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Localization of a tensor category

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南京大学研究生毕业论文中文摘要首页用纸

毕业论文题目：张量范畴的局部化

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摘 要

本文的目标是在带有双正合张量积的阿贝尔幺半范畴的 Serre 商范畴上定义一种张量结构，以使得典范函子为幺半函子。在这个张量积下，我们证明了对双边 Serre 张量理想作商时，多环范畴的商范畴是多环范畴，多张量范畴的商范畴是多张量范畴，以及多融合范畴的商范畴是多融合范畴。在这种情况下，张量范畴的双边 Serre 张量理想总是平凡的。我们将这一结论推广到了任意的张量结构，即如果典范函子是幺半函子，那么对应的张量范畴的 Serre 子范畴是平凡的。

此外，我们发现有左对偶的多环范畴的双边 Serre 张量理想是成分子范畴的直和。因此，对应的商范畴同构于原来范畴的子范畴。我们也定义了分式范畴上的张量积，并证明了当乘法系是由 Serre 子范畴诱导时，Serre 商范畴与分式范畴作为幺半范畴是同构的。

关键词：局部化；商范畴；张量范畴

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THESIS: Localization of a tensor category

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ABSTRACT

The aim of this paper is to introduce a tensor structure for the Serre quotient category of an abelian monoidal category with biexact tensor product in order to make the canonical funtor a monoidal functor. In this tensor product, we prove that the quotient category of a multiring category (resp. a multitensor category, resp. a multifusion category) by a two-sided Serre tensor-ideal is still a multiring category (resp. a multitensor category, resp. a multifusion category). In this case, a two-sided Serre tensor-ideal of a tensor category is always trivial. This result can be generalized to any tensor product. We show that if the canonical functor is a monoidal functor, then the corresponding Serre subcategory of the tensor category is trivial.

Besides, we find that a two-sided Serre tensor-ideal of a multiring category with left duals is a direct sum of the component subcategories. Consequently, the corresponding quotient category is isomorphic to a subcategory of the original category. We also define a tensor product for the category of fractions and show that, when the multiplicative system is induced by a Serre subcategory, the Serre quotient category and the category of fractions are isomorphic as monoidal categories.

KEYWORDS: Localization; Quotient category; Tensor category

CONTENTS

Chapter 1 Introduction

1.1 Background

The quotient category $\mathcal{A}/\mathcal{C}$ of an abelian category $\mathcal{A}$ by a Serre subcategory $\mathcal{C}$ was introduced by Gabriel in [1], which is sometimes called the Serre quotient categories. Gabriel proved that the quotient category of an abelian category is still an abelian category, and the canonical functor $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is an exact functor. The process of obtaining the quotient category of an abelian category is often called the localization of an abelian category.

We collect several recent results about Serre quotient category as follows. An analogue of the the fundamental homomorphism theorem, which builds up on Serre quotient category, is introduced by Mohamed Barakat and Markus Lange-Hegermann in [2]. They also show that the coimage of a Gabriel monad is a Serre quotient category in [3], and research the Ext-computability of Serre quotient categories in [4]. Ramin Ebrahimi shows that the natural map $q^i_{X,A} : \mathrm{Ext}^i_{\mathcal{A}}(X, A) \to \mathrm{Ext}^i_{\mathcal{A}/\mathcal{C}}(q(X), q(A))$ is invertible in [5], where q is the canonical functor and X, A are objects in $\mathcal{A}$.

More generally, localization is a process of adding inverse maps to an algebraic structure. In particular, localization of a category means adding formal inverses to some morphisms. A widely used approach is the calculus of fractions, see Definition 5.2.1 in [6] and Chapter 1 in [7]. In fact, calculus of fractions for triangulated category is of great significance, and it is known as the Verdier quotient, see section 3.2 in [8] and Chapter 2 in [9].

A tensor category is an abelian monoidal category with a more complicated structure. It has a very close relation with Hopf algebras, for example the reconstruction theorem (see Theorem 5.3.12 in [10]). One can refer a systematic theory of tensor category to [10]. Moreover, multitensor category appears in [11]. For more recent results about tensor category, one can refer to [12]. Furthermore, monoidal category is close to topological field theory, see [13] and [14].

The original motivation of this thesis is to construct a new finite dimensional Hopf algebra via the reconstruction theorem for finite dimensional Hopf algebras. For a finite dimensional Hopf algebra H, the reconstruction theorem tell us that the representation category Rep(H) is a finite tensor category with a fiber functor. If the quotient category of Rep(H) by some Serre subcategory is still a finite tensor category, then it may be possible to construct a new finite dimensional Hopf algebra by applying the reconstruction theorem on the quotient category.

1.2 Main results

To answer the original motivation, we show in section 6.2 that if the canonical functor $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is a monoidal functor, where $\mathcal{A}$ is a tensor category and $\mathcal{C}$ is a Serre subcategory of $\mathcal{A}$, then $\mathcal{C}$ is trivial. This declares that our original motivation is trivial in the case of the canonical functor being a monoidal functor.

In addition to this, we define a tensor product for the quotient category of an abelian monoidal category with biexact tensor product, and show that the quotient category of a multiring category (resp. a multitensor category, resp. a multifusion category) by a two-sided Serre tensor-ideal is a multiring category (resp. a multitensor category, resp. a multifusion category). In this case, the canonical functor is a monoidal functor. We also define a tensor product for the category of fractions and show that, when the multiplicative system S is induced by a Serre subcategory, the Serre quotient category and the category of fractions are isomorphic as monoidal categories.

1.3 Organization

In Chapter 2, we present some basic definitions and results as preparation.

In Chapter 3, we reproduce the theory for the quotient category of an abelian category, and we provide more details. We show that the quotient category of an abelian category by a Serre subcategory is also an abelian category, and the canonical funtor is an exact functor.

In Chapter 4, we show that a Serre subcategory can induce a multiplicative system. Based on this, we show that the Serre quotient category and the category of fractions are isomorphic when the multiplicative system is induced by the Serre subcategory. Besides, we show that a multiplicative system can induce a Serre subcategory.

In Chapter 5, we focus on cases of abelian category. We show that the quotient category of a locally finite abelian category is a locally finite abelian category, and that the quotient category of a finite semisimple abelian category is still a finite semisimple abelian category.

In Chapter 6, we focus on cases of monoidal category. We provide a definition for tensor product in the quotient category of an abelian monoidal category with biexact tensor product, and study the quotient category of a multiring category (resp. a multitensor category, resp. a multifusion category) by a two-sided Serre tensor-ideal. Moreover, we show that a two-sided Serre tensor-ideal of a multiring category with left duals must be a direct sum of the component subcategories, and so be the corresponding quotient category. This implies the corresponding quotient category is actually isomorphic to a subcategory of the original category. In section 6.4, we define a tensor product for the category of fractions and show that the isomorphism functor in Chapter 4 is a monoidal functor.

Chapter 2 Preliminaries

Throughout the paper, k is an algebraically closed field. In this chapter, we recall some basic concepts and facts, one can refer to [8], [10], [15], and [16] for more details.

2.1 Finite abelian category

Recall that an additive category $\mathcal{A}$ is said to be k-linear if for any objects X, Y in $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is equipped with a structure of a vector space over k such that the composition of morphisms is k-linear. Besides, we say that an object X has finite length if its Jordan-Hölder series has finite length.

Definition 2.1 (Locally finite)**.** *A k-linear abelian category $\mathcal{A}$ is said to be locally finite if the following two conditions are satisfied:*

1. *for any two objects X, Y in $\mathcal{A}$, the vector space $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is finite dimensional;*
2. *every object in $\mathcal{A}$ has finite length.*

Definition 2.2 (Finite)**.** *A k-linear abelian category $\mathcal{A}$ is said to be finite if it is locally finite and in addition*

1. *$\mathcal{A}$ has enough projectives, i.e. every simple object in $\mathcal{A}$ has a projective cover;*
2. *there are finitely many isomorphism classes of simple objects.*

Recall that a finite k-linear abelian category is equivalent to the category A-mod of finite dimensional modules over a finite dimensional k-algebra A, see [10] Definition 1.8.5.

2.2 Tensor category, ring category and fusion category

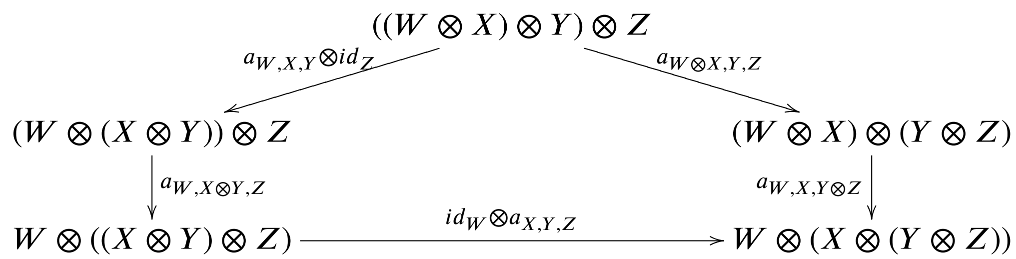
The following definitions refer to [10]

Definition 2.3. *A monoidal category is a quintuple* $(\mathcal{A}, \otimes, a, 1, \iota)$ *where* $\mathcal{A}$ *is a category,* $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *is a bifunctor called the tensor product bifunctor.* $a : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$ *is a natural isomorphism:*

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \quad X, Y, Z \in \mathcal{A}$$

called the associativity constraint (or associativity isomorphism), $1 \in \mathcal{A}$ *is an object of* $\mathcal{A}$ *and* $\iota : 1 \otimes 1 \to 1$ *is an isomorphism, subject the following two axioms.*

1. *The pentagon axiom. The diagram*



   is commutative for all objects W, X, Y, Z *in* $\mathcal{A}$.

2. *The unit axiom. The functors*

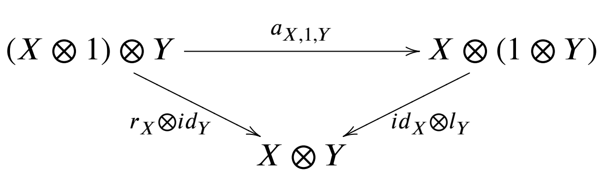
$$L_1 : X \to 1 \otimes X \text{ and}$$

$$R_1 : X \to X \otimes 1$$

   of left and right multiplication by 1 *are autoequivalences of* $\mathcal{A}$.

Recall the Definition 2.2.8 in [10]. It shows that a monoidal category can be alternatively defined as follows

Definition 2.4. *A monoidal category is a sextuple* $(\mathcal{A}, \otimes, a, 1, l, r)$ *satisfying the pentagon axiom, and the triangle axiom (that is, the following diagram is commutative)*



for all $X, Y \in \mathcal{A}$.

Definition 2.5. *Let $\mathcal{A}$ be a monoidal category. For any object X in $\mathcal{A}$, an object X^* is said to be a left dual of X if there exist an evaluation $ev_X : X^* \otimes X \to 1$ and a coevaluation $coev_X : 1 \to X \otimes X^*$ such that the compositions*

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X,$$

$$X^* \xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^*$$

are the identity morphisms.

An object ${}^*X$ is said to be a right dual of X if there exist an evaluation $ev'_X : X \otimes {}^*X \to 1$ and a coevaluation $coev'_X : 1 \to {}^*X \otimes X$ such that the compositions

$$X \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{ev'_X \otimes id_X} X$$

$${}^*X \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} {}^*X$$

are the identity morphisms.

Furthermore, an object X in $\mathcal{A}$ is said to be rigid if it has both left and right duals. $\mathcal{A}$ is said to be rigid if every object of $\mathcal{A}$ is rigid.

Next, we recall the definitions of a multiring category, a multitensor category and a multifusion category.

Definition 2.6. *A multiring category $\mathcal{A}$ over k is a locally finite k-linear abelian monoidal category with bilinear and biexact tensor product. If in addition $\mathrm{End}_{\mathcal{A}}(1) = k$, we will call $\mathcal{A}$ a ring category.*

Definition 2.7. *Let $\mathcal{A}$ be a locally finite k-linear abelian rigid monoidal category. We will call $\mathcal{A}$ a multitensor category over k if the bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is bilinear on morphisms. If in addition $\mathrm{End}_{\mathcal{A}}(1) = k$, and $\mathcal{A}$ is indecomposable i.e. $\mathcal{A}$ is not equivalent to a direct sum of nonzero multitensor categories, then we will call $\mathcal{A}$ a tensor category.*

Definition 2.8. *A multifusion category is a finite semisimple multitensor category. A fusion category $\mathcal{A}$ is a multifusion category with* $\mathrm{End}_{\mathcal{A}}(1) \cong k$*. i.e. a finite semisimple tensor category.*

By Proposition 4.2.1 in [10], any multitensor category is a multiring category, and any tensor category is a ring category. One can easily observe from the definition that any multifusion category ia s multitensor category, and any fusion category is a tensor category.

2.3 Serre subcategory

Definition 2.9 (Serre subcategory)**.** *Let $\mathcal{A}$ be an abelian category. A non-empty full additive subcategory $C \subset \mathcal{A}$ is a Serre subcategory provided that C is closed under taking subobjects, quotients and extensions.*

To be more precise, the definition is to say that for any exact sequence of $\mathcal{A}$

$$0 \to M' \to M \to M'' \to 0,$$

M is an object in C if and only if M' and M'' are objects in C.

Now, we give an equivalent definition for Serre subcategory.

Lemma 2.1. *A non-empty full additive subcategory C of $\mathcal{A}$ is a Serre subcategory if and only if for any $X' \to X \to X''$ exact in $\mathcal{A}$ with $X', X'' \in C$, then also $X \in C$.*

Proof. ($\Longleftarrow$) If $0 \to X' \to X \to X'' \to 0$ exact in $\mathcal{A}$, then $X' \to X \to X''$ exact means $X', X'' \in C \Longrightarrow X \in C$; $0 \to X' \to X$ exact means $X \in C \Longrightarrow X' \in C$; $X \to X'' \to 0$ exact means $X \in C \Longrightarrow X'' \in C$.

($\Longrightarrow$) If

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

is exact in $\mathcal{A}$, then we have the following short exact sequence.

$$0 \longrightarrow \mathrm{Im}\, f \longrightarrow X \xrightarrow{g} \mathrm{Im}\, g \longrightarrow 0$$

Since $\text{Im } f \cong X'/\ker f$ is a quotient of X' and $\text{Im } g$ is a subobject of X'', we have $X'/\ker f, \text{ Im } g \in \mathcal{C}$. Therefore, $X \in \mathcal{C}$. □

See Example 4.2.2 in [17] for the following example.

Example 2.1. *The full subcategory of torsion abelian groups is a Serre subcategory of abelian groups category.*

Proof. Recall that an abelian group G is said to be torsion if every element of G has finite order. Denote the full subcategory of torsion abelian groups by **Tors**. Given exact sequence in the category of abelian groups

$$0 \to G' \to G \to G'' \to 0.$$

If $G \in$ **Tors**, its subgroup G' and quotient G'' are also in **Tors**. If $G', G'' \in$ **Tors**, for any element $g \in G$, the image $\bar{g}$ has finite order in G'' that implies $\exists$ integer n such that $n\bar{g} = 0$ and then $ng \in G'$ i.e. $\exists$ integer m such that $mng = 0$. Thus, G is torsion. □

The next example is a classical result for Noetherian modules as it is closed under taking submodules, quotient modules and extensions. It is also correct for Artin modules.

Example 2.2. *The full subcategory of Noetherian modules is a Serre subcategory of R-module category.*

See Proposition 4.2.3 in [17] for the following example.

Example 2.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories. Then* ker *F is a Serre subcategory of $\mathcal{A}$.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. Suppose M', M'' are in the kernel of F, apply F to the exact sequence and get

$$0 \to 0 \to FM \to 0 \to 0$$

which means M is in the kernel of F. Conversely, if M is in the kernel of F, apply F to the exact sequence and get

$$0 \to FM' \to 0 \to FM'' \to 0$$

so M' and M'' are in ker F. □

In particular, let e be an idempotent element in an algebra A, the functor $res_e : A$-mod → B-mod by $res_e(-) = (-)e$ is an exact functor where $B = eAe \cong \text{End } eA$. The exactness see I.6.8 [18]. Hence, ker(res_e) is a Serre subcategory of A-mod.

Recall also Theorem 4.3.8 in [10] as the following:

Theorem 2.1. *1. In a ring category with left duals, the unit object* 1 *is simple.*

2. In a multiring category with left duals, the unit object 1 *is semisimple, and is a direct sum of pairwise non-isomorphic simple objects* 1_i.

Note that 1_i is the image of p_i, where $\{p_i\}_{i \in I}$ is the set of primitive idempotents of the algebra End(1). And $1 = \bigoplus_{i \in I} 1_i$.

We also write down the definition of two-sided Serre tensor-ideal. In some literature, for example in [19], 'tensor-ideal' is written as '⊗-ideal', and it means one side absorption. In some other literature, for example in [20], the name 'tensor-ideal' is used.

Definition 2.10. *Let $\mathcal{A}$ be an abelian monoidal category. A Serre subcategory C of $\mathcal{A}$ is called a two-sided Serre tensor-ideal of $\mathcal{A}$ if for any $X \in \mathcal{A}$, $Y \in C$, we have $X \otimes Y \in C$ and $Y \otimes X \in C$.*

2.4 Directed quasi-ordered set

Definition 2.11. *A set is said to be a quasi-ordered set if its order is reflexive and transitive. A quasi-ordered set I is said to be directed if for any pair $i, j \in I$, there is an index k with $i \leqslant k$ and $j \leqslant k$.*

The following lemma is common. One can refer to [16].

