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On quantization of Lie algebra $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$

by

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南京大学研究生毕业论文中文摘要首页用纸

毕业论文题目：李代数 $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$ 的量子化

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摘 要

李双代数的概念由 Drinfeld 于 20 世纪 80 年代引入，作为量子群经典极限的代数框架。这一概念不仅统一了经典 Yang-Baxter 方程解的理论，还为量子群的构造提供了系统的代数基础。关于单李代数及其流代数上李双代数结构的分类，目前已取得诸多显著成果。李双代数与其量子化（QUE 代数）之间的联系亦属重要研究课题。最著名的拟三角 QUE 代数被称为量子群，它们是对可对称化 Kac-Moody 代数上标准李双代数结构的量子化。

本文旨在对截断流代数 $\mathfrak{g}_m := \mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$ 的李双代数结构进行初步研究，并且构造标准李双代数 $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$ 的一个量子化 $U_h(\mathfrak{g}_m)$。我们证明了 $\mathfrak{g}_m$ 上的任意李双代数结构均是上边界的，且由唯一的反对称 r-矩阵诱导。我们给出了 $\mathfrak{g}_m$ 上两类（拟）三角李双代数结构，它们可视为 $\mathfrak{sl}_2$ 上两个非平凡李双代数结构的推广。为了揭示 $\mathfrak{g}_m$ 上李双代数结构的复杂性，我们完整描述了一类拟三角李双代数结构。通过建立 $U_h(\mathfrak{g}_m)$ 的拓扑 PBW 基，我们证明了 $U_h(\mathfrak{g}_m)$ 是 $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$ 的量子化。最后，一个细致的上同调论证说明 $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$ 作为 $U(\mathfrak{g}_m)$ 的代数形变在 $(\mathrm{mod}\ h^4)$ 意义下是平凡的。

关键词：李双代数；量子化；截断流代数；Hopf 代数

南京大学研究生毕业论文英文摘要首页用纸

THESIS: On quantization of Lie algebra $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$

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ABSTRACT

The concept of Lie bialgebras was introduced by Drinfeld in the 1980s as the algebraic framework for the classical limit of quantum group. This concept not only unifies the theory of solutions to the classical Yang-Baxter equation but also provides a systematic algebraic foundation for the construction of quantum groups. Many remarkable results on classfying Lie bialgebra structures on simple Lie algebras and their current algebras have been obtained. Connection between Lie bialgebras and their quantizations, QUE algebras, is also an important topic. The most famous quasitriangular QUE algebras are called quantum groups, which are quantizations of the standard Lie bialgebra structures on symmetrizable Kac-Moody algebras.

The aim of this paper is to give a preliminary study on the Lie bialgebra structures on truncated current algebra $\mathfrak{g}_m := \mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$ and establish a quantization $U_h(\mathfrak{g}_m)$ of the so-called standard Lie bialgebra $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$. We show that any Lie bialgebra structure on $\mathfrak{g}_m$ is coboundary and induced by an unique antisymmetric r-matrix. Two types of (quasi)triangular Lie bialgebra structures on $\mathfrak{g}_m$ are constructed, which can be viewed as generalizations of the two nontrivial Lie bialgebra structures on $\mathfrak{sl}_2$. To show the complexity of Lie bialgebra structures on $\mathfrak{g}_m$, a class of quasitriangular Lie bialgebra structures is totally described. We establish a topological PBW basis of $U_h(\mathfrak{g}_m)$ to prove that $U_h(\mathfrak{g}_m)$ is a quantization of $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$. Finally, a careful argument on cohomology illustrates that $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$ is trivial as an algebra deformation $\pmod{h^4}$ of $U(\mathfrak{g}_m)$.

KEYWORDS: Lie bialgebras; Quantization; Truncated current algebras; Hopf algebras

Contents

Chapter 1

Introduction

1.1 Background

The concept of Lie bialgebras was introduced by Drinfeld in the 1980s as the algebraic framework for the classical limit of quantum groups [1]. A Lie bialgebra simultaneously carries a Lie algebra structure and a Lie coalgebra structure, satisfying a certain compatibility condition. This concept not only unifies the theory of solutions to the classical Yang-Baxter equation (CYBE) but also provides a systematic algebraic foundation for the construction of quantum groups.

It is natural to classify Lie bialgebras on a given Lie algebra $\mathfrak{g}$. Many important and remarkable results have been obtained: Belavin and Drinfeld provided a classification of the non-skew-symmetric solutions of CYBE for simple Lie algebras [2]. It is known that the problem of constructing skew-symmetric solutions of CYBE is equivalent to find all quasi-Frobenius Lie subalgebras of $\mathfrak{g}$ if $\dim \mathfrak{g} < \infty$, which is still open even if $\mathfrak{g}$ is simple [3, 4]. Lie bialgebra structures on current algebras $\mathfrak{g}[\mathbf{t}]$ or $\mathfrak{g}[[\mathbf{t}]]$ are also studied for $\mathfrak{g}$ simple: Montaner, Stolin and Zelmanov showed that Lie bialgebra structures on $\mathfrak{g}[[\mathbf{t}]]$ are in one-to-one correspondence with generalized Belavin-Drinfeld triple data on $\mathfrak{g}$, and any Lie bialgebra structure on $\mathfrak{g}[\mathbf{t}]$ either arises from a structure on $\mathfrak{g}[[\mathbf{t}]]$ or is generated by a quasi-trigonometric r-matrix [5]. A work by Abedin, Maximov, Stolin and Zelmanov classified topological Lie bialgebra structures on $\mathfrak{g}[[\mathbf{t}]]$ [6]. Lie bialgebra structures on loop algebras are also studied by Abedin and Maximov [7].

Connection between Lie bialgebras and their quantizations, QUE algebras, is also an important topic. For a finite dimensional Lie algebra $\mathfrak{g}$, any QUE algebra $U_h(\mathfrak{g})$ gives a Lie bialgebra structure $(\mathfrak{g}, \delta)$. Conversely, any Lie bialgebra structure on $\mathfrak{g}$ has a quantization [8, 9]. As quasitriangular Lie bialgebras provide solutions of CYBE, quasitriangular QUE algebras provide solutions of quantum Yang-Baxter equation (QYBE). The most famous quasitriangular QUE algebras are called quantum groups, which are quantizations of the standard Lie bialgebra structures on symmetrizable Kac-Moody algebras. See Example 2.3.6 and 2.4.6 or [3, 10, 11, 12] for more details.

The first truncated current Lie algebra appeared in the work of Takiff [13], and so they are often referred to as Takiff Lie algebras. In [13] Takiff showed that the symmetric invariant algebra $S(\mathfrak{g}[\mathbf{t}]/(\mathbf{t}^2))^{\mathfrak{g}[\mathbf{t}]/(\mathbf{t}^2)}$ is a polynomial algebra on $2\,\mathrm{rank}(\mathfrak{g})$ variables for some simple Lie algebra $\mathfrak{g}$. Later, Raïs and Tauvel extended Takiff's theorem for $S(\mathfrak{g}[\mathbf{t}]/(\mathbf{t}^{m+1}))^{\mathfrak{g}[\mathbf{t}]/(\mathbf{t}^{m+1})}$ and successfully described the center of $U(\mathfrak{g}[\mathbf{t}]/(\mathbf{t}^{m+1}))$ [14]. Based on their work, Chaffe and Topley studied the BGG category $\mathcal{O}$ of $\mathfrak{g}[\mathbf{t}]/(\mathbf{t}^{m+1})$ [15]. There is also a connection between truncated current Lie algebras and finite W-algebras given by Brundan and Kleshchev [16].

1.2 Main results and organizations

In this paper we mainly concern Lie bialgebra structures on the truncated current Lie algebra $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$ and construct a quantization of the so-called standard Lie bialgebra $(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}), \delta_{\mathbf{r}_0})$. The key results will be presented along with the structure of the article.

In Chapter 2 we recall some basic notations and concepts, including Lie algebras and their representations, cohomology of Lie algebras, Lie bialgebras and their quantizations. The standard Lie bialgebra structures on simple Lie algebras and their quantization, quantum groups, are briefly reviewed. Some results used in the following chapters are also proved here.

The main topic of Chapter 3 is to study Lie bialgebra structures on $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$. We first provide a basic observation on modules of $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$ in Section 3.1, which relies on the representation theory of $\mathfrak{sl}_2$. A special submodule of $(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}))^{\otimes n}$ is also totally described here for later application in Section 4.5, which is also a natural realization of the $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$-mod $V(2i) \otimes \mathbb{k}[\mathbf{t}]/(\mathbf{t}^{n+1})$. In Section 3.2 we show that any Lie bialgebra structures on $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$ is coboundary by proving that $\mathrm{H}^1(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}), \mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}) \otimes \mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})) = 0$. Then by generalizing the method in Example 2.3.6 we obtain a class of Lie bialgebra structures on $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$ via Manin triples, which is one-to-one correspondent with invertible elements in $\mathbb{k}[\mathbf{t}]/(\mathbf{t}^{m+1})$. Two more general (quasi)triangular Lie bialgebra structures are immediately deduced and classified up to gauge equivalence, except $\mathbf{r}_0$ and $\mathbf{r}_0 + \mathbf{r}_m$, which are not gauge equivalent in the case $m = 1$. We call $(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}), \mathbf{r}_0)$ the standard Lie bialgebra structure on $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$. To show the complexity of Lie bialgebra structures on $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$, by imitating the approach in [2], in Section 3.3 we find out a class of solutions of CYBE up to orthogonal automorphisms in the case $m = 1$ after a tedious calculation.

Chapter 4 provides a quantization $U_h(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}))$ of the standard Lie bialgebra $(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}), \delta_{\mathbf{r}_0})$. Sections 4.2-4.4 are contributed to prove the construction given in Theorem 4.1.1 is the desired quantization respect to Definition 2.4.4. In particular, Section 4.4 establishes a topological basis of $U_h(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}))$. Finally, a primary research on how trivial $U_h(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}))$ is as an algebra deformation of $U(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}))$ is given in Section 4.5, providing that the coefficients of all h's power in the multiplication of $U_h(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}))$ are 2-coboundaries. A careful argument is applied there to achieve our goal. As a corollary, we have $U_h(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})) \cong U(\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1}))[[h]] \pmod{h^4}$.

Chapter 2

Preliminaries

In this paper $\Bbbk$ is always an algebraically closed field of characteristic zero and all vector spaces are over $\Bbbk$.

2.1 Lie algebras and their representations

In this section we recall some basic results on Lie algebras and their representations (modules) for later application. For details see [17, 18].

Definition 2.1.1. A *Lie algebra* over $\Bbbk$ is a $\Bbbk$-vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that: (1) $[x,x]=0$ for all $x\in\mathfrak{g}$; (2) $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ for all $x,y,z\in\mathfrak{g}$. $[-,-]$ is called the *Lie bracket* of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$*-mod* or a *representation* of $\mathfrak{g}$ is a tuple (V,ϕ), where V is a $\Bbbk$-vector space and $\phi:\mathfrak{g}\to\mathrm{End}_{\Bbbk}(V)$ is a linear map, such that $\phi([x,y])=\phi(x)\phi(y)-\phi(y)\phi(x)$ for all $x,y\in V$. Denote $\phi(x)(v)$ by $x.v$ for all $x\in\mathfrak{g}$ and $v\in V$, then $x.(y.v)-y.(x.v)=[x,y].v$. A $\mathfrak{g}$*-submodule* of V is a subspace W such that $\mathfrak{g}.W\subseteq W$. It is clear that $(\Bbbk,0)$ is a $\mathfrak{g}$-mod, called the trivial $\mathfrak{g}$-mod. There is also an important $\mathfrak{g}$-mod $(\mathfrak{g},\mathrm{ad})$, called the *adjoint representation* of $\mathfrak{g}$, where $\mathrm{ad}(x)(y)=[x,y]$ for all $x,y\in\mathfrak{g}$.

A $\mathfrak{g}$-mod V is called *irreducible* or *simple* if V has no proper submodules. $\mathfrak{g}$ is called *simple* if $(\mathfrak{g},\mathrm{ad})$ is simple and $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$.

The simplest simple Lie algebra is $\mathfrak{sl}_2=\Bbbk\{H,X,Y\}$, whose Lie bracket is as follows:

$$[H,X]=2X,\quad [H,Y]=-2Y,\quad [X,Y]=H.$$

By [17, Section 7], we have

Proposition 2.1.2. *Let n be a non-negative integer. If V is an $(n+1)$-dimensional irreducible $\mathfrak{sl}_2$-mod, then there exists a basis $\{v_0,v_1,\dots,v_n\}$ such that*

$$H.v_i=(n-2i)v_i,\quad X.v_i=(n-i+1)v_{i-1},\quad Y.v_i=(i+1)v_{i+1},$$

where $v_{-1}=v_{n+1}=0$. Therefore any two $(n+1)$-dimensional irreducible $\mathfrak{sl}_2$-mods are isomorphic, and we can say the $(n+1)$-dimensional irreducible $\mathfrak{sl}_2$-mod and denote it by $V(n)$.

The following lemma is a useful method to construct irreducible submodules of a given finite dimensional $\mathfrak{sl}_2$-mod.

Lemma 2.1.3. *Let V be a finite-dimensional $\mathfrak{sl}_2$-mod and n be a non-negative integer. Let $v\in V$ be a nonzero vector such that $H.v=nv$ and $X.v=0$ (such v is called a maximal vector of weight n in V). Then $\mathfrak{sl}_2.v$ is an $(n+1)$-dimensional irreducible submodule of V.*

Proof. Denote $v_j = (j!)^{-1}Y^j.v$ for $j \in \mathbb{N}$ and $v_{-1} = 0$, then an easy computation shows that $H.v_i = (n-2i)v_i$ and $X.v_i = (n-i+1)v_{i-1}$. Therefore $\Bbbk\{v_i \mid i \in \mathbb{N}\}$ is a submodule of V. Since V is finite dimensional, there exists $m \in \mathbb{N}$ such that $v_i \neq 0$ if and only if $0 \leq i \leq m$. In particular, $0 = X.v_{m+1} = (n-m)v_m$, hence $m = n$. Then $\mathfrak{sl}_2.v = \Bbbk\{v_0, \dots, v_n\}$ is isomorphic to the $(n+1)$-dimensional irreducible $\mathfrak{sl}_2$-mod. $\square$

For any $\Bbbk$-vector space V, positive integer n and $\tau \in S_n$, let $p_\tau : V^{\otimes n} \to V^{\otimes n}$ be the linear map such that $p_\tau(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}$. We also denote $w_\tau := p_\tau(w)$ for any $w \in V^{\otimes n}$. Define a linear map $\sigma : V^{\otimes n} \to V^{\otimes n}$ as follows:

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \frac{1}{n!}\sum_{\tau \in S_n} p_\tau(v_1 \otimes v_2 \otimes \cdots \otimes v_n), \quad \forall v_1, v_2, \dots, v_n \in V.$$

Let $S^nV = \ker(\sigma - \mathrm{id}_{V^{\otimes n}})$. Similarly, define a linear map $\Lambda : V^{\otimes n} \to V^{\otimes n}$ as follows:

$$\Lambda(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \frac{1}{n!}\sum_{\tau \in S_n} (-1)^\tau p_\tau(v_1 \otimes v_2 \otimes \cdots \otimes v_n), \quad \forall v_1, v_2, \dots, v_n \in V.$$

Let $\Lambda^nV = \mathrm{Im}\,\Lambda$. We denote $v_1 \wedge v_2 \wedge \cdots \wedge v_n := n!\Lambda(v_1 \otimes \cdots \otimes v_n)$ for any $v_i \in V$. If $\dim V < \infty$, then

$$\dim S^nV = \binom{n + \dim V - 1}{\dim V - 1}, \quad \dim \Lambda^nV = \binom{\dim V}{n}.$$

Suppose V is a $\mathfrak{g}$-mod, then σ, Λ are $\mathfrak{g}$-mod maps and S^nV, Λ^nV are $\mathfrak{g}$-submodules of $V^{\otimes n}$. For later application, we give the decomposition of S^nV, where V is the 3-dimensional irreducible $\mathfrak{sl}_2$-mod. Some combinatorial identities are required.

Lemma 2.1.4. (1) *Let p, q be non-negative integers. Then*

$$\sum_{j=0}^{q}(-1)^j \frac{\binom{2q}{2j}\binom{p+2q-2j}{q-j}\binom{p+2q+1-j}{j}}{\binom{2p+4q-4j}{2q-2j}\binom{2p+4q+2-2j}{2j}} = \delta_{q,0}. \tag{2.1.1}$$

(2) *Let n, i, k be non-negative integers such that $n \geq 2i \geq 2k$. Then*

$$\sum_{j=0}^{k}(-1)^j \frac{\binom{n+1-i-j}{n+1-2i}\binom{k}{j}}{\binom{2n+2-2i-2j}{2n+2-4i}} = (-1)^k 2^{2k} \frac{(2n+2-4i)!(n+1-i)!(n-2i+k)!(2i-2k)!}{(n+1-2i)!(2n+2-2i)!(n-2i)!(i-k)!}. \tag{2.1.2}$$

Proof. (1) It is clear that (2.1.1) holds for $q = 0$. Now suppose $q \geq 1$. Then

$$\begin{aligned}
&\frac{1}{(2q-1)!!}\sum_{j=0}^{q}(-1)^j\frac{\binom{2q}{2j}\binom{p+2q-2j}{q-j}\binom{p+2q+1-j}{j}}{\binom{2p+4q-4j}{2q-2j}\binom{2p+4q+2-2j}{2j}}\\
=&\sum_{j=0}^{q}(-1)^j\binom{q}{j}\frac{2p+4q+1-4j}{(2p+4q+1-2j)(2p+4q-1-2j)\cdots(2p+2q+1-2j)}\\
=&\sum_{j=0}^{q}(-1)^j\left(\frac{1}{(2p+4q+1-2j)(2p+4q-1-2j)\cdots(2p+2q+3-2j)}\binom{q}{j}\right.\\
&\qquad\left.+\frac{2q}{(2p+4q+1-2j)(2p+4q-1-2j)\cdots(2p+2q+1-2j)}\binom{q-1}{j}\right)\\
=&\sum_{j=0}^{q}(-1)^j\left[\frac{1}{(2p+4q+1-2j)(2p+4q-1-2j)\cdots(2p+2q+3-2j)}\left(\binom{q-1}{j}+\binom{q-1}{j-1}\right)\right.\\
&\qquad\left.+\frac{2q}{(2p+4q+1-2j)(2p+4q-1-2j)\cdots(2p+2q+1-2j)}\binom{q-1}{j}\right]\\
=&\sum_{j=0}^{q-1}(-1)^j\frac{1}{(2p+4q-1-2j)(2p+4q-3-2j)\cdots(2p+2q+1-2j)}\binom{q-1}{j}\\
&+\sum_{j=1}^{q}(-1)^j\frac{1}{(2p+4q+1-2j)(2p+4q-1-2j)\cdots(2p+2q+3-2j)}\binom{q-1}{j-1}\\
=&0.
\end{aligned}$$

(2) Denote the LHS and RHS of (2.1.2) by $A_{n,i,k}$ and $B_{n,i,k}$ respectively. It is clear that $A_{n,i,k} = B_{n,i,k}$ if $k = i = 0$. Now suppose $n \geq 2i \geq 2k \geq 2$ and $A_{n',i',k'} = B_{n',i',k'}$ for all possible n', i', k' with $n' + i' + k' < n + i + k$. Then

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j\frac{\binom{n+1-i-j}{n+1-2i}\binom{k}{j}}{\binom{2n+2-2i-2j}{2n+2-4i}}\\
=&\sum_{j=0}^{k-1}(-1)^j\frac{\binom{n+1-i-j}{n+1-2i}}{\binom{2n+2-2i-2j}{2n+2-4i}}\binom{k-1}{j}+\sum_{j=1}^{k}(-1)^j\frac{\binom{n+1-i-j}{n+1-2i}}{\binom{2n+2-2i-2j}{2n+2-4i}}\binom{k-1}{j-1}\\
=&\sum_{j=0}^{k-1}(-1)^j\frac{\binom{n+1-i-j}{n+1-2i}}{\binom{2n+2-2i-2j}{2n+2-4i}}\binom{k-1}{j}-\sum_{j=0}^{k-1}(-1)^j\frac{\binom{n-i-j}{n+1-2i}}{\binom{2n-2i-2j}{2n+2-4i}}\binom{k-1}{j}\\
=&A_{n,i,k-1}-A_{n-2,i-1,k-1}\\
=&B_{n,i,k-1}-B_{n-2,i-1,k-1}\\
=&B_{n,i,k}.
\end{aligned}$$

An induction on $n + i + k$ completes the proof. ☐

The following lemma completely describes the structure of $S^nV(2)$.

