南京大学

研究生毕业论文

（申请博士学位）

论 文 题 目: 有限维具有对偶Chevalley性质的Hopf代数的余表示型分类

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专 业 名 称: 基 础 数 学

研 究 方 向: Hopf代数与张量范畴

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2025 年 3 月

学　　　　号: **DG21210018**
论文答辩日期: **2025** 年 **5** 月
指 导 教 师: （签字）

Dissertation for the Doctoral Degree of Science

Classification of finite-dimensional Hopf algebras with the dual Chevalley property according to their corepresentation type

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Fundamental Mathematics

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March, 2025

Contents

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摘 要

在表示型的观点下，每一个有限维代数恰好属于以下三种类型中：有限表示型，可驯表示型和野表示型。自此，依据表示型去分类给定种类的代数得到了极大的关注。

在 Hopf 代数的情形下，人们目前的主要工作主要集中在研究点Hopf代数或者它们的对偶，基本 Hopf 代数中。而具有（对偶）Chevalley 性质的 Hopf 代数是基本（点）Hopf 代数的自然推广。人们对于研究这类 Hopf 代数的诸多方面都存在着极大的兴趣，其中之一就是它们的分类问题。

在这篇论文中，我们尝试依据余表示型去分类具有对偶 Chevalley 性质的 Hopf 代数。在这个过程中，我们主要运用的工具是连接箭图。其中，这篇论文用到的核心观点就是我们可以利用可乘矩阵和本原矩阵去刻画连接箭图的结构。

我们令 H 是一个定义在代数闭域 $\Bbbk$ 上的有限维具有对偶 Chevalley 性质的 Hopf 代数。下面是我们的主要结果：

首先，如果 H 不是余半单的，我们证明了 H 是有限余表示型当且仅当它是 coNakayama 的，也当且仅当 H 的连接箭图是基本圈的不交并，还当且仅当包含 $\Bbbk 1$ 的连接不可分解分支 $H_{(1)}$ 是一个点 Hopf 代数,并且 $H_{(1)}$ 的连接箭图是一个基本圈。在域 $\Bbbk$ 的特征是 0 时，H 是有限余表示型当且仅当要么它是余半单的，要么它不是余半单的且满足 $H_{(1)} \cong A(n,d,\mu,q)$。在域 $\Bbbk$ 的特征是 p 时，H 是有限余表示型当且仅当要么它是余半单的，要么它不是余半单的且满足 $H_{(1)} \cong C_d(n)$。

最后，在域 $\Bbbk$ 的特征是 0 时，我们证明了 $\mathrm{gr}^c(H)$ 是可驯余表示型当且仅当存在某个有限维半单的 Hopf 代数 H' 和某个特殊理想 I 使得我们有 $\mathrm{gr}^c(H) \cong (\Bbbk\langle x,y\rangle/I)^* \times H'$。之后，利用连接箭图和玻色化的工具，我们讨论了在一些特殊条件下什么样的理想可以使得 $(\Bbbk\langle x,y\rangle/I)^* \times H_0$ 成为一个可驯余表示型的 Hopf 代数。

关键词: Hopf 代数; 对偶Chevalley性质; 余表示型.

THESIS: Classification of finite-dimensional Hopf algebras with the dual Chevalley property according to their corepresentation type

SPECIALIZATION: Fundamental Mathematics

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Abstract

In the view point of representation type, every finite-dimensional algebra exactly belongs to one of following three kinds of algebras: algebras of finite representation type, algebras of tame types and wild algebras. From then on, the classification for a given kind of algebras according to their representation type has received considerable attention.

In the case of Hopf algebras, much effort was put in pointed Hopf algebras or their dual, that is, elementary Hopf algebras. Moreover, Hopf algebras with the (dual) Chevalley property is a kind of natural generalization of elementary (pointed) Hopf algebras. These Hopf algebras are interesting by various reasons, one of which is their classification.

We try to classify Hopf algebras with the dual Chevalley property according to their corepresentation type. The main tool we want to use is the link quiver. One of key points of this thesis is that one can describe the structure of the link quiver by applying multiplicative matrices and primitive matrices.

Let H be a finite-dimensional Hopf algebra over an algebraically closed field $\Bbbk$ with the dual Chevalley property. The main results are described as follows.

At first, if H is non-cosemisimple, we prove that H is of finite corepresentation type if and only if it is coNakayama, if and only if the link quiver $\mathrm{Q}(H)$ of H is a disjoint union of basic cycles, if and only if the link-indecomposable component $H_{(1)}$ containing $\Bbbk 1$ is a pointed Hopf algebra and the link quiver of $H_{(1)}$ is a basic cycle. If $\mathrm{char}(\Bbbk)=0$, then H is of finite corepresentation type if and only if either H is cosemisimple or H is not cosemisimple and $H_{(1)} \cong A(n,d,\mu,q)$. If $\mathrm{char}(\Bbbk)=p$, then H is of finite corepresentation type if and only if either H is cosemisimple or H is not cosemisimple and $H_{(1)} \cong C_d(n)$.

Finally, if $\mathrm{char}(\Bbbk)=0$, we show that $\mathrm{gr}^c(H)$ is of tame corepresentation type if and only if $\mathrm{gr}^c(H) \cong (\Bbbk\langle x,y\rangle/I)^* \times H'$ for some finite-dimensional semisimple Hopf algebra H' and some special ideals I. Then, by the method of link quiver and bosonization, we discuss which of the above ideals will occur when $(\Bbbk\langle x,y\rangle/I)^* \times H_0$ is a Hopf algebra of tame corepresentation type under some assumptions.

Keywords: Hopf algebras; Dual Chevalley property; Corepresentation type.

Chapter 1 Introduction

§1.1 Background

In the view point of representation type, every finite-dimensional algebra exactly belongs to one of following three kinds of algebras: algebras of finite representation type, algebras of tame types and wild algebras (See [24]). From then on, the classification for a given kind of algebras according to their representation type has received considerable attention. See, for example, [8–10, 23, 26, 37, 62, 63]. The category of finite-dimensional left (right) modules over a finite representation type algebra is considered easiest to understand.

In the case of Hopf algebras, much effort was put in pointed Hopf algebras or their dual, that is, elementary Hopf algebras. In the case of modular group algebras of finite groups, the authors in [12, 14, 27, 35] show that a block of such modular group algebra is of finite representation type if and only if the corresponding defect groups are cyclic and while it is tame if and only if $\operatorname{char} \Bbbk = 2$ and its defects groups are dihedral, semidihedral and generalized quaternion. The classification of small quantum groups according to their representation type can be found in [20, 67, 70]. They show that the only tame one is $u_q(\mathfrak{sl}_2)$ and others are all wild. For cocommutative Hopf algebras, Farnsteiner and his collaborators have classify all finite-dimensional cocommutative Hopf algebras, i.e., finite algebraic groups, of finite representation type and tame type [29–33]. Liu and his collaborators get the classification of elementary Hopf algebras according to their representation type from 2006 to 2013 [36, 42–44]. We note that there is indeed a common point in above classification: a finite-dimensional (cocommutative, elementary) Hopf algebra is of finite representation type if and only if it is a Nakayama algebra. We cannot help but hope that this observation holds true for more Hopf algebras.

Among of these results, constructing Hopf algebras structures through using quivers was shown to be a very effective way, which is due to the works of of Cibils-Rosso for pointed case and Green-Solberg for elementary case [19, 21, 22, 34]. As a development, in [16, 57], the authors give a classification of non-semisimple monomial Hopf algebras and get more. In 2007, the third author and Li [44] have classified all finite-dimensional pointed Hopf algebras of finite corepresentation type and show that they are all monomial Hopf algebras [44, Theorem 4.6].

At the same time, it is well known that in the representation of finite-dimensional algebras, the Ext quiver is a fundamental tool. The Ext quiver of a coalgebra has been introduced by Chin and Montgomery in [18] too. Montgomery also introduced the link quiver of coalgebra H by using the wedge of simple subcoalgebras of H (see [55, Definition 1.1]). In [17, Definition 4.1], the definition of link quiver has been modified. In addition, the authors of [17] unified the link quiver of a coalgebra with the Ext quiver. Obviously, these quivers are not limited to elementary or pointed Hopf algebras.

More or less, one mainly focused on the classification of finite-dimensional basic (pointed) Hopf algebras according to their (co)representation type. We know that the Hopf algebras with the (dual) Chevalley property is a kind of natural generalization of elementary (pointed) Hopf algebras. These

Hopf algebras have been studied by many authors. See, for example, [1,3,40,41,45,47]. In [4,15,53,71], the authors present some explicit examples of Hopf algebras with the dual Chevalley property.

Our motivation is to classify finite-dimensional Hopf algebras with the dual Chevalley property according to their corepresentation type. Here by the dual Chevalley property we mean that the coradical H_0 is a Hopf subalgebra. The main tool we want to use is the link quiver. One of key points of this thesis is that one can describe the structure of the link quiver by applying multiplicative matrices and primitive matrices now, which are developed by the Li and his collaborator [40,41,45,47].

§1.2 Main results

Denote the set of all the simple subcoalgebras of a Hopf algebra H with the dual Chevallry property by $\mathcal{S}$. According to Corollary 3.1.16, we can view set ${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$ of a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices as the set of arrows from vertex D to vertex C. Denote ${}^{\mathcal{C}}\mathcal{P} = \bigcup_{\mathcal{D}\in\mathcal{S}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}, \mathcal{P}^{\mathcal{D}} = \bigcup_{\mathcal{C}\in\mathcal{S}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}, \mathcal{P} = \bigcup_{\mathcal{C}\in\mathcal{S}} {}^{\mathcal{C}}\mathcal{P}$. We can also view $\mathcal{P}^{\mathcal{D}}$ as the set of arrows with start vertex D and view ${}^{\mathcal{C}}\mathcal{P}$ as the set of arrows with end vertex C. This means that we can view $\mathrm{Q}(H) = (\mathcal{S},\mathcal{P})$ as the link quiver of H.

At first, we characterize the link quiver of finite-dimensional Hopf algebras with the dual Chevalley property of finite or tame corepresentation type. This appears as Theorem 4.2.1 in this thesis:

Theorem 1.2.1 *Let $\Bbbk$ be an algebraically closed field and H a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property. Denote ${}^1\mathcal{S} = \{C \in \mathcal{S} \mid \Bbbk 1 + C \neq \Bbbk 1 \wedge C\}$.*

(1) H is of finite corepresentation type if and only if $\mid {}^1\mathcal{P} \mid = 1$ and ${}^1\mathcal{S} = \{\Bbbk g\}$ for some group-like element $g \in G(H)$.

(2) If H is of tame corepresentation type, then one of the following two cases appears:

- *(i) $\mid {}^1\mathcal{P} \mid = 2$ and for any $C \in {}^1\mathcal{S}$, $\dim_{\Bbbk}(C) = 1$;*
- *(ii) $\mid {}^1\mathcal{P} \mid = 1$ and ${}^1\mathcal{S} = \{C\}$ for some $C \in \mathcal{S}$ with $\dim_{\Bbbk}(C) = 4$.*

(3) If one of the following holds, H is of wild corepresentation type.

- *(i) $\mid {}^1\mathcal{P} \mid \geq 3$;*
- *(ii) $\mid {}^1\mathcal{P} \mid = 2$ and there exists some $C \in {}^1\mathcal{S}$ such that $\dim_{\Bbbk}(C) \geq 4$;*
- *(iii) $\mid {}^1\mathcal{P} \mid = 1$ and ${}^1\mathcal{S} = \{C\}$ for some $C \in \mathcal{S}$ with $\dim_{\Bbbk}(C) \geq 9$.*

We attempt to generalize above stated result [44, Theorem 4.6] in order to give the structure of finite-dimensional Hopf algebras with the dual Chevalley property of finite corepresentation type. See Corollary 4.1.9, Theorems 5.1.1 and 5.2.1, stating that:

Corollary 1.2.2 *A finite-dimensional Hopf algebra H over an algebraically closed field $\Bbbk$ with the dual Chevalley property is of finite corepresentation type if and only if H is coNakayama.*

Theorem 1.2.3 *Let $\mathbb{k}$ be an algebraically closed field of characteristic 0. Then a finite-dimensional Hopf algebra H over $\mathbb{k}$ with the dual Chevalley property is of finite corepresentation type if and only if either of the following conditions is satisfied:*

(1) H is cosemisimple;

(2) H is not cosemisimple and $H_{(1)} \cong A(n, d, \mu, q)$.

Theorem 1.2.4 *Let $\mathbb{k}$ be an algebraically closed field of positive characteristic p. Then a finite-dimensional Hopf algebra H over $\mathbb{k}$ with the dual Chevalley property is of finite corepresentation type if and only if either of the following conditions is satisfied:*

(1) H is cosemisimple;

(2) H is not cosemisimple and $H_{(1)} \cong C_d(n)$.

Denote $\mathcal{S}^1 = \{C \in \mathcal{S} \mid C + \mathbb{k}1 \neq C \wedge \mathbb{k}1\}$. Note that$\mid {}^1\mathcal{P} \mid=\mid \mathcal{P}^1 \mid$ and $C \in {}^1\mathcal{S}$ if and only if $S(C) \in \mathcal{S}^1$ (see Lemma 3.3.5). Using Theorem 1.2.1, we know that if H is of tame corepresentation type, then one of the following three cases appears:

(i) $\mid \mathcal{P}^1 \mid= 1$ and $\mathcal{S}^1 = \{C\}$ for some $C \in \mathcal{S}$ with $\dim_{\mathbb{k}}(C) = 4$;

(ii) $\mid \mathcal{P}^1 \mid= 2$ and $\mathcal{S}^1 = \{\mathbb{k}g\}$ for some $g \in G(H)$;

(iii) $\mid \mathcal{P}^1 \mid= 2$ and $\mathcal{S}^1 = \{\mathbb{k}g, \mathbb{k}h\}$ for some $g, h \in G(H)$.

Besides, we determine the structures of finite-dimensional coradically graded Hopf algebra with the dual Chevalley property of tame corepresentation type completely. See Theorem 6.1.2, stating that:

Theorem 1.2.5 *Let $\mathbb{k}$ be an algebraically closed field of characteristic 0 and H a finite-dimensional Hopf algebra over $\mathbb{k}$ with the dual Chevalley property. Then $\mathrm{gr}^c(H)$ is of tame corepresentation type if and only if*

$$\mathrm{gr}^c(H) \cong (\mathbb{k}\langle x, y\rangle/I)^* \times H'$$

for some finite-dimensional semisimple Hopf algebra H' and some ideal I which is one of the following forms:

(1) $I = (x^2 - y^2, yx - ax^2, xy)$ for $0 \neq a \in \mathbb{k}$;

(2) $I = (x^2, y^2, (xy)^m - a(yx)^m)$ for $0 \neq a \in \mathbb{k}$ and $m \geq 1$;

(3) $I = (x^n - y^n, xy, yx)$ for $n \geq 2$;

(4) $I = (x^2, y^2, (xy)^m x - (yx)^m y)$ for $m \geq 1$.

According to [13, Theorem 4.1.2], if R is a Hopf algebra in ${}^{H'}_{H'}\mathcal{YD}$, then we can form the bosonization $R \times H'$ which is a Hopf algebra. For an tame algebra A, above theorem does not imply the existence of finite-dimensional semisimple Hopf algebra H' satisfying A^* is a braided Hopf algebra in ${}^{H'}_{H'}\mathcal{YD}$. That is to say, for the ideals I listed in the above theorem, we do not know whether $(\mathbb{k}\langle x, y\rangle/I)^* \times H'$ is a Hopf algebra or not. By the method of link quiver and bosonization, we try to discuss this question in the three cases separately.

We consider case (i) under some assumptions. See Proposition 6.3.8, stating that:

Proposition 1.2.6 *Let* $\mathrm{gr}^c(H) \cong (\mathbb{k}\langle x, y\rangle/I)^* \times H_0$ *be a finite-dimensional coradically graded Hopf algebra over* $\mathbb{k}$ *with the dual Chevalley property of tame corepresentation type. Suppose* $\mathcal{P}^1 = \{\mathcal{X}\}$*,* $\mathcal{S}^1 = \{C\}$ *for some* $C \in \mathcal{S}$ *with* $\dim_{\mathbb{k}}(C) = 4$ *and the invertible matrix* K *in Lemma 6.3.3 is diagonal, namely*

$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

If in addition R_H *is generated by* u, v*, then*

(1) $I = (x^2, y^2, (xy)^m - a(yx)^m)$ *for* $0 \neq a \in \mathbb{k}$ *and* $m \geq 1$*;*

(2) $\alpha_1 = \alpha_4 = -1$*;*

(3) $a = (-1)^{m-1}\alpha_2^m$ *or* $a = (-1)^{m-1}\alpha_3^m$*;*

(4) $\alpha_2\alpha_3$ *is an mth primitive root of unity.*

In fact, when we study the properties for the finite-dimensional coradically graded Hopf algebra over $\mathbb{k}$ with the dual Chevalley property of tame corepresentation type, we only need to focus on its link-indecomposable component containing $\mathbb{k}1$. This appears as Proposition 6.2.5 in this thesis:

Proposition 1.2.7 *Let* H *be a finite-dimensional coradically graded Hopf algebra over* $\mathbb{k}$ *with the dual Chevalley property. Then* H *is of tame corepresentation type if and only if* $H_{(1)}$ *is of tame corepresentation type.*

With the help of the preceding proposition, we can consider cases (ii) and (iii). See Proposition 6.3.10, stating that:.

Proposition 1.2.8 *Let* $\mathrm{gr}^c(H) \cong (\mathbb{k}\langle x, y\rangle/I)^* \times H_0$ *be a finite-dimensional coradically graded Hopf algebra over* $\mathbb{k}$ *with the dual Chevalley property of tame corepresentation type.*

(1) If $| \mathcal{P}^1 |= 2$ *and* $\mathcal{S}^1 = \{\mathbb{k}g\}$ *for some* $g \in G(H)$*, then* $I = (x^2, y^2, xy + yx)$*;*

(2) If $| \mathcal{P}^1 |= 2$ *and* $\mathcal{S}^1 = \{\mathbb{k}g, \mathbb{k}h\}$ *for some* $g, h \in G(H)$*, then* $I = (x^2, y^2, (xy)^m - a(yx)^m)$ *for* $0 \neq a \in \mathbb{k}$ *and* $m \geq 1$*.*

§1.3 Organization

In this section, we give an outline of this thesis.

In Chapter 1 , we provide the research background and main results.

In Chapter 2 , we give a preparation of the following chapters.

In Chapter 3 , we provide the properties of a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices. Besides, we construct a complete family of non-trivial primitive matrices in two ways. Note that the cardinal number of a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices coincides with the number of arrows from vertex D to vertex C in the link quiver. Then we discuss the properties for the link quiver of a Hopf algebra with the dual Chevalley property.

In Chapter 4 , we characterize the link quiver of Hopf algebras with the dual Chevalley property of finite or tame corepresentation type.

In Chapter 5 , we attempt to generalize [44, Theorem 4.6] in order to give the structure of finite-dimensional Hopf algebras with the dual Chevalley property of finite corepresentation type. We give a more accurate description for $H_{(1)}$ in the case that H is a finite-dimensional non-cosemisimple Hopf algebra with the dual Chevalley property of finite corepresentation type.

In Chapter 6 , we determine the structures of coradically graded Hopf algebra H with the dual Chevalley property of tame corepresentation type. We show that H is of tame corepresentation type if and only if the link-indecomposable component $H_{(1)}$ containing $\mathbb{k}1$ is of tame corepresentation type. Next we discuss which ideal will occur when $(\mathbb{k}\langle x,y\rangle/I)^*\times H_0$ is a finite-dimensional coradically graded Hopf algebra with the dual Chevalley property of tame corepresentation type under some assumptions. At last, we give some examples and applications.

Chapter 2 Preliminaries

In this chapter, we recall the most needed knowledge about coalgebras, Hopf algebras and based rings. Throughout this thesis $\Bbbk$ denotes an *algebraically closed field* and all spaces are over $\Bbbk$. The tensor product over $\Bbbk$ is denoted simply by $\otimes$.

§2.1 Multiplicative matrices and primitive matrices

The concept of multiplicative matrices was introduced by Manin in [49]. Later in 2019, Li and Zhu [47] introduced the concept of primitive matrices. Recently, more properties of multiplicative matrices and primitive matrices have been observed. The authors of [40, 41, 45, 47] used these two notions to generalize some results of pointed Hopf algebras to non-pointed ones.

Let us first recall the definition of multiplicative matrices.

Definition 2.1.1 *([40, Definition 2.3]) Let (H, Δ, ε) be a coalgebra over $\Bbbk$.*

(1) A square matrix $\mathcal{G} = (g_{ij})_{r\times r}$ over H is said to be multiplicative, if for any $1 \leq i, j \leq r$, we have $\Delta(g_{ij}) = \sum\limits_{t=1}^{r} g_{it} \otimes g_{tj}$ and $\varepsilon(g_{ij}) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker notation;

(2) A multiplicative matrix $\mathcal{C}$ is said to be basic, if its entries are linearly independent.

Multiplicative matrices over a coalgebra can be understood as a generalization of group-like elements. We know that all the entries of a basic multiplicative matrix $\mathcal{C}$ span a simple subcoalgebra C of H. Conversely, for any simple coalgebra C over $\Bbbk$, there exists a basic multiplicative matrix $\mathcal{C}$ whose entries span C (for details, see [47], [40]). And according to [40, Lemma 2.4], the basic multiplicative matrix of the simple coalgebra C would be unique up to the similarity relation. More specifically, suppose that $\mathcal{C}$ is a basic multiplicative matrix of the simple coalgebra C. Then $\mathcal{C}'$ is also a basic multiplicative matrix of C if and only if there exists an invertible matrix L over $\Bbbk$ such that $\mathcal{C}' = L\mathcal{C}L^{-1}$.

Next we recall the definition of primitive matrices, which is a non-pointed analogue of primitive elements.

Definition 2.1.2 *([47, Definition 3.2] and [41, Definition 4.4]) Let (H, Δ, ε) be a coalgebra over $\Bbbk$. Suppose $\mathcal{C} = (c_{ij})_{r\times r}$ and $\mathcal{D} = (d_{ij})_{s\times s}$ are basic multiplicative matrices over H.*

(1) A matrix $\mathcal{X} = (x_{ij})_{r\times s}$ over H is said to be $(\mathcal{C}, \mathcal{D})$-primitive, if

$$\Delta(x_{ij}) = \sum_{k=1}^{r} c_{ik} \otimes x_{kj} + \sum_{t=1}^{s} x_{it} \otimes d_{tj}$$

holds for any $1 \leq i, j \leq r$;

(2) A primitive matrix $\mathcal{X}$ is said to be non-trivial, if there exists some entry of $\mathcal{X}$ which does not belong to the coradical H_0.

Recall that a finite-dimensional Hopf algebra is said to have the dual Chevalley property, if its coradical H_0 is a Hopf subalgebra. In this thesis, we still use the term *dual Chevalley property* to indicate a Hopf algebra H with its coradical H_0 as a Hopf subalgebra, even if H is infinite-dimensional.

In the following part, let H be a Hopf algebra over $\Bbbk$ with the dual Chevalley property. Let C, D be the simple subcoalgebras spanned by the entries of basic multiplicative matrices $\mathcal{C}$ and $\mathcal{D}$, respectively. For any $(\mathcal{C},\mathcal{D})$-primitive matrix $\mathcal{X}$, it is evident that all the entries of $\mathcal{X}$ must belong to $C \wedge D$ and automatically belong to $H_1 := H_0 \wedge H_0$, where H_0 is the coradical of H.

We say that two matrices $\mathcal{A}$ and $\mathcal{A}'$ over H are *similar*, which is denoted by $\mathcal{A} \sim \mathcal{A}'$ for simplicity, if there exists an invertible matrix L over $\Bbbk$ such that $\mathcal{A}' = L\mathcal{A}L^{-1}$. Next we recall some notations.

For any matrix $\mathcal{A} = (a_{ij})_{r\times s}$ and $\mathcal{B} = (b_{ij})_{u\times v}$ over H, define $\mathcal{A} \odot \mathcal{B}$ and $\mathcal{A} \odot' \mathcal{B}$ as follow

$$\mathcal{A} \odot \mathcal{B} = \begin{pmatrix} a_{11}\mathcal{B} & \cdots & a_{1s}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{r1}\mathcal{B} & \cdots & a_{rs}\mathcal{B} \end{pmatrix}, \quad \mathcal{A} \odot' \mathcal{B} = \begin{pmatrix} \mathcal{A}b_{11} & \cdots & \mathcal{A}b_{1v} \\ \vdots & \ddots & \vdots \\ \mathcal{A}b_{u1} & \cdots & \mathcal{A}b_{uv} \end{pmatrix}.$$

Some evident formulas on $\odot$ and $\odot'$ should be noted for later computations.

Lemma 2.1.3 *Let $\mathcal{A}, \mathcal{B}$ be matrices over H and I be the identity matrix over $\Bbbk$, then*

(1) $(L_1\mathcal{A}L_2) \odot' \mathcal{B} = (L_1 \odot' I)(\mathcal{A} \odot' \mathcal{B})(L_2 \odot' I)$ holds for any invertible matrices L_1, L_2 over $\Bbbk$;

(2) $\mathcal{A} \odot (L_1\mathcal{B}L_2) = (I \odot L_1)(\mathcal{A} \odot \mathcal{B})(I \odot L_2)$ holds for any invertible matrices L_1, L_2 over $\Bbbk$;

(3) There exist invertible matrices K, K' over $\Bbbk$ such that $K(\mathcal{A} \odot \mathcal{B})K' = \mathcal{A} \odot' \mathcal{B}$. Moreover, if $\mathcal{A}, \mathcal{B}$ are square matrices, then $\mathcal{A} \odot' \mathcal{B} \sim \mathcal{A} \odot \mathcal{B}$.

Proof:

(1) Suppose that $\mathcal{B} = \begin{pmatrix} b_{11} & \cdots & b_{1v} \\ \vdots & & \vdots \\ b_{u1} & \cdots & b_{uv} \end{pmatrix}$, then

$$\begin{aligned}
(L_1\mathcal{A}L_2) \odot' \mathcal{B} &= \begin{pmatrix} L_1\mathcal{A}L_2b_{11} & \cdots & L_1\mathcal{A}L_2b_{1v} \\ \vdots & & \vdots \\ L_1\mathcal{A}L_2b_{u1} & \cdots & L_1\mathcal{A}L_2b_{uv} \end{pmatrix} \\
&= \begin{pmatrix} L_1\mathcal{A}b_{11}L_2 & \cdots & L_1\mathcal{A}b_{1v}L_2 \\ \vdots & & \vdots \\ L_1\mathcal{A}b_{u1}L_2 & \cdots & L_1\mathcal{A}b_{uv}L_2 \end{pmatrix} \\
&= \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_1 \end{pmatrix} \begin{pmatrix} \mathcal{A}b_{11} & \cdots & \mathcal{A}b_{1v} \\ \vdots & & \vdots \\ \mathcal{A}bu1 & \cdots & \mathcal{A}b_{uv} \end{pmatrix} \begin{pmatrix} L_2 & & \\ & \ddots & \\ & & L_2 \end{pmatrix} \\
&= (L_1 \odot' I)(\mathcal{A} \odot' \mathcal{B})(L_2 \odot' I).
\end{aligned}$$

(2) Consider the Hopf algebra H^{op}, whose multiplication is opposite to H. Using (1), we can get this result.

(3) By [65, Theorem 8.26], there exist commutation matrices K, K' such that

$$K(\mathcal{A} \odot \mathcal{B})K' = \mathcal{A} \odot' \mathcal{B},$$

where commutation matrix is defined in [65, Definition 8.1]. Moreover, from the proof of [65, Theorem 8.24], we know that if $\mathcal{A}, \mathcal{B}$ are square matrices, then

$$\mathcal{A} \odot' \mathcal{B} \sim \mathcal{A} \odot \mathcal{B}.$$

□

Let $B, C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{B}, \mathcal{C}, \mathcal{D}$ respectively. According to [40, Proposition 2.6], there exists an invertible matrices $L_{\mathcal{B},\mathcal{C}}$ over $\Bbbk$ such that

$$L_{\mathcal{B},\mathcal{C}}(\mathcal{B} \odot' \mathcal{C})L_{\mathcal{B},\mathcal{C}}^{-1} = \begin{pmatrix} \mathcal{E}_1 & & \\ & \ddots & \\ & & \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}} \end{pmatrix},$$

where $\mathcal{E}_1, \cdots, \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}}$ are the basic multiplicative matrices of $E_1, \cdots, E_{u_{(\mathcal{B},\mathcal{C})}}$, respectively. In particular, let $L_{1,\mathcal{C}} = L_{\mathcal{C},1} = I$, where I is the identity matrix over $\Bbbk$. Note that cosemisimple coalgebra BC admits a decomposition into a direct sum of simple subcoalgebras and $u_{(\mathcal{B},\mathcal{C})}$ is exactly the number of such simple subcoalgebras. Thus in fact $u_{(\mathcal{B},\mathcal{C})}$ does not depend on the choices of basic multiplicative matrices $\mathcal{B}$ and $\mathcal{C}$ as well as the invertible matrix $L_{\mathcal{B},\mathcal{C}}$.

For any $(\mathcal{C}, \mathcal{D})$-primitive matrix $\mathcal{X}$, by [40, Proposition 2.6], there exist invertible matrices $L_{\mathcal{B},\mathcal{C}}, L_{\mathcal{B},\mathcal{D}}$ over $\Bbbk$ such that

$$\begin{aligned}
&\begin{pmatrix} L_{\mathcal{B},\mathcal{C}} & \\ & L_{\mathcal{B},\mathcal{D}} \end{pmatrix}\left(\mathcal{B} \odot' \begin{pmatrix} \mathcal{C} & \mathcal{X} \\ 0 & \mathcal{D} \end{pmatrix}\right)\begin{pmatrix} L_{\mathcal{B},\mathcal{C}}^{-1} & \\ & L_{\mathcal{B},\mathcal{D}}^{-1} \end{pmatrix} \\
= &\begin{pmatrix} L_{\mathcal{B},\mathcal{C}} & \\ & L_{\mathcal{B},\mathcal{D}} \end{pmatrix}\begin{pmatrix} \mathcal{B} \odot' \mathcal{C} & \mathcal{B} \odot' \mathcal{X} \\ 0 & \mathcal{B} \odot' \mathcal{D} \end{pmatrix}\begin{pmatrix} L_{\mathcal{B},\mathcal{C}}^{-1} & \\ & L_{\mathcal{B},\mathcal{D}}^{-1} \end{pmatrix} \\
= &\begin{pmatrix} \mathcal{E}_1 & & & \mathcal{X}_{11} & \cdots & \mathcal{X}_{1u_{(\mathcal{B},\mathcal{D})}} \\ & \ddots & & \vdots & & \vdots \\ & & \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}} & \mathcal{X}_{u_{(\mathcal{B},\mathcal{C})}1} & \cdots & \mathcal{X}_{u_{(\mathcal{B},\mathcal{C})}u_{(\mathcal{B},\mathcal{D})}} \\ & & & \mathcal{F}_1 & & \\ & 0 & & & \ddots & \\ & & & & & \mathcal{F}_{u_{(\mathcal{B},\mathcal{D})}} \end{pmatrix},
\end{aligned} \tag{2.1}$$

where $\mathcal{E}_1, \cdots, \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}}, \mathcal{F}_1, \cdots, \mathcal{F}_{u_{(\mathcal{B},\mathcal{D})}}$ are the given basic multiplicative matrices. Combining [40, Remark 2.5 and Lemma 2.7] and [47, Remark 3.2], we can show that each $\mathcal{X}_{ij}$ is a $(\mathcal{E}_i, \mathcal{F}_j)$-primitive matrix.

With the notations above, we have

Lemma 2.1.4 *For any $B, C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{B}, \mathcal{C}, \mathcal{D}$ respectively. If $\mathcal{X}$ is a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix, then*

(1) The set of all row vectors of $\mathcal{B} \odot' \mathcal{X}$ is linearly independent over H_1/H_0;

(2) The set of all column vectors of $\mathcal{B} \odot' \mathcal{X}$ is linearly independent over H_1/H_0;

(3) For each $1 \leq i \leq u_{(\mathcal{B},\mathcal{C})}$, there is some $1 \leq j \leq u_{(\mathcal{B},\mathcal{D})}$ such that $\mathcal{X}_{ij}$ is non-trivial;

(4) For each $1 \leq j \leq u_{(\mathcal{B},\mathcal{D})}$, there is some $1 \leq i \leq u_{(\mathcal{B},\mathcal{C})}$ such that $\mathcal{X}_{ij}$ is non-trivial.

Proof: These four claims are exactly (i), (ii), (I), (II) appearing in the proof of [40, Lemma 3.12] in the case of H^{op}. □

§2.2 Based ring

Let $\mathbb{Z}_+$ be the set of nonnegative integers. Some relevant concepts and results are recalled as follows.

Definition 2.2.1 (*[56, Definitions 2.1 and 2.2]) Let A be an associative ring with unit which is free as a $\mathbb{Z}$-module.*

(1) A $\mathbb{Z}_+$-basis of A is a basis $B = \{b_i\}_{i \in I}$ such that $b_i b_j = \sum_{t \in I} c_{ij}^t b_t$, where $c_{ij}^t \in \mathbb{Z}_+$.

(2) A ring with a fixed $\mathbb{Z}_+$-basis $\{b_i\}_{i \in I}$ is called a unital based ring if the following conditions hold:

(i) 1 is a basis element.

(ii) Let $\tau : A \to \mathbb{Z}$ denote the group homomorphism defined by

$$\tau(b_i) = \begin{cases} 1, & \text{if} \quad b_i = 1, \\ 0, & \text{if} \quad b_i \neq 1. \end{cases}$$

There exists an involution $i \mapsto i^*$ of I such that the induced map

$$a = \sum_{i \in I} a_i b_i \mapsto a^* = \sum_{i \in I} a_i b_{i^*}, \quad a_i \in \mathbb{Z}$$

is an anti-involution of A, and

$$\tau(b_i b_j) = \begin{cases} 1, & \text{if} \quad i = j^*, \\ 0, & \text{if} \quad i \neq j^*. \end{cases}$$

(3) A fusion ring is a unital based ring of finite rank.

It is straightforward to show the following lemma.

Lemma 2.2.2 (*cf. [25, Exercise 3.3.2]*) *Suppose A is a unital based ring with $\mathbb{Z}_+$-basis I, then for any $X, Z \in I$, there exist Y_1, Y_2 such that XY_1 and Y_1Z contain Z with a nonzero coefficient.*

Proof: Since both XX^* and Z^*Z contain 1 with a nonzero coefficient, we can take Y_1 to be a suitable summand of X^*Z and Y_2 to be a suitable summand of ZX^*. □

Example 2.2.3 *Let H be a cosemisimple Hopf algebra and $\mathcal{F}$ be the free abelian group generated by isomorphism classes of finite-dimensional right H-comodules and $\mathcal{F}_0$ the subgroup of $\mathcal{F}$ generated by all expressions $[Y] - [X] - [Z]$, where $0 \to X \to Y \to Z \to 0$ is a short exact sequence of finite-dimensional right H-comodules. Recall that the Grothendieck group $\mathrm{Gr}(H\text{-}comod)$ of the category of finite-dimensional right H-comodules is defined by*

$$\mathrm{Gr}(H\text{-}comod) := \mathcal{F}/\mathcal{F}_0.$$

From [25, Proposition 4.5.4] and [39, Theorem 2.7], $\mathrm{Gr}(H\text{-}comod)$ is a unital based ring with $\mathbb{Z}_+$-basis $\mathcal{V}$, where $\mathcal{V}$ is the set of all the isomorphism classes of simple right H-comodules.