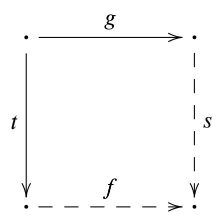
Lemma 2.2. *Let $\{A_i, \varphi^i_j\}$ be a direct system of modules over a directed quasi-ordered set I. Suppose $\varinjlim A_i = (\coprod A_i)/S$, where $S =< \{\lambda_j \varphi^i_j a_i - \lambda_i a_i | a_i \in A_i,$ and $i \leqslant j\} >$ and $\lambda_i : A_i \to \coprod A_i$ is the ith injection, then*

1. $\varinjlim A_i = \{\lambda_i a_i + S | i \in I\}$;
2. $\lambda_i a_i + S = 0 \Longleftrightarrow \varphi^i_t a_i = 0$ *for some* $t \geqslant i$.

2.5 Multiplicative system

Definition 2.12 (Multiplicative system)**.** *Let $\mathcal{A}$ be a category. A set of morphisms S in $\mathcal{A}$ is said to be a left multiplicative system if:*

1. *S is closed under composition of morphisms, and all identity morphisms are in S;*
2. *Every solid diagram*

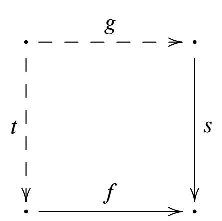


   with $t \in S$ can be completed to a commutative diagram with $s \in S$;

3. *For every pair of morphisms $f, g : X \to Y$ and a morphism $t : \cdot \to X$ in S such that $f \circ t = g \circ t$, there exists a morphism $s : Y \to \cdot$ in S such that $s \circ f = s \circ g$.*

A set of morphisms S in $\mathcal{A}$ is said to be a right multiplicative system if:

1. *S is closed under composition of morphisms, and all identity morphisms are in S;*
2. *Every solid diagram*



   with $s \in S$ can be completed to a commutative diagram with $t \in S$;

3. *For every pair of morphisms $f, g : X \to Y$ and a morphism $s : Y \to \cdot$ in S such that $s \circ f = s \circ g$, there exists a morphism $t : \cdot \to X$ in S such that $f \circ t = g \circ t$.*

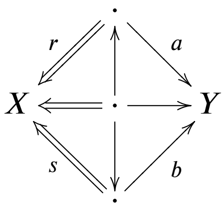
If S is both a left multiplicative system and a right multiplicative system, it is said to be a multiplicative system.

In the following, we will use the double arrow $s : \cdot \Rightarrow \cdot$ to imply $s \in S$.

Definition 2.13. *Let $\mathcal{A}$ be a category, S be a multiplicative system of $\mathcal{A}$, X, Y are objects in $\mathcal{A}$. A right fraction (or a right roof) from X to Y is a morphism diagram*

$$X \xLeftarrow[s]{} \cdot \xrightarrow{b} Y,$$

which is denoted by (b, s). We say two right fractions (a, r) and (b, s) are equivalent if there exists a commutative diagram as following.



We will denote the equivalent class of (b, s) by b/s.

Definition 2.14. *Let $\mathcal{A}$ be a category, S be a multiplicative system of $\mathcal{A}$, we define the quotient category $S^{-1}\mathcal{A}$ by*

1. $\mathrm{ob}(S^{-1}\mathcal{A}) = \mathrm{ob}(\mathcal{A})$;
2. *Morphisms in $S^{-1}\mathcal{A}$ are equivalent classes of right fractions.*

One can also define the morphisms in $S^{-1}\mathcal{A}$ to be equivalent classes of left fractions. In this article, we will use the right fractions. Besides, the quotient category $S^{-1}\mathcal{A}$ is also called as the localization of $\mathcal{A}$. In the following text, we denote the localization functor by $Q : \mathcal{A} \to S^{-1}\mathcal{A}$, which maps f to f/id.

Recall Proposition 3.6 in [7] Chapter 1.

Lemma 2.3. *Let $\mathcal{A}$ be an abelian category. If S is a multiplicative system of $\mathcal{A}$, then the category $S^{-1}\mathcal{A}$ is abelian and the functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ is exact.*

Chapter 3 Quotient category of an abelian category

The aim of this chapter is to systematically recall the definition of localization for an abelian category $\mathcal{A}$ by a Serre subcategory $\mathcal{C}$. One can refer to Part 3 in [1].

3.1 Definition of Serre quotient category

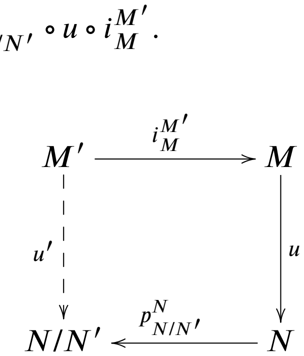
Given two objects M and N in $\mathcal{A}$, let M' and N' be subobjects of M and N respectively. The canonical morphisms

$$i_M^{M'} : M' \to M \text{ and } p_{N/N'}^N : N \to N/N'$$

induce a map

$$\mathrm{Hom}_{\mathcal{A}}(i_M^{M'}, p_{N/N'}^N) : \mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathcal{A}}(M', N/N').$$

To be more specific, for a morphism $u : M \to N$, there is a morphism $u' : M' \to N/N'$ that is actually $u' = p_{N/N'}^N \circ u \circ i_M^{M'}$.



Claim 1. *Given an abelian category $\mathcal{A}$ with a Serre subcategory $\mathcal{C}$, consider two objects M and N in $\mathcal{A}$, The set $\{\mathrm{Hom}_{\mathcal{A}}(M', N/N') | M/M', N' \in \mathcal{C}\}$, in which M' and N' are subobjects of M and N respectively, is a direct system.*

Proof. First, we define an order on the set $\{\mathrm{Hom}_{\mathcal{A}}(M', N/N') | M/M', N' \in \mathcal{C}\}$ by

$$\mathrm{Hom}_{\mathcal{A}}(M_1', N/N_1') \preceq \mathrm{Hom}_{\mathcal{A}}(M_2', N/N_2') \text{ if } M_2' \subset M_1', N_1' \subset N_2'$$

with $M/M_1', N_1', M/M_2', N_2' \in \mathcal{C}$. This definition is reasonable because of the following commutative diagram.

$$\begin{array}{ccccc} M_2' & \xrightarrow{i^{M_2'}_{M_1'}} & M_1' & \xrightarrow{i^{M_1'}_{M}} & M \\ \downarrow & & \downarrow & & \downarrow \\ N/N_2' & \xleftarrow{p^{N/N_1'}_{N/N_2'}} & N/N_1' & \xleftarrow{p^{N}_{N/N_1'}} & N \end{array}$$

By observing the commutative diagram, one can readily define a map

$$\begin{aligned} \phi^{(M_1', N/N_1')}_{(M_2', N/N_2')} : \mathrm{Hom}_{\mathcal{A}}(M_1', N/N_1') &\rightarrow \mathrm{Hom}_{\mathcal{A}}(M_2', N/N_2') \\ f &\rightarrow p^{N}_{N/N'} \circ f \circ i^{M'}_{M} \end{aligned}$$

and obtain following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(M_1', N/N_1') & \xrightarrow{\phi^{(M_1', N/N_1')}_{(M_3', N/N_3')}} & \mathrm{Hom}_{\mathcal{A}}(M_3', N/N_3') \\ {\scriptstyle \phi^{(M_1', N/N_1')}_{(M_2', N/N_2')}} \searrow & & \nearrow {\scriptstyle \phi^{(M_2', N/N_2')}_{(M_3', N/N_3')}} \\ & \mathrm{Hom}_{\mathcal{A}}(M_2', N/N_2') & \end{array}$$

This means $\{\mathrm{Hom}_{\mathcal{A}}(M', N/N') | M/M', N' \in \mathcal{C}\}$ is a direct system. □

Definition 3.1 (Quotient Category)**.** *The category $\mathcal{A}/\mathcal{C}$ is defined as follows:*

1. *the objects of $\mathcal{A}/\mathcal{C}$ coincide with the objects of $\mathcal{A}$.*
2. *the set of morphisms from M to N is defined by*

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) := \varinjlim_{M', N'} \mathrm{Hom}_{\mathcal{A}}(M', N/N')$$

in which M' and N' go through the subobjects of M and N respectively such that $M/M', N \in \mathcal{C}$.

Figuring out the composition in $\mathcal{A}/\mathcal{C}$ is of great significance. Let $\bar{f}$ be an element of $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$, and let $\bar{g}$ be an element of $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(N, P)$. Denote

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \varinjlim_{M',N'} \mathrm{Hom}_{\mathcal{A}}(M', N/N') = (\bigoplus \mathrm{Hom}_{\mathcal{A}}(M', N/N'))/I_1;$$
$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(N, P) = \varinjlim_{N'',P'} \mathrm{Hom}_{\mathcal{A}}(N'', P/P') = (\bigoplus \mathrm{Hom}_{\mathcal{A}}(N'', P/P'))/I_2;$$

Here, I_1 is the subgroup of $\bigoplus \mathrm{Hom}_{\mathcal{A}}(M', N/N')$ generated by

$$\lambda_{(M'_j, N/N'_j)}\phi^{(M'_i, N/N'_i)}_{(M'_j, N/N'_j)}(f_i) - \lambda_{(M'_i, N/N'_i)}(f_i) \text{ for all } f_i \in \mathrm{Hom}_{\mathcal{A}}(M'_i, N/N'_i),$$

where $\mathrm{Hom}_{\mathcal{A}}(M'_i, N/N'_i) \leq \mathrm{Hom}_{\mathcal{A}}(M'_j, N/N'_j)$, and $\lambda_{(M'_i, N/N'_i)} : \mathrm{Hom}_{\mathcal{A}}(M'_i, N/N'_i) \to \bigoplus \mathrm{Hom}_{\mathcal{A}}(M', N/N')$ is the embedding. I_2 is the subgroup of $\bigoplus \mathrm{Hom}_{\mathcal{A}}(N'', P/P'))$ satisfying a similar condition of I_1. Note that the indexed set is directed, thus Lemma 2.2 is valid here, and we obtain a morphism $f : M' \to N/N'$ such that $\bar{f} = f + I_1$ and a morphism $g : N'' \to P/P'$ such that $\bar{g} = g + I_2$ with $M/M', N', N/N'', P'$ are objects of $\mathcal{C}$.

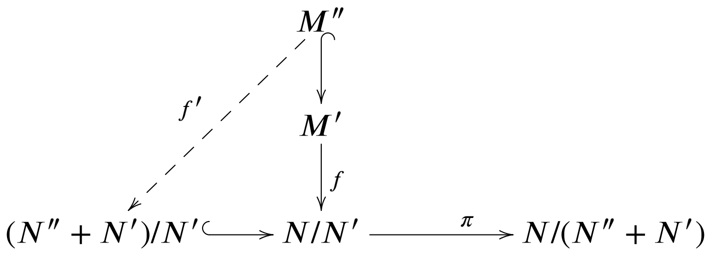
Recall that the sum of a family of subobjects $(X_i)_{i\in I}$ of an object X is defined to be

$$\sum_{i\in I} X_i = \mathrm{Im}(\bigoplus_{i\in I} X_i \to X),$$

and the intersection of a family of subobjects $(X_i)_{i\in I}$ of an object X is defined to be

$$\bigcap_{i\in I} X_i = \ker(X \to \prod_{i\in I} X/X_i).$$

Note that $(N'' + N')/N'$ is the kernel of projection $\pi : N/N' \to N/(N'' + N')$. Let $M'' = \ker(\pi \circ f)$. By the universal property, one can get $f' : M'' \to (N''+N')/N'$ and following commutative diagram.



In some papers, such as Gabriel's thesis [1], M'' is written as $f^{-1}((N''+N')/N')$. Since

$$M'/M'' = M'/\ker(\pi\circ f) \cong \mathrm{Im}(\pi\circ f) \subset N/(N''+N')$$

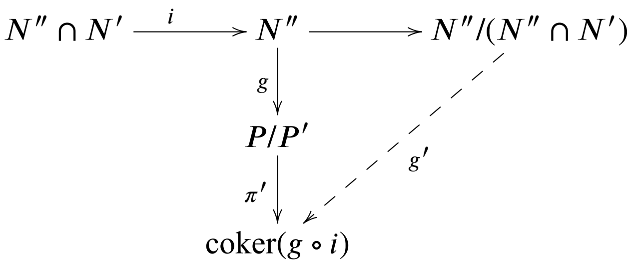
and note that

$$N/(N''+N') \cong \frac{N/N''}{(N''+N')/N''} \in \mathcal{C},$$

we have $M'/M'' \in \mathcal{C}$. Also note that $M/M' \in \mathcal{C}$, the property of closing under taking extension tells that $M/M'' \in \mathcal{C}$ because of the following short exact sequence.

$$0 \to M'/M'' \to M/M'' \to M/M' \to 0$$

On the other hand, by the universal property of cokernel, we can obtain a morphism $g' : N''/(N''\cap N') \to \mathrm{coker}(g\circ i)$ such that the following diagram commute



where $\pi' : P/P' \to P/P''$ is the projection. In fact, $\mathrm{coker}(g\circ i)$ is a quotient object of P/P', and P/P' is a quotient object of P. Hence, $\mathrm{coker}(g\circ i)$ is a quotient object of P. We can write that $\mathrm{coker}(g\circ i) \cong P/P''$ where $P'' = \ker(P \to P/P' \to \mathrm{coker}(g\circ i))$. Note that $g(N''\cap N')$ is an object of $\mathcal{C}$ since $\mathcal{C}$ is closed under taking subobject and quotient. Because

$$P''/P' = \ker(\pi') = g(N''\cap N') \in \mathcal{C}$$

and $P' \in C$, the exact sequence

$$0 \to P' \to P'' \to P''/P' \to 0$$

means that $P'' \in C$. In Gabriel's thesis [1], P'' is written as $P' + g(N'' \cap N')$.

Therefore, we get a composition

$$g' \circ \upsilon \circ f' : M'' \to (N'' + N')/N' \cong N''/(N'' \cap N') \to P/P'',$$

where $\upsilon : (N'' + N')/N' \cong N''/(N'' \cap N')$ is the canonical isomorphism, with $M/M'', P'' \in C$. We want to define $\bar{g} \circ \bar{f}$ to be the image of $g' \circ \upsilon \circ f'$ in $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, P)$. It remains to check that the composition does not rely on the choice of f and g. One can refer to section 3.2 in [21] for another proof.

Proposition 3.1. *Let $\bar{f}$ be an element of* $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$*, and let $\bar{g}$ be an element of* $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(N, P)$*. The composition of $\bar{f}$ and $\bar{g}$ defined in the above manner does not rely on the choice of f and g.*

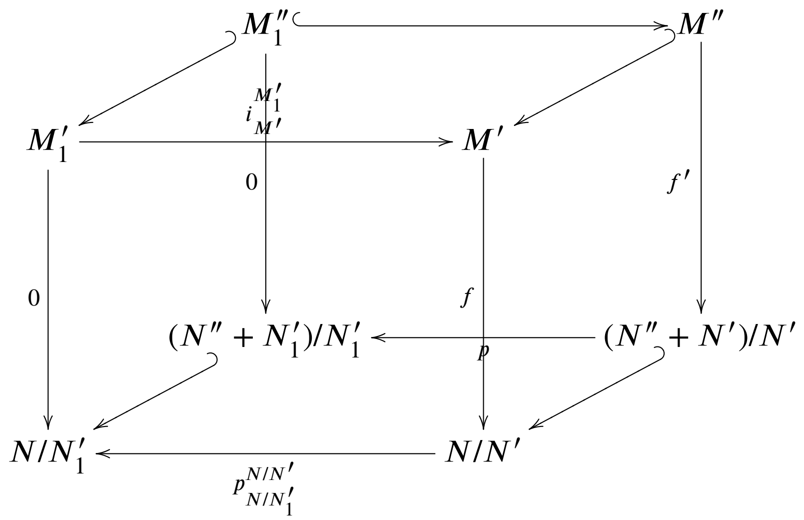
Proof. Denote

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, P) = \varinjlim_{M'', P'} \mathrm{Hom}_{\mathcal{A}}(M'', P/P') = (\bigoplus \mathrm{Hom}_{\mathcal{A}}(M'', P/P'))/I.$$

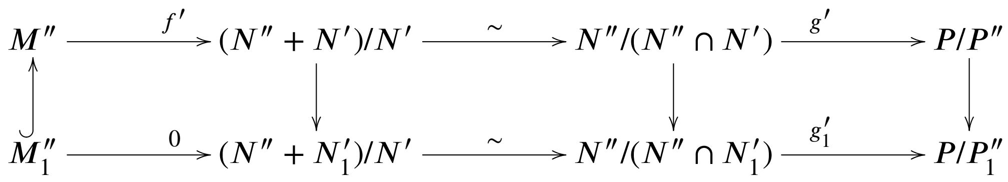
First of all, we prove that the composition of a morphism with a zero morphism is still a zero morphism. Assume $\bar{f}$ is the zero morphism. By Lemma 2.2, there is (M_1', N_1') such that $\phi^{(M', N/N')}_{(M_1', N/N_1')}(f) = 0$. This means the following commutative diagram.

$$\begin{array}{ccc} M_1' & \xrightarrow{i^{M_1'}_{M'}} & M' \\ \downarrow 0 & & \downarrow f \\ N/N_1' & \xleftarrow[p^{N/N'}_{N/N_1'}]{} & N/N' \end{array}$$

One can also find the following commutative diagram, the back square commutes because it is the restriction of the front square.