Lemma 2.1.5. *Let* $V = \Bbbk\{v_0, v_1, v_2\}$ *be the 3-dimensional irreducible* $\mathfrak{sl}_2$*-mod such that*

$$H.v_i = (2-2i)v_i, \quad X.v_i = (3-i)v_{i-1}, \quad Y.v_i = (i+1)v_{i+1},$$

where $v_{-1} = v_3 = 0$. *Let* n *be a positive integer and* $M = S^nV$. *Then*

$$M \cong \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} V(2n-4i).$$

More explicitly, for any partition (n_0, n_1, n_2) *of* n*, denote*

$$u_{n_0,n_1,n_2} := \frac{n!}{n_0!n_1!n_2!}\sigma(v_0 \otimes \cdots \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_2),$$

where $v_0 \otimes \cdots \otimes v_0 \otimes v_1 \otimes \cdots \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_2$ *is the tenser product of* n_i v_i*'s. Let* $w_0 = u_{n,0,0}$ *and for any* $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$*, we define* w_i *recursively as follows:*

$$w_i = u_{n-i,0,i} - \sum_{j=0}^{i-1} \frac{1}{(2i-2j)!}\binom{n-2j}{i-j}\binom{2n-4j}{2i-2j}^{-1} Y^{2i-2j}.w_j.$$

Then $w_i \in M$ *is a maximal vector of weight* $2n-4i$ *and*

$$M = \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathfrak{sl}_2.w_i.$$

Moreover, M *is generated by* $u_{0,n,0} = v_1 \otimes \cdots \otimes v_1$ *and*

$$u_{0,n,0} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i \frac{2^{n-2i}(n-2i)!}{(2n-4i)!} Y^{n-2i}.w_i.$$

Proof. It is clear that $w_i \in M$ for each i by definition. First we show that for any $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$w_i = \sum_{j=0}^{i} \frac{(-1)^j}{(2j)!}\binom{n+1+j-2i}{j}\binom{2n+2+2j-4i}{2j}^{-1} Y^{2j}.u_{n-i+j,0,i-j}. \tag{2.1.3}$$

We use induction on i. The case $i = 0$ is apparent. Now suppose $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and (2.1.3) holds for all $j < i$. Then by definition

$$\begin{aligned}
u_{n-i,0,i} - w_i =& \sum_{j=0}^{i-1} \frac{1}{(2i-2j)!}\binom{n-2j}{i-j}\binom{2n-4j}{2i-2j}^{-1} Y^{2i-2j}.w_j \\
=& \sum_{j=0}^{i-1}\sum_{k=0}^{j} \frac{(-1)^k}{(2i-2j)!(2k)!} \frac{\binom{n-2j}{i-j}\binom{n+1+k-2j}{k}}{\binom{2n-4j}{2i-2j}\binom{2n+2+2k-4j}{2k}} Y^{2i-2j+2k}.u_{n-j+k,0,j-k} \\
=& \sum_{j=0}^{i-1}\sum_{k=0}^{j} \frac{(-1)^{j-k}}{(2i-2j)!(2j-2k)!} \frac{\binom{n-2j}{i-j}\binom{n+1-k-j}{j-k}}{\binom{2n-4j}{2i-2j}\binom{2n+2-2k-2j}{2j-2k}} Y^{2i-2k}.u_{n-k,0,k} \\
=& \sum_{k=0}^{i-1}\sum_{j=0}^{i-k-1} \frac{(-1)^{j}}{(2i-2j-2k)!(2j)!} \frac{\binom{n-2j-2k}{i-j-k}\binom{n+1-2k-j}{j}}{\binom{2n-4j-4k}{2i-2j-2k}\binom{2n+2-4k-2j}{2j}} Y^{2i-2k}.u_{n-k,0,k} \\
=& \sum_{k=0}^{i-1}\sum_{j=0}^{i-k-1} \frac{(-1)^{j}}{(2i-2k)!} \frac{\binom{2i-2k}{2j}\binom{n-2j-2k}{i-j-k}\binom{n+1-2k-j}{j}}{\binom{2n-4j-4k}{2i-2j-2k}\binom{2n+2-4k-2j}{2j}} Y^{2i-2k}.u_{n-k,0,k} \\
=& -\sum_{k=0}^{i-1} \frac{(-1)^{i-k}}{(2i-2k)!}\binom{n+1-i-k}{i-k}\binom{2n+2-2k-2i}{2i-2k}^{-1} Y^{2i-2k}.u_{n-k,0,k},
\end{aligned}$$

where the last equality comes from (2.1.1) by setting $p = n-2i$ and $q = i-k$. Therefore (2.1.3) holds for all i by induction hypothesis.

An easy induction shows that for any $s \geq 0$ and $0 \leq k \leq n$,

$$Y^s.u_{n-k,0,k} = \sum_{\substack{j=0,\\ 2|j+s}}^{s} s!\binom{k+\frac{s-j}{2}}{k} u_{n-k-\frac{s+j}{2},j,k+\frac{s-j}{2}}.$$

Therefore

$$\begin{aligned}
w_i =&\sum_{j=0}^{i} \frac{(-1)^j}{(2j)!}\binom{n+1+j-2i}{j}\binom{2n+2+2j-4i}{2j}^{-1} Y^{2j}.u_{n-i+j,0,i-j}\\
=&\sum_{j=0}^{i} \frac{(-1)^j}{(2j)!}\binom{n+1+j-2i}{j}\binom{2n+2+2j-4i}{2j}^{-1} \sum_{k=0}^{j}(2j)!\binom{i-k}{i-j}u_{n-i-k,2k,i-k}\\
=&\sum_{k=0}^{i}\sum_{j=0}^{i-k}(-1)^{i-j}\binom{n+1-i-j}{i-j}\binom{2n+2-2i-2j}{2i-2j}^{-1}\binom{i-k}{j}u_{n-i-k,2k,i-k}\\
=&\sum_{k=0}^{i}(-1)^k 2^{2(i-k)}\frac{(2n+2-4i)!(n+1-i)!(2k)!(n-i-k)!}{(n+1-2i)!(2n+2-2i)!k!(n-2i)!}u_{n-i-k,2k,i-k},
\end{aligned}$$

where the last equality is obtained from (2.1.2). Now we can compute $H.w_i$ and $X.w_i$:

$$\begin{aligned}
&H.w_i\\
=&\sum_{j=0}^{i} \frac{(-1)^j}{(2j)!}\binom{n+1+j-2i}{j}\binom{2n+2+2j-4i}{2j}^{-1} H.(Y^{2j}.u_{n-i+j,0,i-j})\\
=&\sum_{j=0}^{i} \frac{(-1)^j}{(2j)!}\binom{n+1+j-2i}{j}\binom{2n+2+2j-4i}{2j}^{-1} (2(n-i+j)-2(i-j)-4j)Y^{2j}.u_{n-i+j,0,i-j}\\
=&(2n-4i)w_i,
\end{aligned}$$

and

$$\begin{aligned}
&X.w_i\\
=&\sum_{k=0}^{i}(-1)^k 2^{2(i-k)}\frac{(2n+2-4i)!(n+1-i)!(2k)!(n-i-k)!}{(n+1-2i)!(2n+2-2i)!k!(n-2i)!}X.u_{n-i-k,2k,i-k}\\
=&\sum_{k=0}^{i}(-1)^k 2^{2(i-k)}\frac{(2n+2-4i)!(n+1-i)!(2k)!(n-i-k)!}{(n+1-2i)!(2n+2-2i)!k!(n-2i)!}\\
&\cdot(2(n-i-k+1)u_{n-i-k+1,2k-1,i-k}+(2k+1)u_{n-i-k,2k+1,i-k-1})\\
=&\sum_{k=0}^{i-1}\left((-1)^{k+1}2^{2(i-k-1)}\frac{(2n+2-4i)!(n+1-i)!(2k+2)!(n-i-k-1)!}{(n+1-2i)!(2n+2-2i)!(k+1)!(n-2i)!}2(n-i-k)\right.\\
&\left.+(-1)^k 2^{2(i-k)}\frac{(2n+2-4i)!(n+1-i)!(2k)!(n-i-k)!}{(n+1-2i)!(2n+2-2i)!k!(n-2i)!}(2k+1)\right)u_{n-i-k,2k+1,i-k-1}\\
=&\sum_{k=0}^{i-1}(-1)^k 2^{2(i-k-1)}\frac{(2n+2-4i)!(n+1-i)!}{(n+1-2i)!(2n+2-2i)!(n-2i)!}\\
&\left(-\frac{(2k+2)!(n-i-k-1)!}{(k+1)!}2(n-i-k)+2^2\frac{(2k)!(n-i-k)!}{k!}(2k+1)\right)u_{n-i-k,2k+1,i-k-1}\\
=&0.
\end{aligned}$$

Thanks to Lemma 2.1.3, $\mathfrak{sl}_2.w_i$ is a $(2n-4i+1)$-dimensional irreducible submodule of M for each $0 \le i \le \lfloor \frac{n}{2} \rfloor$. Moreover, since

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\dim \mathfrak{sl}_2.w_i = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(2n-4i+1) = \binom{n+2}{2} = \dim M,$$

we have

$$M = \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathfrak{sl}_2.w_i \cong \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} V(2n-4i).$$

Finally,

$$\begin{aligned}
&\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i \frac{2^{n-2i}(n-2i)!}{(2n-4i)!} Y^{n-2i}.w_i \\
=&\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=0}^{i}(-1)^{i+j}\frac{2^{n-2i}(n-2i)!}{(2n-4i)!(2j)!}\binom{n+1+j-2i}{j}\binom{2n+2+2j-4i}{2j}^{-1} Y^{n-2i+2j}.u_{n-i+j,0,i-j} \\
=&\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=0}^{i}(-1)^{i+j}\frac{2n+1-4i}{(2n+1+2j-4i)!!(2j)!!} Y^{n-2i+2j}.u_{n-i+j,0,i-j} \\
=&\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=0}^{i}(-1)^{j}\frac{2n+1-4i}{(2n+1-2j-2i)!!(2i-2j)!!} Y^{n-2j}.u_{n-j,0,j} \\
=&\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{i=j}^{\lfloor \frac{n}{2} \rfloor}(-1)^{j}\frac{2n+1-4i}{(2n+1-2j-2i)!!(2i-2j)!!} Y^{n-2j}.u_{n-j,0,j} \\
=&\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor-j}(-1)^{j}\frac{2n+1-4i-4j}{(2n+1-4j-2i)!!(2i)!!} Y^{n-2j}.u_{n-j,0,j} \\
=&\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^{j} Y^{n-2j}.u_{n-j,0,j}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor-j}\left(\frac{2n+1-2i-4j}{(2n+1-4j-2i)!!(2i)!!}-\frac{2i}{(2n+1-4j-2i)!!(2i)!!}\right) \\
=&\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^{j}\frac{1}{(n-2j)!} Y^{n-2j}.u_{n-j,0,j} \\
=&\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\sum_{\substack{k=0\\ 2|n+k}}^{n-2j}(-1)^{j}\binom{\frac{n-k}{2}}{j} u_{\frac{n-k}{2},k,\frac{n-k}{2}} \\
=&\sum_{\substack{k=0\\ 2|n+k}}^{n}\sum_{j=0}^{\frac{n-k}{2}}(-1)^{j}\binom{\frac{n-k}{2}}{j} u_{\frac{n-k}{2},k,\frac{n-k}{2}} \\
=&u_{0,n,0}.
\end{aligned}$$

Therefore M is generated by $u_{0,n,0}$. □

2.2 Cohomology of Lie algebras

From now on $\mathfrak{g}$ is always a Lie algebra. In this section we recall the cohomology theory of Lie algebras and establish the equivalence between the cohomology of $\mathfrak{g}$ and the Hochschild cohomology of its universal enveloping algebra $U(\mathfrak{g})$.

Definition 2.2.1. Let $\mathfrak{g}$ be a Lie algebra and M be a $\mathfrak{g}$-mod. The *n-th cohomology group* $\mathrm{H}^n(\mathfrak{g}, M)$ of $\mathfrak{g}$ with coefficients in M is defined as $\mathrm{Ext}^n_{U(\mathfrak{g})}(\mathbb{k}, M)$.

Let $\Lambda^n\mathfrak{g}$ be the n-th exterior product of the $\mathbb{k}$-vector space $\mathfrak{g}$. For $n > 0$ let $C^n(\mathfrak{g}, M) = \mathrm{Hom}_{\mathbb{k}}(\Lambda^n\mathfrak{g}, M)$ be the space of all antisymmetric n-linear maps from $\mathfrak{g}$ to M. Set $C^0(\mathfrak{g}, M) = M$.

The elements of $C^n(\mathfrak{g}, M)$ is called n-cochains on $\mathfrak{g}$ with coefficients in M. By [10, Chapter XVIII], there exists a complex as follows:

$$0 \to C^0(\mathfrak{g}, M) \xrightarrow{d^0_{\mathfrak{g}}} C^1(\mathfrak{g}, M) \xrightarrow{d^1_{\mathfrak{g}}} \cdots \to C^n(\mathfrak{g}, M) \xrightarrow{d^n_{\mathfrak{g}}} C^{n+1}(\mathfrak{g}, M) \to \cdots , \tag{2.2.1}$$

where for $f \in C^n(\mathfrak{g}, M)$,

$$(d^n_{\mathfrak{g}} f)(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i x_i f(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \tag{2.2.2}$$

$$+ \sum_{0 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n) \tag{2.2.3}$$

for all $x_0, \ldots, x_n \in \mathfrak{g}$. The hat $\widehat{\ }$ on a letter means that it has been omitted. The kernel and the image in $C^n(\mathfrak{g}, M)$ of derivation $d_{\mathfrak{g}}$ is denoted by $Z^n(\mathfrak{g}, M)$ and $B^n(\mathfrak{g}, M)$. An element of $Z^n(\mathfrak{g}, M)$ is called an *n-cocycle* on $\mathfrak{g}$ with coefficients in M whereas an element of $B^n(\mathfrak{g}, M)$ is called an *n-coboundary* on $\mathfrak{g}$ with coefficients in M.

Proposition 2.2.2. *The n-th cohomology group $Z^n(\mathfrak{g}, M)/B^n(\mathfrak{g}, M)$ of complex (2.2.1) is exactly $\mathrm{H}^n(\mathfrak{g}, M)$ for any $\mathfrak{g}$-mod M and $n \in \mathbb{N}$.*

Proof. The following projective resolution of trivial $U(\mathfrak{g})$-mod $\Bbbk$ completes the proof:

$$\cdots \to U(\mathfrak{g}) \otimes_{\Bbbk} \Lambda^n \mathfrak{g} \xrightarrow{\delta^{\mathfrak{g}}_n} U(\mathfrak{g}) \otimes_{\Bbbk} \Lambda^{n-1} \mathfrak{g} \xrightarrow{\delta^{\mathfrak{g}}_{n-1}} \cdots \to U(\mathfrak{g}) \otimes_{\Bbbk} \Lambda^1 \mathfrak{g} \xrightarrow{\delta^{\mathfrak{g}}_1} U(\mathfrak{g}) \xrightarrow{\epsilon} \Bbbk \to 0, \tag{2.2.4}$$

where ϵ is the counit of Hopf algebra $U(\mathfrak{g})$ and

$$\delta^{\mathfrak{g}}_n(u \otimes x_1 \wedge \cdots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n.$$

For details see [19, Section 7.7] or [20, Chapter XIII, Section 7]. □

An important result of cohomology of simple Lie algebras must be mentioned.