§2.3 Comonomial Hopf algebras

Let $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1)$ be a finite quiver. Note that we read paths in Q from right to left. Denote by $\Bbbk\mathrm{Q}^a$ and $\Bbbk\mathrm{Q}^c$ the path algebra of Q and the path coalgebra of Q, respectively.

Recall that the counit and comultiplication of path coalgebra $\Bbbk\mathrm{Q}^c$ are defined by $\varepsilon(e) = 1,\ \Delta(e) = e \otimes e$ for each $e \in Q_0$, and for each nontrivial path $p = a_n \cdots a_1, \varepsilon(p) = 0$,

$$\Delta(a_n \cdots a_1) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p.$$

Definition 2.3.1 (*[16, Definition 1.2]*)

(1) An algebra A is called monomial if there exits a quiver Q and an admissible ideal I generated by some paths such that $A \cong \Bbbk\mathrm{Q}^a/I$.

(2) A subcoalgebra C of $\Bbbk\mathrm{Q}^c$ is called comonomial provided that the following conditions are satisfied:

(i) C contains all vertices and arrows in Q;

(ii) C is contained in subcoalgebra $C_d(\mathrm{Q}) := \bigoplus_{i=0}^{d-1} \Bbbk\mathrm{Q}(i)$ for some $d \geq 2$, where $\mathrm{Q}(i)$ is the set of all paths of length i in Q;

(iii) C has a basis consisting of paths.

(3) A finite-dimensional Hopf algebra is called a monomial (resp. comonomial) Hopf algebra if it is monomial (resp. comonomial) as an algebra (resp. coalgebra).

Remark 2.3.2 *Note that comonomial Hopf algebra in [16] was called monomial Hopf algebra. In order to not cause confusion, we recall the defintion of comonomial Hopf algebra.*

One of the key observation we need is the following lemma which was proved in [16], which is true no matter when the characteristic of $\Bbbk$ is equal to 0 or is equal to p.

Lemma 2.3.3 *[16, Corollary 2.4] A non-semisimple Hopf algebra over $\Bbbk$ is a monomial Hopf algebra if and only if it is elementary and Nakayama.*

The authors of [16] classify non-cosemisimple comonomial Hopf algebras via group data when the characteristic of $\Bbbk$ is zero. Let us briefly recall their results.

Let $\Bbbk$ be an algebraicly closed field with characteristic 0, a *group datum* (see [16, Definition 5.3]) over $\Bbbk$ is defined to be a sequence $\alpha = (G, g, \chi, \mu)$ consisiting of

(1) a finite group G, with an element g in its center;

(2) a one-dimensional $\Bbbk$-representation χ of G; and

(3) an element $\mu \in \Bbbk$ such that $\mu = 0$ if $o(g) = o(\chi(g))$, and that if $\mu \neq 0$, then $\chi^{o(\chi(g))} = 1$.

For a group datum $\alpha = (G, g, \chi, \mu)$ over $\Bbbk$, the authors of [16] give the corresponding Hopf algebra structure $A(\alpha)$ as follow (for details, see [16, 5.7]). Define $A(\alpha)$ to be an associative algebra with generators x and all $h \in G$, with relations

$$x^d = \mu(1 - g^d), \;\; xh = \chi(h)hx, \;\; \forall h \in G,$$

with comultiplication Δ, counit ε, and the antipode S given by

$$\Delta(g) = g \otimes g, \;\; \varepsilon(g) = 1, \;\; \Delta(x) = x \otimes 1 + g \otimes x, \;\; \varepsilon(x) = 0, \;\; S(g) = g^{-1}, \;\; S(x) = -xg^{-1}.$$

Let H be a non-semisimple comonomial Hopf algebra over $\Bbbk$, [16, Lemma 5.2] permits us to introduce the following notion. A group datum $\alpha(H) = (G, g, \chi, \mu)$ is called an *induced group datum* of H (see [16, Definition 5.5] for details) provide that

(1) $G = G(H)$, where $G(H)$ is the set of all the group-like elements of H;

(2) there exists a non-trivial $(1, g)$-primitive element x in H such that

$$x^d = \mu(1 - g^d), \;\; xh = \chi(h)hx, \;\; \forall h \in G,$$

where d is the multiplicative order of $\chi(g)$.

It is not difficult to verify that $H \cong A(\alpha(H))$.

Denote by $\mathbb{Z}_n$ the basic cycle of length n, i.e., a quiver with n vertices $e_0, e_1, \cdots, e_{n-1}$ and n arrows $a_0, a_1, \cdots a_{n-1}$, where the arrow a_i goes from the vertex e_i to the vertex e_{i+1}. In the following, denote $C_d(\mathbb{Z}_n)$ by $C_d(n)$. According to [16, Lemma 5.8], we know that $A(G, g, \chi, \mu) \cong C_d(n) \oplus \cdots \oplus C_d(n)$ as coalgebras, where $n = o(g)$ and $d = o(\chi(g))$.

As mentioned above, let us illustrate it with an example.

Example 2.3.4 *Let $q \in \Bbbk$ be an n-th root of unit of order d. In [6] and [59], Radford and Andruskiewitsch-Schneider have considered the following Hopf algebra $A(n, d, \mu, q)$ which as an associative algebra is generated by g and x with relations*

$$g^n = 1, \quad x^d = \mu(1 - g^d), \quad xg = qgx.$$

Its comultiplication Δ, counit ε, and the antipode S are given by

$$\Delta(g) = g \otimes g, \ \ \varepsilon(g) = 1, \ \ \Delta(x) = x \otimes 1 + g \otimes x, \ \ \varepsilon(x) = 0, \ \ S(g) = g^{-1}, \ \ S(x) = -xg^{-1}.$$

In fact, $(\mathbb{Z}_n, \overline{1}, \chi, \mu)$ with $\chi(\overline{1}) = q$ is an induced group datum and $A(n, d, \mu, q) = A(\mathbb{Z}_n, \overline{1}, \chi, \mu)$.

The following results gives a classification of non-semisimple comonomial Hopf algebra over an algebraically filed $\Bbbk$ of characteristic zero.

Lemma 2.3.5 (*[16, Theorem 5.9]) Let $\Bbbk$ be an algebraically closed field with characteristic 0, there is a one-to-one corresponding between sets*

$$\{\text{the isoclasses of non-cosemisimple comonomial Hopf algebras over } \Bbbk\}$$

and

$$\{\text{the isoclasses of group data over } \Bbbk\}.$$

Next, we focus on the above lemma in the case of that $\Bbbk$ is an algebraically closed field of characteristic p. For any quiver Q, we define $C_d\mathrm{Q} := \bigoplus_{i=0}^{d-1} \Bbbk\mathrm{Q}(i)$ for $d \geq 2$, where $\mathrm{Q}(i)$ is the set of all paths of length i in $\mathrm{Q}(i)$. It is not difficult to show that $C_d\mathrm{Q}$ is a subcoalgebra of path coalgabra $\Bbbk\mathrm{Q}$ (see [22] for the definition of path coalgebra). We denote the basic cycle of length n by Z_n and denote $C_d(Z_n)$ by $C_d(n)$. By [48, Theorem 1], we know that $C_d(n)$ admits a Hopf algebra structure if and only if there exists a primitive d_0-th root $q \in \Bbbk$ of unity with $d_0 \mid n$ and a natural number $r \geq 0$ such that $d = p^r d_0$.

Moreover, the authors of [48] have given a description of the structures of comonomial Hopf algebras when the characteristic of $\Bbbk$ is not zero.

Lemma 2.3.6 (*[48, Theorem 4.2]) Let H be a non-cosemisimple comonomial Hopf algebra over an algebraically closed field $\Bbbk$ of character p. Then there exists a d_0-th primitive root $q \in \Bbbk$ of unit with*

$d_0 \mid n$, $r \geq 0$ and $d = p^r d_0$ such that

$$H \cong C_d(n) \oplus \cdots \oplus C_d(n)$$

as coalgebras and

$$H \cong C_d(n)\#\Bbbk(G/N)$$

as Hopf algebras, where $G = G(H)$, the set of group-like elements of H, and $N = G(C_d(n))$, the set of group-like elements of $C_d(n)$.

Chapter 3 Properties for the link quiver

§3.1 Non-trivial primitive matrices and simple bicomodules over a coalgebra

In this section, let (H, Δ, ε) be a coalgebra over $\Bbbk$. Denote the coradical filtration of H by $\{H_n\}_{n\geq 0}$ and the set of all the simple subcoalgebras of H by $\mathcal{S}$. For any simple subcoalgebra $C \in \mathcal{S}$, we fix a basic multiplicative matrix $\mathcal{C}$ of C.

For any matrix $\mathcal{X} = (x_{ij})_{r\times s}$ over H, denote the matrix $(\overline{x_{ij}})_{r\times s}$ by $\overline{\mathcal{X}}$, where $\overline{x_{ij}} = x_{ij} + H_0 \in H/H_0$. Besides, the subspace of H/H_0 spanned by the entries of $\overline{\mathcal{X}}$ is denoted by $\operatorname{span}(\overline{\mathcal{X}})$.

We start this section by giving the following lemma, which describes a property of simple bicomodules.

Lemma 3.1.1 *For any $C, D \in \mathcal{S}$ with $\dim_{\Bbbk}(C) = r^2, \dim_{\Bbbk}(D) = s^2$, if M is a simple C-D-bicomodule, then $\dim_{\Bbbk}(M) = rs$.*

Proof: Since C^* and D^* are central simple algebras, it follows that $D^* \otimes C^{*\mathrm{op}}$ is also a central simple algebra and

$$D^* \otimes C^{*\mathrm{op}} \cong M_{rs}(\Bbbk)$$

as algebras, where $M_{rs}(\Bbbk)$ is a matrix algebra. It is known that the dimension of simple left $M_{rs}(\Bbbk)$-modules is rs. Besides, the category of finite-dimensional left $D^* \otimes C^{*\mathrm{op}}$-modules, the category of finite dimensional D^*-C^*-bimodules and the category of finite-dimensional C-D-bicomodules are isomorphic. And the isomorphisms preserve the dimension. Hence the dimension of the simple C-D-bicomodule M is rs. □

Let

$$\pi : H_1 \longrightarrow H_1/H_0$$

be the quotient map. For any $\bar{h} \in H_1/H_0$, define

$$\rho_L(\bar{h}) = (\operatorname{Id} \otimes \pi)\Delta(h), \quad \rho_R(\bar{h}) = (\pi \otimes \operatorname{Id})\Delta(h). \tag{3.1}$$

It is evident that $(H_1/H_0, \rho_L, \rho_R)$ is an H_0-bicomodule. Now we turn to mention $\operatorname{span}(\overline{\mathcal{X}})$, where $\mathcal{X}$ is a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix.

Lemma 3.1.2 *For any $C, D \in \mathcal{S}$ with $\dim_{\Bbbk}(C) = r^2, \dim_{\Bbbk}(D) = s^2$, if $\mathcal{X}_{r\times s} = (x_{ij})_{r\times s}$ is a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix, then $\operatorname{span}(\overline{\mathcal{X}})$ is a simple C-D-bicomodule. Moreover, $\dim_{\Bbbk}(\operatorname{span}(\overline{\mathcal{X}})) = rs$.*

Proof: By [40, Proposition 2.11], we know that $x_{ij} \notin H_0$ holds for all $1 \leq i \leq r$ and $1 \leq j \leq s$.

Notice that

$$\rho_L(\overline{x_{ij}}) = (\mathrm{Id} \otimes \pi)\Delta(x_{ij}) = \sum_{k=1}^{r} c_{ik} \otimes \overline{x_{kj}},$$

$$\rho_R(\overline{x_{ij}}) = (\pi \otimes \mathrm{Id})\Delta(x_{ij}) = \sum_{t=1}^{s} \overline{x_{it}} \otimes d_{tj}.$$

It is straightforward to show that $(\mathrm{span}(\overline{\mathcal{X}}), \rho_L, \rho_R)$ is a C-D-bicomodule and

$$\dim_{\Bbbk}(\mathrm{span}(\overline{\mathcal{X}})) \leq rs.$$

But according to Lemma 3.1.1, the dimension of any C-D-sub-bicomodule is at least rs. Thus we conclude that

$$\dim_{\Bbbk}(\mathrm{span}(\overline{\mathcal{X}})) = rs$$

and $\mathrm{span}(\overline{\mathcal{X}})$ is a simple C-D-bicomodule. □

A direct consequence of this lemma is:

Corollary 3.1.3 *If $\mathcal{X}$ and $\mathcal{X}'$ are non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices over H, then either* $\mathrm{span}(\overline{\mathcal{X}}) \cap \mathrm{span}(\overline{\mathcal{X}'}) = 0$ *or* $\mathrm{span}(\overline{\mathcal{X}}) = \mathrm{span}(\overline{\mathcal{X}'})$.

Proof: According to Lemma 3.1.2, it follows that $\mathrm{span}(\overline{\mathcal{X}})$ and $\mathrm{span}(\overline{\mathcal{X}'})$ are both C-D-bicomodules. It is clear that $\mathrm{span}(\overline{\mathcal{X}}) \cap \mathrm{span}(\overline{\mathcal{X}'})$ is a sub-C-D-bicomodule of $\mathrm{span}(\overline{\mathcal{X}})$. But since $\mathrm{span}(\overline{\mathcal{X}})$ is simple, its sub-C-D-bicomodule $\mathrm{span}(\overline{\mathcal{X}}) \cap \mathrm{span}(\overline{\mathcal{X}'})$ is either $\mathrm{span}(\overline{\mathcal{X}})$ or 0. In the previous case, $\mathrm{span}(\overline{\mathcal{X}}) \supseteq \mathrm{span}(\overline{\mathcal{X}'})$. By the same taken, we can prove that $\mathrm{span}(\overline{\mathcal{X}}) \subseteq \mathrm{span}(\overline{\mathcal{X}'})$. □

Moreover, there are further properties for non-trivial primitive matrices.

Corollary 3.1.4 *Let $C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{C}_{r\times r}$ and $\mathcal{D}_{s\times s}$, respectively. Suppose $\mathcal{X} := (x_{ij})_{r\times s}$ is a $(\mathcal{C}, \mathcal{D})$-primitive matrix. Then the followings are equivalent:*

(1) $\mathcal{X}$ is non-trivial;

(2) $x_{ij} \notin H_0$ holds for all $1 \leq i \leq r$ and $1 \leq j \leq s$;

(3) $\{x_{ij} \mid 1 \leq j \leq s\}$ are linearly independent in H_1/H_0 (the quotient space) for each $1 \leq i \leq r$, and $\{x_{ij} \mid 1 \leq i \leq r\}$ are linearly independent in H_1/H_0 for each $1 \leq j \leq s$.

(4) $\{x_{ij} \mid 1 \leq j \leq s, 1 \leq i \leq r\}$ are linearly independent in H_1/H_0.

Proof: The equivalence of $(1), (2)$ and (3) is by [40, Proposition 2.11]. And (4) clearly implies $(1), (2)$ and (3). To complete the proof, we only need to show that (1) implies (4). Note that if $\mathcal{X}$ is non-trivial, it follows from Lemma 3.1.2 that $\mathrm{span}(\overline{\mathcal{X}})$ is a simple C-D-bicomodule and $\dim_{\Bbbk}(\mathrm{span}(\overline{\mathcal{X}})) = rs$, which means that $\{x_{ij} \mid 1 \leq j \leq s, 1 \leq i \leq r\}$ are linearly independent in H_1/H_0. □

Recall that $\{e_C\}_{C\in\mathcal{S}}$ is called a family of *coradical orthonormal idempotents* (see [60, Section 1]) in H^*, if

$$e_C|_D = \delta_{C,D}\varepsilon|_D, \quad e_Ce_D = \delta_{C,D}e_C \quad (\text{for any } C, D \in \mathcal{S}), \quad \sum_{C\in\mathcal{S}} e_C = \varepsilon.$$

The existence of a family of coradical orthonormal idempotents is affirmed in [60, Lemma 2]. About more properties of coradical orthonormal idempotents, the reader is referred to [47, Proposition 2.2] for details. We use the notations below for convenience:

$${}^C h = h \leftharpoonup e_C, \quad h^D = e_D \rightharpoonup h, \quad {}^C h^D = e_D \rightharpoonup h \leftharpoonup e_C \quad (\text{for any } h \in H \text{ and } C, D \in \mathcal{S}),$$

where $\rightharpoonup$ and $\leftharpoonup$ are hit actions of H^* on H.

Moreover, let $\{e_C\}_{C\in\mathcal{S}}$ be a family of coradical orthonormal idempotents. If V is an H_0-H_0-bicomodule with left comodule structure δ_L and right comodule structure δ_R, define

$${}^C v = v \leftharpoonup e_C = (e_C \otimes \mathrm{Id})\delta_L(v), \quad v^D = e_D \rightharpoonup v = (\mathrm{Id} \otimes e_D)\delta_R(v),$$

$${}^C v^D = e_D \rightharpoonup v \leftharpoonup e_C \quad (\text{for any } v \in V \text{ and } C, D \in \mathcal{S}).$$

With the notations above, we can establish the following decomposition of H_1/H_0 as a direct sum.

Lemma 3.1.5 *Suppose that V is an H_0-H_0-bicomodule, then $V = \bigoplus_{C,D\in\mathcal{S}} {}^C V^D$, where ${}^C V^D = e_D \rightharpoonup V \leftharpoonup e_C$ is a C-D-bicomodule. In particular, we have $H_1/H_0 = \bigoplus_{C,D\in\mathcal{S}} {}^C(H_1/H_0)^D$.*

Proof: It is straightforward to show that ${}^C V^D$ is a C-D-bicomodule. For any $v \in V$, since $\sum_{C\in\mathcal{S}} e_C = \varepsilon$, we have

$$v = \varepsilon \rightharpoonup v \leftharpoonup \varepsilon = \sum_{C,D\in\mathcal{S}} {}^C v^D.$$

Suppose $0 = \sum_{C,D\in\mathcal{S}} w_{C,D}$, where $w_{C,D} \in {}^C V^D$ for any $C, D \in \mathcal{S}$. Note that for any $E, F \in \mathcal{S}$, we have

$$\begin{aligned} 0 &= e_E \rightharpoonup 0 \leftharpoonup e_F \\ &= e_E \rightharpoonup (\sum_{C,D\in\mathcal{S}} w_{C,D}) \leftharpoonup e_F \\ &= \sum_{C,D\in\mathcal{S}} e_E \rightharpoonup w_{C,D} \leftharpoonup e_F \\ &= w_{F,E}. \end{aligned}$$

Thus we complete the proof. □

Besides, for any $C, D \in \mathcal{S}$, since $\Delta({}^C H_1{}^D) \subseteq C \otimes {}^C H_1{}^D + {}^C H_1{}^D \otimes D$, it follows that $({}^C H_1{}^D + H_0)/H_0$ is exactly a C-D-bicomodule with the bicomodule structure ρ_L, ρ_R defined in (3.1). Thus we have another direct sum decomposition of H_1/H_0 and these two kinds of decomposition are related.

Lemma 3.1.6 *As an H_0-H_0-bicomodule, $H_1/H_0 = \bigoplus_{C,D\in\mathcal{S}} ({}^C H_1{}^D + H_0)/H_0$. Moreover, ${}^C(H_1/H_0)^D = ({}^C H_1{}^D + H_0)/H_0$ holds for any $C, D \in \mathcal{S}$.*

Proof: For any $x \in H_1$, a direct computation follows that

$$\begin{aligned} e_D \rightharpoonup \overline{x} \leftharpoonup e_C &= \sum \langle e_C, x_{(1)}\rangle \overline{x_{(2)}} \langle e_D, x_{(3)}\rangle \\ &= \sum \overline{\langle e_C, x_{(1)}\rangle x_{(2)} \langle e_D x_{(3)}\rangle} \\ &= \overline{e_D \rightharpoonup x \leftharpoonup e_C} \\ &\in ({}^C H_1{}^D + H_0)/H_0, \end{aligned}$$

where we use the Sweedler notation $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ for the comultiplication. So we have

$${}^C(H_1/H_0)^D \subseteq ({}^C H_1{}^D + H_0)/H_0$$

and

$$\bigoplus_{C,D\in\mathcal{S}} {}^C(H_1/H_0)^D = H_1/H_0 = \sum_{C,D\in\mathcal{S}} ({}^C H_1{}^D + H_0)/H_0.$$

The same proof with Lemma 3.1.5 can be applied to H_1/H_0, then we get

$$H_1/H_0 = \bigoplus_{C,D\in\mathcal{S}} ({}^C H_1{}^D + H_0)/H_0,$$

which implies that

$${}^C(H_1/H_0)^D = ({}^C H_1{}^D + H_0)/H_0.$$

□

For the remaining of this section, let $C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{C} = (c_{ij})_{r\times r}$ and $\mathcal{D} = (d_{ij})_{s\times s}$, respectively.

Lemma 3.1.7 *For any $(\mathcal{C}, \mathcal{D})$-primitive matrix $\mathcal{X}$, we have*

$$\operatorname{span}(\overline{\mathcal{X}}) \subseteq ({}^C H_1{}^D + H_0)/H_0.$$

Proof: For any $\overline{x} \in \operatorname{span}(\overline{\mathcal{X}}) \subseteq H_1/H_0$, it follows from Lemma 3.1.1 that

$$\rho_L(\overline{x}) \subseteq C \otimes \operatorname{span}(\overline{\mathcal{X}})$$

and

$$\rho_R(\overline{x}) \subseteq \operatorname{span}(\overline{\mathcal{X}}) \otimes D.$$

According to Lemma 3.1.6, we have

$$\overline{x} \in \bigoplus_{E,F\in\mathcal{S}} ({}^{E}{H_1}^{F} + H_0)/H_0.$$

Note that $({}^{E}{H_1}^{F} + H_0)/H_0$ is a E-F-bicomodule, for any $E, F \in \mathcal{S}$. It follows that

$$\overline{x} \in ({}^{C}{H_1}^{D} + H_0)/H_0,$$

which means that

$$\operatorname{span}(\overline{\mathcal{X}}) \subseteq ({}^{C}{H_1}^{D} + H_0)/H_0.$$

□

Next we consider the inverse.

Lemma 3.1.8 *If W is a subspace of ${}^{C}{H_1}^{D} + H_0$ such that $\overline{W}$ is a simple C-D-sub-bicomodule of $({}^{C}{H_1}^{D} + H_0)/H_0$, then there exists some non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix $\mathcal{W}$ such that* $\operatorname{span}(\overline{\mathcal{W}}) = \overline{W}$.

Proof: For any nonzero $\overline{w} \in \overline{W}$, without the loss of generality, we assume $w \in {}^{C}{H_1}^{D}$.

(1) If $C \neq D$, by [47, Theorem 3.1(1)] and its proof, we know that there exist rs $(\mathcal{C}, \mathcal{D})$-primitive matrices $\mathcal{W}^{(i',j')} = \left(w_{ij}^{(i',j')}\right)_{r\times s}$ $(1 \leq i' \leq r, 1 \leq j' \leq s)$ such that

$$w = \sum_{i=1}^{r}\sum_{j=1}^{s} w_{ij}^{(i,j)},$$

$$\Delta(w) = \sum_{i',i=1}^{r} c_{i'i} \otimes x_i^{(i')} + \sum_{j,j'=1}^{s} y_j^{(j')} \otimes d_{jj'},$$

and

$$\Delta(x_i^{(i')}) = \sum_{k=1}^{r} c_{ik} \otimes x_k^{(i')} + \sum_{j,j'=1}^{s} w_{ij}^{(i',j')} \otimes d_{jj'},$$

where $x_i^{(i')}, y_j^{(j')} \in {}^{C}{H_1}^{D} \cap \ker\varepsilon$ for all $1 \leq i', i \leq r, 1 \leq j', j \leq s$. Observe $\overline{W}$ is a C-D-sub-bicomodule whose comodule structure is induced by comultiplication, namely,

$$\rho_L(\overline{w}) = (\mathrm{Id}\otimes\pi)\Delta(w) \in C \otimes \overline{W}, \quad \rho_R(\overline{w}) = (\pi\otimes\mathrm{Id})\Delta(w) \in \overline{W} \otimes D.$$

As $\{c_{ii'} \mid 1 \leq i, i' \leq r\}$ and $\{d_{jj'} \mid 1 \leq j, j' \leq s\}$ are linearly independent, thus

$$\overline{x_i^{(i')}}, \overline{y_j^{(j')}} \in \overline{W}$$

for all i, i', j, j'. According to a similar argument, for any i', i, we have

$$\rho_R(\overline{x_i^{(i')}}) \in \overline{W} \otimes D.$$

This means that $\overline{w_{ij}^{(i',j')}} \in \overline{W}$ for all i, i', j, j'. Hence we have

$$\operatorname{span}(\overline{\mathcal{W}^{(i',j')}}) \subseteq \overline{W}$$

for all i', j'. Since $\overline{w}$ is nonzero, there must be some pair (i'_0, j'_0) such that $\mathcal{W}^{(i'_0,j'_0)}$ is a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix. However, note that $\overline{W}$ is a simple C-D-sub-bicomodule. It follows that

$$\operatorname{span}(\overline{\mathcal{W}^{(i'_0,j'_0)}}) = \overline{W}.$$

(2) If $C = D$, according to [47, Theorem 3.1] or [45, Lemma 2.14(2)], we choose $\mathcal{C} = \mathcal{D}$, and there exist rs $(\mathcal{C}, \mathcal{D})$-primitive matrices $\mathcal{W}^{(i',j')} = \left(w_{ij}^{(i',j')}\right)_{r\times s}$ $(1 \leq i' \leq r, 1 \leq j' \leq s)$ such that

$$w - \sum_{i=1}^{r}\sum_{j=1}^{s} w_{ij}^{(i,j)} \in C.$$

Using [47, Lemma 3.1], we know that there exists an element $c \in C$ such that

$$\Delta(w - c) \in C \otimes ({}^{C}H_1{}^{C})^+ + ({}^{C}H_1{}^{C})^+ \otimes C,$$

where $({}^{C}H_1{}^{C})^+ = {}^{C}H_1{}^{C} \cap \ker\varepsilon$. Then the same proof of (1) can be applied to the element $w - c$. Thus we can find a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix $\mathcal{W}^{(i'_0,j'_0)}$ such that

$$\operatorname{span}(\overline{\mathcal{W}^{(i'_0,j'_0)}}) = \overline{W}.$$

□

Clearly, a coalgebra H is cosemisimple if and only if the category of left (resp. right) H-comodules is a semisimple category. This means that any C-D-bicomodule is cosemisimple. Applying Lemma 3.1.6 to the cosemisimple C-D-bicomodule $({}^{C}H_1{}^{D} + H_0)/H_0$, we can decompose it into the direct sum of simple C-D-sub-bicomodules as the following.

Corollary 3.1.9 *There exists a family $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ of non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices such that*

$$^{C}(H_1/H_0)^{D} = ({}^{C}H_1{}^{D} + H_0)/H_0 = \bigoplus_{\gamma\in\Gamma} \operatorname{span}(\overline{\mathcal{X}^{(\gamma)}}). \tag{3.2}$$

Definition 3.1.10 *A family of non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ satisfying the property of (3.2) in Corollary 3.1.9 is said to be complete.*

The corollary below is followed immediately by Lemma 3.1.1 and Corollary 3.1.9.

Corollary 3.1.11 *If $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ is a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices, where $\mathcal{X}_\gamma = (x_{ij}^{(\gamma)})_{r\times s}$, then $\{\overline{x_{ij}^{(\gamma)}} \mid \gamma\in\Gamma, 1\leq i\leq r, 1\leq j\leq s\}$ is a linear basis of $({}^C{H_1}^D + H_0)/H_0$.*

A complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices is the main tool to help us characterize the link quiver of H in the subsequent sections. Thus some of its properties should be noticed.

Lemma 3.1.12 *Suppose $\{\mathcal{X}^{(\lambda)}\}_{\lambda\in\Lambda}$ is a family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices such that the sum $\sum\limits_{\lambda\in\Lambda}\operatorname{span}(\overline{\mathcal{X}^{(\lambda)}})$ in $({}^C{H_1}^D + H_0)/H_0$ is direct. Then we can find a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ such that $\{\mathcal{X}^{(\lambda)}\}_{\lambda\in\Lambda}$ is a subset of $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$.*

Proof: Let M be a complement of $\bigoplus_{\lambda\in\Lambda}\operatorname{span}(\overline{\mathcal{X}^{(\lambda)}})$ in $({}^C{H_1}^D + H_0)/H_0$. According to Lemma 3.1.8, we can show that

$$M = \bigoplus_{\gamma'\in\Gamma'}\operatorname{span}(\overline{\mathcal{X}^{(\gamma')}})$$

for some non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices $\{\mathcal{X}^{(\gamma')}\}_{\gamma'\in\Gamma'}$. Let

$$\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma} = \{\mathcal{X}^{(\lambda)}\}_{\lambda\in\Lambda} \cup \{\mathcal{X}^{(\gamma')}\}_{\gamma'\in\Gamma'}.$$

Then $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ is a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices. □

The important property of a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices in the vector space spanned by all $(\mathcal{C},\mathcal{D})$-primitive matrices is summarized in the following proposition.

Proposition 3.1.13 *Suppose $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ is a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices. Then for any $(\mathcal{C},\mathcal{D})$-primitive matrix $\mathcal{Y}$, we have $\overline{\mathcal{Y}} = \sum\limits_{\gamma\in\Gamma}\alpha_\gamma\overline{\mathcal{X}^{(\gamma)}}$, where $\alpha_\gamma\in\Bbbk$ $(\gamma\in\Gamma)$ and only a finite number of them are nonzero.*

Proof: Suppose that $\mathcal{X}^{(\gamma)} = (x_{ij}^{(\gamma)})_{r\times s}$ and $\mathcal{Y} = (y_{ij})_{r\times s}$. By the definition of $(\mathcal{C},\mathcal{D})$-primitive matrix, we have

$$\Delta(y_{ij}) = \sum_{k=1}^{r} c_{ik}\otimes y_{kj} + \sum_{l=1}^{s} y_{il}\otimes d_{lj}.$$

According to Corollary 3.1.11, for any $1\leq i\leq r, 1\leq j\leq s$, we can assume

$$\overline{y_{ij}} = \sum_{\gamma\in\Gamma}\sum_{p=1}^{r}\sum_{q=1}^{s}\beta_{pq}^{(ij,\gamma)}\overline{x_{pq}^{(\gamma)}},$$

where $\beta_{pq}^{(ij,\gamma)}\in\Bbbk$ for any $1\leq p\leq r, 1\leq q\leq s, \gamma\in\Gamma$, and only a finite number of them are nonzero.

Then

$$\begin{aligned}\rho_L(\overline{y_{ij}}) &= (\mathrm{Id}\otimes\pi)\Delta(y_{ij}) = \sum_{k=1}^{r} c_{ik}\otimes\overline{y_{kj}} \\ &= \rho_L(\sum_{\gamma\in\Gamma}\sum_{p=1}^{r}\sum_{q=1}^{s}\beta_{pq}^{(ij,\gamma)}\overline{x_{pq}^{(\gamma)}}) \\ &= \sum_{p=1}^{r}\sum_{k=1}^{r} c_{pk}\otimes(\sum_{\gamma\in\Gamma}\sum_{q=1}^{s}\beta_{pq}^{(ij,\gamma)}\overline{x_{kq}^{(\gamma)}}).\end{aligned}$$

Since the entries of $\mathcal{C}$ are linearly independent, it follows that

$$\overline{y_{kj}} = \sum_{\gamma\in\Gamma}\sum_{q=1}^{s}\beta_{iq}^{(ij,\gamma)}\overline{x_{kq}^{(\gamma)}}$$

holds for any $1\le k\le r$. Using the same argument as above, when we consider the right comodule structure of $\operatorname{span}(\overline{\mathcal{Y}})$, we get

$$\overline{y_{il}} = \sum_{\gamma\in\Gamma}\sum_{p=1}^{r}\beta_{pj}^{(ij,\gamma)}\overline{x_{pl}^{(\gamma)}}$$

holds for any $1\le l\le s$. It follows that

$$\overline{y_{kl}} = \sum_{\gamma\in\Gamma}\sum_{q=1}^{s}\beta_{iq}^{(il,\gamma)}\overline{x_{kq}^{(\gamma)}} = \sum_{\gamma\in\Gamma}\sum_{p=1}^{r}\beta_{pj}^{(kj,\gamma)}\overline{x_{pl}^{(\gamma)}},$$

for any $1\le k\le r, 1\le l\le s$. Because of the linear independence of

$$\{\overline{x_{ij}^{(\gamma)}} \mid 1\le i\le r, 1\le j\le s, \gamma\in\Gamma\}$$

in H_1/H_0, for any $1\le i,k\le r, 1\le j,l\le s, \gamma\in\Gamma$, we have

$$\beta_{iq}^{(il,\gamma)} = 0$$

when $q\neq l$, and

$$\beta_{pj}^{(kj,\gamma)} = 0$$

when $p\neq k$. Moreover, when $p=k, q=l$,

$$\beta_{il}^{(il,\gamma)} = \beta_{kj}^{(kj,\gamma)}$$

holds for all $1\le i,k\le r, 1\le j,l\le s, \gamma\in\Gamma$. This means that

$$\overline{\mathcal{Y}} = \sum_{\gamma\in\Gamma}\alpha_\gamma\overline{\mathcal{X}^{(\gamma)}},$$

where $\alpha_\gamma = \beta_{11}^{(11,\gamma)}\in\Bbbk$ for any $\gamma\in\Gamma$, and only a finite number of them are nonzero. □

With the help of the preceding proposition, we can now prove:

Corollary 3.1.14 *Let $\mathcal{X}$ be a non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrix. Suppose $P\mathcal{X}Q$ is also a $(\mathcal{C},\mathcal{D})$-primitive matrix, where P and Q are invertible matrices over $\Bbbk$. Then $P\overline{\mathcal{X}}Q = \alpha\overline{\mathcal{X}}$ for some $\alpha \in \Bbbk$.*

Proof: Using Lemma 3.1.12, we can find a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ with some element $\mathcal{X}^{(\gamma_1)} = \mathcal{X}$. Then by Proposition 3.1.13,

$$P\overline{\mathcal{X}}Q = \sum_{\gamma\in\Gamma} \alpha_\gamma \overline{\mathcal{X}^{(\gamma)}},$$

where $\alpha_\gamma \in \Bbbk$ $(\gamma \in \Gamma)$ and only a finite number of them are nonzero. However,

$$\operatorname{span}(P\overline{\mathcal{X}}Q) \cap \big(\sum_{\gamma\in\Gamma\setminus\{\gamma_1\}} \operatorname{span}(\overline{\mathcal{X}^{(\gamma)}})\big) = 0.$$

This implies that $\alpha_\gamma = 0$ for all $\gamma \neq \gamma_1$. Therefore,

$$P\overline{\mathcal{X}}Q = \alpha_{\gamma_1}\overline{\mathcal{X}}.$$

□

Note that by [17, Theorem 4.1], we have

$$H_1/H_0 \cong \bigoplus_{C,D\in\mathcal{S}} (C\wedge D)/(C+D),$$

where $(C\wedge D)/(C+D)$ is isomorphic to the following C-D-bicomodule

$$\{h \in H_1/H_0 \mid \rho_L(h) \in C\otimes H_1/H_0,\ \rho_R(h) \in H_1/H_0 \otimes D\},$$

which is exactly ${}^C(H_1/H_0)^D$. So we can now obtain the following lemma:

Lemma 3.1.15 *If $C, D \in \mathcal{S}$, then we have a C-D-bicomodule isomorphism:*

$$({}^C{H_1}^D + H_0)/H_0 \cong (C\wedge D)/(C+D).$$

Combining Corollary 3.1.9 and Lemma 3.1.15, we obtain the following corollary.