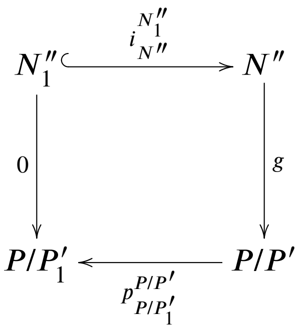


Thus, we have the following commutative diagram.

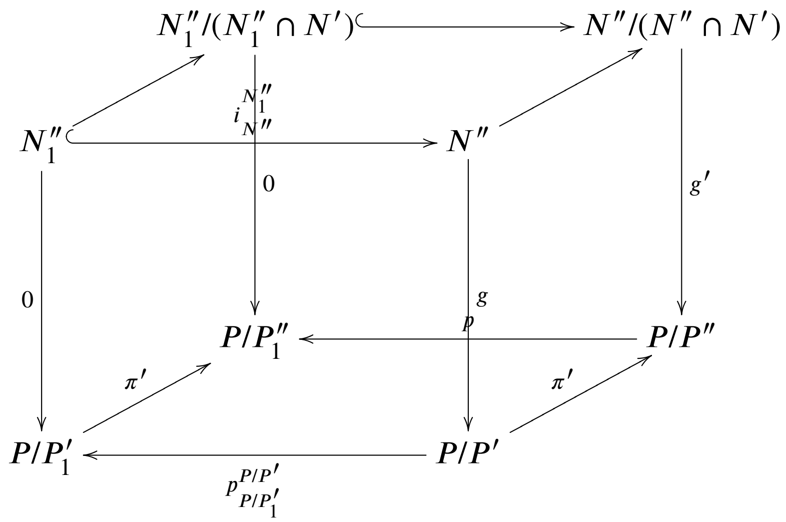


This means $g' \circ f' + I = g_1' \circ 0 + I = I$ i.e. $\bar{g} \circ \bar{f} = 0$.

On the other hand, suppose $\bar{g} = 0$, then $g : N'' \to P/P' \in I_2$. By Lemma 2.2, there is (N_1'', P_1') such that $\phi^{(N'', P/P')}_{(N_1'', P/P_1')}(g) = 0$ i.e. the following diagram commutes.



By the construction of composition, we have the following commutative diagram. The back square commutes because it is the projection of the front square.



Consequently, we can induce a commutative diagram from the back square as following.

$$
\begin{array}{ccccccc}
M'' & \xrightarrow{f'} & (N'' + N')/N' & \xrightarrow{\sim} & N''/(N'' \cap N') & \xrightarrow{g'} & P/P'' \\
\uparrow & & \uparrow & & \uparrow & & \downarrow p \\
M_1'' & \xrightarrow{f_1'} & (N_1'' + N')/N' & \xrightarrow{\sim} & N_1''/(N_1'' \cap N') & \xrightarrow{0} & P/P_1''
\end{array}
$$

This implies $g' \circ f' + I = 0 \circ f_1' + I = I$ i.e. $\bar{g} \circ \bar{f} = 0$.

Now, we consider the composition of morphisms in general. Suppose $\bar{f}_1 = \bar{f}_2$ and $\bar{g_1} = \bar{g_2}$. We want to show

$$\bar{g_1} \circ \bar{f}_1 = \bar{g_2} \circ \bar{f}_2.$$

Since $\bar{f}_1 - \bar{f}_2 = \bar{g_2} - \bar{g_1} = 0$, it is clear that

$$\bar{g_1} \circ (\bar{f}_1 - \bar{f}_2) = 0 = (\bar{g_2} - \bar{g_1}) \circ \bar{f}_2.$$

Consequently,

$$\bar{g_1} \circ \bar{f}_1 - \bar{g_1} \circ \bar{f}_2 = \bar{g_2} \circ \bar{f}_2 - \bar{g_1} \circ \bar{f}_2.$$

Thus,

$$\bar{g_1} \circ \bar{f}_1 = \bar{g_2} \circ \bar{f}_2.$$

□

3.2 Serre quotient category is an abelian category

Proposition 3.2. *Given an abelian category $\mathcal{A}$ with a Serre subcategory C, the quotient category $\mathcal{A}/C$ is an additive category.*

Proof.

1. The Hom-sets have structure of abelian group, because the direct limit of abelian groups is also an abelian group.
2. The distributive law is also correct, because Proposition 3.1 has proven the composition does not rely on the choice.
3. Because $\mathcal{A}$ has all the finite coproducts, so does $\mathcal{A}/C$.

□

The quotient functor (or say the canonical functor) T from $\mathcal{A}$ to $\mathcal{A}/C$ is defined to be

$$TM = M \text{ for any object } M \in \mathcal{A}$$

and for any morphism $f \in \mathrm{Hom}_{\mathcal{A}}(M, N)$

$$\begin{aligned} T : \mathrm{Hom}_{\mathcal{A}}(M, N) &\to \mathrm{Hom}_{\mathcal{A}/C}(M, N) \\ f &\to f + I \end{aligned}$$

Here, I is the subgroup of $\bigoplus \mathrm{Hom}_{\mathcal{A}}(M', N/N')$ generated by

$$\lambda_{(M'_j, N/N'_j)} \phi^{(M'_i, N/N'_i)}_{(M'_j, N/N'_j)}(f_i) - \lambda_{(M'_i, N/N'_i)}(f_i) \text{ for all } f_i \in \mathrm{Hom}_{\mathcal{A}}(M'_i, N/N'_i),$$

where $\mathrm{Hom}_{\mathcal{A}}(M'_i, N/N'_i) \leq \mathrm{Hom}_{\mathcal{A}}(M'_j, N/N'_j)$, and $\lambda_{(M'_i, N/N'_i)} : \mathrm{Hom}_{\mathcal{A}}(M'_i, N/N'_i) \to \bigoplus \mathrm{Hom}_{\mathcal{A}}(M', N/N')$ is the embedding. Note that $f + I$ is indeed in $\mathrm{Hom}_{\mathcal{A}/C}(M, N)$, because one can choose $M' = M$ and $N' = 0$, hence $M/M' = N' = 0 \in C$ and the following diagram commutes

$$\begin{array}{ccc} M' = M & \xrightarrow{f} & N = N/0 = N/N' \\ \Big\downarrow {\scriptstyle i^M_M} & & \Big\uparrow {\scriptstyle p^N_N} \\ M & \xrightarrow{\quad f \quad} & N. \end{array}$$

This implies that the quotient functor is well-defined.

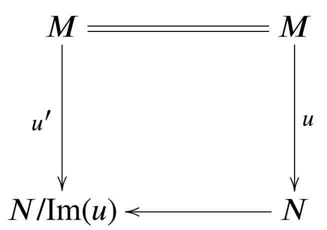
In fact, the quotient functor T is an additive functor. One can refer to [1] Lemma 1 in Part 3. Additionally, Lemma 2.2 means that every morphism $\bar{f}$ can be written as Tf for some f in the direct system.

Next, we use the approach of [1] to show that $\mathcal{A}/\mathcal{C}$ is an abelian category. Firstly, we introduce a Lemma which is Lemma 2 in [1].

Lemma 3.1. *Let $u : M \to N$ be a morphism of $\mathcal{A}$, then*

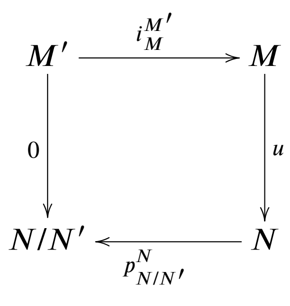
1. *Tu is zero* $\Longleftrightarrow \operatorname{Im} u \in \mathcal{C}$*;*
2. *Tu is a monomorphism* $\Longleftrightarrow \ker u \in \mathcal{C}$*;*
3. *Tu is an epimorphism* $\Longleftrightarrow \operatorname{coker} u \in \mathcal{C}$*;*

Proof. 1. ($\Longleftarrow$) Suppose $\operatorname{Im} u \in \mathcal{C}$, then we have $u' : M \to N/\operatorname{Im}(u)$ that is located in the direct system satisfying the following commutative diagram.



Thus, $u' = 0$, which means $Tu = Tu' = 0$.

($\Longrightarrow$) $Tu = 0$ means the following commutative diagram.



That is $0 = \operatorname{Im}(0) = \operatorname{Im}(p^N_{N/N'} \circ u \circ i^{M'}_M) = (u(M') + N')/N'$. This implies $u(M') \subset N'$ and then $u(M') \in \mathcal{C}$. Note that

$$u(M') = \operatorname{Im}(u \circ i^{M'}_M) \cong M'/\ker(u \circ i^{M'}_M) = M'/(\ker u \cap M') \cong (\ker u + M')/\ker u,$$

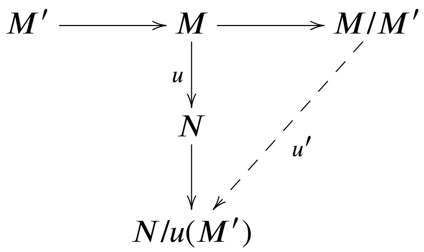
Therefore, we get a short exact sequence

$$0 \to u(M') \to \operatorname{Im} u \to M/(\ker u + M') \to 0.$$

Note that $M/(\ker u + M')$ is in $\mathcal{C}$ because it is a quotient of M/M'. Since $\mathcal{C}$ is closed under taking extensions, $\operatorname{Im} u$ is in $\mathcal{C}$.

2.($\Longrightarrow$) Denote the embedding by $i : \ker u \to M$. It is clear $u \circ i = 0$, which means $Tu \circ Ti = 0$. With the knowing that Tu is a monomorphism, the equation implies $Ti = 0$. By 1., we have $\ker u = \operatorname{Im} i \in \mathcal{C}$.

($\Longleftarrow$) Given a morphism $0 \neq \bar{f} : P \to M$ in $\mathcal{A}/\mathcal{C}$, we want to show $Tu \circ \bar{f} \neq 0$. Choose an image $f : P' \to M/M'$ in direct system with $P/P',\ M' \in \mathcal{C}$. We can assume M' contains $\ker u$ since it is lawful to replace M' by $M' + \ker u$. Therefore, u induces a monomorphism $u' : M/M' \to N/u(M')$ such that the following diagram commute.



Note that $\bar{f} \neq 0$, which implies $\operatorname{Im} f \notin \mathcal{C}$ by 1.. This means $\operatorname{Im}(u' \circ f) \notin \mathcal{C}$ since $\operatorname{Im} f$ is a subobject of $\operatorname{Im}(u' \circ f)$. Thus, $Tu \circ \bar{f} \neq 0$.

3.Similar to the proof of 2. □

The following Lemma is Lemma 3 in [1].

Lemma 3.2. *Let $f : M \to N$ be a morphism in $\mathcal{A}$, then*

$$\ker(Tf) = T(\ker f) \textit{ and } \operatorname{coker}(Tf) = T(\operatorname{coker} f).$$

Proof. Denote $\ker f$ by (K, k), we have

$$(Tf)(Tk) = T(fk) = T(0) = 0.$$

We now prove the universal property of kernel for (TK, Tk). If $\bar{p} : TP \to TM$ satisfies $Tf \circ \bar{p} = 0$, we want to find a $\bar{q}$ realizing the following commutative diagram.

$$\begin{array}{ccccc} & & TP & & \\ & \swarrow^{\bar{q}} & \downarrow{\scriptstyle \bar{p}} & & \\ TK & \xrightarrow{Tk} & TM & \xrightarrow{Tf} & TN \end{array}$$

Note that $\bar{p} = Tp$ is the image of a morphism $p : P' \to M/M'$ where $P/P',\ M' \in \mathcal{C}$, and f induces $f' : M/M' \to N/f(M')$. Because of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \\ 0 & \longrightarrow & f(M') & \longrightarrow & N & \longrightarrow & N/f(M') & \longrightarrow & 0, \end{array}$$

the snake lemma indicates the following exact sequence

$$0 \longrightarrow K \cap M' \longrightarrow K \longrightarrow \ker(f') \longrightarrow 0.$$

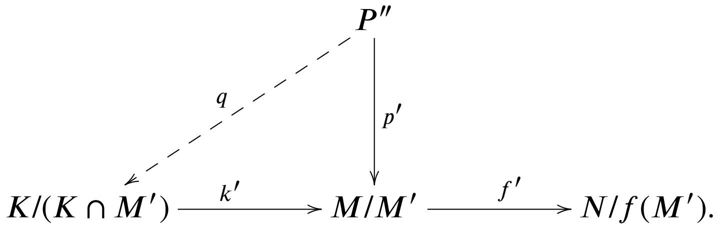
This means that $\ker(f') = K/(K \cap M')$. Also note that k induces $k' : K/(K \cap M') \to M/M'$, then $(K/(K \cap M'), k')$ is the kernel of f'.

Let $P'' = \ker(f' \circ p)$ and $p' : P'' \to M/M'$ that is induced by p. Note that $T(f' \circ p) = (Tf)\bar{p} = 0$, which means $\mathrm{Im}(f' \circ p) \in \mathcal{C}$ by Lemma 3.1. Consequently, $P'/P'' = P'/\ker(f' \circ p) \cong \mathrm{Im}(f' \circ p) \in \mathcal{C}$ and $P/P' \in \mathcal{C}$. The short exact sequence

$$0 \to P'/P'' \to P/P'' \to P/P' \to 0$$

implies $P/P'' \in \mathcal{C}$.

Denote by $i : P'' \to P'$ the injection. It is clear that $f'p' = (f'p)i = 0$ and there is a morphism $q : P'' \to K/(K \cap M')$ making the following diagram commute



Thus, Tq is a morphism such that

$$\bar{p} = T(p') = T(k'q) = T(k')T(q) = T(k)T(q).$$

This means we have proven the universal property, and then $\ker(Tf) = T(\ker f)$.

One can prove $\mathrm{coker}(Tf) = T(\mathrm{coker}\, f)$ by a similar procedure. □

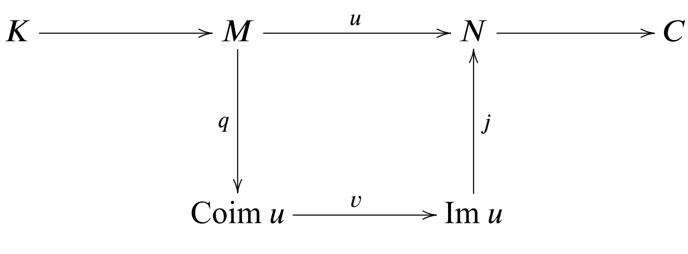
We give an equivalent description for isomorphisms in $\mathcal{A}/\mathcal{C}$, which is Lemma 4 in [1].

Proposition 3.3. *Let $u : M \to N$ be a morphism in $\mathcal{A}$, then*

$$Tu \textit{ is an isomorphism } \iff \ker u \textit{ and } \mathrm{cokernel}\, u \textit{ belong to } \mathcal{C}.$$

Proof. ($\Longrightarrow$) By Lemma 3.1, it is obvious.

($\Longleftarrow$) Consider the canonical factorization of u.



Note that $\mathrm{Coim}\, u = M/\ker q = M/K$. We know that $id_{\mathrm{Coim}\, u} \in \mathrm{Hom}_{\mathcal{A}}(\mathrm{Coim}\, u, M/K)$. Since $K \in \mathcal{C}$, the image $Tid_{\mathrm{Coim}\, u}$ is in $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(\mathrm{Coim}\, u, M)$. The following composition means that $Tid_{\text{Coim } u}$ is actually an inverse of Tq in $\text{Hom}_{\mathcal{A}/\mathcal{C}}(\text{Coim } u, M)$.

$$\begin{array}{ccccc} & & M & \xrightarrow{q} & \text{Coim u} \\ & & \downarrow & & \parallel \\ M/K & \xrightarrow{id_{\text{Coim } u}} & M/K & \xrightarrow{id} & M/K \end{array}$$

This shows that Tq is an isomorphism. Similarly, one can show that Tj is an isomorphism. Thus

$$Tu = Tj \circ Tv \circ Tq$$

is an isomorphism. □

Theorem 3.1. *Given an abelian category $\mathcal{A}$ with a Serre subcategory $\mathcal{C}$, the quotient category $\mathcal{A}/\mathcal{C}$ is an abelian category.*

Proof. By Proposition 3.2, $\mathcal{A}/\mathcal{C}$ is an additive category. Given a morphism $\bar{f} : M \to N$ in $\mathcal{A}/\mathcal{C}$. We know that $\bar{f}$ is the image of $f : M' \to N/N'$ where $M/M', N' \in \mathcal{C}$. By Lemma 3.2, $\ker(\bar{f}) = \ker(Tf) = T(\ker f)$ and $\text{coker}(\bar{f}) = \text{coker}(Tf) = T(\text{coker } f)$ which exist since $\mathcal{A}$ is an abelian category. Also note that

$$\text{coker ker } \bar{f} = T(\text{coker ker } f) \cong T(\text{ker coker } f) = \text{ker coker } \bar{f}.$$

Consequently, $\mathcal{A}/\mathcal{C}$ is an abelian category. □

In fact, we apply T on the canonical factorization of f, and then we can obtain the canonical factorization of $\bar{f}$ as shown in the right side of the following diagram

$$\begin{array}{ccc} M' & \xrightarrow{f} & N/N' \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle j} \\ \text{Coim } f & \xrightarrow{v} & \text{Im } f, \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\bar{f}} & N \\ {\scriptstyle Tq}\downarrow & & \uparrow{\scriptstyle Tj} \\ T(\text{Coim } f) & \xrightarrow{Tv} & T(\text{Im } f). \end{array}$$

By Lemma 3.1, Tv is an isomorphism, Tq is an epimorphism, and Tj is a monomorphism. In addition,

$$\begin{aligned}\mathrm{Coim}(\bar{f}) &= \mathrm{Coim}(Tf) = \mathrm{coker}(\ker(Tf)) \\ &= T(\mathrm{coker}(\ker f)) = T(\mathrm{Coim}(f)),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Im}(\bar{f}) &= \mathrm{Im}(Tf) = \ker(\mathrm{coker}(Tf)) \\ &= T(\ker(\mathrm{coker}\, f)) = T(\mathrm{Im}(f)).\end{aligned}$$

Now, we have already known that the quotient category of an abelian category by a Serre subcategory is also an abelian category. It follows that the canonical functor is an exact functor.