Proposition 2.2.3. *Let $\mathfrak{g}$ be a simple Lie algebra and M be a finite dimensional $\mathfrak{g}$-mod. Then $\mathrm{H}^1(\mathfrak{g}, M) = \mathrm{H}^2(\mathfrak{g}, M) = 0$. More generally, $\mathrm{H}^n(\mathfrak{g}, M) \cong \mathrm{H}^n(\mathfrak{g}, \Bbbk) \otimes M^{\mathfrak{g}}$ as vector spaces for any $n \geq 0$.*

Let us review the definition of Hochschild cohomology of an algebra. Let A be a $\Bbbk$-algebra and $A^e := A \otimes_{\Bbbk} A^{op}$ be the enveloping algebra of A. Any A-bimodule is a left A^e-mod and vice versa. Suppose N is an A-bimodule, for $n > 0$ let $C^n(A, N) = \mathrm{Hom}_{\Bbbk}(A^{\otimes n}, N)$ be the space of all n-bilinear maps from A^n to N. Set $C^0(A, N) = N$. By [19, Chapter 9] or [21, Chapter 11], there exists a complex as follows:

$$0 \to C^0(A, N) \xrightarrow{d^0_A} C^1(A, N) \xrightarrow{d^1_A} \cdots \to C^n(A, N) \xrightarrow{d^n_A} C^{n+1}(A, N) \to \cdots , \tag{2.2.5}$$

where for $f \in C^n(A, N)$,

$$(d^n_A f)(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1})$$

$$+ \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}$$

for all $x_1, \ldots, x_{n+1} \in A$. Denote the kernel and the image in $C^n(A, N)$ of derivation d_A by $Z^n(A, N)$ and $B^n(A, N)$ respectively. Elements of $Z^n(A, N)$ and $B^n(A, N)$ are called *n-th cocycles* and *n-th coboundaries* on A with coefficients in N respectively. The *n-th Hochschild cohomology group* of A with coefficients in N is defined as $\mathrm{HH}^n(A, N) := Z^n(A, N)/B^n(A, N)$.

Proposition 2.2.4. *Let A be a A-bimodule in the natural way. Then $\mathrm{HH}^n(A,N)=\mathrm{Ext}^n_{A^e}(A,N)$ for any A-bimodule N and $n\in\mathbb{N}$.*

Proof. The desired result follows from the following projective resolution of A as a left A^e-mod:

$$\cdots\to A^{\otimes(n+2)}\xrightarrow{\delta^A_n}A^{\otimes(n+1)}\xrightarrow{\delta^A_{n-1}}\cdots\to A^{\otimes 3}\xrightarrow{\delta^n_1}A^e\xrightarrow{\mu}A\to 0, \tag{2.2.6}$$

where μ is the multiplication of A and

$$\delta^A_n(x_0\otimes\cdots\otimes x_{n+1})=\sum_{i=0}^{n}(-1)^i x_0\otimes\cdots\otimes(x_ix_{i+1})\otimes\cdots\otimes x_{n+1}.$$

For details see [20, Chapter IX, Section 6] . ☐

Now we suppose A is a Hopf algebra. Let V,W be two left A-mods, then $\mathrm{Hom}_{\Bbbk}(V,W)$ and $V\otimes_{\Bbbk}W$ are also left A-mods via:

$$(x.f)(v):=\sum x_{(1)}f(S(a_{(2)})v),\quad x.(v\otimes w):=\sum x_{(1)}v\otimes x_{(2)}w.$$

There are natural isomorphisms:

$$\mathrm{Hom}_{\Bbbk}(U\otimes_{\Bbbk}V,W)\cong\mathrm{Hom}_{\Bbbk}(U,\mathrm{Hom}_{\Bbbk}(V,W)),\quad \mathrm{Hom}_A(U\otimes_{\Bbbk}V,W)\cong\mathrm{Hom}_A(U,\mathrm{Hom}_{\Bbbk}(V,W)).$$

If A is cocommutative then $U\otimes_{\Bbbk}V\cong V\otimes_{\Bbbk}U$ as A-mods, hence

$$\mathrm{Hom}_{\Bbbk}(U\otimes_{\Bbbk}V,W)\cong\mathrm{Hom}_{\Bbbk}(V,\mathrm{Hom}_{\Bbbk}(U,W)),\quad \mathrm{Hom}_A(U\otimes_{\Bbbk}V,W)\cong\mathrm{Hom}_A(V,\mathrm{Hom}_{\Bbbk}(U,W)).$$

Suppose P and V are left A-mods and P is projective, then $P\otimes_{\Bbbk}V$ is a projective left A-mod since $\mathrm{Hom}_A(P\otimes_{\Bbbk}V,*)\cong\mathrm{Hom}_A(P,\mathrm{Hom}_{\Bbbk}(V,*))$. If A is cocommutative then $V\otimes_{\Bbbk}P$ is also projective. For right A-mods there is also a similar argument.

Lemma 2.2.5. *Let $(A,m,u,\Delta,\epsilon,S)$ be a Hopf algebra. Then:*
(1) *there exists an injective algebra homomorphism $\eta:A\to A^e$ defined by:*

$$\eta(x)=\sum x_{(1)}\otimes S(x_{(2)}).$$

(2) *Consider A^e to be a right A-mod via right multiplication by elements of $\eta(A)$. Then $A^e\otimes_A\Bbbk\cong A$ as left A^e-mods.*
(3) *If the antipode S is bijective, then A^e is a projective right A-mod.*

Proof. The proof of statement (1) is a routine verification. For statement (2), we give isomorphisms $f:A\to A^e\otimes_A\Bbbk$ and $g:A^e\otimes\Bbbk\to A$ as follows:

$$f(x):=x\otimes 1\otimes 1,\quad g(x\otimes y\otimes 1):=xy.$$

One can easily check that $g\circ f=\mathrm{id}_A$ and $f\circ g=\mathrm{id}_{A^e\otimes\Bbbk}$.

Now suppose S is bijective. Equip A and A^{op} right A-mod structures by multiplication on A and by multiplication by $S(A)$ on A^{op}. Then S becomes an isomorphism of right A-mods since $S(b.a)=S(ba)=S(a)S(b)=S(b).a$ for all $a,b\in A$. Moreover, S induces an isomorphism of right A-mods from $A\otimes A$ to $A\otimes A^{op}=A^e$. Therefore A^e is a projective right A-mod. ☐

Suppose N is an A-bimodule, let N^{ad} be the left A-mod whose module structure is defined by

$$x.a:=\sum x_{(1)}aS(x_2),\quad \forall x\in A,a\in N.$$

Proposition 2.2.6. *Let A be a Hopf algebra with invertible antipode. Then for any A-bimodule N and $n\in\mathbb{N}$, $\mathrm{HH}^n(A,N)\cong\mathrm{Ext}^n_A(\Bbbk,N^{\mathrm{ad}})$.*

Proof. Take a projective resolution $P^\bullet$ of left A-mod $\Bbbk$. Since A^e is a projective right A-mod, then $A^e \otimes_A P^\bullet$ is a projective resolution of left A^e-mod $A^e \otimes_A \Bbbk$. By Lemma 2.2.5 $A^e \otimes_A \Bbbk \cong A$ as left A^e-mods. For any A-bimodule M, there holds $\mathrm{Hom}_{A^e}(A^e, M) \cong M^{\mathrm{ad}}$ as left A-mod. Therefore $\mathrm{Hom}_{A^e}(A^e \otimes_A P^\bullet, M) \cong \mathrm{Hom}_A(P^\bullet, \mathrm{Hom}_{A^e}(A^e, M)) \cong \mathrm{Hom}_A(P^\bullet, M^{\mathrm{ad}})$. Finally we have $\mathrm{HH}^n(A, N) \cong \mathrm{Ext}^n_{A^e}(A, N) \cong \mathrm{Ext}^n_A(\Bbbk, N^{\mathrm{ad}})$. □

Now let A be the universal enveloping algebra $U(\mathfrak{g})$ of some Lie algebra $\mathfrak{g}$. Any $\mathfrak{g}$-mod M is also a left $U(\mathfrak{g})$-mod and vice versa. Hence $\mathrm{HH}^n(U(\mathfrak{g}), M) \cong \mathrm{H}^n(\mathfrak{g}, M^{\mathrm{ad}})$ by Proposition 2.2.6. For latter application we give an isomorphism between $\mathrm{HH}^n(U(\mathfrak{g}), M)$ and $\mathrm{H}^n(\mathfrak{g}, M^{\mathrm{ad}})$.

Proposition 2.2.7. *Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$ and M be a $U(\mathfrak{g})$-bimodule. For any $n \in \mathbb{N}$, there exists an isomorphism ϕ^n as vector spaces:*

$$\phi^n : \mathrm{HH}^n(U(\mathfrak{g}), M) \to \mathrm{H}^n(\mathfrak{g}, M^{\mathrm{ad}})$$

$$f + B^n(U(\mathfrak{g}), M) \mapsto f^\sigma + B^n(\mathfrak{g}, M^{\mathrm{ad}}),$$

where

$$f^\sigma(x_1, \ldots, x_n) = \sum_{\tau \in S_n} (-1)^\tau f(x_{\tau(1)}, \ldots, x_{\tau(n)}), \quad \forall x_1, \ldots, x_n \in \mathfrak{g}.$$

Proof. (2.2.4) and (2.2.6) give projective resolution $X^\bullet$ of left $U(\mathfrak{g})$-mod $\Bbbk$ and projective resolution $P^\bullet$ of left $U(\mathfrak{g})^e$-mod $U(\mathfrak{g})$ respectively. We only need to give a chain map $\varphi^\bullet : U(\mathfrak{g})^e \otimes_{U(\mathfrak{g})} X^\bullet \to P^\bullet$ which yields the expected isomorphisms. Indeed, φ is as follows: for any $a \otimes c \otimes b \otimes x_1 \wedge \cdots \wedge x_n \in U(\mathfrak{g})^e \otimes_{U(\mathfrak{g})} U(\mathfrak{g}) \otimes_\Bbbk \Lambda^n \mathfrak{g}$,

$$\begin{aligned}\varphi^n(a \otimes c \otimes b \otimes x_1 \wedge \cdots \wedge x_n) =& \varphi^n(ab_{(1)} \otimes S(b_{(2)})c \otimes 1 \otimes x_1 \wedge \cdots \wedge x_n) \\ =& ab_{(1)} \otimes \left(\sum_{\tau \in S_n} (-1)^\tau x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)} \right) \otimes S(b_{(2)})c.\end{aligned}$$

It is clear that each φ^n is a $U(\mathfrak{g})^e$-mod map. Now we prove $\varphi^\bullet$ is a chain map. If $n = 0$, then for any $b \in U(\mathfrak{g})$,

$$\mu\varphi^1(1 \otimes 1 \otimes b) = \mu(b_{(1)} \otimes S(b_{(2)})) = \epsilon(b).$$

Now suppose $n \geq 1$. For convenience we denote $v_1 \otimes \cdots \otimes v_n$ by $\{v_1, \ldots, v_n\}$ and $v_1 \wedge \cdots \wedge v_n$ by $\langle v_1, \ldots, v_n \rangle$. On one hand,

$$\begin{aligned}&\delta_n^{U(\mathfrak{g})} \varphi^n(1 \otimes 1 \otimes b \otimes \langle x_1, \ldots, x_n \rangle) \\ =& \sum_{\tau \in S_n} (-1)^\tau \Bigg(\{b_{(1)}x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(n)}, S(b_{(2)})\} + (-1)^n \{b_{(1)}, x_{\tau(1)}, \ldots, x_{\tau(n-1)}, x_{\tau(n)}S(b_{(2)})\} \\ &+ \sum_{i=1}^{n-1} (-1)^i \{b_{(1)}, x_{\tau(1)}, \ldots, x_{\tau(i)}x_{\tau(i+1)}, \ldots, x_{\tau(n)}, S(b_{(2)})\} \Bigg).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\varphi^{n-1}(\mathrm{id}_{U(\mathfrak{g})^e} \otimes \delta_n^{\mathfrak{g}})(1 \otimes 1 \otimes b \otimes \langle x_1, \ldots, x_n \rangle) \\ =& \varphi^{n-1} \Bigg(1 \otimes 1 \otimes \sum_{i=1}^n (-1)^{i+1} b x_i \otimes \langle x_1, \ldots, \widehat{x_i}, \ldots, x_n \rangle \\ &+ 1 \otimes 1 \otimes \sum_{1 \leq i < j \leq n} (-1)^{i+j} b \otimes \langle [x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n \rangle \Bigg).\end{aligned}$$

For each i, denote $\langle x_1, \dots, \widehat{x_i}, \dots, x_n\rangle = \langle y_1^i, \dots, y_{n-1}^i\rangle$, then

$$\begin{aligned}
&\varphi^{n-1}\left(1 \otimes 1 \otimes \sum_{i=1}^{n} (-1)^{i+1} b x_i \otimes \langle x_1, \dots, \widehat{x_i}, \dots, x_n\rangle\right)\\
=&\sum_{i=1}^{n} (-1)^{i+1} \sum_{\pi \in S_{n-1}} (-1)^{\pi} \left(\{b_{(1)}x_i, y^i_{\pi(1)}, \dots, y^i_{\pi(n-1)}, S(b_{(2)})\} - \{b_{(1)}, y^i_{\pi(1)}, \dots, y^i_{\pi(n-1)}, x_i S(b_{(2)})\}\right)\\
=&\sum_{i=1}^{n} (-1)^{i+1} \left(\sum_{\substack{\tau \in S_n\\ \tau(1)=i}} (-1)^{\tau}(-1)^{i-1}\{b_{(1)}x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(n)}, S(b_{(2)})\}\right.\\
&\qquad\qquad\left. - \sum_{\substack{\tau \in S_n\\ \tau(n)=i}} (-1)^{\tau}(-1)^{n-i}\{b_{(1)}, x_{\tau(1)}, \dots, x_{\tau(n-1)}, x_{\tau(n)} S(b_{(2)})\}\right)\\
=&\sum_{\tau \in S_n} (-1)^{\tau} \left(\{b_{(1)}x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(n)}, S(b_{(2)})\} + (-1)^n \{b_{(1)}, x_{\tau(1)}, \dots, x_{\tau(n-1)}, x_{\tau(n)} S(b_{(2)})\}\right).
\end{aligned}$$

Similarly, denote $\langle [x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n\rangle$ by $\langle z_1^{i,j}, \dots, z_{n-1}^{i,j}\rangle$ for $1 \le i < j \le n$, then

$$\begin{aligned}
&\varphi^{n-1}\left(1 \otimes 1 \otimes \sum_{1 \le i<j \le n} (-1)^{i+j} b \otimes \langle [x_i, x_j], x_1, \dots, \widehat{x_i}, \dots, \widehat{x_j}, \dots, x_n\rangle\right)\\
=&\sum_{1 \le i<j \le n} (-1)^{i+j} \sum_{\pi \in S_{n-1}} (-1)^{\pi}\{b_{(1)}, z^{i,j}_{\pi(1)}, z^{i,j}_{\pi(2)}, \dots, z^{i,j}_{\pi(n-1)}, S(b_{(2)})\}\\
=&\sum_{1 \le i<j \le n} (-1)^{i+j} \sum_{k=1}^{n-1} \sum_{\substack{\pi \in S_{n-1}\\ \pi(k)=1}} (-1)^{\pi}\{b_{(1)}, z^{i,j}_{\pi(1)}, \dots, z^{i,j}_{\pi(k-1)}, x_i x_j - x_j x_i, z^{i,j}_{\pi(k+1)}, \dots, z^{i,j}_{\pi(n-1)}, S(b_{(2)})\}\\
=&\sum_{1 \le i<j \le n} (-1)^{i+j}\\
&\cdot \sum_{k=1}^{n-1} \left(\sum_{\substack{\tau \in S_n, \tau(k)=i\\ \tau(k+1)=j}} (-1)^{\tau}(-1)^{i+j+k}\{b_{(1)}, x_{\tau(1)}, \dots, x_{\tau(k-1)}, x_{\tau(k)}x_{\tau(k+1)}, x_{\tau(k+2)}, \dots, x_{\tau(n)}, S(b_{(2)})\}\right.\\
&\qquad\left. - \sum_{\substack{\tau \in S_n, \tau(k)=j\\ \tau(k+1)=i}} (-1)^{\tau}(-1)^{i+j+k}\{b_{(1)}, x_{\tau(1)}, \dots, x_{\tau(k-1)}, x_{\tau(k)}x_{\tau(k+1)}, x_{\tau(k+2)}, \dots, x_{\tau(n)}, S(b_{(2)})\}\right)\\
=&\sum_{k=1}^{n-1} (-1)^k \sum_{\tau \in S_n} (-1)^{\tau}\{b_{(1)}, x_{\tau(1)}, \dots, x_{\tau(k)}x_{\tau(k+1)}, \dots, x_{\tau(n)}, S(b_{(2)})\}.
\end{aligned}$$

Therefore φ is a chain map.

For any $f \in \mathrm{Hom}_{\Bbbk}(U(\mathfrak{g})^{\otimes n}, M) \cong \mathrm{Hom}_{U(\mathfrak{g})^e}(U(\mathfrak{g})^{\otimes(n+2)}, M)$, we have

$$\mathrm{Hom}(M, \varphi^n)(f)(1 \otimes 1 \otimes 1 \otimes x_1 \wedge \cdots \wedge x_n) = f\left(1 \otimes \left(\sum_{\tau \in S_n} (-1)^{\tau} x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}\right) \otimes 1\right),$$

which exactly yields the isomorphism ϕ^n since $\mathrm{Hom}_{U(\mathfrak{g})^e}(U(\mathfrak{g})^e \otimes_{U(\mathfrak{g})} U(\mathfrak{g}) \otimes_{\Bbbk} \Lambda^n \mathfrak{g}, M)$ is isomorphic to $\mathrm{Hom}_{\Bbbk}(\Lambda^n \mathfrak{g}, M^{\mathrm{ad}})$. □

2.3 Lie bialgebras

In this section we recall some basic definitions and results in Lie bialgebra theory. For more details see [3].

Definition 2.3.1. Let $\mathfrak{g}$ be a Lie algebra. A *Lie bialgebra structure* on $\mathfrak{g}$ is a skew-symmetric linear map $\delta_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, called the *cocommutator*, such that
(i) $\delta_{\mathfrak{g}}^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$.
(ii) $\delta_{\mathfrak{g}}$ is an 1-cocycle on $\mathfrak{g}$ with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$.

A Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is called *coboundary* if δ is an 1-coboundary, that is, there is some $r \in \mathfrak{g}\otimes\mathfrak{g}$ such that $\delta_{\mathfrak{g}}(x) = x.r$ for all $x \in \mathfrak{g}$. In this case we say $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is induced by r. Conversely, let $r \in \mathfrak{g} \otimes \mathfrak{g}$. If 1-coboundary $\delta_r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}, x \mapsto x.r$ is a Lie bialgebra structure on $\mathfrak{g}$, we say r induces a Lie bialgebra structure on $\mathfrak{g}$.