Corollary 3.1.16 *Let $C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{C}_{r\times r}$ and $\mathcal{D}_{s\times s}$, respectively. If $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ is a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices, then the cardinal number*

$$\mid \Gamma \mid = \frac{1}{rs}\dim_{\Bbbk}\left((C\wedge D)/(C+D)\right). \tag{3.3}$$

The corollary above will help us transform the problem of number of arrows from vertex D to vertex C in the link quiver of H to the problem of cardinal number of a complete family of non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices in the subsequent sections.

Note that the number (3.3) in Corollary 3.1.16 does not depend on the choices of basic multiplicative matrices $\mathcal{C}$ and $\mathcal{D}$ as well as a complete family of non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices.

§3.2 Constructions of a complete family of non-trivial primitive matrices

In this section, let H be a Hopf algebra over $\Bbbk$ with the dual Chevalley property. Denote the coradical filtration of H by $\{H_n\}_{n\geq 0}$ and the set of all the simple subcoalgebras of H by $\mathcal{S}$. Let $\mathcal{M}$ denote the set of representative elements of basic multiplicative matrices over H for the similarity class. It is clear that there is a bijection from $\mathcal{S}$ to $\mathcal{M}$, mapping each simple subcoalgebra to its basic multiplicative matrix, and $\mathcal{S} = \{\mathrm{span}(\mathcal{C}) \mid \mathcal{C} \in \mathcal{M}\}$, where $\mathrm{span}(\mathcal{C})$ is the subspace of H_0 spanned by the entries of $\mathcal{C}$.

The aim of this section is to construct a complete family of non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices over H for any $C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{C}, \mathcal{D}$, respectively.

§3.2.1 The first construction

Denote ${}^1\mathcal{S} = \{C \in \mathcal{S} \mid \Bbbk 1 + C \neq \Bbbk 1 \wedge C\}$. For any $C \in {}^1\mathcal{S}$ with basic multiplicative matrix $\mathcal{C} \in \mathcal{M}$, using Corollary 3.1.9, we can fix a complete family $\{\mathcal{X}_C^{(\gamma_C)}\}_{\gamma_C \in \Gamma_C}$ of non-trivial $(1, \mathcal{C})$-primitive matrices.

Denote

$$ {}^1\mathcal{P} := \bigcup_{C \in {}^1\mathcal{S}} \{\mathcal{X}_C^{(\gamma_C)} \mid \gamma_C \in \Gamma_C\}. \tag{3.4} $$

Then for any non-trivial $(1, \mathcal{C})$-primitive matrix $\mathcal{Y} \in {}^1\mathcal{P}$ and $\mathcal{B} \in \mathcal{M}$, we have

$$ \begin{pmatrix} I & 0 \\ 0 & L_{\mathcal{B},\mathcal{C}} \end{pmatrix} \left(\mathcal{B} \odot' \begin{pmatrix} 1 & \mathcal{Y} \\ 0 & \mathcal{C} \end{pmatrix} \right) \begin{pmatrix} I & 0 \\ 0 & L_{\mathcal{B},\mathcal{C}}^{-1} \end{pmatrix} = \begin{pmatrix} \mathcal{B} & \mathcal{Y}_1 & \cdots & \mathcal{Y}_{u(\mathcal{B},\mathcal{C})} \\ & \mathcal{E}_1 & & \\ 0 & & \ddots & \\ & & & \mathcal{E}_{u(\mathcal{B},\mathcal{C})} \end{pmatrix}, \tag{3.5} $$

where $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{u(\mathcal{B},\mathcal{C})} \in \mathcal{M}$. According to Lemma 2.1.4, we know that $\mathcal{Y}_1, \mathcal{Y}_2, \cdots, \mathcal{Y}_{u(\mathcal{B},\mathcal{C})}$ are non-trivial.

Denote

$$ {}^{\mathcal{B}}\mathcal{P}_{\mathcal{Y}} := \{\mathcal{Y}_i \mid 1 \leq i \leq u_{(\mathcal{B},\mathcal{C})}\}, \tag{3.6} $$

$$ {}^{\mathcal{B}}\mathcal{P} := \bigcup_{\mathcal{Y} \in {}^1\mathcal{P}} {}^{\mathcal{B}}\mathcal{P}_{\mathcal{Y}}, \quad \mathcal{P}_{\mathcal{Y}} := \bigcup_{\mathcal{B} \in \mathcal{M}} {}^{\mathcal{B}}\mathcal{P}_{\mathcal{Y}}. \tag{3.7} $$

We remark that $\bigcup_{\mathcal{Y}\in {}^1\mathcal{P}} {}^1\mathcal{P}_{\mathcal{Y}}$ coincides with ${}^1\mathcal{P}$ defined in (3.4).

Moreover, denote

$$\mathcal{P} := \bigcup_{\mathcal{B}\in\mathcal{M}} {}^{\mathcal{B}}\mathcal{P} = \bigcup_{\mathcal{Y}\in {}^1\mathcal{P}} \mathcal{P}_{\mathcal{Y}}. \tag{3.8}$$

Note that the elements in the set ${}^{\mathcal{B}}\mathcal{P}_{\mathcal{Y}}$ depends on the choice of the invertible matrix $L_{\mathcal{B},\mathcal{C}}$ in (3.5). It will be shown in the following Lemma that the cardinal number $|\ {}^{\mathcal{B}}\mathcal{P}_{\mathcal{Y}}\ |$ does not depend on the choice of $L_{\mathcal{B},\mathcal{C}}$.

Lemma 3.2.1 *The sum* $\sum_{1\le i\le u_{(\mathcal{B},\mathcal{C})}} \mathrm{span}(\overline{\mathcal{Y}_i})$ *is direct, where each* $\mathcal{Y}_i$ *appears in (3.5).*

Proof: Without loss of generality, assume that

$$E_1 = E_2 = \cdots = E_t$$

for some $1 \le t \le u_{\mathcal{B},\mathcal{C}}$, and

$$E_j \neq E_1$$

when $t < j \le u_{(\mathcal{B},\mathcal{C})}$. In fact $\sum_{1\le i\le t} \mathrm{span}(\overline{\mathcal{Y}_i})$ is a B-E_1-sub-bicomodule of H_1/H_0. Let T_0 be a maximal subset of $\{1, 2, \cdots, t\}$ such that $\sum_{j\in T_0} \mathrm{span}(\overline{\mathcal{Y}_j})$ is direct. Suppose

$$\bigoplus_{j\in T_0} \mathrm{span}(\overline{\mathcal{Y}_j}) \subsetneqq \sum_{1\le i\le t} \mathrm{span}(\overline{\mathcal{Y}_i}).$$

Since for any $1 \le i \le t$, $\mathrm{span}(\overline{\mathcal{Y}_i})$ is simple, there exists some $s \notin T_0$,

$$\mathrm{span}(\overline{\mathcal{Y}_s}) \cap (\bigoplus_{j\in T_0} \mathrm{span}(\overline{\mathcal{Y}_j})) = 0.$$

Thus

$$\mathrm{span}(\overline{\mathcal{Y}_s}) + (\bigoplus_{j\in T_0} \mathrm{span}(\overline{\mathcal{Y}_j})) = \mathrm{span}(\overline{\mathcal{Y}_s}) \oplus (\bigoplus_{j\in T_0} \mathrm{span}(\overline{\mathcal{Y}_j})),$$

which is a contradiction. Now we can get a subset $\{w_1, \cdots, w_r\}$ of $\{1, \cdots, t\}$ such that

$$\sum_{i=1}^{t} \mathrm{span}(\overline{\mathcal{Y}_i}) = \bigoplus_{i=j}^{r} \mathrm{span}(\overline{\mathcal{Y}_{w_j}}).$$

Without loss of generality, assume that

$$\{w_1, w_2, \cdots, w_r\} = \{1, 2, \cdots, r\}.$$

According to Lemma 3.1.12, there exists a complete family $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$ of non-trivial $(\mathcal{B}, \mathcal{E}_1)$-primitive matrices such that $\{\mathcal{Y}_i\}_{1\le i\le r}$ is a subset of $\{\mathcal{X}^{(\gamma)}\}_{\gamma\in\Gamma}$. It follows from Proposition 3.1.13 that $\overline{\mathcal{Y}_t}$ is the linear combination of $\{\overline{\mathcal{X}^{(\gamma)}}\}_{\gamma\in\Gamma}$. Note that if $t > r$, then

$$\text{span}(\overline{\mathcal{Y}_t}) \subseteq \bigoplus_{1\leq i\leq r} \text{span}(\overline{\mathcal{Y}_i}).$$

According to Corollary 3.1.11, $\overline{\mathcal{Y}_t}$ is the linear combination of $\{\overline{\mathcal{Y}_i}\}_{1\leq i\leq r}$. This implies that the column vectors of $\begin{pmatrix} \mathcal{Y}_1 & \mathcal{Y}_2 & \cdots & \mathcal{Y}_t \end{pmatrix}$ are linearly dependent over H/H_0, which is in contradiction with Lemma 2.1.4. Thus we have $t = r$ and the sum $\sum\limits_{1\leq i\leq t} \text{span}(\overline{\mathcal{Y}_i})$ is direct. Then by Corollary 3.1.9, the proof is completed. □

Remark 3.2.2 *The cardinal number* $| \, {}^{\mathcal{B}}\mathcal{P}_{\mathcal{Y}} \, | = u_{B,C}$*, where* ${}^{\mathcal{B}}\mathcal{P}_{\mathcal{Y}}$ *appears in (3.6).*

Now we define an H_0-bimodule structure on H_1/H_0 as follows:

$$h \otimes \overline{x} \mapsto h \cdot \overline{x} := \overline{hx}, \;\; \overline{x} \otimes h \mapsto \overline{x} \cdot h := \overline{xh} \; (h \in H_0, x \in H_1).$$

Thus H_1/H_0 becomes an H_0-Hopf bimodule with the bicomodule structure defined in (3.1) and bimodule structure defined obove.

Lemma 3.2.3 *With the notations in (3.8), we have* $H_1/H_0 = \sum\limits_{\mathcal{X}\in\mathcal{P}} \text{span}(\overline{\mathcal{X}})$.

Proof: It suffices us to prove that

$$H_1/H_0 \subseteq \sum_{\mathcal{X}\in\mathcal{P}} \text{span}(\overline{\mathcal{X}}).$$

Applying the fundamental theorem of Hopf modules ([68, Theorem 4.1.1]), we know that as a left H_0-Hopf module,

$$H_0 \otimes {}^{coH_0}(H_1/H_0) \cong H_1/H_0,$$

where $(H_1/H_0)^{coH_0}$ is the left coinvariants of H_0 in H_1/H_0. This isomorphism maps $h\otimes\overline{x}$ to $\overline{hx}$, where $h \in H_0, \overline{x} \in {}^{coH_0}(H_1/H_0)$. Using [45, Proposition 2.6(4)], we can obtain the direct sum decomposition

$${}^1H_1 = \bigoplus_{C\in\mathcal{S}} {}^1{H_1}^C.$$

It follows that

$$\begin{aligned} ({}^1H_1 + H_0)/H_0 &= \big((\bigoplus_{C\in\mathcal{S}} {}^1{H_1}^C) + H_0\big)/H_0 \\ &= (\sum_{C\in\mathcal{S}} {}^1{H_1}^C + H_0)/H_0 \\ &= \sum_{C\in\mathcal{S}} ({}^1{H_1}^C + H_0)/H_0. \end{aligned}$$

Note that $\sum_{C\in\mathcal{S}}({}^1H_1{}^C + H_0)/H_0$ is direct and according to Corollary 3.1.9, we have

$$({}^1H_1 + H_0)/H_0 = \bigoplus_{C\in\mathcal{S}}({}^1H_1{}^C + H_0)/H_0 = \bigoplus_{\mathcal{Y}\in{}^1\mathcal{P}} \mathrm{span}(\overline{\mathcal{Y}}). \tag{3.9}$$

From the proof of [45, Proposition 3.9], we know that

$$(H_1/H_0)^{coH_0} = \bigoplus_{\mathcal{Y}\in{}^1\mathcal{P}} \mathrm{span}(\overline{\mathcal{Y}}). \tag{3.10}$$

Moreover, the definition of $\mathcal{P}$ yields that

$$\sum_{\mathcal{X}\in\mathcal{P}} \mathrm{span}(\overline{\mathcal{X}}) = \sum_{\mathcal{B}\in\mathcal{M}}\sum_{\mathcal{Y}\in{}^1\mathcal{P}} \mathrm{span}(\overline{\mathcal{B}\odot'\mathcal{Y}}).$$

In fact

$$H_0 = \sum_{\mathcal{B}\in\mathcal{M}} \mathrm{span}(\mathcal{B}). \tag{3.11}$$

According to (3.9) and (3.11), one can get

$$\begin{aligned} H_1/H_0 &= \{h\cdot\overline{x} \mid h\in H_0, x\in {}^1H_1 + H_0\} \\ &\subseteq \Big(\sum_{\mathcal{B}\in\mathcal{M}} \mathrm{span}(\mathcal{B})\Big)\cdot\Big(\bigoplus_{\mathcal{Y}\in{}^1\mathcal{P}} \mathrm{span}(\overline{\mathcal{Y}})\Big) \\ &\subseteq \sum_{\mathcal{B}\in\mathcal{M}}\sum_{\mathcal{Y}\in{}^1\mathcal{P}} \mathrm{span}(\overline{\mathcal{B}\odot'\mathcal{Y}}) \\ &= \sum_{\mathcal{X}\in\mathcal{P}} \mathrm{span}(\overline{\mathcal{X}}). \end{aligned}$$

□

Lemma 3.2.4 *With the notations in (3.7), for any $\mathcal{C}\in\mathcal{M}$ and non-trivial $(1,\mathcal{C})$-primitive matrix $\mathcal{X}\in{}^1\mathcal{P}$, we have*

$$\left(\sum_{\mathcal{W}\in\mathcal{P}_\mathcal{X}} \mathrm{span}(\overline{\mathcal{W}})\right)\cap\left(\sum_{\mathcal{Z}\in{}^1\mathcal{P},\mathcal{Z}\neq\mathcal{X}}\ \sum_{\mathcal{W}\in\mathcal{P}_\mathcal{Z}} \mathrm{span}(\overline{\mathcal{W}})\right) = 0.$$

Proof: According to 3.10, we have

$$H_0\otimes(H_1/H_0)^{coH_0} = \bigoplus_{\mathcal{Y}\in{}^1\mathcal{P}} H_0\otimes\mathrm{span}(\overline{\mathcal{Y}}).$$

Moreover, the isomorphism

$$H_0\otimes(H_1/H_0)^{coH_0}\cong H_1/H_0$$

maps $h \otimes \overline{x}$ to $h \cdot \overline{x}$, where $h \in H_0, \overline{x} \in {}^{coH_0}(H_1/H_0)$. By the definition of ${}^1\mathcal{P}$, we know that

$$\mathrm{span}(\overline{\mathcal{X}}) \cap \big(\sum_{\mathcal{Z} \in {}^1\mathcal{P}, \mathcal{Z} \neq \mathcal{X}} \mathrm{span}(\overline{\mathcal{Z}}) \big) = 0.$$

Then

$$\big(H_0 \otimes \mathrm{span}(\overline{\mathcal{Y}})\big) \cap \left(H_0 \otimes \big(\sum_{\mathcal{Z} \in {}^1\mathcal{P}, \mathcal{Z} \neq \mathcal{X}} \mathrm{span}(\overline{\mathcal{Z}}) \big) \right) = 0,$$

which suggests that

$$\big(H_0 \cdot \mathrm{span}(\overline{\mathcal{Y}})\big) \cap \left(H_0 \cdot \big(\sum_{\mathcal{Z} \in {}^1\mathcal{P}, \mathcal{Z} \neq \mathcal{X}} \mathrm{span}(\overline{\mathcal{Z}}) \big) \right) = 0,$$

where

$$\begin{aligned} H_0 \cdot \mathrm{span}(\overline{\mathcal{Y}}) &= \{ h \cdot \overline{y} \mid h \in H_0, y \in \mathcal{Y} \}, \\ H_0 \cdot \big(\sum_{\mathcal{Z} \in {}^1\mathcal{P}, \mathcal{Z} \neq \mathcal{X}} \mathrm{span}(\overline{\mathcal{Z}}) \big) &= \{ h \cdot \overline{z} \mid h \in H_0, \overline{z} \in \sum_{\mathcal{Z} \in {}^1\mathcal{P}, \mathcal{Z} \neq \mathcal{X}} \mathrm{span}(\overline{\mathcal{Z}}) \}. \end{aligned}$$

Therefore, by the definition of $\mathcal{P}_{\mathcal{X}}$, we conclude that

$$\left(\sum_{\mathcal{W} \in \mathcal{P}_{\mathcal{X}}} \mathrm{span}(\overline{\mathcal{W}}) \right) \cap \left(\sum_{\mathcal{Z} \in {}^1\mathcal{P}, \mathcal{Z} \neq \mathcal{X}} \sum_{\mathcal{W} \in \mathcal{P}_{\mathcal{Z}}} \mathrm{span}(\overline{\mathcal{W}}) \right) = 0.$$

□

A direct consequence of this lemma is:

Corollary 3.2.5 *With the notations in (3.8), then the union* $\mathcal{P} = \bigcup_{\mathcal{Y} \in {}^1\mathcal{P}} \mathcal{P}_{\mathcal{Y}}$ *is disjoint.*

Now it is not difficult to verify the following theorem.

Theorem 3.2.6 *Let* $C, D \in \mathcal{S}$ *with basic multiplicative matrices* $\mathcal{C}, \mathcal{D} \in \mathcal{M}$ *respectively. Denote*

$${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}} := \{ \mathcal{X} \in \mathcal{P} \mid \mathcal{X} \text{ is a non-trivial } (\mathcal{C}, \mathcal{D})\text{-primitive matrix} \}.$$

Then it is a complete family of non-trivial $(\mathcal{C}, \mathcal{D})$*-primitive matrices. Moreover, we have* $H_1/H_0 = \bigoplus_{\mathcal{X} \in \mathcal{P}} \mathrm{span}(\overline{\mathcal{X}})$.

Proof: By the definition of $\mathcal{P}$, we have

$$\mathcal{P} = \bigcup_{\mathcal{C}, \mathcal{D} \in \mathcal{M}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}},$$

which means that

$$\sum_{\mathcal{X} \in \mathcal{P}} \mathrm{span}(\overline{\mathcal{X}}) = \sum_{\mathcal{C}, \mathcal{D} \in \mathcal{M}} \sum_{\mathcal{X} \in {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}} \mathrm{span}(\overline{\mathcal{X}}).$$

According to Lemma 3.1.6, we know that

$$H_1/H_0 = \bigoplus_{C,D\in\mathcal{S}} ({}^C H_1{}^D + H_0)/H_0.$$

It follows from Lemma 3.1.7 that

$$\sum_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}} \mathrm{span}(\overline{\mathcal{X}}) \subseteq ({}^C H_1{}^D + H_0)/H_0,$$

which implies that

$$\begin{aligned}\sum_{\mathcal{X}\in\mathcal{P}} \mathrm{span}(\overline{\mathcal{X}}) &= \bigoplus_{\mathcal{C},\mathcal{D}\in\mathcal{M}} \Big(\sum_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}} \mathrm{span}(\overline{\mathcal{X}})\Big)\\ &\subseteq \bigoplus_{C,D\in\mathcal{S}} ({}^C H_1{}^D + H_0)/H_0\\ &= H_1/H_0.\end{aligned}$$

By Lemma 3.2.3, we have

$$H_1/H_0 = \sum_{\mathcal{X}\in\mathcal{P}} \mathrm{span}(\overline{\mathcal{X}}).$$

Therefore one can get

$$({}^C H_1{}^D + H_0)/H_0 = \sum_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}} \mathrm{span}(\overline{\mathcal{X}}). \tag{3.12}$$

Note that

$${}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}} \subseteq \mathcal{P}_{\mathcal{Y}}.$$

Combining Lemmas 3.2.1 and 3.2.4, we can get

$$\begin{aligned}\sum_{\mathcal{Y}\in {}^1\mathcal{P}} \Big(\sum_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}}} \mathrm{span}(\overline{\mathcal{X}})\Big) &= \sum_{\mathcal{Y}\in {}^1\mathcal{P}} \Big(\bigoplus_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}}} \mathrm{span}(\overline{\mathcal{X}})\Big)\\ &= \bigoplus_{\mathcal{Y}\in {}^1\mathcal{P}} \Big(\bigoplus_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}}} \mathrm{span}(\overline{\mathcal{X}})\Big).\end{aligned}$$

Thus it follows from

$${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}} \subseteq \bigcup_{\mathcal{Y}\in {}^1\mathcal{P}} {}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}}$$

that

$$\sum_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}} \mathrm{span}(\overline{\mathcal{X}}) = \bigoplus_{\mathcal{X}\in {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}} \mathrm{span}(\overline{\mathcal{X}}),$$

and ${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$ is a family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices. Moreover, we have

$$H_1/H_0 = \bigoplus_{\mathcal{X}\in\mathcal{P}} \mathrm{span}(\overline{\mathcal{X}}).$$

□

Corollary 3.2.7 *Let $C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{C}, \mathcal{D} \in \mathcal{M}$ respectively. Denote*

$$\mathcal{P}^{\mathcal{D}} := \bigcup_{\mathcal{C}\in\mathcal{M}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}},$$

which is a disjoint union of ${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$ ($\mathcal{C} \in \mathcal{M}$) defined in Theorem 3.2.6. Then

$$({H_1}^{D} + H_0)/H_0 = \bigoplus_{\mathcal{X}\in\mathcal{P}^{\mathcal{D}}} \operatorname{span}(\overline{\mathcal{X}}).$$

Moreover, we have

$${}^{\mathcal{C}}\mathcal{P} = \bigcup_{\mathcal{D}\in\mathcal{M}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}},$$

which is also a disjoint union, and that

$$({}^{C}H_1 + H_0)/H_0 = \bigoplus_{\mathcal{X}\in{}^{\mathcal{C}}\mathcal{P}} \operatorname{span}(\overline{\mathcal{X}}).$$

Proof: Note that by the definition of ${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$, we know that ${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$ contains all the non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices in $\mathcal{P}$. It follows that

$${}^{\mathcal{C}}\mathcal{P} = \bigcup_{\mathcal{D}\in\mathcal{M}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}.$$

According to Lemma 3.1.6, we have

$$H_1/H_0 = \bigoplus_{C,D\in\mathcal{S}} ({}^{C}{H_1}^{D} + H_0)/H_0.$$

By (3.12) in the proof of Theorem 3.2.6, we know that

$$({}^{C}{H_1}^{D} + H_0)/H_0 = \sum_{\mathcal{X}\in{}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}} \operatorname{span}(\overline{\mathcal{X}}).$$

This means that these two unions

$$\mathcal{P}^{\mathcal{D}} = \bigcup_{\mathcal{C}\in\mathcal{M}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$$

and

$${}^{\mathcal{C}}\mathcal{P} = \bigcup_{\mathcal{D}\in\mathcal{M}} {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$$

are both disjoint union.

Using [45, Proposition 2.6(4)], we can obtain the direct sum decomposition

$$H_1{}^D = \bigoplus_{C\in\mathcal{S}} {}^C H_1{}^D.$$

It follows that

$$\begin{aligned}(H_1{}^D + H_0)/H_0 &= \big((\bigoplus_{C\in\mathcal{S}} {}^C H_1{}^D) + H_0\big)/H_0 \\ &= (\sum_{C\in\mathcal{S}} {}^C H_1{}^D + H_0)/H_0 \\ &= \sum_{C\in\mathcal{S}} ({}^C H_1{}^D + H_0)/H_0.\end{aligned}$$

Then it follows from (3.12) in the proof of Theorem 3.2.6 that

$$\begin{aligned}(H_1{}^D + H_0)/H_0 &= \sum_{C\in\mathcal{S}} \sum_{\mathcal{X}\in {}^C\mathcal{P}^{\mathcal{D}}} \operatorname{span}(\overline{\mathcal{X}}) \\ &= \sum_{\mathcal{X}\in\mathcal{P}^{\mathcal{D}}} \operatorname{span}(\overline{\mathcal{X}}) \\ &= \bigoplus_{\mathcal{X}\in\mathcal{P}^{\mathcal{D}}} \operatorname{span}(\overline{\mathcal{X}}),\end{aligned}$$

the last equation is due to Theorem 3.2.6. The proof of

$$({}^C H_1 + H_0)/H_0 = \bigoplus_{\mathcal{X}\in {}^C\mathcal{P}} \operatorname{span}(\overline{\mathcal{X}})$$

is similar. □

§3.2.2 The second construction

In this subsection, for any $C, D \in \mathcal{S}$, we give another construction of a complete family of non-trivial $(\mathcal{C},\mathcal{D})$-primitive matrices over H. We construct a set $\mathcal{P}'$ in the following way. Denote $\mathcal{S}^1 = \{C \in \mathcal{S} \mid C + \Bbbk 1 \neq C \wedge \Bbbk 1\}$. For any $C \in \mathcal{S}^1$ with basic multiplicative matrix $\mathcal{C} \in \mathcal{M}$, according to Corollary 3.1.9, we can fix a complete family $\{\mathcal{X}_C'^{(\gamma_C')}\}_{\gamma_C'\in\Gamma_C'}$ of non-trivial $(\mathcal{C},1)$-primitive matrices.

Denote

$$\mathcal{P}'^1 = \bigcup_{C\in\mathcal{S}_1} \{\mathcal{X}_C'^{(\gamma_C')} \mid \gamma_C' \in \Gamma_C'\}. \tag{3.13}$$

Then for any non-trivial $(\mathcal{C},1)$-primitive matrix $\mathcal{Y}' \in \mathcal{P}'^1$, we have

$$\begin{pmatrix} L_{\mathcal{B},\mathcal{C}} & 0 \\ 0 & I \end{pmatrix} \left(\mathcal{B} \odot' \begin{pmatrix} \mathcal{C} & \mathcal{Y}' \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} L_{\mathcal{B},\mathcal{C}}^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \mathcal{E}_1 & & & \mathcal{Y}_1' \\ & \ddots & & \vdots \\ & & \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}} & \mathcal{Y}'_{u_{(\mathcal{B},\mathcal{C})}} \\ & 0 & & \mathcal{B} \end{pmatrix}, \tag{3.14}$$

where I is the identity matrix over $\Bbbk$ and $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}} \in \mathcal{M}$.

Denote

$$\mathcal{P}'^{\mathcal{B}}_{\mathcal{Y}'} = \{\mathcal{Y}'_i \mid 1 \leq i \leq u_{(\mathcal{B},\mathcal{C})}\}, \tag{3.15}$$

$$\mathcal{P}'^{\mathcal{B}} = \bigcup_{\mathcal{Y}' \in \mathcal{P}'^1} \mathcal{P}'^{\mathcal{B}}_{\mathcal{Y}'}, \quad \mathcal{P}'_{\mathcal{Y}'} = \bigcup_{\mathcal{B} \in \mathcal{M}} \mathcal{P}'^{\mathcal{B}}_{\mathcal{Y}'}, \tag{3.16}$$

and

$$\mathcal{P}' = \bigcup_{\mathcal{B} \in \mathcal{M}} \mathcal{P}'^{\mathcal{B}} = \bigcup_{\mathcal{Y}' \in \mathcal{P}^1} \mathcal{P}'_{\mathcal{Y}'}. \tag{3.17}$$

The same proof with Remark 3.2.2 can be applied to $\mid \mathcal{P}'^{\mathcal{B}}_{\mathcal{Y}'} \mid$.

Remark 3.2.8 *The cardinal number* $\mid \mathcal{P}'^{\mathcal{B}}_{\mathcal{Y}'} \mid = u_{B,C}$, *where* $\mathcal{P}'^{\mathcal{B}}_{\mathcal{Y}'}$ *appears in (3.15).*

According to [59, Corollary 3.6], since H has the dual Chevalley property, the antipode S of H is bijective. Then for the mixed Hopf module H_1/H_0 in ${}_H\mathcal{M}^H$, we have

$$H_0 \otimes (H_1/H_0)^{coH_0} \cong H_1/H_0,$$

where $(H_1/H_0)^{coH_0}$ is the right coinvariants of H_0 in H_1/H_0. And the isomorphism maps $h \otimes \overline{x}$ to $h \cdot \overline{x}$, where $h \in H_0, \overline{x} \in (H_1/H_0)^{coH_0}$.

The proofs of the following theorem and corollary can be completed by the method analogous to that used in the proofs of Theorem 3.2.6 and Corollary 3.2.7.

Theorem 3.2.9 *Let* $C, D \in \mathcal{S}$ *with basic multiplicative matrices* $\mathcal{C}, \mathcal{D} \in \mathcal{M}$ *respectively. Denote*

$${}^{\mathcal{C}}\mathcal{P}'^{\mathcal{D}} := \{\mathcal{X}' \in \mathcal{P}' \mid \mathcal{X}' \textit{ is a non-trivial } (\mathcal{C}, \mathcal{D})\textit{-primitive matrix}\}.$$

Then it is a complete family of non-trivial $(\mathcal{C}, \mathcal{D})$*-primitive matrices. Moreover, we have* $H_1/H_0 = \bigoplus_{\mathcal{X}' \in \mathcal{P}'} \operatorname{span}(\overline{\mathcal{X}'})$.

Corollary 3.2.10 *Let* $C, D \in \mathcal{S}$ *with basic multiplicative matrices* $\mathcal{C}, \mathcal{D} \in \mathcal{M}$ *respectively. Denote*

$${}^{\mathcal{C}}\mathcal{P}' := \bigcup_{\mathcal{D} \in \mathcal{M}} {}^{\mathcal{C}}\mathcal{P}'^{\mathcal{D}},$$

which is a disjoint union of ${}^{\mathcal{C}}\mathcal{P}'^{\mathcal{D}}$ $(\mathcal{D} \in \mathcal{M})$ *defined in Theorem 3.2.6. Then*

$$({}^{C}H_1 + H_0)/H_0 = \bigoplus_{\mathcal{X} \in {}^{\mathcal{C}}\mathcal{P}'} \operatorname{span}(\overline{\mathcal{X}}).$$

Moreover, we have

$$\mathcal{P}'^{\mathcal{D}} = \bigcup_{\mathcal{C}\in\mathcal{M}} {}^{\mathcal{C}}\mathcal{P}'^{\mathcal{D}},$$

which is also a disjoint union, and that

$$({H_1}^D + H_0)/H_0 = \bigoplus_{\mathcal{X}\in\mathcal{P}'^{\mathcal{D}}} \operatorname{span}(\overline{\mathcal{X}}).$$

So far, for any $C, D \in \mathcal{S}$ with $\dim_{\Bbbk}(C) = r^2, \dim(D)_{\Bbbk} = s^2$, we have already constructed two complete families of non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrices over H. According to Corollary 3.1.16, the cardinal number $\mid {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}} \mid=\mid {}^{\mathcal{C}}\mathcal{P}'^{\mathcal{D}} \mid= \frac{1}{rs}\dim_{\Bbbk}((C\wedge D)/(C+D))$. Thus we can determine the number $\frac{1}{rs}\dim_{\Bbbk}((C\wedge D)/(C+D))$ by studying ${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$ and ${}^{\mathcal{C}}\mathcal{P}'^{\mathcal{D}}$.

§3.3 Link quiver

Let H be a Hopf algebra over $\Bbbk$ with the dual Chevalley property. For convenience, we still use the notations in Section §3.2.

Let $\mathbb{Z}\mathcal{S}$ be the free additive abelian group generated by the elements of $\mathcal{S}$. For our purpose, let us start by giving a unital based $\mathbb{Z}_+$-ring structure on $\mathbb{Z}\mathcal{S}$. The related definitions and properties of $\mathbb{Z}_+$rings can be found in [56, Section 2] and [25, Chapter 3].

For any $B, C \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{B}, \mathcal{C} \in \mathcal{M}$ respectively. Since H has the dual Chevalley property, it follows from [40, Proposition 2.6(2)] that there exists an invertible matrix L over $\Bbbk$ such that

$$L(\mathcal{B}\odot'\mathcal{C})L^{-1} = \begin{pmatrix} \mathcal{E}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{E}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{E}_t \end{pmatrix}, \tag{3.18}$$

where $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_t$ are basic multiplicative matrices over H.

Define a multiplication on $\mathbb{Z}\mathcal{S}$ as follow: for $B, C \in \mathcal{S}$,

$$B\cdot C = \sum_{i=1}^{t} E_i,$$

where $E_1, \cdots, E_t \in \mathcal{S}$ are well-defined with basic multiplicative matrices $\mathcal{E}_i \in \mathcal{M}$ as in (3.18).

With the multiplication defined above, now we can prove the following proposition by using Lemma 2.1.3.