Proposition 3.4. *Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ be a Serre subcategory, then the canonical functor $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is an exact functor.*

Proof. Recall that T is left exact if and only if T preserves kernels and T is right exact if and only if T preserves cokernels. By Lemma 3.2, we obtain that T is an exact functor. □

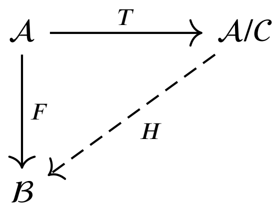
In some viewpoints, the purpose of localization is to make some objects become isomorphic. Now, we say that the zero objects in $\mathcal{A}/\mathcal{C}$ are actually those objects in $\mathcal{C}$. This proposition is a direct corollary of Lemma 3.1.

Proposition 3.5. *Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$. For any M in $\mathcal{A}$, $M \cong 0$ in $\mathcal{A}/\mathcal{C}$ if and only if $M \in \mathcal{C}$.*

Proof. $M \cong 0$ in $\mathcal{A}/\mathcal{C}$ if and only if $T(id_M) = 0$ in $\mathcal{A}/\mathcal{C}$. By Lemma 3.1, $T(id_M) = 0$ in $\mathcal{A}/\mathcal{C}$ if and only if $M = \mathrm{Im}(id_M) \in \mathcal{C}$. □

For two abelian categories $\mathcal{A}$ and $\mathcal{B}$, let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. By Example 2.3, we know that $\ker F$ is a Serre subcategory of $\mathcal{A}$. As a result, we can obtain a quotient category $\mathcal{A}/\ker F$ with a canonical functor $T : \mathcal{A} \to \mathcal{A}/\ker F$. By proposition 3.5, $\ker T = \ker F$. A natural question is under what conditions $\mathcal{B}$ is equivalent to $\mathcal{A}/\ker F$. In order to study their relation, Gabriel proved the universal property for quotient category of an abelian category, refer to Corollary 2 and Corollary 3 in [1].

Proposition 3.6. *Let $\mathcal{A}$ be an abelian category, C be a Serre subcategory of $\mathcal{A}$. Let F be an exact functor from $\mathcal{A}$ to an abelian category $\mathcal{B}$. If $F(M) = 0$ for any object M in C, then there is a unique exact functor $H : \mathcal{A}/C \to \mathcal{B}$ such that $F = H \circ T$ where T is the canonical functor.*



However, the universal property does not answer the question directly. The following proposition is a rewrite of Proposition 5 in [1], which provides a condition to make $\mathcal{B} \cong \mathcal{A}/\ker F$.

Proposition 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ be an exact functor. Denote the canonical functor by $T : \mathcal{A} \to \mathcal{A}/\ker F$. If there is a functor S right adjoint to F such that $F \circ S \cong id_{\mathcal{B}}$ is a natural isomorphism, then F induces an equivalence between $\mathcal{A}/\ker F$ and $\mathcal{B}$.*

In fact, for abelian categories $\mathcal{A}$ and $\mathcal{B}$, a functor $S : \mathcal{B} \to \mathcal{A}$ is said to be a section functor of an exact functor $F : \mathcal{A} \to \mathcal{B}$ if S is a right adjoint to F such that $F \circ S \cong id_{\mathcal{B}}$ is a natural isomorphism. We note that Proposition 3.3 in [2] is a more recent result, as shown below.

Proposition 3.8. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact and essentially surjective functor of abelian categories which admits a section functor up to extension. Then F induces an equivalence between $\mathcal{A}/\ker F \cong \mathcal{B}$ where $\ker F$ is a thick torsion subcategory of $\mathcal{A}$.*

Chapter 4 Calculus of fractions

4.1 Multiplicative system from a Serre subcategory

Recall that proposition 3.3 said Tu is an isomorphism if and only if $\ker u,\ \operatorname{coker} u \in \mathcal{C}$. In calculus of fractions, we choose a set of morphisms to construct a multiplicative system in order to make them into isomorphisms. One can refer the following proposition to Lemma 2.2.4 in [8].

Proposition 4.1. *Let $\mathcal{A}$ be an abelian category and $\mathcal{C}$ be a Serre subcategory. Denote by $S = \{f \in \mathrm{Mor}\mathcal{A} | \ker f,\ \operatorname{coker} f \in \mathcal{C}\}$. Then S is a multiplicative system.*

Proof. 1).

1. $\forall X \in \mathcal{A}$, $\ker id_X = \operatorname{coker} id_X = 0$, thus $id_X \in S$.
2. Given

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with $f, g \in S$. Note that there is a commutative diagram

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \longrightarrow & \operatorname{coker} f & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle gf} & & \downarrow{\scriptstyle g} & & \downarrow & & \\ 0 & \longrightarrow & C & \xrightarrow{id} & C & \longrightarrow & 0 & & \end{array}$$

By snake lemma, this implies an exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker(gf) \longrightarrow \ker g \longrightarrow \operatorname{coker} f \longrightarrow \operatorname{coker}(gf) \longrightarrow \operatorname{coker} g \longrightarrow 0$$

which means

$$\begin{cases} \ker f \to \ker gf \to \ker g \\ \operatorname{coker} f \to \operatorname{coker} gf \to \operatorname{coker} g \end{cases}$$

exact. Note that $\ker f$, $\ker g$, $\operatorname{coker} f$, $\operatorname{coker} g \in \mathcal{C}$ as $f, g \in S$. By the above lemma, $\ker gf$ and $\operatorname{coker} gf$ i.e. $gf \in S$.

2).Given

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{f} & A \end{array}$$

with $s \in S$. Consider its pullback

$$\begin{array}{ccc} X & \xrightarrow{g} & B \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{f} & A \end{array}$$

which means an isomorphism $\ker t \cong \ker s \in \mathcal{C}$, and a monomorphism $\operatorname{coker} t \to \operatorname{coker} s \in \mathcal{C}$. This implies $\operatorname{coker} t \in \mathcal{C}$ and then $t \in S$.

On the other hand, given

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle t}\downarrow & & \\ C & & \end{array}$$

with $t \in S$. Consider its pushout

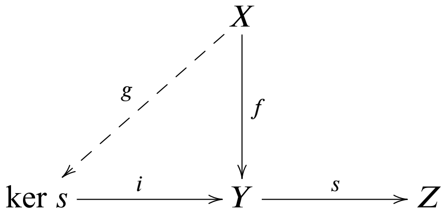
$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ C & \xrightarrow{f} & X \end{array}$$

which means an isomorphism $\operatorname{coker} s \cong \operatorname{coker} t \in \mathcal{C}$, and an epimorphism $\ker t \to \ker s$. This implies $\ker s \in \mathcal{C}$, and then $s \in S$.

3).On one hand, suppose

$$X \xrightarrow{f} Y \xRightarrow{s} Z$$

with $sf = 0$ and $s \in S$. By universal property, there is a g satisfying following commutative diagram.

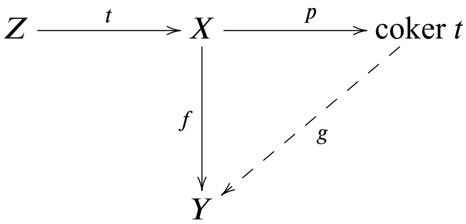


Denote the inclusion by $t : \ker g \hookrightarrow X$, we have $ft = igt = 0$. Note that $\operatorname{coker} t = X/\ker g \hookrightarrow \ker s$. This means $t \in S$.

On the other hand, suppose

$$Z \overset{t}{\Longrightarrow} X \overset{f}{\longrightarrow} Y$$

with $ft = 0$ with $s \in S$. By universal property, there is a g satisfying following commutative diagram.



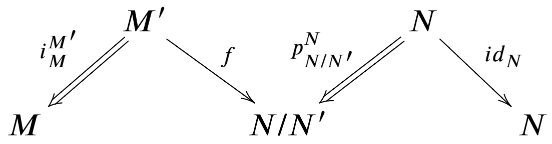
Denote the projection by $s : Y \to \operatorname{coker} g$, we have $sf = sgp = 0$. Note that $\ker s = \operatorname{Im} g = g(\operatorname{coker} t)$, thus $s \in S$. □

Remark that if $M/M',\ N' \in C$ then $i_M^{M'}, p_{N/N'}^N \in S$. We now prove that $\mathcal{A}/C$ is isomorphic to $S^{-1}\mathcal{A}$ by giving functors between them. In fact, this result is well-known. It was mentioned in [7] Chapter 1 2.5 d).

Theorem 4.1. *Let $\mathcal{A}$ be an abelian category, C be a Serre subcategory of $\mathcal{A}$. Let $S = \{f \in \operatorname{Mor}\mathcal{A} | \ker f,\ \operatorname{coker} f \in C\}$ be the multiplicative system induced by C, then $\mathcal{A}/C$ is isomorphic to $S^{-1}\mathcal{A}$.*

Proof. Given a morphism $\bar{f} : M \to N$ in $\mathcal{A}/C$, there is an $f : M' \to N/N'$ with $M/M',\ N' \in C$. Define $F : \mathcal{A}/C \to S^{-1}\mathcal{A}$ preserving objects and mapping $\bar{f}$ to

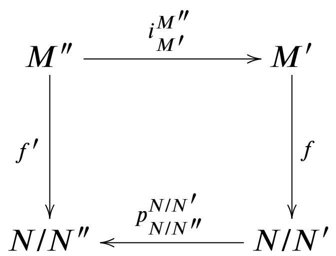
$id_N/p^N_{N/N'} \circ f/i^{M'}_M$ as following.



Define $G : S^{-1}\mathcal{A} \to \mathcal{A}/\mathcal{C}$ preserving objects and mapping f/s to $(Tf) \circ (Ts)^{-1}$.

Firstly, let us check that the definitions of F and G does not rely on the choice of representative elements.

Suppose f, f' are in the direct system such that $\bar{f} = Tf = Tf'$. Without loss of generality, we may always assume the following diagram is commutative.

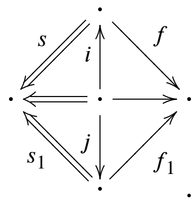


Note that

$$
\begin{aligned}
& id_N/p^N_{N/N''} \circ f'/i^{M''}_M \\
=& id_N/p^N_{N/N''} \circ (p^{N/N'}_{N/N''} \circ f \circ i^{M''}_{M'})/i^{M''}_M \\
=& id_N/p^N_{N/N''} \circ p^{N/N'}_{N/N''}/id_{N/N'} \circ f/id_{M'} \circ i^{M''}_{M'}/i^{M''}_M \\
=& id_N/p^N_{N/N'} \circ id_{N/N'}/p^{N/N'}_{N/N''} \circ p^{N/N'}_{N/N''}/id_{N/N'} \circ f/id_{M'} \circ i^{M''}_{M'}/i^{M''}_{M'} \circ id_{M'}/i^{M'}_M \\
=& id_N/p^N_{N/N'} \circ f/id_{M'} \circ id_{M'}/i^{M'}_M \\
=& id_N/p^N_{N/N'} \circ f/i^{M'}_M
\end{aligned}
$$

This means the definition of F does not rely on the choice of representative elements.

Suppose $f/s = f_1/s_1$, we have the following commutative diagram



In fact, this means that there exists i and j such that $f \circ i = f_1 \circ j$, and $s \circ i = s_1 \circ j \in S$. Therefore,

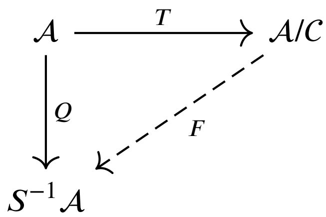
$$(Tf) \circ (Ts)^{-1} = Tf \circ Ti \circ (Ti)^{-1} \circ (Ts)^{-1}$$
$$= T(f \circ i) \circ (T(s \circ i))^{-1} = T(f_1 \circ j) \circ (T(s_1 \circ j))^{-1}$$
$$= Tf_1 \circ Tj \circ (Tj)^{-1} \circ (Ts_1)^{-1} = (Tf_1) \circ (Ts_1)^{-1}.$$

This means the definition of G does not rely on the choice of representative elements.

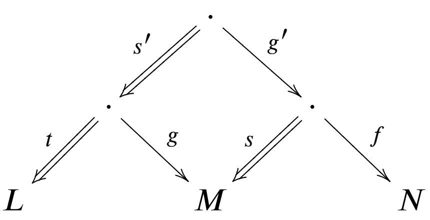
Secondly, we want to show that F and G are functors. Because

$$F(\bar{f}) = id_N/p^N_{N/N'} \circ f/i^{M'}_M = Q(p^N_{N/N'})^{-1} \circ Q(f) \circ Q(i^{M'}_M)^{-1},$$

F is actually the functor induced by the universal property in Proposition 3.6 satisfying the following commutative diagram.



It remains to check G is a functor. For $g/t : L \to M$, $f/s : M \to N$ in $S^{-1}\mathcal{A}$, consider their composition, there are two morphisms g' and $s' \in S$ such that the following diagram commute.



In fact, this implies $s \circ g' = g \circ s'$. Hence, $Ts \circ Tg' = Tg \circ Ts'$. Consequently, $Tg' \circ (Ts')^{-1} = (Ts)^{-1} \circ Tg$.

Because

$$\begin{aligned} G(f/s \circ g/t) = G(fg'/ts') &= T(fg') \circ T(ts')^{-1} \\ &= Tf \circ Tg' \circ T(s')^{-1} \circ (Tt)^{-1} \\ &= Tf \circ (Ts)^{-1} \circ Tg \circ (Tt)^{-1} = G(f/s) \circ G(g/t), \end{aligned}$$

we obtain that G is a well-defined functor.

Thirdly, we show the compositions of F and G are identities. On one hand,

$$\begin{aligned} GF(\bar{f}) &= G(id_N/p^N_{N/N'} \circ f/i^{M'}_M) \\ &= (Tp^N_{N/N'})^{-1} \circ Tf \circ (Ti^{M'}_M)^{-1} = \bar{f}. \end{aligned}$$

On the other hand, given

$$M \overset{s}{\Longleftarrow} X \overset{f}{\longrightarrow} N$$

we have

$$\begin{aligned} FG(f/s) &= F((Tf)(Ts)^{-1}) = F(Tf) \circ F((Ts)^{-1}) \\ &= F(Tf) \circ (FT(s))^{-1} = f/id_X \circ id_X/s = f/s. \end{aligned}$$

This means F and G are isomorphism, and then $\mathcal{A}/\mathcal{C}$ is isomorphic to $S^{-1}\mathcal{A}$. □

By example 2.3, ker Q is a Serre subcategory since Q is an exact functor. We claim that ker Q is equal to $\mathcal{C}$.

Proposition 4.2. *Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$. Let $S = \{f \in \mathrm{Mor}\mathcal{A} | \ker f,\ \mathrm{coker}\, f \in \mathcal{C}\}$ be the multiplicative system induced by $\mathcal{C}$, then*

$\ker Q = \mathcal{C}$.

Proof. On one hand, for any $X \in \mathcal{C}$, consider the zero morphism $0_{X,0} : X \to 0$. Because $\ker 0_{X,0} = X \in \mathcal{C}$ and $\operatorname{coker} 0_{X,0} = 0 \in \mathcal{C}$, we know that $0_{X,0} \in S$ and $Q(0_{X,0}) = 0_{X,0}/id_X$ is an isomorphism in $S^{-1}\mathcal{A}$. Thus, $X \cong 0$ in $S^{-1}\mathcal{A}$ i.e. $X \in \ker Q$. This means $\mathcal{C} \subset \ker Q$.

On the other hand, if $X \in \ker Q$ i.e. $Q(X) \cong 0$. It is clear that

$$\operatorname{Hom}_{S^{-1}\mathcal{A}}(X, X) = 0,$$

in particular, $id_X/id_X = 0$. It follows that $G(id_X/id_X) = 0$ i.e. $T(id_X) \circ T(id_X)^{-1} = 0$, and then $T(id_X) = 0$. By Lemma 3.1, this means $X = \operatorname{Im} id_X \in \mathcal{C}$. Thus, $\ker Q \subset \mathcal{C}$. □

We end this section by giving a negligible proposition. This proposition rejects the assumption that a multiplicative system induced by a Serre subcategory contains only finitely many non-identity elements.

Proposition 4.3. *Let S be a multiplicative system induced by a Serre subcategory $\mathcal{C}$. If S contains a non-identity element, then S contains infinite non-identity elements.*

Proof. Suppose $s : X \to Y$ is a non-identity element in S. Note that $\ker(s \oplus s) \cong \ker s \oplus \ker s$, and $\operatorname{coker}(s \oplus s) \cong \operatorname{coker} s \oplus \operatorname{coker} s$. It follows that $s \oplus s \in S$. Similarly, one can show that all finite direct sum of s are in S that implies S contains at least infinite non-identity elements. □

4.2 Serre subcategory from Multiplicative system

Let $\mathcal{A}$ be an abelian category, S be a multiplicative system of $\mathcal{A}$. Recall Lemma 2.3 that the localization functor $Q : \mathcal{A} \to S^{-1}\mathcal{A}$ is an exact functor. Example 2.3 tells us that $\ker Q$ is a Serre subcategory of $\mathcal{A}$. The following proposition describes $\ker Q$ accurately.