A *homomorphism* (an *isomorphism*) of Lie bialgebras $f : (\mathfrak{g}, \delta_{\mathfrak{g}}) \to (\mathfrak{h}, \delta_{\mathfrak{h}})$ is a homomorphism (an isomorphism) of Lie algebras such that $(f \otimes f) \circ \delta_{\mathfrak{g}} = \delta_{\mathfrak{h}} \circ f$. Two Lie bialgebras $(\mathfrak{g}, \delta_{\mathfrak{g}}), (\mathfrak{h}, \delta_{\mathfrak{h}})$ are said to be *gauge equivalent* if there exists a nonzero scalar c such that $(\mathfrak{g}, c\delta_{\mathfrak{g}})$ and $(\mathfrak{h}, \delta_{\mathfrak{h}})$ are isomorphic. In particular, if $(\mathfrak{g}, \delta_{\mathfrak{g}}), (\mathfrak{h}, \delta_{\mathfrak{h}})$ are coboundary Lie bialgebras induced by $r_1 \in \mathfrak{g} \otimes \mathfrak{g}$ and $r_2 \in \mathfrak{h} \otimes \mathfrak{h}$ respectively, then $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and $(\mathfrak{h}, \delta_{\mathfrak{h}})$ are gauge equivalent if and only if there exists a nonzero scalar c and an isomorphism $\alpha : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras such that $r_2 - (\alpha \otimes \alpha)(cr_1)$ is $\mathfrak{h}$-invariant.

Proposition 2.2.3 provides that any Lie bialgebra structure of a simple Lie algebra is coboundary. Let $r = \sum_{i=1}^{m} a_i \otimes b_i$ and $r' = \sum_{j=1}^{n} a_i' \otimes b_i'$ be elements in $\mathfrak{g} \otimes \mathfrak{g}$. Denote

$$r_{1,2} = r, \quad r_{2,1} = \sum_{i=1}^{n} b_i \otimes a_i,$$

$$[r_{1,2}, r'_{1,3}] = \sum_{i=1}^{m}\sum_{j=1}^{n}[a_i, a_j'] \otimes b_i \otimes b_j',$$

$$[r_{1,2}, r'_{2,3}] = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i \otimes [b_i, a_j'] \otimes b_j',$$

$$[r_{1,3}, r'_{2,3}] = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i \otimes a_j' \otimes [b_i, b_j']$$

and $[[r, r']] = [r_{1,2}, r'_{1,3}] + [r_{1,2}, r'_{2,3}] + [r_{1,3}, r'_{2,3}]$. An easy computation shows that $[[r, r]] \in \Lambda^3\mathfrak{g}$ if $r \in \Lambda^2\mathfrak{g}$.

Proposition 2.3.2. *[3, Proposition 2.1.2] Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$. Then r induces a Lie bialgebra structure if and only if both $r_{1,2} + r_{2,1}$ and $[[r, r]]$ are $\mathfrak{g}$-invariant. Therefore any coboundary Lie bialgebra structure on $\mathfrak{g}$ can be induced by an element in $\Lambda^2\mathfrak{g}$.*

Definition 2.3.3. A coboundary Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is called *quasitriangular* if $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is induced by some $r \in \mathfrak{g}\otimes\mathfrak{g}$ such that $[[r, r]] = 0$. Moreover, if in this case $r_{1,2} + r_{2,1} = 0$, then $(\mathfrak{g}, \delta_{\mathfrak{g}})$ is called *triangular*.

The equation $[[r, r]] = 0$ is called the classical Yang-Baxter equation (CYBE). From now on we suppose $\mathfrak{g}$ is finite dimensional.

Definition 2.3.4. A *Manin triple* is a triple of finite dimensional Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ together with a nondegenerate symmetric invariant bilinear form $(-, -)_{\mathfrak{g}}$ on $\mathfrak{g}$, such that
(i) $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Lie subalgebras;
(ii) $\mathfrak{g}_+, \mathfrak{g}_-$ are isotropic for $(-, -)_{\mathfrak{g}}$.

Proposition 2.3.5. *[3, Proposition 1.3.4] For any finite dimensional Lie algebra $\mathfrak{g}$, there is an 1-1 correspondence between Lie bialgebra structures on $\mathfrak{g}$ and Manin triples $(\mathfrak{h}, \mathfrak{h}_+, \mathfrak{h}_-)$ such that $\mathfrak{h}_+ = \mathfrak{g}$.*

Proposition 2.3.5 provides a strengthful method to find nontrivial Lie bialgebra structures. See following example.

Example 2.3.6. (1) Let $A = (a_{i,j})_{1\leq i,j,\leq n}$ be a indecomposable generalized Cartan matrix of finite type and $\mathfrak{g} = \mathfrak{g}(A)$ be the simple Lie algebra associated to A, with following notations (for details see [22]):
(i) Let Φ be the root system of $\mathfrak{g}$ and $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots. Then $\mathfrak{g}$ is generated by H_i, X_i, Y_i and relations

$$[H_i, H_j] = 0, \quad [H_i, X_j] = a_{i,j}X_j, \quad [H_i, Y_j] = -a_{i,j}Y_j, \quad [X_i, Y_j] = \delta_{i,j}H_i,$$

$$\mathrm{ad}(X_i)^{1-a_{i,j}}(X_j) = 0, \quad \mathrm{ad}(Y_i)^{1-a_{i,j}}(Y_j) = 0, \quad \forall i \neq j.$$

(ii) $\mathrm{diag}(d_1, \cdots)A$ is symmetric and d_i are coprime positive integers.
(iii) $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is the triangular decomposition of $\mathfrak{g}$. Denote $\mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{h}$.
(iv) Let $(-,-)_\mathfrak{g}$ be the standard nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Then $(-,-)_\mathfrak{g}$ is nondegenerate on $\mathfrak{h}$ and $\mathfrak{n}_+, \mathfrak{n}_-$ are isotropic for $(-,-)_\mathfrak{g}$. Moreover, $(\mathfrak{h}, \mathfrak{n}_\pm)_\mathfrak{g} = 0$.

Let $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{p}_+ = \{(x,x) | x \in \mathfrak{g}\}$ be the diagonal subalgebra of $\mathfrak{p}$,

$$\mathfrak{p}_- = \{(x,y) \in \mathfrak{p} | x \in \mathfrak{b}_-, y \in \mathfrak{b}_+ \text{ and the } \mathfrak{h}\text{-component of } x+y \text{ is zero }\}.$$

Define the nondegenerate symmetric invariant bilinear form $(-,-)_\mathfrak{p}$ on $\mathfrak{p}$ as follows:

$$((x_1, y_1), (x_2, y_2))_\mathfrak{p} = (x_1, x_2)_\mathfrak{g} - (x_2, y_2)_\mathfrak{g}, \quad \forall x_1, x_2, y_1, y_2 \in \mathfrak{g}.$$

It is easy to see $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ together with $(-,-)_\mathfrak{p}$ is a Manin triple.

We obtain a Lie bialgebra $(\mathfrak{g}, \delta_\kappa)$ via Manin triple $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$ such that:

$$\delta_\kappa(H) = 0, \quad \delta_\kappa(X_i) = d_i X_i \wedge H_i, \quad \delta_\kappa(Y_i) = d_i Y_i \wedge H_i, \quad \forall H \in \mathfrak{h} \text{ and } 1 \leq i \leq n.$$

$(\mathfrak{g}, \delta_\kappa)$ is called the *standard Lie bialgebra structure* on $\mathfrak{g}$.

In particular, if $\mathfrak{g} = \mathfrak{sl}_2$, then $(\mathfrak{sl}_2, \delta_\kappa)$ is quasitriangular since $(\mathfrak{sl}_2, \delta_\kappa)$ can be induced by $\frac{1}{2}H \otimes H + 2X \otimes Y$, which is a solution of CYBE. $(\mathfrak{sl}_2, \delta_\kappa)$ can also be induced by $X \wedge Y$.

(2) Since $\mathrm{H}^1(\mathfrak{sl}_2, M) = 0$ for all finite dimensional $\mathfrak{sl}_2$-mod M, any Lie bialgebra structure on $\mathfrak{sl}_2$ is coboundary. Moreover, there are only three Lie bialgebra structures on $\mathfrak{sl}_2$ up to gauge equivalence, which are induced by $r_1 = X \wedge Y$, $r_2 = H \wedge X$ and $r_3 = 0$ respectively.

It suffices to find elements $r \in \Lambda^2\mathfrak{sl}_2$ such that $[[r, r]] \in \Lambda^3\mathfrak{sl}_2$ is $\mathfrak{sl}_2$-invariant by Proposition 2.3.2. Note that $\Lambda^3\mathfrak{sl}_2$ is a trivial $\mathfrak{sl}_2$-mod sine it is 1-dimensional, therefore any elements of $\Lambda^2\mathfrak{sl}_2$ induces a Lie bialgebra structure on $\mathfrak{sl}_2$. On the other hand, since $\Lambda^2\mathfrak{sl}_2$ is isomorphic to the adjoint representation of $\mathfrak{sl}_2$, there is no nontrivial $\mathfrak{sl}_2$-invariant element in $\Lambda^2\mathfrak{sl}_2$. Therefore an 1-1 correspondence between the Lie bialgebra structures on $\mathfrak{sl}_2$ and elements of $\Lambda^2\mathfrak{sl}_2$ is established.

Let $r_1, r_2 \in \Lambda^2\mathfrak{sl}_2$. Then the Lie bialgebra structures induced by r_1, r_2 are isomorphic if and only if there exists an automorphism α of $\mathfrak{sl}_2$ such that $r_2 = (\alpha \otimes \alpha)(r_1)$. Regarding $\mathfrak{sl}_2$ as the subalgebra of $\mathfrak{gl}_2$ consisting of matrices with zero trace, each automorphism of $\mathfrak{sl}_2$ is given by the similarity transformation of some invertible matrix in $\mathfrak{gl}_2$. Therefore the isomorphism classes of Lie bialgebra structures on $\mathfrak{sl}_2$ correspond one-to-one with the similarity orbits of $\Lambda^2\mathfrak{sl}_2 \cong \mathfrak{sl}_2$. These orbits are of three types by an ordinary argument:

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \ \forall \lambda \neq 0 \in \Bbbk; \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad 0,$$

which correspond to λr_1, $\frac{1}{2}r_2$ and r_3 respectively.

2.4 Quantization of Lie bialgebras

Definition 2.4.1. A *co-Poisson algebra* over $\Bbbk$ is a cocommutative $\Bbbk$-coalgebra (A, Δ, ϵ) equipped with a skew-symmetric linear map $\delta : A \to A \otimes A$, called the *Poisson co-bracket*, satisfying:
(i) the co-Jacobi identity $(\mathrm{id}_{A^{\otimes 3}} + p_{(1,2,3)} + p_{(1,3,2)}) \circ (\delta \otimes \mathrm{id}_A) \circ \delta = 0$;
(ii) the co-Leibniz identity $(\Delta \otimes \mathrm{id}_A) \circ \delta = (\mathrm{id}_A \otimes \delta) \circ \Delta + p_{(2,3)} \circ (\delta \otimes \mathrm{id}) \circ \Delta$.

A *co-Poisson-Hopf algebra* is a co-Poisson algebra $(A, \Delta, \epsilon, \delta)$ which is also a Hopf algebra $(A, m, u, \Delta, \epsilon, S)$ such that:
(iii) $\delta(a_1 a_2) = \delta(a_1)\Delta(a_2) + \Delta(a_1)\delta(a_2)$ for any $a_1, a_2 \in A$.

There is an important result:

Proposition 2.4.2. *[3, Proposition 6.2.3] Let $\mathfrak{g}$ be a Lie algebra. If its universal enveloping algebra $U(\mathfrak{g})$ has a co-Poisson structure δ, making it a co-Poisson-Hopf algebra, then $\delta(\mathfrak{g}) \subseteq \mathfrak{g} \otimes \mathfrak{g}$ and $\delta|_{\mathfrak{g}}$ is a Lie bialgebra structure on $\mathfrak{g}$. Conversely, any Lie bialgebra structure δ on $\mathfrak{g}$ extends uniquely to a Poisson co-bracket on $U(\mathfrak{g})$, which makes $U(\mathfrak{g})$ into a co-Poisson-Hopf algebra.*

Definition 2.4.3. A *deformation* of a Hopf algebra $(A, m, u, \Delta, \epsilon, S)$ is a topological Hopf algebra $(A_h, m_h, u, \Delta_h, \epsilon, S_h)$ over the ring $\Bbbk[[h]]$ of formal power series in an indeterminate h over $\Bbbk$, such that
(i) $A_h \cong A[[h]]$ as $\Bbbk[[h]]$-mods;
(ii) $m_h \equiv m \pmod{h}$, $\Delta_h \equiv \Delta \pmod{h}$.

Two deformations A_h and A'_h are equivalent if there is an isomorphism $f_h : A_h \to A'_h$ of Hopf algebras over $\Bbbk[[h]]$ and $f \equiv \mathrm{id}_A \pmod{h}$.

Definition 2.4.4. Let A be a co-Poisson-Hopf algebra with Poisson co-bracket δ, a *quantization* of A is a Hopf algebra deformation A_h of A such that

$$\delta(x) \equiv \frac{\Delta_h(a) - \Delta_h^{op}(a)}{h} \pmod{h}, \quad \forall x \in A \text{ and } a \in A_h \text{ with } a \equiv x \pmod{h}.$$

A *quantization* of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a quantization $U_h(\mathfrak{g})$ of the co-Poisson-Hopf algebra $U(\mathfrak{g})$, whose co-Poisson bracket is extended by δ, see Proposition 2.4.2. $(\mathfrak{g}, \delta)$ is called the *classical limit* of the quantized universal enveloping algebra (QUE algebra) $U_h(\mathfrak{g})$.

Indeed, any deformation of Hopf algebra $U(\mathfrak{g})$ also provides a Lie bialgebra structure on $\mathfrak{g}$.

Proposition 2.4.5. *[3, Proposition 6.2.7] Let $\mathfrak{g}$ be a Lie algebra and let $U_h(\mathfrak{g})$ be a Hopf algebra deformation of $U(\mathfrak{g})$. Define $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by*

$$\delta(x) \equiv \frac{\Delta_h(a) - \Delta_h^{op}(a)}{h} \pmod{h},$$

where $x \in U(\mathfrak{g}), a \in U_h(\mathfrak{g})$ with $a \equiv x \pmod{h}$. Then $(U(\mathfrak{g}), \delta)$ is a co-Poisson-Hopf algebra.

Example 2.4.6. Let $A = (a_{i,j})_{1 \leq i,j, \leq n}$ be a indecomposable generalized Cartan matrix of finite type and $\mathfrak{g} = \mathfrak{g}(A)$ be the simple Lie algebra associated to A. Let $U_h(\mathfrak{g})$ be the algebra over $\Bbbk[[h]]$ topologically generated by elements $H_i, X_i, Y_i, i = 1, \ldots, n$, subject to the following defining relations:

$$[H_i, H_j] = 0, \quad [H_i, X_j] = a_{i,j} X_j, \quad [H_i, Y_j] = -a_{i,j} Y_j,$$

$$[X_i, Y_j] = \delta_{i,j} \frac{e^{d_i h H_i} - e^{-d_i h H_i}}{e^{d_i h} - e^{-d_i h}},$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1 - a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} X_i^r X_j X_i^{1-a_{i,j}-r} = \sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1 - a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}} Y_i^r Y_j Y_i^{1-a_{i,j}-r} = 0, \ \forall i \neq j,$$

where $\begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{e^{d_i h}}$ is the quantum binomial coefficient on $e^{d_i h}$. Then $U_h(\mathfrak{g})$ is a topological Hopf algebra over $\Bbbk[[h]]$ with comultiplication defined by

$$\Delta_h(H_i) = 1 \otimes H_i + H_i \otimes 1,$$

$$\Delta_h(X_i) = X_i \otimes e^{d_i h H_i} + 1 \otimes X_i, \quad \Delta_h(Y_i) = Y_i \otimes 1 + e^{-d_i h H_i} \otimes Y_i,$$

antipode defined by

$$S_h(H_i) = -H_i, \quad S_h(X_i) = -X_i e^{-d_i h H_i}, \quad S_h(Y_i) = -e^{d_i h H_i} Y_i,$$

and counit defined by

$$\epsilon_h(H_i) = \epsilon(X_i) = \epsilon(Y_i) = 0.$$

Moreover, $U_h(\mathfrak{g})$ is a quantization of the standard Lie bialgebra structure on $\mathfrak{g}$ given in Example 2.3.6.

Chapter 3

Lie bialgebra structures on $\mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$

From now on m is a positive integer and $\mathfrak{g}_m := \mathfrak{sl}_2 \otimes \Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1}) = \mathfrak{sl}_2[\mathbf{t}]/(\mathbf{t}^{m+1})$. Denote $H_i = H \otimes \mathbf{t}^i$, $X_i = X \otimes \mathbf{t}^i$ and $Y_i = Y \otimes \mathbf{t}^i$.

This chapter is organized as follows. For later application, a basic research on $\mathfrak{g}_m$-mods is given in Section 3.1. In particular, a totally description of a special submodule of $S^n\mathfrak{g}_m$ is provided as an example. In Section 3.2 we illustrates that each Lie bialgebra structure on $\mathfrak{g}_m$ is coboundary. Indeed, we prove that $\mathrm{H}^1(\mathfrak{g}_m, \mathfrak{g}_m \otimes \mathfrak{g}_m) = 0$. Inspired by Example 2.3.6, two classes of quasitriangular Lie bialgebra structures on $\mathfrak{g}_m$ are studied, one of which is called the standard Lie bialgebra structure on $\mathfrak{g}_m$. Finally, to show how complex to classify Lie bialgebra structures on $\mathfrak{g}_m$, we compute a class of quasitriangular Lie bialgebra structures when $m = 1$ in Section 3.3.