Proposition 3.3.1 *Let H be a Hopf algebra over $\Bbbk$ with the dual Chevalley property and $\mathcal{S}$ be the set of all the simple subcoalgebras of H. Then $\mathbb{Z}\mathcal{S}$ is a unital based ring with $\mathbb{Z}_+$-basis $\mathcal{S}$.*

Proof: For any $B, C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{B}, \mathcal{C}, \mathcal{D}$ respectively. With the notations in (3.18), we know that

$$(B \cdot C) \cdot D = \sum_{i=1}^{t} E_i \cdot D.$$

By Lemma 2.1.3, we have

$$\begin{aligned}
\begin{pmatrix} \mathcal{E}_1 \odot' \mathcal{D} & 0 & \cdots & 0 \\ 0 & \mathcal{E}_2 \odot' \mathcal{D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{E}_t \odot' \mathcal{D} \end{pmatrix} &\sim \begin{pmatrix} \mathcal{E}_1 \odot \mathcal{D} & 0 & \cdots & 0 \\ 0 & \mathcal{E}_2 \odot \mathcal{D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{E}_t \odot \mathcal{D} \end{pmatrix} \\
&= (L(\mathcal{B} \odot' \mathcal{C})L^{-1}) \odot \mathcal{D} \\
&\sim (L(\mathcal{B} \odot' \mathcal{C})L^{-1}) \odot' \mathcal{D} \\
&\sim (\mathcal{B} \odot' \mathcal{C}) \odot' \mathcal{D}.
\end{aligned}$$

Suppose that

$$C \cdot D = \sum_{i=1}^{s} F_i,$$

where $F_i \in \mathcal{S}$ for any $1 \leq i \leq s$, which means that

$$L'(\mathcal{C} \odot' \mathcal{D})L'^{-1} = \begin{pmatrix} \mathcal{F}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{F}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{F}_s \end{pmatrix}$$

for some invertible matrix L' over $\Bbbk$. Then

$$B \cdot (C \cdot D) = \sum_{i=1}^{s} B \cdot F_i.$$

By Lemma 2.1.3, we have

$$\begin{aligned}
\begin{pmatrix} \mathcal{B} \odot' \mathcal{F}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{B} \odot' \mathcal{F}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{B} \odot' \mathcal{F}_s \end{pmatrix} &= \mathcal{B} \odot' \begin{pmatrix} \mathcal{F}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{F}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{F}_s \end{pmatrix} \\
&= \mathcal{B} \odot' \left(L'(\mathcal{C} \odot' \mathcal{D})L'^{-1}\right) \\
&\sim \mathcal{B} \odot \left(L'(\mathcal{C} \odot' \mathcal{D})L'^{-1}\right) \\
&\sim \mathcal{B} \odot (\mathcal{C} \odot' \mathcal{D}) \\
&\sim \mathcal{B} \odot' (\mathcal{C} \odot' \mathcal{D}).
\end{aligned}$$

As a conclusion, we have

$$\begin{pmatrix} \mathcal{E}_1 \odot' \mathcal{D} & 0 & \cdots & 0 \\ 0 & \mathcal{E}_2 \odot' \mathcal{D} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{E}_t \odot' \mathcal{D} \end{pmatrix} \sim \begin{pmatrix} \mathcal{B} \odot' \mathcal{F}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{B} \odot' \mathcal{F}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{B} \odot' \mathcal{F}_s \end{pmatrix}.$$

It follows that the traces of these two matrices are equal. Thus a direct verification gives rise to the fact that $\mathbb{Z}\mathcal{S}$ is a unital $\mathbb{Z}_+$-ring. Let S be the antipode of H, then according to [39, Theorem 3.3], we get an anti-involution $C \mapsto S(C)$ of $\mathcal{S}$. It follows from [39, Theorem 2.7] that there is only one 1 in the summand of $C \cdot S(C)$, which means that $\mathbb{Z}\mathcal{S}$ is a based ring. □

Given a finite-dimensional right comodule M over a coalgebra H', with comultiplication map $\rho : M \to M \otimes H'$, let $\mathrm{cf}(M)$ be the coefficient coalgebra of M, which is the smallest subcoalgebra of H' such that $\rho(M) \subseteq M \otimes \mathrm{cf}(M)$. One can show that:

Lemma 3.3.2 *Let H be a Hopf algebra over $\Bbbk$ with the dual Chevalley property and $\mathcal{S}$ be the set of all the simple subcoalgebras of H. Then* $\mathrm{Gr}(H_0$*-comod) is isomorphic to $\mathbb{Z}\mathcal{S}$ as unital based rings.*

Proof: Define

$$\begin{aligned} F : \mathrm{Gr}(H_0\text{-comod}) &\to \mathbb{Z}\mathcal{S}, \\ M &\mapsto \mathrm{cf}(M). \end{aligned}$$

Next we show that F is a ring isomorphism. In fact, since H_0 is cosemisimple, it follows that M is a completely irreducible right H_0-comodule. In other words, there are simple right H_0-comodules $V_1, V_2, \cdots, V_t$ such that $M = \bigoplus_{1\leq i\leq t} V_i$. Note that for any simple right H_0-comodule V_i, its coefficient coalgebra $\mathrm{cf}(V_i)$ is a simple subcoalgebra of H. If V_i and V_j are non-isomorphic as right H_0-comodules, it is apparent that $\mathrm{cf}(V_i)$ and $\mathrm{cf}(V_j)$ are non-isomorphic as subcoalgebras. This means that F is injective. Furthermore, for any $C \in \mathcal{S}$, any simple right C-comodule X is a simple H_0-comodule and the coefficient coalgebra of X is C. One can show that F is surjective. Using the fact that the coefficient coalgebra $\mathrm{cf}(V_i \otimes V_j)$ of $V_i \otimes V_j$ is $\mathrm{cf}(V_i)\,\mathrm{cf}(V_j)$, we get that F is a ring isomorphism. □

Note that if in addition H_0 is finite-dimensional, it is clear that $\mathbb{Z}\mathcal{S}$ is a fusion ring. In this situation, we can study *Frobenius-Perron dimensions* in $\mathbb{Z}\mathcal{S}$. The reader is referred to [25, Chapter 3] and [28, Section 3] for details.

For any $C \in \mathcal{S}$, let $\mathrm{FPdim}(C)$ be the maximal non-negative eigenvalue of the matrix of left multiplication by C. Since this matrix has non-negative entries, it follows from the Frobenius-Perron theorem that $\mathrm{FPdim}(C)$ exists. Furthermore, FPdim is the unique character of $\mathbb{Z}\mathcal{S}$ which takes non-negative values on $\mathcal{S}$.

Lemma 3.3.3 *If H_0 is finite-dimensional, then for any $C \in \mathcal{S}$, we have* $\mathrm{FPdim}(C) = \sqrt{\dim_{\Bbbk}(C)}$.

Proof: This is because

$$C \mapsto \sqrt{\dim_{\Bbbk}(C)}$$

is exactly the unique character of $\mathbb{Z}\mathcal{S}$ which take non-negative values. □

In this section, let H be a non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property and H_1/H_0 is finite-dimensional. Note that according to Lemma 3.2.3, we know that $\mathcal{P}$ is a finite set in this situation. Besides, for any matrix $\mathcal{A} = (a_{ij})_{m\times n}$ over H, denote the matrix $\mathcal{A}^T := (a_{ji})_{n\times m}$ and $S(\mathcal{A}) := (S(a_{ij}))_{m\times n}$, where S is the antipode of H.

Now let us recall the concept of link quiver.

Definition 3.3.4 (*[17, Definition 4.1]*) *Let H be a coalgebra over $\Bbbk$. The link quiver $\mathrm{Q}(H)$ of H is defined as follows: the vertices of $\mathrm{Q}(H)$ are the elements of $\mathcal{S}$; for any simple subcoalgebra $C, D \in \mathcal{S}$ with $\dim_{\Bbbk}(C) = r^2, \dim_{\Bbbk}(D) = s^2$, there are exactly $\frac{1}{rs}\dim_{\Bbbk}((C \wedge D)/(C + D))$ arrows from D to C.*

With the notations in Section §3.2, we can view ${}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}}$ as the set of arrows from vertex D to vertex C, view $\mathcal{P}^{\mathcal{D}}$ as the set of arrows with start vertex D and view ${}^{\mathcal{C}}\mathcal{P}$ as the set of arrows with end vertex C. Similar statements can also be applied to $\mathcal{P}'$.

Now we start to study the properties for the link quiver of a Hopf algebra with the dual Chevalley property.

Lemma 3.3.5 *Let H be a non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property. Denote ${}^1\mathcal{S} = \{C \in \mathcal{S} \mid \Bbbk 1 + C \neq \Bbbk 1 \wedge C\}$, $\mathcal{S}^1 = \{C \in \mathcal{S} \mid C + \Bbbk 1 \neq C \wedge \Bbbk 1\}$. Then*

(1) $\mid {}^1\mathcal{P} \mid \geq 1$;

(2) $\mid {}^1\mathcal{P} \mid = \mid \mathcal{P}^1 \mid$;

(3) $C \in {}^1\mathcal{S}$ *if and only if* $S(C) \in \mathcal{S}^1$.

Proof:

(1) At first, we try to find a non-trivial $(1, \mathcal{F})$-primitive matrix for some $\mathcal{F} \in \mathcal{M}$. This can be obtained by the same reason in the proof of [45, Lemma 4.7(1)], but here we prove it in another way. When $H \neq H_0$, it follows from Lemma 3.2.3 that $\mathcal{P} \neq 0$. So there exists some non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix $\mathcal{X} \in \mathcal{P}$. Let $KS(\mathcal{C})^T K^{-1} \in \mathcal{M}$ be the basic multiplicative matrix of $S(C)$, where K is some invertible matrix over $\Bbbk$. Since $S(C) \cdot C$ contains $\Bbbk 1$ with a nonzero

coefficient, by Lemma 2.1.3, we have

$$\begin{aligned}
& (KS(\mathcal{C})^T K^{-1}) \odot' \begin{pmatrix} \mathcal{C} & \mathcal{X} \\ 0 & \mathcal{D} \end{pmatrix} \\
= & (K \odot' I) \left(S(\mathcal{C})^T \odot' \begin{pmatrix} \mathcal{C} & \mathcal{X} \\ 0 & \mathcal{D} \end{pmatrix} \right) (K^{-1} \odot' I) \\
\sim & \begin{pmatrix}
\mathcal{E}_1 & & & & \mathcal{X}_{1,1} & \cdots & \mathcal{X}_{1,u} \\
& \ddots & & & \vdots & & \vdots \\
& & \mathcal{E}_t & & \mathcal{X}_{t,1} & \cdots & \mathcal{X}_{t,u} \\
& & & 1 & \mathcal{X}_{t+1,1} & \cdots & \mathcal{X}_{t+1,u} \\
& & & & F_1 & & \\
& 0 & & & & \ddots & \\
& & & & & & \mathcal{F}_u
\end{pmatrix},
\end{aligned}$$

where I is the identity matrix over $\Bbbk$, $\mathcal{E}_i, \mathcal{F}_j \in \mathcal{M}$ for any $1 \leq i \leq t, 1 \leq j \leq u$. By Lemma 2.1.4, there exists some k such that $\mathcal{X}_{t+1,k}$ is non-trivial, where $1 \leq k \leq u$. Thus we get a non-trivial $(1, \mathcal{F}_k)$-primitive matrix $\mathcal{X}_{t+1,k}$.
According to Lemma 3.1.7, we have

$$0 \neq \operatorname{span}(\overline{\mathcal{X}_{t+1,k}}) \subseteq ({}^1H_1{}^{\mathcal{F}_k} + H_0)/H_0 \subseteq ({}^1H_1 + H_0)/H_0.$$

Consequently, it follows from Corollary 3.2.7 that

$$| \, {}^1\mathcal{P} \, | \geq 1.$$

(2) For any $\mathcal{C} \in \mathcal{M}$, it is not difficult to verify that $S(\mathcal{Y})^T$ is a non-trivial $(S(\mathcal{C})^T, 1)$-primitive matrix, where $\mathcal{Y} \in {}^1\mathcal{P}$ is a non-trivial $(1, \mathcal{C})$-primitive matrix. According to (3.12) in the proof of Theorem 3.2.6, we have

$$\begin{aligned}
S\big(({}^1H_1{}^C + H_0)/H_0\big) &= S\Big(\bigoplus_{\mathcal{Y} \in {}^1\mathcal{P}^\mathcal{C}} \operatorname{span}(\overline{\mathcal{Y}})\Big) \\
&\subseteq \sum_{\mathcal{Y} \in {}^1\mathcal{P}^\mathcal{C}} \operatorname{span}(\overline{S(\mathcal{Y})^T}) \\
&\subseteq ({}^{S(C)}H_1{}^1 + H_0)/H_0,
\end{aligned}$$

the last inclusion is due to Lemma 3.1.7. It follows from [59, Corollay 3.6] that S is bijective. Hence we have

$$\dim_\Bbbk(({}^1H_1{}^C + H_0)/H_0)) \leq \dim_\Bbbk(({}^{S(C)}H_1{}^1 + H_0)/H_0).$$

This implies that

$$\begin{aligned} |\, {}^1\mathcal{P}^{\mathcal{C}} \,| &= \frac{1}{\sqrt{\dim_{\Bbbk}(C)}} \dim_{\Bbbk}(({}^1{H_1}^C + H_0)/H_0)) \\ &\leq \frac{1}{\sqrt{\dim_{\Bbbk}(S(C))}} \dim_{\Bbbk}(({}^{S(C)}{H_1}^1 + H_0)/H_0)) \\ &= |\, {}^{KS(\mathcal{C})^T K^{-1}}\mathcal{P}^1 \,|, \end{aligned}$$

where K is some invertible matrix over $\Bbbk$ such that $KS(\mathcal{C})^T K^{-1} \in \mathcal{M}$ is a basic multiplicative matrix of $S(C)$. By Corollary 3.2.7 and the fact that S is a permutation on $\mathcal{S}$, we have

$$\begin{aligned} |\, {}^1\mathcal{P} \,| &= \sum_{\mathcal{C}\in\mathcal{M}} |\, {}^1\mathcal{P}^{\mathcal{C}} \,| \\ &\leq \sum_{\mathcal{C}\in\mathcal{M}} |\, {}^{KS(\mathcal{C})^T K^{-1}}\mathcal{P}^1 \,| \\ &= \sum_{\mathcal{C}\in\mathcal{M}} |\, {}^{\mathcal{C}}\mathcal{P}^1 \,| \\ &= |\, \mathcal{P}^1 \,| . \end{aligned}$$

Next we adopt the same procedure to deal with $\mathcal{P}'^1$, we get

$$|\, \mathcal{P}'^1 \,| \leq |\, {}^1\mathcal{P}' \,| .$$

According to Corollaries 3.2.7 and 3.2.10,

$$|\, {}^1\mathcal{P} \,| = \sum_{\mathcal{C}\in\mathcal{M}} |\, {}^1\mathcal{P}^{\mathcal{C}} \,| = \sum_{\mathcal{C}\in\mathcal{M}} |\, {}^1\mathcal{P}'^{\mathcal{C}} \,| = |\, {}^1\mathcal{P}' \,| .$$

A similar arguments shows that

$$|\, \mathcal{P}^1 \,| = |\, \mathcal{P}'^1 \,| . \tag{3.19}$$

Thus the proof is completed.

(3) It is straightforward to know that

$$\begin{aligned} C \in {}^1\mathcal{S} &\iff \Bbbk 1 + C \neq \Bbbk 1 \wedge C \\ &\iff S(C) + \Bbbk 1 \neq S(C) \wedge \Bbbk 1 \\ &\iff S(C) \in \mathcal{S}^1. \end{aligned}$$

□

Lemma 3.3.6 *Suppose that $\mathcal{X}$ is a non-trivial $(\mathcal{C}, \mathcal{D})$-primitive matrix. For any $B \in \mathcal{S}$,*

(1) if $B \cdot C$ contains E with a nonzero coefficient, then there exists some arrow in $\mathrm{Q}(H)$ *with end*

vertex E;*

(2) if $B \cdot D$ contains F with a nonzero coefficient, then there exists some arrow in $\mathrm{Q}(H)$ with start vertex F.

Proof: We only prove (1); the proof of (2) is similar. By [40, Proposition 2.6], there exist invertible matrices $L_{\mathcal{B},\mathcal{C}}, L_{\mathcal{B},\mathcal{D}}$ over $\Bbbk$ such that

$$\begin{aligned}
&\begin{pmatrix} L_{\mathcal{B},\mathcal{C}} & \\ & L_{\mathcal{B},\mathcal{D}} \end{pmatrix}\left(\mathcal{B}\odot'\begin{pmatrix} \mathcal{C} & \mathcal{X} \\ 0 & \mathcal{D}\end{pmatrix}\right)\begin{pmatrix} L_{\mathcal{B},\mathcal{C}}^{-1} & \\ & L_{\mathcal{B},\mathcal{D}}^{-1} \end{pmatrix} \\
=&\begin{pmatrix} L_{\mathcal{B},\mathcal{C}} & \\ & L_{\mathcal{B},\mathcal{D}} \end{pmatrix}\begin{pmatrix} \mathcal{B}\odot'\mathcal{C} & \mathcal{B}\odot'\mathcal{X} \\ 0 & \mathcal{B}\odot'\mathcal{D}\end{pmatrix}\begin{pmatrix} L_{\mathcal{B},\mathcal{C}}^{-1} & \\ & L_{\mathcal{B},\mathcal{D}}^{-1} \end{pmatrix} \\
=&\begin{pmatrix} \mathcal{E}_1 & & & \mathcal{X}_{11} & \cdots & \mathcal{X}_{1u_{(\mathcal{B},\mathcal{D})}} \\ & \ddots & & \vdots & & \vdots \\ & & \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}} & \mathcal{X}_{u_{(\mathcal{B},\mathcal{C})}1} & \cdots & \mathcal{X}_{u_{(\mathcal{B},\mathcal{C})}u_{(\mathcal{B},\mathcal{D})}} \\ & & & \mathcal{F}_1 & & \\ & 0 & & & \ddots & \\ & & & & & \mathcal{F}_{u_{(\mathcal{B},\mathcal{D})}} \end{pmatrix},
\end{aligned} \tag{3.20}$$

where $\mathcal{E}_1, \cdots, \mathcal{E}_{u_{(\mathcal{B},\mathcal{C})}}, \mathcal{F}_1, \cdots, \mathcal{F}_{u_{(\mathcal{B},\mathcal{D})}}$ are the given basic multiplicative matrices. According to Lemma 2.1.4, we know that for each $1 \leq i \leq u_{(\mathcal{B},\mathcal{C})}$, there is some $1 \leq j \leq u_{(\mathcal{B},\mathcal{D})}$ such that $\mathcal{X}_{ij}$ is non-trivial; and for each $1 \leq j \leq u_{(\mathcal{B},\mathcal{D})}$, there is some $1 \leq i \leq u_{(\mathcal{B},\mathcal{C})}$ such that $\mathcal{X}_{ij}$ is non-trivial. Without loss of generality, for any E_i contained in $B \cdot C$, suppose that $\mathcal{X}_{i1}$ is non-trivial. Note that

$$\operatorname{span}(\overline{\mathcal{X}_{i1}}) \subseteq ({}^{E_i}H_1{}^{F_1} + H_0)/H_0).$$

It follows from Lemma 3.1.16 that

$$\dim_{\Bbbk}((E_i \wedge F_1)/(E_i + F_1)) > 0.$$

This means that there exists some arrow from F_1 to E_i, the proof of (1) is complete. □

For convenience, denote $\mathcal{S} = \{C_i \mid i \in I\}$ be the set of all the simple subcoalgebras of H. For any $C_i, C_j \in \mathcal{S}$, let $C_i \cdot C_j = \sum\limits_{t\in I} \alpha_{i,j}^t C_t$, where $\alpha_{i,j}^t \in \mathbb{Z}_+$. Moreover, we denote $\mathcal{M} = \{\mathcal{C}_j \mid i \in I\}$, such that each $\mathcal{C}_j \in \mathcal{M}$ is the basic multiplicative matrix of $C_j \in \mathcal{S}$. By Remarks 3.2.2 and 3.2.8, we can show the following results.

Lemma 3.3.7 *Let H be a non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property.*

(1) For any $\mathcal{Y} \in {}^1\mathcal{P}$, where $\mathcal{Y}$ is a non-trivial $(1, \mathcal{C}_j)$-primitive matrix and $\mathcal{C}_j \in \mathcal{M}$, let β_{ij} be the cardinal number of ${}^{\mathcal{C}_i}\mathcal{P}_{\mathcal{Y}}$. Then $\beta_{ij} = \sum\limits_{t\in I} \alpha_{i,j}^t \geq 1$;

(2) *For any* $\mathcal{Y}' \in \mathcal{P}'^1$, *where* $\mathcal{Y}'$ *is a non-trivial* $(\mathcal{C}_j, 1)$*-primitive matrix and* $\mathcal{C}_j \in \mathcal{M}$, *let* β'_{ij} *be the cardinal number of* $\mathcal{P}'^{\mathcal{C}_i}_{\mathcal{Y}'}$. *Then* $\beta'_{ij} = \sum\limits_{t \in I} \alpha^t_{i,j} \geq 1$.

For any $\mathcal{Y} \in {}^1\mathcal{P}$ and $\mathcal{C}_i \in \mathcal{M}$, denote

$$\mathcal{P}^{\mathcal{C}_i}_{\mathcal{Y}} := \mathcal{P}^{\mathcal{C}_i} \cap \mathcal{P}_{\mathcal{Y}}.$$

Using the lemma above, we can acquire further properties.

Corollary 3.3.8 *Let* H *be a non-cosemisimple Hopf algebra over* $\Bbbk$ *with the dual Chevalley property. Then for any non-trivial* $(1, \mathcal{C}_j)$*-primitive matrix* $\mathcal{Y} \in {}^1\mathcal{P}$, *where* $\mathcal{C}_j \in \mathcal{M}$, *we have*

(1) $\mid {}^{\mathcal{C}_i}\mathcal{P}_{\mathcal{Y}} \mid \geq 1$, $\mid \mathcal{P}^{\mathcal{C}_i}_{\mathcal{Y}} \mid \geq 1$ *hold for all* $\mathcal{C}_i \in \mathcal{M}$;

(2) $\mid \mathcal{P}^1_{\mathcal{Y}} \mid = 1$.

Proof:

(1) For any $\mathcal{C}_i \in \mathcal{M}$, it is apparent from Lemma 3.3.7 that

$$\beta_{ij} = \mid {}^{C_i}\mathcal{P}_{\mathcal{Y}} \mid \geq 1.$$

Since $\mathbb{Z}\mathcal{S}$ is a unital based ring, according to Lemma 2.2.2, there exists some simple subcolagebra $C_t \in \mathcal{S}$ such that $C_t \cdot C_j$ contains C_i with a nonzero coefficient. Now we consider

$$\begin{pmatrix} I & 0 \\ 0 & L_{\mathcal{C}_t, \mathcal{C}_j} \end{pmatrix} \left(\mathcal{C}_t \odot' \begin{pmatrix} 1 & \mathcal{Y} \\ 0 & \mathcal{C}_j \end{pmatrix} \right) \begin{pmatrix} I & 0 \\ 0 & L^{-1}_{\mathcal{C}_t, \mathcal{C}_j} \end{pmatrix},$$

where I is the identity matrix over $\Bbbk$ and $L_{\mathcal{C}_t, \mathcal{C}_j}$ is an invertible matrix over $\Bbbk$ which is defined in Section §2.1. It follows from Lemma 2.1.4 that there exists some non-trivial $(\mathcal{C}_t, \mathcal{C}_i)$-primitive matrix $\mathcal{Z} \in {}^{C_t}\mathcal{P}_{\mathcal{Y}} \subseteq \mathcal{P}_{\mathcal{Y}}$, where $\mathcal{C}_t \in \mathcal{M}$. On the other hand, we know that $\mathcal{Z} \in {}^{C_t}\mathcal{P}^{\mathcal{C}_i} \subseteq \mathcal{P}^{\mathcal{C}_i}$, where the last inclusion is due to Corollary 3.2.7. It follows that

$$\mathcal{Z} \in \mathcal{P}^{\mathcal{C}_i} \cap \mathcal{P}_{\mathcal{Y}}.$$

Thus

$$\mid \mathcal{P}^{\mathcal{C}_i}_{\mathcal{Y}} \mid \geq 1.$$

(2) Choosing $C_i = \Bbbk 1$ in (1), we know that

$$\mid \mathcal{P}^1_{\mathcal{Y}} \mid \quad \geq \quad 1 = \mid {}^1\mathcal{P}_{\mathcal{Y}} \mid,$$

where

$${}^1\mathcal{P}_{\mathcal{Y}} = \{\mathcal{Y}\}.$$

Since

$$\mathcal{P}^1 = \mathcal{P}^1 \cap \mathcal{P} = \bigcup_{\mathcal{Z}\in{}^1\mathcal{P}} (\mathcal{P}^1 \cap \mathcal{P}_{\mathcal{Z}}),$$

it follows from Corollary 3.2.5 that

$$\mid \mathcal{P}^1 \mid = \sum_{\mathcal{Z}\in{}^1\mathcal{P}} \mid \mathcal{P}^1_{\mathcal{Z}} \mid \geq \sum_{\mathcal{Z}\in{}^1\mathcal{P}} \mid {}^1\mathcal{P}_{\mathcal{Z}} \mid = \mid {}^1\mathcal{P} \mid .$$

But by Lemma 3.3.5, we have

$$\mid \mathcal{P}^1 \mid = \mid {}^1\mathcal{P} \mid,$$

which follows that the cardinal number $\mid \mathcal{P}^1_{\mathcal{Z}} \mid$ can only equal to 1 for each $\mathcal{Z} \in {}^1\mathcal{P}$. □

For any $C, D \in \mathcal{S}$ with basic multiplicative matrices $\mathcal{C}, \mathcal{D} \in \mathcal{M}$ respectively. Recall that C, D are said to be *directly linked* in H if $C + D$ is a proper subspace of $C \wedge D + D \wedge C$. Note that by [40, Lemma 3.6 (2)] that C, D are directly linked in H if and if only there exists some $(\mathcal{C}, \mathcal{D})$-primitive or $(\mathcal{D}, \mathcal{C})$-primitive matrix, which is non-trivial. We can use this notion to describe ${}^1\mathcal{S}$ and $\mathcal{S}^1$.

The following proposition exactly recovers the definition of Hopf quiver.

Proposition 3.3.9 *Let H be a non-cosemisimple Hopf algebra over $\mathbb{k}$ with the dual Chevalley property. If all the simple subcoalgebras directly linked to $\mathbb{k}1$ are 1-dimensional, then we have $\mid {}^{\mathcal{C}}\mathcal{P} \mid = \mid \mathcal{P}^{\mathcal{C}} \mid = \mid {}^1\mathcal{P} \mid$, for any $\mathcal{C} \in \mathcal{M}$.*

Proof: Suppose that $y \in {}^1\mathcal{P}$ is a non-trivial $(1, g)$-primitive matrix of size 1 for some $g \in G(H)$. For any $C \in \mathcal{S}$, it is straightforward to show that Cg is also a simple subcoalgebra of H. By (3.6), we know that for any $\mathcal{C} \in \mathcal{M}$,

$${}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}} = \{\mathcal{C}y\} \quad \text{and} \quad {}^1\mathcal{P}_{\mathcal{Y}} = \{y\}.$$

This means that

$$\mid {}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}} \mid = 1 = \mid {}^1\mathcal{P}_{\mathcal{Y}} \mid .$$

Therefore, we have

$$\mid {}^{\mathcal{C}}\mathcal{P} \mid = \sum_{\mathcal{Y}\in{}^1\mathcal{P}} \mid {}^{\mathcal{C}}\mathcal{P}_{\mathcal{Y}} \mid = \sum_{\mathcal{Y}\in{}^1\mathcal{P}} \mid {}^1\mathcal{P}_{\mathcal{Y}} \mid = \mid {}^1\mathcal{P} \mid . \tag{3.21}$$

By the same method as employed above, we can show that

$$\mid \mathcal{P}'^{\mathcal{C}} \mid = \mid \mathcal{P}'^1 \mid . \tag{3.22}$$

According to Corollaries 3.2.7 and 3.2.10, for any $\mathcal{D} \in \mathcal{M}$, we have

$$\mid \mathcal{P}^{\mathcal{D}} \mid = \sum_{\mathcal{C}\in\mathcal{M}} \mid {}^{\mathcal{C}}\mathcal{P}^{\mathcal{D}} \mid = \sum_{\mathcal{C}\in\mathcal{M}} \mid {}^{\mathcal{C}}\mathcal{P}'^{\mathcal{D}} \mid = \mid \mathcal{P}'^{\mathcal{D}} \mid . \tag{3.23}$$

It follows from Lemma 3.3.5 that

$$| {}^{\mathcal{C}}\mathcal{P} | \overset{(3.21)}{=} | {}^{1}\mathcal{P} | = | \mathcal{P}^{1} | \overset{(3.23)}{=} | \mathcal{P}'^{1} | \overset{(3.22)}{=} | \mathcal{P}'^{\mathcal{C}} | \overset{(3.23)}{=} | \mathcal{P}^{\mathcal{C}} | .$$

□

Now let us focus on a special situation.

Corollary 3.3.10 *Let H be a non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property. Then the followings are equivalent:*

(1) $| {}^{\mathcal{C}}\mathcal{P} | = | \mathcal{P}^{\mathcal{C}} | = 1$ holds for all $C \in \mathcal{M}$;

(2) $| {}^{1}\mathcal{P} | = 1$ and the unique subcoalgebra $C \in {}^{1}\mathcal{S}$ is 1-dimensional.

Proof: Indeed, it follows from Proposition 3.3.9 that (2) implies (1). Conversely, assume that there exists some simple subcoalgebra $C \in \mathcal{S}_1$ such that

$$\dim_{\Bbbk}(C) > 1.$$

Suppose $KS(\mathcal{C})^T K^{-1} \in \mathcal{M}$ is a basic multiplicative matrix of $S(C)$, where K is some invertible matrix over $\Bbbk$. Since there is only one 1 in the summand of $S(C) \cdot C$ in $\mathbb{Z}\mathcal{S}$, it follows from Lemma 3.3.7 that

$$| {}^{KS(C)K^{-1}}\mathcal{P} | > 1,$$

which is a contradiction to the assumption in (1). □

According to Proposition 3.3.9, we know that if all the simple subcoalgebras directly linked to $\Bbbk 1$ are 1-dimensional, we have

$$| {}^{1}\mathcal{P} | \mid | {}^{\mathcal{C}}\mathcal{P} |$$

and

$$| {}^{\mathcal{C}}\mathcal{P} | = | \mathcal{P}^{\mathcal{C}} |,$$

for any $\mathcal{C} \in \mathcal{M}$.

According to [25, Propositions 3.3.6(2) and 3.3.11], there exists a unique, up to scaling, nonzero element $R \in \mathbb{Z}\mathcal{S} \otimes_{\mathbb{Z}} \mathbb{C}$ such that $X \cdot R = \mathrm{FPdim}(X)R$ for all $X \in \mathbb{Z}\mathcal{S}$, and it satisfies the equality $R \cdot Y = \mathrm{FPdim}(Y)R$ for all $Y \in \mathbb{Z}\mathcal{S}$. Such an element R is called a regular element of $\mathbb{Z}\mathcal{S}$. It is straightforward to show that the element $R = \sum\limits_{Y \in I} \mathrm{FPdim}(Y)Y$ is a regular element. We can obtain a useful equation, which is needed in the next section.

Lemma 3.3.11 *With the notations in Lemma 3.3.7, suppose that H_0 is finite-dimensional. Then we have the following equation*

$$\sqrt{\dim_{\Bbbk}(C_k)} \left(\sum_{i \in I} \sqrt{\dim_{\Bbbk}(C_i)} \right) = \sum_{i \in I} \sqrt{\dim_{\Bbbk}(C_i)} \beta_{ik}.$$

Proof: By Lemma 3.3.3 and Lemma 3.3.7, we know that

$$\beta_{ik} = \sum_{t\in I} \alpha_{i,k}^t$$

and

$$\mathrm{FPdim}(C_i) = \sqrt{\dim_{\Bbbk}(C_i)}$$

hold for any $i \in I$. It follows from [25, Propositions 3.3.6(2) and 3.3.11] that

$$R = \sum_{i\in I} \sqrt{\dim_{\Bbbk}(C_i)} C_i$$

is a regular element and

$$\sqrt{\dim_{\Bbbk}(C_k)} R = R \cdot C_k.$$

This means that

$$\begin{aligned}
\sqrt{\dim_{\Bbbk}(C_k)} \left(\sum_{i\in I} \sqrt{\dim_{\Bbbk}(C_i)} C_i \right) &= \left(\sum_{i\in I} \sqrt{\dim_{\Bbbk}(C_i)} C_i \right) \cdot C_k \\
&= \sum_{i\in I} \sum_{t\in I} \sqrt{\dim_{\Bbbk}(C_i)} \alpha_{i,k}^t C_t,
\end{aligned}$$

which follows that

$$\sqrt{\dim_{\Bbbk}(C_k)} \sqrt{\dim_{\Bbbk}(C_t)} = \sum_{i\in I} \sqrt{\dim_{\Bbbk}(C_i)} \alpha_{i,k}^t.$$

Thus we have

$$\sqrt{\dim_{\Bbbk}(C_k)} \left(\sum_{i\in I} \sqrt{\dim_{\Bbbk}(C_i)} \right) = \sum_{i\in I} \sqrt{\dim_{\Bbbk}(C_i)} \beta_{ik}.$$

□

At the end of this section, we recall the concept of link-indecomposable components of coalgebra H.

Definition 3.3.12 (*[55, Definition 1.1]) A subcoalgebra H' of coalgebra H is called link-indecomposable if the link quiver $\mathcal{Q}(H')$ of H' is connected (as an undirected graph). A link-indecomposable component of H is a maximal link-indecomposable subcoalgebra.*

As a consequence, we obtain the following proposition.

Proposition 3.3.13 *Let H be a non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property. If all the simple subcoalgebras directly linked to $\Bbbk 1$ are 1-dimensional, then the link-indecomposable component $H_{(1)}$ containing $\Bbbk 1$ is a pointed Hopf algebra.*

Proof: Suppose there exists a simple subcoalgebra B with $\dim_{\Bbbk}(B) > 1$ such that some 1-dimensional simple subcoalgebra $\Bbbk g$ contained in $H_{(1)}$ is directly linked to B, where $g \in G(H)$. Without the loss of generality, we can assume that there exists some non-trivial $(g, \mathcal{B})$-primitive matrix for some $\mathcal{B} \in \mathcal{M}$. Now we consider

$$g^{-1} \odot' \begin{pmatrix} g & \mathcal{X} \\ 0 & \mathcal{B} \end{pmatrix}.$$

We can get a $(1, g^{-1}\mathcal{B})$-primitive matrix $g^{-1}\mathcal{X}$, where $g^{-1}\mathcal{B}$ is a multiplicative matrix of $g^{-1}B \in \mathcal{S}$. We know that $\Bbbk 1$ is directly linked to the simple subcoalgebra $g^{-1}B$, which is a contradiction. Therefore, it is directly from [40, Theorem 4.8 (3)] that $H_{(1)}$ is a pointed Hopf algebra. $\square$

Let $\mathrm{Q}(H)$ be the link quiver of H. For each arrow $\mathcal{X} : C \to D$ in $\mathrm{Q}(H)$, let $\mathcal{X}^{-1} : D \to C$ be the formal reverse. Recall that a walk from C to D is a nonempty sequence of arrows $\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_m$ such that there exists a family of $\{\lambda_i\}_{1 \leq i \leq m}$ such that $\mathcal{X}_1^{\lambda_1}\mathcal{X}_2^{\lambda_2} \cdots \mathcal{X}_m^{\lambda_m}$ is a path from C to D, where $\{\lambda_i \mid 1 \leq i \leq m\} \subseteq \{-1, 1\}$.

For each $C_i \in \mathcal{S}$, $\lambda_i \in \{-1, 1\}$, define

$$C_i^{\lambda_i} = \begin{cases} C_i \quad , & \text{if} \quad \lambda_i = 1; \\ S(C_i), & \text{if} \quad \lambda_i = -1. \end{cases}$$

Recall that in [22, section 3], a Hopf quiver $\mathcal{Q}(G, \chi)$ is connected if and only if the union $\cup_{\chi_C \neq 0} \mathcal{C}$ generates G. The following proposition generalizes this result.