Proposition 4.4. *Let S be a multiplicative system of an abelian category $\mathcal{A}$, denote the localization functor of $\mathcal{A}$ to $S^{-1}\mathcal{A}$ by Q. We have*

$$\ker Q = \{X \cong \operatorname{coker} s \text{ for some } s \in S\} = \{X | X \cong \ker t \text{ for some } t \in S\}.$$

Proof. For $X \cong \operatorname{coker} s$, consider

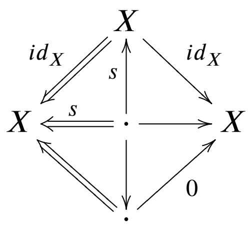
$$A \overset{s}{\Longrightarrow} B \overset{\pi}{\longrightarrow} \operatorname{coker} s,$$

there exists a morphism $t \in S$ such that $t\pi = 0$ which can be seen in the following diagram.

$$A \overset{s}{\Longrightarrow} B \overset{\pi}{\longrightarrow} \operatorname{coker} s \overset{t}{\Longrightarrow} \cdot$$

Since π is an epimorphism, we have $t = 0 \in S$. Consequently, $\operatorname{coker} s \cong 0$ in $S^{-1}\mathcal{A}$ i.e. $X \cong \operatorname{coker} s \in \ker Q$. In addition, $\operatorname{coker} s = \ker t$ for $t = 0 \in S$ i.e. $X \cong \ker t$.

Conversely, suppose $X \in \ker Q$. It means id_X/id_X is zero in $S^{-1}\mathcal{A}$. There is a commutative diagram



which implies $s : \cdot \to X$ is zero. Since $\operatorname{coker} s = X/\operatorname{Im} s \cong X$, we conclude that $X \cong \operatorname{coker} s$ for some $s \in S$.

Besides, suppose $X \cong \ker t$, consider

$$\ker t \overset{i}{\longrightarrow} A \overset{t}{\Longrightarrow} B.$$

Since $ti = 0$ there is a morphism $s \in S$ such that $is = 0$.

$$\cdot \overset{s}{\Longrightarrow} \ker t \overset{i}{\longrightarrow} A \overset{t}{\Longrightarrow} B$$

Because i is a monomorphism, this means $s = 0 \in S$. Thus, $\ker t \cong \operatorname{coker} s$. □

Let S be a multiplicative system of an abelian category $\mathcal{A}$, denote the localization functor of $\mathcal{A}$ to $S^{-1}\mathcal{A}$ by Q. Since ker Q is a Serre subcategory, and we know that we can get a new multiplicative system from the Serre subcategory ker Q, which is

$$S' = \{f \in \mathrm{Mor}\mathcal{A} | \ker f, \ \mathrm{coker}\, f \in \ker Q\}.$$

It is routine to show $S \subset S'$. However, it is unexpected that this inclusion could be proper because it is not necessary for S to contain all isomorphisms.

However, a saturated multiplicative system contains all isomorphisms. It may be beneficial to mention that 3.6 in [7] chapter 1, there is a one-to-one correspondence between the set of thick subcategories of $\mathcal{A}$ and the set of saturated subsets of morphisms in $\mathcal{A}$ which admit a calculus of left and right fractions, if $\mathcal{A}$ is an abelian category.

Chapter 5 Localization and finiteness

The purpose of this chapter is to provide a basis for studying the quotient category of a multiring category (resp. multitensor category, resp. multifusion category). We know that multiring category, multitensor category and multifusion category are locally finite k-linear abelian categories. Besides, a multifusion category is finite semisimple. Therefore, it is inevitable to study the quotient category of a locally finite k-linear abelian category and the quotient category of a finite semisimple abelian category.

5.1 Localization preserves locally finiteness

Given an abelian category $\mathcal{A}$ with a Serre subcategory $\mathcal{C}$. Theorem 3.1 says that $\mathcal{A}/\mathcal{C}$ is an abelian category. This allows us to discuss the length of an object in $\mathcal{A}/\mathcal{C}$.

We have the following lemma about simple objects in $\mathcal{A}/\mathcal{C}$.

Lemma 5.1. *Every simple object in $\mathcal{A}$ has length* 1 *or* 0 *in $\mathcal{A}/\mathcal{C}$.*

Proof. Suppose X is a simple object in $\mathcal{A}$, consider its subobject Y in $\mathcal{A}/\mathcal{C}$, and denote the corresponding monomorphism by $\bar{f} : Y \to X$.

We also view Y as an object in $\mathcal{A}$. Because

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(Y, X) = \varinjlim_{Y', X'} \mathrm{Hom}_{\mathcal{A}}(Y', X/X'),$$

where $Y/Y',\ X' \in \mathcal{C}$, there exists a subobject Y_1 of Y such that $Y/Y_1 \in \mathcal{C}$ such that the following diagram commute

$$\begin{array}{ccc} TY_1 & \xrightarrow{Ti} & TY \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ TX & = & TX \end{array}$$

where T is the canonical functor, i is the injection, and $f : Y_1 \to X$ is an epimorphism since X is simple in $\mathcal{A}$. Thus, we have the following exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow Y_1 \xrightarrow{f} X \longrightarrow 0.$$

Because the canonical functor T is exact, we have the following exact sequence

$$0 \longrightarrow T\operatorname{Ker} f \longrightarrow TY_1 \xrightarrow{Tf} TX \longrightarrow 0.$$

This means Tf is an epimorphism in $\mathcal{A}/\mathcal{C}$. Note that Ti is an isomorphism in $\mathcal{A}/\mathcal{C}$ and $\bar{f}$ is a monomorphism, we know $Tf = \bar{f} \circ Ti$ is a monomorphism. Therefore, Tf is an isomorphism. Consequently, $\bar{f}$ is an isomorphism. Thus, every monomorphism in $\mathcal{A}/\mathcal{C}$ to X is an isomorphism in $\mathcal{A}/\mathcal{C}$. It shows that X has length 1 or 0 in $\mathcal{A}/\mathcal{C}$. Note that the case of length 0 means X is in $\mathcal{C}$, and it is zero in $\mathcal{A}/\mathcal{C}$. □

Next, we prove that the localization preserves finite length.

Proposition 5.1. *Suppose every object in $\mathcal{A}$ has finite length, then every object in $\mathcal{A}/\mathcal{C}$ has finite length.*

Proof. For an arbitrary object X in $\mathcal{A}$, it has finite length in $\mathcal{A}$ and one can assume its Jordan-Hölder series as following without loss of generality.

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

Since X_{i+1}/X_i is a simple object in $\mathcal{A}$, it has length 1 or 0 in $\mathcal{A}/\mathcal{C}$ by the above lemma. Because

$$l(X_{i+1}) = l(X_i) + l(X_{i+1}/X_i), \quad \text{for all } 0 \leqslant i \leqslant n-1,$$

we know that

$$\begin{aligned} l(X) = l(X_n) &= l(X_{n-1}) + l(X_n/X_{n-1}) \\ &= l(X_{n-2}) + l(X_{n-1}/T_{n-2}) + l(X_{n-1}/T_n) \\ &= \cdots \\ &= \sum_{i=0}^{n-1} l(X_{i+1}/X_i) \\ &\leqslant n. \end{aligned}$$

This means X has finite length in $\mathcal{A}/\mathcal{C}$, which is actually smaller than its length in $\mathcal{A}$. □

In order to show that the quotient category of a locally finite abelian category is locally finite, we are required to prove that the Hom-spaces of the quotient category are finite dimensional.

Lemma 5.2. *Let $\mathcal{A}$ be a locally finite k-linear abelian category, then* $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ *is a finite dimensional vector space, where M, N are objects in $\mathcal{A}$.*

Proof. By definition, $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is obviously a vector space. We now prove $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is finite dimensional by induction on the lengths of M, N in $\mathcal{A}$. Firstly, suppose $l(M) = l(N) = 1$. If $M \in \mathcal{C}$ or $N \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = 0$. If $M, N \notin \mathcal{C}$, it is clear that $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \mathrm{Hom}_{\mathcal{A}}(M, N)$, which is finite dimensional.

Secondly, we want to show $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is a finite dimensional vector space for $l(N) = 1$. We prove it by induction on the length of M, suppose $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is finite dimensional for all M, N such that $l(M) \leqslant m$ and $l(N) = 1$. Now we consider the case of $l(M) = m + 1$ and $l(N) = 1$. On one hand, if there is a subobject X of M such that $0 \leqslant l(X) \leqslant m$ and $M/X \in \mathcal{C}$, we know that Ti_M^X is an isomorphism in $\mathcal{A}/\mathcal{C}$, where $i_M^X : X \to M$ is the monomorphism. This means that $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X, N)$, which is finite dimensional by the induction assumption. On the other hand, if for any subobject X of M satisfying $0 \leqslant l(X) \leqslant m$, $M/X \notin \mathcal{C}$. It follows that $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \mathrm{Hom}_{\mathcal{A}}(M, N)$, which is finite dimensional. Therefore, $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is a finite dimensional vector space for $l(N) = 1$.

Similarly, $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is a finite dimensional vector space for $l(M) = 1$.

Thirdly, given two positive integers $m, n \geqslant 1$, suppose $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is a finite dimensional vector space for $l(M) \leqslant m$ and $l(N) \leqslant n + 1$ and for $l(M) \leqslant m + 1$ and $l(N) \leqslant n$. We consider the case of $l(M) = m + 1$ and $l(N) = n + 1$.

If there is a subobject X of M such that $0 \leqslant l(X) \leqslant m$ and $M/X \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X, N)$ since Ti_M^X is an isomorphism in $\mathcal{A}/\mathcal{C}$, where $i_M^X : X \to M$ is the monomorphism. By the induction assumption, $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X, N)$ is finite dimensional.

If there is a subobject Y of N such that $1 \leqslant l(Y) \leqslant n + 1$ and $Y \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N/Y)$ since $Tp_{N/Y}^N$ is an isomorphism in $\mathcal{A}/\mathcal{C}$, where $p_{N/Y}^N : N \to N/Y$ is the epimorphism. By the induction assumption, $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \cong \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N/Y)$ is finite dimensional.

If for any subobject X of M and any subobject Y of N satisfying $0 \leqslant l(X) \leqslant m$ and $1 \leqslant l(Y) \leqslant n + 1$, $M/X, Y \notin \mathcal{C}$. It follows that $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \mathrm{Hom}_{\mathcal{A}}(M, N)$, which is finite dimensional.

In summary, $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is a finite dimensional vector space for $l(M) = m+1$ and $l(N) = n+1$. Consequently, we obtain that $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M, N)$ is a finite dimensional vector space for any objects M, N in $\mathcal{A}$. □

It follows directly from Proposition 5.1 and Lemma 5.2 that $\mathcal{A}/\mathcal{C}$ is locally finite.

Proposition 5.2. *The quotient category $\mathcal{A}/\mathcal{C}$ of a locally finite k-linear abelian category $\mathcal{A}$ is a locally finite k-linear abelian category.*

5.2 Localization of a finite semisimple abelian category

In this section, we show that the quotient category of a finite semisimple abelian category is still finite semisimple. Firstly, we give a description for the structure of a Serre subcategory of a finite semisimple abelian category.

Lemma 5.3. *Let $\mathcal{A}$ be a finite semisimple abelian category. Any Serre subcategory $\mathcal{C}$ of $\mathcal{A}$ consists of all finite direct sums of some simple objects in $\mathcal{A}$.*

Proof. We denote the isomorphism classes of simple objects in $\mathcal{A}$ by $\{M_1, \cdots, M_n\}$. Because $\mathcal{A}$ is finite, there is no infinite direct sums in $\mathcal{A}$. Let

$$I = \{i | M_i \in \mathcal{C}\}.$$

For any $X \in \mathcal{C}$, since $\mathcal{A}$ is semisimple, X can be written as

$$X = \bigoplus_{i \in J} M_i^{n_i}, \text{where } J \text{ is a subset of } \{1, \cdots, n\}.$$

Here, $M_i^{n_i}$ is the direct sum of n_i copies of M_i. For any $j \in J$, M_j is a subobject of X, whence $M_j \in \mathcal{C}$. This means $j \in I$. Thus, $J \subset I$. Consequently, we may always write

$$X = \bigoplus_{i \in I} M_i^{n_i}$$

where some $n_i' s$ can be 0. Thus,

$$\mathcal{C} \subset \text{all finite direct sums of } \{M_i\}_{i \in I}.$$

On the other hand, it is clear that

$$\text{all finite direct sums of } \{M_i\}_{i \in I} \subset \mathcal{C}.$$

Therefore,

$$\mathcal{C} = \text{all finite direct sums of } \{M_i\}_{i \in I}.$$

□

Proposition 5.3. *Let $\mathcal{A}$ be a finite semisimple abelian category. For any Serre subcategory $\mathcal{C}$ of $\mathcal{A}$, $\mathcal{A}/\mathcal{C}$ is a finite semisimple abelian category.*

Proof. Consider an arbitrary object Y in $\mathcal{A}$,

$$Y = (\bigoplus_{i \in I} M_i^{n_i}) \bigoplus (\bigoplus_{j \notin I} M_j^{n_j}).$$

Let

$$Y' = \bigoplus_{j \notin I} M_j^{n_j},$$

it follows that

$$Y/Y' = \bigoplus_{i \in I} M_i^{n_i} \in \mathcal{C}.$$

This means

$$Ti_Y^{Y'} : Y' \to Y \text{ is an isomorphism in } \mathcal{A}/\mathcal{C}.$$

Consequently, every object in $\mathcal{A}/\mathcal{C}$ can be written as a finite direct sum of elements in $\{M_j | j \notin I\}$. This implies $\mathcal{A}/\mathcal{C}$ is also a semisimple abelian category.

In fact, $\mathcal{A}/\mathcal{C}$ is finite. Because $\mathcal{A}$ is a finite semisimple abelian category, it is the category of finite dimensional modules over a finite dimensional semisimple k-algebra A. Hence, by the Wedderburn-Artin theorem,

$$A \cong \mathrm{Mat}_{n_1} k \times \cdots \times \mathrm{Mat}_{n_t} k$$

Because a left module of $\mathrm{Mat}_{n_1} k \times \cdots \times \mathrm{Mat}_{n_t} k$ is actually a direct sum of $N_1, \cdots, N_t$, where N_i is a left module of $\mathrm{Mat}_{n_i} k$, we know that a simple left module of $\mathrm{Mat}_{n_1} k \times \cdots \times \mathrm{Mat}_{n_t} k$ is a simple left module S_{n_i} of $\mathrm{Mat}_{n_i} k$ for some i. Let D be the set of all mutually non-isomorphic simple modules of $\mathrm{Mat}_{n_1} k \times \cdots \times \mathrm{Mat}_{n_t} k$. As mentioned above, a Serre subcategory $\mathcal{C}$ of $\mathcal{A}$ consists of all finite direct sums of elements in a subset E of D. Then $\mathcal{A}/\mathcal{C}$ consists of all finite direct sum of simple modules in the complement of E in D.

Write the complement of E in D as $\{S_{r_1}, \cdots, S_{r_l}\}$, then $\mathcal{A}/\mathcal{C}$ is the representation category of $\mathrm{Mat}_{n_{r_1}} k \times \cdots \times \mathrm{Mat}_{n_{r_l}} k$, which is finite dimensional. Hence, $\mathcal{A}/\mathcal{C}$ is a finite abelian category. □

Example 5.1. *Let S_3 be the symmetric group on 3 letters, k be an algebraically closed field satisfying* char $k \nmid |S_3|$, *we know kS_3 is semisimple by Maschke's Theorem. It is clear that the category of finite dimensional modules of kS_3, which we denote by $\mathcal{A}$, is a finite semisimple abelian category. As a special case of the above example, we know* *that $\mathcal{A}/C$ is also a finite semisimple abelian category for any Serre subcategory C. We now discuss more details in this case.*

It is well-known that there are three irreducible representations in the category of finite dimensional representations of kS_3. They are trivial representation V_1, alternating representation V_2, and standard representation V_3. Using the approach mentioned in [22] gives that

$$kS_3 = V_1 \bigoplus V_2 \bigoplus V_3^2.$$

Similar to the steps in the proof of Wedderburn-Artin theorem,

$$\begin{aligned} kS_3 &\cong \mathrm{End}(V_1) \times \mathrm{End}(V_2) \times \mathrm{End}(V_3^2) \\ &\cong \mathrm{Mat}_1(k) \times \mathrm{Mat}_1(k) \times \mathrm{Mat}_2(k). \end{aligned}$$

Let C be the Serre subcategory of $\mathcal{A}$ consists of finite direct sums of V_3. Then $\mathcal{A}/C$ consists of all finite direct sums of V_1 and V_2. Therefore, $\mathcal{A}/C$ is the category of finite dimensional representations of $M_1(k) \times M_1(k) = k \times k$. Note that $k \times k$ is isomorphic to $k\mathbb{Z}_2$, because we can construct an isomorphism as following:

$$\begin{aligned} \varphi : k \times k &\to k\mathbb{Z}_2 \\ (a, b) &\to \frac{a+b}{2} + \frac{a-b}{2} g. \end{aligned}$$

Also recall that $\mathbb{Z}_2 \cong S_3/A_3$. Thus, in this case, $\mathcal{A}/C$ is actually the category of finite dimensional representations of $k\mathbb{Z}_2$.

Note that, in the above example, we did not take the tensor structure into consideration.

Chapter 6 Tensor product in quotient category

6.1 Tensor product in quotient category

In order to give a definition for tensor product in the quotient category of an abelian monoidal category, we first introduce a lemma.