3.1 An observation on $\mathfrak{g}_m$-mods

Obviously each $\mathfrak{g}_m$-mod is a $\mathfrak{sl}_2$-mod by restriction. The aim of this section is to give a basic result on the structure of $\mathfrak{g}_m$-mods. For details,

Proposition 3.1.1. *Let V be a $\mathfrak{g}_m$-mod such that $V = \oplus_{\alpha \in J} V_\alpha$ as $\mathfrak{sl}_2$-mods, where V_α is isomorphic to the $(n_\alpha + 1)$-dimensional irreducible $\mathfrak{sl}_2$-mod for some non-negative integer n_α. Then for any $\alpha \in J$,*

$$H_1.V_\alpha \subseteq \bigoplus_{\substack{\beta \in J \\ |n_\alpha - n_\beta| = 0 \text{ or } 2}} V_\beta.$$

Moreover, if V is finite dimensional and $n_\alpha = n_\beta = p$ for all $\alpha, \beta \in J$, then there exists $0 \le k_1 \le k_2 \le \cdots \le k_r \le m$ such that

$$V \cong \bigoplus_{j=1}^{r} V(p) \otimes \Bbbk[\mathbf{t}]/(\mathbf{t}^{k_j+1}).$$

We prove Proposition 3.1.1 step by step. Thanks to Proposition 2.1.2, for each $\alpha \in J$ we can take a basis $\{v^\alpha_{n_\alpha}, v^\alpha_{n_\alpha-2}, \ldots, v^\alpha_{-n_\alpha}\}$ of V_α such that

$$H_0.v^\alpha_k = kv^\alpha_k, \quad X_0.v^\alpha_k = \frac{n_\alpha + k + 2}{2} v^\alpha_{k+2}, \quad Y_0.v^\alpha_k = \frac{n_\alpha - k + 2}{2} v^\alpha_{k-2}, \quad \forall |k| \le n_\alpha \text{ and } 2 \mid n_\alpha + k,$$

where $v^\alpha_k = 0$ if $k \in \mathbb{Z} \backslash \{n_\alpha, n_\alpha - 2, \ldots, -n_\alpha\}$. Since $[H_0, H_1] = 0$, we can write

$$H_1.v^\alpha_k = \sum_{\beta \in J} b^{\alpha,\beta}_k v^\beta_k, \tag{3.1.1}$$

where $b^{\alpha,\beta}_k = 0$ for all but finitely $\beta \in J$ if α and k are fixed and $b^{\alpha,\beta}_k = 0$ if $v^\beta_k = 0$. The structure of $\mathfrak{g}_m$-mod V is determined by (3.1.1) since H_0, X_0, Y_0 and H_1 generate $\mathfrak{g}_m$. Let $p_\beta : V \to V_\beta$ be the projection. From now on we fix $\alpha, \beta \in J$ such that n_α and n_β have same parity.

Since $2H_1 = 2[X_1, Y_0] = [[H_1, X_0], Y_0]$, one gets

$$2p_\beta(H_1.v_k^\alpha) = p_\beta([[H_1, X_0], Y_0].v_k^\alpha).$$

That is, by direct computation,

$$\begin{aligned} 2b_k^{\alpha,\beta} =& \frac{(n_\alpha - k + 2)(n_\alpha + k)}{4} b_k^{\alpha,\beta} - \delta_{k\neq -n_\alpha}\delta_{k\neq n_\beta+2}\frac{(n_\alpha - k + 2)(n_\beta + k)}{4} b_{k-2}^{\alpha,\beta} \\ & - \delta_{k\neq n_\alpha}\delta_{k\neq -n_\beta-2}\frac{(n_\alpha + k + 2)(n_\beta - k)}{4} b_{k+2}^{\alpha,\beta} + \frac{(n_\beta + k + 2)(n_\beta - k)}{4} b_k^{\alpha,\beta}. \end{aligned} \tag{3.1.2}$$

(i) $n_\alpha \le n_\beta$. In this case, (3.1.2) means that

$$L^{\alpha,\beta} \cdot \begin{pmatrix} b_{n_\alpha}^{\alpha,\beta} \\ b_{n_\alpha-2}^{\alpha,\beta} \\ \vdots \\ b_{-n_\alpha}^{\alpha,\beta} \end{pmatrix} = 0,$$

where $L^{\alpha,\beta}$ is an $(n_\alpha + 1) \times (n_\alpha + 1)$ matrix such that

$$(L^{\alpha,\beta})_{i,j} = \begin{cases} \frac{(n_\beta+n_\alpha)(n_\beta-n_\alpha+2)}{4} + 2n_\alpha i - 2i^2 - 2, & j = i; \\ -\frac{(i+1)(n_\beta+n_\alpha-2i)}{2}, & j = i+1; \\ -\frac{(n_\alpha-i+1)(n_\beta-n_\alpha+2i)}{2}, & j = i-1; \\ 0, & \text{otherwise} \end{cases}$$

for any $0 \le i, j \le n_\alpha$.

Lemma 3.1.2. *Let n_α be a non-negative integer, $n_\beta \in \Bbbk$ and $L^{\alpha,\beta}$ be the $(n_\alpha+1)\times(n_\alpha+1)$ matrix defined above. Then*

$$\begin{aligned} \det L^{\alpha,\beta} =& 2^{-2n_\alpha-2}[(n_\beta + n_\alpha + 4)(n_\beta + n_\alpha + 2)\cdots(n_\beta - n_\alpha + 4)] \\ & \cdot [(n_\beta + n_\alpha - 2)(n_\beta + n_\alpha - 4)\cdots(n_\beta - n_\alpha - 2)]. \end{aligned}$$

Moreover,
(1) *if $n_\beta = n_\alpha \ge 1$, then the solution space of $L^{\alpha,\beta}x = 0$ is 1-dimensional, which is spanned by*

$$u_0 = (n_\alpha, n_\alpha - 2, \ldots, -n_\alpha + 2, -n_\alpha)^T;$$

(2) *if $n_\beta = n_\alpha + 2$, then the solution space of $L^{\alpha,\beta}x = 0$ is 1-dimensional, which is spanned by*

$$u_1 = (n_\alpha + 1, 2n_\alpha, 3(n_\alpha - 1), \ldots, n_\alpha \cdot 2, n_\alpha + 1)^T.$$

Proof. For convenience we denote $p = n_\alpha$, $q = n_\beta$ and $L := L^{\alpha,\beta}$. Consider the vector space $\Bbbk[z]/(z^{p+1})$. We represent any column vector $v = (v_0, v_1, \ldots, v_N)^T$ as a polynomial $P(z) = \sum_{i=0}^r v_i z^i$.

Our first step is to construct a differential operator $\mathcal{L}_z$ such that the i-th component of Lv is the coefficient of z^i in $\mathcal{L}_z P(z)$. We have the following easy correspondences between linear transformations of $(p + 1)$-column vector space and differential operators on $P(z)$:

$$\begin{aligned} \sum_{i=0}^{p} iE_{i,i} &\leftrightarrow zP'(z), & \sum_{i=0}^{p} i^2 E_{i,i} &\leftrightarrow z^2P''(z) + zP'(z), \\ \sum_{i=0}^{p-1}(i+1)E_{i,i+1} &\leftrightarrow P'(z), & \sum_{i=0}^{p-1} i(i+1)E_{i,i+1} &\leftrightarrow zP''(z), \\ \sum_{i=1}^{p}(i-1)E_{i,i-1} &\leftrightarrow z^2P'(z), & \sum_{i=1}^{p}(i-1)^2E_{i,i-1} &\leftrightarrow z^3P''(z) + z^2P'(z). \end{aligned}$$

Using these correspondences we translate L into the operator $\mathcal{L}_z$ as follows:

$$\begin{aligned}\mathcal{L}_z :=&\frac{(p+q)(q-p+2)-8}{4}P(z)+2pzP'(z)-2(z^2P''(z)+zP'(z))\\ &-\frac{p+q}{2}P'(z)+zP''(z)+(z^3P''(z)+z^2P'(z))+\frac{q-3p+2}{2}z^2P'(z)-\frac{p(q-p+2)}{2}zP(z)\\ =&z(z-1)^2P''(z)+\left(\frac{q-3p+4}{2}z^2+2(p-1)z-\frac{p+q}{2}\right)P'(z)\\ &-\left(\frac{p(q-p+2)}{2}z-\frac{(p+q)(q-p+2)-8}{4}\right)P(z).\end{aligned}$$

Let $w=z-1$ and $Q(w)=P(w+1)$. Substituting $z=w+1$ transforms the operator $\mathcal{L}_z$ into

$$\begin{aligned}\mathcal{L}_w =&w^2(w+1)Q''(w)+\left(\frac{q-3p+4}{2}w^2+(q-p+2)w\right)Q'(w)\\ &-\left(\frac{p(q-p+2)}{2}w+\frac{(p-q)(q-p+2)}{4}+2\right)Q(w).\end{aligned}$$

An easy observation shows that $\mathcal{L}_w$ is represented by a lower triangular $(p+1)\times(p+1)$ matrix under the basis $\{1,w,\dots,w^p\}$, whose (i,i)-entry is

$$i(i-1)+i(q-p+2)-\frac{(p-q)(q-p+2)}{4}-2=\frac{1}{4}(q-p+2i+4)(q-p+2i-2).$$

Therefore

$$\begin{aligned}\det L =&\prod_{i=0}^{p}\frac{1}{4}(q-p+2i+4)(q-p+2i-2)\\ =&2^{-2p-2}[(q+p+4)(q+p+2)\cdots(q-p+4)][(q+p-2)(q+p-4)\cdots(q-p-2)].\end{aligned}$$

Now suppose $p\geq 1$ and $q=p$ or $q=p+2$. In this case, it is clear that $\det L=0$ and the $(0,p)$-algebraic complement minor of L is not zero. Therefore the solution space of $L^{\alpha,\beta}x=0$ is 1-dimensional. A routine verification illustrates that $Lu_i=0$ if $q=p+2i$, which completes the proof. □

Note that $p_\beta(H_1.V_\alpha)\neq 0$ if and only if $b^{\alpha,\beta}_{n_\alpha}, b^{\alpha,\beta}_{n_\alpha-2},\dots,b^{\alpha,\beta}_{-n_\alpha}$ are not all zero, which implies that $\det L^{\alpha,\beta}=0$. Therefore $n_\beta=n_\alpha$ or $n_\beta=n_\alpha+2$ by Lemma 3.1.2 if $p_\beta(H_1.V_\alpha)\neq 0$ since $n_\beta\geq n_\alpha$. In this case, there exist $\lambda^{\alpha,\beta}\in\Bbbk$ such that

$$b^{\alpha,\beta}_k=\begin{cases}\lambda^{\alpha,\beta}\frac{(n_\alpha+2)^2-k^2}{4}, & n_\beta=n_\alpha+2;\\ \lambda^{\alpha,\beta}k, & n_\beta=n_\alpha.\end{cases}$$

(ii) $\underline{n_\alpha>n_\beta}$. Then $b^{\alpha,\beta}_k=0$ if $|k|>n_\beta$. In this case, (3.1.2) means that

$$G^{\alpha,\beta}\cdot\begin{pmatrix}b^{\alpha,\beta}_{n_\beta}\\ b^{\alpha,\beta}_{n_\beta-2}\\ \vdots\\ b^{\alpha,\beta}_{-n_\beta}\end{pmatrix}=0,$$

where $G^{\alpha,\beta}$ is an $(n_\beta+1)\times(n_\beta+1)$ matrix such that

$$(G^{\alpha,\beta})_{i,j}=\begin{cases}\frac{(n_\alpha+n_\beta)(n_\alpha-n_\beta+2)}{4}-2i^2+2n_\beta i-2, & j=i;\\ -\frac{(n_\beta-i)(n_\alpha-n_\beta+2i+2)}{2}, & j=i+1;\\ -\frac{i(n_\alpha+n_\beta-2i+2)}{2}, & j=i-1;\\ 0, & \text{otherwise}\end{cases}$$

for any $0\leq i,j\leq n_\beta$. Since $G^{\alpha,\beta}=(L^{\beta,\alpha})^T$, we have

Lemma 3.1.3. *Let n_β be a non-negative integer, $n_\alpha \in \Bbbk$ and $G^{\alpha,\beta}$ be the $(n_\beta+1)\times(n_\beta+1)$ matrix defined above. Then*

$$\begin{aligned}\det G^{\alpha,\beta} =& 2^{-2n_\beta-2}[(n_\alpha+n_\beta+4)(n_\alpha+n_\beta+2)\cdots(n_\alpha-n_\beta+4)]\\ &\cdot[(n_\alpha+n_\beta-2)(n_\alpha+n_\beta-4)\cdots(n_\alpha-n_\beta-2)].\end{aligned}$$

Moreover, if $n_\alpha = n_\beta+2$, then the solution space of $G^{\alpha,\beta}x=0$ is 1-dimensional, which is spanned by

$$(1,1,\dots,1)^T.$$

Therefore $p_\beta(H_1.V_\alpha)\neq 0$ if and only if $n_\beta+2=n_\alpha$ since $n_\beta<n_\alpha$. In this case, there exists $\lambda^{\alpha,\beta}\in\Bbbk$ such that

$$b_k^{\alpha,\beta}=\lambda^{\alpha,\beta},\quad \forall|k|\le n_\beta \text{ such that } 2\mid k+n_\beta.$$

Now suppose V is finite dimensional and $n_\alpha=n_\beta=p$ for all $\alpha,\beta\in J$. Let $\phi:\mathfrak{g}_m\to\operatorname{End}_\Bbbk(V)$ represents the action of $\mathfrak{g}_m$ on V. Then there exists a matrix $\Theta=(\lambda^{\alpha,\beta})_{\alpha,\beta\in J}$ such that under the basis $\{b_k^\alpha \mid \alpha\in J, k=p,p-2,\dots,-p\}$,

$$\phi(H_0)=I\otimes\operatorname{diag}(p,p-2,\dots,-p+2,-p),$$

$$\phi(X_0)=I\otimes\begin{pmatrix}0 & p & & & \\ & 0 & p-1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1\\ & & & & 0\end{pmatrix},\quad \phi(Y_0)=I\otimes\begin{pmatrix}0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & p & 0\end{pmatrix},$$

$$\phi(H_1)=\Theta\otimes\operatorname{diag}(p,p-2,\dots,-p+2,-p),$$

where I is the identity matrix. An easy induction shows that

$$\phi(H_i)=\Theta^i\otimes\operatorname{diag}(p,p-2,\dots,-p+2,-p),$$

$$\phi(X_i)=\Theta^i\otimes\begin{pmatrix}0 & p & & & \\ & 0 & p-1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1\\ & & & & 0\end{pmatrix},\quad \phi(Y_i)=\Theta^i\otimes\begin{pmatrix}0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & p & 0\end{pmatrix}.$$

Then $\Theta^{m+1}=0$ since $[X_1,Y_m]=0$. By a suitable change of basis, we could suppose there exist $0\le k_1\le k_2\le\cdots\le k_r\le m$ such that Θ is the direct sum of matrices Θ_j, where Θ_j is the $(k_j+1)\times(k_j+1)$ canonical Jordan block

$$\begin{pmatrix}0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0\end{pmatrix}.$$

It is clear that $V\cong\oplus_{j=1}^r V(p)\otimes\Bbbk[\mathbf{t}]/(\mathbf{t}^{k_j+1})$. We have already proved Proposition 3.1.1.

Let n be a positive integer. For later application, we discuss the structure of a special $\mathfrak{g}_m$-submodule of $S^n\mathfrak{g}_m$. For $0\le i\le m$, let $\varrho_i: V(2)\to\mathfrak{g}_m$ be the $\mathfrak{sl}_2$-mod map as follows:

$$\varrho_i(v_0)=-X_i,\quad \varrho_i(v_1)=H_i,\quad \varrho(v_2)=Y_i.$$

Recall $u_{n_0,n_1,n_2} \in S^nV(2)$ defined in Lemma 2.1.5. For any non-negative integer N, denote

$$u_{n_0,n_1,n_2;N} = \sum_{\substack{a_1+\cdots+a_n=N\\0\le a_i\le m}} (\varrho_{a_1}\otimes\varrho_{a_2}\otimes\cdots\otimes\varrho_{a_n})(u_{n_0,n_1,n_2}).$$

Clearly $u_{n_0,n_1,n_2;N} \in S^n\mathfrak{g}_m$ and $u_{n_0,n_1,n_2;N} = 0$ if $N > nm$. Let $M_N = \mathfrak{sl}_2.u_{0,n,0;N}$ be a $\mathfrak{sl}_2$-submodule of $S^n\mathfrak{g}_m$. Then there exists unique $\mathfrak{sl}_2$-submodules $M_N^0, M_N^1, \dots, M_N^{\lfloor\frac{n}{2}\rfloor}$ of M_N such that

$$M_N = \bigoplus_{i=0}^{\lfloor\frac{n}{2}\rfloor} M_N^i$$

and $M_N^i \cong V(2n-4i)$ by applying Lemma 2.1.5.

