensor product $\underline{\otimes}$ is well-defined. Similarly, the unit object, the associativity constraint, left and right unit constraints are given by

$$\underline{\mathbf{1}} := F(\mathbf{1}), \quad \underline{a}_{X,Y,Z} := F(a_{X,Y,Z}), \quad \underline{l}_X := F(l_X), \quad \underline{r}_X := F(r_X) \quad (X \in \underline{\mathcal{C}}).$$

Finally, the establishment of "the pentagon axiom" and "the triangle axiom" are obvious since F is a functor.

Owing to Lemma 2.3.1, we conclude that $- \underline{\otimes} Y$ is an exact functor of $\underline{\mathcal{C}}$ [22, Lemma 2.8] and the exactness in the second variable is similar. Thus $- \underline{\otimes} -$ is biexact in $\underline{\mathcal{C}}$. Hence $\underline{\mathcal{C}}$ is a tensor triangulated category. □

Before proceeding further, we point out that the stable category of a finite tensor category also has some rich compatibility between tensor functors and cosygyzy functors. The following commutative diagram is usually called "coherence".

Proposition 2.3.3. *Let $\mathcal{C}$ be a finite tensor category. Then for any objects $X, Y, Z \in \underline{\mathcal{C}}$, we deduce the following commutative diagrams in $\underline{\mathcal{C}}$:*

$$\begin{array}{ccccc}
\Omega^{-1}((X \underline{\otimes} Y) \underline{\otimes} Z) & \xrightarrow{\underline{e}_{X \underline{\otimes} Y, Z}} & \Omega^{-1}(X \underline{\otimes} Y) \underline{\otimes} Z & \xrightarrow{\underline{e}_{X,Y} \underline{\otimes} \underline{\mathrm{id}}} & (\Omega^{-1}(X) \underline{\otimes} Y) \underline{\otimes} Z \\
\downarrow{\scriptstyle \Omega^{-1}(\underline{a}_{X,Y,Z})} & & & & \downarrow{\scriptstyle \underline{a}_{\Omega^{-1}(X),Y,Z}} \\
\Omega^{-1}(X \underline{\otimes} (Y \underline{\otimes} Z)) & & \xrightarrow{\underline{e}_{X, Y \underline{\otimes} Z}} & & \Omega^{-1}(X) \underline{\otimes} (Y \underline{\otimes} Z)
\end{array}$$

$$\begin{array}{ccccc}
\Omega^{-1}((X \underline{\otimes} Y) \underline{\otimes} Z) & \xrightarrow{\underline{\theta}_{X \underline{\otimes} Y, Z}} & (X \underline{\otimes} Y) \underline{\otimes} \Omega^{-1}(Z) & \xrightarrow{\underline{a}_{X,Y,\Omega^{-1}(Z)}} & X \underline{\otimes} (Y \underline{\otimes} \Omega^{-1}(Z)) \\
\downarrow{\scriptstyle \Omega^{-1}(\underline{a}_{X,Y,Z})} & & & & \downarrow{\scriptstyle \underline{\mathrm{id}} \underline{\otimes} \underline{\theta}_{X,Y}} \\
\Omega^{-1}(X \underline{\otimes} (Y \underline{\otimes} Z)) & & \xrightarrow{\underline{\theta}_{X, Y \underline{\otimes} Z}} & & X \underline{\otimes} \Omega^{-1}(Y \underline{\otimes} Z)
\end{array}$$

Proof. We only give proof details of the first diagram. For any $X, Y, Z \in \mathcal{C}$, consider the following three exact sequences:

$$0 \to X \rightarrowtail I(X) \twoheadrightarrow \Omega^{-1}(X) \to 0,$$

$$0 \to X \otimes Y \rightarrowtail I(X \otimes Y) \twoheadrightarrow \Omega^{-1}(X \otimes Y) \to 0,$$

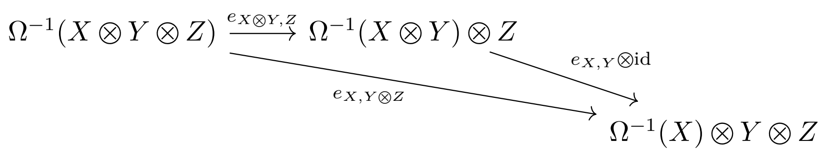
$$0 \to X \otimes Y \otimes Z \rightarrowtail I(X \otimes Y \otimes Z) \twoheadrightarrow \Omega^{-1}(X \otimes Y \otimes Z) \to 0.$$

Since $-\otimes-$ is biexact in $\mathcal{C}$, there are exact sequences:

$$0 \to X \otimes Y \otimes Z \rightarrowtail I(X) \otimes Y \otimes Z \twoheadrightarrow \Omega^{-1}(X) \otimes Y \otimes Z \to 0,$$

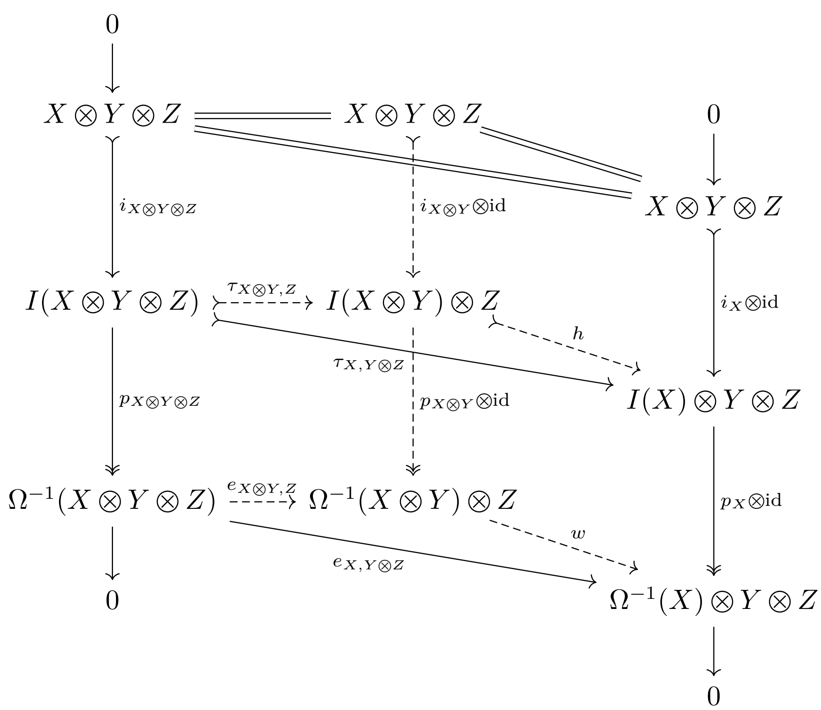
$$0 \to X \otimes Y \otimes Z \rightarrowtail I(X \otimes Y) \otimes Z \twoheadrightarrow \Omega^{-1}(X \otimes Y) \otimes Z \to 0.$$

According to the definition of $-\underline{\otimes}-$, we only need to verify the following triangle is commutative in $\underline{\mathcal{C}}$. Here the associativity constraints are suppressed:



which implies the commutativity of the desired diagram.

Consider the following commutative diagram where all the three columns are exact sequences:



Note that monomorphisms

$$\tau_{X\otimes Y,Z} : I(X \otimes Y \otimes Z) \to I(X \otimes Y) \otimes Z,$$

$$\tau_{X,Y\otimes Z} : I(X \otimes Y \otimes Z) \to I(X) \otimes Y \otimes Z$$

are obtained since $I(X \otimes Y \otimes Z)$ is the injective hull of $X \otimes Y \otimes Z$. As $I(X) \otimes Y \otimes Z$ is injective by Lemma 2.1.11, there is a morphism

$$h : I(X \otimes Y) \otimes Z \to I(X) \otimes Y \otimes Z$$

such that the middle triangle commute. An argument similer to one used in Lemma 2.3.1 shows that all remaining parts are commutative. However, the morphism

$$w : \Omega^{-1}(X \otimes Y) \otimes Z \to \Omega^{-1}(X) \otimes Y \otimes Z$$

is unique in $\underline{\mathcal{C}}$ and besides

$$e_{X,Y} \otimes \mathrm{id} : \Omega^{-1}(X \otimes Y) \otimes Z \to \Omega^{-1}(X) \otimes Y \otimes Z$$

makes the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \otimes Y \otimes Z & \xrightarrow{i_{X\otimes Y}\otimes \mathrm{id}} & I(X \otimes Y) \otimes Z & \xrightarrow{p_{X\otimes Y}\otimes \mathrm{id}} & \Omega^{-1}(X \otimes Y) \otimes Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \tau_{X,Y}\otimes \mathrm{id}} & & \downarrow{\scriptstyle e_{X,Y}\otimes \mathrm{id}} & & \\ 0 & \longrightarrow & X \otimes Y \otimes Z & \xrightarrow{i_X\otimes \mathrm{id}} & I(X) \otimes Y \otimes Z & \xrightarrow{p_X\otimes \mathrm{id}} & \Omega^{-1}(X) \otimes Y \otimes Z & \longrightarrow & 0 \end{array}$$

Hence, $w = e_{X,Y} \otimes \mathrm{id}$ in $\underline{\mathcal{C}}$. Thus we arrive at the conclusion. □

Remark 2.3.4. A stable equivalence induced by an exact monoidal functor is obviously a tensor triangulated equivalence. We will call it a *stable tensor equivalence.*

Stable Grothendieck groups are the Grothendieck groups of stable categories as triangulated categories. They are invariant under stable equivalences. Moreover, in the case of the stable category of a finite tensor category, the following lemma shows that a stable Grothendieck ring becomes an invariant of a stable tensor equivalence.

Lemma 2.3.5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-semisimple finite tensor categories. Assume there is a stable equivalence between them induced by an exact $\Bbbk$-linear monoidal functor F. Then $\mathrm{Gr}(\underline{\mathcal{C}})$ and $\mathrm{Gr}(\underline{\mathcal{C}'})$ are ring isomorphism.*

Proof. Define $\varphi : \mathrm{Gr}(\underline{\mathcal{C}}) \to \mathrm{Gr}(\underline{\mathcal{C}'})$ such that $\varphi([X]) = [F(X)]$ for all $X \in \mathcal{C}$.

- φ is well-defined, that is if $[X] = [Y]$, then $[F(X)] = [F(Y)]$.

- φ is a group homomorphism, that is

$$\varphi([X]+[Y]) = \varphi([X]) + \varphi([Y]).$$

Since there is a distinguished triangle

$$X \to X \oplus Y \to Y \to \Omega^{-1}(X),$$

we know

$$\begin{aligned}\varphi([X]+[Y]) &= \varphi([X \oplus Y]) = [F(X \oplus Y)] = [F(X) \oplus F(Y)] \\ &= [F(X)] + [F(Y)] = \varphi([X]) + \varphi([Y]).\end{aligned}$$

- φ is a ring homomorphism, that is

$$\varphi([X][Y]) = \varphi([X])\varphi([Y]).$$

For the reason that F is a monoidal functor,

$$\begin{aligned}\varphi([X][Y]) &= \varphi([X \underline{\otimes} Y]) = [F(X \underline{\otimes} Y)] = [F(X) \underline{\otimes} F(Y)] \\ &= [F(X)][F(Y)] = \varphi([X])\varphi([Y]).\end{aligned}$$

- φ is an isomorphism.

(1) φ is injective. If $\varphi(X) = [F(X) = 0]$, then $F(X)$ is a projective object. Since F induces a stable equivalence, X is a projective objective, that is $[X] = 0$.

(2) φ is surjective. For any $[Y'] \in \mathrm{Gr}(\underline{\mathcal{C}'})$, there is $Y \in \underline{\mathcal{C}}$ such that $F(Y) \cong Y'$, that is $[F(Y)] = [Y']$ and $\varphi([Y]) = [Y']$.

□

§2.3.2 Derived tensor categories

This subsection provides a detail exposition of an example of tensor triangulated categories: the derived categories of finite tensor categories. In this case, we can also see some compatible relations between tensor products and shift functors.

Lemma 2.3.6. *Given a finite tensor category $\mathcal{C} = (\otimes, a, l, r, \mathbf{1})$, the category of complexes $C^b(\mathcal{C})$ is a monoidal category.*

Proof. Let $X, Y \in C^b(\mathcal{C})$,

$$X = \cdots \longrightarrow X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} X^{n+2} \longrightarrow \cdots$$

$$Y = \cdots \longrightarrow Y^n \xrightarrow{d_Y^n} Y^{n+1} \xrightarrow{d_Y^{n+1}} Y^{n+2} \longrightarrow \cdots$$

we define the tensor product of objects $X \tilde{\otimes} Y$ to be the total complex:

$$X\tilde{\otimes}Y = \cdots \longrightarrow (X\tilde{\otimes}Y)^n \xrightarrow{d^n_{X\tilde{\otimes}Y}} (X\tilde{\otimes}Y)^{n+1} \xrightarrow{d^{n+1}_{X\tilde{\otimes}Y}} (X\tilde{\otimes}Y)^{n+2} \longrightarrow \cdots$$

where $(X\tilde{\otimes}Y)^n := \bigoplus_{i+j=n} X^i \otimes Y^j$ and $d^n_{X\tilde{\otimes}Y} := \bigoplus_{i+j=n} d_X^i \otimes \mathrm{id}_{Y^j} + (-1)^i \mathrm{id}_{X^i} \otimes d_Y^j$ for any $n \in \mathbb{Z}$.

For two chain maps $f : X \to Y$ and $g : Z \to L$ in $C^b(\mathcal{C})$, we define the tensor product of chain maps to be $f\tilde{\otimes}g$ where $(f\tilde{\otimes}g)^n := \bigoplus_{i+j=n} f^i \otimes g^j$ for any $n \in \mathbb{Z}$, which is a chain map. Indeed, in each degree n, there are

$$\begin{aligned}&\bigoplus_{i+j=n+1} (f^i \otimes g^j) \circ (\bigoplus_{i+j=n} d_X^i \otimes \mathrm{id}_{Z^j} + (-1)^i \mathrm{id}_{X^i} \otimes d_Z^j) \\ &= \bigoplus_{i+j=n} (f^{i+1} \circ d_X^i) \otimes g^j + (-1)^i f^i \otimes (g^{j+1} \circ d_Z^j)\end{aligned}$$

and

$$\begin{aligned}&(\bigoplus_{i+j=n} d_Y^i \otimes \mathrm{id}_{L^j} + (-1)^i \mathrm{id}_{Y^i} \otimes d_L^j \circ) \bigoplus_{i+j=n} (f^i \otimes g^j) \\ &= \bigoplus_{i+j=n} (d_Y^i \circ f^i) \otimes g^j + (-1)^i f^i \otimes (d_L^j \circ g^j).\end{aligned}$$

They are equal since f and g are chain maps. Next, we follow the definition to show $C^b(\mathcal{C})$ is a monoidal category.

(1) Claim: $-\tilde{\otimes}-$ is a bifunctor.

By the definition above, $-\tilde{\otimes}-$ maps objects into objects and maps chain maps into chain maps.

Given $f : X \to Y$, $g : Y \to Z$, $h : U \to V$, $l : V \to W$, it happens

$$(g \circ f)\tilde{\otimes}(l \circ h) = (g\tilde{\otimes}l) \circ (f\tilde{\otimes}h).$$

Indeed, $((g \circ f)\tilde{\otimes}(l \circ h))^n = \bigoplus_{i+j=n} (g \circ f)^i \otimes (l \circ h)^j$ and

$$((g\tilde{\otimes}l) \circ (f\tilde{\otimes}h))^n = (\bigoplus_{i+j=n} g^i \otimes l^j) \circ (\bigoplus_{i+j=n} f^i \otimes h^j) = \bigoplus_{i+j=n} (g \circ f)^i \otimes (l \circ h)^j.$$

So $-\widetilde{\otimes}-$ preserves composition.

Given $\mathrm{id}_X : X \to X$ and any object Z in $C^b(\mathcal{C})$, it happens

$$\mathrm{id}_X \widetilde{\otimes} \mathrm{id}_Z = \mathrm{id}_{X\widetilde{\otimes}Z}.$$

Indeed, $(\mathrm{id}_X \widetilde{\otimes} \mathrm{id}_X)^n = \bigoplus_{i+j=n} \mathrm{id}_{X^i} \otimes \mathrm{id}_{Z^j} = \mathrm{id}^n_{X\widetilde{\otimes}Z}$. Hence $-\widetilde{\otimes}-$ preserves unit morphism:

(2) Associativity constraint.

Define $\widetilde{a}_{X,Y,Z} : (X\widetilde{\otimes}Y)\widetilde{\otimes}Z \longrightarrow X\widetilde{\otimes}(Y\widetilde{\otimes}Z)$, where $\widetilde{a}^n_{X,Y,Z} := \bigoplus_{i+j+k=n} a_{X^i,Y^j,Z^k}$. Then $\widetilde{a}_{X,Y,Z}$ is an isomorphism as a_{X^i,Y^j,Z^k} is an isomorphism for arbitrary $i,j,k \in \mathbb{Z}$. We need to verify $\widetilde{a}_{X,Y,Z}$ is a chain map and the naturality. It is enough to verify in each degree $n \in \mathbb{Z}$ the following diagram is commutative

$$\begin{array}{ccc} ((X\widetilde{\otimes}Y)\widetilde{\otimes}Z)^n & \xrightarrow{d^n_{(X\widetilde{\otimes}Y)\widetilde{\otimes}Z}} & ((X\widetilde{\otimes}Y)\widetilde{\otimes}Z)^{n+1} \\ \downarrow{\scriptstyle \widetilde{a}^n_{X,Y,Z}} & & \downarrow{\scriptstyle \widetilde{a}^{n+1}_{X,Y,Z}} \\ (X\widetilde{\otimes}(Y\widetilde{\otimes}Z))^n & \xrightarrow{d^n_{X\widetilde{\otimes}(Y\widetilde{\otimes}Z)}} & (X\widetilde{\otimes}(Y\widetilde{\otimes}Z))^{n+1} \end{array}$$

Actually,

$$d^n_{(X\widetilde{\otimes}Y)\widetilde{\otimes}Z} = \bigoplus_{i+j+k=n} (d^i_X \otimes \mathrm{id}_{Y^j}) \otimes \mathrm{id}_{Z^k} + (-1)^i (\mathrm{id}_{X^i} \otimes d^j_Y) \otimes \mathrm{id}_{Z^k} + (-1)^{i+j} (\mathrm{id}_{X^i} \otimes \mathrm{id}_{Y^j}) \otimes d^k_Z$$

$$d^n_{X\widetilde{\otimes}(Y\widetilde{\otimes}Z)} = \bigoplus_{i+j+k=n} d^i_X \otimes (\mathrm{id}_{Y^j} \otimes \mathrm{id}_{Z^k}) + (-1)^i \mathrm{id}_{X^i} \otimes (d^j_Y \otimes \mathrm{id}_{Z^k}) + (-1)^{i+j} \mathrm{id}_{X^i} \otimes (\mathrm{id}_{Y^j} \otimes d^k_Z)$$

Hence, the above diagram is commutative by the naturality of each a_{X^i,Y^j,Z^k}.