Lemma 6.1. *Let $\mathcal{A}$ be an abelian monoidal category, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$. Suppose f, g are morphisms in $\mathcal{A}$ such that* $\ker f$, $\ker g$ *belong to $\mathcal{C}$, then* $\ker(f \otimes g)$ *belongs to $\mathcal{C}$. Similarly, if* $\operatorname{coker} f$, $\operatorname{coker} g$ *belong to $\mathcal{C}$, then* $\operatorname{coker}(f \otimes g)$ *belongs to $\mathcal{C}$.*

Proof. Consider

$$f : M \to N \text{ and } g : X \to Y.$$

Note that there is a composition

$$f \otimes g : M \otimes X \xrightarrow{f\otimes id_X} N \otimes X \xrightarrow{id_N \otimes g} N \otimes Y,$$

and there is a commutative diagram

$$\begin{array}{ccccccccc} & & M \otimes X & \xrightarrow{f\otimes id} & N \otimes X & \longrightarrow & \operatorname{coker}(f \otimes id) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f\otimes g} & & \downarrow{\scriptstyle id\otimes g} & & \downarrow & & \\ 0 & \longrightarrow & N \otimes Y & \xrightarrow{id} & N \otimes Y & \longrightarrow & 0 & & \end{array}$$

By snake lemma, this implies an exact sequence

$$0 \longrightarrow \ker(f \otimes id) \longrightarrow \ker(f \otimes g) \longrightarrow \ker(id \otimes g) \longrightarrow$$
$$\operatorname{coker}(f \otimes id) \longrightarrow \operatorname{coker}(f \otimes g) \longrightarrow \operatorname{coker}(id \otimes g) \longrightarrow 0$$

which means

$$\begin{cases}\ker(f \otimes id) \to \ker(f \otimes g) \to \ker(id \otimes g) \\ \operatorname{coker}(f \otimes id) \to \operatorname{coker}(f \otimes g) \to \operatorname{coker}(id \otimes g)\end{cases}$$

exact. Note that $\ker(f \otimes id) = \ker f \otimes X \in \mathcal{C}$, and $\ker(id \otimes g) = N \otimes \ker g \in \mathcal{C}$ provided $\ker f$, $\ker g$ belong to $\mathcal{C}$. By Lemma 2.1, we know $\ker(f \otimes g) \in \mathcal{C}$. Similarly, $\operatorname{coker}(f \otimes g) \in \mathcal{C}$ provided $\operatorname{coker} f$, $\operatorname{coker} g$ belong to $\mathcal{C}$. □

In the following, we study the tensor product of two subobjects (resp. quotient objects).

Lemma 6.2. *Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product. Let M, X be two objects in $\mathcal{A}$, M' be a subobject of M, X' be a subobject of X, then $M' \otimes X'$ is a subobject of $M \otimes X$.*

Proof. Consider monomorphisms $i_1 : M' \to M$ and $i_2 : X' \to X$, and exact sequences

$$0 \longrightarrow M' \xrightarrow{i_1} M \longrightarrow \operatorname{coker} i_1 \longrightarrow 0;$$

$$0 \longrightarrow X' \xrightarrow{i_2} X \longrightarrow \operatorname{coker} i_2 \longrightarrow 0.$$

Because the tensor product is biexact, we have two following exact sequences

$$0 \longrightarrow M' \otimes X' \xrightarrow{i_1 \otimes id} M \otimes X' \longrightarrow \operatorname{coker} i_1 \otimes X' \longrightarrow 0;$$

$$0 \longrightarrow M \otimes X' \xrightarrow{id \otimes i_2} M \otimes X \longrightarrow M \otimes \operatorname{coker} i_2 \longrightarrow 0.$$

This means

$$i_1 \otimes i_2 : M' \otimes X' \xrightarrow{i_1 \otimes id} M \otimes X' \xrightarrow{id \otimes i_2} M \otimes X$$

is a monomorphism. Hence, $M' \otimes X'$ is a subobject of $M \otimes X$. □

Lemma 6.3. *Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product. Let N, Y be two objects in $\mathcal{A}$, N/N' be a quotient object of N, Y/Y' be a quotient object* *of Y, then $N/N' \otimes Y/Y'$ is a quotient object of $N \otimes Y$.*

Proof. Consider epimorphisms $p_1 : N \to N/N'$ and $p_2 : Y \to Y/Y'$, and exact sequences

$$0 \longrightarrow N' \longrightarrow N \xrightarrow{p_1} N/N' \longrightarrow 0;$$

$$0 \longrightarrow Y' \longrightarrow Y \xrightarrow{p_2} Y/Y' \longrightarrow 0;$$

Because the tensor product is biexact, we have two following exact sequences

$$0 \longrightarrow N' \otimes Y/Y' \longrightarrow N \otimes Y/Y' \xrightarrow{p_1 \otimes id} N/N' \otimes Y/Y' \longrightarrow 0;$$

$$0 \longrightarrow N \otimes Y' \longrightarrow N \otimes Y \xrightarrow{id \otimes p_2} N \otimes Y/Y' \longrightarrow 0.$$

This means

$$p_1 \otimes p_2 : N \otimes Y \xrightarrow{id \otimes p_2} N \otimes Y/Y' \xrightarrow{p_1 \otimes id} N/N' \otimes Y/Y'$$

is an epimorphism. Hence, $N/N' \otimes Y/Y'$ is a quotient object of $N \otimes Y$. □

In fact, we have an isomorphism

$$N/N' \otimes Y/Y' \cong N \otimes Y/\ker(p_1 \otimes p_2).$$

Now, we can prove that $T(f \otimes g) \in \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M \otimes X, N \otimes Y)$ for any $\bar{f} = Tf : M \to N, \bar{g} = Tg : X \to Y$ in $\mathcal{A}/\mathcal{C}$.

Proposition 6.1. *Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$. Let $\bar{f} : M \to N$, $\bar{g} : X \to Y$ be two morphisms in $\mathcal{A}/\mathcal{C}$. Then $T(f \otimes g)$ is a morphism in $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M \otimes X, N \otimes Y)$ where f, g are in direct systems, and T is the canonical functor.*

Proof. Suppose $f : M' \to N/N', g : X' \to Y/Y'$ with $M/M', N', X/X', Y'$ are in $\mathcal{C}$. It follows that

$$f \otimes g : M' \otimes X' \to N/N' \otimes Y/Y'.$$

Denote $i_1 : M' \to M, i_2 : X' \to X$ to be inclusions. Since Ti_1, Ti_2 are isomorphisms in $\mathcal{A}/\mathcal{C}$, we know that coker i_1, coker i_2 belong to $\mathcal{C}$. By Lemma 6.1, coker$(i_1 \otimes i_2) \in \mathcal{C}$. This means $(M \otimes X)/(M' \otimes X') \in \mathcal{C}$.

Denote $p_1 : N \to N/N', p_2 : Y \to Y/Y'$. We have already known that $N/N' \otimes Y/Y' \cong N \otimes Y/\ker(p_1 \otimes p_2)$. Since Tp_1, Tp_2 are isomorphisms in $\mathcal{A}/\mathcal{C}$, we know that ker p_1, ker p_2 are in $\mathcal{C}$. By Lemma 6.1, it follows that $\ker(p_1 \otimes p_2) \in \mathcal{C}$.

In summary, $f \otimes g$ is in the direct system. Therefore, $T(f \otimes g) \in \mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(M \otimes X, N \otimes Y)$. □

Next, we show that $T(f \otimes g)$ does not rely on the choice of representative elements.

Proposition 6.2. *Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$. Let $\bar{f} : M \to N$, $\bar{g} : X \to Y$ be two morphisms in $\mathcal{A}/\mathcal{C}$. Suppose $\bar{f} = Tf = Tf_1$ and $\bar{g} = Tg = Tg_1$, then $T(f \otimes g) = T(f_1 \otimes g_1)$.*

Proof. First of all, we claim that

$$T(id \otimes g) = T(id \otimes g_1).$$

Suppose $g : X' \to Y/Y'$, $g_1 : X_1' \to Y/Y_1'$. Since the direct system in the definition of quotient categroty is directed, we can obtain a morphism $g_2 : X_2' \to Y/Y_2'$ such that the following diagrams commute:

$$\begin{array}{ccc} X_2' & \xrightarrow{i^{X_2'}_{X'}} & X' \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g} \\ Y/Y_2' & \xleftarrow[p^{Y/Y'}_{Y/Y_2'}]{} & Y/Y' \end{array}$$

and

$$\begin{array}{ccc} X_2' & \xrightarrow{i^{X_2'}_{X_1'}} & X_1' \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} \\ Y/Y_2' & \xleftarrow[p^{Y/Y_1'}_{Y/Y_2'}]{} & Y/Y_1'. \end{array}$$

Thus, we obtain two commutative diagrams:

$$\begin{array}{ccc} M \otimes X_2' & \xrightarrow{id \otimes i_{X'}^{X_2'}} & M \otimes X' \\ \downarrow{\scriptstyle id \otimes g_2} & & \downarrow{\scriptstyle id \otimes g} \\ M \otimes Y/Y_2' & \xleftarrow[id \otimes p_{Y/Y_2'}^{Y/Y'}]{} & M \otimes Y/Y' \end{array}$$

and

$$\begin{array}{ccc} M \otimes X_2' & \xrightarrow{id \otimes i_{X_1'}^{X_2'}} & M \otimes X_1' \\ \downarrow{\scriptstyle id \otimes g_2} & & \downarrow{\scriptstyle id \otimes g_1} \\ M \otimes Y/Y_2' & \xleftarrow[id \otimes p_{Y/Y_2'}^{Y/Y_1'}]{} & M \otimes Y/Y_1'. \end{array}$$

Because $id \otimes i_{X'}^{X_2'} = i_{M \otimes X'}^{M \otimes X_2'}$, $id \otimes i_{X_1'}^{X_2'} = i_{M \otimes X_1'}^{M \otimes X_2'}$, $id \otimes p_{Y/Y_2'}^{Y/Y'} = p_{M \otimes Y/Y_2'}^{M \otimes Y/Y'}$ and $id \otimes p_{Y/Y_2'}^{Y/Y_1'} = p_{M \otimes Y/Y_2'}^{M \otimes Y/Y_1'}$, we obtain two commutative diagrams:

$$\begin{array}{ccc} M \otimes X_2' & \xrightarrow{i_{M \otimes X'}^{M \otimes X_2'}} & M \otimes X' \\ \downarrow{\scriptstyle id \otimes g_2} & & \downarrow{\scriptstyle id \otimes g} \\ M \otimes Y/Y_2' & \xleftarrow[p_{M \otimes Y/Y_2'}^{M \otimes Y/Y'}]{} & M \otimes Y/Y' \end{array}$$

and

$$\begin{array}{ccc} M \otimes X_2' & \xrightarrow{i_{M \otimes X_1'}^{M \otimes X_2'}} & M \otimes X_1' \\ \downarrow{\scriptstyle id \otimes g_2} & & \downarrow{\scriptstyle id \otimes g_1} \\ M \otimes Y/Y_2' & \xleftarrow[p_{M \otimes Y/Y_2'}^{M \otimes Y/Y_1'}]{} & M \otimes Y/Y_1'. \end{array}$$

This means

$$T(id \otimes g) = T(id \otimes g_2) = T(id \otimes g_1).$$

The claim has been proven, and similarly one can obtain that

$$T(f \otimes id) = T(f_1 \otimes id).$$

Therefore,

$$\begin{aligned} T(f \otimes g) &= T((f \otimes id) \circ (id \otimes g)) \\ &= T(f \otimes id) \circ T(id \otimes g) \\ &= T(f_1 \otimes id) \circ T(id \otimes g_1) \\ &= T((f_1 \otimes id) \circ (id \otimes g_1)) \\ &= T(f_1 \otimes g_1). \end{aligned}$$

□

Hence, we can define the tensor product of two morphisms in the quotient category. The propositions above guarantee that the following definition is well-defined.

Definition 6.1. *Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product, C be a two-sided Serre tensor-ideal of $\mathcal{A}$. Define the tensor product of objects in $\mathcal{A}/C$ by the same tensor product in $\mathcal{A}$, and define the tensor product of morphisms in $\mathcal{A}/C$ by*

$$\bar{f} \otimes \bar{g} := T(f \otimes g)$$

where f, g are in direct systems, and T is the canonical functor. It is clear from the definition that

$$Tf \otimes Tg = T(f \otimes g).$$

Additionally, we define the associativity constraint in $\mathcal{A}/C$ by

$$\bar{a}_{X,Y,Z} = Ta_{X,Y,Z}.$$

It is also clear that the left and right unit isomorphisms in $\mathcal{A}/C$ are Tl_X and Tr_X where l_X and r_X are the left and right unit isomorphisms in $\mathcal{A}$.

We finish this section by considering an example of two-sided Serre tensor-ideal.

Example 6.1. *Let I be a finite indexed set, $\mathcal{A} = \bigoplus_{i\in I} \mathrm{Mat}_{n_i}(\mathrm{Vec})$, where $\mathrm{Mat}_{n_i}(\mathrm{Vec})$ is the category whose objects are n_i-by-n_i matrices of finite dimensional vector spaces. The tensor product of two objects from distinct direct summands is defined to be zero. One can observe that $\mathcal{A}$ is a multitensor category, and each $\mathrm{Mat}_{n_i}(\mathrm{Vec})$ is a two-sided Serre tensor-ideal of $\mathcal{A}$. In fact, let $\mathcal{C} = \mathrm{Mat}_{n_i}(\mathrm{Vec})$ to be such a two-sided Serre tensor-ideal, then $\mathcal{A}/\mathcal{C} = \bigoplus_{j\in I\setminus\{i\}} \mathrm{Mat}_{n_i}(\mathrm{Vec})$*

6.2 Localization of multiring categories

In this section, we will discuss the quotient categories of a multiring category, a multitensor category, and a multifusion category, respectively.

Proposition 6.3. *Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{C}$ is a monoidal category.*

Proof. Consider the following pentagon axiom diagram of $\mathcal{A}$:

$$
\begin{array}{ccccc}
 & & ((W\otimes X)\otimes Y)\otimes Z & & \\
 & \swarrow^{a_{W,X,Y}\otimes id_Z} & & \searrow^{a_{W\otimes X,Y,Z}} & \\
(W\otimes(X\otimes Y))\otimes Z & & & & (W\otimes X)\otimes(Y\otimes Z) \\
\downarrow a_{W,X\otimes Y,Z} & & & & a_{W,X,Y\otimes Z}\downarrow \\
W\otimes((X\otimes Y)\otimes Z) & & \xrightarrow{id_W\otimes a_{X,Y,Z}} & & W\otimes(X\otimes(Y\otimes Z))
\end{array}
$$

Applying the canonical functor T gives that

$$
\begin{array}{ccccc}
 & & ((W\otimes X)\otimes Y)\otimes Z & & \\
 & \swarrow^{\bar{a}_{W,X,Y}\otimes id_Z} & & \searrow^{\bar{a}_{W\otimes X,Y,Z}} & \\
(W\otimes(X\otimes Y))\otimes Z & & & & (W\otimes X)\otimes(Y\otimes Z) \\
\downarrow \bar{a}_{W,X\otimes Y,Z} & & & & \bar{a}_{W,X,Y\otimes Z}\downarrow \\
W\otimes((X\otimes Y)\otimes Z) & & \xrightarrow{id_W\otimes \bar{a}_{X,Y,Z}} & & W\otimes(X\otimes(Y\otimes Z)).
\end{array}
$$

This is actually the pentagon axiom of $\mathcal{A}/\mathcal{C}$.

Consider the following triangle axiom diagram of $\mathcal{A}$:

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{a_{X,1,Y}} & X \otimes (1 \otimes Y) \\ & {\scriptstyle r_X \otimes id_Y} \searrow \quad \swarrow {\scriptstyle id_X \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

Applying the canonical functor T gives that

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{\bar{a}_{X,1,Y}} & X \otimes (1 \otimes Y) \\ & {\scriptstyle Tr_X \otimes id_Y} \searrow \quad \swarrow {\scriptstyle id_X \otimes Tl_Y} & \\ & X \otimes Y & \end{array} .$$

Thus, $\mathcal{A}/C$ is a monoidal category. □

Recall that $Tf \otimes Tg = T(f \otimes g)$, this implies that T is a monoidal functor. Next, we study the quotient category of a multiring category (resp. a multitensor category, resp. a multifusion category) by a two-sided Serre tensor-ideal.

Proposition 6.4. *Let $\mathcal{A}$ be a multiring category, C be a two-sided Serre tensor-ideal of $\mathcal{A}$. Then $\mathcal{A}/C$ is a multiring category.*

Proof. By Proposition 5.2 and Proposition 6.3, the quotient category of a locally finite k-linear abelian monoidal category is a locally finite k-linear abelian monoidal category. Therefore, it suffices to show that the tensor product $\otimes : \mathcal{A}/C \times \mathcal{A}/C \to \mathcal{A}/C$ is bilinear and biexact.