Proposition 3.3.14 *Let H be a finite-dimensional non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property. The link quiver $\mathcal{Q}(H)$ of H is connected if and only if for any $D \in \mathcal{S}$, there exist $C_1, \cdots, C_n \in {}^1\mathcal{S}$ such that $C_1^{\lambda_1} \cdot C_2^{\lambda_2} \cdots C_n^{\lambda_n}$ contains D with a nonzero coefficient, where $\{\lambda_i \mid 1 \leq i \leq n\} \subseteq \{-1, 1\}$.*

Proof: For any $D \in \mathcal{S}$, suppose that there exist $C_1, \cdots, C_s \in {}^1\mathcal{S}$ such that $C_1^{\lambda_1} C_2^{\lambda_2} \cdots C_s^{\lambda_s}$ contains D with a nonzero coefficient. We are going to find a walk from $\Bbbk 1$ to D.
When $\lambda_1 = 1$ or $\lambda_1 = -1$, we can find a walk from $\Bbbk 1$ to $C_1^{\lambda_1}$.
When $\lambda_2 = 1$, there exists a non-trivial $(1, \mathcal{C}_2)$-primitive matrix $\mathcal{X}_2$. According to Lemma 3.3.6, we know that for any summand E_2 contained in $C_1^{\lambda_1} \cdot C_2$ with a nonzero coefficient, there exists some arrow from E_2 to $C_1^{\lambda_1}$. When $\lambda_2 = -1$, there exists a non-trivial $(K_1 S(\mathcal{C}_2) K_1^{-1}, 1)$-primitive matrix $\mathcal{Y}_2$, where K_1 is some invertible matrix over $\Bbbk$ such that $K_1 S(\mathcal{C}_2) K_1^{-1} \in \mathcal{M}$. It is a consequence of Lemma 3.3.6 that for any summand E_2 contained in $C_1^{\lambda_1} \cdot S(C_2)$ with a nonzero coefficient, there exists some arrow from $C_1^{\lambda_1}$ to E_2. It turns out that for any summand E_2 contained in $C_1^{\lambda_1} \cdot C_2^{\lambda_2}$ with a nonzero coefficient, we can find a walk from $\Bbbk 1$ to E_2.
Continuing by induction, we can finally find a walk from $\Bbbk 1$ to D. Therefore, $\mathrm{Q}(H)$ is connected.
Next we show the inverse. If $\mathrm{Q}(H)$ is connected, then for any $D \in \mathcal{S}$, we can find a walk from $\Bbbk 1$ to D which goes through vertices $E_0, E_1, \cdots, E_n$, where

$$E_0 = \Bbbk 1, E_n = D.$$

Now we claim that for each E_i, $i \geq 1$, there exists a family of $\{C_j\}_{1\leq j\leq i}$ such that $C_1^{\lambda_1}C_2^{\lambda_2}\cdots C_i^{\lambda_i}$ contains E_i with a nonzero coefficient, where $C_1, \cdots, C_i \in {}^1\mathcal{S}$, $\lambda_1, \cdots, \lambda_i \in \{1, -1\}$. We prove the claim by induction.

When $i = 1$, in the link quiver $\mathrm{Q}(H)$, there exists some arrow from E_1 to $\Bbbk 1$ or from $\Bbbk 1$ to E_1. If there exists some arrow from E_1 to $\Bbbk 1$, the claim is evident. If there exists some arrow from $\Bbbk 1$ to E_1, then by Lemma 3.3.5, we have $S(E_1) \in {}^1\mathcal{S}$. Let

$$C_1 = S(E_1),$$

the claim is proved.

Suppose that the claim holds for E_i, which means that there exists a family of $\{C_j\}_{1\leq j\leq i}$ such that $C_1^{\lambda_1}C_2^{\lambda_2}\cdots C_i^{\lambda_i}$ contains E_i with a nonzero coefficient. Now we consider E_{i+1}. We know that there must be some arrow from E_i to E_{i+1} or from E_{i+1} to E_i. If there exists some arrow from E_{i+1} to E_i, it follows from Lemma 3.2.6 that there exists some non-trivial $(\mathcal{E}_i, \mathcal{E}_{i+1})$-primitive matrix $\mathcal{X}_i \in \mathcal{P}$. By the definition of $\mathcal{P}$, we know that there exists some non-trivial $(1, \mathcal{F})$-primitive matrix $\mathcal{Y} \in {}^1\mathcal{P}$ such that $\mathcal{X}_i \in {}^{\mathcal{E}_i}\mathcal{P}_{\mathcal{Y}}$, where $\mathcal{F} \in \mathcal{M}$. Let

$$C_{i+1} = F,$$

it follows that $E_i \cdot C_{i+1}$ contains E_{i+1} with a nonzero coefficient.

If there exists some arrow from E_i to E_{i+1}, we can find some non-trivial $(\mathcal{E}_i, \mathcal{E}_{i+1})$-primitive matrix $\mathcal{X}_i$. It is straightforward to show that $S(\mathcal{X}_i)$ is a non-trivial $(S(\mathcal{E}_{i+1}), S(\mathcal{E}_i))$-primitive matrix. This means that

$$({}^{S(E_{i+1})}H_1{}^{S(E_i)} + H_0)/H_0 \neq 0.$$

Let $K_1S(\mathcal{E}_i)K_1^{-1}, K_2S(\mathcal{E}_{i+1})K_2^{-1} \in \mathcal{M}$ be the basic multiplicative matrices of $S(E_i), S(E_{i+1})$, respectively, where K_1, K_2 are invertible matrices over $\Bbbk$. From Lemma 3.2.6, there exists some non-trivial $(K_1S(\mathcal{E}_i)K_1^{-1}, K_2S(\mathcal{E}_{i+1})K_2^{-1})$-primitive matrix $\mathcal{X}_i' \in \mathcal{P}$. By the definition of $\mathcal{P}$, we know that there exists some non-trivial $(1, \mathcal{F})$-primitive matrix $\mathcal{Y} \in {}^1\mathcal{P}$ such that $\mathcal{X}_i' \in {}^{K_1S(\mathcal{E}_i)K_1^{-1}}\mathcal{P}_{\mathcal{Y}}$. This means that $S(E_i) \cdot F$ contains $S(E_{i+1})$ with a nonzero coefficient. Let

$$C_0 = S(F),$$

applying Lemma 3.3.1 yields that $C_0^{-1}E_i$ contains E_{i+1} with a nonzero coefficient. The proof is completed. □

Chapter 4 Corepresentation type

§4.1 Finite corepresentation type

One of the most important topics in representation theory is the classification of indecomposable (co)modules over a (co)algebra. The reader is referred to [5] and [11] for general background knowledge of representation theory.

Recall that a finite-dimensional algebra A is said to be of *finite representation type* provided there are finitely many non-isomorphic indecomposable A-modules. We say that A is of *tame representation type* or A is a *tame* algebra if A is not of finite representation type, whereas for any dimension $d > 0$, there are finite number of A-$\Bbbk[T]$-bimodules M_i which are free of finite rank as right $\Bbbk[T]$-modules such that all but finite number of indecomposable A-modules of dimension d are isomorphic to $M_i \otimes_{\Bbbk[T]} \Bbbk[T]/(T-\lambda)$ for $\lambda \in \Bbbk$. A is of *wild representation type* or A is a *wild* algebra if there is a finitely generated A-$\Bbbk[T]$-bimodules B which is free as a right $\Bbbk(X,Y)$-module such that the functor $B \otimes_{\Bbbk(X,Y)} -$ from the category of finitely generated $\Bbbk(X,Y)$-modules to the category of finitely generated A-modules, preserves indecomposability and reflects isomorphisms. A finite-dimensional coalgebra C is said to be of *finite corepresentation type*, if the dual algebra C^* is of finite representation type. C is defined to be of *tame corepresentation type*, if C^* is a tame algebra. We say that C is of *wild corepresentation type*, if the dual algebra C^* is a wild algebra. See [27, 64].

Besides, an algebra A is said to be of *infinite representation type*, if A is not of finite representation type. A finite-dimensional coalgebra C is defined to be of *infinite corepresentation type*, if C^* is of infinite representation type.

Let A (resp. C) be an algebra (resp. coalgebra) over $\Bbbk$ and $\{M_i\}_{i\in I}$ be the complete set of isoclasses of simple left A-modules (resp. right C-comodules). The *Ext quiver* $\Gamma(A)$ (resp. $\Gamma(C)$) of A (resp. C) is an oriented graph with vertices indexed by I, and there are $\dim_\Bbbk \operatorname{Ext}^1(M_i, M_j)$ arrows from i to j for any $i, j \in I$. To avoid confusion, for any Hopf algebra H over $\Bbbk$, we denote the algebra's version of Ext quiver of H by $\Gamma(H)^{\mathrm{a}}$ and denote the coalgebra's version of Ext quiver of H by $\Gamma(H)^{\mathrm{c}}$.

Now let us recall the definition of separated quiver.

Definition 4.1.1 (*cf. [5, §X. 2]*) *Let A be a finite-dimensional algebra over $\Bbbk$ and $\Gamma(A) = (\Gamma_0, \Gamma_1)$ be its Ext quiver, where $\Gamma_0 = \{1, 2, \cdots, n\}$. The separated quiver $\Gamma(A)_s$ of A has $2n$ vertices $\{1, 2, \cdots, n, 1', 2', \cdots, n'\}$ and an arrow $i \to j'$ for every arrow $i \to j$ of $\Gamma(A)$.*

Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property. As mentioned in Section §3.3, for any simple subcoalgebra $C, D \in \mathcal{S}$ with $\dim_\Bbbk(C) = r^2, \dim_\Bbbk(D) = s^2$, there are exactly $\frac{1}{rs}\dim_\Bbbk((C \wedge D)/(C + D))$ arrows from D to C in the link quiver $\mathrm{Q}(H)$ of H. Moreover, we have

$$| {}^{C}\mathcal{P}^{\mathcal{D}} | = | {}^{C}\mathcal{P}'^{D} | = \frac{1}{rs}\dim_\Bbbk((C \wedge D)/(C + D)),$$

and there are exactly $| {}^{C}\mathcal{P} |$ $(= | {}^{C}\mathcal{P}' |)$ arrows with end vertex C and $| \mathcal{P}^{\mathcal{D}} |$ $(= | \mathcal{P}'^{D} |)$ arrows with start vertex D in the link quiver $\mathrm{Q}(H)$ of H.

In order to solve the classification problems, we divide it into several different situations. Let us consider the first case.

Proposition 4.1.2 *Let H be a finite-dimensional non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property. If $|\ {}^1\mathcal{P}\ | \geq 2$, then H is of infinite corepresentative type.*

Proof: We know that the $\Bbbk$-linear abelian category of finite-dimensional comodules over H is isomorphic to the category of finite-dimensional modules over H^*. This means that the coalgebra's version of Ext quiver $\Gamma(H)^{\mathrm{c}}$ of H is the same as the algebra's version of Ext quiver $\Gamma(H^*)^{\mathrm{a}}$ of H^*. According to [17, Theorem 2.1 and Corollary 4.4], the link quiver $\mathrm{Q}(H)$ of H coincides with the algebra's version of Ext quiver $\Gamma(H^*)^{\mathrm{a}}$ of H^*.
Note that H^* is Morita equivalent to a basic algebra $\mathcal{B}(H^*)$. It suffices to prove that the basic algebra $\mathcal{B}(H^*)$ of H^* is of infinite representative type. Let J be the ideal generated by all the arrows in $\mathrm{Q}(H)$. By the Gabriel's theorem, there exists an admissible ideal I such that

$$\Bbbk\mathrm{Q}(H)/I \cong \mathcal{B}(H^*),$$

where $J^t \subseteq I \subseteq J^2$ for some integer $t \geq 2$. Thus there exists an algebra epimorphism

$$f : \mathcal{B}(H^*) \to \Bbbk\mathrm{Q}(H)/J^2.$$

It is enough to show that $\Bbbk\mathcal{Q}(H)/J^2$ is of infinite representative type. Since the Jacobson radical of $\Bbbk\mathcal{Q}(H)/J^2$ is J/J^2, we know that $\Bbbk\mathrm{Q}(H)/J^2$ is an artinian algebra with radical square zero.
Now assume on the countrary that $\Bbbk\mathrm{Q}(H)/J^2$ is of finite representation type. It follows from [5, X.2 Theorem 2.6] that the separated quiver $\mathrm{Q}(H)_s$ of $\Bbbk\mathrm{Q}(H)/J^2$ is a finite disjoint union of Dynkin diagrams.
Since $|\ {}^1\mathcal{P}\ | \geq 2$, it follows from Corollary 3.2.5 that

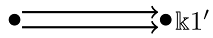
$$|\ {}^C\mathcal{P}\ | = \sum_{\mathcal{Y}\in {}^1\mathcal{P}} |\ {}^C\mathcal{P}_{\mathcal{Y}}\ | \geq |\ {}^1\mathcal{P}\ | \geq 2. \tag{4.1}$$

According to Corollary 3.3.8, we have

$$|\ \mathcal{P}^C\ | = \sum_{\mathcal{Y}\in {}^1\mathcal{P}} |\ \mathcal{P}^C_{\mathcal{Y}}\ | \geq |\ {}^1\mathcal{P}\ | \geq 2 \tag{4.2}$$

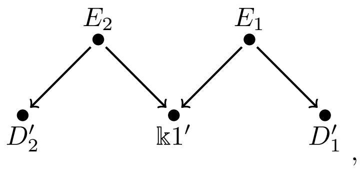
for all $\mathcal{C} \in \mathcal{M}$.
By a discussion on $|\ {}^1\mathcal{S}\ |$, we aim to find a contradiction to $\mathrm{Q}(H)_s$ being a finite disjoint union of Dynkin diagrams.

i) When $|\,{}^1\mathcal{S}\,| = 1$, then the separated quiver $\mathrm{Q}(H)_s$ must contain

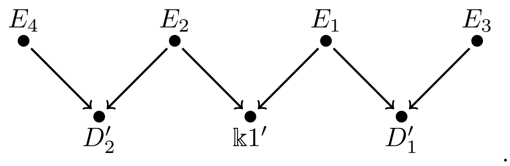


as a sub-quiver. The above quiver is a Kronecker quiver whose underlying graph is not a Dynkin diagram. We know that $\Bbbk\mathrm{Q}(H)/J^2$ is of infinite representation.

ii) When $|\,{}^1\mathcal{S}\,| \geq 2$, by (4.1) and (4.2), the separated quiver $\mathrm{Q}(H)_s$ contains a sub-quiver of the form



where $E_1 \neq E_2$ and there are 3 possible situation.
If $D_1' = \Bbbk 1'$ or $D_2' = \Bbbk 1'$, the separated quiver of $\mathrm{Q}(H)$ contains a Kronecker quiver as a sub-quiver.
If $D_1' = D_2' \neq \Bbbk 1'$, the separated quiver of $\mathrm{Q}(H)$ contains a sub-quiver whose underlying graph is $\tilde{A}_n$ for some $n \geq 3$ and it is a Euclidean diagram. Therefore, $\Bbbk\mathrm{Q}(H)/J^2$ is of infinite representation type.
If $D_1', D_2', \Bbbk 1'$ are distinct from each other, the separated quiver $\mathrm{Q}(H)_s$ contains the following sub-quiver



Then if $E_4 = E_i$ for some $i = 1, 2, 3$, it is evident that $\Bbbk\mathrm{Q}(H)/J^2$ is of infinite representation type. Otherwise, we repeat above process. Since S is a finite set, the separated quiver $\mathrm{Q}(H)_s$ either contains the Kronecker quiver as a sub-quiver or contains a sub-quiver whose underlying graph is $\tilde{A}_n$ for some $n \geq 3$.

As a conclusion, $\Bbbk\mathrm{Q}(H)/J^2$ is of infinite representation type, and this implies that H is of infinite corepresentative type. □

Recall that an algebra is said to be *Nakayama*, if each indecomposable projective left and right module has a unique composition series. It is well-known that a basic algebra A is Nakayama if and only if every vertex of the Ext quiver of A is the start vertex of at most one arrow and the end vertex of at most one arrow (see [11, §V. 2. Theorem 2.6]).

Next we consider the second case.

Proposition 4.1.3 *Let H be a finite-dimensional non-cosemisimple Hopf algebra over $\mathbb{k}$ with the dual Chevalley property. If $|\ {}^1\mathcal{P}\ |= 1$ and the unique subcoalgebra $C \in {}^1\mathcal{S}$ is 1-dimensional, then H is of finite corepresentative type.*

Proof: From the proof of Proposition 4.1.2, we know that the link quiver $\mathrm{Q}(H)$ of H is the same as the Ext quiver $\Gamma(H^*)^{\mathrm{a}}$ of H^*. Using Lemma 3.3.10, we can find

$$|\ {}^{\mathcal{C}}\mathcal{P}\ |=|\ \mathcal{P}^{\mathcal{C}}\ |= 1, \quad \mathcal{C} \in \mathcal{M},$$

which means that the basic algebra $\mathcal{B}(H^*)$ is a Nakayama algebra. It follows from [5, §VI. Theorem 2.1] that the Nakayama algebra $\mathcal{B}(H^*)$ is of finite representation type, which implies that H is of finite corepresentation type. □

Note that since H is finite-dimensional, $\mathbb{Z}\mathcal{S}$ is a fusion ring with $\mathbb{Z}_+$-basis $\mathcal{S} = \{C_i\}_{i\in I}$. Suppose that $C_i \cdot C_j = \sum\limits_{t\in I} \alpha_{ij}^t C_t$, for any $C_i, C_j \in \mathcal{S}$. By the proof of Proposition 3.3.1, the involution of I is decided by S, that is $C_{i^*} = S(C_i)$.

Before proceeding further, let us give the following lemma.

Lemma 4.1.4 *Let H be a finite-dimensional non-cosemisimple Hopf algebra over $\mathbb{k}$ with the dual Chevalley property. Let $|\ {}^1\mathcal{P}\ |= 1$ and C_k be the unique subcoalgebra contained in ${}^1\mathcal{S}$.*

(1) The number of arrows with end vertex C_i in $\mathrm{Q}(H)$ is equal to $\sum\limits_{t\in I} \alpha_{ik}^t$, and the number of arrows with start vertex C_i in $\mathrm{Q}(H)$ is equal to $\sum\limits_{t\in I} \alpha_{ik^*}^t$;

(2) The number of arrows from C_t to C_i in $\mathrm{Q}(H)$ is equal to α_{ik}^t and we have $\alpha_{ik}^t = \alpha_{tk^*}^i$.

Proof:

(1) According to Lemma 3.3.5 (2) and (3.19) in its proof, we know that

$$|\ \mathcal{P}'^1\ |=|\ \mathcal{P}^1\ |=|\ {}^1\mathcal{P}\ |= 1.$$

Suppose

$${}^1\mathcal{P} = \{\mathcal{Y}\}$$

and

$$\mathcal{P}'^1 = \{\mathcal{Y}'\}.$$

Combining (3.7) and Lemma 3.3.7, we have

$$|\ {}^{\mathcal{C}_i}\mathcal{P}\ |=|\ {}^{\mathcal{C}_i}\mathcal{P}_{\mathcal{Y}}\ |= \sum_{t\in I} \alpha_{ik}^t.$$

This means that the number of arrows with end vertex C_i in $\mathrm{Q}(H)$ is equal to $\sum\limits_{t\in I}\alpha_{ik}^t$. A similar argument shows that the number of arrows with start vertex C_i in $\mathrm{Q}(H)$ is equal to $\sum\limits_{t\in I}\alpha_{ik^*}^t$.

(2) In $\mathbb{Z}\mathcal{S}$, we have

$$S(C_k)\cdot S(C_i)=\sum_{t\in I}\alpha_{ik}^t S(C_t)=\sum_{t\in I}\alpha_{k^*i^*}^{t^*}S(C_t).$$

It follows from [25, Proposition 3.1.6] that

$$\alpha_{ik}^t=\alpha_{k^*i^*}^{t^*}=\alpha_{tk^*}^i.$$

Moreover, by (3.7), we can find that

$$^{C_i}\mathcal{P}={}^{C_i}\mathcal{P}_{\mathcal{Y}}.$$

It follows from Theorem 3.2.6 that

$$\mid {}^{C_i}\mathcal{P}^{C_t}\mid=\alpha_{ik}^t.$$

Thus the number of arrows from C_t to C_i in $\mathrm{Q}(H)$ is equal to α_{ik}^t. □

Using Lemma 4.1.4, now we can turn to the last situation.

Proposition 4.1.5 *Let H be a finite-dimensional non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property. Let $\mid {}^1\mathcal{P}\mid=1$ and C_k be the unique subcoalgebra contained in ${}^1\mathcal{S}$. If $\dim_{\Bbbk}(C_k)\geq 4$, then H is of infinite corepresentative type.*

Proof: Proceeding as in the proof of Proposition 4.1.2, we need only to observe the separated quiver of $\Bbbk\mathrm{Q}(H)/J^2$.
If $\dim_{\Bbbk}(C_k)\geq 9$, since $\beta_{1k}=1$, it follows from Lemma 3.3.11 that there exists at least one subcoalgebra C_u such that

$$\beta_{uk}=\sum_{t\in I}\alpha_{uk}^t\geq 4.$$

The separated quiver of $\mathrm{Q}(H)$ contains a vertex which is the end vertex of at least 4 arrows. Evidently, the underlying graph of this separated quiver is not the union of Dynkin diagrams, thus H is of infinite corepresentative type.
If $\dim_{\Bbbk}(C_k)=4$, we deal with this situation through classified discussion. In the following part, for any $n\geq 2$, let $\mathcal{S}(n)$ be the set of all the n^2-dimensional simple subcoalgebras of H and let $G(H)$ be the set of all the group-like elements. According to Lemma 4.1.4 (2), we know that the following two numbers are equal:

- The number of C_t contained in $C_i\cdot C_k$;

- The number of C_i contained in $C_t\cdot S(C_k)$.

Now let us start discussing different situations.

(I) Suppose that

$$S(C_k) \cdot C_k = \Bbbk 1 + \Bbbk g_1 + \Bbbk g_2 + \Bbbk g_3$$

in $\mathbb{Z}\mathcal{S}$, where $g_1, g_2, g_3 \in G(H)$. According to Lemma 4.1.4 (1), the separated quiver $\mathrm{Q}(H)_s$ contains a vertex which is the end vertex of 4 arrows and it can not be a finite disjoint union of Dynkin diagrams. We know that H is of infinite corepresentation type.
Note that if there exists some vertex in $\mathrm{Q}(H)_s$ which is the end vertex or the start vertex of at least 4 arrows, then a similar arguments shows that H is of infinite corepresentation type. For simplicity, in the following proof, we will no longer consider the occurrence of this situation.

(II) Suppose that in $\mathbb{Z}\mathcal{S}$, we have

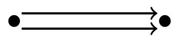
$$S(C_k) \cdot C_k = \Bbbk 1 + \Bbbk g_1 + D_1^{(2)},$$

for some $g_1 \in G(H)$ and $D_1^{(2)} \in \mathcal{S}(2)$.

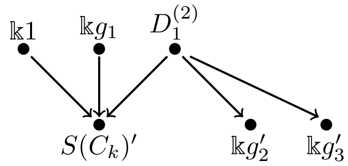
i) If

$$D_1^{(2)} \cdot S(C_k) = S(C_k) + \Bbbk g_2 + \Bbbk g_3,$$

where $g_2, g_3 \in G(H)$. Using Lemma 4.1.4 (2), the separated quiver of $\Bbbk\mathrm{Q}(H)/J^2$ either contains a sub-quiver of the form


,

or contains



as a sub-quiver. The underlying graph of the sub-quiver in the latter case is $\tilde{D_5}$ and it is an Euclidean graph. Since the underlying graph of both of them are not Dynkin diagrams, it follows that H is of infinite corepresentation type.

ii) If

$$D_1^{(2)} \cdot S(C_k) = S(C_k) + D_2^{(2)}$$

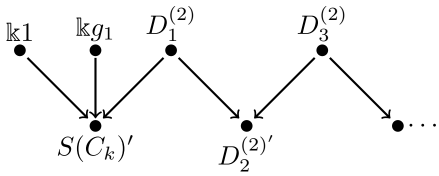
for some $D_2^{(2)} \in \mathcal{S}(2)$, a similar argument shows that if

$$D_2^{(2)} \cdot C_k = D_1^{(2)} + \Bbbk g_4 + \Bbbk g_5,$$

where $g_4, g_5 \in G(H)$, then H is of infinite corepresentation. If not, we can consider the case that

$$D_2^{(2)} \cdot C_k = D_1^{(2)} + D_3^{(2)},$$

where $D_3^{(2)} \in \mathcal{S}(2)$. Continue the process, we know that either H is of infinite corepresentation type, or we can get a sub-quiver which contains infinite vertexes of $\Bbbk Q(H)/J^2$ of the following form



.

For the latter case, it is in contradiction with the fact that H is finite-dimensional.

(III) Finally, we focus on the case that

$$S(C_k) \cdot C_k = \Bbbk 1 + D_1^{(3)}$$

in $\mathbb{Z}\mathcal{S}$ for some $D_1^{(3)} \in \mathcal{S}(3)$.

i) If

$$D_1^{(3)} \cdot S(C_k) = S(C_k) + D_1^{(2)} + D_2^{(2)},$$

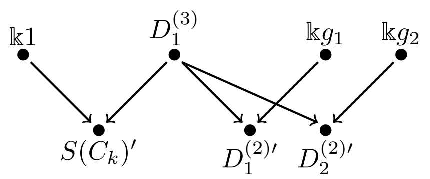
where $D_1^{(2)}, D_2^{(2)} \in \mathcal{S}(2)$, then

$$D_1^{(2)} \cdot C_k = D_1^{(3)} + \Bbbk g_1$$

and

$$D_2^{(2)} \cdot C_k = D_1^{(3)} + \Bbbk g_2,$$

where $g_1, g_2 \in G(H)$. It follows from Lemma 4.1.4 (2) that the separated quiver for $\Bbbk Q(H)/J^2$ either contains the Kronecker quiver as a sub-quiver, a sub-quiver whose underlying graph is $\tilde{A}_n$ for some $n \geq 3$, or a sub-quiver of the following form



.

The underlying graph of the quiver in the latter case is $\tilde{E}_6$, which is an Euclidean graph. This means that H is of infinite corepresentation type.

ii) If

$$D_1^{(3)} \cdot S(C_k) = S(C_k) + D_2^{(3)} + \Bbbk g_1,$$

where $g_1 \in G(H)$ and $D_2^{(3)} \in \mathcal{S}(3)$, we know that $\Bbbk g_1 \cdot C_k$ contains $D_1^{(3)}$ with a nonzero coefficient in $\mathbb{Z}\mathcal{S}$. But

$$\sqrt{\dim_{\Bbbk}(\Bbbk g_1)}\sqrt{\dim_{\Bbbk}(C_k)} < \sqrt{\dim_{\Bbbk}(D_1^{(3)})},$$

this leads to a contradiction. Therefore, this situation never happen.

iii) Suppose that

$$D_1^{(3)} \cdot S(C_k) = S(C_k) + D_1^{(4)},$$

where $D_1^{(4)} \in \mathcal{S}(4)$, we can continue this process. Since H is finite-dimensional, an argument similar to the one used in (2)(II) shows that there exists some $n \geq 3$ such that

$$D_1^{(i)} \cdot S^{\alpha_i}(C_k) = D_1^{(i-1)} + D_1^{(i+1)},$$

holds for all $3 \leq i \leq n$, and

$$D_1^{(n+1)} \cdot S^{\alpha_{n+1}}(C_k) = D_1^{(n)} + E + F,$$

where $E, F \in \mathcal{S}, D_1^{(2)} = S(C_k), D_1^{(i)} \in \mathcal{S}(i)$ for $3 \leq i \leq n+1$ and $\alpha_i = 0$ when i is even, $\alpha_i = 1$ when i is odd.

When $n = 2m$ for some $m \geq 2$, a similar argument shows that

$$\sqrt{\dim_{\Bbbk}(E)} = m + 1, \quad \sqrt{\dim_{\Bbbk}(F)} = m + 1.$$

Notice that $E \cdot C_k$ contains at least one subcoalgebra G with a nonzero coefficient besides $D_1^{(2m+1)}$, where $\sqrt{\dim_{\Bbbk}(G)} = 1$. Then we know that $G \cdot S(C_k)$ contains E, which is in contradiction with $\sqrt{\dim_{\Bbbk}(E)} \geq 3$.

When $n = 2m+1$ for some $m \geq 1$, since $E \cdot S(C_k)$ and $F \cdot S(C_k)$ contain $D_1^{(2m+2)}$ with a nonzero coefficient, it follows that

$$\sqrt{\dim_{\Bbbk}(E)} \geq m + 1, \quad \sqrt{\dim_{\Bbbk}(F)} \geq m + 1.$$

Without loss of generality, we can assume

$$\sqrt{\dim_{\Bbbk}(E)} = m + 2, \quad \sqrt{\dim_{\Bbbk}(F)} = m + 1.$$

Note that $E \cdot S(C_k)$ contains at least one subcoalgebra G with a nonzero coefficient besides $D_1^{(2m+2)}$, where $\sqrt{\dim_{\Bbbk}(G)} \leq 2$. Then we know that $G \cdot S(C_k)$ contains E, which means that $m = 1$ or $m = 2$ and $\sqrt{\dim_{\Bbbk}(G)} = 2$.

Based on the consideration above, we need only to consider the situations of $n = 3$ and $n = 5$.

When $n = 3$,

$$D_1^{(4)} \cdot C_k = D_1^{(3)} + E + F,$$

where $E, F \in \mathcal{S}$. Since $E \cdot S(C_k)$ and $F \cdot S(C_k)$ contains $D_1^{(4)}$ with a nonzero coefficient, it follows that

$$\sqrt{\dim_{\Bbbk}(E)} \geq 2, \quad \sqrt{\dim_{\Bbbk}(F)} \geq 2.$$

Without loss of generality, suppose that

$$\sqrt{\dim_{\Bbbk}(E)} = 3, \quad \sqrt{\dim_{\Bbbk}(F)} = 2.$$

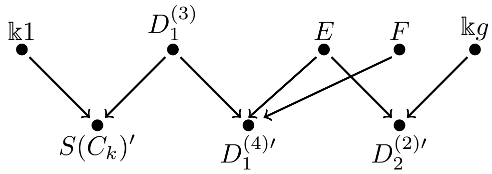
Then we have

$$E \cdot S(C_k) = D_1^{(4)} + D_2^{(2)},$$

and

$$D_2^{(2)} \cdot S(C_k) = E + \Bbbk g_1,$$

where $g_1 \in G(H)$, $D_2^{(2)} \in \mathcal{S}(2)$. According to Lemma 4.1.4 (2), the separated quiver of $\Bbbk Q(H)/J^2$ either contains the Kronecker quiver as a sub-quiver, a sub-quiver whose underlying graph is $\tilde{A}_n$ for some $n \geq 3$ or a sub-quiver of the following type



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The underlying graph of the sub-quiver in the latter case is $\tilde{E}_7$ and it is an Euclidean graph, which means that H is of infinite corepresentation type.
When $n = 5$, we have

$$D_1^{(4)} \cdot C_k = D_1^{(3)} + D_1^{(5)},$$

$$D_1^{(5)} \cdot S(C_k) = D_1^{(4)} + D_1^{(6)}$$

and

$$D_1^{(6)} \cdot C_k = D_1^{(5)} + E + F.$$

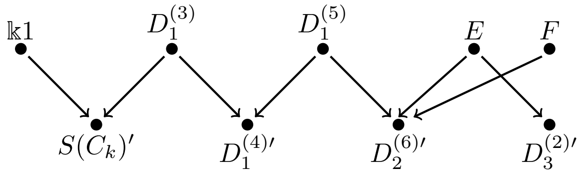
Without loss of generality, we can assume

$$\sqrt{\dim_{\Bbbk}(E)} = 4, \quad \sqrt{\dim_{\Bbbk}(F)} = 3.$$

It follows that

$$E \cdot S(C_k) = D_1^{(6)} + D_3^{(2)}.$$

This means that the separated quiver for $\Bbbk Q(H)/J^2$ either contains a sub-quiver whose underlying graph is $\tilde{A}_n$ for some $n \geq 3$ or a sub-quiver of the following type



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The underlying graph of the sub-quiver in the latter case is $\tilde{E}_8$ and it is still an Euclidean graph, which means that H is of infinite corepresentation type.

In conclusion, H is of infinite corepresentation type. □

Recall that a basic cycle of length n is a quiver with n vertices $e_0, e_1, \cdots, e_{n-1}$ and n arrows $a_0, a_1, \cdots a_{n-1}$, where the arrow a_i goes from the vertex e_i to the vertex e_{i+1}. With the help of the proceeding three propositions and Corollary 3.3.10, we can now obtain the following theorem.

Theorem 4.1.6 *Let H be a finite-dimensional non-cosemisimple Hopf algebra over an algebraically closed field $\Bbbk$ with the dual Chevalley property and $\mathrm{Q}(H)$ be the link quiver of H. Then the following statements are equivalent:*

(1) H is of finite corepresentation type;

(2) Every vertex in $\mathrm{Q}(H)$ is both the start vertex of only one arrow and the end vertex of only one arrow, that is, $\mathrm{Q}(H)$ is a disjoint union of basic cycles;

(3) There is only one arrow $C \to \Bbbk 1$ in $\mathrm{Q}(H)$ whose end vertex is $\Bbbk 1$ and $\dim_{\Bbbk}(C) = 1$;

(4) There is only one arrow $\Bbbk 1 \to D$ in $\mathrm{Q}(H)$ whose start vertex is $\Bbbk 1$ and $\dim_{\Bbbk}(D) = 1$.

Proof: Combining Propositions 4.1.2, 4.1.3 and 4.1.5, we can prove the equivalence of (1) and (3). According to Lemma 3.3.5 and Corollary 3.3.10, we know the equivalence of (2), (3), and (4). □

Let $H_{(1)}$ be the link-indecomposable component containing $\Bbbk 1$. Combining Proposition 3.3.13 and Theorem 4.1.6, we have:

Corollary 4.1.7 *A finite-dimensional non-cosemisimple Hopf algebra H over $\Bbbk$ with the dual Chevalley property is of finite corepresentation type if and only if $H_{(1)}$ is a pointed Hopf algebra and the link quiver of $H_{(1)}$ is a basic cycle.*

Recall that a finite-dimensional Hopf algebra H over $\Bbbk$ is said to have the *Chevalley property*, if radical $Rad(H)$ is a Hopf ideal. According to [3, Propersition 4.2], we know that H has the Chevalley property if and only if H^* has the dual Chevalley property.

Theorem 4.1.8 *A finite-dimensional Hopf algebra H over an algebraically closed field $\Bbbk$ with the Chevalley property is of finite representation type if and only if H is a Nakayama algebra.*

Proof: The sufficiency follows immediately since it is known that every Nakayama algebra is of finite representation type. Next we show the necessity. In fact if H has the Chevalley property, we know that H^* has the dual Chevalley property. According to the proof of Proposition 4.1.2, the Ext quiver of H is the same as the link quiver of H^*. If H is semisimple, the Ext quiver of H contains no arrows. If H is not semisimple, it follows from Theorem 4.1.6 that the Ext quiver of H is a finite union of basic cycles. Thus H is a Nakayama algebra. □

Recall that a coalgebra C is said to be coNakayama, if the dual algebra C^* is a Nakayama algebra. It is direct to see the following corollary.