Next given $f : X \to U$, $g : Y \to V$, $h : Z \to W$, the following diagram is commutative.

$$\begin{array}{ccc} ((X\widetilde{\otimes}Y)\widetilde{\otimes}Z)^n & \xrightarrow{\widetilde{a}^n_{X,Y,Z}} & (X\widetilde{\otimes}(Y\widetilde{\otimes}Z))^n \\ \downarrow{\scriptstyle ((f\widetilde{\otimes}g)\widetilde{\otimes}h)^n} & & \downarrow{\scriptstyle (f\widetilde{\otimes}(g\widetilde{\otimes}h))^n} \\ ((U\widetilde{\otimes}V)\widetilde{\otimes}W)^n & \xrightarrow{\widetilde{a}^n_{U,V,W}} & (U\widetilde{\otimes}(V\widetilde{\otimes}W))^n \end{array}$$

Indeed,

$$((f\widetilde{\otimes}g)\widetilde{\otimes}h)^n = \bigoplus_{i+j+k=n} (f^i \otimes g^j) \otimes h^k$$

and

$$(f\widetilde{\otimes}(g\widetilde{\otimes}h))^n = \bigoplus_{i+j+k=n} f^i \otimes (g^j \otimes h^k).$$

The above diagram is commutative by the naturality of each a_{X^i,Y^j,Z^k}.

(3) Unit object: $\mathbf{1} = \cdots \longrightarrow 0 \longrightarrow \Bbbk \longrightarrow 0 \longrightarrow \cdots$ where $\Bbbk$ is in the zero degree.

(4) Left and right unit constraints.

Define $\widetilde{l}_X : \mathbf{1}\widetilde{\otimes}X \to X$ and $\widetilde{r}_X : X\widetilde{\otimes}\mathbf{1} \to X$ where $\widetilde{l}_X^n = l_{X^n}$, $\widetilde{r}_X^n = r_{X^n}$. By the naturality of each l_{X^n} and r_{X^n}, we can get $\widetilde{l}_X$ and $\widetilde{r}_X$ are chain maps and the naturality. Take $\widetilde{l}_X$ for example, the following diagram is commutative.

$$\begin{array}{ccc} (\mathbf{1}\widetilde{\otimes}X)^n & \xrightarrow{d^n_{\mathbf{1}\widetilde{\otimes}X}} & (\mathbf{1}\widetilde{\otimes}X)^{n+1} \\ \downarrow{\scriptstyle \widetilde{l}^n_X} & & \downarrow{\scriptstyle \widetilde{l}^{n+1}_X} \\ X^n & \xrightarrow{d^n_X} & X^{n+1} \end{array}$$

Given any $f : X \to U$, $k : \mathbf{1} \to \mathbf{1}$ where $(k)^0 \in \Bbbk$ and $(k)^n = 0$ for $n \neq 0$, the following diagram is commutative.

$$\begin{array}{ccc} (\mathbf{1}\widetilde{\otimes}X)^n & \xrightarrow{\widetilde{l}^n_X} & X^n \\ \downarrow{\scriptstyle (k\widetilde{\otimes}f)^n} & & \downarrow{\scriptstyle (f)^n} \\ (\mathbf{1}\widetilde{\otimes}U)^n & \xrightarrow{\widetilde{l}^n_U} & U^n \end{array}$$

(5) The pentagon axiom.

In each degree, the following diagram is commutative by the pentagon axiom in $\mathcal{C}$.

$$\begin{array}{ccccc} & & (((W\widetilde{\otimes}X)\widetilde{\otimes}Y)\widetilde{\otimes}Z)^n & & \\ & \swarrow{\scriptstyle (\widetilde{a}_{W,X,Y}\widetilde{\otimes}\,\mathrm{id}_Z)^n} & & \searrow{\scriptstyle \widetilde{a}^n_{W\widetilde{\otimes}X,Y,Z}} & \\ ((W\widetilde{\otimes}(X\widetilde{\otimes}Y)\widetilde{\otimes}Z))^n & & & & ((W\widetilde{\otimes}X)\widetilde{\otimes}(Y\widetilde{\otimes}Z))^n \\ \downarrow{\scriptstyle \widetilde{a}^n_{W,X\widetilde{\otimes}Y,Z}} & & & & \downarrow{\scriptstyle \widetilde{a}^n_{W,X,Y\widetilde{\otimes}Z}} \\ (W\widetilde{\otimes}((X\widetilde{\otimes}Y)\widetilde{\otimes}Z))^n & & \xrightarrow[(\mathrm{id}_W\,\widetilde{\otimes}\widetilde{a}_{X,Y,Z})^n]{} & & (W\widetilde{\otimes}(X\widetilde{\otimes}(Y\widetilde{\otimes}Z)))^n \end{array}$$

(6) The triangle diagram.

In each degree, the following diagram is commutative by the triangle axiom in $\mathcal{C}$.

$$\begin{array}{ccc} ((X\widetilde{\otimes}\mathbf{1})\widetilde{\otimes}Y)^n & \xrightarrow{\widetilde{a}^n_{X,1,Y}} & (X\widetilde{\otimes}(\mathbf{1}\widetilde{\otimes}Y))^n \\ & {\scriptstyle (\widetilde{r}_X\widetilde{\otimes}\,\mathrm{id}_Y)^n}\searrow \quad \swarrow{\scriptstyle (\mathrm{id}_X\,\widetilde{\otimes}\,\widetilde{l}_Y)^n} & \\ & (X\widetilde{\otimes}Y)^n & \end{array}$$

To sum up, $C^b(\mathcal{C})$ is a monodial category. $\square$

Lemma 2.3.7. *Given a finite tensor category $\mathcal{C} = (\otimes, a, l, r, \mathbf{1})$, there are natural isomorphisms in $C^b(\mathcal{C})$:*

$$e_{X,Y} : X[1]\tilde{\otimes}Y \to (X\tilde{\otimes}Y)[1] \quad (X, Y \in C^b(\mathcal{A})),$$

and

$$\theta_{X,Y} : X\tilde{\otimes}Y[1] \to (X\tilde{\otimes}Y)[1] \quad (X, Y \in C^b(\mathcal{A})).$$

Proof. Define

$$(e_{X,Y})^n := \bigoplus_{i+j=n, i'+j'=n+1} \delta_{i+1,i'}\delta_{j,j'} \operatorname{id}_{X^{i+1}\otimes Y^j} : \bigoplus_{i+j=n} X^{i+1} \otimes Y^j \to \bigoplus_{i'+j'=n+1} X^{i'} \otimes Y^{j'}$$

which is a canonical isomorphism. It is direct to see $e_{-,-}$ is a chain map and the naturality of $e_{-,-}$.

While for $\theta_{X,Y}$, we define

$$(\theta_{X,Y})^n := \bigoplus_{i+j=n, i'+j'=n+1} (-1)^i \delta_{i,i'}\delta_{j+1,j'} \operatorname{id}_{X^i\otimes Y^{j+1}} : \bigoplus_{i+j=n} X^i \otimes Y^{j+1} \to \bigoplus_{i'+j'=n+1} X^{i'} \otimes Y^{j'}.$$

The additional sign makes θ a chain map. Indeed, for fixed i, j there are

$$\begin{aligned} &\bigoplus_{i+j=n+1} (-1)^i \operatorname{id}_{X^i\otimes Y^{j+1}} \circ \bigoplus_{i+j=n} d_X^i \otimes \operatorname{id}_{Y^{j+1}} + (-1)^i \operatorname{id}_{X^i} \otimes (-d_Y^{j+1}) \\ &= \bigoplus_{i+j=n+1} (-1)^{i+1} d_X^i \otimes \operatorname{id}_{Y^j} + (-1)^{2i} \operatorname{id}_{X^i} \otimes (-d_Y^j) \end{aligned}$$

and

$$\begin{aligned} &-(\bigoplus_{i+j=n+1} d_X^i \otimes \operatorname{id}_{Y^j} + (-1)^i \operatorname{id}_{X^i} \otimes d_Y^j) \circ \bigoplus_{i+j=n} (-1)^i \operatorname{id}_{X^i\otimes Y^{j+1}} \\ &= \bigoplus_{i+j=n+1} (-1)^{i+1} d_X^i \otimes \operatorname{id}_{Y^j} + (-1)^{2i} \operatorname{id}_{X^i} \otimes (-d_Y^j) \end{aligned}$$

In other words, the following diagram is commuting.

$$\begin{array}{ccc} (X\tilde{\otimes}Y[1])^n & \xrightarrow{\bigoplus_{i+j=n} d_X^i \otimes \operatorname{id}_{Y^{j+1}} + (-1)^i \operatorname{id}_{X^i} \otimes (-d_Y^{j+1})} & (X\tilde{\otimes}Y[1])^{n+1} \\ \downarrow (\theta_{X,Y})^n & & \downarrow (\theta_{X,Y})^{n+1} \\ ((X\tilde{\otimes}Y)[1])^n & \xrightarrow{-(\bigoplus_{i+j=n+1} d_X^i \otimes \operatorname{id}_{Y^j} + (-1)^i \operatorname{id}_{X^i} \otimes d_Y^j)} & ((X\tilde{\otimes}Y)[1])^{n+1} \end{array}$$

The natuality of θ is obvious. □

Lemma 2.3.8. *Let $\mathcal{C}$ be a finite tensor category. For $X, Y \in C^b(\mathcal{C})$, the following diagram is anti-commutative in $C^b(\mathcal{C})$.*

$$\begin{array}{ccc} X[1]\widetilde{\otimes}Y[1] & \xrightarrow{e_{X,Y[1]}} & (X\widetilde{\otimes}Y[1])[1] \\ \downarrow{\scriptstyle \theta_{X[1],Y}} & & \downarrow{\scriptstyle \theta_{X,Y}[1]} \\ (X[1]\widetilde{\otimes}Y)[1] & \xrightarrow{e_{X,Y}[1]} & (X\widetilde{\otimes}Y)[2] \end{array}$$

Proof. We only need to verify in each degree $n \in \mathbb{Z}$ the following diagram is anti-commutative.

$$\begin{array}{ccc} (X[1]\widetilde{\otimes}Y[1])^n & \xrightarrow{(e_{X,Y[1]})^n} & ((X\widetilde{\otimes}Y[1])[1])^n \\ \downarrow{\scriptstyle (\theta_{X[1],Y})^n} & & \downarrow{\scriptstyle (\theta_{X,Y}[1])^n} \\ ((X[1]\widetilde{\otimes}Y)[1])^n & \xrightarrow{(e_{X,Y}[1])^n} & ((X\widetilde{\otimes}Y)[2])^n \end{array}$$

In fact,

$$\begin{aligned} (\theta_{X,Y}[1])^n \circ (e_{X,Y[1]})^n &= -(\bigoplus_{i+j=n+1} (-1)^i \operatorname{id}_{X^i \otimes Y^{j+1}}) \circ \bigoplus_{i+j=n} \operatorname{id}_{X^{i+1} \otimes Y^{j+1}} \\ &= \bigoplus_{i+j=n} (-1)^{i+1} \operatorname{id}_{X^{i+1} \otimes Y^{j+1}} \end{aligned}$$

and

$$\begin{aligned} (e_{X,Y}[1])^n \circ (\theta_{X[1],Y})^n &= -(\bigoplus_{i+j=n+1} \operatorname{id}_{X^{i+1} \otimes Y^{j}}) \circ \bigoplus_{i+j=n} (-1)^{i+1} \operatorname{id}_{X^{i+1} \otimes Y^{j+1}} \\ &= \bigoplus_{i+j=n} (-1)^{i+2} \operatorname{id}_{X^{i+1} \otimes Y^{j+1}} . \end{aligned}$$

which complete the proof. □

Lemma 2.3.9. *Suppose that $\mathcal{C} = (\otimes, a, l, r, \mathbf{1})$ is a finite tensor category, the bounded derived category $D^b(\mathcal{C})$ is a tensor triangulated category.*

Proof. First, we show $K^b(\mathcal{C})$ is a monoidal category whose tensor structure inherits $-\widetilde{\otimes}-$ from $C^b(\mathcal{C})$. It suffices to check that $-\widetilde{\otimes}-$ preserves null-homotopy.

Let $f : X \to Y$ be homotopy to zero. That means there is $s = \{s^n\}_{n\in\mathbb{Z}}$, where $s^n : X^n \to Y^{n-1}$, such that $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$. For any chain map $g : V \to W$, we define $\widetilde{s} = \{\widetilde{s}^n\}_{n\in\mathbb{Z}}$ where $\widetilde{s}^n = \bigoplus_{i+j=n} s^i \otimes g^j$. Then

$$(f\widetilde{\otimes}g)^n = \widetilde{s}^{n+1} \circ d^n_{X\widetilde{\otimes}V} + d^{n-1}_{Y\widetilde{\otimes}W} \circ \widetilde{s}^n,$$

that is $f\widetilde{\otimes}g$ is null-homotopy. Similarly, so is $g\widetilde{\otimes}f$. Thus $K^b(\mathcal{C})$ is a tensor triangulated category.

We have already known $D^b(\mathcal{C})$ is a triangulated category. Besides $-\widetilde{\otimes}-$ is an exact functor by Lemma 2.3.7 and $-\otimes-$ is exact in $\mathcal{C}$. Next, we show $D^b(\mathcal{C})$ is a monoidal category whose tensor structure inherits $-\widetilde{\otimes}-$ from $K^b(\mathcal{C})$. The task is now to show that $-\widetilde{\otimes}-$ preserves zero morphisms in $D^b(\mathcal{C})$.

Given a zero morphism $\alpha = d\circ t^{-1} : V \Leftarrow U \to W \in D^b(\mathcal{C})$, where t is a quasi-isomorphism. Then there is a quasi-isomorphism $\rho : T \to U$ such that $d\circ\rho$ is a zero morphism in $K^b(\mathcal{C})$ by Fact 2.2.3. Hence there exists $\epsilon = \{\epsilon^n\}_{n\in\mathbb{Z}}$, where $\epsilon^n : T^n \to W^{n-1}$, such that $(d\circ\rho)^n = \epsilon^{n+1}\circ d_T^n + d_W^{n-1}\circ\epsilon^n$.

Let $f = b\circ u^{-1} : X \Leftarrow Z \to Y$ be any morphism in $D^b(\mathcal{C})$ where u is a quasi-isomorphism. We need to show $\alpha\widetilde{\otimes} f$ is a zero morphism in $D^b(\mathcal{C})$ (The proof of $f\widetilde{\otimes}\alpha$ is similar). Before that, we should define the tensor product of right roofs. Let

$$(d\circ t^{-1})\widetilde{\otimes}(b\circ u^{-1}) := (d\widetilde{\otimes} b)\circ(t\widetilde{\otimes} u)^{-1},$$

where $t\widetilde{\otimes} u$ is a quasi-isomorphism by Acyclic Assembly Lemma in [54, Lemma 2.7.3]. This definition is well-defined.

Hence we can find a quasi-isomorphism $\widetilde{\rho} = \{\widetilde{\rho}^n\}_{n\in\mathbb{Z}}$ and a morphism $\widetilde{\epsilon} = \{\widetilde{\epsilon}^n\}_{n\in\mathbb{Z}}$ where $\widetilde{\rho}^n = \bigoplus_{i+j=n}\rho^i\otimes \mathrm{id}_{Z^j}$, $\widetilde{\epsilon}^n = \bigoplus_{i+j=n}\epsilon^i\otimes b^j$ such that

$$((d\widetilde{\otimes} b)\circ\widetilde{\rho})^n = \widetilde{\epsilon}^{n+1}\circ d^n_{T\widetilde{\otimes} Z} + d^{n-1}_{W\widetilde{\otimes} Y}\circ\widetilde{\epsilon}^n.$$

Indeed,

$$\begin{aligned}
\widetilde{\epsilon}^{n+1}\circ d^n_{T\widetilde{\otimes} Z} + d^{n-1}_{W\widetilde{\otimes} Y}\circ\widetilde{\epsilon}^n &= \Big(\bigoplus_{i+j=n+1}\epsilon^i\otimes b^j\Big)\circ\Big(\bigoplus_{i+j=n} d_T^i\otimes \mathrm{id}_{Z^j} + (-1)^i\,\mathrm{id}_{T^i}\otimes d_Z^j\Big)\\
&\quad + \Big(\bigoplus_{i+j=n-1} d_W^i\otimes\mathrm{id}_{Y^j} + (-1)^i\,\mathrm{id}_{W^i}\otimes d_Y^j\Big)\circ\Big(\bigoplus_{i+j=n}\epsilon^i\otimes b^j\Big)\\
&= \bigoplus_{i+j=n}(\epsilon^{i+1}\circ d_T^i)\otimes b^j + (-1)^i\epsilon^i\otimes(b^{j+1}\circ d_Z^j)\\
&\quad + \bigoplus_{i+j=n}(d_W^{i-1}\circ\epsilon^i)\otimes b^j + (-1)^{i-1}\epsilon^i\otimes(d_Y^j\circ b^j)\\
&= \bigoplus_{i+j=n}(d^i\circ\rho^i)\otimes b^j = ((d\widetilde{\otimes} b)\circ\widetilde{\rho})^n.
\end{aligned}$$

That is $\alpha\widetilde{\otimes} f$ is a zero morphism in $D^b(\mathcal{C})$ which completes the proof. □

Remark 2.3.10. A tensor triangulated equivalence between two derived categories is called a *derived tensor equivalence.*

According to [7, Definition 1.2], a *thick tensor ideal* $\mathcal{X}$ of a tensor triangulated category $\mathcal{C}$ is a thick triangulated subcategory such that $\mathcal{X}$ is a (*two-sided*) *tensor ideal*: if $X \in \mathcal{X}$, $A \in \mathcal{C}$ then $X \otimes A \in \mathcal{X}$ and $A \otimes X \in \mathcal{X}$.

Lemma 2.3.11. ([40, Remark 4.0.6]) *Let $\mathcal{X}$ be a thick tensor ideal of a tensor triangulated category $\mathcal{C}$, then the Verdier quotient category $\mathcal{C}/\mathcal{X}$ is still a tensor triangulated category.*

Recall that $D^b(\mathcal{C}) := K^b(\mathcal{C})/\mathcal{D}$ where $\mathcal{D} = \{X \in K^b(\mathcal{C}) \mid H^n(X) = 0, \ \forall n \in \mathbb{Z}\}$. By Acyclic Assembly Lemma in [54, Lemma 2.7.3], we know $\mathcal{D}$ is a thick tensor ideal of $K^b(\mathcal{C})$. Then by Lemma $2.3.11, we can also know D^b(\mathcal{C})$ is a tensor triangulated category.

Chapter 3 Stable equivalences between finite tensor categories

Our aim in this Chapter is to show that a tensor equivalence can be recovered by a stable equivalence induced by an exact $\Bbbk$-linear monoidal functor , as a special form of tensor triangulated equivalences, under some certain conditions. Almost all the results in this Chapter can be found in [56].

In Section 3.1, I rephrased a statement about a functor inducing a stable equivalence between non-semisimple finite Frobenius categories $\mathcal{C}$ and $\mathcal{C}'$ gives the following one to one correspondence (see Lemma 3.1.2):

$$\left\{ \begin{array}{c} \text{Isoclasses of indecomposable} \\ \text{non-projective objects in } \mathcal{C} \end{array} \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Isoclasses of indecomposable} \\ \text{non-projective objects in } \mathcal{C}' \end{array} \right\}.$$

Furthermore, if $\mathcal{C}$ and $\mathcal{C}'$ have no projective simple objects, we can deduce that for two simple objects $L \in \mathcal{C}$ and $L' \in \mathcal{C}'$, L is a subobject of $\Psi(L')$ if and only if L' is a quotient object of $\Phi(L)$ (see Lemma 3.1.4). Additionally, the set of the indexes of the isoclasses of simple objects as a quotent object of $\Phi(L)$ in $\mathcal{C}$ can cover the set of the indexes of the isoclasses of simple objects in $\mathcal{C}'$ (see Lemma 3.1.5).