Firstly, we show that the tensor product is bilinear. Note that the canonical functor T is linear, thus

$$\begin{aligned} &(\bar{f}_1 + \bar{f}_2) \otimes \bar{g} = (Tf_1 + Tf_2) \otimes Tg = T(f_1 + f_2) \otimes Tg \\ =&T((f_1 + f_2) \otimes g) = T(f_1 \otimes g + f_2 \otimes g) = T(f_1 \otimes g) + T(f_2 \otimes g) \\ =&T(f_1) \otimes Tg + T(f_2) \otimes Tg = \bar{f}_1 \otimes \bar{g} + \bar{f}_2 \otimes \bar{g} \end{aligned}$$

and similarly

$$\bar{f} \otimes (\bar{g_1} + \bar{g_2}) = Tf \otimes (Tg_1 + Tg_2) = Tf \otimes T(g_1 + g_2)$$
$$= T(f \otimes (g_1 + g_2)) = T(f \otimes g_1 + f \otimes g_2) = T(f \otimes g_1) + T(f \otimes g_2)$$
$$= Tf \otimes Tg_1 + Tf \otimes Tg_2 = \bar{f} \otimes \bar{g_1} + \bar{f} \otimes \bar{g_2}.$$

Furthermore, for any $a \in k$, we have

$$a\bar{f} \otimes \bar{g} = aTf \otimes Tg = T(af) \otimes Tg = T(af \otimes g)$$
$$= T(f \otimes ag) = Tf \otimes T(ag) = Tf \otimes aTg = \bar{f} \otimes a\bar{g}.$$

This implies $\otimes : \mathcal{A}/\mathcal{C} \times \mathcal{A}/\mathcal{C} \to \mathcal{A}/\mathcal{C}$ is bilinear on morphisms.

Secondly, we show that the tensor product is biexact. Consider the following exact sequence

$$L \xrightarrow{\bar{f}} M \xrightarrow{\bar{g}} N.$$

Because we can write $\bar{g} = Tg$, $\bar{f} = Tf$, the exact sequence means that

$$\ker(Tg) = \mathrm{Im}(Tf).$$

By Lemma 3.2, it follows that

$$T(\ker g) = T(\mathrm{Im}\, f).$$

Because T is an exact functor, the short exact sequence

$$0 \longrightarrow \mathrm{Im} f \xrightarrow{i} \ker g \xrightarrow{\pi} \ker g/\mathrm{Im}\, f \longrightarrow 0$$

implies that

$$0 \longrightarrow T\mathrm{Im} f \xrightarrow{Ti} T\ker g \xrightarrow{T\pi} T(\ker g/\mathrm{Im}\, f) \longrightarrow 0$$

is exact. Consequently,

$$T(\ker(g)/\mathrm{Im}(f)) \cong T(\ker g)/T(\mathrm{Im}\ f) = 0 \text{ in } \mathcal{A}/\mathcal{C},$$

which means $\ker(g)/\mathrm{Im}(f) \in \mathcal{C}$.

For any object A in $\mathcal{A}$, since $A \otimes -$ is exact in $\mathcal{A}$, we have the following short exact sequence

$$0 \longrightarrow A \otimes \mathrm{Im} f \xrightarrow{id_A \otimes i} A \otimes \ker g \xrightarrow{id_A \otimes \pi} A \otimes (\ker g/\mathrm{Im} f) \longrightarrow 0.$$

Note that there is a short exact sequence

$$0 \longrightarrow A \otimes \mathrm{Im} f \xrightarrow{id_A \otimes i} A \otimes \ker g \xrightarrow{\pi'} (A \otimes \ker g)/(A \otimes \mathrm{Im} f) \longrightarrow 0.$$

Therefore,

$$(A \otimes \ker g)/(A \otimes \mathrm{Im} f) \cong A \otimes (\ker g/\mathrm{Im} f) \in \mathcal{C}.$$

Now, we claim that

$$A \otimes \ker g = \ker(id_A \otimes\ g).$$

Consider the following exact sequence

$$0 \longrightarrow \ker g \longrightarrow M \xrightarrow{g} N,$$

applying $A \otimes -$ gives the following exact sequence

$$0 \longrightarrow A \otimes \ker g \longrightarrow A \otimes M \xrightarrow{id_A \otimes g} A \otimes N.$$

This indicates that

$$A \otimes \ker g = \ker(id_A \otimes g).$$

Besides, note that

$$A \otimes \mathrm{Im} f = \mathrm{Im}(id_A \otimes f).$$

It follows that

$$\ker(id_A \otimes g)/\mathrm{Im}(id_A \otimes f) = (A \otimes \ker g)/(A \otimes \mathrm{Im} f) \in \mathcal{C}.$$

This implies that

$$T(\ker(id_A \otimes g)/\mathrm{Im}(id_A \otimes f)) = 0$$

i.e.

$$T(\ker(id_A \otimes g))/T(\mathrm{Im}(id_A \otimes f)) = 0$$

i.e.

$$T(\ker(id_A \otimes g)) = T(\mathrm{Im}(id_A \otimes f))$$

By Lemma 3.2, this means

$$\ker(T(id_A \otimes g)) = \mathrm{Im}(T(id_A \otimes f))$$

i.e.

$$\ker(id_A \otimes Tg) = \mathrm{Im}(id_A \otimes Tf)$$

i.e.

$$\ker(id_A \otimes \bar{g}) = \mathrm{Im}(id_A \otimes \bar{f}).$$

This means $A \otimes -$ is exact in $\mathcal{A}/\mathcal{C}$. Similarly, one can show $- \otimes A$ is exact in $\mathcal{A}/\mathcal{C}$. As a result, the tensor product is biexact in $\mathcal{A}/\mathcal{C}$. Thus, $\mathcal{A}/\mathcal{C}$ is a multiring category. □

Proposition 6.5. *Let $\mathcal{A}$ be a multitensor category, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{C}$ is a multitensor category.*

Proof. As proved in the above proposition, we know that $\mathcal{A}/\mathcal{C}$ is a locally finite k-linear abelian monoidal category with bilinear tensor product. Therefore, it suffices to show $\mathcal{A}/\mathcal{C}$ is rigid.

For any object X in $\mathcal{A}/\mathcal{C}$, it has a left dual X^*, which means there exist an evaluation $ev_X : X^* \otimes X \to 1$ and a coevaluation $coev_X : 1 \to X \otimes X^*$ such that the compositions

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X,$$

$$X^* \xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^*$$

are the identity morphisms. Applying T gives that the compositions

$$X \xrightarrow{Tcoev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{\bar{a}_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes Tev_X} X,$$

$$X^* \xrightarrow{id_{X^*} \otimes Tcoev_X} X^* \otimes (X \otimes X^*) \xrightarrow{\bar{a}^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{Tev_X \otimes id_{X^*}} X^*$$

are the identity morphisms. As a result, every object in $\mathcal{A}/\mathcal{C}$ has a left dual. Similarly, one can show every object in $\mathcal{A}/\mathcal{C}$ has a right dual. Thus, $\mathcal{A}/\mathcal{C}$ is rigid. Consequently, $\mathcal{A}/\mathcal{C}$ is a multitensor category. □

Proposition 6.6. *Let $\mathcal{A}$ be a multifusion category, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{C}$ is a multifusion category.*

Proof. We have already know that $\mathcal{A}/\mathcal{C}$ is a multitensor category. By proposition 5.3, $\mathcal{A}/\mathcal{C}$ is a finite semisimple abelian category. As a result, $\mathcal{A}/\mathcal{C}$ is a multifusion category. □

The following proposition shows that a two-sided Serre tensor-ideal of a tensor category is always trivial.

Proposition 6.7. *Let $\mathcal{A}$ be a tensor category, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$, then $\mathcal{C}$ is trivial.*

Proof. Suppose $\mathcal{C}$ is not zero, choose a non-zero object B in $\mathcal{C}$. Because $\mathcal{C}$ is a two-sided Serre tensor-ideal, $B^* \otimes B \in \mathcal{C}$. Note that

$$ev_B : B^* \otimes B \to 1$$

is a non-zero epimorphism in $\mathcal{A}$, since 1 is simple and the composition $(id_B \otimes ev_B) \circ a_{B,B^*,B} \circ (coev_B \otimes id_B)$ is identity. Therefore, $1 \in \mathcal{C}$. Consequently, $\mathcal{A} = \mathcal{C}$. □

This proposition means that $\mathcal{A}/\mathcal{C}$ could only be zero or $\mathcal{A}$ itself.

However, someone may be interested in other definitions on tensor structures of $\mathcal{A}/\mathcal{C}$ in order to make $\mathcal{A}/\mathcal{C}$ be a monoidal category. Interestingly, we will prove next that no matter how we define the tensor structure, the canonical functor being monoidal implies the Serre subcategory is trivial.

Proposition 6.8. *Let $\mathcal{A}$ be a tensor category, $\mathcal{C}$ be a Serre subcategory of $\mathcal{A}$. Suppose the canonical functor T is a monoidal functor, then $\mathcal{C}$ is trivial.*

Proof. We prove it by contradiction. Assume $\mathcal{C}$ is non-trivial. Choose a non-zero object B in $\mathcal{C}$, we know that $TB = 0$ in $\mathcal{A}/\mathcal{C}$. This means $T(B^* \otimes B) \cong TB^* \otimes TB = 0$ in $\mathcal{A}/\mathcal{C}$. However,

$$ev_B : B^* \otimes B \to 1$$

is a non-zero epimorphism in $\mathcal{A}$ because 1 is a simple object in $\mathcal{A}$. Therefore,

$$T(ev_B) : T(B^* \otimes B) \cong TB^* \otimes TB = 0 \to T1$$

is an epimorphism by Lemma 3.1. Since T is a monoidal functor, $T1 \neq 0$ in $\mathcal{A}/\mathcal{C}$. However, this contradicts $T(ev_B)$ is an epimorphism in $\mathcal{A}/\mathcal{C}$. □

Now, we consider the representation category of $k\mathbb{Z}_2$.

Example 6.2. *Consider $\mathbb{Z}_2 = \{1, g\}$ and a field k such that* char $k \nmid 2$, *and denote the representation category of $k\mathbb{Z}_2$ by $\mathcal{A}$. As is well known, $\mathcal{A}$ is semisimple and it has only two irreducible representations $W_1 = k(1-g)$ and $W_2 = k(1+g)$. Define a homomorphism by*

$$\begin{aligned} \varphi : k(1-g) \otimes k(1-g) &\to k(1+g) \\ a(1-g) \otimes (1-g) &\mapsto a(1+g) \end{aligned}$$

where $a \in k$.

Let $\mathcal{C}$ be a Serre subcategory containing W_1. We now show that if $\mathcal{A}/\mathcal{C}$ is a tensor category, then $\mathcal{C} = \mathcal{A}$. Since $W_1 \in \mathcal{C}$, we know that $TW_1 = 0$ in $\mathcal{A}/\mathcal{C}$. It follows that $0 = TW_1 \otimes TW_1 \cong T(W_1 \otimes W_1)$ in $\mathcal{A}/\mathcal{C}$. This implies $W_1 \otimes W_1 \in \mathcal{C}$. Consequently, $W_2 = k(1+g) \in \mathcal{C}$ because $\varphi : W_1 \otimes W_1 \to W_2$ is a $k\mathbb{Z}_2$-isomorphism. Thus, both W_1 and W_2 are in $\mathcal{C}$, and $\mathcal{C} = \mathcal{A}$.

In order to make $\mathcal{A}/\mathcal{C}$ be a tensor category, it is clear that $W_2 = k(1+g)$ cannot be in $\mathcal{C}$, because $W_2 = k(1+g)$ is the unit object in $\mathcal{A}$.

6.3 Two-sided Serre tensor-ideal of a multiring category

The goal of this section is to show that a two-sided Serre tensor-ideal of a multiring category is actually a direct sum of some component subcategories. First of all, recall that a multiring category $\mathcal{A}$ can be written as a direct sum of its component subcategories $\mathcal{A} = \bigoplus_{i,j\in I} \mathcal{A}_{i,j}$, where $\mathcal{A}_{i,j} = 1_i \otimes \mathcal{A} \otimes 1_j$ and I is an indexed set such that $1 = \bigoplus_{i\in I} 1_i$.

Lemma 6.4. *Let $\mathcal{A}$ be a multiring category with left duals, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$, then $\mathcal{C} \cap \mathcal{A}_{i,j}$ is either $\mathcal{A}_{i,j}$ or 0, where $\mathcal{A}_{i,j}$ is a component subcategory of $\mathcal{A}$.*

Proof. Suppose $\mathcal{C}\cap\mathcal{A}_{i,j} \neq 0$. Choose $X \neq 0$ in $\mathcal{C}\cap\mathcal{A}_{i,j}$, we know that $X = 1_i \otimes X \otimes 1_j$. Because $X^* = (1_i \otimes X \otimes 1_j)^* = 1_j^* \otimes X^* \otimes 1_i^* = 1_j \otimes X^* \otimes 1_i$, one can observe that

$$(1_i \otimes X \otimes 1_j)^* \otimes 1_i \otimes X \otimes 1_j \in \mathcal{C} \cap \mathcal{A}_{j,j}.$$

We claim that $\mathrm{Im}(ev_X) = 1_j$. Assume 1_k is a direct summand of $\mathrm{Im}(ev_X)$, where $k \neq j$. Consider the exact sequence

$$X^* \otimes X \xrightarrow{ev_X} \mathrm{Im}(ev_X) \longrightarrow 0.$$

Tensoring this sequence with 1_k on the left, we obtain an exact sequence

$$1_k \otimes X^* \otimes X \xrightarrow{id_{1_k}\otimes ev_X} 1_k \longrightarrow 0.$$

Since

$$1_k \otimes X^* \otimes X = 1_k \otimes 1_j \otimes X^* \otimes 1_i \otimes 1_i \otimes X \otimes 1_j = 0,$$

the above exact sequence means $1_k = 0$, which is absurd. Therefore, 1_k is not a direct summand of $\mathrm{Im}(ev_X)$. Because $ev_X : X^* \otimes X \to 1$ is non-zero, we get that $\mathrm{Im}(ev_X) = 1_j$. Consequently, $1_j \in \mathcal{C}$. This implies $\mathcal{A}_{i,j} \subset \mathcal{C}$, and thus $\mathcal{C} \cap \mathcal{A}_{i,j} = \mathcal{A}_{i,j}$. □

In fact, if $\mathcal{A}_{i,j} \subset \mathcal{C}$, one can know $1_j \in \mathcal{C}$ from the above proof. As a result, $\mathcal{A}_{l,j} \subset \mathcal{C}$ and $\mathcal{A}_{j,l} \subset \mathcal{C}$ for all $l \in I$. Similar to the process of the above proof, if we consider

$$coev_X : 1 \to X \otimes X^* = 1_i \otimes X \otimes 1_j \otimes 1_j \otimes X^* \otimes 1_i,$$

then we can obtain $1_i \in \mathcal{C}$. Consequently, $\mathcal{A}_{l,i} \subset \mathcal{C}$ and $\mathcal{A}_{i,l} \subset \mathcal{C}$ for all $l \in I$.

In particular, if $0 \neq \mathcal{A}_{i,j} \subset \mathcal{C}$, then $0 \neq \mathcal{A}_{i,i} \subset \mathcal{C}$, $0 \neq \mathcal{A}_{j,j} \subset \mathcal{C}$, and $0 \neq \mathcal{A}_{j,i} \subset \mathcal{C}$. Besides,

$$\begin{aligned} \mathcal{C} &= \mathcal{C} \cap \mathcal{A} \\ &= \mathcal{C} \cap \bigoplus_{i,j \in I} \mathcal{A}_{i,j} \\ &= \bigoplus_{i,j \in I} (\mathcal{C} \cap \mathcal{A}_{i,j}). \end{aligned}$$

It follows from the above lemma that $\mathcal{C}$ is a direct sum of $\mathcal{A}_{i,j}$'s.

The following proposition provides a deeper understanding for two-sided Serre tensor-ideal.

Proposition 6.9. *Let $\mathcal{A}$ be a multiring category with left duals, $\mathcal{C}$ be a two-sided Serre tensor-ideal of $\mathcal{A}$. Let $J = \{i \in I \mid 1_i \in \mathcal{C}\}$, then $\mathcal{A}_{i,k} = 0$, $\mathcal{A}_{k,i} = 0$ for all $k \notin J$, $i \in J$.*

Proof. Assume $\mathcal{A}_{i,k} \neq 0$ for a given $i \in J$ and $k \notin J$. Since $1_i \in \mathcal{C}$, we know that $\mathcal{A}_{i,k} \subset \mathcal{C}$. For any $0 \neq X \in \mathcal{A}_{i,k}$, we can obtain $\mathrm{Im}(ev_X) = 1_k \in \mathcal{C}$ from a process similar to the proof of Lemma 6.4. This contradicts $k \notin J$. Thus, $\mathcal{A}_{i,k} = 0$. Similarly, $\mathcal{A}_{k,i} = 0$. □

Let $\mathcal{A}$ be a multiring category with left duals, the above proposition implies that a two-sided Serre tensor-ideal $\mathcal{C}$ of $\mathcal{A}$ can be written as $\mathcal{C} = \bigoplus_{i,j\in J} \mathcal{A}_{i,j}$, where $J = \{i \in I | 1_i \in \mathcal{C}\}$. Furthermore, for any $0 \neq X \in \mathcal{A}$, the above proposition indicates that

$$\begin{aligned} X &= \bigoplus_{i,j\in I}(1_i \otimes X \otimes 1_j) \\ &= \bigoplus_{i,j\notin J}(1_i \otimes X \otimes 1_j) \bigoplus \bigoplus_{i,j\in J}(1_i \otimes X \otimes 1_j). \end{aligned}$$

Let

$$X' = \bigoplus_{i,j\notin J}(1_i \otimes X \otimes 1_j) \text{ and } X'' = \bigoplus_{i,j\in J}(1_i \otimes X \otimes 1_j),$$

it is clear that $X'' \in \mathcal{C}$ and $X' \cong X$ in $\mathcal{A}/\mathcal{C}$. Therefore, $\mathcal{A}/\mathcal{C} \cong \bigoplus_{i,j\notin J} \mathcal{A}_{i,j}$.

Conversely, given a suitable subset J of I, is $\bigoplus_{i,j\in J} \mathcal{A}_{i,j}$ a two-sided Serre tensor-ideal? We show next that the answer is yes.