Proposition 3.1.4. *Let n be a positive integer. Then*

$$M = \bigoplus_{j=0}^{m} M_{n(m-1)+j}$$

is a $\mathfrak{g}_m$-submodule of $S^n\mathfrak{g}_m$. Moreover,

$$M \cong \bigoplus_{i=0}^{\lfloor\frac{n}{2}\rfloor} V(2n-4i)\otimes\Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1}).$$

Proof. Suppose $N \ge (n-1)m$. We compute $H_1.u_{n_0,n_1,n_2;N}$:

$$\begin{aligned}
&H_1.u_{n_0,n_1,n_2;N}\\
=&\frac{1}{n_0!n_1!n_2!}\sum_{\substack{a_1+\cdots+a_n=N\\0\le a_i\le m}} H_1.(\varrho_{a_1}\otimes\varrho_{a_2}\otimes\cdots\otimes\varrho_{a_n})\left(\sum_{\tau\in S_n}\mathbf{v}_{\tau(1)}\otimes\cdots\otimes\mathbf{v}_{\tau(n)}\right)\\
=&\frac{1}{n_0!n_1!n_2!}\sum_{\substack{a_1+\cdots+a_n=N\\0\le a_i\le m}}\sum_{j=1}^{n}(\varrho_{a_1}\otimes\cdots\otimes\varrho_{a_j+1}\otimes\cdots\otimes\varrho_{a_n})\left(\sum_{\tau\in S_n}\mathbf{v}_{\tau(1)}\otimes\cdots\otimes H_0.\mathbf{v}_{\tau(j)}\otimes\cdots\otimes\mathbf{v}_{\tau(n)}\right),
\end{aligned}$$

where

$$\mathbf{v}_1=\cdots=\mathbf{v}_{n_0}=v_0,\quad \mathbf{v}_{n_0+1}=\cdots=\mathbf{v}_{n_0+n_1}=v_1,\quad \mathbf{v}_{n_0+n_1+1}=\cdots=\mathbf{v}_n=v_2.$$

Note that if $a_1+\cdots+a_n=(n-1)m+r$ and $0\le a_i\le m$, then each $a_i\ge r$. Therefore

$$\begin{aligned}
H_1.u_{n_0,n_1,n_2;N} =&\frac{1}{n_0!n_1!n_2!}\sum_{\substack{a_1+\cdots+a_n=N+1\\0\le a_i\le m}}(\varrho_{a_1}\otimes\cdots\otimes\varrho_{a_n})\left(H_0.\sum_{\tau\in S_n}\mathbf{v}_{\tau(1)}\otimes\cdots\otimes\mathbf{v}_{\tau(j)}\otimes\cdots\otimes\mathbf{v}_{\tau(n)}\right)\\
=&2(n_0-n_2)\sum_{\substack{a_1+\cdots+a_n=N+1\\0\le a_i\le m}}\sum_{j=1}^{n}(\varrho_{a_1}\otimes\cdots\otimes\varrho_{a_2}\otimes\cdots\otimes\varrho_{a_n})\sigma(\mathbf{v}_1\otimes\cdots\otimes\mathbf{v}_n)\\
=&2(n_0-n_2)u_{n_0,n_1,n_2;N+1}.
\end{aligned}$$

According to Lemma 2.1.5, we have $M_N^i = \mathfrak{sl}_2.w_N^i$, where each w_N^i is a linear combination of $\{u_{n-i-k,2k,i-k;N} \mid 0\le k\le i\}$. Thus $Y_0.w_N^i$ is a linear combination of $\{u_{n-i-k-1,2k+1,i-k;N} \mid 0\le k\le i\}$, and

$$H_1.(Y_0.w_N^i) = (2n-4i-2)Y_0.w_{N+1}^i \in M_{N+1}^i.$$

Therefore Proposition 3.1.1 implies that for any $0\le i\le\lfloor\frac{n}{2}\rfloor$,

$$M^i =: \bigoplus_{j=0}^{m} M^i_{(n-1)m+j} \cong V(2n-4i)\otimes\Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1})$$

is a $\mathfrak{g}_m$-submodule of $S^n\mathfrak{g}_m$. □

3.2 Lie bialgebra structures on $\mathfrak{g}_m$

Proposition 3.2.1. *Any Lie bialgebra structure on $\mathfrak{g}_m$ is coboundary.*

Proof. We claim that $\mathrm{H}^1(\mathfrak{g}_m, \mathfrak{g}_m \otimes \mathfrak{g}_m) = 0$. Let $\delta : \mathfrak{g}_m \to \mathfrak{g}_m \otimes \mathfrak{g}_m$ be an 1-cocycle. Then $\delta\mid_{\mathfrak{sl}_2}$ is an 1-cocycle on $\mathfrak{sl}_2$ with coefficients in $\mathfrak{g}_m \otimes \mathfrak{g}_m$. Since $\mathrm{H}^1(\mathfrak{sl}_2, \mathfrak{g}_m \otimes \mathfrak{g}_m) = 0$, there exists $r_0 \in \mathfrak{g}_m \otimes \mathfrak{g}_m$ such that $\delta(a) = a.r_0$ for any $a \in \mathfrak{sl}_2$. Denote $\alpha = \delta(H_1) - H_1.r_0$. Since δ is an 1-cocycle, we have

$$\delta(X_1) = \frac{1}{2}(H_1.\delta(X_0) - X_0.\delta(H_1)) = X_1.r - \frac{1}{2}X_0.\alpha,$$

$$\delta(Y_1) = -\frac{1}{2}(H_1.\delta(Y_0) - Y_0.\delta(H_1)) = Y_1.r + \frac{1}{2}Y_0.\alpha,$$

$$0 = \delta([H_0, H_1]) = H_0.\delta(H_1) - H_1.\delta(H_0) = H_0.\alpha,$$

$$0 = \delta([X_0, X_1]) = X_0.\delta(X_1) - X_1.\delta(X_0) = -\frac{1}{2}X_0^2.\alpha,$$

$$0 = \delta([Y_0, Y_1]) = Y_0.\delta(Y_1) - Y_1.\delta(Y_0) = \frac{1}{2}Y_0^2.\alpha,$$

$$H_1.r_0 + \alpha = \delta(H_1) = X_1.\delta(Y_0) - Y_0.\delta(X_1) = H_1.r + \frac{1}{2}Y_0X_0.\alpha.$$

Therefore $H_0.\alpha = X_0^2.\alpha = Y_0^2.\alpha = 0$, $\alpha = \frac{1}{2}Y_0X_0.\alpha$, that is, $\alpha = 0$ or α generates a 3-dimensional irreducible $\mathfrak{sl}_2$-submodule of $\mathfrak{g}_m \otimes \mathfrak{g}_m$. The decomposition of the $\mathfrak{sl}_2$-mod $\mathfrak{g}_m \otimes \mathfrak{g}_m$ forces that α is a linear combination of $\{X_s \otimes Y_t - Y_s \otimes X_t \mid 0 \le s, t \le m\}$. Using δ is an 1-cocycle again, an routine induction shows that

$$\delta(X_s) = X_s.r_0 - \sum_{i=0}^{s-1} \frac{1}{2^{i+1}} H_1^i X_{s-1-i}.\alpha, \quad \delta(Y_s) = Y_s.r_0 - \sum_{i=0}^{s-1} \frac{1}{(-2)^{i+1}} H_1^i Y_{s-1-i}.\alpha.$$

A calculation provides

$$\left(\frac{1}{2}X_mY_0 + \sum_{i=0}^{m-1} \frac{1}{2^{i+1}} Y_1 H_1^i X_{m-1-i}\right).(X_s \otimes Y_t - Y_s \otimes X_t) = \sum_{i+j=m+s+t} X_i \wedge Y_j.$$

Let $\alpha = \sum_{0 \le s,t \le m} \lambda^{s,t}(X_s \otimes Y_t - Y_s \otimes X_t)$. Then the equality $X_m.\delta(Y_1) = Y_1.\delta(X_m)$ implies that

$$\sum_{0 \le s,t \le m} \lambda^{s,t} \sum_{i+j=m+s+t} X_i \wedge Y_j = 0.$$

Then for $0 \le k \le m$,

$$\sum_{\substack{0 \le s,t \le m \\ s+t=k}} \lambda^{s,t} = 0.$$

Since

$$H_1.(H_s \otimes H_t + 2X_s \otimes Y_t + 2Y_s \otimes X_t) = 4[(X_{s+1} \otimes Y_t - Y_{s+1} \otimes X_t) - (X_s \otimes Y_{t+1} - Y_s \otimes X_{t+1})],$$

we have $\alpha = H_1.r_1$ for some $r_1 \in (\mathfrak{g}_m \otimes \mathfrak{g}_m)^{\mathfrak{sl}_2}$. Then $\delta(a) = a.(r_0 + r_1)$ for $a = H_0, X_0, Y_0$ and H_1, hence for any $a \in \mathfrak{g}_m$. Therefore δ is an 1-coboundary. □

In the rest of this section, we construct the so-called standard Lie bialgebra structure on $\mathfrak{g}_m$ following Example 2.3.6. To do this, a triangular decomposition of $\mathfrak{g}_m$ and a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}_m$ being suitable with each other are required. More explicitly, we need to construct

(i) a decomposition $\mathfrak{g}_m = \mathfrak{n}_m^- \oplus \mathfrak{h}_m \oplus \mathfrak{n}_m^+$, such that $\mathfrak{n}_m^\pm$ are subalgebras of $\mathfrak{g}_m$ and $[\mathfrak{h}_m, \mathfrak{h}_m] = 0$,

$[\mathfrak{h}_m, \mathfrak{n}_m^\pm] \subseteq \mathfrak{n}_m^\pm$;
(ii) a nondegenerate invariant symmetric bilinear form $(-,-)_{\mathfrak{g}_m}$ on $\mathfrak{g}_m$;
(iii) $\mathfrak{n}_m^\pm$ are isotropic for $(-,-)_{\mathfrak{g}_m}$ and $(\mathfrak{h}_m, \mathfrak{n}_m^\pm)_{\mathfrak{g}_m} = 0$.

Indeed, the triangular decomposition of $\mathfrak{sl}_2$ naturally gives a triangular decomposition of $\mathfrak{g}_m$: Let $\mathfrak{h}_m = H \otimes \Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1})$, $\mathfrak{n}_m^+ = X \otimes \Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1})$, $\mathfrak{n}_m^- = Y \otimes \Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1})$, then $\mathfrak{g}_m = \mathfrak{n}_m^- \oplus \mathfrak{h}_m \oplus \mathfrak{n}_m^+$. Denote $\mathfrak{b}^\pm = \mathfrak{h}_m \oplus \mathfrak{n}_m^\pm$. A simple argument shows that a bilinear form $(-,-)_{\mathfrak{g}_m}$ on $\mathfrak{g}_m$ is invariant symmetric and satisfies (iii), if and only if, there exists $\mu_0, \mu_1, \ldots, \mu_m \in \Bbbk$ such that

$$(H_r, H_s)_{\mathfrak{g}_m} = 2(X_r, Y_s)_{\mathfrak{g}_m} = 2\mu_{m-r-s}, \quad \forall 0 \leq r, s \leq m,$$

where $\mu_k = 0$ if $k < 0$. Moreover, $(-,-)_{\mathfrak{g}_m}$ is nondegenerate if and only if $\mu_0 \neq 0$. Such a bilinear form determined by $\widehat{\mu} = (\mu_0, \mu_1, \ldots, \mu_m)$ is denoted by $(-,-)_{\mathfrak{g}_m}^{\widehat{\mu}}$.

Suppose $\mu_0 \neq 0$. Let $\mathfrak{p}_m = \mathfrak{g}_m \oplus \mathfrak{g}_m$, $\mathfrak{p}_m^+ = \{(x,x) \mid x \in \mathfrak{g}_m\}$ be the diagonal subalgebra of $\mathfrak{p}_m$,

$$\mathfrak{p}_m^- = \{(x,y) \in \mathfrak{p} \mid x \in \mathfrak{b}_m^+, y \in \mathfrak{b}_m^- \text{ and the } \mathfrak{h}_m\text{-component of } x+y \text{ is zero}\}.$$

Define the nondegenerate symmetric invariant bilinear form $(-,-)_{\mathfrak{p}_m}$ on $\mathfrak{p}_m$ as follows:

$$((x_1,y_1),(x_2,y_2))_{\mathfrak{p}_m}^{\widehat{\mu}} = (x_1,x_2)_{\mathfrak{g}_m}^{\widehat{\mu}} - (y_1,y_2)_{\mathfrak{g}_m}^{\widehat{\mu}}, \quad \forall x_1, x_2, y_1, y_2 \in \mathfrak{g}_m.$$

Then $(\mathfrak{p}_m, \mathfrak{p}_m^+, \mathfrak{p}_m^-)$ together with $(-,-)_{\mathfrak{p}_m}^{\widehat{\mu}}$ is a Manin triple, which yields a Lie bialgebra structure on $\mathfrak{g}_m$ completely determined by $\widehat{\mu}$. Denote this Lie bialgebra by $(\mathfrak{g}_m, \delta_{\widehat{\mu}})$.

Proposition 3.2.2. *For any $k \geq 0$, denote*

$$\mathbf{r}_k = \sum_{r+s=m+k} X_r \wedge Y_s \in \Lambda^2 \mathfrak{g}_m.$$

Let $\delta_{\widehat{\mu}}$ be the cocommutator defined as above. Then for any $x \in \mathfrak{g}_m$,

$$\delta_{\widehat{\mu}}(x) = -\frac{1}{2} x. \sum_{k=0}^m \lambda_k \mathbf{r}_k,$$

where $g(\mathbf{t}) = \sum_{i=0}^m \lambda_i \mathbf{t}^i$ is the inverse of $f(\mathbf{t}) = \sum_{i=0}^m \mu_i \mathbf{t}^i$ in $\Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1})$. Therefore there is an 1-1 correspondence between Lie bialgebras $(\mathfrak{g}_m, \delta_{\widehat{\mu}})$ and invertible elements in $\Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1})$.

Proof. Consider $\mathfrak{g}_m$ as $\mathfrak{p}_m^+$ and $\mathfrak{g}_m^*$ as $\mathfrak{p}_m^-$ via $(-,-)_{\mathfrak{p}_m}$, then an calculation shows that

$$H_i^* = \frac{1}{4}(H \otimes \mathbf{t}^{m-i} g(\mathbf{t}), -H \otimes \mathbf{t}^{m-i} g(\mathbf{t})),$$

$$X_i^* = -(0, Y \otimes \mathbf{t}^{m-i} g(\mathbf{t})), \quad Y_i^* = (X \otimes \mathbf{t}^{m-i} g(\mathbf{t}), 0).$$

Hence in $\mathfrak{g}_m^*$,

$$[H_i^*, X_j^*] = \frac{1}{2} \sum_{k=0}^m \lambda_k X_{i+j-k-m}^*, \quad [H_i^*, Y_j^*] = \frac{1}{2} \sum_{k=0}^m \lambda_k Y_{i+j-k-m}^*,$$

and other brackets are zero. Therefore

$$\delta_{\widehat{\mu}}(H_i) = 0,$$

$$\delta_{\widehat{\mu}}(X_i) = \frac{1}{2} \sum_{k=0}^{m-i} \lambda_k \sum_{r+s=m+k+i} H_r \wedge X_s, \quad \delta_{\widehat{\mu}}(Y_i) = \frac{1}{2} \sum_{k=0}^{m-i} \lambda_k \sum_{r+s=m+k+i} H_r \wedge Y_s.$$

Since

$$H_i.\mathbf{r}_k = 0, \quad X_i.\mathbf{r}_k = \sum_{r+s=m+k+i} X_r \wedge H_s, \quad Y_i.\mathbf{r}_k = \sum_{r+s=m+k+i} Y_r \wedge H_s,$$

we have the desired result. □

Inspired by Proposition 3.2.2 and the classification of Lie bialgebra structures on $\mathfrak{sl}_2$, we give two more general classes of Lie bialgebra structures on $\mathfrak{g}_m$. A technical lemma should be given here.

Lemma 3.2.3. *Let m be a positive integer and $A_m = \Bbbk[\mathbf{t}]/(\mathbf{t}^{m+1})$. For any non-negative integer N, define $\Phi_N^m : A_m \to A_m \otimes A_m$ by*

$$\Phi_N^m(a) = \sum_{i=0}^{N} a^i \otimes a^{N-i}.$$

Let W_N be the subspace of $A_m \otimes A_m$ spanned by $\{\Phi_s^m(\mathbf{t}) \mid s \geq N\}$. Suppose $N \geq m$, then
(i) *$\Phi_N^m(a) \in W_N$ for any $a \in \mathbf{t}A_m$.*
(ii) *Conversely, for any $l_N, l_{N+1}, \ldots, l_{2m} \in \Bbbk$ with $l_N \neq 0$, there exists an element $a \in \mathbf{t}A_m$ such that the coefficient of $\Phi_s^m(\mathbf{t})$ in $\Phi_N(a)$ is exactly l_s for any $N \leq s \leq \min\{2N-1, 2m\}$. In this case if $N = m$, then the coefficient of $\Phi_{2m}^m(\mathbf{t})$ in $\Phi_m^m(a)$ is determined by $l_N, \ldots, l_{2m-1}$.*

For convenience, two elements $\mathbf{r}$ and $\mathbf{r}'$ in $\mathfrak{g}_m \otimes \mathfrak{g}_m$ are called equivalent (gauge equivalent) if the Lie bialgebra structures on $\mathfrak{g}_m$ induced by $\mathbf{r}, \mathbf{r}'$ (if exist) are isomorphic (gauge equivalent). It is clear that $\mathbf{r}$ and $\mathbf{r}'$ are gauge equivalent if and only if there exists an automorphism φ of $\mathfrak{g}_m$ and a nonzero scalar c such that $\mathbf{r}' - (\varphi \otimes \varphi)(cr)$ is $\mathfrak{g}_m$-invariant. Moreover, we have

Lemma 3.2.4. *There is no nonzero $\mathfrak{g}_m$-invariant element in $\Lambda^2\mathfrak{g}_m$. Therefore any Lie bialgebra structure on $\mathfrak{g}_m$ is induced by an unique element in $\Lambda^2\mathfrak{g}_m$.*

Proof. Suppose $a \in \Lambda^2\mathfrak{g}_m$ is $\mathfrak{g}_m$-invariant. Then $\mathfrak{sl}_2.a = 0$ and we can write $a = \sum_{0\leq r,s\leq m} \lambda_{r,s}(w_{r,s} - w_{s,r})$ with $\lambda_{r,s} + \lambda_{s,r} = 0$, where

$$w_{r,s} = H_r \otimes H_s + 2X_r \otimes Y_s + 2Y_r \otimes X_s.$$

We have

$$\begin{aligned} H_1.a =& 4\sum_{0\leq r,s\leq m} \lambda_{r,s}(X_{r+1} \wedge Y_s - X_r \wedge Y_{s+1} - X_{s+1} \wedge Y_r + X_s \wedge Y_{r+1}) \\ =& 4\sum_{0\leq r,s\leq m} \lambda_{r,s}(X_{r+1} \wedge Y_s - X_r \wedge Y_{s+1}) - 4\sum_{0\leq r,s\leq m} \lambda_{s,r}(-X_{s+1} \wedge Y_r + X_s \wedge Y_{r+1}) \\ =& 8\sum_{k=0}^{2m}\sum_{r+s=k} \lambda_{r,s}(X_{r+1} \wedge Y_s - X_r \wedge Y_{s+1}). \end{aligned}$$

Then for each $0 \leq k \leq 2m$,

$$\sum_{r+s=k} \lambda_{r,s}(X_{r+1} \wedge Y_s - X_r \wedge Y_{s+1}) = 0.$$

An observation implies that $\lambda_{r,s} = 0$ if $r + s < m$ and $\lambda_{r,s} = \lambda_{r+1,s-1}$ if $r + s \geq m$. Therefore $\lambda_{r,s}$ are all zero sine $\lambda_{r,s} + \lambda_{s,r} = 0$ and the desired result follows. ☐

Proposition 3.2.5. *For any $k \geq 0$, denote*

$$\mathbf{r}_k' = \sum_{r+s=m+k} X_r \wedge H_s \in \Lambda^2\mathfrak{g}_m.$$

(i) *Any nonzero linear combination of $\{\mathbf{r}_k \mid 0 \leq k \leq m\}$ induces a quasitriangular Lie bialgebra structure on $\mathfrak{g}_m$. Moreover, any such Lie bialgebra structure on $\mathfrak{g}_m$ is gauge equivalent to the Lie bialgebra structure induced by one of $\mathbf{r}_0$, $\mathbf{r}_1, \ldots, \mathbf{r}_m$, $\mathbf{r}_0 + \mathbf{r}_m$.*
(ii) *Any linear combination of $\{\mathbf{r}_k' \mid 0 \leq k \leq m\}$ induces a triangular Lie bialgebra structure on $\mathfrak{g}_m$. Moreover, any such Lie bialgebra structure on $\mathfrak{g}_m$ is gauge equivalent to the Lie bialgebra structure induced by one of 0, $\mathbf{r}_0'$, $\mathbf{r}_1', \ldots, \mathbf{r}_m'$.*

Proof. (i) Let $\mathbf{r}=\sum_{k=0}^m \lambda_k \mathbf{r}_k$ be nonzero. There is no difficult to verify that for any $a,b,c,d\in\mathfrak{g}_m$,

$$[[a\wedge b, c\wedge d]]+[[c\wedge d, a\wedge b]]=[a,c]\wedge b\wedge d-[a,d]\wedge b\wedge c-[b,d]\wedge a\wedge d+[b,d]\wedge a\wedge c.$$

Therefore

$$\begin{aligned}[[\mathbf{r}_k,\mathbf{r}_l]]+[[\mathbf{r}_l,\mathbf{r}_k]]=&\sum_{i=k}^{m}\sum_{j=l}^{m} H_{i+j}\wedge X_{m+l-j}\wedge Y_{m+k-i}+\sum_{i=k}^{m}\sum_{j=l}^{m} H_{2m+k+l-i-j}\wedge X_i\wedge Y_j\\ =&2\sum_{\substack{r+s+t\\=2m+k+l}} H_r\wedge X_s\wedge Y_t.\end{aligned}$$

For any $N\geq 2m$, we have

$$\begin{aligned}H_1.\sum_{r+s+t=N} H_r\wedge X_s\wedge Y_t=&2\sum_{r+s+t=N} H_r\wedge X_{s+1}\wedge Y_t-2\sum_{r+s+t=N} H_r\wedge X_s\wedge Y_{t+1}\\ =&2\sum_{\substack{r+s+t\\=2m+N+1}} H_r\wedge X_s\wedge Y_t-2\sum_{\substack{r+s+t\\=2m+N+1}} H_r\wedge X_s\wedge Y_t\\ =&0.\end{aligned}$$

Since each $H_r\wedge X_s\wedge Y_t$ is $\mathfrak{sl}_2$-invariant, we have $\mathfrak{g}_m.[[\mathbf{r},\mathbf{r}]]=0$. Therefore $\mathbf{r}$ induces a Lie bialgebra structure $\delta_{\mathbf{r}}$ on $\mathfrak{g}_m$ by Proposition 2.3.2. Moreover, $(\mathfrak{g}_m,\delta_{\mathbf{r}})$ may not be triangular since $\mathbf{r}$ is the unique element in $\Lambda^2\mathfrak{g}_m$ which induces $(\mathfrak{g}_m,\delta_{\mathbf{r}})$ by Lemma 3.2.4.