Section 3.2 is devoted to prove our first two main results (Proposition 3.2.3 and Theorem 3.2.12), which establish the relation between tenor equivalences and stable tensor equivalences by utilizing the invertibility of simple objects and the restriction of Frobenius-Perron dimensions..

§3.1 The isoclasses of simple objects under a stable equivalence

First, let us make some basic observations.

Lemma 3.1.1. *Let $\mathcal{C}$ be a non-semisimple finite Frobenius $\Bbbk$-linear abelian category.*

(1) *Let $f : X \to Y$ be an epimorphism in $\mathcal{C}$. If $\underline{f} = 0$ in $\underline{\mathcal{C}}$, then f has the following form:*

$$f : X \xrightarrow{i} P(Y) \overset{p}{\twoheadrightarrow} Y,$$

where $(P(Y), p)$ is a projective cover of Y and $f = p \circ i$.

(2) *Let $g : X \to Y$ be a monomorphism in $\mathcal{C}$. If $\underline{g} = 0$ in $\underline{\mathcal{C}}$, then f has the following form:*

$$g : X \overset{i'}{\rightarrowtail} I(X) \overset{p'}{\to} Y,$$

where $(I(X), i')$ is an injective hull of X and $g = p' \circ i'$.

Proof.

(1) According to $\underline{f} = 0$ in $\underline{\mathcal{C}}$, we can find a projective object P such that $f = \beta \circ \alpha$, where

$$f : X \overset{\alpha}{\to} P \overset{\beta}{\twoheadrightarrow} Y.$$

Moreover, since f is an epimorphism, so is β. By the universal property of projective cover, there exists an epimorphism $h : P \twoheadrightarrow P(Y)$ such that $p \circ h = \beta$.

As a result, we know:

$$f : X \overset{h\alpha}{\to} P(Y) \overset{p}{\twoheadrightarrow} Y.$$

(2) We omit the proof, which is similar to (1). □

Next result is a categorical version of a result in representation theory of artin algebras.

Lemma 3.1.2. ([4, cf. Proposition 1.1, p.336]) *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-semisimple finite $\mathbb{k}$-linear abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ be a $\mathbb{k}$-linear functor inducing a stable equivalence $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{C}'}$. Then F gives a one to one correspondence between the isoclasses of indecomposable non-projective objects in $\mathcal{C}$ and $\mathcal{C}'$.*

Proof. For the reason that $\mathcal{C}$ and $\mathcal{C}'$ are finite $\mathbb{k}$-linear abelian categories, we can assume $\mathcal{C} \cong A$-mod, $\mathcal{C}' \cong A'$-mod as $\mathbb{k}$-linear abelian categories, where A and A' are finite-dimensional $\mathbb{k}$-algebras. For any A-module X, we deduce the result that $\mathcal{P}(X, X) \subseteq \operatorname{rad} \operatorname{End}_A(X)$ if and only if X has no non-zero projective direct summand (See [3, Proposition 2.5].). It follows that $\operatorname{End}_A(X)$ is local if and only if $\operatorname{End}_{A'}(X')$ is local where $F(X) \cong X' \oplus P'$ satisfying that X' has no non-zero projective direct summand and P' is projective. That is, X is indecomposable if only if X' is indecomposable.

Hence we get the following one to one correspondence

$$\left\{ \begin{array}{c} \text{Isoclasses of indecomposable} \\ \text{non-projective } A\text{-modules} \end{array} \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Isoclasses of indecomposable} \\ \text{non-projective } A'\text{-modules} \end{array} \right\}.$$

Specifically, for any indecomposable non-projective A-module X, we define $\Phi(X) = X'$ satisfying that $F(X) \cong X' \oplus P'$ for some projective A'-module P'. Conversely, for any indecomposable non-projective A'-module Y', we define $\Psi(Y') = Y$ satisfying that $F(Y) \cong Y' \oplus Q'$ for some projective A'-module Q'. It is directly to see Φ and Ψ are well-defined by Krull-Schmidt Theorem. Moreover, $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$. The proof is completed. $\square$

Under the assumption of Lemma 3.1.2, there is a pair of mutually inverse maps still denoted by Φ and Ψ

$$\left\{ \begin{array}{c} \text{Isoclasses of indecomposable} \\ \text{non-projective objects in } \mathcal{C} \end{array} \right\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \left\{ \begin{array}{c} \text{Isoclasses of indecomposable} \\ \text{non-projective objects in } \mathcal{C}' \end{array} \right\}.$$

Using the above lemma, we deduce the following result.

Lemma 3.1.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be non-semisimple finite Frobenius $\Bbbk$-linear abelian categories. Suppose a $\Bbbk$-linear functor $F : \mathcal{C} \to \mathcal{C}'$ induces a stable equivalence between $\mathcal{C}$ and $\mathcal{C}'$.*

(1) *For any indecomposable non-projective object $X \in \mathcal{C}$ and any simple object $L' \in \mathcal{C}'$, we get L' is a quotient object of $\Phi(X)$ if and only if $\underline{\mathrm{Hom}}_{\mathcal{C}'}(\Phi(X), L') \neq 0$.*

(2) *For any indecomposable non-projective object $Y' \in \mathcal{C}'$ and any simple object $L \in \mathcal{C}$, we get L is a subobject of $\Psi(Y')$ if and only if $\underline{\mathrm{Hom}}_{\mathcal{C}}(L, \Psi(Y')) \neq 0$.*

Proof.

(1) "Only if" part: We claim the epimorphism $f : \Phi(X) \to L'$ satisfies $\underline{f} \neq 0$, which would follow that $\underline{\mathrm{Hom}}_{\mathcal{C}'}(\Phi(X), L') \neq 0$. First, we note that L' must be non-projective as $\Phi(X)$ is indecomposable and non-projective. Assume on the contrary that f has the following form:

$$f : \Phi(X) \xrightarrow{i} P(L') \overset{j}{\twoheadrightarrow} L'$$

where $P(L')$ can be chosen as a projective cover of L' by Lemma 3.1.1 (1). Let us consider the following commuting diagram:

$$\begin{array}{ccccc} \Phi(X) & \xrightarrow{i} & P(L') & \overset{t}{\twoheadrightarrow} & \mathrm{Coker}(i) \\ & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \beta} \\ & & L' & \overset{\alpha}{\twoheadrightarrow} & N \end{array}$$

where $(\mathrm{Coker}(i), t)$ is the cokernel of i and (β, α) is the pushout of (j, t).

There are two cases which may happen:

(i) If $N = 0$, then there is an epimorphism:

$$P(L') \twoheadrightarrow \mathrm{Coker}(i) \bigoplus L',$$

which follows another epimorphism:

$$P(L') = P(P(L')) \twoheadrightarrow P(\mathrm{Coker}(i)) \bigoplus P(L')$$

where $P(P(L'))$ and $P(\mathrm{Coker}(i))$ denote projective covers of $P(L')$ and $\mathrm{Coker}(i)$ respectively. Thus, $\mathrm{Coker}(i) = 0$ and consequently $P(L')$ is a direct summand of $\Phi(X)$, which contradicts to the fact that $\Phi(X)$ is indecomposable and non-projective.

(ii) If $N \neq 0$, since $\alpha \circ f = \alpha \circ j \circ i = \beta \circ t \circ i = 0$, we find $\alpha = 0$. This leads to a contradiction that $N = \mathrm{Im}(\alpha)$.

In conclusion, $\underline{f} \neq 0$ and thus $\underline{\mathrm{Hom}}_{\mathcal{C}'}(\Phi(X), L') \neq 0$.

"If" part: Conversely, $\underline{\mathrm{Hom}}_{\mathcal{C}'}(\Phi(X), L') \neq 0$ makes $\mathrm{Hom}_{\mathcal{C}}(\Phi(X), L') \neq 0$ which deduces that L' is a quotient object of $\Phi(X)$.

(2) The proof of this result is dual to that given above by using pullback instead and so is omitted. □

Corollary 3.1.4. *Let $\mathcal{C}$ and $\mathcal{C}'$ be non-semisimple finite Frobenius $\Bbbk$-linear abelian categories having no projective simple objects. Suppose a $\Bbbk$-linear functor $F : \mathcal{C} \to \mathcal{C}'$ induces a stable equivalence between $\mathcal{C}$ and $\mathcal{C}'$. For two simple objects $L \in \mathcal{C}$ and $L' \in \mathcal{C}'$, L is a subobject of $\Psi(L')$ if and only if L' is a quotient object of $\Phi(L)$.*

Proof. Since F induces a stable equivalence,

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(L, \Psi(L')) \cong \underline{\mathrm{Hom}}_{\mathcal{C}'}(F(L), F(\Psi(L'))) \cong \underline{\mathrm{Hom}}_{\mathcal{C}'}(\Phi(L), \Phi(\Psi(L'))) \cong \underline{\mathrm{Hom}}_{\mathcal{C}'}(\Phi(L), L').$$

Therefore $\underline{\mathrm{Hom}}_{\mathcal{C}}(L, \Psi(L')) \neq 0$ if and only if $\underline{\mathrm{Hom}}_{\mathcal{C}'}(\Phi(L), L') \neq 0$. The conclusion is obtained by Lemma 3.1.3. □

Let $\mathcal{C}$ and $\mathcal{C}'$ be non-semisimple finite Frobenius $\Bbbk$-linear abelian categories having no projective simple objects. Let $\{L_i\}_{i\in I}$ and $\{L'_j\}_{j\in J}$ be the isoclasses of simple objects in $\mathcal{C}$ and $\mathcal{C}'$ respectively. We introduce the following notation

$$J_i = \{j \in J \mid L'_j \text{ is a quotient object of } \Phi(L_i)\} \quad (i \in I).$$

Corollary 3.1.5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be non-semisimple finite Frobenius $\Bbbk$-linear abelian categories having no projective simple objects. Suppose a $\Bbbk$-linear functor $F : \mathcal{C} \to \mathcal{C}'$ induces a stable equivalence between $\mathcal{C}$ and $\mathcal{C}'$. Then $J = \bigcup_{i\in I} J_i$.*

Proof. It is suffices to prove $J \subset \bigcup_{i\in I} J_i$. Indeed, let L'_j be a simple object in $\mathcal{C}'$ and suppose that L_i is a simple subobject of $\Psi(L'_j)$. Therefore L'_j is a simple quotient object of $\Phi(L_i)$ by Corollary 3.1.4. In other words, $j \in J_i$ for some $i \in I$. □

§3.2 The main theorems

As we can see in Lemma 2.2.5, a stable equivalence induced by an exact functor F between two self-injective algebras can recover the original equivalence between module categories, if and only if F maps simple modules to simple modules. So the crucial point to prove Proposition 3.2.3 and Theorem 3.2.12 boils down to the following question:

Question: When does an exact $\Bbbk$- linear functor maps simple objects to simple object?

§3.2.1 Invertibility of simple objects induces a tensor equivalence

In the begining, we turn to mention the relation between the Chevalley property and the existence of simple projective objects. A Hopf algebra is said to have *the Chevalley property*, if the tensor product of two simple modules is semisimple. Generally, let us say a tensor category has *the Chevalley property* if the category of semisimple objects is a tensor subcategory [2, Definition 4.1].

The following lemma is contributed to simplify the assumptions of our results.

Lemma 3.2.1. *Let $\mathcal{C}$ be a non-semisimple finite tensor category with the Chevalley property. Then $\mathcal{C}$ has no simple projective objects.*

Proof. Otherwise, let L be a simple projective object in $\mathcal{C}$. Since $L \otimes L^*$ is semisimple, $\mathbf{1}$ is a direct summand of it. Moreover, Lemma 2.1.11 tells us that $L \otimes L^*$ is projective as L is projective. This implies $\mathbf{1}$ is also projective, then $\mathcal{C}$ is semisimple by [18, Corollary 4.2.13], a contradiction. □

A direct consequence of this lemma is:

Corollary 3.2.2. *Let H be a finite-dimensional non-semisimple Hopf algebra with the Chevalley property. Then H-mod has no simple projective modules.*

Let $\mathcal{C}$ be a tensor category. An object X in $\mathcal{C}$ is *invertible* if $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ are isomorphisms ([18, Definition 2.11.1]). A tensor category in which every simple object is invertible has the Chevalley property ([18, Proposition 4.12.4]). With this observation, we are in a position to show our first main conclusion now:

Proposition 3.2.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-semisimple finite tensor categories. Suppose $F : \mathcal{C} \to \mathcal{C}'$ is an exact $\Bbbk$-linear monoidal functor inducing a stable equivalence $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{C}'}$. If all simple objects in $\mathcal{C}$ and $\mathcal{C}'$ are invertible, then F is a tensor equivalence.*

Proof. We claim F maps simple objects to simple objects. Actually, for any simple object $L \in \mathcal{C}$, we know

$$F(L^*) \otimes F(L) \cong F(L^* \otimes L) \cong F(\Bbbk) \cong \Bbbk.$$

Then

$$\mathrm{length}(F(L^*))\mathrm{length}(F(L)) \leqslant \mathrm{length}(F(L^*) \otimes F(L)) = \mathrm{length}(\Bbbk) = 1,$$

where length(-) denotes the length of the Jordan-Hölder series. Hence $\mathrm{length}(F(L)) = 1$, that is, $F(L)$ is a simple object.

Since $\mathcal{C}$ and $\mathcal{C}'$ are finite, we may assume $\mathcal{C} \cong A$-mod, $\mathcal{C}' \cong A'$-mod as $\Bbbk$-linear abelian categories, where A and A' are finite-dimensional $\Bbbk$-algebras. In addition, as $\mathcal{C}$ and $\mathcal{C}'$ are tensor categories, A and A' also can be self-injective according to Lemma 2.1.10. Moreover, $\mathcal{C}$ and $\mathcal{C}'$ have no projective simple objects by Lemma 3.2.1. As a result, F is a $\Bbbk$-linear equivalence by Lemma 2.2.5. Consequently it is a tensor equivalence. □

Note that a Hopf algebra H is *basic* if and only if every simple object in the tensor category of finite-dimensional H-modules is invertible. So the following conclusion is directly obtained.

Corollary 3.2.4. *Let H and H' be finite-dimensional non-semisimple basic Hopf algebras. Suppose $F : H\text{-}\mathsf{mod} \to H'\text{-}\mathsf{mod}$ is an exact $\Bbbk$-linear monoidal functor inducing a stable equivalence $\underline{F} : H\text{-}\underline{\mathsf{mod}} \to H'\text{-}\underline{\mathsf{mod}}$. Then H and H' are gauge equivalent.*

Remark 3.2.5. We conclude this subsection by pointing out that: the inverse of Lemma 3.2.1 is false.

Actually, consider a finite-dimensional Hopf algebra H_1 without the Chevalley property such as A''_{C_4} [51] and a finite-dimensional non-semisimple Hopf algebra H_2 having the Chevalley property such as Taft algebra, the tensor product of the two Hopf algebras has no the Chevalley property and all projective modules are not simple. Indeed, the simple modules of $H_1 \otimes H_2$ is the form $V \otimes W$ for unique simple modules V, W of H_1,H_2 respectively [17, Theorem 3.10.2]. Besides, since H_2 has no simple projective modules by Lemma 3.2.1, neither does $H_1 \otimes H_2$.

§3.2.2 The restriction of Frobenius-Perron dimensions induces a tensor equivalence

An important technical tool in the study of tensor categories is Frobenius-Perron dimensions. We mainly follow [18] for the standard notion of Frobenius-Perron dimension.

Let $\mathbb{Z}_+$ denote the semi-ring of non-negative integers. A basis $B = \{b_i\}_{i\in I}$ of a ring A which is free as a $\mathbb{Z}$-module is called a $\mathbb{Z}_+$-*basis* if $b_ib_j = \sum_{k\in I} c_{ij}^k b_k$ with $c_{ij}^k \in \mathbb{Z}_+$. A $\mathbb{Z}_+$-*ring* is a ring with a fixed $\mathbb{Z}_+$-basis and with identity 1. Furthermore, A *unital* $\mathbb{Z}_+$*ring* is a $\mathbb{Z}_+$-ring such that $1 \in B$. Let A be a *transitive* unital $\mathbb{Z}_+$-ring of finite rank, namely a unital $\mathbb{Z}_+$-ring of finite rank satisfies the property: For any $X, Z \in B$ there are $Y_1, Y_2 \in B$ such that XY_1 and Y_2X contain Z with a nonzero coefficient.

Now we are ready to give the definition of Frobenius-Perron dimension. Let A be a transitive unitial $\mathbb{Z}_+$-ring of finite rank with basis $B = \{b_i\}_{i\in I}$ and $|I| < \infty$. Each b_i induces a linear operator

$$\widehat{b_i} : A \to A, \quad a \mapsto b_i \cdot a$$

The classical Frobenius-Perron theorem in [20, VIII.2] tells us the existence of maximal eigenvalue spectral radius $\rho(\widehat{b}_i)$ of $\widehat{b}_i$, that is

$$\rho(\widehat{b}_i) := \max\{\ |\mu|\ \ |\ \mu \text{ is an eigenvalue of } \widehat{b}_i\}$$

is an eigenvalue of the linear operator $\widehat{b}_i$.

Here I only introduce the Frobenius-Perron theorem described in [18].

Theorem 3.2.6. ([18, Theorem 3.2.1]) *Let M be a square matrix with non-negtiva real entries. Then M has a non-negative real eigenvalue and the spectral radius of B is an eigenvalue.*

Define a group homomorphism

$$\mathrm{FPdim} : A \to \mathbb{C}, \quad \mathrm{FPdim}(\sum_{i \in I} \alpha_i b_i) := \sum_{i \in I} \alpha_i \rho(\widehat{b}_i)$$

where $\alpha_i \in \mathbb{Z}$ for $i \in I$. The function FPdim is called the *Frobenius-Perron dimension.*

Regarding Frobenius-Perron dimension, here are some properties.

Lemma 3.2.7. ([18, Proposition 3.3.6])

(1) *The function* $\mathrm{FPdim} : A \to \mathbb{C}$ *is a ring homomorphism.*

(2) FPdim *is the unique character of A which takes non-negative values on I, and* $\mathrm{FPdim}(X) \leqslant 1$ *for any $X \in I$.*

Due to the following result, one can define the Frobenius-Perron dimensions of objects in a finite tensor category $\mathcal{C}$.

Lemma 3.2.8. ([18, Proposition 4.5.4]) *If $\mathcal{C}$ is a finite tensor category, then* $\mathrm{Gr}(\mathcal{C})$ *is a transitive unital $\mathbb{Z}_+$-ring of finite rank.*

To be specific, for each object $X \in \mathcal{C}$, $\mathrm{FPdim}(X)$ is the largest positive eigenvalue of the matrix of left or right multiplication by X on the set of isomorphisc classes of simple objects. Furthermore, FPdim is the unique additive and multiplicative map which takes positive values on all simple objects of $\mathcal{C}$. Here is a lemma which we will need later.

Lemma 3.2.9. ([18, Proposition 4.5.7]) *Let $\mathcal{C}$ and $\mathcal{C}'$ be finite tensor categories. If $F : \mathcal{C} \to \mathcal{C}'$ is an exact $\Bbbk$-linear monoidal functor, then* $\mathrm{FPdim}(F(X)) = \mathrm{FPdim}(X)$ *for any $X \in \mathcal{C}$.*

Let $\{L_i\}_{i\in I}$ be the set of isomorphic classes of simple objects of $\mathcal{C}$, and P_i denotes the projective cover of L_i for each i.