Corollary 4.1.9 *A finite-dimensional Hopf algebra H over an algebraically closed field $\mathbb{k}$ with the dual Chevalley property is of finite corepresentation type if and only if H is coNakayama.*

§4.2 Tame corepresentation type

Let H be a finite-dimensional non-cosemisimple Hopf algebra $\mathbb{k}$ with the dual Chevalley property. Now we can characterize the link quiver of H when it is of finite or tame corepresentation type.

Theorem 4.2.1 *Let $\mathbb{k}$ be an algebraically closed field and H a finite-dimensional Hopf algebra over $\mathbb{k}$ with the dual Chevalley property. Denote ${}^1\mathcal{S} = \{C \in \mathcal{S} \mid \mathbb{k}1 + C \neq \mathbb{k}1 \wedge C\}$.*

(1) H is of finite corepresentation type if and only if $\mid {}^1\mathcal{P} \mid = 1$ and ${}^1\mathcal{S} = \{\mathbb{k}g\}$ for some group-like element $g \in G(H)$.

(2) If H is of tame corepresentation type, then one of the following two cases appears:

(i) $\mid {}^1\mathcal{P} \mid = 2$ and for any $C \in {}^1\mathcal{S}$, $\dim_{\mathbb{k}}(C) = 1$;

(ii) $\mid {}^1\mathcal{P} \mid = 1$ and ${}^1\mathcal{S} = \{C\}$ for some $C \in \mathcal{S}$ with $\dim_{\mathbb{k}}(C) = 4$.

(3) If one of the following holds, H is of wild corepresentation type.

(i) $\mid {}^1\mathcal{P} \mid \geq 3$;

(ii) $\mid {}^1\mathcal{P} \mid = 2$ and there exists some $C \in {}^1\mathcal{S}$ such that $\dim_{\mathbb{k}}(C) \geq 4$;

(iii) $\mid {}^1\mathcal{P} \mid = 1$ and ${}^1\mathcal{S} = \{C\}$ for some $C \in \mathcal{S}$ with $\dim_{\mathbb{k}}(C) \geq 9$.

Proof: Indeed, (1) follows directly from Theorem 4.1.6. Clearly, (2) $\Leftrightarrow$ (3). So it is enough to prove (3).

We know that the $\mathbb{k}$-linear abelian category of finite-dimensional comodules over H is isomorphic to the category of finite-dimensional modules over H^*. This means that the coalgebra's version of Ext quiver $\Gamma(H)^{\mathrm{c}}$ of H is the same as the algebra's version of Ext quiver $\Gamma(H^*)^{\mathrm{a}}$ of H^*. According to [17, Theorem 2.1 and Corollary 4.4], the link quiver $\mathrm{Q}(H)$ of H coincides with the algebra's version of Ext quiver $\Gamma(H^*)^{\mathrm{a}}$ of H^*. Note that H^* is Morita equivalent to a basic algebra $\mathcal{B}(H^*)$. Let J be the ideal generated by all the arrows in $\mathrm{Q}(H)$. By the Gabriel's theorem, there exists an admissible ideal I such that

$$\mathbb{k}\mathrm{Q}(H)/I \cong \mathcal{B}(H^*),$$

where $J^t \subseteq I \subseteq J^2$ for some integer $t \geq 2$. Thus there exists an algebra epimorphism

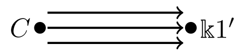
$$f : \mathcal{B}(H^*) \to \mathbb{k}\mathrm{Q}(H)/J^2.$$

It is enough to show that $\Bbbk\mathrm{Q}(H)/J^2$ is of wild representation type. Since the Jacobson radical of $\Bbbk\mathcal{Q}(H)/J^2$ is J/J^2, we know that $\Bbbk\mathrm{Q}(H)/J^2$ is an artinian algebra with radical square zero.

Now assume on the countrary that $\Bbbk\mathrm{Q}(H)/J^2$ is of tame representation type. It follows from the proof of [5, X.2 Theorem 2.6] that the separated quiver of $\Bbbk\mathrm{Q}(H)/J^2$ coincides with the quiver of the hereditary algebra $\sum = \begin{pmatrix} (\Bbbk\mathcal{Q}(H)/J^2)/(J/J^2) & 0 \\ J/J^2 & (\Bbbk\mathcal{Q}(H)/J^2)/(J/J^2) \end{pmatrix}$. Note that $\Bbbk\mathrm{Q}(H)/J^2$ and $\sum$ are stably equivalent, it follows that $\Bbbk\mathrm{Q}(H)/J^2$ is of tame representation type if and only if $\sum$ is of tame representation type. This means that $\mathrm{Q}(H)_s$ of $\Bbbk\mathrm{Q}(H)/J^2$ is a finite disjoint union of Euclidean diagrams.

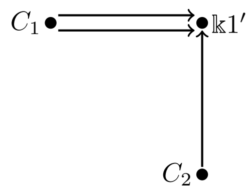
(i) If $\mid {}^1\mathcal{P} \mid \geq 3$, we deal with this situation through classified discussion.

  (a) Suppose that there exists some $C \in {}^1\mathcal{S}$ such that $\mid {}^1\mathcal{P}^{\mathcal{C}} \mid \geq 3$. Then the separated quiver $\mathrm{Q}(H)_s$ must contain



  as a sub-quiver. The underlying graph of this sub-quiver is not a Euclidean diagram. It turns out that H is of wild corepresentation type.

  (b) Suppose that there exist some $C_1, C_2 \in {}^1\mathcal{S}$ such that $\mid {}^1\mathcal{P}^{\mathcal{C}_1} \mid \geq 2$ and $\mid {}^1\mathcal{P}^{\mathcal{C}_2} \mid \geq 1$. Then the separated quiver $\mathrm{Q}(H)_s$ must contain



  as a sub-quiver. The underlying graph of this sub-quiver is not a Euclidean diagram and thus H is of wild corepresentation type.

  (c) Suppose that there exist some $C_1, C_2, C_3 \in {}^1\mathcal{S}$ such that $\mid {}^1\mathcal{P}^{\mathcal{C}_i} \mid \geq 1$ for any $1 \leq i \leq 3$. This means that for any $1 \leq i \leq 3$, there exists some non-trivial $(1, \mathcal{C}_i)$-primitive matrix $\mathcal{X}_i \in {}^1\mathcal{P}$. Combining Lemmas 3.2.5 and 3.3.8, for any $1 \leq i \leq 3$, we know that

$$\mid \mathcal{P}^{\mathcal{C}_i} \mid \geq \mid \mathcal{P}^{\mathcal{C}_i}_{\mathcal{X}_1} \mid + \mid \mathcal{P}^{\mathcal{C}_i}_{\mathcal{X}_2} \mid + \mid \mathcal{P}^{\mathcal{C}_i}_{\mathcal{X}_3} \mid \geq 3.$$

  In such a case, there exist at least 3 vertexes which are the start vertex of 3 arrows and 1 vertex which is the end vertex of 3 arrows in the separated quiver $\mathrm{Q}(H)_s$. As a result, the underlying diagram of $\mathrm{Q}(H)_s$ is not a Euclidean diagram and H is of wild corepresentation type.

(ii) Suppose that

$${}^1\mathcal{P} = \{\mathcal{X}, \mathcal{Y}\},$$

where $\mathcal{X}$ is a non-trivial $(1,\mathcal{C})$-primitive matrix and $\mathcal{Y}$ is a non-trivial $(1,\mathcal{D})$-primitive matrix for some $C, D \in \mathcal{S}$. With loss of generality, assume $\dim_{\mathbb{k}}(C) \geq 4$.

(a) If $\dim_{\mathbb{k}}(C) \geq 9$, it follows from Lemma 3.3.11 that there exists some $E \in \mathcal{S}$ such that $|\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{X}}\ | \geq 4$. According to Lemmas 3.2.5 and 3.3.8, we know that

$$|\ {}^{\mathcal{E}}\mathcal{P}\ | = |\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{X}}\ | + |\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{Y}}\ | \geq 5.$$

This implies that $\mathrm{Q}(H)_s$ contains at least one vertex E which is the end vertex of at least 5 arrows. It follows that the underlying graph of this sub-quiver is not a union of Euclidean diagram, and consequently H is of wild corepresentation type.

(b) If $\dim_{\mathbb{k}}(C) = 4$, Lemma 3.3.11 implies that there exists some $E \in \mathcal{S}$ such that $|\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{X}}\ | \geq 3$. If $|\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{X}}\ | \geq 4$, as in the case of $\dim_{\mathbb{k}}(C) \geq 9$, $\mathrm{Q}(H)_s$ contains at least one vertex E which is the end vertex of at least 5 arrows. This indicates H is of wild corepresentation type. If $|\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{X}}\ | = 3$, using Lemma 3.3.11, we have

$$E \cdot C = C_1 + C_2 + C_3 \tag{4.3}$$

for some $C_1, C_2, C_3 \in \mathcal{S}$. According to Lemma 4.1.4, we know that for any $1 \leq i \leq 3$, $C_i \cdot S(C)$ contains E with a nonzero coefficient. Suppose that $\sqrt{\dim_{\mathbb{k}}(E)} = n$. If for any $1 \leq i \leq 3$, we have

$$C_i \cdot S(C) = E.$$

It means that

$$\sqrt{\dim_{\mathbb{k}}(C_1)} = \sqrt{\dim_{\mathbb{k}}(C_2)} = \sqrt{\dim_{\mathbb{k}}(C_3)} = \frac{n}{2}.$$

But (4.3) implies that $2n = \frac{3}{2}n$, which is impossible. Thus there exists at least one C_j such that $C_j \cdot S(C)$ contains some $F \in \mathcal{S}$ with a nonzero coefficient besides E, where $1 \leq j \leq 3$. Combining Lemmas 3.2.5 and 3.3.8, we have

$$|\ {}^{\mathcal{E}}\mathcal{P}\ | \geq \sum_{i=1}^{3} |\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{X}}^{C_i}\ | + |\ {}^{\mathcal{E}}\mathcal{P}_{\mathcal{Y}}\ | \geq 4$$

and

$$|\ \mathcal{P}^{\mathcal{C}_j}\ | \geq |\ {}^{\mathcal{E}}\mathcal{P}^{\mathcal{C}_j}\ | + |\ {}^{\mathcal{F}}\mathcal{P}^{\mathcal{C}_j}\ | \geq 2.$$

As a result, there exist at least one vertex which is the end vertex of 4 arrows and one vertex which is the start vertex of 4 arrows in $\mathrm{Q}(H)_s$. It is easy to see that H is of wild corepresentation type.

(iii) (a) Note that if $\dim_{\mathbb{k}}(C) \geq 16$, it follows from Lemma 3.3.11 that there exists some $E \in \mathcal{S}$ such that $|\ {}^{\mathcal{E}}\mathcal{P}\ | \geq 5$. This means that the separated quiver $\mathrm{Q}(H)_s$ contains a vertex which is the end vertex of 5 arrows and it cannot be a finite disjoint union of Euclidean diagram. We know that H is of wild corepresentation type.

(b) If $\dim_{\Bbbk}(C) = 9$, it follows from Lemma 3.3.11 that there exists some $E \in \mathcal{S}$ such that $| {}^{\mathcal{E}}\mathcal{P} | \geq 4$. If $| {}^{\mathcal{E}}\mathcal{P} | \geq 5$, a similar argument shows that H is of wild corepresentation type. We only need to consider the case that $| {}^{\mathcal{E}}\mathcal{P} | = 4$. In this case, Lemma 3.3.11 implies that

$$E \cdot C = C_1 + C_2 + C_3 + C_4, \tag{4.4}$$

where $C_i \in \mathcal{S}$ for $1 \leq i \leq 4$. Applying Lemma 4.1.4 yields that for any $1 \leq i \leq 4$, $C_i \cdot S(C)$ contains E with a nonzero coefficient. Suppose that $\sqrt{\dim_{\Bbbk}(E)} = n$. If for any $1 \leq i \leq 4$, we have

$$C_i \cdot S(C) = E.$$

It means that

$$\sqrt{\dim_{\Bbbk}(C_i)} = \frac{n}{3},$$

for $1 \leq i \leq 4$. But (4.4) implies that $3n = \frac{4}{3}n$, which leads to a contradiction. Thus there exists at least one C_j such that $C_j \cdot S(C)$ contains some $F \in \mathcal{S}$ with a nonzero coefficient besides E, where $1 \leq j \leq 4$. A similar argument shows that $\mathrm{Q}(H)_s$ contains at least one vertex which is the end vertex of 4 arrows and one vertex which is the start vertex of 4 arrows. Clearly, the underlying graph of this sub-quiver is not a Euclidean graph. Consequently, H is of wild corepresentation type. □

As a corollary, we have

Corollary 4.2.2 *Let H be a finite-dimensional non-cosemisimple Hopf algebra over $\Bbbk$ with the dual Chevalley property of finite or tame corepresentation type. Then we have $| {}^{1}\mathcal{P} | \mid | {}^{\mathcal{C}}\mathcal{P} |$, for any $\mathcal{C} \in \mathcal{M}$.*

Proof: Note that for any $C \in {}^{1}\mathcal{S}$, if $\dim_{\Bbbk}(C) = 1$, it follows from Proposition 3.3.9 that

$$| {}^{1}\mathcal{P} | = | {}^{\mathcal{C}}\mathcal{P} | .$$

If $| {}^{1}\mathcal{P} | = 1$ and $\dim_{\Bbbk}(C) = 4$, where $C \in {}^{1}\mathcal{S}$. According to Lemma 3.3.11, we have

$$1 = | {}^{1}\mathcal{P} | \mid | {}^{\mathcal{C}}\mathcal{P} | .$$

The proof is completed. □

Chapter 5 Hopf algebras with the dual Chevalley property of finite corepresentation type

Next we give a more accurate description for $H_{(1)}$ in the case that H is a finite-dimensional non-cosemisimple Hopf algebra with the dual Chevalley property of finite corepresentation type.

It is known that a finite-dimensional Hopf algebra H over an algebraically closed field is pointed if and only if H^* is elementary. And according to the proof of Proposition 4.1.2, the link quiver $\mathrm{Q}(H_{(1)})$ of $H_{(1)}$ agrees with the Ext quiver $\Gamma(H^*)^{\mathrm{a}}$ of H^*. Thus the following corollary is a direct consequence of Corollary 4.1.7 and Lemma 2.3.3.

Corollary 5.0.3 *Let H be a finite-dimensional non-cosemisimple Hopf algebra with the dual Chevalley property over an algebraically closed field $\Bbbk$. Then H is of finite corepresentation type if and only if $H_{(1)}$ is a comonomial Hopf algebra.*

§5.1 Char($\Bbbk$) = 0

Now we prove the following theorem of finite-dimensional Hopf algebras over an algebraically closed field $\Bbbk$ with characteristic 0 with the dual Chevalley property of finite corepresentation type, which is a generalization of [44, Theorem 4.6].

Theorem 5.1.1 *Let $\Bbbk$ be an algebraically closed field with characteristic zero. Then a finite-dimensional Hopf algebra H over $\Bbbk$ with the dual Chevalley property is of finite corepresentation type if and only if either of the following conditions is satisfied:*

(1) H is cosemisimple;

(2) H is not cosemisimple and $H_{(1)} \cong A(n, d, \mu, q)$.

Proof: Since $A(n, d, \mu, q)$ is a comonomial Hopf algebra, the if implication follows immediately from Corollary 5.0.3. It suffices to prove the only if part. According to Lemma 2.3.5, it is enough to find the induced datum of $H_{(1)}$. Let $G(H_{(1)})$ be the set of group-like elements of $H_{(1)}$. If H is a non-cosemisimple Hopf algebra of finite corepresntation type, it follows from Theorem 4.1.6 and Corollary 4.1.7 that $H_{(1)}$ is a pointed Hopf algebra and there exists a unique non-trivial $(1, g)$-primitive element x for some $g \in G(H_{(1)})$. Without loss of generality, assume

$$| G(H_{(1)}) |= n.$$

Due to $H_{(1)}$ is link-indecomposable, the link quiver of $H_{(1)}$ is connected. This means that $G(H_{(1)})$ is a cyclic group whose generator is g. Thus the induced group datum of $H_{(1)}$ is $(\langle g\rangle, g, \chi, \mu)$ and we have

$$H_{(1)} \cong A(\langle g\rangle, g, \chi, \mu) \cong A(n, d, \mu, q).$$

□

Remark 5.1.2 *Let $\Bbbk$ be an algebraically closed field with characteristic zero.*

(1) Andruskiewitsch and Schneider conjectured that any finite-dimensional pointed Hopf algebra over $\Bbbk$ is generated in degree 1 of its coradical filtration, i.e., by grouplike and skew-primitive elements [7]. Suppose H is a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property of finite corepresentation type. According to Theorem 5.1.1 and [40, Corollary 4.10], we can show that H is generated in degree 1 of its coradical filtration.

(2) Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property. Recall that the rank of H is defined to be n if $\dim_{\Bbbk}(\Bbbk \otimes_{H_0} H_1) = n + 1$ and H is generated by H_1 as an algebra [38]. It is not difficult to show that H is of rank one if and only if H is of finite corepresentation type.

Let us first give a example which is of finite corepresentation type.

Example 5.1.3 *Let $\Bbbk$ be an algebraically closed field of characteristic zero and H be the Hopf algebra over $\Bbbk$ of dimension 16 appeared in [15, Theorem 5.1]. As an algebra, H is generated by c, b, x, y with relations:*

$$c^2 = 1, \ \ b^2 = 1, \ \ x^2 = \frac{1}{2}(1 + c + b - cb),$$

$$cb = bc, \ \ xc = bx, \ \ xb = cx,$$

$$y^2 = 0, \ \ yc = -cy, \ \ yb = -by, \ \ yx = \sqrt{-1}cxy.$$

The coalgebra structure and antipode are given by:

$$\Delta(c) = c \otimes c, \ \ \varepsilon(c) = 1, \ \ S(c) = c,$$

$$\Delta(b) = b \otimes b, \ \ \varepsilon(b) = 1, \ \ S(b) = b,$$

$$\Delta(x) = \frac{1}{2}(x \otimes x + bx \otimes x + x \otimes cx - bx \otimes cx), \ \ \varepsilon(x) = 1, \ \ S(x) = x,$$

$$\Delta(y) = c \otimes y + y \otimes 1, \ \ \varepsilon(y) = 0, \ \ S(y) = -c^{-1}y.$$

Denote $E = \operatorname{span}\{x, bx, cx, bcx\}$, then $\mathcal{S} = \{\Bbbk 1, \Bbbk c, \Bbbk b, \Bbbk bc, E\}$. We give the corresponding multiplicative matrix $\mathcal{E}$ of E, where

$$\mathcal{E} = \frac{1}{2}\begin{pmatrix} x + bx & x - bx \\ cx - bcx & cx + bcx \end{pmatrix}.$$

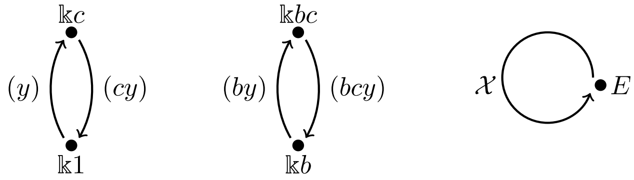
We know that $\mathbb{Z}\mathcal{S}$ is a unital based ring and its multiplication table is shown below:

left action	$\Bbbk 1$	$\Bbbk c$	$\Bbbk b$	$\Bbbk bc$	E
$\Bbbk 1$	$\Bbbk 1$	$\Bbbk c$	kb	$\Bbbk bc$	E
$\Bbbk c$	$\Bbbk c$	$\Bbbk 1$	$\Bbbk bc$	$\Bbbk b$	E
$\Bbbk b$	$\Bbbk b$	$\Bbbk bc$	$\Bbbk 1$	$\Bbbk c$	E
$\Bbbk bc$	$\Bbbk bc$	$\Bbbk b$	$\Bbbk c$	$\Bbbk 1$	E
E	E	E	E	E	$\Bbbk 1+\Bbbk c+\Bbbk b+\Bbbk bc$

And in this example, $\mathcal{P}=\{(y),(cy),(by),(bcy),\mathcal{X}\}$*, where*

$$\mathcal{X}=\frac{1}{2}\begin{pmatrix} xy+bxy & xy-bxy \\ bcxy-cxy & -cxy-bcxy \end{pmatrix}$$

is a non-trivial $(\mathcal{E},\mathcal{E})$*-primitive matrix. In this example, the link quiver of* H *is shown below:*



It follows from Theorem 4.1.6 that H *is of finite corepresentation type. Moreover, due to*

$$ad_r(\frac{x+bx}{2})(c)=b\notin H_{(1)},$$

we know that $H_{(1)}$ *is not normal in* H*. Thus this example gives a negative answer to [40, Question 4.13].*

§5.2 $\mathrm{Char}(\Bbbk)=p$

Finally, we focus on the above theorem in the case of that $\Bbbk$ is an algebraically closed field of characteristic p. We can obtain the following theorem immediately.

Theorem 5.2.1 *Let* $\Bbbk$ *be an algebraically closed field of positive characteristic* p*. Then a finite-dimensional Hopf algebra* H *over* $\Bbbk$ *with the dual Chevalley property is of finite corepresentation type if and only if either of the following conditions is satisfied:*

(1) H *is cosemisimple;*

(2) H *is not cosemisimple and* $H_{(1)}\cong C_d(n)$*.*

Chapter 6 Coradically graded Hopf algebras with the dual Chevalley property of tame corepresentation type

In this chapter, we work over an algebraically closed field $\mathbb{k}$ of characteristic zero. The main aim of this chapter is to describe the structures of coradically graded Hopf algebras with the dual Chevalley property of tame corepresentation type.

§6.1 Characterization

Let H, H' be Hopf algebras and $\pi : H \to H'$ and $i : H' \to H$ Hopf homomorphisms. Assume that $\pi \circ i = id_{H'}$, so that π is surjective and i is injective. Define

$$R := \{h \in H \mid (id \otimes \pi)\Delta(h) = h \otimes 1\}.$$

According to [61, Theorem 3], we know that

$$H \cong R \times H'$$

as Hopf algebras, where "$\times$" was called *biproduct* in [61] and bosonization in [50]. Note that as a linear space,

$$H \cong R \times H' = R \otimes H'.$$

Its multiplication and comultiplication are usual smash product and smash coproduct respectively. In addition, R is a braided Hopf algebra in ${}^{H'}_{H'}\mathcal{YD}$, the category of Yetter-Drinfeld modules over H'. See, for example, [6, 50, 61].

Let H be a finite-dimensional Hopf algebra with the Chevalley property and J_H its Jacobson radical. Denote $\mathrm{gr}^a(H)$ its radically graded algebra, i.e.,

$$\mathrm{gr}^a(H) = H/J_H \oplus J_H/J_H^2 \oplus \cdots \oplus J_H^{m-1},$$

if $J_H^m = 0$. According to [42, Lemma 5.1], we know that $\mathrm{gr}^a(H)$ is a radically graded Hopf algebra. Clearly, $H/J_H = \mathrm{gr}^a(H)(0)$ is a Hopf subalgebra of $\mathrm{gr}^a(H)$ and there exists a natural Hopf algebra epimorphism

$$\pi^a : \mathrm{gr}^a(H) \to H/J_H$$

with a retraction of the inclusion. Define

$$A_H := \{h \in \mathrm{gr}^a(H) \mid (id \otimes \pi^a)\Delta(h) = h \otimes 1\}.$$

By [61, Theorem 3], we know that

$$\mathrm{gr}^a(H) \cong A_H \times H/J_H$$

as Hopf algebras.

Proposition 6.1.1 *Let $\Bbbk$ be an algebraically closed field of characteristic 0 and H a finite-dimensional Hopf algebra over $\Bbbk$ with the Chevalley property. Then*

(1) A_H and $\mathrm{gr}^a(H)$ have the same representation type;

(2) A_H is a local graded Frobenius algebra.

Proof:

(1) Note that as an algebra,

$$\mathrm{gr}^a(H) \cong A_H \# H/J_H,$$

and the multiplication of $A_H \# H/J_H$ is usual smash product. Since H/J_H is a finite-dimensional semisimple Hopf algebra, it follows from [46, Theorem 3.3] that H/J_H is cosemisimple. Thus (1) is a direct consequence of [42, Theorem 4.5].

(2) This can be obtained by the same reason in the proof of [42, Proposition 5.3 (ii)]. □

Let H be a finite-dimensional Hopf algebra with the dual Chevalley property. Denote $\mathrm{gr}^c(H)$ by the coradically graded Hopf algebra of H, i.e., $\mathrm{gr}^c(H) = \bigoplus_{n\geq 0} H_n/H_{n-1}$, where $H_{-1} = 0$. In fact, there exists a natural Hopf algebra epimorphism

$$\pi^c : \mathrm{gr}^c(H) \to H_0$$

with a retraction of the inclusion. Define

$$R_H := \{h \in \mathrm{gr}^c(H) \mid (id \otimes \pi^c)\Delta(h) = h \otimes 1\}.$$

It follows from [61, Theorem 3] that

$$\mathrm{gr}^c(H) \cong R_H \times H_0$$

as Hopf algebras.

The next conclusion will give us the structure of coradically graded Hopf algebras with the dual Chevalley property of tame corepresentation type.

Theorem 6.1.2 *Let $\Bbbk$ be an algebraically closed field of characteristic 0 and H a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property. Then $\mathrm{gr}^c(H)$ is of tame corepresentation type if and only if*

$$\mathrm{gr}^c(H) \cong (\Bbbk\langle x, y\rangle/I)^* \times H'$$

for some finite-dimensional semisimple Hopf algebra H' and some ideal I which is one of the following forms:

(1) $I = (x^2 - y^2, yx - ax^2, xy)$ *for* $0 \neq a \in \Bbbk$*;*

(2) $I = (x^2, y^2, (xy)^m - a(yx)^m)$ *for* $0 \neq a \in \Bbbk$ *and* $m \geq 1$*;*

(3) $I = (x^n - y^n, xy, yx)$ *for* $n \geq 2$*;*

(4) $I = (x^2, y^2, (xy)^m x - (yx)^m y)$ *for* $m \geq 1$*.*

Proof: “If part": Combining [42, Theorem 3.1] and [43, Lemma 4.2], we know that $\Bbbk\langle x, y\rangle/I$ is a tame algebra. Because of the fact that a finite-dimensional Hopf algebra H' is semisimple if and only if it is cosemisimple, the desired conclusion is got from [42, Theorem 4.5].

“Only if part":Using Proposition 6.1.1, we know that $\mathrm{gr}^a(H^*)$ is of tame representation type if and only if A_{H^*} is of tame representation type. Since

$$\mathrm{gr}^c(H) \cong (\mathrm{gr}^a(H^*))^*$$

as Hopf algebra, one can conclude that $\mathrm{gr}^c(H)$ is of tame corepresentation type if and only if A_{H^*} is of tame representation type. According to [42, Theorem 3.1] and [43, Lemma 4.2], as a tame local graded Frobenius algebra,

$$A_{H^*} \cong \Bbbk\langle x, y\rangle/I.$$

It follows from [52, Theorem 5.1] that

$$\mathrm{gr}^c(H) \cong (\mathrm{gr}^a(H^*))^* \cong (A_{H^*} \times H^*/J_{H^*})^* \cong (A_{H*})^* \times H_0.$$

□

According to [13, Theorem 4.1.2], if R is a Hopf algebra in ${}^{H'}_{H'}\mathcal{YD}$, then we can form the bosonization $R \times H'$ which is a Hopf algebra. For a tame local graded Frobenius algebra A, above theorem does not imply the existence of finite-dimensional semisimple Hopf algebra H' satisfying A^* is a braided Hopf algebra in ${}^{H'}_{H'}\mathcal{YD}$. That is to say, for the ideals I listed in the above theorem, we do not know weather $(\Bbbk\langle x, y\rangle/I)^* \times H'$ is a Hopf algebra or not.

Question 6.1.3 *For a tame local graded Frobenius algebra* A*, give an efficient method to determine that whether there is a cosemisimple Hopf algebra* H' *satisfying* A *is a braided Hopf algebra in* ${}^{H'}_{H'}\mathcal{YD}$*. If such* H' *exists, then find all of them.*

The question above exactly recovers [42, Problem 5.1]. We will discuss this question in the subsequent sections.

§6.2 Link-indecomposable component containing $\Bbbk1$

We have the following characterization of the coradical of the link-indecomposable component $H_{(1)}$ containing $\Bbbk1$.

Lemma 6.2.1 *Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property. Then the coradical of the link-indecomposable component $H_{(1)}$ containing $\Bbbk 1$ is generated by* $\{\mathrm{span}(C) \mid C \in {}^1\mathcal{S}\} \cup \{\mathrm{span}(S(C)) \mid C \in {}^1\mathcal{S}\}$.

Proof: It is directly from [40, Theorem 4.8 (3)] that $H_{(1)}$ is a link-indecomposable Hopf algebra. This means that the link quiver $\mathcal{Q}(H_{(1)})$ of $H_{(1)}$ is connected. Using Proposition 3.3.14, we can complete the proof. □

Now we discuss the relation between the corepresentation type of H and $H_{(1)}$.

Lemma 6.2.2 *Let H be a finite-dimensional Hopf algebra over $\Bbbk$ with the dual Chevalley property of tame corepresentation type. Then the link-indecomposable component $H_{(1)}$ contating $\Bbbk 1$ is of tame corepresentation type.*

Proof: Since H is of tame corepresentation type, it follows from Theorem 4.2.1 that either

$$| {}^1\mathcal{P} | > 1$$

or

$$\dim_{\Bbbk}(C) > 1$$

for $C \in {}^1\mathcal{S}$. This implies that $H_{(1)}$ is not of finite corepresentation type. On the other hand, there is an inclusion from the category of finite-dimensional right $H_{(1)}$-comodules to the category of finite-dimensional right H-comodules. Suppose that $H_{(1)}$ is of wild corepresentation type. It follows that $H_{(1)}^*$ is a wild algebra. Hence by [64, Theorem 1.11], H^* is a wild algebra, which means that H is of wild corepresentation type. This leads to a contradiction. We remark that $H_{(1)}$ is of tame corepresentation type by the fundamental result of [24]. □

In the following part, let $H = \bigoplus_{i=0}^{n} H(i)$ be a finite-dimensional coradically graded Hopf algebra over $\Bbbk$ with the dual Chevalley property. Denote the coradical filtration of H by $\{H_n\}_{n\geq 0}$ and the set of all the simple subcoalgebras of H by $\mathcal{S}$. Note that there exists a natural Hopf algebra epimorphism

$$\pi : H \to H_0$$

with a retraction of the inclusion. Next we give a more accurate description for the structures of R_H, where

$$R_H = \{h \in H \mid (id \otimes \pi)\Delta(h) = h \otimes 1\}.$$

Firstly, we have the following lemma.

Lemma 6.2.3 *Let H be a finite-dimensional coradically graded Hopf algebra over $\Bbbk$ with the dual Chevalley property. Then we have $R_H \subseteq H_{(1)}$.*

Proof: At first, let us introduce an equivalence relation on $\mathcal{S}$, defining that C and D are related if $CH_{(1)} = DH_{(1)}$. Let $\mathcal{S}_0 \subseteq \mathcal{S}$ be a full set of chosen non-related representatives with respect to this equivalence relation. By [40, Corollary 4.10], we have

$$H = \bigoplus_{C \in \mathcal{S}_0} CH_{(1)}.$$

For any non-zero $x \in CH_{(1)}$, where $C \in \mathcal{S}_0 \setminus \{\Bbbk 1\}$. According to [40, Theorem 4.8 (3)], we know that

$$\begin{aligned}(id \otimes \pi)\Delta(x) &= (id \otimes \pi)\Delta(\sum_{i=1}^{n} c_i y_i) \\ &= (id \otimes \pi)(\sum_{i=1}^{n} \Delta(c_i)\Delta(y_i)) \\ &\subseteq (id \otimes \pi)(CH_{(1)} \otimes CH_{(1)})\end{aligned}$$

Using the fact that H is a coradical graded Hopf algebra, we obtain

$$\pi(CH_{(1)}(i)) = 0$$

for $i \geq 1$, where

$$H_{(1)}(i) = H_{(1)} \cap H(i).$$

According to Lemma 3.3.1, we know that $\mathbb{Z}\mathcal{S}$ is a unital based ring. It follows that

$$1 \notin CH_{(1)},$$

which means that $x \notin R_H$ and thus $R_H \subseteq H_{(1)}$. □

In fact, $H_{(1)} = \bigoplus_{i=0}^{n} H_{(1)}(i)$ is also a finite-dimensional coradically graded Hopf algebra over $\Bbbk$ with the dual Chevalley property, where $H_{(1)}(i) = H_{(1)} \cap H(i)$. Let

$$\pi' : H_{(1)} \to (H_{(1)})_0$$

be a natural Hopf algebra epimorphism with a retraction of the inclusion and

$$R' = \{r \in H_{(1)} \mid (id \otimes \pi')\Delta(r) = r \otimes 1\}.$$

Lemma 6.2.4 *With the notations above, we have $R' = R_H$ and $H_{(1)} \cong R_H \times (H_{(1)})_0$.*

Proof: Because of the fact that

$$\pi' = \pi \mid_{H_{(1)}},$$

we can show that

$$R' \subseteq R_H.$$

It is a concequence of Lemma 6.2.3 that

$$R' = R_H.$$

Now the lemma follows directly by [61, Theorem 3]. □

With the help of the preceding lemmas, we can now prove:

Proposition 6.2.5 *Let H be a finite-dimensional coradically graded Hopf algebra over $\Bbbk$ with the dual Chevalley property. Then H is of tame corepresentation type if and only if $H_{(1)}$ is of tame corepresentation type.*

Proof: The "only if" implication follows immediately by Lemma 6.2.2. Next we show the "if" implication. Since $H_{(1)}$ is of tame corepresentation type, it follows from Theorem 6.1.2 that

$$H_{(1)} \cong (\Bbbk\langle x, y\rangle/I)^* \times (H_{(1)})_0$$

for some I listed in Theorem 6.1.2. According to Lemma 6.2.4, one can show that

$$H \cong (\Bbbk\langle x, y\rangle/I)^* \times H_0.$$

From Theorem 6.1.2, we have H is of tame corepresentation type. □

The above proposition implies that when we study the properties for the finite-dimensional coradically graded Hopf algebra over $\Bbbk$ with the dual Chevalley property of tame corepresentation type, we only need to focus on its link-indecomposable component contained $\Bbbk 1$.

§6.3 Characterization of R_H

In this section, we discuss which ideal in Theorem 6.1.2 will occur when $(\Bbbk\langle x, y\rangle/I)^* \times H_0$ is a finite-dimensional coradically graded Hopf algebra with the dual Chevalley property of tame corepresentation type.