Now let N be the minimal integer such that $\lambda_N\neq 0$. We claim that: $\mathbf{r}$ is gauge equivalent to $\mathbf{r}_0$ or $\mathbf{r}_0+\mathbf{r}_m$ if $N=0$, and is equivalent to $\mathbf{r}_N$ if $N>0$. Indeed, take any $f(\mathbf{t})\in\mathbf{t}A_m\backslash\mathbf{t}^2A_m$, then $f(t)$ determines an algebra automorphism φ_f of A_m via $\mathbf{t}\mapsto f(\mathbf{t})$. Hence $f(t)$ also induces an automorphism of $\mathfrak{g}_m$ which act as $\mathrm{id}_{\mathfrak{sl}_2}\otimes\varphi_f$ and is still denoted by φ_f. Since $\mathfrak{g}_m\otimes\mathfrak{g}_m\cong(\Lambda^2\mathfrak{sl}_2\otimes S^2A_m)\oplus(S^2\mathfrak{sl}_2\otimes\Lambda^2A_m)$, one gets

$$(\varphi_f\otimes\varphi_f)(\mathbf{r}_k)=(X\wedge Y)\otimes\left(\sum_{r+s=m+k} f(\mathbf{t})^r\otimes f(\mathbf{t})^s\right)=(X\wedge Y)\otimes\Phi^m_{m+k}(f(\mathbf{t})).$$

Thanks to Lemma 3.2.3, there exists $f(\mathbf{t})$ such that $(\varphi_f\otimes\varphi_f)(\mathbf{r}_N)=\mathbf{r}$ if $N>0$. Now suppose $N=0$, applying Lemma 3.2.3 again we have $\mathbf{r}$ is equivalent to $\mathbf{r}_0+\lambda\mathbf{r}_m$ for some $\lambda\in\Bbbk$. Suppose $\lambda\neq 0$, let $f(\mathbf{t})=\lambda^{-\frac{1}{m}}\mathbf{t}$, then

$$(\varphi_f\otimes\varphi_f)(\mathbf{r}_0+\lambda\mathbf{r}_m)=\lambda^{-1}(\mathbf{r}_0+\mathbf{r}_m).$$

Therefore $\mathbf{r}$ is gauge equivalent to one of $\mathbf{r}_0,\ \mathbf{r}_1,\dots,\mathbf{r}_m,\ \mathbf{r}_0+\mathbf{r}_m$.

Finally, recall the $\mathfrak{sl}_2$-invariant element $w_{r,s}\in\mathfrak{g}_m\otimes\mathfrak{g}_m$ given in Lemma 3.2.4. For any $0\leq k\leq m$ let

$$\mathbf{s}_k=\frac{1}{2}\sum_{r+s=m+k} w_{r,s}\in S^2\mathfrak{g}_m.$$

Since

$$H_1.\mathbf{s}_k=2\sum_{r+s=m+k}(X_{r+1}\otimes Y_s-X_r\otimes Y_{s+1}-Y_{r+1}\otimes X_s+Y_r\otimes X_{s+1})=0,$$

$\mathbf{s}_k$ are $\mathfrak{g}_m$-invariant. Therefore $\mathbf{r}_k+\mathbf{s}_k$ and $\mathbf{r}_k$ induce same Lie bialgebra structure. For any $a,b,c,d\in\mathfrak{g}_m$, denote

$$\psi(a;b,c)=a\otimes(b\wedge c)+b\otimes a\otimes c-c\otimes a\otimes b+(b\wedge c)\otimes a=a\wedge b\wedge c+2(b\otimes a\otimes c-c\otimes a\otimes b)\in\mathfrak{g}_m^{\otimes 3}.$$

Then

$$\begin{aligned}&[[a\otimes b+b\otimes a, c\otimes d+d\otimes c]]+[[c\otimes d+d\otimes c, a\otimes b+b\otimes a]]\\ =&\psi([a,c];b,d)+\psi([a,d];b,c)+\psi([b,c];a,d)+\psi([b,d];a,c).\end{aligned}$$

Thus

$$\begin{aligned}&[[w_{r,s}+w_{s,r}, w_{r',s'}+w_{s',r'}]]\\
=&2(2X_{r+r'}\wedge H_s\wedge Y_{s'}-2Y_{r+s'}\wedge H_s\wedge X_{r'}+2X_{s+r'}\wedge H_r\wedge Y_{s'}-2Y_{s+s'}\wedge H_r\wedge X_{r'})\\
&+4(2H_s\otimes X_{r+r'}\otimes Y_{s'}-2Y_{s'}\otimes X_{r+r'}\otimes H_s-2H_s\otimes Y_{r+s'}\otimes X_{r'}+2X_{r'}\otimes Y_{r+s'}\otimes H_s\\
&+2H_r\otimes X_{s+r'}\otimes Y_{s'}-2Y_{s'}\otimes X_{s+r'}\otimes H_r-2H_r\otimes Y_{s+s'}\otimes X_{r'}+2X_{r'}\otimes Y_{s+s'}\wedge H_r)\\
&+2(2X_{r+s'}\wedge H_s\wedge Y_{r'}-2Y_{r+r'}\wedge H_s\wedge X_{s'}+2X_{s+s'}\wedge H_r\wedge Y_{r'}-2Y_{s+r'}\wedge H_r\wedge X_{s'})\\
&+4(2H_s\otimes X_{r+s'}\otimes Y_{r'}-2Y_{r'}\otimes X_{r+s'}\otimes H_s-2H_s\otimes Y_{r+r'}\otimes X_{s'}+2X_{s'}\otimes Y_{r+r'}\otimes H_s\\
&+2H_r\otimes X_{s+s'}\otimes Y_{r'}-2Y_{r'}\otimes X_{s+s'}\otimes H_r-2H_r\otimes Y_{s+r'}\otimes X_{s'}+2X_{s'}\otimes Y_{s+r'}\wedge H_r)\\
&+2(2X_{r'+r}\wedge H_{s'}\wedge Y_s-2Y_{r'+s}\wedge H_{s'}\wedge X_r+2X_{s'+r}\wedge H_{r'}\wedge Y_s-2Y_{s'+s}\wedge H_{r'}\wedge X_r)\\
&+4(2H_{s'}\otimes X_{r'+r}\otimes Y_s-2Y_s\otimes X_{r'+r}\otimes H_{s'}-2H_{s'}\otimes Y_{r'+s}\otimes X_r+2X_r\otimes Y_{r'+s}\otimes H_{s'}\\
&+2H_{r'}\otimes X_{s'+r}\otimes Y_s-2Y_s\otimes X_{s'+r}\otimes H_{r'}-2H_{r'}\otimes Y_{s'+s}\otimes X_r+2X_r\otimes Y_{s'+s}\wedge H_{r'})\\
&+2(2X_{r'+s}\wedge H_{s'}\wedge Y_r-2Y_{r'+r}\wedge H_{s'}\wedge X_s+2X_{s'+s}\wedge H_{r'}\wedge Y_r-2Y_{s'+r}\wedge H_{r'}\wedge X_s)\\
&+4(2H_{s'}\otimes X_{r'+s}\otimes Y_r-2Y_r\otimes X_{r'+s}\otimes H_{s'}-2H_{s'}\otimes Y_{r'+r}\otimes X_s+2X_s\otimes Y_{r'+r}\otimes H_{s'}\\
&+2H_{r'}\otimes X_{s'+s}\otimes Y_r-2Y_r\otimes X_{s'+s}\otimes H_{r'}-2H_{r'}\otimes Y_{s'+r}\otimes X_s+2X_s\otimes Y_{s'+r}\wedge H_{r'})\\
&+4(H_{r+s'}\wedge Y_s\wedge X_{r'}-H_{s+r'}\wedge X_r\wedge Y_{s'})\\
&+8(Y_s\otimes H_{r+s'}\otimes X_{r'}-X_{r'}\otimes H_{r+s'}\otimes Y_s-X_r\otimes H_{s+r'}\otimes Y_{s'}+Y_{s'}\otimes H_{s+r'}\otimes X_r)\\
&+4(H_{r+r'}\wedge Y_s\wedge X_{s'}-H_{s+s'}\wedge X_r\wedge Y_{r'})\\
&+8(Y_s\otimes H_{r+r'}\otimes X_{s'}-X_{s'}\otimes H_{r+r'}\otimes Y_s-X_r\otimes H_{s+s'}\otimes Y_{r'}+Y_{r'}\otimes H_{s+s'}\otimes X_r)\\
&+4(H_{s+s'}\wedge Y_r\wedge X_{r'}-H_{r+r'}\wedge X_s\wedge Y_{s'})\\
&+8(Y_r\otimes H_{s+s'}\otimes X_{r'}-X_{r'}\otimes H_{s+s'}\otimes Y_r-X_s\otimes H_{r+r'}\otimes Y_{s'}+Y_{s'}\otimes H_{r+r'}\otimes X_s)\\
&+4(H_{s+r'}\wedge Y_r\wedge X_{s'}-H_{r+s'}\wedge X_s\wedge Y_{r'})\\
&+8(Y_r\otimes H_{s+r'}\otimes X_{s'}-X_{s'}\otimes H_{s+r'}\otimes Y_r-X_s\otimes H_{r+s'}\otimes Y_{r'}+Y_{r'}\otimes H_{r+s'}\wedge X_s).\end{aligned}$$

An observation illustrates

$$\begin{aligned}&16([[\mathbf{s}_k,\mathbf{s}_l]]+[[\mathbf{s}_l,\mathbf{s}_k]])\\
=&\sum_{r+s=m+k}\sum_{r'+s'=m+l}([[w_{r,s}+w_{s,r}, w_{r',s'}+w_{s',r'}]]+[[w_{r',s'}+w_{s',r'}, w_{r,s}+w_{s,r}]])\\
=&-32\sum_{r+s+t=2m+k+l}H_r\wedge X_s\wedge Y_t.\end{aligned}$$

Since for any $a\in\Lambda^2\mathfrak{g}_m$ and $b\in(\mathfrak{g}_m\otimes\mathfrak{g}_m)^{\mathfrak{g}_m}$, $[[a+b,a+b]]=[[a,a]]+[[b,b]]$, we have

$$\left[\left[\sum_{k=0}^m\lambda_k(\mathbf{r}_k+\mathbf{s}_k),\sum_{k=0}^m\lambda_k(\mathbf{r}_k+\mathbf{s}_k)\right]\right]=0.$$

Therefore any linear combination of $\{\mathbf{r}_k\mid 0\le k\le m\}$ induces a quasitriangular Lie bialgebra structure on $\mathfrak{g}_m$.

(ii) Since

$$[[\mathbf{r}'_k,\mathbf{r}'_l]]+[[\mathbf{r}'_l,\mathbf{r}'_k]]=2\sum_{i=k}^m\sum_{j=l}^m X_{i+j}\wedge H_{m+k-i}\wedge X_{m+l-j}-2\sum_{i=k}^m\sum_{j=l}^m X_{2m+k+l-i-j}\wedge X_i\wedge H_j=0,$$

any linear combination of $\{\mathbf{r}'_k \mid 0 \leq k \leq m\}$ induces a triangular Lie bialgebra structure on $\mathfrak{g}_m$. The proof of rest results is similar as (i), but at this time we claim that $\mathbf{r}'_0$ and $\mathbf{r}'_0 + \mathbf{r}'_m$ are equivalent. Indeed, let φ be the automorphism of $\mathfrak{g}_m$ such that

$$\varphi(a) = a + \frac{1}{2}[a, H]\mathbf{t}^m, \quad \varphi(a\mathbf{t}^i) = a\mathbf{t}^i, \quad \forall a \in \mathfrak{sl}_2, i \geq 1,$$

where $\lambda = 2^{\frac{1}{m}}$. Then

$$(\varphi \otimes \varphi)(\mathbf{r}'_0 + \mathbf{r}'_m) = \mathbf{r}'_0 - \mathbf{r}'_m + \mathbf{r}'_m = \mathbf{r}'_0.$$

Therefore $\mathbf{r}'_0$ and $\mathbf{r}'_0 + \mathbf{r}'_m$ induce isomorphic Lie bialgebra structures. □

A natural question is that whether $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_m, \mathbf{r}_0 + \mathbf{r}_m$ induce gauge equivalent Lie bialgbera structures on $\mathfrak{g}_m$. Let φ be an automorphism of $\mathfrak{g}_m$, then it is clear that $\varphi(\mathfrak{a}_k) = \mathfrak{a}_k$ for any $0 \leq k \leq m$, where $\mathfrak{a}_k = \Bbbk\{H_i, X_i, Y_i \mid i \geq k\}$ is the ideal of $\mathfrak{g}_m$ generated by H_k. It is clear that φ induces automorphisms $\overline{\varphi}_k : \mathfrak{g}_m/\mathfrak{a}_k \to \mathfrak{g}_m/\mathfrak{a}_k$. Let $\pi_k : \mathfrak{g}_m \to \mathfrak{g}_m/\mathfrak{a}_k$ be the natural quotient map. Then $\pi_k \circ \varphi = \overline{\varphi}_k \circ \pi_k$, and

$$\begin{aligned}(\pi_{k+1} \otimes \mathrm{id}_{\mathfrak{g}_m})(\varphi \otimes \varphi)(\mathbf{r}_k) =& (\overline{\varphi}_{k+1} \otimes \varphi)(\pi_{k+1} \otimes \mathrm{id}_{\mathfrak{g}_m})(\mathbf{r}_k) \\ =& (\overline{\varphi}_{k+1} \otimes \varphi)(\pi_{k+1} \otimes \mathrm{id}_{\mathfrak{g}_m})(X_k \otimes Y_m - Y_k \otimes X_m)\end{aligned}$$

is nonzero. Therefore $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_m, \mathbf{r}_0 + \mathbf{r}_m$ are not gauge equivalent to each other, expect $\mathbf{r}_0$ and $\mathbf{r}_0 + \mathbf{r}_m$. Similarly, $\mathbf{r}'_0, \mathbf{r}'_1, \ldots, \mathbf{r}'_m$ are not gauge equivalent to each other.

Example 3.2.6. Let $m = 1$. In this example we show $\mathbf{r}_0$ and $\mathbf{r}_0 + \mathbf{r}_1$ are not gauge equivalent. Take any automorphism φ of $\mathfrak{g}_1$. An easy argument provides that there exists an automorphism φ_0 of $\mathfrak{sl}_2$, an element $\gamma \in \mathfrak{sl}_2$ and a nonzero scalar λ, such that

$$\varphi(a) = \varphi_0(a) + [\varphi_0(a), \gamma]\mathbf{t}, \quad \varphi(a\mathbf{t}) = \lambda\varphi_0(a)\mathbf{t}, \quad \forall a \in \mathfrak{sl}_2.$$

Then

$$\begin{aligned}(\varphi \otimes \varphi)(\mathbf{r}_0 + \mathbf{r}_1) =& \lambda(\varphi_0(X) + [\varphi_0(X), \gamma]\mathbf{t}) \wedge \varphi_0(Y)\mathbf{t} + \lambda\varphi_0(X)\mathbf{t} \wedge (\varphi_0(Y) + [\varphi_0(Y), \gamma]\mathbf{t}) \\ &+ \lambda^2\varphi_0(X)\mathbf{t} \wedge \varphi_0(Y)\mathbf{t} \\ =& \lambda(\varphi_0(X) \wedge \varphi_0(Y)\mathbf{t} + \varphi_0(X)\mathbf{t} \wedge \varphi_0(Y)) \\ &+ \lambda([\varphi_0(X), \gamma]\mathbf{t} \wedge \varphi_0(Y)\mathbf{t} + \varphi_0(X)\mathbf{t} \wedge [\varphi_0(Y), \gamma]\mathbf{t} + \lambda\varphi_0(X)\mathbf{t} \wedge \varphi_0(Y)\mathbf{t}).\end{aligned}$$

Denote $\gamma' = \varphi_0^{-1}(\gamma)$, then $(\varphi \otimes \varphi)(\mathbf{r}_0 + \mathbf{r}_1)$ is a scalar of $\mathbf{r}_0$ only if

$$\gamma'.X \wedge Y = \lambda X \wedge Y. \tag{3.2.1}$$

Note that $\Lambda^2\mathfrak{sl}_2$ is isomorphic to the adjoint representation of $\mathfrak{sl}_2$, thus (3.2.1) can not hold since $\lambda \neq 0$. Therefore $\mathbf{r}_0$ and $\mathbf{r}_0 + \mathbf{r}_1$ are not gauge equivalent.