Definition 3.2.10. ([18, Definition 6.1.6]) Let $\mathcal{C}$ be a finite tensor category. Then the Frobenius-Perron dimension of $\mathcal{C}$ is defined by

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{i\in I} \mathrm{FPdim}(L_i)\,\mathrm{FPdim}(P_i)$$

For a finite dimensional Hopf algebra H, it is clear that $\mathrm{FPdim}(H\text{-}\mathsf{mod}) = \dim_{\Bbbk}(H)$, which can be found in [18, Example 6.1.9].

The following theorem shows that Frobenius-Perron dimensions are invariant under tensor equivalences, which gives us the hint to add the assumption of Frobenius-Perron dimensions.

Theorem 3.2.11. ([18, Proposition 6.3.3]) *Let $\mathcal{C}$ and $\mathcal{C}'$ be finite tensor categories. A tensor functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence if and only if* $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}')$.

We now turn to prove one of the main theorems.

Theorem 3.2.12. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two non-semisimple finite tensor categories having no projective simple objects such that* $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}')$. *Suppose $F : \mathcal{C} \to \mathcal{C}'$ is an exact $\Bbbk$-linear monoidal functor inducing a stable equivalence $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{C}'}$, then F is a tensor equivalence.*

Proof. Let $\{L_i\}_{i\in I}$ and $\{L'_j\}_{j\in J}$ be the set of isoclasses of simple objects in $\mathcal{C}$ and $\mathcal{C}'$ respectively, where I and J are finite sets. Moreover, we use P_i (resp. P'_j) to represent a projective cover of each simple object L_i (resp. L'_j).

The trick of the proof is to show F maps simple objects to simple objects. For any simple object L_i, we know $F(L_i) \cong \Phi(L_i) \oplus Q'_i$ for some projective object Q'_i by Lemma 3.1.2. In addition, as F is an exact functor, there is an epimorphism $F(P_i) \twoheadrightarrow P(\Phi(L_i))$ for any $i \in I$, where $P(\Phi(L_i))$ denotes a projective cover of $\Phi(L_i)$.

Consequently, we can get the following formula:

$$\begin{aligned}
\mathrm{FPdim}(\mathcal{C}) &= \sum_{i\in I} \mathrm{FPdim}(L_i)\,\mathrm{FPdim}(P_i) = \sum_{i\in I} \mathrm{FPdim}(F(L_i))\,\mathrm{FPdim}(F(P_i)) \\
&= \sum_{i\in I} \mathrm{FPdim}(\Phi(L_i)\oplus Q_i')\,\mathrm{FPdim}(F(P_i)) \\
&\geqslant \sum_{i\in I} \mathrm{FPdim}(\Phi(L_i))\,\mathrm{FPdim}(P(\Phi(L_i))) \\
&\geqslant \sum_{i\in I} (\sum_{j\in J_i} \mathrm{FPdim}(L_j'))(\sum_{j\in J_i} \mathrm{FPdim}(P_j'))) \\
&\geqslant \sum_{j\in J} \mathrm{FPdim}(L_j')\,\mathrm{FPdim}(P_j') \quad \text{(by Corollary 3.1.5)} \\
&= \mathrm{FPdim}(\mathcal{C}').
\end{aligned}$$

By the condition that $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}')$, all the " $\geqslant$ " above are in fact equalities. Due to

$$\sum_{i\in I} \mathrm{FPdim}(\Phi(L_i)\oplus Q_i')\,\mathrm{FPdim}(F(P_i)) = \sum_{i\in I} \mathrm{FPdim}(\Phi(L_i))\,\mathrm{FPdim}(P(\Phi(L_i))),$$

we can deduce that $Q_i' = 0$ for any $i \in I$. Moreover, by

$$\sum_{i\in I} (\sum_{j\in J_i} \mathrm{FPdim}(L_j'))(\sum_{j\in J_i} \mathrm{FPdim}(P_j')) = \sum_{j\in J} \mathrm{FPdim}(L_j')\,\mathrm{FPdim}(P_j'),$$

it is clear that each J_i has just one element for $i \in I$. Without loss of generality, let $J_i = \{L'_{\varphi(i)}\}$ where $\varphi : I \to J$ is a surjection given by Corollary 3.1.5. At last,

$$\begin{aligned}
\sum_{i\in I} \mathrm{FPdim}(L'_{\varphi(i)})\,\mathrm{FPdim}(P'_{\varphi(i)}) &= \sum_{j\in J} \mathrm{FPdim}(L_j')\,\mathrm{FPdim}(P_j') \\
&= \sum_{i\in I} \mathrm{FPdim}(\Phi(L_i))\,\mathrm{FPdim}(P(\Phi(L_i))),
\end{aligned}$$

it follows that $\mathrm{FPdim}(\Phi(L_i)) = \mathrm{FPdim}(L'_{\varphi(i)})$ for $i \in I$. Hence $F(L_i) \cong \Phi(L_i) \cong L'_{\varphi(i)}$ for any $i \in I$.

Since $\mathcal{C}$ and $\mathcal{C}'$ are finite, using the same method used in proof of Proposition 3.2.3, we can assume $\mathcal{C} \cong A\text{-}\mathsf{mod}$, $\mathcal{C}' \cong A'\text{-}\mathsf{mod}$ as $\Bbbk$-linear abelian categories, where A and A' are finite-dimensional self-injective $\Bbbk$-algebras. By Lemma 2.2.5, F is a $\Bbbk$-linear equivalence. Consequently it is a tensor equivalence. □

It is direct to see the following corollary.

Corollary 3.2.13. *Let H and H' be finite-dimensional non-semisimple Hopf algebras having no simple projective modules such that* $\dim_{\Bbbk}(H) = \dim_{\Bbbk}(H')$. *If an exact* $\Bbbk$*-linear monoidal functor* $F : H\text{-}\mathsf{mod} \to H'\text{-}\mathsf{mod}$ *induces a stable equivalence* $\underline{F} : H\text{-}\underline{\mathsf{mod}} \to H'\text{-}\underline{\mathsf{mod}}$, *then H and H' are gauge equivalent.*

Proof. By $\mathrm{FPdim}(H\text{-}\mathsf{mod}) = \dim_{\Bbbk}(H)$ we can get the conclusion. □

Remark 3.2.14. We end this section by pointing out that: The condition "the functor F is monoidal" can not be removed in Theorem 3.2.12. Let us illustrate it with an example. Consider the n^2-dimensional Taft algebras $H_n(q_1)$ and $H_n(q_2)$, where q_1 and q_2 are primitive n-th roots of unity (The specific definition of Taft algebras will be introduced in Chapter 5). [28, Corollary 3.3] tells us that $H_n(q_1)$ and $H_n(q_2)$ are gauge equivalent if and only if $q_1 = q_2$. As the fact that $H_n(q_1)$ and $H_n(q_2)$ are isomorphic as algebras, they are Morita equivalent inducing a functor from $H_n(q_1)\text{-}\mathsf{mod}$ to $H_n(q_2)\text{-}\mathsf{mod}$. This functor satisfies the assumptions of Theorem 3.2.12, except that F is a monoidal functor when $q_1 \neq q_2$.

Chapter 4 Monoidal t-structures on tensor triangulated categories

The concept of t-structure was introduced by Beilinson, Bernstein and Deligne to construct the category of perverse sheaves over an algebraic or analytic variety [9]. Many evidences show that t-structures play a key role in under standing the structure of triangulated categories such as derived categories. Alonso Tarrío, Jeremías López and Souto Salorio showed that Rickard' theorem, that characterizes when two bounded derived categories of rings are equivalent, can be deduced from t-structure constructed in [53]. Psaroudakis and Vitória established a derived Morita theory for abelian categories with a projective generator or injective cogenerator by using realization functor associated to t-structure in triangulated categories [41].

In [59] Zhang and Zhou defined mtt-structures (i.e. monoidal triangulated t-structure) on tensor triangulated categories. They observed that under certain conditions of strength, the heart of a mtt-structure manifests as a tensor category. By using this result, they also gave an statement that derived tensor equivalences between two finite-dimensional hereditary weak bialgebras can recover the monoidal abelian equivalences between categories of modules if one the the weak bialgebras is bialgebra. Inspired by these ideas, we would like to understand if we can give a new description of t-structures in tensor triangulated categories, in order to express the equivalences of derived tensor equivalences without adding conditions of hereditary.

Section 4.1 is intended to motivate the investigation of monoidal t-structures. It is worth pointing out that the Grothendieck ring of the heart of a monoidal t-structure carries all the information of the Grothendieck ring of the triangulated category itself (see Proposition 4.1.9). We emphasize that the property of integral make two equivalent monoidal t-structures equal (see Theorem 4.1.15). Section 4.2 established the relations between tensor equivalences (gauge equivalences), derived tensor equivalences and stable tensor equivalences by using the tool of monoidal t-structures defined in Section 4.1 (see Theorem 4.2.3 and Theorem 4.2.10).

§4.1 Deviation and uniqueness of monoidal t-structures

I firstly recalled some basic definitions and properties related to t-structures. References are made to [9, 13, 30]. Let $(\mathcal{T}, \Sigma)$ be a triangulated category where Σ is the translation functor.

Definition 4.1.1. ([9, Definition 1.3.1]) A pair of full subcategories $\mathfrak{t} = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ in $\mathcal{T}$ is said to be a *t-structure* on $\mathcal{T}$, if they satisfy the following conditions:

(T1) $\Sigma\mathcal{D}^{\leqslant 0} \subseteq \mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 1} \subseteq \Sigma\mathcal{D}^{\geqslant 1}$;

(T2) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1}) = 0$;

(T3) For any object $X \in \mathcal{T}$, there is a distinguished triangle

$$X^{\leqslant 0} \to X \to X^{\geqslant 1} \to \Sigma X^{\leqslant 0},$$

where $X^{\leqslant 0} \in \mathcal{D}^{\leqslant 0}$ and $X^{\geqslant 1} \in \mathcal{D}^{\geqslant 1}$.

The subcategories $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 1}$ are called the *aisle* and *coaisle* of $\mathfrak{t}$ respectively.

Let $\mathfrak{t} = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ be a t-structure on $\mathcal{T}$. The following are some definitions and notations we will need later.

- For any $n \in \mathbb{Z}$, let $\mathcal{D}^{\leqslant n} := \Sigma^{-n}\mathcal{D}^{\leqslant 0}$, $\mathcal{D}^{\geqslant n+1} := \Sigma^{-n}\mathcal{D}^{\geqslant 1}$ and $\Sigma^{-n}\mathfrak{t} := (\mathcal{D}^{\leqslant n}, \mathcal{D}^{\geqslant n+1})$. If $\mathfrak{t}$ is a t-structure, so is $\Sigma^{n}\mathfrak{t}$ for any $n \in \mathbb{Z}$ ([30, Remark 10.1.2]).

- $\mathcal{H}_{\mathfrak{t}} := \mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0}$ is called the heart of $\mathfrak{t}$, which is an abelian category ([30, Proposition 10.1.11]). There is a *cohomological functor* $H^0_{\mathfrak{t}} : \mathcal{T} \to \mathcal{H}_{\mathfrak{t}}$ (i.e. a functor sending distinguished triangles in $\mathcal{T}$ to long exact sequences in $\mathcal{H}_{\mathfrak{t}}$) defined by:

  $$H^0_{\mathfrak{t}}(X) := \tau^{\leqslant 0}_{\mathfrak{t}}\tau^{\geqslant 0}_{\mathfrak{t}}(X) \cong \tau^{\geqslant 0}_{\mathfrak{t}}\tau^{\leqslant 0}_{\mathfrak{t}}(X) \text{ for any } X \in \mathcal{T},$$

  where $\tau^{\leqslant 0}_{\mathfrak{t}}$ and $\tau^{\geqslant 0}_{\mathfrak{t}}$ are the *truncation functors* (i.e. the left and right adjoint functor respectively of the inclusions of $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 0}$ in $\mathcal{T}$). In the same way, one can also define functors $\tau^{\leqslant n}_{\mathfrak{t}}$, $\tau^{\geqslant n}_{\mathfrak{t}}$ and $H^n_{\mathfrak{t}} := \tau^{\leqslant 0}_{\mathfrak{t}}\tau^{\geqslant 0}_{\mathfrak{t}}\Sigma^n \cong \Sigma^n\tau^{\leqslant n}_{\mathfrak{t}}\tau^{\geqslant n}_{\mathfrak{t}}$ [30].

- Let $\mathfrak{t}^+ := \bigcup_{n\in\mathbb{Z}} \mathcal{D}^{\geqslant n}$, $\mathfrak{t}^- := \bigcup_{n\in\mathbb{Z}} \mathcal{D}^{\leqslant n}$ and $\mathfrak{t}^b := \mathfrak{t}^+ \cap \mathfrak{t}^-$. $\mathfrak{t}$ is called *bounded below* (resp. *bounded above, bounded*) if $\mathfrak{t}^+ = \mathcal{T}$(resp. $\mathfrak{t}^- = \mathcal{T}$, $\mathfrak{t}^b = \mathcal{T}$).

- Let $\mathfrak{t} = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ and $\mathfrak{t}_1 = (\mathcal{D}_1^{\leqslant 0}, \mathcal{D}_1^{\geqslant 1})$ be t-structures on $\mathcal{T}$. $\mathfrak{t}$ and $\mathfrak{t}_1$ are called *equivalent* if there exist $m \leqslant n \in \mathbb{Z}$ such that $\mathcal{D}^{\leqslant m} \subseteq \mathcal{D}_1^{\leqslant 0} \subseteq \mathcal{D}^{\leqslant n}$ (if and only if $\mathcal{D}^{\geqslant m} \supseteq \mathcal{D}_1^{\geqslant 0} \supseteq \mathcal{D}^{\geqslant n}$ see [13, Lemma 4.1]). It is clear that $\mathfrak{t}$ is equivalent to $\Sigma^n\mathfrak{t}$ for any $n \in \mathbb{Z}$.

Example 4.1.2. Let $\mathcal{A}$ be an abelian category. We consider the *standard t-structure* $\mathfrak{t}_{\mathcal{A}} := (\mathcal{D}_{\mathcal{A}}^{\leqslant 0}, \mathcal{D}_{\mathcal{A}}^{\geqslant 1})$ on its derived category $D^*(\mathcal{A})$ as follow where $* \in \{\emptyset, +, -, b\}$:

$$\mathcal{D}_{\mathcal{A}}^{\leqslant 0} := \{X \in \mathcal{D}^*(\mathcal{A}) \mid H^i(X) = 0, \forall i \geqslant 1\}\ ,\ \mathcal{D}_{\mathcal{A}}^{\geqslant 1} := \{X \in D^*(\mathcal{A}) \mid H^i(X) = 0, \forall i \leqslant 0\}.$$

In this case, the heart of $\mathfrak{t}_{\mathcal{A}}$ is equivalent to $\mathcal{A}$ [30]. When $*$ is $+$ (resp. $-$, b), the standard t-structure $\mathfrak{t}_{\mathcal{A}}$ is bounded below (resp. bounded above, bounded).

A t-structure on $D^*(\mathcal{A})$ is called *intermediate* if it is equivalent to the standard t-structure.

Lemma 4.1.3. ([30, Proposition 10.1.6]) *Let $\mathfrak{t} = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ be a t-structure on a triangulated category $\mathcal{T}$.*

(1) *If $X \in \mathcal{D}^{\leqslant n}$ (resp. $X \in \mathcal{D}^{\geqslant n}$), then $\tau_{\mathfrak{t}}^{\leqslant n}X \cong X$ (resp. $\tau_{\mathfrak{t}}^{\geqslant n}X \cong X$).*

(2) *Let $X \in \mathcal{T}$. Then $X \in \mathcal{D}^{\leqslant n}$ (resp. $X \in \mathcal{D}^{\geqslant n}$) if and only if $\tau_{\mathfrak{t}}^{> n}X = 0$ (resp. $\tau_{\mathfrak{t}}^{< n}X = 0$).*

Lemma 4.1.4. *Let $\mathfrak{t}$ be a bounded t-structure on a triangulated category $\mathcal{T}$ with heart $\mathcal{H}_{\mathfrak{t}}$ and $n \in \mathbb{Z}$. Then*

(1) *$X = 0$ if and only if $H_{\mathfrak{t}}^i(X) = 0$ for any $i \in \mathbb{Z}$.*

(2) *$X \in \mathcal{D}^{\leqslant n}$ (resp. $X \in \mathcal{D}^{\geqslant n}$) if and only if $H_{\mathfrak{t}}^i(X) = 0$ (resp. $H_{\mathfrak{t}}^i(X) = 0$) for any $i \geqslant n+1$ (resp. $i \leqslant n-1$).*

Proof. By [13, Lemma 2.4], (1) holds. Here we only prove the first statement in (2). Let $X \in \mathcal{D}^{\leqslant n}$ and $i \geqslant n+1$. Then $\Sigma^i X \in \mathcal{D}^{\leqslant -1}$. Hence

$$H_{\mathfrak{t}}^i(X) = H_{\mathfrak{t}}^0(\Sigma^i X) = 0.$$

For another direction, we suppose that $X \in \mathcal{T}$ satisfying $H_{\mathfrak{t}}^i(X) = 0$ for any $i \geqslant n+1$. There is a distinguish triangle:

$$X^{\leqslant n} \longrightarrow X \longrightarrow X^{\geqslant n+1} \longrightarrow \Sigma X^{\leqslant n}.$$

By taking the cohomology funtor we get the exact sequence for any $i \in \mathbb{Z}$

$$H^i(X^{\leqslant n}) \longrightarrow H^i(X) \longrightarrow H^i(X^{\geqslant n+1}) \longrightarrow H^{i+1}(X^{\leqslant n}) \text{ in } \mathcal{H}_{\mathfrak{t}}.$$

For any $i \geqslant n+1$, $H^i(X^{\leqslant n}) = 0$ by Lemma 4.1.3. Hence $H^i(X) = H^i(X^{\geqslant n+1})$. It follows by the assumption $H^i(X^{\geqslant n+1}) = 0$ for any $i \geqslant n+1$. We also know $H^i(X^{\geqslant n+1}) = 0$ for any $i \leqslant n$ by Lemma 4.1.3. Hence $H^i(X^{\geqslant n+1}) = 0$ for all $i \in \mathbb{Z}$, and $X^{\geqslant n+1} = 0$ by (1), which implies $X \cong X^{\leqslant n} \in \mathcal{D}^{\leqslant n}$. $\square$

The following lemma states that the information of Grothendieck group of the heart of a bounded t-structure can cover the information of the Grothendieck group of the triangulated category.

Lemma 4.1.5. ([1, Proposition A.9.5]) *Let $\mathfrak{t}$ be a bounded t-structure on a triangulated category $\mathcal{T}$ with heart $\mathcal{H}_{\mathfrak{t}}$. Then $\mathrm{Gr}(\mathcal{H}_{\mathfrak{t}}) \cong \mathrm{Gr}(\mathcal{T})$ as groups.*

Now we are in the position to give one of the main definitions in this Chapter.

Definition 4.1.6. A bounded t-structure $\mathfrak{t} = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ on a tensor triangulated category $\mathcal{C}$ is called *monoidal t-structure* if there exists $n \in \mathbb{Z}$ such that

(M1) $\mathcal{D}^{\leqslant 0} \otimes \mathcal{D}^{\leqslant n} \subseteq \mathcal{D}^{\leqslant 0}$;

(M2) $\mathcal{D}^{\geqslant 0} \otimes \mathcal{D}^{\geqslant n} \subseteq \mathcal{D}^{\geqslant 0}$.

The set of integer satisfies the conditions in Definition 4.1.6 is called the *deviation* of $\mathfrak{t}$, and is denoted by $\mathrm{dev}(\mathfrak{t})$.