Let H be a finite-dimensional Hopf algebra with the dual Chevalley property. Denote the coradical filtration of H by $\{H_n\}_{n\geq 0}$ and the set of all the simple subcoalgebras of H by $\mathcal{S}$. In fact, there exists a natural Hopf algebra epimorphism

$$\pi : \mathrm{gr}^c(H) \to H_0$$

with a retraction of the inclusion

$$i : H_0 \to \mathrm{gr}^c(H).$$

Denote

$$R_H := \{h \in \mathrm{gr}^c(H) \mid (id \otimes \pi)\Delta(h) = h \otimes 1\}.$$

We will give a more accurate description for the structure of R_H.

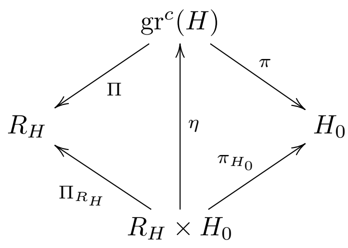
Firstly, let us recall some properties of biproduct.

Set $\Pi = id * (i \circ S \circ \pi)$, where S is the antipode of $\mathrm{gr}^c(H)$ and $*$ is the convolution product. According to [61, Theorem 3], we know that $R_H = \Pi(\mathrm{gr}^c(H))$ and R_H has a unique coalgebra structure such that Π is a coalgebra map. Let $j : R_H \to \mathrm{gr}^c(H)$ be the inclusion. Then the map

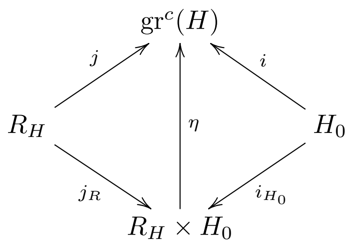
$$\eta : R_H \times H_0 \to \mathrm{gr}^c(H), \quad r \times h \mapsto rj(h)$$

is an isomorphism of Hopf algebras.

Moreover, it follows from [61, Theorem 2 (b)] that the following diagrams



and



commute, where

$$\begin{aligned} \Pi_{R_H} &: r \times h \mapsto r\varepsilon(h), \\ j_{R_H} &: r \mapsto r \times 1, \\ i_{H_0} &: h \mapsto 1 \times h, \\ \pi_{H_0} &: r \times h \mapsto \varepsilon(r)h, \end{aligned}$$

for any $h \in H_0, r \in R_H$.

With the notations above, we have

Lemma 6.3.1 *For any $r \in R_H$, we know that*

$$\Delta_{R_H}(r) = ((\Pi_R \circ \eta^{-1}) \otimes id)\Delta(r),$$

where Δ and Δ_{R_H} are the comultiplications of H and R_H, respectively.

Proof: According to the proof of [61, Theorem 3], we know that

$$\Delta_{R_H}(r) = (\Pi \otimes id)\Delta(r).$$

This means that

$$\begin{aligned}
\Delta_{R_H}(r) &= (\Pi \otimes id)\Delta(r) \\
&= (\Pi \otimes id)\Delta(\eta(r \times 1)) \\
&= (\Pi \otimes id)(\eta \otimes \eta)\Delta'(r \times 1) \\
&= (\Pi_{R_H} \otimes \eta)\Delta'(r \times 1) \\
&= (\Pi_{R_H} \otimes \eta)(\eta^{-1} \otimes \eta^{-1})\Delta(r) \\
&= ((\Pi_{R_H} \circ \eta^{-1}) \otimes id)\Delta(r),
\end{aligned}$$

where Δ' is the comultiplications of $R_H \times H_0$. □

As stated in the previous section, we know that $\mathrm{gr}^a(H^*)$ is a finite-dimensional radically graded Hopf algebra over $\Bbbk$ with the Chevalley property. There exists a natural Hopf algebra epimorphism

$$\tau : \mathrm{gr}^a(H^*) \to H^*/J_{H^*}$$

with a retraction of the inclusion, where J_{H^*} is the radical of H^*. Furthermore, we have

$$\mathrm{gr}^a(H^*) \cong A_{H^*} \times H^*/J_{H^*},$$

where

$$A_{H^*} := \{h \in \mathrm{gr}^a(H^*) \mid (id \otimes \tau)\Delta(h) = h \otimes 1\}.$$

Lemma 6.3.2 *With the notations above, we have*

$$R_H \cong (A_{H^*})^*$$

as coalgebras.

Proof: We have

$$\mathrm{gr}^c(H) \cong (\mathrm{gr}^a(H^*))^*$$

as Hopf algebra. It follows from [52, Theorem 5.1] that

$$R_H \times H_0 \cong (A_{H^*} \times H^*/J_{H^*})^* \cong (A_{H*})^* \times H_0.$$

According to [61, Theorem 3], we know that

$$R_H \cong (A_{H^*})^*$$

as coalgebras. $\square$

In the following part, let $\mathrm{gr}^c(H)$ be a finite-dimensional Hopf algebra with the dual Chevalley property of tame corepresentation type. Combining Lemma 3.3.5 and Theorem 4.2.1, we know that one of the following three cases appears:

(i) $\mid \mathcal{P}^1 \mid = 1$ and $\mathcal{S}^1 = \{C\}$ for some $C \in \mathcal{S}$ with $\dim_{\Bbbk}(C) = 4$;

(ii) $\mid \mathcal{P}^1 \mid = 2$ and $\mathcal{S}^1 = \{\Bbbk g\}$ for some $g \in G(H)$;

(iii) $\mid \mathcal{P}^1 \mid = 2$ and $\mathcal{S}^1 = \{\Bbbk g, \Bbbk h\}$ for some $g, h \in G(H)$.

We need to determine which ideal in Theorem 6.1.2 can make $R_H \cong (\Bbbk\langle x, y\rangle / I)^*$ as coalgebras in the three cases. Next, we discuss these three cases separately.

§6.3.1 Cases (i)

Suppose $\mathcal{P}^1 = \{\mathcal{X}\}$ and $\mathcal{S}^1 = \{C\}$, where

$$\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix}$$

and C is a 4-dimensional simple subcoalgebra with basic multiplicative matrix

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

By the definition of primitive matrix, we have

$$\begin{aligned} \Delta(u) &= c_{11} \otimes u + c_{12} \otimes v + u \otimes 1, \\ \Delta(v) &= c_{21} \otimes u + c_{22} \otimes v + v \otimes 1. \end{aligned}$$

It is apparent that the subalgebra U of $\mathrm{gr}^c(H)$ generated by u, v is contained in R_H. We need to know $\Delta_{R_H}(r)$ for any $r \in U$.

Before proceeding further, let us give the following lemma.

Lemma 6.3.3 *With the notations above, we have* $\{c_{ij}u \mid 1 \leq i, j \leq 2\} \cup \{c_{ij}v \mid 1 \leq i, j \leq 2\}$ *are linearly independent in* $\mathrm{gr}^c(H)$*. Moreover, there exists an invertible matrix* $K = (k_{ij})_{4\times 4}$ *over* $\Bbbk$ *such that*

$$\mathcal{C} \odot' \mathcal{X} = K(\mathcal{X} \odot \mathcal{C}),$$

namely,

$$\begin{pmatrix} c_{11}u & c_{12}u \\ c_{21}u & c_{22}u \\ c_{11}v & c_{12}v \\ c_{21}v & c_{22}v \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} \begin{pmatrix} uc_{11} & uc_{12} \\ uc_{21} & uc_{22} \\ vc_{11} & vc_{12} \\ vc_{21} & vc_{22} \end{pmatrix}.$$

Proof: By [40, Proposition 2.6], there exists an invertible matrix L over $\Bbbk$ such that

$$\begin{aligned} & \begin{pmatrix} L & \\ & I \end{pmatrix} (\mathcal{C} \odot' \begin{pmatrix} \mathcal{C} & \mathcal{X} \\ & 1 \end{pmatrix}) \begin{pmatrix} L^{-1} & \\ & I \end{pmatrix} \\ = & \begin{pmatrix} L(\mathcal{C} \odot' \mathcal{C})L^{-1} & L(\mathcal{C} \odot' \mathcal{X}) \\ & \mathcal{C} \end{pmatrix} \\ = & \begin{pmatrix} \mathcal{D}_1 & & & & \\ & \mathcal{D}_2 & & & L(\mathcal{C} \odot' \mathcal{X}) \\ & & \ddots & & \\ & & & \mathcal{D}_u & \\ & & & & \mathcal{C} \end{pmatrix}, \end{aligned}$$

where $\mathcal{D}_1, \cdots, \mathcal{D}_u$ are the given basic multiplicative matrices. Using Corollary 3.1.4 and Lemma 3.2.1, we can show that $\{c_{ij}u \mid 1 \leq i, j \leq 2\} \cup \{c_{ij}v \mid 1 \leq i, j \leq 2\}$ are linearly independent in $\mathrm{gr}^c(H)$.
Let

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be an invertible matrix over $\Bbbk$, we know that

$$J(\mathcal{C} \odot' \mathcal{C})J^{-1} = \mathcal{C} \odot \mathcal{C}.$$

It follows that

$$\begin{aligned} & \begin{pmatrix} LJ^{-1} & \\ & I \end{pmatrix} (\begin{pmatrix} \mathcal{C} & \mathcal{X} \\ & 1 \end{pmatrix} \odot \mathcal{C}) \begin{pmatrix} JL^{-1} & \\ & I \end{pmatrix} \\ = & \begin{pmatrix} LJ^{-1}(\mathcal{C} \odot \mathcal{C})JL^{-1} & LJ^{-1}(\mathcal{X} \odot \mathcal{C}) \\ & \mathcal{C} \end{pmatrix} \\ = & \begin{pmatrix} \mathcal{D}_1 & & & & \\ & \mathcal{D}_2 & & & LJ^{-1}(\mathcal{X} \odot \mathcal{C}) \\ & & \ddots & & \\ & & & \mathcal{D}_u & \\ & & & & \mathcal{C} \end{pmatrix}. \end{aligned}$$

(1) Suppose

$$C \cdot C = E^{(4)},$$

where $E^{(4)} \in \mathcal{S}$ is a 16-dimensional simple subcoalgebra. We know that both $L(\mathcal{C} \odot' \mathcal{X})$ and $LJ^{-1}(\mathcal{X} \odot \mathcal{C})$ are non-trivial $(\mathcal{E}^{(4)}, \mathcal{C})$-primitive matrices, where $\mathcal{E}^{(4)} \in \mathcal{M}$ is the basic multiplicative matrix of $E^{(4)}$. From Corollary 3.1.14, there exists an invertible matrix $P_1 = \alpha I$ over $\Bbbk$ such that

$$P_1(L(\mathcal{C} \odot' \mathcal{X})) = LJ^{-1}(\mathcal{X} \odot \mathcal{C}).$$

(2) Suppose

$$C \cdot C = \Bbbk g + E^{(3)}$$

for some group-like element $g \in G(H)$ and some 9-dimensional simple subcoalgebra $E^{(3)} \in \mathcal{S}$. According to Corollary 3.1.14, there exists an invertible matrix

$$P_2 = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_2 & \\ & & & \alpha_2 \end{pmatrix}$$

over $\Bbbk$ such that

$$P_2(L(\mathcal{C} \odot' \mathcal{X})) = LJ^{-1}(\mathcal{X} \odot \mathcal{C}).$$

(3) Suppose

$$C \cdot C = E_1^{(2)} + E_2^{(2)}$$

for some 4-dimensional simple subcoalgebras $E_1^{(2)}, E_2^{(2)} \in \mathcal{S}$ and $E_1^{(2)} \neq E_2^{(2)}$. Using Corollary 3.1.14, we obtain an invertible matrix

$$P_3 = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_1 & & \\ & & \alpha_2 & \\ & & & \alpha_2 \end{pmatrix}$$

over $\Bbbk$ such that

$$P_3(L(\mathcal{C} \odot' \mathcal{X})) = LJ^{-1}(\mathcal{X} \odot \mathcal{C}).$$

(4) Suppose

$$C \cdot C = 2E^{(2)}$$

for some 4-dimensional simple subcoalgebra $E^{(2)} \in \mathcal{S}$. It follows from Proposition 3.1.13 that there exists an invertible matrix

$$P_4 = \begin{pmatrix} \alpha_1 & & \alpha_2 & \\ & \alpha_1 & & \alpha_2 \\ \alpha_3 & & \alpha_4 & \\ & \alpha_3 & & \alpha_4 \end{pmatrix}$$

over $\Bbbk$ such that

$$P_4(L(\mathcal{C} \odot' \mathcal{X})) = LJ^{-1}(\mathcal{X} \odot \mathcal{C}).$$

(5) Suppose

$$C \cdot C = \Bbbk g_1 + \Bbbk g_2 + \Bbbk g_3 + \Bbbk g_4$$

for some group-like elements $g_1, g_2, g_3, g_4 \in G(H)$. Note that g_1, g_2, g_3, g_4 are different with each other, otherwise the link quiver of $\mathrm{gr}^c(H)$ is not a Euclid diagram. By Corollary 3.1.14, there exists an invertible matrix

$$P_5 = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}$$

over $\Bbbk$ such that

$$P_5(L(\mathcal{C} \odot' \mathcal{X})) = LJ^{-1}(\mathcal{X} \odot \mathcal{C}).$$

Based on the above argument, there exists some $1 \leq i \leq 5$ such that invertible matrix $K = L^{-1}P_iLJ^{-1}$ over $\Bbbk$ satisfying

$$\mathcal{C} \odot' \mathcal{X} = K(\mathcal{X} \odot \mathcal{C}).$$

□

In fact, for any $r \in U$, $\Delta_{R_H}(r)$ is determined by the invertible matrix K in Lemma 6.3.3. Next we consider case (i) under the assumption that K is a diagonal matrix.

Lemma 6.3.4 *Let* $\mathrm{gr}^c(H) \cong (\Bbbk\langle x, y\rangle/I)^* \times H_0$ *be a finite-dimensional coradically graded Hopf algebra over* $\Bbbk$ *with the dual Chevalley property of tame corepresentation type. If* $\mathcal{P}^1 = \{\mathcal{X}\}$, $\mathcal{S}^1 = \{C\}$ *for some* $C \in \mathcal{S}$ *with* $\dim_{\Bbbk}(C) = 4$ *and the invertible matrix* K *in Lemma 6.3.3 is diagonal, namely*

$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

Then $I \neq (x^2 - y^2, yx - ax^2, xy)$, *where* $0 \neq a \in \Bbbk$.

Proof: It follows by direct computations that

$$\begin{aligned}
\Delta(uv) &= c_{11}c_{21}\otimes u^2 + c_{11}c_{22}\otimes uv + c_{12}c_{21}\otimes vu + c_{12}c_{22}\otimes v^2 + uv\otimes 1\\
&\quad + c_{11}v\otimes u + c_{12}v\otimes v + uc_{21}\otimes u + uc_{22}\otimes v,\\
\Delta(vu) &= c_{21}c_{11}\otimes u^2 + c_{21}c_{12}\otimes uv + c_{22}c_{11}\otimes vu + c_{22}c_{12}\otimes v^2 + vu\otimes 1\\
&\quad + c_{21}u\otimes u + c_{22}u\otimes v + vc_{11}\otimes u + vc_{12}\otimes v,\\
\Delta(u^2) &= c_{11}^2\otimes u^2 + c_{11}c_{12}\otimes uv + c_{12}c_{11}\otimes vu + c_{12}^2\otimes v^2 + u^2\otimes 1\\
&\quad + c_{11}u\otimes u + c_{12}u\otimes v + uc_{11}\otimes u + uc_{12}\otimes v,\\
\Delta(v^2) &= c_{21}^2\otimes u^2 + c_{21}c_{22}\otimes uv + c_{22}c_{21}\otimes vu + c_{22}^2\otimes v^2 + v^2\otimes 1\\
&\quad + c_{21}v\otimes u + c_{22}v\otimes v + vc_{21}\otimes u + vc_{22}\otimes v.
\end{aligned}$$

According to Lemma 6.3.1, we have

$$\Delta_{R_H}(uv) = 1\otimes uv + uv\otimes 1 + \alpha_3 v\otimes u + u\otimes v, \tag{6.1}$$

$$\Delta_{R_H}(vu) = 1\otimes vu + vu\otimes 1 + \alpha_2 u\otimes v + v\otimes u, \tag{6.2}$$

$$\Delta_{R_H}(u^2) = 1\otimes u^2 + u^2\otimes 1 + (\alpha_1+1)u\otimes u, \tag{6.3}$$

$$\Delta_{R_H}(v^2) = 1\otimes v^2 + v^2\otimes 1 + (\alpha_4+1)v\otimes v. \tag{6.4}$$

If

$$\dim_{\Bbbk}(R_H) = \dim_{\Bbbk}((\Bbbk\langle x,y\rangle/(x^2-y^2, yx-ax^2, xy))^*) = 4,$$

then

$$u^2, v^2, uv, vu \in \Bbbk\{(x^2)^*\}.$$

It follows that

$$\alpha_1 = \alpha_4 = -1,\ \ \alpha_2 = \frac{1}{\alpha_3}.$$

Thus we have

$$u^2 = v^2 = 0,\ \ uv = \alpha_2 vu.$$

We know that

$$(u^*)^2 = (v^*)^2 = 0,\ \ u^*v^* = \alpha_2 v^*u^*.$$

Hence

$$R_H^* \cong \Bbbk\langle x,y\rangle/(x^2, y^2, xy-\alpha_2 yx),$$

which indicates that

$$I \neq (x^2 - y^2, yx - ax^2, xy),$$

where $0 \neq a \in \Bbbk$. □

Next we consider whether or not $I = (x^n - y^n, xy, yx)$ in this case.

Lemma 6.3.5 *Let* $\mathrm{gr}^c(H) \cong (\Bbbk\langle x, y\rangle/I)^* \times H_0$ *be a finite-dimensional coradically graded Hopf algebra over* $\Bbbk$ *with the dual Chevalley property of tame corepresentation type. Suppose* $\mathcal{P}^1 = \{\mathcal{X}\}$, $\mathcal{S}^1 = \{C\}$ *for some* $C \in \mathcal{S}$ *with* $\dim_{\Bbbk}(C) = 4$ *and the invertible matrix* K *in Lemma 6.3.3 is diagonal, namely*

$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

If in addition R_H *is generated by* u, v, *then* $I \neq (x^n - y^n, xy, yx)$, *where* $n \geq 2$.

Proof: If $n = 2$, using the same argument as in the proof of Lemma 6.3.4, we can easily carry out the proof of this lemma.
If $n \geq 3$, we know that

$$(\Bbbk\langle x, y\rangle/I)^*(2) = \Bbbk\{(x^2)^*, (y^2)^*\}$$

and

$$\begin{aligned} \Delta((x^2)^*) &= (x^2)^* \otimes 1 + 1 \otimes (x^2)^* + x^* \otimes x^*, \\ \Delta((y^2)^*) &= (y^2)^* \otimes 1 + 1 \otimes (y^2)^* + y^* \otimes y^*. \end{aligned}$$

Without loss of generality, suppose that

$$u = k_1 x^* + k_2 y^*, \tag{6.5}$$

$$v = k_3 x^* + k_4 y^*, \tag{6.6}$$

$$u^2 = k_5 (x^2)^* + k_6 (y^2)^*, \tag{6.7}$$

$$v^2 = k_7 (x^2)^* + k_8 (y^2)^*, \tag{6.8}$$

$$uv = k_9 (x^2)^* + k_{10} (y^2)^*, \tag{6.9}$$

$$vu \quad = \quad k_{11}(x^2)^* + k_{12}(y^2)^*, \tag{6.10}$$

where $k_i \in \Bbbk$ for $1 \le i \le 12$. By substituting (6.5-6.10) into (6.1-6.4), we obtain

$$\begin{aligned}
(\alpha_1+1)k_1^2 x^* \otimes x^* &= k_5 x^* \otimes x^*,\\
(\alpha_1+1)k_1k_2 x^* \otimes y^* &= 0,\\
(\alpha_1+1)k_1k_2 y^* \otimes x^* &= 0,\\
(\alpha_1+1)k_2^2 y^* \otimes y^* &= k_6 y^* \otimes y^*,\\
(\alpha_4+1)k_3^2 x^* \otimes x^* &= k_7 x^* \otimes x^*,\\
(\alpha_4+1)k_3k_4 x^* \otimes y^* &= 0,\\
(\alpha_4+1)k_3k_4 y^* \otimes x^* &= 0,\\
(\alpha_4+1)k_4^2 y^* \otimes y^* &= k_8 y^* \otimes y^*.\\
(\alpha_2+1)k_1k_3 x^* \otimes x^* &= k_9 x^* \otimes x^*,\\
(\alpha_2k_1k_4+k_2k_3) x^* \otimes y^* &= 0,\\
(\alpha_2k_2k_3+k_1k_4) y^* \otimes x^* &= 0,\\
(\alpha_2+1)k_2k_4 y^* \otimes y^* &= k_{10} y^* \otimes y^*,\\
(\alpha_3+1)k_1k_3 x^* \otimes x^* &= k_{11} x^* \otimes x^*,\\
(\alpha_3k_2k_3+k_1k_4) x^* \otimes y^* &= 0,\\
(\alpha_3k_1k_4+k_2k_3) y^* \otimes x^* &= 0,\\
(\alpha_3+1)k_2k_4 y^* \otimes y^* &= k_{12} y^* \otimes y^*.
\end{aligned}$$

Comparing the coefficients of the both side, we have

$$(\alpha_1+1)k_1k_2 = 0. \tag{6.11}$$

If

$$k_1 = 0,$$

since

$$(\alpha_3k_1k_4 + k_2k_3) = 0,$$

it follows that

$$k_2 = 0$$

or

$$k_3 = 0,$$

which is in contradiction with the fact that u and v are linearly independent. A similar argument shows that

$$k_i \neq 0$$

for $1 \leq i \leq 4$. It follows from (6.11) that
$$\alpha_1 = -1.$$
Moreover, because of the fact that
$$(\alpha_4 + 1)k_3k_4 = 0,$$
we obtain
$$\alpha_4 = -1.$$
This indicates that
$$u^2 = v^2 = 0.$$
We claim that
$$\alpha_2 \neq -1.$$
Otherwise
$$k_9 = k_{10} = 0.$$
Hence
$$uv = 0,$$
a contradiction. Note that
$$\alpha_2(\alpha_2k_2k_3 + k_1k_4) - (\alpha_2k_1k_4 + k_2k_3) = 0,$$
direct computations shows that
$$\alpha_2 = 1.$$
Using the same argument, we can obtain
$$\alpha_3 = 1.$$
Thus we have
$$uv = vu,$$
which is a contradiction to $\dim_{\Bbbk}(R_H(2)) = 2$. The proof is completed. □

Now we turn to $I = (x^2, y^2, (xy)^m x - (yx)^m y)$.

Lemma 6.3.6 *Let* $\mathrm{gr}^c(H) \cong (\Bbbk\langle x, y\rangle/I)^* \times H_0$ *be a finite-dimensional coradically graded Hopf algebra over* $\Bbbk$ *with the dual Chevalley property of tame corepresentation type. Suppose* $\mathcal{P}^1 = \{\mathcal{X}\}$, $\mathcal{S}^1 = \{C\}$ *for some* $C \in \mathcal{S}$ *with* $\dim_{\Bbbk}(C) = 4$ *and the invertible matrix* K *in Lemma 6.3.3 is diagonal, namely*
$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

If in addition R_H *is generated by* u, v*, then* $I \neq (x^2, y^2, (xy)^m x - (yx)^m y)$*, where* $m \geq 1$.

Proof: Suppose that

$$\begin{aligned} u &= k_1 x^* + k_2 y^*, \\ v &= k_3 x^* + k_4 y^*, \\ u^2 &= k_5 (x^2)^* + k_6 (y^2)^*, \\ v^2 &= k_7 (x^2)^* + k_8 (y^2)^*, \end{aligned}$$

where $k_i \in \Bbbk$ for $1 \leq i \leq 8$. Similar to the proof of Lemma 6.3.5, we have

$$\begin{aligned} (\alpha_1 + 1)k_1^2 &= 0, \\ (\alpha_1 + 1)k_1 k_2 &= k_5, \\ (\alpha_1 + 1)k_1 k_2 &= k_6, \\ (\alpha_1 + 1)k_2^2 &= 0, \\ (\alpha_4 + 1)k_3^2 &= 0, \\ (\alpha_4 + 1)k_3 k_4 &= k_7, \\ (\alpha_4 + 1)k_3 k_4 &= k_8, \\ (\alpha_4 + 1)k_4^2 &= 0. \end{aligned}$$

It is straightforward to show that

$$\alpha_1 = \alpha_4 = -1$$

and thus

$$u^2 = v^2 = 0.$$

Since $(uv)^m u, (vu)^m v \in \Bbbk\{((xy)^m x)^*\}$, it follows that

$$(uv)^m u = k_9 (vu)^m v \tag{6.12}$$

for some $k_9 \in \Bbbk$. Note that

$$\begin{aligned} \Delta((uv)^m u) &= (\Delta(uv))^m \Delta(u) \\ &= (c_{11}c_{21} \otimes u^2 + c_{11}c_{22} \otimes uv + c_{12}c_{21} \otimes vu + c_{12}c_{22} \otimes v^2 + uv \otimes 1 \\ &\quad + c_{11} v \otimes u + c_{12} v \otimes v + u c_{21} \otimes u + u c_{22} \otimes v)^m (c_{11} \otimes u + c_{12} \otimes v + u \otimes 1), \\ \Delta((vu)^m v) &= (\Delta(vu))^m \Delta(v) \\ &= (c_{21}c_{11} \otimes u^2 + c_{21}c_{12} \otimes uv + c_{22}c_{11} \otimes vu + c_{22}c_{12} \otimes v^2 + vu \otimes 1 \\ &\quad + c_{21} u \otimes u + c_{22} u \otimes v + v c_{11} \otimes u + v c_{12} \otimes v)^m (c_{21} \otimes u + c_{22} \otimes v + v \otimes 1). \end{aligned}$$

It follows from (6.12) that

$$((uv)^m c_{11} + c_{11}(vu)^m) \otimes u = k_9((vu)^m c_{21} + c_{21}(uv)^m) \otimes u.$$

This means that

$$((\Pi_R \circ \eta^{-1}) \otimes id)(((uv)^m c_{11} + c_{11}(vu)^m) \otimes u) = k_9((\Pi_R \circ \eta^{-1}) \otimes id)(((vu)^m c_{21} + c_{21}(uv)^m) \otimes u).$$

It turns out that

$$((uv)^m + (-1)^m \alpha_3^m (vu)^m) = 0.$$

This contradicts the fact that R_H is generated by u, v and

$$\dim_{\Bbbk}(R_H(2m)) = \dim_{\Bbbk}((\Bbbk\langle x, y\rangle / I_2)(2m)) = 2.$$

Thus

$$I \neq (x^2, y^2, (xy)^m x - (yx)^m y),$$

where $m \geq 1$. □

For our purpose, we need to consider the following combinatorial functors:

$$\begin{aligned} H_1(m,l,t) &= \sum_{0 \leq m_1 \leq m_2 \leq \cdots \leq m_l \leq m-l} t^{\sum_{i=1}^{l} m_i}, \\ H_2(m,l,t) &= \sum_{0 \leq n_1 + n_2 + \cdots + n_l \leq m-l} t^{\sum_{i=1}^{l}(l+1-i)n_i}, \\ H_3(m,l,t) &= t^{m-l} \sum_{0 \leq n_1 + n_2 + \cdots + n_{l-1} \leq m-l} t^{\sum_{i=1}^{l-l}(l-i)n_i} + \sum_{0 \leq n_1 + n_2 + \cdots + n_l \leq m-l} t^{\sum_{i=1}^{l}(l+1-i)n_i}. \end{aligned}$$

Here $m, l \in \mathbb{Z}_+, 0 < l < m, m_1, \cdots, m_l, n_1, \cdots, n_l \in \mathbb{N}$ and t is an indeterminant.

Lemma 6.3.7 *([36, Lemma 3.1, Proposition 3.2]) We have*

(1) $H_1(m,l,t) = H_2(m,l,t) = H_3(m,l,t)$*;*

(2) $H_1(m,l,t) = 0$ *for all* $0 < l < m$ *if and only if* t *is an* m*th primitive root of unit.*

With the help of the preceding lemmas, we can get the main result for case (i).

Proposition 6.3.8 *Let* $\mathrm{gr}^c(H) \cong (\Bbbk\langle x, y\rangle / I)^* \times H_0$ *be a finite-dimensional coradically graded Hopf algebra over* $\Bbbk$ *with the dual Chevalley property of tame corepresentation type. Suppose* $\mathcal{P}^1 = \{\mathcal{X}\}$, $\mathcal{S}^1 = \{C\}$ *for some* $C \in \mathcal{S}$ *with* $\dim_{\Bbbk}(C) = 4$ *and the invertible matrix* K *in Lemma 6.3.3 is diagonal, namely*

$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

If in addition R_H is generated by u, v, then

(1) $I = (x^2, y^2, (xy)^m - a(yx)^m)$ *for* $0 \neq a \in \Bbbk$ *and* $m \geq 1$*;*

(2) $\alpha_1 = \alpha_4 = -1$*;*

(3) $a = (-1)^{m-1}\alpha_2^m$ *or* $a = (-1)^{m-1}\alpha_3^m$*;*

(4) $\alpha_2\alpha_3$ *is an mth primitive root of unity.*

Proof:

(1) Combining Theorem 6.1.2, Lemmas 6.3.4, 6.3.5 and 6.3.6, we know that

$$I = (x^2, y^2, (xy)^m - a(yx)^m)$$

for $0 \neq a \in \Bbbk$ and $m \geq 1$.

(2) An argument similar to the one used in the proof of Lemma 6.3.6 shows that

$$\alpha_1 = \alpha_4 = -1$$

and

$$u^2 = v^2 = 0.$$

(3) Note that

$$\begin{aligned}
\Delta((uv)^m) &= (\Delta(uv))^m \\
&= (c_{11}c_{21} \otimes u^2 + c_{11}c_{22} \otimes uv + c_{12}c_{21} \otimes vu + c_{12}c_{22} \otimes v^2 + uv \otimes 1 \\
&\quad + c_{11}v \otimes u + c_{12}v \otimes v + uc_{21} \otimes u + uc_{22} \otimes v)^m, \\
\Delta((vu)^m) &= (\Delta(vu))^m \\
&= (c_{21}c_{11} \otimes u^2 + c_{21}c_{12} \otimes uv + c_{22}c_{11} \otimes vu + c_{22}c_{12} \otimes v^2 + vu \otimes 1 \\
&\quad + c_{21}u \otimes u + c_{22}u \otimes v + vc_{11} \otimes u + vc_{12} \otimes v)^m.
\end{aligned}$$

Besides, in $(\Bbbk\langle x, y\rangle/(x^2, y^2, (xy)^m - a(yx)^m))^*$, we have

$$\begin{aligned}
\Delta(((xy)^m)^*) &= 1 \otimes ((xy)^m)^* + x^* \otimes (y(xy)^{m-1})^* + (xy)^* \otimes ((xy)^{m-1})^* \\
&+ \cdots + ((xy)^i)^* \otimes ((xy)^{m-i})^* + ((xy)^i x)^* \otimes (y(xy)^{m-1-i})^* \\
&+ + \cdots + ((xy)^{m-1}x)^* \otimes y^* + ((xy)^m)^* \otimes 1 \\
&+ \frac{1}{a}(1 \otimes ((yx)^m)^* + y^* \otimes (x(yx)^{m-1})^* + (yx)^* \otimes ((yx)^{m-1})^* \\
&+ \cdots + ((yx)^i)^* \otimes ((yx)^{m-i})^* + ((yx)^i y)^* \otimes (x(yx)^{m-1-i})^* \\
&+ + \cdots + ((yx)^{m-1}y)^* \otimes x^* + ((yx)^m)^* \otimes 1).
\end{aligned}$$

Suppose that

$$u = k_1 x^* + k_2 y^*, \tag{6.13}$$

$$v = k_3 x^* + k_4 y^*, \tag{6.14}$$

$$(vu)^m = k_5(((xy)^m)^*), \tag{6.15}$$

where $k_i \in \Bbbk$ for $1 \leq i \leq 5$. By substituting (6.13) and (6.14) into (6.15), we obtain

$$\begin{aligned}
(k_1(vu)^{m-1}v + k_3(\alpha_3)^m(-1)^{m-1}u(vu)^{m-1}) \otimes x^* &= k_5\frac{1}{a}((yx)^{m-1}y)^* \otimes x^*, \\
x^* \otimes (k_1(\alpha_3)^m(-1)^{m-1}(vu)^{m-1}v + k_3u(vu)^{m-1}) &= x^* \otimes k_5((yx)^{m-1}y)^*.
\end{aligned}$$

It follows that

$$k_1(\alpha_3)^m(-1)^{m-1} = \frac{1}{a}k_1$$

and

$$k_3 = k_3(\alpha_3)^m(-1)^{m-1}\frac{1}{a}.$$

If $k_1 = 0$ and $k_3 \neq 0$, then

$$a = (-1)^{m-1}(\alpha_3)^m.$$

If $k_1 \neq 0$ and $k_3 = 0$, then

$$a = (-1)^{m-1}(\alpha_2)^m.$$

If $k_1 \neq 0$ and $k_3 \neq 0$, then

$$a = (-1)^{m-1}(\alpha_3)^m = (-1)^{m-1}(\alpha_2)^m.$$

(4) We shall adopt the same procedure as in the proof of Lemma 6.3.6. Suppose that

$$(uv)^m = k_6(vu)^m,$$

for some $k_6 \in \Bbbk$. It follows from

$$\Delta((uv)^m) = k_6\Delta((vu)^m)$$

that

$$((uv)^{m-1}uc_{21} + c_{11}(vu)^{m-1}v) \otimes u = k_6((vu)^{m-1}vc_{11} + c_{21}(uv)^{m-1}v) \otimes u$$

and

$$((uv)^{m-1}uc_{22} + c_{12}(vu)^{m-1}v) \otimes v = k_6((vu)^{m-1}vc_{12} + c_{22}(uv)^{m-1}u) \otimes v.$$

Thus we have

$$
\begin{aligned}
& ((\Pi_R \circ \eta^{-1}) \otimes id)((((uv)^{m-1}uc_{21} + c_{11}(vu)^{m-1}v) \otimes u) \\
= \; & k_0((\Pi_R \circ \eta^{-1}) \otimes id)(k_6((vu)^{m-1}vc_{11} + c_{21}(uv)^{m-1}v) \otimes u),
\end{aligned}
$$

and

$$
\begin{aligned}
& ((\Pi_R \circ \eta^{-1}) \otimes id)((((uv)^{m-1}uc_{22} + c_{12}(vu)^{m-1}v) \otimes v) \otimes u) \\
= \; & k_0((\Pi_R \circ \eta^{-1}) \otimes id)(k_6((vu)^{m-1}vc_{12} + c_{22}(uv)^{m-1}u) \otimes v).
\end{aligned}
$$

Direct computations shows that

$$(-1)^{m-1}\alpha_3^m = k_6,$$

$$1 = k_6(-1)^{m-1}\alpha_2^m.$$

It follows that

$$(\alpha_2\alpha_3)^m = 1.$$

Note that for any element $f(u,v)$ generated by u, v, we can always write uniquely $\Delta(f(u,v))$ in the following form:

$$
\begin{aligned}
& f(u,v) \otimes 1 + (f(u,v))_u \otimes u + (f(u,v))_v \otimes v + (f(u,v))_{uv} \otimes uv + \cdots \\
+ \; & (f(u,v))_{(uv)^i} \otimes (uv)^i + (f(u,v))_{vu}^i \otimes (vu)^i + (f(u,v))_{(uv)^i u} \otimes (uv)^i u \\
+ \; & (f(u,v))_{(vu)^i v} \otimes (vu)^i v + \cdots .
\end{aligned}
$$

Since

$$(uv)^m = (-1)^{m-1}\alpha_3^m (vu)^m,$$

it follows that

$$(\Pi_R \circ \eta^{-1} \otimes id)\Delta((uv)^m) = (\Pi_R \circ \eta^{-1} \otimes id)\Delta((-1)^{m-1}\alpha_3^m (vu)^m).$$

But

$$\varepsilon(c_{12}) = \varepsilon(c_{21}) = 0,$$

this means that we only need to focus on

$$(c_{11}c_{22} \otimes uv + uv \otimes 1 + c_{11}v \otimes u + uc_{22} \otimes v)^m$$

and

$$(c_{22}c_{11} \otimes vu + vu \otimes 1 + c_{22}u \otimes v + vc_{11} \otimes u)^m.$$

Note that for any $0 < l < m$, u and v should appear alternately in the left items in $(uv)^m_{(uv)^l}$. By this observation, the items starting with u in $(uv)^m_{(uv)^l}$ are just

$$\sum_{0 \le n_1+n_2+\cdots+n_l \le m-l} (uv)^{n_1} c_{11}c_{22}(uv)^{n_2} c_{11}c_{22} \cdots c_{11}c_{22}(uv)^{n_l} c_{11}c_{22}(uv)^{n_{l+1}}.$$

But the items starting with u in $(vu)^m_{(vu)^l}$ is 0. This indicates that

$$\begin{aligned} & \sum_{0 \le n_1+n_2+\cdots+n_l \le m-l} (uv)^{n_1} c_{11}c_{22}(uv)^{n_2} c_{11}c_{22} \cdots c_{11}c_{22}(uv)^{n_l} c_{11}c_{22}(uv)^{n_{l+1}} \\ = & \sum_{0 \le n_1+n_2+\cdots+n_l \le m-l} (\alpha_2\alpha_3)^{n_1} (\alpha_2\alpha_3)^{n_1+n_2} \cdots (\alpha_2\alpha_3)^{n_1+n_2+\cdots+n_l} (c_{11}c_{22})^l (uv)^{m-l} \\ = & H_2(m, l, \alpha_2\alpha_3)(c_{11}c_{22})^l (uv)^{m-l} \\ = & 0. \end{aligned}$$

Using Lemma 6.3.7, we know that $\alpha_2\alpha_3$ is an mth primitive root of unity. □

Corollary 6.3.9 *With the notations in Proposition 6.3.8, if $m \ge 2$, then*

$$c_{11}c_{12} = c_{12}c_{11} = c_{21}c_{22} = c_{22}c_{21} = 0.$$

Proof: According to the proof of Proposition 6.3.8, we know that $u^2 = v^2 = 0$. This means that

$$\begin{aligned} \Delta(u^2) &= c_{11}^2 \otimes u^2 + c_{11}c_{12} \otimes uv + c_{12}c_{11} \otimes vu + c_{12}^2 \otimes v^2 + u^2 \otimes 1 \\ &\quad + c_{11}u \otimes u + c_{12}u \otimes v + uc_{11} \otimes u + uc_{12} \otimes v \\ &= 0, \\ \Delta(v^2) &= c_{21}^2 \otimes u^2 + c_{21}c_{22} \otimes uv + c_{22}c_{21} \otimes vu + c_{22}^2 \otimes v^2 + v^2 \otimes 1 \\ &\quad + c_{21}v \otimes u + c_{22}v \otimes v + vc_{21} \otimes u + vc_{22} \otimes v \\ &= 0. \end{aligned}$$

Since $m \ge 2$, it follows that uv, vu are linearly independent. Thus we have

$$c_{11}c_{12} = c_{12}c_{11} = c_{21}c_{22} = c_{22}c_{21} = 0.$$

□

To conclude, we only consider case (i) under the assumption that K in Lemma 6.3.3 is a diagonal matrix in this subsection. Indeed, at present, we do not know which ideal in Theorem 6.1.2 will occur without this assumption. But if K is given, we can solve it by the same way.