Proposition 6.10. *Let $\mathcal{A}$ be a multiring category with left duals, $J \subset I$ satisfying that $\mathcal{A}_{i,k} = 0$, $\mathcal{A}_{k,i} = 0$ for all $k \notin J$, $i \in J$. Then $\mathcal{C} = \bigoplus_{i,j\in J} \mathcal{A}_{i,j}$ is a two-sided Serre tensor-ideal of $\mathcal{A}$.*

Proof. It suffices to show $\mathcal{C}$ is a Serre subcategory. For any $X \in \mathcal{C}$, let Y be a subobject of X. We know that

$$0 \longrightarrow Y \longrightarrow X$$

is exact. For $p \notin J$ or $q \notin J$, tensoring 1_p and 1_q on the left and right respectively gives that

$$0 \longrightarrow 1_p \otimes Y \otimes 1_q \longrightarrow 1_p \otimes X \otimes 1_q$$

is exact. Because $1_p \otimes X \otimes 1_q$ is 0, we obtain $1_p \otimes Y \otimes 1_q = 0$. This means $Y \in \mathcal{C}$. Hence, $\mathcal{C}$ is closed under taking subobjects. Similarly, $\mathcal{C}$ is closed under taking quotient objects. Now, suppose there is an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

where $X, Z \in \mathcal{C}$. For $p \notin J$ or $q \notin J$, tensoring 1_p and 1_q on the left and right respectively gives that

$$0 \longrightarrow 1_p \otimes X \otimes 1_q \longrightarrow 1_p \otimes Y \otimes 1_q \longrightarrow 1_p \otimes Z \otimes 1_q \longrightarrow 0$$

is exact. Because $1_p \otimes X \otimes 1_q = 0 = 1_p \otimes Z \otimes 1_q$, we obtain that $1_p \otimes Y \otimes 1_q = 0$. This means $Y \in \mathcal{C}$. Hence, $\mathcal{C}$ is closed under taking extensions. It follows that $\mathcal{C}$ is a Serre subcategory. □

In summary, on one hand for $J \subset I$ satisfying that $\mathcal{A}_{i,k} = 0$, $\mathcal{A}_{k,i} = 0$ for all $k \notin J$, $i \in J$, $\mathcal{C} = \bigoplus_{i,j \in J} \mathcal{A}_{i,j}$ is a two-sided Serre tensor-ideal of $\mathcal{A}$. On the other hand, every two-sided Serre tensor-ideal of $\mathcal{A}$ can be written as $\mathcal{C} = \bigoplus_{i,j \in J} \mathcal{A}_{i,j}$ for some $J \subset I$.

Consider the two-sided Serre tensor-ideal $\mathcal{C} = \bigoplus_{i,j \in J} \mathcal{A}_{i,j}$. One can observe that the restriction of the canonical functor T on $\bigoplus_{i,j \notin J} \mathcal{A}_{i,j}$ is both an isomorphism and a monoidal functor. This implies the corresponding quotient category $\mathcal{A}/\mathcal{C}$ is actually isomorphic to $\bigoplus_{i,j \notin J} \mathcal{A}_{i,j}$ which is a subcategory of $\mathcal{A}$. Furthermore, it is easy to see that $T(1_{\mathcal{A}}) = T(\bigoplus_i 1_i) = \bigoplus_{i \notin J} 1_i$. Since T is a monoidal functor, $1_{\mathcal{A}/\mathcal{C}} = T(1_{\mathcal{A}}) = \bigoplus_{i \notin J} 1_i$. In fact, for $i, j \notin J$, $\mathcal{A}_{i,j}$ is a component subcategory of $\mathcal{A}/\mathcal{C}$.

Now, we claim that the image of another two-sided Serre tensor-ideal $\mathcal{C}'$ is a two-sided Serre tensor-ideal of the quotient category. Let $\mathcal{C} = \bigoplus_{i,j \in J} \mathcal{A}_{i,j}$, $\mathcal{C}' = \bigoplus_{i,j \in J'} \mathcal{A}_{i,j}$ be two two-sided Serre tensor-ideals of $\mathcal{A}$. Because $\mathcal{A}/\mathcal{C} = \bigoplus_{i,j \notin J} \mathcal{A}_{i,j}$, $T(\mathcal{C}') \cong \bigoplus_{i,j \in J' \setminus J} \mathcal{A}_{i,j}$ in $\mathcal{A}/\mathcal{C}$. We know that $\mathcal{A}_{i,k} = 0$, $\mathcal{A}_{k,i} = 0$ for all $k \notin J'$, $i \in J'$. Hence $\mathcal{A}_{i,k} = 0$, $\mathcal{A}_{k,i} = 0$ for all $k \notin J' \setminus J$, $i \in J' \setminus J$. This means the image of $\mathcal{C}'$ is a two-sided Serre tensor-ideal of $\mathcal{A}/\mathcal{C}$.

We end this chapter by discussing two-sided Serre tensor-ideals from a groupoid, one can refer the following example to section 4.13 in [10].

Example 6.3. *Let $\mathcal{G} = (X, G, \mu, s, t, u, i)$ be a groupoid whose set of objects X is finite and let $\mathcal{C}(\mathcal{G})$ be the category of finite dimensional vector spaces graded by the set G of morphisms of $\mathcal{G}$ i.e. vector spaces of the form $V = \bigoplus_{g \in G} V_g$. Introduce a tensor product* *on $\mathcal{C}(\mathcal{G})$ by the formula*

$$(V \otimes W)_g = \bigoplus_{(g_1,g_2):g_1g_2=g} V_{g_1} \otimes W_{g_2},$$

where V, W are objects in $\mathcal{C}(\mathcal{G})$.

We know that $\mathcal{C}(\mathcal{G})$ is a multitensor category. Suppose $A \in X$ is an object in $\mathcal{G}$ such that $\mathrm{Hom}_{\mathcal{G}}(A, B)$ *is empty for all $B \neq A$. For convenience, we denote $G(A) =$* $\mathrm{Hom}_{\mathcal{G}}(A, A)$*. Now, we show that $\{V = \bigoplus_{g \in G(A)} V_g\}$ is a two-sided Serre tensor-ideal of $\mathcal{C}(\mathcal{G})$. For any object W in $\mathcal{C}(\mathcal{G})$,*

$$((\bigoplus_{g \in G(A)} V_g) \otimes W)_k = \bigoplus_{(g_1,g_2):g_1g_2=k} (\bigoplus_{g \in G(A)} V_g)_{g_1} \otimes W_{g_2}.$$

Note that $(\bigoplus_{g \in G(A)} V_g)_{g_1} \otimes W_{g_2}$ is not zero only if $g_1 \in G(A)$. Consequently, it is not zero only if $k \in G(A)$. This means that $(\bigoplus_{g \in G(A)} V_g) \otimes W \in \{V = \bigoplus_{g \in G(A)} V_g\}$. Similarly, $W \otimes (\bigoplus_{g \in G(A)} V_g) \in \{V = \bigoplus_{g \in G(A)} V_g\}$. Thus, $\{V = \bigoplus_{g \in G(A)} V_g\}$ is a two-sided Serre tensor-ideal of $\mathcal{C}(\mathcal{G})$.

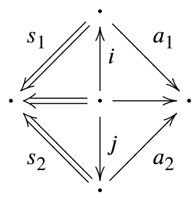
In fact, $\{V = \bigoplus_{g \in G'} V_g\}$ is also a two-sided Serre tensor-ideal of $\mathcal{C}(\mathcal{G})$ if G' is the morphism set of a connected component of $\mathcal{G}$. The above example is the case of a connected component consisting of one object.

6.4 Another view of tensor product in quotient category

Now, we turn our attention to the localization of a monoidal category by a multiplicative system, and we would like to define the tensor product on the quotient category. In this section, we still consider the right fraction.

Definition 6.2. *Let $\mathcal{A}$ be a monoidal category, and S be a multiplicative system of $\mathcal{A}$, and S is closed under tensor product. Define the tensor product of objects in $S^{-1}\mathcal{A}$ by the same tensor product in $\mathcal{A}$, and define the tensor product of two morphisms a/s and b/t in $S^{-1}\mathcal{A}$ by $(a \otimes b)/(s \otimes t)$.*

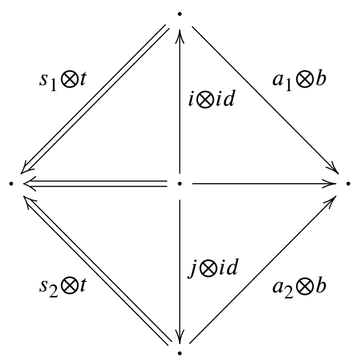
Next, we show that the tensor product for $S^{-1}\mathcal{A}$ is well-defined. If $a_1/s_1 = a_2/s_2$, which means that there is a commutative diagram



i.e. there exist two morphisms i, j such that

$$\begin{cases} a_1 i = a_2 j \\ s_1 i = s_2 j \in S \end{cases}$$

then the following diagram is commutative



where b/t is another morphism in $S^{-1}\mathcal{A}$. As a result, $(a_1 \otimes b)/(s_1 \otimes t) = (a_2 \otimes b)/(s_2 \otimes t)$. This indicates that $a_1/s_1 \otimes b/t = a_2/s_2 \otimes b/t$. Similarly, one can show $b/t \otimes a_1/s_1 = b/t \otimes a_2/s_2$. Consequently, the tensor product is well-defined.

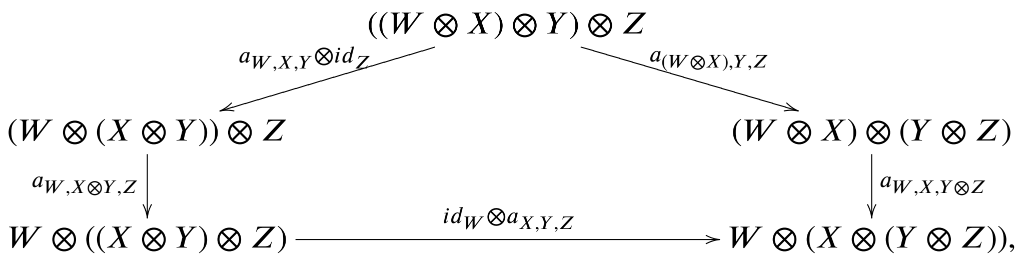
In addition, we define the associativity constraint in $S^{-1}\mathcal{A}$ by

$$a_{W,X,Y}/id_{(W \otimes X) \otimes Y} : (W \otimes X) \otimes Y \to W \otimes (X \otimes Y),$$

where W, X, Y are objects in $\mathcal{A}$ and $a_{W,X,Y}$ is the associativity constraint in $\mathcal{A}$. It is also clear that the left and right unit isomorphisms in $S^{-1}\mathcal{A}$ are $l_X/id_{1 \otimes X}$ and $r_X/id_{X \otimes 1}$ where l_X and r_X are the left and right unit isomorphisms in $\mathcal{A}$.

Proposition 6.11. *Let $\mathcal{A}$ be a monoidal category, and S be a multiplicative system of $\mathcal{A}$, and S is closed under tensor product. Then $S^{-1}\mathcal{A}$ is a monoidal category.*

Proof. Recall the pentagon axiom for $\mathcal{A}$



we want to show that the pentagon axiom for $S^{-1}\mathcal{A}$ is also held, i.e.

$$\begin{aligned}&\frac{id_W \otimes a_{X,Y,Z}}{id_{W\otimes((X\otimes Y)\otimes Z)}} \circ \frac{a_{W,X\otimes Y,Z}}{id_{(W\otimes(X\otimes Y))\otimes Z}} \circ \frac{a_{W,X,Y} \otimes id_Z}{id_{((W\otimes X)\otimes Y)\otimes Z}} \\ &= \frac{a_{W,X,Y\otimes Z}}{id_{(W\otimes X)\otimes(Y\otimes Z)}} \circ \frac{a_{W\otimes X,Y,Z}}{id_{((W\otimes X)\otimes Y)\otimes Z}}\end{aligned}$$

Because of the composition rule, the above equation is

$$\frac{(id_W \otimes a_{X,Y,Z}) \circ a_{W,X\otimes Y,Z} \circ (a_{W,X,Y} \otimes id_Z)}{id_{((W\otimes X)\otimes Y)\otimes Z}} = \frac{a_{W,X,Y\otimes Z} \circ a_{W\otimes X,Y,Z}}{id_{((W\otimes X)\otimes Y)\otimes Z}}.$$

The pentagon axiom for $\mathcal{A}$ states that

$$(id_W \otimes a_{X,Y,Z}) \circ a_{W,X\otimes Y,Z} \circ (a_{W,X,Y} \otimes id_Z) = a_{W,X,Y\otimes Z} \circ a_{W\otimes X,Y,Z},$$

thus the pentagon axiom for $S^{-1}\mathcal{A}$ is proved. As to the triangle axiom for $S^{-1}\mathcal{A}$, we need to show that

$$(id_X/id_X \otimes l_Y/id_{1\otimes Y}) \circ a_{X,1,Y}/id_{(X\otimes 1)\otimes Y} = r_X/id_{(X\otimes 1)} \otimes id_Y/id_Y,$$

which is

$$(id_X \otimes l_Y) \circ a_{X,1,Y}/id_{(X\otimes 1)\otimes Y} = r_X \otimes id_Y/id_{(X\otimes 1)\otimes Y}.$$

This follows from the triangle axiom for $\mathcal{A}$. □

Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product, C be a two-sided Serre tensor-ideal of $\mathcal{A}$. We can obtain a multiplicative system $S = \{f \in \mathrm{Mor}\mathcal{A} | \ker f,\ \mathrm{coker}\, f \in C\}$ from C by Proposition 4.1. Recall Theorem 4.1 that $S^{-1}\mathcal{A}$ is isomorphic to $\mathcal{A}/C$ via

$$F : \mathcal{A}/C \to S^{-1}\mathcal{A}$$
$$\bar{f} \to id_N/p^N_{N/N'} \circ f/i^{M'}_M$$

and

$$G : S^{-1}\mathcal{A} \to \mathcal{A}/C$$
$$f/s \to T(f) \circ T(s)^{-1}$$

where T is the canonical functor.

Next, we show that the tensor product for $S^{-1}\mathcal{A}$ coincide with the tensor product for $\mathcal{A}/C$ that we gave in Definition 6.1.

Proposition 6.12. *Let $\mathcal{A}$ be an abelian monoidal category with biexact tensor product, C be a two-sided Serre tensor-ideal of $\mathcal{A}$, and S be the multiplicative system induced by C. Then F and G are monoidal functors.*

Proof. It suffices to show that $F(\bar{f} \otimes \bar{g}) = F(\bar{f}) \otimes F(\bar{g})$ and $G(a/s \otimes b/t) = G(a/s) \otimes G(b/t)$,where $\bar{f} : A_1 \to A_2, \bar{g} : B_1 \to B_2$ are morphisms in $\mathcal{A}/C$, a/s, b/t are morphisms in $S^{-1}\mathcal{A}$.

Firstly, $\bar{f}$ is the image of $f : A_1' \to A_2/A_2'$ and $\bar{g}$ is the image of $g : B_1' \to B_2/B_2'$ i.e. $\bar{f} = Tf$, $\bar{g} = Tg$ where T is the canonical functor. For convenience, we denote $i^{A_1'}_{A_1} : A_1' \to A_1$ by i_1, $i^{B_1'}_{B_1} : B_1' \to B_1$ by i_2, $p^{A_2}_{A_2/A_2'} : A_2 \to A_2/A_2'$ by p_1, and $p^{B_2}_{B_2/B_2'} : B_2 \to B_2/B_2'$ by p_2. We know that

$$F(T(f \otimes g)) = id/(p_1 \otimes p_2) \circ (f \otimes g)/(i_1 \otimes i_2)$$

and

$$F(Tf) \otimes F(Tg) = (id/p_1 \circ f/i_1) \otimes (id/p_2 \circ g/i_2).$$

Because

$$(p_1 \otimes p_2) \circ ((id/p_1 \circ f/i_1) \otimes (id/p_2 \circ g/i_2)) = f/i_1 \otimes g/i_2 = (f \otimes g)/(i_1 \otimes i_2),$$

it follows that

$$((id/p_1 \circ f/i_1) \otimes (id/p_2 \circ g/i_2)) = id/(p_1 \otimes p_2) \circ (f \otimes g)/(i_1 \otimes i_2)$$

i.e.

$$F(Tf) \otimes F(Tg) = F(T(f \otimes g)).$$

Secondly, we know that

$$G(a/s \otimes b/t) = G((a \otimes b)/(s \otimes t)) = T(a \otimes b) \circ T(s \otimes t)^{-1}$$

and

$$G(a/s) \otimes G(b/t) = (Ta \circ (Ts)^{-1}) \otimes (Tb \circ (Tt)^{-1}).$$

Because

$$\begin{aligned} &((Ta \circ (Ts)^{-1}) \otimes (Tb \circ (Tt)^{-1})) \circ T(s \otimes t) \\ =&((Ta \circ (Ts)^{-1}) \otimes (Tb \circ (Tt)^{-1})) \circ (Ts \otimes Tt) \\ =&Ta \otimes Tb = T(a \otimes b), \end{aligned}$$

this means

$$(Ta \circ (Ts)^{-1}) \otimes (Tb \circ (Tt)^{-1}) = T(a \otimes b) \circ T(s \otimes t)^{-1}$$

i.e.

$$G(a/s) \otimes G(b/t) = G(a/s \otimes b/t).$$

□

Therefore, $S^{-1}\mathcal{A}$ and $\mathcal{A}/\mathcal{C}$ are isomorphic as monoidal categories when the multiplicative system S is induced by the Serre subcategory $\mathcal{C}$.

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攻读硕士学位期间研究成果

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