Lemma 4.1.7. *Let $\mathfrak{t}$ be a monoidal t-structure on a tensor triangulated category $\mathcal{C}$. For any $k \in \mathbb{Z}$, $\Sigma^k \mathfrak{t}$ is also a monoidal t-structure on $\mathcal{C}$. Moreover, if $n \in \mathrm{dev}(\mathfrak{t})$, then $n-k \in \mathrm{dev}(\Sigma^{-k}\mathfrak{t})$ for any $k \in \mathbb{Z}$.*

Proof. Since $\mathfrak{t}$ is a monoidal t-structure, there exist $n \in \mathrm{dev}(\mathfrak{t})$ such that

$$\mathcal{D}^{\leqslant 0} \otimes \mathcal{D}^{\leqslant n} \subseteq \mathcal{D}^{\leqslant 0} \text{ and } \mathcal{D}^{\geqslant 0} \otimes \mathcal{D}^{\geqslant n} \subseteq \mathcal{D}^{\geqslant 0}.$$

Hence for any $k \in \mathbb{Z}$

$$\mathcal{D}^{\leqslant k} \otimes \mathcal{D}^{\leqslant k+n-k} \subseteq \mathcal{D}^{\leqslant k} \text{ and } \mathcal{D}^{\geqslant k} \otimes \mathcal{D}^{\geqslant k+n-k} \subseteq \mathcal{D}^{\geqslant k},$$

which means that $\Sigma^k \mathfrak{t}$ is also a monoidal t-structure with $n-k \in \mathrm{dev}(\Sigma^{-k}\mathfrak{t})$. $\square$

Due to Lemma 4.1.7, even if the deviation of a monoidal t-structure $\mathbb{t}$ may not contain 0, we can always find a integer k such that $0 \in \mathrm{dev}(\Sigma^{-k}\mathbb{t})$. Hence it is not harmful for us to assume that $0 \in \mathrm{dev}(\mathbb{t})$.

If $\mathbb{t} = (\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ is a monoidal t-structure with $0 \in \mathrm{dev}(\mathbb{t})$, it is clear that the heart $\mathcal{H}_t$ is closed under $\otimes$, namely $\mathcal{H}_{\mathbb{t}} \otimes \mathcal{H}_{\mathbb{t}} \subseteq \mathcal{H}_{\mathbb{t}}$. Thus I rephrase the following useful Künneth formula with respect to $\mathbb{t}$.

Theorem 4.1.8. ([10, Theorem 4.1]) *Let $\mathcal{C}$ be a tensor triangulated category and $\mathbb{t}$ be a monoidal t-structure on $\mathcal{C}$ with $0 \in \mathrm{dev}(\mathbb{t})$. Then for any $n \in \mathbb{Z}$, there is a natural isomorphism*

$$H^n_{\mathbb{t}}(X \otimes Y) \cong \bigoplus_{i+j=n} H^i_{\mathbb{t}}(X) \otimes H^j_{\mathbb{t}}(Y) \ \textit{ for all } \ X, Y \in \mathcal{C}.$$

Proposition 4.1.9. *Let $\mathbb{t}$ be a monoidal t-structure on a triangulated category $\mathcal{C}$ with $0 \in \mathrm{dev}(\mathbb{t})$. Then $\mathcal{H}_{\mathbb{t}}$ is a monoidal abelian category with unit $H^0_{\mathbb{t}}(\mathbb{1})$. Furthermore, the Grothendieck group $\mathrm{Gr}(\mathcal{H}_{\mathbb{t}})$ is a ring and $\mathrm{Gr}(\mathcal{H}_{\mathbb{t}}) \cong \mathrm{Gr}(\mathcal{T})$ as rings.*

Proof. Let $X \in \mathcal{H}_{\mathbb{t}}$. We know $H^i_{\mathbb{t}}(X) = 0$ for any $i \neq 0$. According to Proposition 4.1.8 and Lemma 4.1.3, we deduce

$$X \cong H^0_{\mathbb{t}}(X) \cong H^0_{\mathbb{t}}(\mathbb{1} \otimes X) \cong H^0_{\mathbb{t}}(\mathbb{1}) \otimes H^0_{\mathbb{t}}(X) \cong H^0_{\mathbb{t}}(\mathbb{1}) \otimes X.$$

Since $\mathcal{H}_{\mathbb{t}}$ is closed under $\otimes$, it is a monoidal category. By Lemma 4.1.5 we can get $K_0(\mathcal{H}_{\mathbb{t}}) = K_0(\mathcal{T})$ as rings. □

Remark 4.1.10. For a monoidal t-structure $\mathbb{t}$, we know $H^0_{\mathbb{t}}(\mathbb{1}) \neq 0$. Otherwise for any $X \in \mathcal{H}_{\mathbb{t}}$, $X \cong H^0_{\mathbb{t}}(\mathbb{1}) \otimes X \cong 0$, hence $\mathcal{H}_{\mathbb{t}} = 0$, which is impossible for a bounded t-structure on $\mathcal{T}$.

Following by [35, Definition 9.14], a t-structure is called *stable* if $\Sigma\mathcal{D}^{\leqslant 0} = \mathcal{D}^{\leqslant 0}$. But the heart of a stable t-structure is $\{0\}$ (see [13, Lemma 2.5]). Hence a monoidal t-structure must not be stable. Here I remind readers that for a t-structure $\mathbb{t}$ which is not stable, there are $a, b \in \mathbb{Z}$ such that $a < b$ if and only if $\mathcal{D}^{\leqslant a} \subsetneq \mathcal{D}^{\leqslant b}$.

Definition 4.1.11. A monoidal category $\mathcal{C}$ is called *integral* if $X \otimes Y = 0$ if and only if $X = 0$ or $Y = 0$ for any object $X, Y \in \mathcal{C}$.

Example 4.1.12. ([18, Example 4.3.12]) We consider the category $\mathcal{C}$ of finite-dimensional representations of the quiver of type A_2, which is a monidal category. Such representations are triples (V, W, τ) where V, W are finite-dimensional vector space, and $\tau : V \to W$ is a linear operator. The tensor product on such triples is defined by

$$(V, W, \tau) \otimes (V', W', \tau') = (V \otimes_{\Bbbk} V', W \otimes_{\Bbbk} W', \tau \otimes_{\Bbbk} \tau')$$

with unit object $(\Bbbk, \Bbbk, \mathrm{id})$. Then $\mathcal{C}$ is not integral.

Lemma 4.1.13. *We assume that* $\mathfrak{t}$ *is a monoidal t-structure on a tensor triangulated category* $\mathcal{C}$ *with* $0 \in \mathrm{dev}(\mathfrak{t})$ *and heart* $\mathcal{H}_{\mathfrak{t}}$ *is an integral monoidal category.*

(1) *Let* $X \in \mathcal{D}^{\leqslant m}$, $Y \in \mathcal{D}^{\leqslant n}$ *where* $m, n \in \mathbb{Z}$ *such that* $H^m_{\mathfrak{t}}(X) \neq 0$ *and* $H^n_{\mathfrak{t}}(Y) \neq 0$. *Then* $X \otimes Y \in \mathcal{D}^{\leqslant m+n+1}$ *with* $H^{m+n}_{\mathfrak{t}}(X \otimes Y) \neq 0$..

(2) *Let* $X \in \mathcal{D}^{\geqslant m}$ *and* $Y \in \mathcal{D}^{\geqslant n}$ *where* $m, n \in \mathbb{Z}$ *such that* $H^m_{\mathfrak{t}}(X) \neq 0$ *and* $H^n_{\mathfrak{t}}(Y) \neq 0$. *Then* $X \otimes Y \in \mathcal{D}^{\geqslant m+n-1}$ *with* $H^{m+n}_{\mathfrak{t}}(X \otimes Y) \neq 0$..

Proof. We only show the first statement. By using Proposition 4.1.8, we know

$$H^n_{\mathfrak{t}}(X \otimes Y) \cong \bigoplus_{i+j=n} H^i_{\mathfrak{t}}(X) \otimes H^j_{\mathfrak{t}}(Y).$$

Since $X \in \mathcal{D}^{\leqslant m}$, we know $H^j_{\mathfrak{t}}(X) = 0$ for any $j \geqslant m+1$. Likewise $H^j_{\mathfrak{t}}(Y) = 0$ for any $j \geqslant n+1$. Hence $H^j_{\mathfrak{t}}(X \otimes Y) = 0$ for any $j \geqslant m+n+2$, which implies $X \otimes Y \in \mathcal{D}^{\leqslant m+n+1}$ by Lemma 4.1.4. Note that $\mathcal{H}_{\mathfrak{t}}$ is integral and $H^m_{\mathfrak{t}}(X)$, $H^n_{\mathfrak{t}}(Y)$ are not zero, so $H^{m+n}_{\mathfrak{t}}(X \otimes Y) \neq 0$. □

Proposition 4.1.14. *Let* $\mathfrak{t}$ *be a monoidal t-structure on a tensor triangulated category* $\mathcal{C}$ *such that* $0 \in \mathrm{dev}(\mathfrak{t})$ *and* $\mathcal{H}_{\mathfrak{t}}$ *is an integral monoidal category. Then* $\mathrm{dev}(\mathfrak{t})$ *contains only one element* 0.

Proof. Otherwise, assume $a \in \mathrm{dev}(\mathfrak{t})$ and $a \neq 0$. If $a > 0$, then we know $\mathcal{D}^{\leqslant 0} \otimes \mathcal{D}^{\leqslant a} \subseteq \mathcal{D}^{\leqslant 0}$. Let $X \in \mathcal{D}^{\leqslant 0}$ with $H^0_{\mathfrak{t}}(X) \neq 0$ and $Y \in \mathcal{D}^{\leqslant a}$ with $H^a_{\mathfrak{t}}(X) \neq 0$. By Lemma 4.1.13, we know $H^a_{\mathfrak{t}}(X \otimes Y) \neq 0$. Hence $X \otimes Y$ can not be in $\mathcal{D}^{\leqslant 0}$. Dually we can show that the case $a < 0$ is also impossible. □

The following result will prove extremely useful in Section 4.2.

Theorem 4.1.15. *Let $\mathfrak{t}$ be a monoidal t-structure on a tensor triangulated category $\mathcal{C}$ such that $0 \in \operatorname{dev}(\mathfrak{t})$ and $\mathcal{H}_{\mathfrak{t}}$ is an integral monoidal category. If $\mathfrak{t}$ is equivalent to any monoidal t-structure $\mathfrak{t}_1$ on $\mathcal{C}$, then $\mathfrak{t} = \mathfrak{t}_1$.*

Proof. We can assume that $0 \in \operatorname{dev}(\mathfrak{t}_1)$. Since $\mathfrak{t}$ and $\mathfrak{t}_1$ are equivalent, there exist $a \leqslant b \in \mathbb{Z}$ such that $\mathcal{D}^{\leqslant a} \subseteq \mathcal{D}_1^{\leqslant 0} \subseteq \mathcal{D}^{\leqslant b}$. Here we can choose b (resp. a)to be the minimum (resp. maximal) integer satisfies this condition, namely, there doesn't exist a integer $k < b$ (resp. $k > a$) such that $\mathcal{D}_1^{\leqslant 0} \subseteq \mathcal{D}^{\leqslant k}$ (resp. $\mathcal{D}^{\leqslant k} \subseteq \mathcal{D}_1^{\leqslant 0}$). In this case, there are $X, Y \in \mathcal{D}_1^{\leqslant 0}$ such that both $H^a_{\mathfrak{t}}(X)$ and $H^b_{\mathfrak{t}}(Y)$ are non zero.

If $a = b$, then $\mathfrak{t}_1 = \Sigma^{-a}\mathfrak{t}$. We can deduce that $a = 0$. Indeed, $\mathfrak{t}_1 = \Sigma^{-a}\mathfrak{t}$ and $\mathfrak{t}_1$ is a monoidal t-structure with $0 \in \operatorname{dev}(\mathfrak{t}_1)$, which implies $-a \in \operatorname{dev}(\mathfrak{t})$. By Proposition 4.1.14, $a = 0$.

If $a < b$, then one of the following two cases must occur: $b > 0$ or $a < 0$.

(i) Let $b > 0$. By Lemma 4.1.13, $H^{2b}_{\mathfrak{t}}(Y \otimes Y) \neq 0$. Hence $Y \otimes Y$ is not contained in $\mathcal{D}_1^{\leqslant 0}$, which is a contradiction.

(ii) Let $a < 0$. By Lemma 4.1.13, $H^{2a}_{\mathfrak{t}}(X \otimes X) \neq 0$. Hence $X \otimes X$ is not contained in $\mathcal{D}_1^{\geqslant 0}$, which is a contradiction.

To sum up, the only possible case is $a = b = 0$. □

§4.2 Derived equivalences between finite tensor categories

In this section, we will see finite tensor categories are integral. So all the statements in Section 4.1 are available in the case of derived categories of finite tensor categories.

§4.2.1 Reconstruction of a finite tensor category from a derived tensor category

Lemma 4.2.1. *Suppose that $\mathcal{C}$ is a finite tensor category and $X, Y \in \mathcal{C}$. Then is integral.*

Proof. By using Frobenius-Perron dimension, it is clear to get $\mathcal{C}$ is integral. □

The lemma below is used to prove our main Theorem 4.2.3 in this subsection.

Lemma 4.2.2. ([13, Example 4.5]) *Let $\mathfrak{t}$ be a bounded t-structure on a triangulated category $\mathcal{C}$ such that heart $\mathcal{H}_{\mathfrak{t}}$ is a finite abelian category. Then all the bounded t-structures on $\mathcal{T}$ are equivalent to $\mathfrak{t}$. In particular, for a finite dimensional algebra A, all bounded t-structures on $D^b(A\text{-}\mathsf{mod})$ are equivalent to the standard t-structure.*

Proof. Let $\{S_i\}_{i\in I}$ be the set of non-isomorphism simple object in $\mathcal{H}_{\mathfrak{t}}$ where I is a finite set, and let $\mathfrak{t}_1$ be another bounded t-structure on $\mathcal{T}$. Since I is a finite set, there exists $n \leqslant m \in \mathbb{Z}$ such that $\{S_i\}_{i\in I} \subseteq \mathcal{D}_1^{\leqslant m} \cap \mathcal{D}_1{}^{\geqslant n}$. Note that $\mathcal{D}_1^{\leqslant m} \cap \mathcal{D}_1{}^{\geqslant n}$ is closed under extensions. Hence $\mathcal{H}_{\mathfrak{t}} \subseteq \mathcal{D}_1^{\leqslant m} \cap \mathcal{D}_1{}^{\geqslant n}$, which means $\mathcal{D}^{\leqslant n} \subseteq \mathcal{D}_1^{\leqslant 0} \subseteq \mathcal{D}^{\leqslant m}$ and then $\mathfrak{t}_1$ is equivalent to $\mathfrak{t}$. □

Theorem 4.2.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be finite tensor categories. If $D^b(\mathcal{C})$ and $D^b(\mathcal{C}')$ are tensor triangulated equivalent, then $\mathcal{C}$ and $\mathcal{C}'$ are equivalent as tensor categories.*

Proof. Let $F : D^b(\mathcal{C}) \longrightarrow D^b(\mathcal{C}')$ be a tensor triangulated equivalence. A pair of full subcategories in $D^b(\mathcal{C}')$ is denoted by

$$\mathcal{U} := \{X \in D^b(\mathcal{C}') \mid \exists \tilde{X} \in \mathcal{D}_{\mathcal{C}}^{\leqslant 0} \text{ such that } F(\tilde{X}) \cong X\},$$

$$\mathcal{V} := \{Y \in D^b(\mathcal{C}') \mid \exists \tilde{Y} \in \mathcal{D}_{\mathcal{C}}^{\geqslant 1} \text{ such that } F(\tilde{Y}) \cong Y\}$$

where $(\mathcal{D}_{\mathcal{C}}^{\leqslant 0}, \mathcal{D}_{\mathcal{C}}^{\geqslant 1})$ is the standard t-structure which is a monoidal t-structure in $\mathcal{C}$. Since F induces a tensor triangulated equivalence, $\mathfrak{t} = (\mathcal{U}, \mathcal{V})$ is a monoidal t-structure on $D^b(\mathcal{C}')$ and the restriction

$$F|_{\mathcal{C}} : \mathcal{C} \to \mathcal{H}_{\mathfrak{t}}$$

is a monoidal abelian equivalence. By Lemma 4.2.2 and Theorem 4.1.15, $\mathfrak{t}$ is nothing but the standard t-structure $\mathfrak{t}_{\mathcal{C}'}$, hence $\mathcal{H}_{\mathfrak{t}} = \mathcal{C}'$. □

Corollary 4.2.4. *Let H and H' be finite dimensional Hopf algebras. If $D^b(H\text{-}\mathsf{mod})$ and $D^b(H'\text{-}\mathsf{mod})$ are tensor triangulated equivalent, then H and H' are gauge equivalent.*

§4.2.2 Stable tensor equivalences induced by derived tensor equivalences

Rickard's Morita theorem for derived categories gives a necessary and sufficient condition for two algebras to be derived equivalent 4.2.6. This condition is based on the existence of a tilting complex. If the algebras are self-injective, then the derived equivalence is closely connected with a stable equivalence 4.2.7. The purpose of this Chapter is to explore whether Rickard's theorems are still true under the setting on derived tensor categories. I will only state Rickard's theorems in category version as follows without mentioning tilting complexes.

Theorem 4.2.5. ([45, Theorem 6.4]) *Let $\mathcal{A}$ and $\mathcal{A}'$ be two finite k-linear abelian categories. We use $\mathcal{P}$ and $\mathcal{P}'$ to denote the full subcategories consisting of projective objects, then the following are equivalent:*

(1) *$K^b(\mathcal{P})$ and $K^b(\mathcal{P}')$ are equivalent as triangulated categories;*

(2) *$D^b(\mathcal{A})$ and $D^b(\mathcal{A}')$ are equivalent as triangulated categories.*

Theorem 4.2.6. ([47, Theorem 2.1]) *Let $\mathcal{A}$ be a finite Frobenius $\Bbbk$-linear abelian category. The essential image of the natural embedding*

$$K^b(\mathcal{P}) \to D^b(\mathcal{A})$$

is a thick subcategory. The quotient category $D^b(\mathcal{A})/K^b(\mathcal{P})$ is equivalent to the stable category $\underline{\mathcal{A}}$ as a triangulated category.

Theorem 4.2.7. ([47, Corollary 2.2]) *Let $\mathcal{A}$ and $\mathcal{A}'$ be finite Frobenius $\Bbbk$-linear abelian categories. If $\mathcal{A}$ and $\mathcal{A}'$ are derived equivalent then they are stably equivalent.*

Firstly, Theorem 4.2.6 will be realized in the case of derived tensor categories.

Lemma 4.2.8. *Let $\mathcal{C}$ be a finite tensor category. The essential image of the natural embedding*

$$K^b(\mathcal{P}) \to D^b(\mathcal{C})$$

is a thick tensor ideal. Moreover, $D^b(\mathcal{C})/K^b(\mathcal{P})$ is a tensor triangulated category.

Proof. We already known that $K^b(\mathcal{P})$ is a thick subcategory by Theorem 4.2.6. The only thing we need to verify is that $K^b(\mathcal{P})$ is a tensor ideal. As the localization functor is a monoidal functor, so we deduce that $D^b(\mathcal{C})$ is equivalent to $K^{-,b}(\mathcal{P})$ as tensor triangulated categories.