§6.3.2 Cases (ii) and (iii)

Proposition 6.3.10 *Let* $\mathrm{gr}^c(H) \cong (\Bbbk\langle x, y\rangle/I)^* \times H_0$ *be a finite-dimensional coradically graded Hopf algebra over* $\Bbbk$ *with the dual Chevalley property of tame corepresentation type.*

(1) If $|\mathcal{P}^1| = 2$ *and* $\mathcal{S}^1 = \{\Bbbk g\}$ *for some* $g \in G(H)$*, then* $I = (x^2, y^2, xy + yx)$*;*

(2) If $|\mathcal{P}^1| = 2$ *and* $\mathcal{S}^1 = \{\Bbbk g, \Bbbk h\}$ *for some* $g, h \in G(H)$*, then* $I = (x^2, y^2, (xy)^m - a(yx)^m)$*.*

Proof: It follows from Proposition 6.2.5 that the link-indecomposable component $(\mathrm{gr}(H))_{(1)}$ containing $\Bbbk 1$ is of tame corepresentation type. According to Proposition 3.3.13, in case (ii) and (iii), we have $(\mathrm{gr}(H))_{(1)}$ is a pointed Hopf algebra. So the desire conclusion comes from [36, Theorems 4.9 and 4.16]. □

Indeed, Proposition 6.3.10 can be obtained by the same reason in the proof of Lemmas 6.3.4, 6.3.5, 6.3.6 and Proposition 6.3.8. Moreover, using the same argument as in the proof of Proposition 6.3.8, we can easily carry out the proof of the following remark.

Remark 6.3.11 *Let* $\mathrm{gr}^c(H) \cong (\Bbbk\langle x, y\rangle/I)^* \times H_0$ *be a finite-dimensional coradical graded Hopf algebra with the dual Chevalley property of tame corepresentation type.*

(1) If $|\mathcal{P}^1| = 2$ *and* $\mathcal{S}^1 = \{\Bbbk g\}$ *for some* $g \in G(H)$*, suppose that* $gu = \alpha_1 ug + \alpha_2 vg, gv = \alpha_3 ug + \alpha_4 vg$ *for some* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Bbbk$*. Then* $\alpha_1 = \alpha_4 = -1, \alpha_2 = \alpha_3 = 0$*;*

(2) If $|\mathcal{P}^1| = 2$ *and* $\mathcal{S}^1 = \{\Bbbk g, \Bbbk h\}$ *for some* $g, h \in G(H)$*, assume that* $gu = \beta_1 ug, gv = \beta_2 vg, hu = \beta_3 uh, hv = \beta_4 vh$ *for some* $\beta_1, \beta_2, \beta_3, \beta_4 \in \Bbbk$*. Then*

  (i) $\beta_1 = \beta_4 = -1$*;*

  (ii) a in Proposition 6.3.10 equals $(-1)^{m-1}\beta_2$ *or* $(-1)^{m-1}\beta_3$*;*

  (iii) $\beta_2\beta_3$ *is an mth primitive root of unity.*

It should be pointed out that the above remark coincides with [36, Lemma 4.8, Proposition 4.15].

§6.4 Examples

As stated in the previous section, if H is a finite-dimensional coradically graded Hopf algebra over $\Bbbk$ with the dual Chevalley property of tame corepresentation type, one of the following three cases appears:

(i) $|\mathcal{P}^1| = 1$ and $\mathcal{S}^1 = \{C\}$ for some $C \in \mathcal{S}$ with $\dim_{\Bbbk}(C) = 4$;

(ii) $|\mathcal{P}^1| = 2$ and $\mathcal{S}^1 = \{\Bbbk g\}$ for some $g \in G(H)$;

(iii) $|\mathcal{P}^1| = 2$ and $\mathcal{S}^1 = \{\Bbbk g, \Bbbk h\}$ for some $g, h \in G(H)$.

Using Proposition 6.2.5, we know that H is of tame corepresentation type if and only if $H_{(1)}$ is of tame corepresentation type. In this section, we attempt to give several examples of finite-dimensional coradically graded link-indecomposable Hopf algebras over $\Bbbk$ with the dual Chevalley property of tame corepresentation type in the three cases.

In fact, if H is link-indecomposable, it follows from Lemma 6.2.1 that the coradical of H is generated by $\{\mathrm{span}(C) \mid C \in {}^1\mathcal{S}\} \cup \{\mathrm{span}(S(C)) \mid C \in {}^1\mathcal{S}\}$. In particular, combining [36, Lemma 2.1] and Proposition 3.3.13, we know that $(H_{(1)})_0$ is an abelian group in cases (ii) and (iii).

According to [36, Remark 4.10], we have

Lemma 6.4.1 *Let H be the algebra which is generated by g, u, v satisfying the following relations:*

$$gu = -ug, \;\; gv = -vg, \;\; uv = -vu, \;\; u^2 = v^2 = 0,$$

$$g^n = 1,$$

where n is an even number.
Moreover, the coalgebra structure and antipode are given by:

$$\Delta(g) = g \otimes g, \;\; \varepsilon(g) = 1, \;\; S(g) = g^{-1},$$

$$\Delta(u) = g \otimes u + u \otimes 1, \;\; \varepsilon(u) = 0, \;\; S(u) = -g^{-1}u,$$

$$\Delta(v) = g \otimes v + v \otimes 1, \;\; \varepsilon(v) = 0, \;\; S(v) = -g^{-1}v.$$

Then H is a coradically graded Hopf algebra of tame corepresentation type with $\mid \mathcal{P}^1 \mid = 2$ and $\mathcal{S}^1 = \{\Bbbk g\}$. Moreover, we have

$$H \cong (\Bbbk\langle x, y\rangle/(x^2, y^2, xy + yx))^* \times \Bbbk\langle g\rangle.$$

From [36, Remark 4.17(2)], we know that

Example 6.4.2 *Let H be the algebra which is generated by g, h, u, v satisfying the following relations:*

$$gh = hg, \;\; g^{n_1} = h^{n_2} = 1,$$

$$gu = -ug, \;\; gv = \alpha vg, \;\; hu = \beta uh, \;\; hv = -vh,$$

$$u^2 = v^2 = 0, (uv)^m = (-1)^{m-1}\beta^m (vu)^m,$$

where $n_1, n_2 \in \mathbb{Z}$, $\alpha\beta$ is an mth primitive root of unit and $m \mid l.c.m(n_1, n_2)$.
Moreover, the coalgebra structure and antipode are given by:

$$\Delta(g) = g \otimes g, \;\; \varepsilon(g) = 1, \;\; S(g) = g^{-1},$$

$$\Delta(h) = h \otimes h, \;\; \varepsilon(h) = 1, \;\; S(h) = h^{-1},$$

$$\Delta(u) = g \otimes u + u \otimes 1, \ \ \varepsilon(u) = 0, \ \ S(u) = -g^{-1}u,$$

$$\Delta(v) = h \otimes v + v \otimes 1, \ \ \varepsilon(v) = 0, \ \ S(v) = -h^{-1}v.$$

Then H is a coradically graded Hopf algebra of tame corepresentation type with $\mid \mathcal{P}^1 \mid = 2$ and $\mathcal{S}^1 = \{\Bbbk g, \Bbbk h\}$. Moreover, we have

$$H \cong (\Bbbk\langle x, y\rangle/(x^2, y^2, (xy)^m - (-1)^{m-1}\beta^m(yx)^m))^* \times \Bbbk\langle g, h\rangle.$$

In case (ii) and (iii), according to Proposition 6.3.10 and Remark 6.3.11, we know that only some special ideals of $\{(x^2, y^2, (xy)^m - a(yx)^m) \mid 0 \neq a \in \Bbbk, m \geq 1\}$ can appear and if one of them appears, then we can construct coradically graded Hopf algebra of tame corepresentation type over $H' = \Bbbk G$ for some $G = G(H)$ in Examples 6.4.1 and 6.4.2. However, in case (i), we do not know how to find all H' such that $(\Bbbk\langle x, y\rangle/I)^* \times H'$ is a Hopf algebra for some special ideals I listed in Theorem 6.1.2, even if the invertible matrix K in Lemma 6.3.3 is diagonal.

In the following part, we will give some examples of link-indecomposable coradically graded Hopf algebras of tame corepresentation type over 8-dimensional non-pointed cosemisimple Hopf algebras, such that the invertible matrix K in Lemma 6.3.3 is diagonal.

According to [51, Theorem 2. 13], we have

Lemma 6.4.3 *Non-pointed 8-dimensional semisimple Hopf algebras over $\Bbbk$ consist of 3 isomorphic classes, which are represented by*

$$(\Bbbk D_8)^*, \ \ (\Bbbk Q_8)^* \ \ , H_8,$$

where $D_8 = \langle x, y \mid x^4 = y^2 = 1, yx = x^{-1}y\rangle$ is the dihedral group and $Q_8 = \langle x, y \mid x^4 = 1, y^2 = x^2, yx = x^{-1}y\rangle$ is the quaternion group. Among these H_8 is the unique one that is neither commutative nor cocommutative, and is generated as an algebra by x, y, z with relations

$$x^2 = y^2 = 1, \ z^2 = \frac{1}{2}(1 + x + y - xy), \ yx = xy, \ zx = yz, \ zy = xz; \tag{6.16}$$

the coalgebra structure and antipode are given by:

$$\Delta(x) = x \otimes x, \ \Delta(y) = y \otimes y, \ \varepsilon(x) = \varepsilon(y) = 1, \tag{6.17}$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \ \varepsilon(z) = 1, \tag{6.18}$$

$$S(x) = x, \ S(y) = y, \ S(z) = z. \tag{6.19}$$

According to Lemma 6.2.1, when we consider link-indecomposable coradically graded Hopf algebras of tame corepresentaion type over 8-dimensional non-pointed cosemisimple Hopf algebras, we only need to consider case (i).

§6.4.1 Hopf algebras of tame corepresentation type over $(\Bbbk D_8)^*$

Let $\{e_{pq}\}_{p=0,1,2,3;q=0,1}$ be the basis of $(\Bbbk D_8)^*$, dual to the basis $\{x^p y^q\}_{p=0,1,2,3;q=0,1}$ of $\Bbbk D_8$. The multiplication and unit are given, respectively, by

$$e_{p_1q_1}e_{p_2q_2} = \delta_{p_1,p_2}\delta_{q_1,q_2}e_{p_1q_1}, \quad 1 = \sum_{p,q} e_{pq}, \tag{6.20}$$

the coalgebra structure and antipode are given by

$$\Delta(e_{pq}) \quad = \sum_{\substack{p_1+p_2+2q_1p_2 \equiv\ p\ mod\ 4 \\ q_1+q_2\equiv\ q\ mod\ 2}} e_{p_1q_1} \otimes e_{p_2q_2}, \tag{6.21}$$

$$\varepsilon(e_{pq}) \quad = \quad \delta_{p,0}\delta_{q,0}, \tag{6.22}$$

$$S(e_{pq}) \quad = \quad e_{p'q'}, \text{where } p + p' + 2qp' \equiv\ 0\ \ mod\ \ 4, \ \ q + q' \equiv\ 0\ \ mod\ \ 2. \tag{6.23}$$

It is easy to check that elements

$$\begin{aligned} X &= \sum_{pq}(-1)^p e_{pq}, \\ Y &= \sum_{pq}(-1)^q e_{pq} \end{aligned}$$

are group-like elements of order 2. Let

$$\begin{aligned} c_{11} &= e_{00} - \sqrt{-1}e_{10} - e_{20} + \sqrt{-1}e_{30}, \\ c_{12} &= \sqrt{-1}e_{01} + e_{11} - \sqrt{-1}e_{21} - e_{31}, \\ c_{21} &= -\sqrt{-1}e_{01} + e_{11} + \sqrt{-1}e_{21} - e_{31}, \\ c_{22} &= e_{00} + \sqrt{-1}e_{10} - e_{20} - \sqrt{-1}e_{30}, \end{aligned}$$

then

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

is a basic multiplicative matrix of C, where $C = \text{span}\{c_{11}, c_{12}, c_{21}, c_{22}\}$ Thus the simple subcoalgebras in $(\Bbbk D_8)^*$ are $\Bbbk 1, \Bbbk X, \Bbbk Y, \Bbbk XY, C$.

Next we try to construct a link-indecomposable coradically graded Hopf algebra H of tame corepresentaion type over $(\Bbbk D_8)^*$ such that the invertible matrix K in Lemma 6.3.3 is diagonal. Namely, suppose there exists an diagonal invertible matrix $K = (k_{ij})_{4\times 4}$ over $\Bbbk$ such that

$$\mathcal{C} \odot' \mathcal{X} = K(\mathcal{X} \odot \mathcal{C}),$$

where

$$\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix}$$

is a non-trivial $(\mathcal{C}, \Bbbk 1)$-primitive matrix, and

$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

According to Proposition 6.3.8, if

$$R_H = \{h \in H \mid (id \otimes \pi)\Delta(h) = h \otimes 1\}$$

is generated by u, v, we know that

$$\alpha_1 = \alpha_4 = -1.$$

Since

$$c_{11}c_{22} + c_{12}c_{21} = 1,$$

then

$$\begin{aligned} (c_{11}c_{22} + c_{12}c_{21})u &= -\alpha_2 u(c_{11}c_{22} + c_{12}c_{21}) \\ &= u(c_{11}c_{22} + c_{12}c_{21}). \end{aligned}$$

It follows that

$$\alpha_2 = -1.$$

Next we consider

$$(c_{11}c_{22} + c_{12}c_{21})v,$$

a similar argument shows that

$$\alpha_3 = -1.$$

Besides, we also have

$$c_{11}c_{22} - c_{12}c_{21} = Y,\ c_{11}^2 - c_{12}^2 = X,$$

thus $(\Bbbk D_8)^*$ is generated by span(C) and

$$Xu = uX,\ Yu = uY,\ Xv = vX,\ Yv = vY.$$

As a summary, we have

Example 6.4.4 *Let H be a Hopf algebra generated as an algebra by $\{e_{pq}\}_{p=0,1,2,3;q=0,1}, u, v$ satisfying*

(6.20) and the following relations:

$$\mathcal{C} \odot' \mathcal{X} = K(\mathcal{X} \odot \mathcal{C}),$$

$$u^2 = v^2 = 0, \ \ uv + vu = 0,$$

where

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} e_{00} - \sqrt{-1}e_{10} - e_{20} + \sqrt{-1}e_{30} & \sqrt{-1}e_{01} + e_{11} - \sqrt{-1}e_{21} - e_{31} \\ -\sqrt{-1}e_{01} + e_{11} + \sqrt{-1}e_{21} - e_{31} & e_{00} + \sqrt{-1}e_{10} - e_{20} - \sqrt{-1}e_{30} \end{pmatrix},$$

$$\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix},$$

and

$$K = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

The coalgebra structure and antipode are given by (6.21-6.23) and

$$\Delta(u) = c_{11} \otimes u + c_{12} \otimes v + u \otimes 1,$$

$$\Delta(v) = c_{21} \otimes u + c_{22} \otimes v + v \otimes 1,$$

$$\varepsilon(u) = \varepsilon(v) = 0,$$

$$S(u) = -(e_{00} - \sqrt{-1}e_{30} - e_{20} - \sqrt{-1}e_{10})u - (\sqrt{-1}e_{01} + e_{11} - \sqrt{-1}e_{21} - e_{31})v,$$

$$S(v) = -(-\sqrt{-1}e_{01} + e_{11} + \sqrt{-1}e_{21} - e_{31})u - (e_{00} + \sqrt{-1}e_{30} - e_{20} - \sqrt{-1}e_{10})v.$$

One can show that $H \cong (\Bbbk\langle x, y\rangle/(x^2, y^2, (xy)^2 + (yx)^2))^* \times (\Bbbk D_8)^*$*, and it is a link-indecomposable coradically graded Hopf algebra of tame corepresentaion type over* $(\Bbbk D_8)^*$*.*

§6.4.2 Hopf algebras of tame corepresentation type over $(\Bbbk Q_8)^*$

Let $\{e_{pq}\}_{p=0,1,2,3;q=0,1}$ be the basis of $(\Bbbk Q_8)^*$, dual to the basis $\{x^p y^q\}_{p=0,1,2,3;q=0,1}$ of $\Bbbk Q_8$. The multiplication and unit are given, respectively, by

$$e_{p_1q_1}e_{p_2q_2} = \delta_{p_1,p_2}\delta_{q_1,q_2}e_{p_1q_1}, \ \ 1 = \sum_{p,q} e_{pq}, \tag{6.24}$$

the coalgebra structure and antipode are given by

$$\Delta(e_{pq}) = \sum_{\substack{p_1+p_2+2q_1(p_2+q_2) \equiv \ p \ mod \ 4 \\ q_1+q_2 \equiv \ q \ mod \ 2}} e_{p_1q_1} \otimes e_{p_2q_2}, \tag{6.25}$$

$$\varepsilon(e_{pq}) = \delta_{p,0}\delta_{q,0}, \tag{6.26}$$

$$S(e_{pq}) = e_{p'q'}, \text{where } p+p'+2q(p'+q') \equiv 0 \ mod \ 4, \ \ q+q' \equiv 0 \ mod \ 2. \tag{6.27}$$

It is easy to check that elements

$$\begin{aligned} X &= \sum_{pq}(-1)^p e_{pq}, \\ Y &= \sum_{pq}(-1)^q e_{pq} \end{aligned}$$

are group-like elements of order 2. Let

$$\begin{aligned} c_{11} &= e_{00} + \sqrt{-1}e_{01} - e_{20} - \sqrt{-1}e_{21}, \\ c_{12} &= \sqrt{-1}e_{10} + e_{11} - \sqrt{-1}e_{30} - e_{31}, \\ c_{21} &= \sqrt{-1}e_{10} - e_{11} - \sqrt{-1}e_{30} + e_{31}, \\ c_{22} &= e_{00} - \sqrt{-1}e_{01} - e_{20} + \sqrt{-1}e_{21}, \end{aligned}$$

then

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

is a basic multiplicative matrix of C, where $C = \text{span}\{c_{11}, c_{12}, c_{21}, c_{22}\}$ Thus the simple subcoalgebras in $(\Bbbk Q_8)^*$ are $\Bbbk 1, \Bbbk X, \Bbbk Y, \Bbbk XY, C$.

Next we try to construct a link-indecomposable coradically graded Hopf algebra H of tame corepresentaion type over $(\Bbbk D_8)^*$ such that the invertible matrix K in Lemma 6.3.3 is diagonal. Namely, there exists an diagonal invertible matrix $K = (k_{ij})_{4\times 4}$ over $\Bbbk$ such that

$$\mathcal{C} \odot' \mathcal{X} = K(\mathcal{X} \odot \mathcal{C}),$$

where

$$\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix}$$

is a non-trivial $(\mathcal{C}, \Bbbk 1)$-primitive matrix, and

$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

Suppose that

$$R_H = \{h \in H \mid (id \otimes \pi)\Delta(h) = h \otimes 1\}$$

is generated by u, v. Since

$$c_{11}c_{22} - c_{12}c_{21} = 1,$$

an argument similar to the one used in Example 6.4.4 shows that

$$\alpha_i = -1$$

for $1 \leq i \leq 4$. We also have

$$c_{11}c_{22} + c_{12}c_{21} = X,\ c_{11}^2 - c_{12}^2 = Y,$$

it follows that $(\Bbbk D_8)^*$ is generated by $\mathrm{span}(C)$ and

$$Xu = uX,\ Yu = uY,\ Xv = vX,\ Yv = vY.$$

Based on the above argument, we have

Example 6.4.5 *Let H be a Hopf algebra generated as an algebra by $\{e_{pq}\}_{p=0,1,2,3;q=0,1}, u, v$ satisfying (6.24) and the following relations:*

$$\mathcal{C} \odot' \mathcal{X} = K(\mathcal{X} \odot \mathcal{C}),$$

$$u^2 = v^2 = 0,\ \ uv + vu = 0,$$

where

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} e_{00} + \sqrt{-1}e_{01} - e_{20} - \sqrt{-1}e_{21} & \sqrt{-1}e_{10} + e_{11} - \sqrt{-1}e_{30} - e_{31} \\ \sqrt{-1}e_{01} - e_{11} - \sqrt{-1}e_{30} + e_{31} & e_{00} - \sqrt{-1}e_{01} - e_{20} + \sqrt{-1}e_{21} \end{pmatrix},$$

$$\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix},$$

and

$$K = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

The coalgebra structure and antipode are given by (6.21-6.23) and

$$\Delta(u) = c_{11} \otimes u + c_{12} \otimes v + u \otimes 1,$$

$$\Delta(v) = c_{21} \otimes u + c_{22} \otimes v + v \otimes 1,$$

$$\varepsilon(u) = \varepsilon(v) = 0,$$

$$S(u) = -(e_{00} + \sqrt{-1}e_{21} - e_{20} - \sqrt{-1}e_{01})u - (\sqrt{-1}e_{30} + e_{31} - \sqrt{-1}e_{10} - e_{11})v,$$

$$S(v) = -(\sqrt{-1}e_{30} - e_{31} - \sqrt{-1}e_{10} + e_{11})u - (e_{00} - \sqrt{-1}e_{21} - e_{20} + \sqrt{-1}e_{01})v.$$

One can show that $H \cong (\Bbbk\langle x,y\rangle/(x^2,y^2,(xy)^2+(yx)^2))^* \times (\Bbbk Q_8)^*$, *and it is a link-indecomposable coradically graded Hopf algebra of tame corepresentaion type over* $(\Bbbk Q_8)^*$.

§6.4.3 Hopf algebras of tame corepresentation type over H_8

Note that the simple subcoalgebras in H_8 are $\Bbbk 1, \Bbbk c, \Bbbk b, \Bbbk bc, C$, where $C = \text{span}\{x, bx, cx, bcx\}$. We give a corresponding basic multiplicative matrix $\mathcal{C}$ of C, where

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x+bx & x-bx \\ cx-bcx & cx+bcx \end{pmatrix}. \tag{6.28}$$

Suppose there exists a link-indecomposable coradically graded Hopf algebra H of tame corepresentaion type over H_8 such that the invertible matrix K in Lemma 6.3.3 is diagonal. Namely, there exists an diagonal invertible matrix $K = (k_{ij})_{4\times 4}$ over $\Bbbk$ such that

$$\mathcal{C} \odot' \mathcal{X} = K(\mathcal{X} \odot \mathcal{C}),$$

where

$$\mathcal{X} = \begin{pmatrix} u \\ v \end{pmatrix}$$

is a non-trivial $(\mathcal{C}, \Bbbk 1)$-primitive matrix, and

$$K = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix}.$$

Note that

$$\begin{aligned} \Delta(c_{11}u) &= (c_{11}\otimes c_{11} + c_{12}\otimes c_{21})(c_{11}\otimes u + c_{12}\otimes v + u\otimes 1) \\ &= c_{11}^2 \otimes c_{11}u + c_{12}c_{11}\otimes c_{21}u + c_{11}c_{12}\otimes c_{11}v + c_{12}^2\otimes c_{21}v \\ &\quad + c_{11}u\otimes c_{11} + c_{12}u\otimes c_{21}, \end{aligned}$$

$$\begin{aligned} \Delta(uc_{11}) &= (c_{11}\otimes u + c_{12}\otimes v + u\otimes 1)(c_{11}\otimes c_{11} + c_{12}\otimes c_{21}) \\ &= c_{11}^2 \otimes uc_{11} + c_{11}c_{12}\otimes uc_{21} + c_{12}c_{11}\otimes vc_{11} + c_{12}^2\otimes vc_{21} \\ &\quad + uc_{11}\otimes c_{11} + uc_{12}\otimes c_{21}. \end{aligned}$$

According to Lemma 6.3.3, we know that $c_{21}u, c_{11}v$ are linearly independent. It follows that

$$c_{11}c_{12} = c_{12}c_{11},$$

which is a contradiction. Thus there exists no link-indecomposable coradically graded Hopf algebra H of tame corepresentaion type over H_8 such that the invertible matrix K in Lemma 6.3.3 is diagonal.

However, we have a link-indecomposable coradically graded Hopf algebra of tame corepresentaion type over H_8 such that K in Lemma 6.3.3 is not diagonal.

Example 6.4.6 *([66, Definition 5.18]) Let H be a Hopf algebra generated as an algebra by x, y, z, p_1, p_2 with relations (6.4.3) and*

$$p_1^2 = p_2^2 = 0, \;\; p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0,$$

$$xp_1 = p_1x, \;\; yp_1 = p_1y, \;\; xp_2 = -p_2x, \;\; yp_2 = -p_2y,$$

$$zp_1 = -p_1z, \;\; zp_2 = \sqrt{-1}p_2xz.$$

The coalgebra structure and antipode of H are given by (6.17-6.19) and

$$\Delta(p_1) = (f_{00} - \sqrt{-1}f_{11})z \otimes p_1 + (f_{10} + \sqrt{-1}f_{01})z \otimes p_2 + p_1 \otimes 1,$$

$$\Delta(p_2) = (f_{00} + \sqrt{-1}f_{11})z \otimes p_2 + (f_{10} - \sqrt{-1}f_{01})z \otimes p_1 + p_2 \otimes 1,$$

$$\varepsilon(p_1) = \varepsilon(p_2) = 0,$$

$$S(p_1) = -z(f_{00} - \sqrt{-1}f_{11}) - z(f_{10} + \sqrt{-1}f_{01})p_2,$$

$$S(p_2) = -z(f_{00} + \sqrt{-1}f_{11})p_2 - z(f_{10} - \sqrt{-1}f_{01})p_1,$$

where $f_{ij} = \frac{1}{4}[1 + (-1)^i x][1 + (-1)^k y], i, j = 0, 1$.
We know that

$$\mathcal{X} = \begin{pmatrix} p_1 + p_2 \\ -\sqrt{-1}(p_1 - p_2) \end{pmatrix}$$

is a non-trivial $(\mathcal{C}, 1)$-primitive matrix, where $\mathcal{C}$ is defined in (6.28). In this case,

$$K = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{-1}}{2} & -\frac{\sqrt{-1}}{2} & \frac{1}{2} \\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ \frac{\sqrt{-1}}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{-1}}{2} \\ \frac{1}{2} & \frac{\sqrt{-1}}{2} & -\frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{pmatrix},$$

and we can show that

$$H \cong (\Bbbk\langle x, y\rangle/(x^2, y^2, (xy)^2 + (yx)^2))^* \times H_8.$$

This means that H is of tame corepresentation type.

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博士期间科研成果

1 Jing Yu, Kangqiao Li, Gongxiang Liu, Hopf algebras with the dual Chevalley property of finite corepresentation type, Algebr. Represent. Theory 27 (2024), no. 5, 1821-1867.

2 Jing Yu, Gongxiang Liu, Coradically graded Hopf algebras with the dual Chevalley property of tame corepresentation type, arXiv:2407.21389.

3 Jing Yu, Gongxiang Liu, Hopf algebras with the dual Chevalley property of discrete corepresentation type, arXiv:2409.20292.

4 Jing Yu, Gongxiang Liu, Derived discrete Hopf algebras with the Chevalley property, arXiv:2412.08093.

致　　谢

四年的博士生涯即将画上句号，我为自己能够在南京大学学习感到幸运和自豪，同时我也由衷地感谢这段求学路上给予我支持、鼓励和帮助的老师、同学和朋友们。

首先，我要衷心感谢我的导师刘公祥教授，感谢他这几年在学习和生活上对我的关心、支持和帮助。这四年里，他带着我和我的同门将一本本晦涩难懂的教材从头到尾细致地学完，从 Hopf 代数到张量范畴，从三角范畴到倾斜理论，通过不断地学习，我的知识储备因此不断增加。在我报告讨论班的过程中，刘老师总能提出一些相关的问题让我去考虑，引导着我进行深入思考，这让我学习的更加透彻，也让我一直有着可以研究的问题。特别是在知道我对 quiver 很感兴趣后，刘老师就给了我 Hopf 代数表示型分类的相关问题，让我可以研究自己感兴趣的问题。而且在我做问题的过程中，每一次和刘老师的讨论，他都能给我下一步的指引，让我知道了后续的研究方向，少走了许多弯路。当我的研究止步不前的时候，刘老师往往能站在很高的角度给我提示，很多时候都可以让我豁然开朗。

我要感谢我的硕士导师韩刚副教授，感谢他对我读博士的关心与帮助。在他的严格要求下，我硕士阶段学习了李群李代数，代数表示论和同调代数等等知识，为我博士期间的学习奠定了基础。我要感谢丁南庆教授，黄兆泳教授在代数大组讨论班上提出的宝贵建议，拓宽了我的知识面，同时我也非常感激他们为我写的推荐信。我要感谢汪正方副教授，他的报告让我接触到了 gentle 代数，这对我刻画 Hopf 代数导出范畴中不可分解对象起到了至关重要的帮助。同时，也感谢汪老师对我们出去参加会议的经费支持。我要感谢我的授课老师杨东教授，张明敬老师，感谢我做助教时的老师李春副教授，感谢李方教授，任伟教授，杨涛副教授，感谢他们在不同方面对我的指导和帮助。我要感谢南京大学数学学院和数学学院全体工作人员，感谢他们创造了如此优越的学习环境和氛围。

我要感谢跟我一起参加讨论班的同学们，感谢周坤，李凤昌，王冬，徐玉莹，刘绵涛，张永亮，李康桥，李博文，何宇韬，左臻邦，王旭，苏杭，王梦君，李孟高，吴玥玥，刘子丰，阿依古丽，张梦蝶，赵卓一，甄翔钧，倪志扬，陈骏，白子正，朱红玉，卢星原，王德民，王聿澄，彭洪伟等等，他们的报告令我受益匪浅。我还要感谢所有和我一起报告张量范畴的博士和博士后们，感谢这个充满挑战的讨论班。

在我的同门师兄中，我要特别感谢李康桥副教授，他刨根问底的态度让我记忆尤新，每次和他讨论问题都让我收获满满。也是在与他的合作下，我才完成了我的第一篇论文。我还要感谢在学习之余和我一起打羽毛球的郑志伟，刘绵涛，徐玉莹，陈骏，彭洪伟，一起跑步的甄翔钧，一起买奶茶的李博文，吴玥玥，一起吃自助餐的李凤昌，一起玩桌游的李孟高……这些经历使我的身心得到了放松，让我能够以更饱满的精神状态投入到繁重的学习中。

最后感谢我的家人，感谢我的父母，他们是我坚强的后盾，无论我遇到什么困难，他们始终给予我无条件的支持。感谢我的女朋友段倩文，感谢她多年以来的忍耐，支持，关爱与陪伴。每当我因为研究的停滞不前而负面情绪缠身时，都是她耐心地安慰与鼓励才给了我继续研究的动力。感谢家人的理解、信任与支持，让我可以无所顾虑地完成我的学业！

俞 靖

2025 年 3 月