Let $P \in K^b(\mathcal{P})$ and $Q \in K^{-,b}(\mathcal{P})$. Then P, Q can be written as follows respectively:

$$P: \quad 0 \to P^{-l} \to P^{-l+1} \to \cdots \to P^0 \to \cdots \to P^s \to 0,$$

$$Q: \quad \cdots \to Q^{-t} \to Q^{-t+1} \to \cdots \to Q^0 \to \cdots \to Q^m \to 0$$

where the n-th homology object of Q is zero when $n < -t$. By using Lemma 2.1.11 and Lemma 4.1.8 we know that $Q \otimes P$ is quasi-isomorphic to a complex G in $K^b(\mathcal{P})$. Moreover, Lemma 2.2.17 tells us that $P \otimes Q$ and G are also homotopy equivalent, which means $P \otimes Q \in K^b(\mathcal{P})$.

Next, we consider $O \in K^{-,b}(\mathcal{P})$ which is homotopic equivalent to a complex $P \in K^b(\mathcal{P})$ and any $Q \in K^{-,b}(\mathcal{P})$. The tensor product functor preserves the quasi-isomorphism (also homotopy equivalence in this case) due to Lemma 2.2.17 and Lemma 4.1.8. Hence $O \otimes Q$ is homotopy equivalent to $P \otimes Q$. we know known that $P \otimes Q \in K^b(\mathcal{P})$ which shows that $O \otimes Q \in K^b(\mathcal{P})$.

Applying the same process on the other side, we get the conclusion the essential image of the natural embedding $K^b(\mathcal{P}) \to D^b(\mathcal{C})$ is a thick tensor ideal. Lemma 2.3.11 helps us complete the proof. $\square$

Lemma 4.2.9. *Let $\mathcal{C}$ be a finite tensor category. The essential image of the natural embedding*

$$K^b(\mathcal{P}) \to D^b(\mathcal{C})$$

is a thick tensor ideal. The Verdier quotient category $D^b(\mathcal{C})/K^b(\mathcal{P})$ is equivalent as a tensor triangulated category to the stable tensor category $\underline{\mathcal{C}}$.

Proof. By Theorem 4.2.6 and Lemma 4.2.8, it remains to prove that the equivalence functor $F : \underline{\mathcal{C}} \to D^b(\mathcal{C})/K^b(\mathcal{P})$ in the proof of Theorem 4.2.6 is also a monoidal functor. Recall that F is given by the following diagram:

$$\begin{array}{ccccc} F' : \mathcal{C} & \rightarrowtail & D^b(\mathcal{C}) & \twoheadrightarrow & D^b(\mathcal{C})/K^b(\mathcal{P}) \\ & \searrow & & \nearrow_{F} & \\ & & \underline{\mathcal{C}} & & \end{array}$$

where F' is obtained by composing the natiral embedding of $\mathcal{C}$ into $D^b(\mathcal{C})$ with the Verdier functor. Note that F' is obviously a monoidal functor, and the tensor products in $D^b(\mathcal{C})/K^b(\mathcal{P})$ and $\underline{\mathcal{C}}$ are derived from the $\otimes$ in $\mathcal{C}$. It is a routine to verify F is a monoidal functor.

□

Let $F : \mathcal{C} \to \mathcal{C}'$ be an tensor triangulated functor between two tensor triangulated $\mathcal{C}$ and $\mathcal{C}'$. Suppose that I, I' are thick tensor ideal of $\mathcal{C}$, $\mathcal{C}'$ respectively such that $F(I) \subset I'$, F induces a tensor triangulated functor $\underline{F} : \mathcal{C}/I \to \mathcal{C}'/I'$ such that the following diagram commute by Theorem 2.2.15:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathcal{C}/I & \xrightarrow[\underline{F}]{} & \mathcal{C}'/I' \end{array}$$

If moreover, $F(I) \simeq I'$, then we know $\underline{F}$ is also equivalent.

Theorem 4.2.10. *Given two finite tensor categories $\mathcal{C}$ and $\mathcal{C}'$. If $D^b(\mathcal{C}) \simeq D^b(\mathcal{C}')$ as tensor triangulated categories, then*

$$\underline{\mathcal{C}} \simeq \underline{\mathcal{C}'}$$

as tensor triangulated categories.

Proof. By Theorem 4.2.3 or directly by the above statement and Lemma 4.2.9. □

Given two finite-dimensional non-semisimple Hopf algebras H and H'.

Corollary 4.2.11. *If $D^b(H\text{-}\mathsf{mod}) \simeq D^b(H'\text{-}\mathsf{mod})$ as tensor triangulated categories, then*

$$H\text{-}\underline{\mathsf{mod}} \simeq H'\text{-}\underline{mod}$$

as tensor triangulated categories.

Chapter 5 The bounded derived categories of Taft algebras

Throughout this Chapter, we work over a fixed field $\Bbbk$ with an n-th primitive root of unity q for some positive integer n. For $n \geqslant 1$, the n^2-dimensional Hopf algebra $H_n(q)$ constructed by Taft in [52] is one of the frequently used Hopf algebras called Taft alegbras. When $n = 2$, $H_2(q)$ is known as Sweedler's 4-dimensional Hopf algebras [36]. See [44] for more details about Taft algebras. Cibils in [15] gave the indecomposable modules over the Taft algebra $H_n(q)$, and the decomposition formula of the tensor product of two indecomposable modules over $H_n(q)$. In [14], the authors described the structure of the Green rings of the Taft algebras, and it turns out that the Green rings of Taft algebras are commutative even if Taft algebras are not quasitriangular in the case $n > 2$ (not even almost cocommutative [15]). Derived Green rings (or derived representation rings) of a class of finite-dimensional Hopf algebras constructed from the Nakayama truncated algebras $\Bbbk\mathbb{Z}_n/J^2$ were introduced in [25]. However, the authors did not determine the final form of the rings. When $n = 2$, the Nakayama truncated algebra $\Bbbk\mathbb{Z}_2/J^2$ is the Sweedler's 4-dimensional Hopf algebras.

In Section 5.1 I will discuss the case of bounded derived categories of Taft algebras $D^b(H_n(q))$, with an intention to give all the indecomposable complexes in $D^b(H_n(q))$. Scetion 5.2 deals with the derived Green ring of Sweedler's 4-dimensional Hopf algebras $D^b(H_2(q))$, Theorem 5.2.3 gives a description of the ring structure of $\mathrm{G}_0(D^b(H_2(q)))$.

§5.1 Indecomposable objects in the bounded derived categories of Taft algebras

Given an integer $n \geqslant 2$. The *Taft algebra* $H_n(q)$ is generated by two elements g and x subject to the relations

$$g^n = 1, \;\; x^n = 0, \;\; xg = qgx.$$

The coalgebra structure and antipode of $H_n(q)$ are determined by

$$\Delta(g) = g \otimes g, \;\; \Delta(x) = x \otimes g + 1 \otimes x,$$

$$\epsilon(g) = 1,\ \epsilon(x) = 0,$$

$$S(g) = g^{-1} = g^{n-1},\ S(x) = -q^{-1}g^{n-1}x.$$

We know a decomposition of the regular module as follows:

$$H_n(q) = \bigoplus_{i=0}^{n-1} H_n(q)e_i$$

where for any $0 \leqslant i \leqslant n-1$

$$e_i = \frac{1}{n}\sum_{j=0}^{n-1} q^{-ij}g^j,$$

and $H_n(q)e_i = \mathrm{span}\{e_i, xe_i, \cdots, x^{n-1}e_i\}$. Indeed $\{e_0, e_1, \cdots, e_{n-1}\}$ is a set of orthogonal idempotents such that

$$\sum_{i=0}^{n-1} e_i = 1$$

and

$$ge_i = q^i e_i,\ x^{n-1}e_i \neq 0.$$

Let $M_i^j = H_n(q)e_i/\Bbbk\{x^{n-1}e_i, \cdots, x^j e_i\}$. Hence up to isomorphism,

$$\{M_i^j \mid 1 \leqslant j \leqslant n, 0 \leqslant i \leqslant n-1\}$$

are n^2 indecomposable finite-dimensional $H_n(q)$-modules. Specially, $\{M_i^n\}_{0\leqslant i\leqslant n-1}$ are all non-isomorphic indecomposable projective $H_n(q)$-modules denoted by $P_i := M_i^n$ for any $0 \leqslant i \leqslant n-1$, and they are the projective covers of all the simple $H_n(q)$-modules which are denoted by $\{S_i\}_{0\leqslant i\leqslant n-1}$.

Note that for any $0 \leqslant i \leqslant n-1$, we know the following short exact sequences:

$$0 \longrightarrow M_{\overline{i-k}}^{j-k} \longrightarrow M_i^j \longrightarrow M_i^k \longrightarrow 0 \quad (0 \leqslant k \leqslant j),$$

and

$$0 \longrightarrow M_i^k \longrightarrow M_{\overline{i-k+j}}^j \longrightarrow M_{\overline{i-k+j}}^{j-k} \longrightarrow 0 \quad (0 \leqslant k \leqslant j),$$

where $\overline{i-k}$ denotes module n residue class. In particular, there are exact sequences:

$$0 \longrightarrow M_{\overline{i-k}}^{n-k} \longrightarrow P_i \xrightarrow{\pi_i^k} M_i^k \longrightarrow 0 \quad (0 \leqslant k \leqslant n),$$

and

$$0 \longrightarrow M_i^k \xrightarrow{\tau_i^k} P_{\overline{i-k}} \longrightarrow M_{\overline{i-k}}^{n-k} \longrightarrow 0 \quad (0 \leqslant k \leqslant n).$$

Lemma 5.1.1. *For any $0 \leqslant i \leqslant n-1$ and $1 \leqslant k \leqslant j \leqslant n$, we know*

$$\dim \operatorname{Hom}_{H_n(q)}(M_i^j, M_i^k) = 1.$$

Proof. For any $f \in \operatorname{Hom}_{H_n(q)}(M_i^j, M_i^k)$ and $l \geqslant k$, $f(\overline{x^l e_i}) = 0$ as $x^l f(\overline{e_i}) = 0$. While for $l \leqslant k-1$, we claim $f(\overline{x^l e_i}) = \alpha \overline{x^l e_i}$ for some $\alpha \in \Bbbk$. First, we deal with $l = k-1$. By $x f(\overline{x^{k-1} e_i}) = 0$, we can deduce that

$$f(\overline{x^{k-1} e_i}) \in \Bbbk \overline{x^{k-1} e_i}.$$

Hence

$$f(\overline{x^{k-1} e_i}) = \alpha_{k-1} \overline{x^{k-1} e_i}.$$

Next, since $x^2 f(\overline{x^{k-2} e_i}) = 0$, we know

$$f(\overline{x^{k-2} e_i}) \in \Bbbk \overline{x^{k-1} e_i} \bigoplus \Bbbk \overline{x^{k-2} e_i}.$$

We assume

$$f(\overline{x^{k-2} e_i}) = \beta \overline{x^{k-1} e_i} + \gamma \overline{x^{k-2} e_i}. \tag{5.1.1}$$

where $\alpha, \gamma \in \Bbbk$. Then we action g on both sides

$$g \cdot f(\overline{x^{k-2} e_i}) = g \cdot (\beta \overline{x^{k-1} e_i} + \gamma \overline{x^{k-2} e_i}).$$

we know

$$q^{-(k-2)+i} f(\overline{x^{k-2} e_i}) = q^{-(k-1)+i} \beta \overline{x^{k-1} e_i} + q^{-(k-2)+i} \gamma \overline{x^{k-2} e_i}.$$

Hence

$$f(\overline{x^{k-2} e_i}) = q^{-1} \beta \overline{x^{k-1} e_i} + \gamma \overline{x^{k-2} e_i}.$$

Combine with formula 2.1.10, we can get

$$(1 - q^{-1}) \beta \overline{x^{k-1} e_i} = 0$$

that is $\beta = 0$. Moreover we know $\gamma = \alpha_{k-1}$ by $x f(\overline{x^{k-2} e_i}) = f(\overline{x^{k-1} e_i})$. Similarly, $f(\overline{x^l e_i}) = \alpha_{k-1} \overline{x^l e_i}$ for $l \leqslant k-1$. Let the quotient map $h : M_i^j \twoheadrightarrow M_i^k$ be $\overline{x^l e_i} \mapsto \overline{x^l e_i}$ where $0 \leqslant l \leqslant n-1$. Thus $f = \alpha_{k-1} h$, in other words

$$\dim \operatorname{Hom}_{H_n(q)}(M_i^j, M_i^k) = 1.$$

□

Remark 5.1.2. we have konwn that the category of modules over Taft algebra $H_n(q)$ is a tensor category. Besides, the left dual functor is exact and fully faithful. As linear space, there is an isomorphism for $0 \leqslant k \leqslant j \leqslant n, 0 \leqslant i \leqslant n-1$

$$\mathrm{Hom}_{H_n(q)}(M_i^k, M^j_{\overline{i-k+j}}) \cong \mathrm{Hom}_{H_n(q)}((M^j_{\overline{i-k+j}})^*, (M_i^k)^*).$$

Hence

$$\dim \mathrm{Hom}_{H_n(q)}(M_i^k, M^j_{\overline{i-k+j}}) = 1.$$

We use the following morphisms to represent the "standard" basis:

$$\pi_i^{\overline{i-j}} : P_i \twoheadrightarrow M_i^{\overline{i-j}} \quad x^l e_i \mapsto \overline{x^l e_i} \quad 0 \leqslant l \leqslant n-1.$$

$$\tau_i^{\overline{i-j}} : M_i^{\overline{i-j}} \rightarrowtail P_j \quad \overline{x^l e_i} \mapsto x^{n-\overline{i-j}}\overline{x^l e_j} \quad 0 \leqslant l \leqslant \overline{i-j}-1.$$

Then any non-zero morphism $f \in \mathrm{Hom}_{H_n(q)}(P_i, P_j)$, f can factor through $M_i^{\overline{i-j}}$ and have the following form

$$\mathrm{Hom}_{H_n(q)}(P_i, P_j) = \Bbbk(P_i \overset{\pi_i^{\overline{i-j}}}{\twoheadrightarrow} M_i^{\overline{i-j}} \overset{\tau_i^{\overline{i-j}}}{\rightarrowtail} P_j).$$

Indeed, f has the following decomposation

$$f : P_i \twoheadrightarrow \mathrm{Im}(f) \rightarrowtail P_j,$$

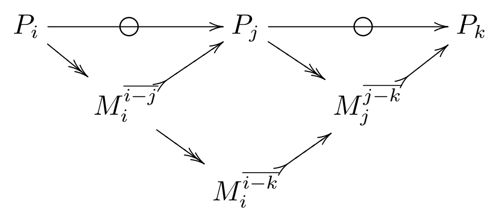
we can deduce $\mathrm{Im}(f)$ must be $M_i^{\overline{i-j}}$ and choose the standard basis $\tau_i^{\overline{i-j}} \circ \pi_i^{\overline{i-j}}$ of $\mathrm{Hom}_{H_n(q)}(P_i, P_j)$.

Notation 5.1.1. For convenience, let us introduce a new notation "$P_i \longrightarrow\!\!\!\!\!\!\!\ominus\;\; P_j$" to denote the linear basis $P_i \overset{\pi_i^{\overline{i-j}}}{\twoheadrightarrow} M_i^{\overline{i-j}} \overset{\tau_i^{\overline{i-j}}}{\rightarrowtail} P_j$ for $0 \leqslant i, j \leqslant n-1$.

Lemma 5.1.3. *$P_i \longrightarrow\!\!\!\!\!\!\!\ominus\;\; P_j \longrightarrow\!\!\!\!\!\!\!\ominus\;\; P_k = P_i \longrightarrow\!\!\!\!\!\!\!\ominus\;\; P_k$ if and only if one of the three conditions is satisfied*

$$\begin{cases} i \leqslant j \leqslant k, \\ k < i \leqslant j, \\ j \leqslant k < i. \end{cases}$$

Proof. In fact, the equality related to the composition of morphisms can be esteblished iff there is one of the following commutative diagrams where the composition of morphism is non zero



iff there are $n - \overline{i-j} < \overline{j-k}$, $\overline{i-k} \leqslant \overline{i-j}$ and $\overline{i-k} \leqslant \overline{j-k}$. There are six cases may happen:

(1) $i \leqslant j \leqslant k$:

$$\begin{cases} n-(n+i-j) < n+j-k \Leftrightarrow k < n+i, \\ n+(i-k) \leqslant n+(i-j) \Leftrightarrow j \leqslant k, \\ n+(i-k) \leqslant n+(j-k) \Leftrightarrow i \leqslant j. \end{cases}$$

(2) $i \leqslant k < j$:

$$\begin{cases} n+(i-k) \leqslant n+(i-j) \Leftrightarrow j \leqslant k \ \text{ which is a contradiction,} \\ n+(i-k) \leqslant j-k \Leftrightarrow n+i \leqslant j \ \text{ which is a contradiction.} \end{cases}$$

(3) $j \leqslant k < i$:

$$\begin{cases} n-(i-j) < n+j-k \Leftrightarrow k < i, \\ i-k \leqslant i-j \Leftrightarrow j \leqslant k, \\ i-k \leqslant n+(j-k) \Leftrightarrow i \leqslant n+j. \end{cases}$$

(4) $j < i \leqslant k$:

$$\begin{cases} n+(i-k) \leqslant i-j \Leftrightarrow n \leqslant k-j \ \text{ which is a contradiction,} \\ n+(i-k) \leqslant n+(j-k) \Leftrightarrow i \leqslant j \ \text{ which is a contradiction.} \end{cases}$$

(5) $k < i \leqslant j$:

$$\begin{cases} n-(n+i-j) < j-k \Leftrightarrow k < i, \\ i-k \leqslant n+(i-j) \Leftrightarrow j \leqslant n+k, \\ i-k \leqslant j-k \Leftrightarrow i \leqslant j. \end{cases}$$

(6) $k < j < i$:

$$\begin{cases} i-k \leqslant i-j \Leftrightarrow j \leqslant k \ \text{ which is a contradiction,} \\ i-k \leqslant j-k \Leftrightarrow i \leqslant j \ \text{ which is a contradiction.} \end{cases}$$

Thus the three cases satisfied are what we need.

□

Lemma 5.1.4. *Let $I = \{0, 1, \cdots, n-1\}$ and A, B be two nonempty subsets of I satisfying $A \cap B = \varnothing$. There is $(j, l) \in A \times B$ such that if $k \in I$ satisfies one of the following cases*

$$\begin{cases} j < k < l, & (*) \\ l < j < k, & (**) \\ k < l < j, & (***) \end{cases}$$

then $k \notin A \cup B$.

Proof. Otherwise, for any $j \in A$ and $l \in B$, there is $k \in I$ such that $\{(j,k),(k,l)\} \cap (A \times B) \neq \varnothing$. There are three cases $(*), (**)$ and $(***)$.

Case 1: If k satisfies $(*)$. There is $k_1 \in I$ satisfies $(*)$, whether $(j,k) \in A \times B$ or $(k,l) \in A \times B$. Repeating this process we get uncountable numbers which is in contradiction to the finiteness of I.

Case 2: If k satisfies $(**)$. Two cases are going to happen. One is $(j,k) \in A \times B$. Due to the Case 1, contradiction can be obtained. The other is $(k,l) \in A \times B$. Then there is $k_1 \in I$ such that $l < k < k_1$ or $k_1 < l < k$.

If $l < k < k_1$ and $(k, k_1) \in A \times B$. Back to Case 1, we get contradiction. If $(k_1, l) \in A \times B$, we repeat Case 2 until $(**)$ happens. If only $(**)$ appears in the next steps, we repeat this process getting uncountable numbers which is in contradiction to the finiteness of I. Otherwise we repeat this process until $(***)$ is happening which is also in contradiction to the finiteness of I.

If $k_1 < l < k$ is happening that is case 3.

Case 3: If k satisfies $(***)$. Two cases are going to happen. One is $(k,l) \in A \times B$. Back to Case 1, contradiction can be obtained. The other is $(j,k) \in A \times B$. Then there is $k_1 \in I$ such that $k < j < k_1$ or $k_1 < k < j$. We can get contradiction from the same reason in Case 2. □

Lemma 5.1.5. *Let $I = \{0, 1, \cdots, n-1\}$ and A, B, C be three nonempty subsets of I satisfying $A \cap B = \varnothing$ and $B \cap C = \varnothing$. Suppose that $(j,k) \in B \times C$ is given by Lemma 5.1.4 and (i,j,k) does not satisfy Lemma 5.1.3 for any $i \in A$, then we can find $l \in A$ such that (l, j) satisfies Lemma 5.1.4.*

Proof. As (i, j, k) does not satisfy Lemma 5.1.3, we deduce the following three situations:

$$\begin{cases} j < i \leqslant k, & (*) \\ i \leqslant k < j, & (**) \\ k < j < i. & (***) \end{cases}$$

If $j < k$, only $(*)$ will happen. We choose the the largest $l \in A$. Then (l, j) must satisfy Lemma 5.1.4.

If $j > k$ and there is $i \in A$ satisfying $(**)$. We choose the largest $l \in A$ such that $l < k$. Then (l, j) must satisfy Lemma 5.1.4. Otherwise, each $i \in A$ satisfies $(***)$. We also choose the largest $l \in A$, then (l, j) must satisfy Lemma 5.1.4. □

Proposition 5.1.6. *Let P_i $i \in \{0, \cdots, n-1\}$ be all non-isomorphic indecomposable $H_n(q)$-modules. All non-isomorphism indecomposable objects in $D^b(H_n(q)\text{-}\mathsf{mod})$ are like (adding shift ones):*

If X is a bounded complex, X must be isomorphic to

$$0 \longrightarrow P_{k_{-t}} \xrightarrow{\circ} P_{k_{-t-1}} \xrightarrow{\circ} \cdots P_{k_{-1}} \xrightarrow{\circ} P_{k_0} \longrightarrow 0;$$

If X is a unbounded complex, X must be isomorphism to

$$\cdots \longrightarrow P_{k_{-3}} \xrightarrow{\circ} P_{k_{-2}} \xrightarrow{\circ} P_{k_{-1}} \xrightarrow{\circ} P_{k_0} \longrightarrow 0$$

where all $k_i \in \{0, \cdots, n-1\}$ and any (k_s, k_{s+1}, k_{s+2}) in X does not satisfy the conditions in Lemma 5.1.3.

Proof. Denoted by $\mathcal{P}$ the additive full subcategory of all projective objects in $H_n(q)$-mod. For the reason that $D^b(H_n(q)\text{-}\mathsf{mod}) \simeq K^{-,b}(\mathcal{P})$, it is equivalent to determine all the indecompasoble objects in $K^{-,b}(\mathcal{P})$.

Let

$$X : \cdots \longrightarrow \bigoplus_{k=0}^{n-1} P_k^{v_k} \longrightarrow \bigoplus_{k=0}^{n-1} P_k^{i_k} \xrightarrow{d} \bigoplus_{k=0}^{n-1} P_k^{j_k} \longrightarrow 0$$

be the indecomposable object in $K^{-,b}(\mathcal{P})$. We might assume that the differentials between non-zero homogeneous components are non-zero, otherwise X is decompasoble.

Step 1: There is no harm in supposing that X has no direct summand being shaped like

$$0 \longrightarrow P_k \xrightarrow{\text{id}} P_k \longrightarrow 0$$

which is zero object in $K^{-,b}(\mathcal{P})$.

Step 2: Any differential d in X does not contain isomorphic component $\alpha\,\mathrm{id}_{P_h}(\alpha \in \Bbbk^\times)$ for some $0 \leqslant h \leqslant n-1$. Otherwise, d can be written as

$$\begin{pmatrix} \alpha\,\mathrm{id}_{P_h} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

There is an isomorphism between complexes by applying elementary transformations to d.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \bigoplus_{k=0}^{n-1} P_k^{\gamma_k} & \longrightarrow & P_h \oplus (\bigoplus_{k=0}^{n-1} P_k^{l'_k}) & \xrightarrow{\left(\begin{smallmatrix} \alpha\,\mathrm{id}_{P_h} & d_{12} \\ d_{21} & d_{22} \end{smallmatrix}\right)} & P_h \oplus (\bigoplus_{k=0}^{n-1} P_k^{m'_k}) & \longrightarrow & \bigoplus_{k=0}^{n-1} P_k^{u_k} & \longrightarrow \cdots \\
& \| & & \downarrow \cong & & \downarrow \cong & & \| & \\
\cdots \longrightarrow & \bigoplus_{k=0}^{n-1} P_k^{\gamma_k} & \longrightarrow & P_h \oplus (\bigoplus_{k=0}^{n-1} P_k^{l'_k}) & \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_{P_h} & 0 \\ 0 & * \end{smallmatrix}\right)} & P_h \oplus (\bigoplus_{k=0}^{n-1} P_k^{m'_k}) & \longrightarrow & \bigoplus_{k=0}^{n-1} P_k^{u_k} & \longrightarrow \cdots
\end{array}$$

where

$$\begin{cases} l'_k = l_k (k \neq h), & l'_h = l_h - 1, \\ m'_k = m_k (k \neq h), & m'_h = m_h - 1. \end{cases}$$

Then X can be written as

$$\cdots \longrightarrow \bigoplus_{k=0}^{n-1} P_k^{\gamma_k} \xrightarrow{\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)} P_h \oplus (\bigoplus_{k=0}^{n-1} P_k^{l'_k}) \xrightarrow{\left(\begin{smallmatrix} \mathrm{id}_{P_h} & 0 \\ 0 & * \end{smallmatrix}\right)} P_h \oplus (\bigoplus_{k=0}^{n-1} P_k^{m'_k}) \xrightarrow{(a\ b)} \bigoplus_{k=0}^{n-1} P_k^{u_k} \longrightarrow \cdots$$

$$\begin{pmatrix} \mathrm{id}_{P_h} & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0, \quad \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} \mathrm{id}_{P_h} & 0 \\ 0 & * \end{pmatrix} = 0,$$

So $f = 0$ and $a = 0$. Hence X has direct summand

$$0 \longrightarrow P_h \xrightarrow{\mathrm{id}} P_h \longrightarrow 0$$

which contradicts to Step 1.

Step 3: There is no such direct summand P_h in $\bigoplus_{k=0}^{n-1} P_k^{j_k}$ such that the corresponding component is $\bigoplus_{k=0}^{n-1} P_k^{i_k} \xrightarrow{0} P_h$. Otherwise, there exist

$$X : \cdots \longrightarrow \bigoplus_{k=0}^{n-1} P_k^{v_k} \longrightarrow \bigoplus_{k=0}^{n-1} P_k^{i_k} \xrightarrow{d} P_h \oplus (\bigoplus_{k=0}^{n-1} P_k^{j'_k}) \longrightarrow 0$$

where $j'_k = j_k$ when $k \neq h$ and $j'_h = j_h - 1$. Then the direct summand $0 \to P_h \longrightarrow 0$ appears in X.

Step 4: $d:\bigoplus\limits_{k=0}^{n-1}P_k^{i_k}\longrightarrow\bigoplus\limits_{k=0}^{n-1}P_k^{j_k}\longrightarrow 0$ must be shaped like

$$\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & 0 \\ 0 & * \end{pmatrix}$$

where j,l are given by Lemma 5.1.4 and $*$ is consisted of some elements in $\Bbbk(\cdot \xrightarrow{\ominus} \cdot)$.

Indeed, from Step 2 we can deduce that any two objects in different degrees are disjoint. Here by Lemma 5.1.4 we can shoose $0\leqslant j,l\leqslant n-1$ such that P_j,P_l are direct summand of $\bigoplus\limits_{k=0}^{n-1}P_k^{i_k},\bigoplus\limits_{k=0}^{n-1}P_k^{j_k}$ respectively. And for $k\in\{0,1,\cdots,n-1\}$ satisfying one of cases $(*),(**),(***)$ in Lemma 5.1.4, we know the coefficient of $P_j \xrightarrow{\ominus} P_k$ and $P_k \xrightarrow{\ominus} P_l$ are all zero. Then d can be shaped like

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}:\ \bigoplus_{k=0}^{n-1}P_k^{i_k}\longrightarrow P_l\oplus(\bigoplus_{k=0}^{n-1}P_k^{j'_k})\longrightarrow 0$$

where $j'_k=j_k$ when $k\neq l$ and $j'_l=j_l-1$. By Step 2 and Step 3, $d_1:\bigoplus\limits_{k=0}^{n-1}P_k^{i_k}\to P_l$ can be written as

$$d_1=(\alpha_?(P_? \xrightarrow{\ominus} P_l),\cdots,\alpha_?(P_j \xrightarrow{\ominus} P_l),\cdots,\alpha_?(P_j \xrightarrow{\ominus} P_l),0,\cdots,0,\alpha_?(P_i \xrightarrow{\ominus} P_l),\cdots,*)$$

or

$$d_1=(0,\cdots,0,\alpha_?(P_j \xrightarrow{\ominus} P_l),\cdots,\alpha_?(P_j \xrightarrow{\ominus} P_l),\alpha_?(P_i \xrightarrow{\ominus} P_l),\cdots,*)$$

or

$$d_1=(\alpha_?(P_? \xrightarrow{\ominus} P_l),\cdots,\alpha_?(P_j \xrightarrow{\ominus} P_l),\cdots,\alpha_?(P_j \xrightarrow{\ominus} P_l),0,\cdots,0,)$$

where not all $\alpha_?\in\Bbbk$ are equal to zero. By Lemma 5.1.3 there is $\alpha_?(P_j \xrightarrow{\ominus} P_l)\neq 0$ which can eliminate all the other terms through column transformation. In other words, d_1 can be transformed into

$$d_1=(P_j \xrightarrow{\ominus} P_l,0,\cdots,0).$$

Furthermore, for any direct summand P_k of $\bigoplus\limits_{k=0}^{n-1}P_k^{j_k}$ such that $P_j \xrightarrow{\ominus} P_k\neq 0$, we know (j,l,k) meets conditions in Lemma 5.1.3. Then d can be transformed into

$$\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & 0 \\ 0 & * \end{pmatrix}$$

where $*$ is consisted of some elements in $\Bbbk(\cdot \xrightarrow{\ominus} \cdot)$.

Actually there is an isomorphism between complexes by applying elementary transformations to d.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \bigoplus\limits_{k=0}^{n-1} P_k^{v_k} & \longrightarrow & P_j \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{i'_k}) & \xrightarrow{\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & * \\ * & * \end{pmatrix}} & P_l \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{j'_k}) & \longrightarrow 0 \\
& \Vert & & \downarrow \cong & & \downarrow \cong & \\
\cdots \longrightarrow & \bigoplus\limits_{k=0}^{n-1} P_k^{v_k} & \longrightarrow & P_j \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{i'_k}) & \xrightarrow{\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & 0 \\ 0 & * \end{pmatrix}} & P_l \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{j'_k}) & \longrightarrow 0
\end{array}$$

Step 5: According to previous discussion, we assume X is an indecomposable object in $K^{-,b}(\mathcal{P})$

$$X : \cdots \longrightarrow X_t \xrightarrow{d_X^t} X_{t-1} \xrightarrow{d_X^{t-1}} \cdots \longrightarrow X_0 \to 0$$

and for $s < t$, all d_X^s are shaped like

$$\begin{pmatrix} P_{k_s} \xrightarrow{\ominus} P_{k_{s+1}} & 0 \\ 0 & * \end{pmatrix} \tag{5.1.2}$$

where $*$ is consisted of some elements in $\Bbbk(\cdot \xrightarrow{\ominus} \cdot)$. If $X_t = 0$, as X is indecomposable then X must be isomorphic to complex like

$$X : 0 \longrightarrow P_{k_{-t}} \xrightarrow{\ominus} P_{k_{-t-1}} \xrightarrow{\ominus} \cdots P_{k_{-1}} \xrightarrow{\ominus} P_{k_0} \longrightarrow 0$$

where $0 \leqslant k_0, k_1, \cdots, k_t \leqslant n-1$ and any (k_s, k_{s+1}, k_{s+2}) in X does not satisfy the conditions in Lemma 5.1.3. Moreover, $\mathrm{End}_{\Bbbk}(X) \cong \Bbbk$.

Next we claim: If $X_n \neq 0$, then d_X^t can be transformed into a form similar to the matrix (5.1.2).

By Step 4, X is isomorphism to the following complex

$$\cdots \longrightarrow P_j \oplus (\bigoplus_{k=0}^{n-1} P_k^{\eta'_k}) \xrightarrow{\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & * \\ * & * \end{pmatrix}} P_l \oplus (\bigoplus_{k=0}^{n-1} P_k^{\beta'_k}) \xrightarrow{\begin{pmatrix} P_l \xrightarrow{\ominus} P_m & 0 \\ 0 & * \end{pmatrix}} P_m \oplus (\bigoplus_{k=0}^{n-1} P_k^{\gamma'_k}) \longrightarrow \cdots$$

where X_t, X_{t-1}, X_{t-2} are isomorphism to $\bigoplus\limits_{k=0}^{n-1} P_k^{\eta_k}, \bigoplus\limits_{k=0}^{n-1} P_k^{\beta_k}, \bigoplus\limits_{k=0}^{n-1} P_k^{\gamma_k}$ respectively and $(j,l), (l,m)$ satisfy the premise of lemma 5.1.4. Note that we can find (j,l) due to Lemma 5.1.5 and the composition of differential is zero. Then by applying column transformation there is an isomorphism of complexes:

$$\begin{array}{cccccccc}
\cdots \longrightarrow & P_j \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{\eta'_k}) & \xrightarrow{\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & * \\ * & * \end{pmatrix}} & P_l \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{\beta'_k}) & \xrightarrow{\begin{pmatrix} P_l \xrightarrow{\ominus} P_m & 0 \\ 0 & * \end{pmatrix}} & P_m \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{\gamma'_k}) & \longrightarrow \cdots \\
& \downarrow & & \Vert & & \Vert & \\
\cdots \longrightarrow & P_j \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{\eta'_k}) & \xrightarrow{\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & 0 \\ * & * \end{pmatrix}} & P_l \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{\beta'_k}) & \xrightarrow{\begin{pmatrix} P_l \xrightarrow{\ominus} P_m & 0 \\ 0 & * \end{pmatrix}} & P_m \oplus (\bigoplus\limits_{k=0}^{n-1} P_k^{\gamma'_k}) & \longrightarrow \cdots
\end{array}$$

Next we apply row transformation to

$$\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & 0 \\ * & * \end{pmatrix}.$$

In the meanwhile, we need to make sure that the differentials on the right side still are diagonal matrixes. We get the following commutative diagram:

$$\begin{array}{ccccccc}
P_j \oplus (\bigoplus_{k=0}^{n-1} P_k^{\eta'_k}) & \xrightarrow{\begin{pmatrix} P_j \twoheadrightarrow P_l & 0 \\ * & * \end{pmatrix}} & P_l \oplus (\bigoplus_{k=0}^{n-1} P_k^{\beta'_k}) & \xrightarrow{\begin{pmatrix} P_l \twoheadrightarrow P_m & 0 \\ 0 & *_1 \end{pmatrix}} & P_m \oplus (\bigoplus_{k=0}^{n-1} P_k^{\gamma'_k}) & \longrightarrow & \cdots \\
\Big\| & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}} & & \Big\downarrow {\scriptstyle \begin{pmatrix} 1 & 0 \\ d_1 & 1 \end{pmatrix}} & & \\
P_j \oplus (\bigoplus_{k=0}^{n-1} P_k^{\eta'_k}) & \xrightarrow{\begin{pmatrix} P_j \twoheadrightarrow P_l & 0 \\ 0 & * \end{pmatrix}} & P_l \oplus (\bigoplus_{k=0}^{n-1} P_k^{\beta'_k}) & \xrightarrow{\begin{pmatrix} P_l \twoheadrightarrow P_m & 0 \\ 0 & *_1 \end{pmatrix}} & P_m \oplus (\bigoplus_{k=0}^{n-1} P_k^{\gamma'_k}) & \longrightarrow & \cdots
\end{array}$$

where $c = \begin{pmatrix} \alpha_{s_1}(P_l \twoheadrightarrow P_{s_1}) \\ \alpha_{s_2}(P_l \twoheadrightarrow P_{s_2}) \\ \vdots \\ \alpha_{s_e}(P_l \twoheadrightarrow P_{s_e}) \end{pmatrix}$, $\{P_{s_1}, \cdots, P_{s_e}\}$ are all direct summands in $\bigoplus_{k=0}^{n-1} P_k^{\beta'_k}$ and 1 are identity matrixes of different orders. In addition, we need to find d_1 such that

$$\begin{pmatrix} 1 & 0 \\ d_1 & 1 \end{pmatrix} \begin{pmatrix} P_l \twoheadrightarrow P_m & 0 \\ 0 & *_1 \end{pmatrix} = \begin{pmatrix} P_l \twoheadrightarrow P_m & 0 \\ 0 & *_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix},$$

that is

$$\begin{pmatrix} P_l \twoheadrightarrow P_m & 0 \\ d_1(P_l \twoheadrightarrow P_m) & *_1 \end{pmatrix} = \begin{pmatrix} P_l \twoheadrightarrow P_m & 0 \\ *_1 c & *_1 \end{pmatrix}.$$

In other words, matrix d_1 need satisfies

$$d_1(P_l \twoheadrightarrow P_m) = *_1 c.$$

Since for any $P_{k'}$ as a direct summand of $\bigoplus_{k=0}^{n-1} P_k^{\gamma'_k}$ or $\bigoplus_{k=0}^{n-1} P_k^{\beta'_k}$, we know (l, m, k') meets conditions in Lemma 5.1.3. Moreover,

$$*_1 c = \begin{pmatrix} \alpha_?(P_{s_1} \twoheadrightarrow P_{k_1}) & \cdots & \alpha_?(P_{s_e} \twoheadrightarrow P_{k_1}) \\ \alpha_?(P_{s_1} \twoheadrightarrow P_{k_2}) & \cdots & \alpha_?(P_{s_e} \twoheadrightarrow P_{k_2}) \\ \vdots & \cdots & \vdots \\ \alpha_?(P_{s_1} \twoheadrightarrow P_{k_q}) & \cdots & \alpha_?(P_{s_e} \twoheadrightarrow P_{k_q}) \end{pmatrix} \cdot \begin{pmatrix} \alpha_{s_1}(P_l \twoheadrightarrow P_{s_1}) \\ \alpha_{s_2}(P_l \twoheadrightarrow P_{s_2}) \\ \vdots \\ \alpha_{s_e}(P_l \twoheadrightarrow P_{s_e}) \end{pmatrix} = \begin{pmatrix} \alpha_{k_1}(P_l \twoheadrightarrow P_{k_1}) \\ \alpha_{k_2}(P_l \twoheadrightarrow P_{k_2}) \\ \vdots \\ \alpha_{k_q}(P_l \twoheadrightarrow P_{k_q}) \end{pmatrix}.$$

Here $\{P_{k_1}, \cdots, P_{k_q}\}$ are all direct summands in $\bigoplus\limits_{k=0}^{n-1} P_k^{\gamma'_k}$ and it may happen

$$P_l \xrightarrow{\ominus} P_{s_i} \xrightarrow{\ominus} P_{k_j} = 0$$

where $i \in \{1, \cdots, e\}, j \in \{1, \cdots, q\}$, then some α_{k_j} could be zero. Therefore, we can choose

$$d_1 = \begin{pmatrix} \alpha_{k_1}(P_m \xrightarrow{\ominus} P_{k_1}) \\ \alpha_{k_2}(P_m \xrightarrow{\ominus} P_{k_2}) \\ \vdots \\ \alpha_{k_q}(P_m \xrightarrow{\ominus} P_{k_q}) \end{pmatrix}$$

which will make the second square commute. Continue the discussion above, there are $d_2, d_3, \cdots$ making squares commute. Thus X is isomorphism to the following complex:

$$\cdots \longrightarrow P_j \oplus (\bigoplus_{k=0}^{n-1} P_k^{\eta'_k}) \xrightarrow{\begin{pmatrix} P_j \xrightarrow{\ominus} P_l & 0 \\ 0 & * \end{pmatrix}} P_l \oplus (\bigoplus_{k=0}^{n-1} P_k^{\beta'_k}) \xrightarrow{\begin{pmatrix} P_l \xrightarrow{\ominus} P_m & 0 \\ 0 & * \end{pmatrix}} P_m \oplus (\bigoplus_{k=0}^{n-1} P_k^{\gamma'_k}) \longrightarrow \cdots$$

where all differentials are shaped like

$$\begin{pmatrix} P_{k_i} \xrightarrow{\ominus} P_{k_j} & 0 \\ 0 & * \end{pmatrix}.$$

If X is unbounded complex, X must be isomorphism to

$$\cdots \longrightarrow P_{k_{-3}} \xrightarrow{\ominus} P_{k_{-2}} \xrightarrow{\ominus} P_{k_{-1}} \xrightarrow{\ominus} P_{k_0} \longrightarrow 0$$

where all $k_i \in \{0, \cdots, n-1\}$ and any (k_s, k_{s+1}, k_{s+2}) in X does not satisfy the conditions in Lemma 5.1.3. $\square$

§5.2 Derived Green ring of Sweedler's 4-dimensional Hopf algebra

Since Taft algebras $H_n(q)$ are finite-dimensional Hopf algebras, $H_n(q)$-mod are finite tensor categories. Then $D^b(H_n(q)\text{-mod})$ is a tensor triangulated categories by Lemma 2.3.9. In [43], Radford proved that Sweedler's 4-dimensional Hopf algebra is quasitriangular (see [16] for definition). Since the module category of a quasitriangular Hopf algebra H is braided, there is a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ for any $X, Y \in H\text{-mod}$ such that hexagonal diagrams commute (see [18, Chapter

8]). In this section, I will only consider the case of Sweedler's 4-dimensional Hopf algebra $H_2(-1)$-mod, and compute the Green ring of $D^b(H_2(-1)\text{-mod})$.

Proposition 5.1.6 describes the indecomposable complexes in $D^b(H_n(-1)\text{-mod})$. I present all the indecomposable complexes in the case of $D^b(H_2(-1)\text{-mod})$ as follows.

Let P_0, P_1 be the projective cover of two simple modules S_0 and S_1 in $H_2(-1)$-mod. I use the notations X_l and Y_l to denote the bounded indecomposable complexes with the first non-zero object (from the right) in degree zero:

$$X_l:\ 0 \longrightarrow P_? \xrightarrow{\tau_?\pi_?} \cdots \xrightarrow{\tau_1\pi_1} P_0 \xrightarrow{\tau_0\pi_0} P_1 \xrightarrow{\tau_1\pi_1} P_0 \xrightarrow{\tau_0\pi_0} P_1 \longrightarrow 0,$$

$$Y_l:\ 0 \longrightarrow P_? \xrightarrow{\tau_?\pi_?} \cdots \xrightarrow{\tau_0\pi_0} P_1 \xrightarrow{\tau_1\pi_1} P_0 \xrightarrow{\tau_0\pi_0} P_1 \xrightarrow{\tau_1\pi_1} P_0 \longrightarrow 0$$

where

$$\pi_i : P_i \twoheadrightarrow S_i, \quad x^k e_i \mapsto \overline{x^k e_i} \qquad i,\ k \in \{0,1\}$$

$$\tau_i : S_i \rightarrowtail P_j, \quad \overline{e_i} \mapsto x^{2-\overline{i-j}} e_j \qquad i,\ j \in \{0,1\}$$

and l is the number of non-zero objects in each indecomposable complex.

For the unbounded case

$$X_\infty:\ \cdots \xrightarrow{\tau_1\pi_1} P_0 \xrightarrow{\tau_0\pi_0} P_1 \xrightarrow{\tau_1\pi_1} P_0 \xrightarrow{\tau_0\pi_0} P_1 \longrightarrow 0,$$

$$Y_\infty:\ \cdots \xrightarrow{\tau_0\pi_0} P_1 \xrightarrow{\tau_1\pi_1} P_0 \xrightarrow{\tau_0\pi_0} P_1 \xrightarrow{\tau_1\pi_1} P_0 \longrightarrow 0$$

which are the projective resolutions of simple objects S_1 and S_0 respectively. That means X_∞, Y_∞ are isomorphic to S_1, S_0 in $D^b(H_2(-1)\text{-mod})$. So all non-isomorphic indecomposable complexes in $D^b(H_2(-1)\text{-mod})$ are

$$X_l\,[i],\ Y_l\,[i],\ S_1\,[i],\ S_0\,[i] \qquad (l \geqslant 0,\ i \in \mathbb{Z}).$$

Folloing the same notation in Section 5.1, the following statement tells us that the tensor product of two projective objects in $H_n(q)$-mod is isomorphic $H_n(q)$ itself.

Proposition 5.2.1. ([14, Proposition 3.2]) *Let $i,\ j \in \mathbb{Z}_n$ and $1 \leqslant k \leqslant n$. There are isomorphisms of $H_n(q)$-modules*

(i) $M_i^k \otimes S_j \cong S_j \otimes M_i^k \cong M^k_{\overline{i+j}}$,

(ii) $P_i \otimes P_j \cong P_j \otimes P_i \cong \bigoplus_{k=0}^{n-1} P_k$.

Applying Proposition 5.2.1 to $H_2(-1)$-mod, we know the following isomorphisms where $j \in \mathbb{Z}_2$:

$$S_0 \otimes S_0 \cong S_0 \cong S_1 \otimes S_1,$$

$$S_1 \otimes S_0 \cong S_1 \cong S_0 \otimes S_1,$$

$$S_0 \otimes P_j \cong P_j \cong P_j \otimes S_0,$$

$$S_1 \otimes P_j \cong P_{\overline{j+1}} \cong P_j \otimes S_1.$$

That is there are isomorphisms in $D^b(H_2(-1)\text{-mod})$:

$$Y_\infty \widetilde{\otimes} Y_\infty \cong Y_\infty \cong X_\infty \widetilde{\otimes} X_\infty,$$

$$X_\infty \widetilde{\otimes} Y_\infty \cong X_\infty \cong Y_\infty \widetilde{\otimes} X_\infty,$$

$$Y_\infty \widetilde{\otimes} X_l \cong X_l \cong X_l \widetilde{\otimes} Y_\infty, \;\; Y_\infty \widetilde{\otimes} Y_l \cong Y_l \cong Y_l \widetilde{\otimes} Y_\infty$$

$$X_\infty \widetilde{\otimes} X_l \cong Y_l \cong X_l \widetilde{\otimes} X_\infty, \;\; X_\infty \widetilde{\otimes} Y_l \cong X_l \cong Y_l \widetilde{\otimes} X_\infty.$$

Lemma 5.2.2. *For Sweedler's 4-dimensional Hopf algebra $H_2(-1)$-mod and $X, Y \in D^b(H_2(-1)\text{-mod})$, there is an isomorphim*

$$X \widetilde{\otimes} Y \cong Y \widetilde{\otimes} X$$

in $D^b(H_2(-1)\text{-mod})$ where $\widetilde{\otimes}$ is the tensor product inherited from $H_2(-1)$-mod.

Proof. It is sufficient to consider the indecomposable case. So we assume X and Y are indecomposable complexes. Firstly, if either of X or Y is unbounded, we have already known the results by above statements. Next, we only need to deal with bounded cases. Let X_l, X_m, Y_s and Y_t be indecomposable bounded complexes in $D^b(H_2(-1)\text{-mod})$ where $l \leqslant m \leqslant s \leqslant t$. There are three different cases: $X_l \widetilde{\otimes} X_m \cong X_m \widetilde{\otimes} X_l$, $Y_s \widetilde{\otimes} Y_t \cong Y_t \widetilde{\otimes} Y_s$ and $X_m \widetilde{\otimes} Y_s \cong Y_s \widetilde{\otimes} X_m$. I only verify the first case, the others are the same.

Claim: $X_l \widetilde{\otimes} X_m \cong X_m \widetilde{\otimes} X_l$.

If

$$X_l: \; 0 \longrightarrow P_? \xrightarrow{\tau_? \pi_?} \cdots \xrightarrow{\tau_1 \pi_1} P_0 \xrightarrow{\tau_0 \pi_0} P_1 \xrightarrow{\tau_1 \pi_1} P_0 \xrightarrow{\tau_0 \pi_0} P_1 \longrightarrow 0,$$

$$X_m: \; 0 \longrightarrow P_? \xrightarrow{\tau_? \pi_?} \cdots \xrightarrow{\tau_1 \pi_1} P_0 \xrightarrow{\tau_0 \pi_0} P_1 \xrightarrow{\tau_1 \pi_1} P_0 \xrightarrow{\tau_0 \pi_0} P_1 \longrightarrow 0,$$

Recall that $X_l \widetilde{\otimes} X_m$ and $X_m \widetilde{\otimes} X_l$ are total complexes

$$X_l \widetilde{\otimes} X_m: \ 0 \longrightarrow (X_l \widetilde{\otimes} X_m)^{-l-m+2} \xrightarrow{d^{-l-m+2}_{X_l \widetilde{\otimes} X_m}} \cdots \xrightarrow{d^{-2}_{X_l \widetilde{\otimes} X_m}} P_0 \otimes P_1 \oplus P_0 \otimes P_1 \xrightarrow{d^{-1}_{X_l \widetilde{\otimes} X_m}} P_1 \otimes P_1 \longrightarrow 0$$

$$X_m \widetilde{\otimes} X_l: \ 0 \longrightarrow (X_m \widetilde{\otimes} X_l)^{-l-m+2} \xrightarrow{d^{-l-m+2}_{X_m \widetilde{\otimes} X_l}} \cdots \xrightarrow{d^{-2}_{X_m \widetilde{\otimes} X_l}} P_0 \otimes P_1 \oplus P_0 \otimes P_1 \xrightarrow{d^{-1}_{X_m \widetilde{\otimes} X_l}} P_1 \otimes P_1 \longrightarrow 0$$

where

$$(X_l \widetilde{\otimes} X_m)^n = \bigoplus_{i+j=n} P_{\overline{i+1}} \otimes P_{\overline{j+1}}$$

$$d^n_{X_l \widetilde{\otimes} X_m} = \bigoplus_{i+j=n} (\tau_{\overline{i+1}} \pi_{\overline{i+1}}) \otimes \mathrm{id}_{P_{\overline{j+1}}} + (-1)^i \, \mathrm{id}_{P_{\overline{i+1}}} \otimes (\tau_{\overline{j+1}} \pi_{\overline{j+1}})$$

for any $n \in \mathbb{Z}$. Similar for $X_m \widetilde{\otimes} X_l$. Notice that there are natural isomorphisms

$$c_{P_i, P_j}: P_i \otimes P_j \to P_j \otimes P_i, \quad i, \ j \in \mathbb{Z}_2$$

which give a isomorphic chain map between $X_l \widetilde{\otimes} X_m$ and $X_m \widetilde{\otimes} X_l$.

□

By the above lemma, we know that $\mathrm{G}_0(D^b(H_2(-1)\text{-}\mathsf{mod}))$ is commutative ring.

Theorem 5.2.3. *The derived Green ring $G_0(D^b(H_2(-1)\text{-}\mathsf{mod}))$ of Sweedler's 4-dimensional Hopf algebra $H_2(-1)\text{-}\mathsf{mod}$ is commutative and is generated by elements $s[k]$, $y_i[k]$ where $k \in \mathbb{Z}$ subject to the relations*

$$s^2 = 1,$$

$$y_i[k] y_j[l] = y_i y_j[k+l] = y_j[k+l] + s^i y_j[i-1+k+l], \ \textit{if} \ i \geqslant j \geqslant 1.$$

Proof. In order to decompose the tensor product of indecomposable complexes into the direct sum of indecomposable complexes, we should describe the morphisms of the decomposition in $H_2(-1)\text{-}\mathsf{mod}$.

$$f_{0,1}: P_0 \otimes P_1 \longrightarrow P_0 \oplus P_1, \qquad f_{1,0}: P_1 \otimes P_0 \longrightarrow P_0 \oplus P_1$$

$$e_0 \otimes xe_1 \longmapsto (e_0, 0) \qquad xe_1 \otimes e_0 \longmapsto (e_0, 0)$$

$$xe_0 \otimes xe_1 \longmapsto (xe_0, 0) \qquad xe_1 \otimes xe_0 \longmapsto (xe_0, 0)$$

$$e_0 \otimes e_1 \longmapsto (0, e_1) \qquad e_1 \otimes e_0 \longmapsto (0, e_1)$$

$$xe_0 \otimes e_1 \longmapsto (e_0, -xe_1) \qquad e_1 \otimes xe_0 \longmapsto (e_0, -xe_1)$$

$$
\begin{aligned}
f_{0,0} : P_0 \otimes P_1 &\longrightarrow P_0 \oplus P_1, & f_{0,0} : P_1 \otimes P_1 &\longrightarrow P_0 \oplus P_1 \\
xe_0 \otimes e_0 &\longmapsto (0, e_1) & e_1 \otimes xe_1 &\longmapsto (0, e_1) \\
xe_0 \otimes xe_0 &\longmapsto (0, xe_1) & xe_1 \otimes xe_1 &\longmapsto (0, xe_1) \\
e_0 \otimes e_0 &\longmapsto (e_0, 0) & e_1 \otimes e_1 &\longmapsto (e_0, 0) \\
e_0 \otimes xe_0 &\longmapsto (xe_0, -e_1) & e_1 \otimes xe_1 &\longmapsto (-xe_0, e_1).
\end{aligned}
$$

We only deal with the case $X_i \widetilde{\otimes} X_j$. Using $f_{i,j}$ one can get $X_1 \widetilde{\otimes} X_2$ is isomorphic to the following complex

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 \otimes P_0 & \xrightarrow{\mathrm{id}_{P_1} \otimes \tau_0\pi_0} & P_1 \otimes P_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{1,0}} & & \downarrow{\scriptstyle f_{1,1}} & & \\
0 & \longrightarrow & P_0 \oplus P_1 & \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ \tau_0\pi_0 & \mathrm{id}_{P_1} \end{smallmatrix}\right)} & P_0 \oplus P_1 & \longrightarrow & 0
\end{array}
$$

Repeating this process, we know all the correspondence between the differentials in the total complexes and the differentials in the "new" complexes. Namely, there are the following correspondence:

$$
\begin{pmatrix} \mathrm{id}_{P_1} \otimes \tau_0\pi_0 \\ \tau_1\pi_1 \otimes \mathrm{id}\, p_0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 0 \\ \tau_0\pi_0 & \mathrm{id}_{P_1} \\ 0 & 0 \\ 0 & \mathrm{id}_{P_1} \end{pmatrix}, \qquad \begin{pmatrix} -\mathrm{id}_{P_1} \otimes \tau_1\pi_1 \\ \tau_0\pi_0 \otimes \mathrm{id}\, p_1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & -\tau_1\pi_1 \\ 0 & \mathrm{id}_{P_1} \\ 0 & \text{-}\tau_1\pi_1 \\ \tau_0\pi_0 & \mathrm{id}_{P_1} \end{pmatrix}
$$

$$
\begin{pmatrix} \mathrm{id}_{P_1} \otimes \tau_1\pi_1 \\ \tau_1\pi_1 \otimes \mathrm{id}\, p_0 \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathrm{id}_{P_0} & 0 \\ \tau_0\pi_0 & 0 \\ \mathrm{id}_{P_0} & \tau_1\pi_1 \\ -\tau_0\pi_0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -\mathrm{id}_{P_0} \otimes \tau_0\pi_0 \\ \tau_1\pi_1 \otimes \mathrm{id}\, p_0 \end{pmatrix} \leftrightarrow \begin{pmatrix} -\mathrm{id}_{P_0} & \tau_1\pi_1 \\ 0 & 0 \\ \mathrm{id}_{P_0} & 0 \\ 0 & 0 \end{pmatrix}
$$

Regarding the "new" complex, it is more convenient for us to deduce the decomposition. Using the same method for other total complexes and applying Künneth formula in Theorem 4.1.8, we get the following relations $i \geqslant j \geqslant 1$:

(1) $X_i \widetilde{\otimes} Y_j \cong \begin{cases} X_j \otimes X_j[i-1], & \text{when } i \text{ is even,} \\ X_j \otimes Y_j[i-1], & \text{when } i \text{ is odd.} \end{cases}$

(2) $Y_i \widetilde{\otimes} X_j \cong \begin{cases} X_j \otimes X_j[i-1], & \text{when } i \text{ is even,} \\ X_j \otimes Y_j[i-1], & \text{when } i \text{ is odd.} \end{cases}$

(3) $X_i \widetilde{\otimes} X_j \cong \begin{cases} Y_j \otimes Y_j[i-1], & \text{when } i \text{ is even,} \\ Y_j \otimes X_j[i-1], & \text{when } i \text{ is odd.} \end{cases}$

(4) $Y_i \widetilde{\otimes} Y_j \cong \begin{cases} Y_j \otimes Y_j[i-1], & \text{when } i \text{ is even,} \\ Y_j \otimes X_j[i-1], & \text{when } i \text{ is odd.} \end{cases}$

Hence for $i \geqslant j \geqslant 1.$, we can deduce the following isomorphisms:

(1) $X_i \widetilde{\otimes} Y_j \cong X_j \oplus (S_1^{\widetilde{\otimes}^i} \widetilde{\otimes} X_j[i-1])$;

(2) $Y_i \widetilde{\otimes} X_j \cong X_j \oplus (S_1^{\widetilde{\otimes}^i} \widetilde{\otimes} X_j[i-1])$;

(3) $X_i \widetilde{\otimes} X_j \cong Y_j \oplus (S_1^{\widetilde{\otimes}^i} \widetilde{\otimes} Y_j[i-1])$;

(4) $Y_i \widetilde{\otimes} Y_j \cong Y_j \oplus (S_1^{\widetilde{\otimes}^i} \widetilde{\otimes} Y_j[i-1])$.

That means $X_i \widetilde{\otimes} Y_j \cong Y_i \widetilde{\otimes} X_j$ and $X_i \widetilde{\otimes} X_j \cong Y_i \widetilde{\otimes} Y_j$. Then we view $S_1[k]$, $Y_i[k]$ as generators s and $y_i[k]$, which completes the proof. $\square$

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Papers published during my PhD program

1. Y. Y. Xu; G. X. Liu, On the stable equivalences between finite tensor categories. Proc. Amer. Math. Soc. 151 (2023), no.5, 1867-1876.

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