Conjecture 3.2.7. *The Lie bialgebra structures on* $\mathfrak{g}_m$ *induced by* $\mathbf{r}_0$ *and* $\mathbf{r}_0 + \mathbf{r}_m$ *are not gauge equivalent.*

We call $\delta_{\mathbf{r}_0}$ the *standard Lie bialgebra structure* on $\mathfrak{g}_m$.

3.3 A class of quasitriangular Lie bialgebra structures on $\mathfrak{g}_1$

Let $m = 1$. In this section, we imitate the method of [2] to give a class of quasitriangular Lie bialgebra structures on $\mathfrak{g}_1$ by a direct but tedious calculation. More explicitly, we will find out all $\mathbf{r} \in \mathfrak{g}_m \otimes \mathfrak{g}_m$ such that

$$\mathbf{r}_{1,2} + \mathbf{r}_{2,1} = \frac{1}{2}(H_0 \otimes H_1 + H_1 \otimes H_0) + X_0 \otimes Y_1 + X_1 \otimes Y_0 + Y_0 \otimes X_1 + Y_1 \otimes X_0, \tag{3.3.1}$$

$$[[\mathbf{r}, \mathbf{r}]] = 0. \tag{3.3.2}$$

Let $\widehat{\mu} = (1, 0)$ and $\langle -, - \rangle := (-, -)^{\widehat{\mu}}_{\mathfrak{g}_1}$ be the nondegenerate symmetric invariant bilinear form on $\mathfrak{g}_1$ defined in Section 3.2. Let $e_1, e_2, \ldots, e_6$ be an orthogonal basis respect to $\langle -, - \rangle$, then the right side of (3.3.1) is exactly $C := \sum_{i=1}^{6} e_i \otimes e_i$. Any $\mathbf{r} \in \mathfrak{g}_1 \otimes \mathfrak{g}_1$ determines an unique linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_1$, such that

$$\mathbf{r} = \sum_{i=1}^{6} f(e_i) \otimes e_i.$$

Then (3.3.1) and (3.3.2) are equivalent to the following equations for f respectively:

$$f + f^* = \mathrm{id}_{\mathfrak{g}_1}, \tag{3.3.3}$$

$$(f - \mathrm{id}_{\mathfrak{g}_1})[f(a), f(b)] = f\left([(f - \mathrm{id}_{\mathfrak{g}_1})(a), (f - \mathrm{id}_{\mathfrak{g}_1})(b)]\right), \tag{3.3.4}$$

where f^* is the adjoint operator of f, i.e., $\langle f(a), b \rangle = \langle a, f^*(b) \rangle$ for all $a, b \in \mathfrak{g}_1$. By (3.3.3), f induces a well-defined linear map $\theta : \mathrm{Im}(f - \mathrm{id}_{\mathfrak{g}_1})/\ker f \to \mathrm{Im}\, f/\ker(f - \mathrm{id}_{\mathfrak{g}_1})$ by taking the coset $(f - \mathrm{id}_{\mathfrak{g}_1})(w) + \ker f$ into $f(w) + \ker(f - \mathrm{id}_{\mathfrak{g}_1})$. Moreover, θ is an isomorphism as vector spaces.

Lemma 3.3.1. *[2, Theorem 6.3] Let $f : \mathfrak{g}_1 \to \mathfrak{g}_1$ be a linear map satisfying* (3.3.3). *Denote $C_1 = \mathrm{Im}(f - \mathrm{id}_{\mathfrak{g}_1})$ and $C_2 = \mathrm{Im}\, f$. Then*
(i) $C_1^{\perp} = \ker f$, $C_2^{\perp} = \ker(f - \mathrm{id}_{\mathfrak{g}_1})$.
(ii) *The linear map $\theta : C_1/C_1^{\perp} \to C_2/C_2^{\perp}$ defined above is orthogonal respect to the nondegenerate symmetric bilinear forms on $C_1/C_1^{\perp}$ and $C_2/C_2^{\perp}$ induced by $\langle -, - \rangle$.*
(iii) *f satisfies* (3.3.4) *if and only if C_1, C_2 are Lie subalgebras of $\mathfrak{g}_1$ and θ is an isomorphism of Lie algebras.*

Lemma 3.3.2. *[2, Lemma 12.3] Let $f : \mathfrak{g}_1 \to \mathfrak{g}_1$ be a linear map satisfying* (3.3.3) *and* (3.3.4). *For any $\lambda \in \mathbb{k}$, denote $\mathfrak{g}_1^{\lambda} = \cup_n \ker(f - \lambda \mathrm{id}_{\mathfrak{g}_1})^n$ and $\mathfrak{g}_1' = \oplus_{\lambda \neq 0,1} \mathfrak{g}_1^{\lambda}$. Then*
(i) $(\mathfrak{g}_1^0)^{\perp} = \mathfrak{g}_1^0 \oplus \mathfrak{g}_1'$, $(\mathfrak{g}_1^1)^{\perp} = \mathfrak{g}_1^1 \oplus \mathfrak{g}_1'$. *Therefore $\mathfrak{g}_1' \subseteq C_1 \cap C_2$ and $2 \mid \dim \mathfrak{g}_1/\mathfrak{g}_1'$, $\dim \mathfrak{g}_1^0 = \dim \mathfrak{g}_1^1$.*
(ii) *$\mathfrak{g}_1^0 \oplus \mathfrak{g}_1'$ and $\mathfrak{g}_1^1 \oplus \mathfrak{g}_1'$ are Lie subalgebras of $\mathfrak{g}_1$, and $\mathfrak{g}_1^i$ is an ideal of $\mathfrak{g}_1^i \oplus \mathfrak{g}_1'$.*

Before our computation, an observation on automorphisms of $\mathfrak{g}_1$ is required.

Lemma 3.3.3. *Let φ is an automorphism of $\mathfrak{g}_1$. Then by Example 3.2.6, there exists a unique triple $(\varphi_0, \gamma, \lambda) \in \mathrm{Aut}\, \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathbb{k}^{\times}$, such that*

$$\varphi(a) = \varphi_0(a) + [\varphi_0(a), \gamma]\mathbf{t}, \quad \varphi(a\mathbf{t}) = \lambda \varphi_0(a)\mathbf{t}, \quad \forall a \in \mathfrak{sl}_2.$$

Indeed, this is an 1-1 correspondence between $\mathrm{Aut}\, \mathfrak{g}_1$ and $\mathrm{Aut}\, \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathbb{k}^{\times}$. Moreover,
(i) $\langle \varphi(a), \varphi(b) \rangle = \lambda \langle a, b \rangle, \quad \forall a, b \in \mathfrak{g}_1$.
(ii) If φ is an inner automorphism, then φ_0 is an inner automorphism of $\mathfrak{sl}_2$ and $\lambda = 1$.

Proof. (i) There exists a nondegenerate symmetric invariant bilinear form κ on $\mathfrak{sl}_2$ defined by $\kappa(a,b)=\mathrm{Tr}(ab)$, where elements of $\mathfrak{sl}_2$ is considered as matrices in $\mathfrak{gl}_2$ with zero trace, i.e.,

$$H\mapsto\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X\mapsto\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y\mapsto\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

It is a well-known fact all automorphisms of $\mathfrak{sl}_2$ is orthogonal respect to κ. For any $a_1,a_2\in\mathfrak{sl}_2$ and $g_1(\mathbf{t}),g_2(\mathbf{t})\in\Bbbk[\mathbf{t}]/(\mathbf{t}^2)$, we have

$$\langle a_1g_1(\mathbf{t}),a_2g_2(\mathbf{t})\rangle=\kappa(a_1,a_2)\left(\frac{\mathrm{d}}{\mathrm{d}\mathbf{t}}g_1(\mathbf{t})g_2(\mathbf{t})\right)(0).$$

Hence $\langle\varphi(a),\varphi(b)\rangle=\lambda\langle a,b\rangle$ if any one of a,b is in $\mathfrak{sl}_2\mathbf{t}$. Now suppose $a,b\in\mathfrak{sl}_2$, then

$$\langle\varphi(a),\varphi(b)\rangle=\kappa(\varphi_0(a),[\varphi_0(b),\gamma])+\kappa([\varphi_0(a),\gamma],\varphi_0(b))=0=\langle a,b\rangle.$$

(ii) Suppose $\varphi=\exp\mathrm{ad}(a+b\mathbf{t})$ for some $a,b\in\mathfrak{sl}_2$. Then for any $x\in\mathfrak{sl}_2$,

$$\varphi(x)=\exp\mathrm{ad}(a+b\mathbf{t})(x)=\exp\mathrm{ad}(a)(x)+c\mathbf{t},$$

$$\varphi(x\mathbf{t})=\exp\mathrm{ad}(a)(x\mathbf{t})=\exp\mathrm{ad}(a)(x)\mathbf{t}.$$

Hence $\varphi_0=\exp\mathrm{ad}(a)$ and $\lambda=1$. ☐

Let φ be an automorphism of $\mathfrak{g}_1$ such that $\langle\varphi(a),\varphi(b)\rangle=\lambda\langle a,b\rangle$ for any $a,b\in\mathfrak{g}_m$. Then

$$(\varphi\otimes\varphi)(\mathbf{r}_{1,2}+\mathbf{r}_{2,1})=(\varphi\otimes\varphi)C=\lambda C$$

and

$$(\varphi\otimes\varphi)(\mathbf{r})=(\varphi\otimes\varphi)(f\otimes\mathrm{id}_{\mathfrak{g}_1})C=\lambda(\varphi\otimes\varphi)(f\otimes\mathrm{id}_{\mathfrak{g}_1})(\varphi^{-1}\otimes\varphi^{-1})C=\lambda(\varphi f\varphi^{-1}\otimes\mathrm{id}_{\mathfrak{g}_1})C.$$

Therefore $(\varphi\otimes\varphi)(\mathbf{r})$ still satisfies (3.3.1) and (3.3.2) if φ is an inner automorphism.

From now on we suppose f satisfies (3.3.3) and (3.3.4). Here is the argument on f up to orthogonal automorphisms of $\mathfrak{g}_1$.

(1) $\underline{C_1\text{ or }C_2\text{ is not solvable}}$. First let C_2 be not solvable. Then there exists an orthogonal automorphism φ of $\mathfrak{g}_1$ such that $\mathfrak{sl}_2\subseteq\varphi(C_2)$. Up to orthogonal automorphisms, we set $\mathfrak{sl}_2\subseteq C_2$. Hence $C_2^{\perp}\subseteq\mathfrak{sl}_2^{\perp}=\mathfrak{sl}_2$. Since $C_2^{\perp}$ is an ideal of C_2, we have either $C_2^{\perp}=0$ or $C_2^{\perp}=C_2$. If $C_2^{\perp}=0$ then $C_2=\mathfrak{g}_1$. Recall $\theta:C_1/C_1^{\perp}\to C_2/C_2^{\perp}$ is an Lie algebra isomorphism, therefore $C_1=\mathfrak{g}_1$ and $C_1^{\perp}=0$. Thus $\theta=f\circ(f-\mathrm{id}_{\mathfrak{g}_1})^{-1}$ becomes an orthogonal automorphism of $\mathfrak{g}_1$. Note that $\det(\theta-\mathrm{id}_{\mathfrak{g}_1})\neq0$, by the corollary of [2, Theorem 9.2] $\mathfrak{g}_1$ is solvable, a contradiction! Then $C_2=C_2^{\perp}=\mathfrak{sl}_2$.

Suppose $C_1=C_1^{\perp}=\Bbbk(X_1-g(X))+\Bbbk(H_1-g(H))+\Bbbk(Y_1-g(Y))$ for some $g(X),g(H),g(Y)\in\mathfrak{sl}_2$. Then $\kappa(a,g(b))+\kappa(g(a),b)=0$ for any $a,b\in\mathfrak{sl}_2$. Hence there exists an unique $c\in\mathfrak{sl}_2$ such that $g=-\mathrm{ad}(c)$. The condition C_1 is a subalgebra forces that $\kappa(c,c)=0$. In this case,

$$\mathbf{r}=\frac{1}{2}H_0\otimes H_1+\frac{1}{2}[H_0,c]\otimes H_0+X_0\otimes Y_1+Y_0\otimes X_1+[Y_0,c]\otimes X_0.$$

(2) $\underline{C_1\text{ and }C_2\text{ are solvable}}$. Then there exist Borel subalgberas $\mathfrak{b}_1,\mathfrak{b}_2$ of $\mathfrak{g}_1$ such that $C_i\subseteq\mathfrak{b}_i$. Then $\mathfrak{b}_i/\mathfrak{a}_1$ are Borel subalgebras of $\mathfrak{sl}_2\cong\mathfrak{g}_1/\mathfrak{a}_1$. Since $\mathfrak{b}_1+\mathfrak{b}_2=\mathfrak{g}_1$, there is an inner automorphism φ_0 such that $\varphi_0(\mathfrak{b}_1/\mathfrak{a}_1)=\Bbbk H+\Bbbk X$ and $\varphi_0(\mathfrak{b}_2/\mathfrak{a}_1)=\Bbbk H+\Bbbk Y$. Therefore up to an inner automorphism of $\mathfrak{g}_1$, we can suppose $\mathfrak{b}_1=\Bbbk H_0+\Bbbk X_0+\mathfrak{a}_1$ and $\mathfrak{b}_2=\Bbbk H_0+\Bbbk Y_0+\mathfrak{a}_1$. By Lemma 3.3.2, $\mathfrak{g}_1'\subseteq C_1\cap C_2\subseteq\mathfrak{b}_1\cap\mathfrak{b}_2=\Bbbk H_0+\mathfrak{a}_1$. Since $\Bbbk X_1=\mathfrak{b}_1^{\perp}\subseteq C_1^{\perp}$, we have $f(X_1)=0$. Similarly, $f(Y_1)=Y_1$. Note that $2\mid\dim\mathfrak{g}_1'$, hence $\mathfrak{g}_1'=0$ or $\mathfrak{g}_1'=\Bbbk H_0+\Bbbk H_1$.

If $\mathfrak{g}'=\Bbbk H_0+\Bbbk H_1$, then a routine argument shows that $C_i=\mathfrak{b}_i$ or $C_1=\Bbbk H_0+\Bbbk H_1+\Bbbk X_0+\Bbbk X_1$ and $C_2=\Bbbk H_0+\Bbbk H_1+\Bbbk Y_0+\Bbbk Y_1$. Since $\theta:C_1/C_1^{\perp}\to C_2/C_2^{\perp}$ is an orthogonal Lie algebra isomorphism, there must be $C_1 = \Bbbk H_0 + \Bbbk H_1 + \Bbbk X_0 + \Bbbk X_1$ and $C_2 = \Bbbk H_0 + \Bbbk H_1 + \Bbbk Y_0 + \Bbbk Y_1$ by a tedious calculation. In this case, $f(X_i) = 0$, $f(Y_i) = Y_i$, $f(H_0) = \lambda H_0$ and $f(H_1) = (1-\lambda)H_1$ for some $\lambda \neq 0, 1$. Therefore

$$\mathbf{r} = \frac{1}{2}\lambda H_0 \otimes H_1 + \frac{1}{2}(1-\lambda)H_1 \otimes H_0 + Y_1 \otimes X_0 + Y_0 \otimes X_1.$$

If $\mathfrak{g}' = 0$, then $\dim \mathfrak{g}_1^0 = \dim \mathfrak{g}_1^1 = 3$, $(\mathfrak{g}_1^0)^\perp = \mathfrak{g}_1^0$ and $(\mathfrak{g}_1^1)^\perp = \mathfrak{g}_1^1$. The condition C_i are solvable implies that $\mathfrak{g}_1^0$ and $\mathfrak{g}_1^1$ are solvable. A boring argument works out that $\mathfrak{g}_1^0 = \Bbbk H_0 + \Bbbk X_0 + \Bbbk X_1$, $\mathfrak{g}_1^1 = \Bbbk H_1 + \Bbbk Y_0 + \Bbbk Y_1$ or $\mathfrak{g}_1^0 = \Bbbk H_1 + \Bbbk X_0 + \Bbbk X_1$, $\mathfrak{g}_1^1 = \Bbbk H_0 + \Bbbk Y_0 + \Bbbk Y_1$. Indeed, these two cases are equivalent, since $\omega : H_i \mapsto -H_i, X_i \mapsto Y_i, Y_i \mapsto X_i$ is an orthogonal automorphism of $\mathfrak{g}_1$ respect to $\langle -, - \rangle$ and one can replacing f by $\mathrm{id}_{\mathfrak{g}_1} - f$. We set $\mathfrak{g}_1^0 = \Bbbk H_0 + \Bbbk X_0 + \Bbbk X_1$, $\mathfrak{g}_1^1 = \Bbbk H_1 + \Bbbk Y_0 + \Bbbk Y_1$. If $C_1 = \mathfrak{g}_0$ and $C_2 = \mathfrak{g}_1$, then $f(H_0) = f(X_0) = f(X_1) = (f - \mathrm{id}_{\mathfrak{g}_1})(H_1) = (f - \mathrm{id}_{\mathfrak{g}_1})(Y_0) = (f - \mathrm{id}_{\mathfrak{g}_1})(Y_1) = 0$, and

$$\mathbf{r} = \frac{1}{2}H_1 \otimes H_0 + Y_0 \otimes X_1 + Y_1 \otimes X_0.$$

If $\dim C_1 = \dim C_2 = 4$, then an uninteresting computation provides that $C_1 = \Bbbk H_0 + \Bbbk H_1 + \Bbbk X_0 + \Bbbk X_1$, $C_2 = \Bbbk H_1 + \Bbbk X_1 + \Bbbk Y_0 + \Bbbk Y_1$, and there is a nonzero scalar λ such that

$$f(X_0) = f(X_1) = (f - \mathrm{id}_{\mathfrak{g}-1})(H_1) = (f - \mathrm{id}_{\mathfrak{g}-1})(Y_1) = 0,$$

$$f(H_0) = -2\lambda X_1, \quad (f - \mathrm{id}_{\mathfrak{g}_1})(Y_0) = \lambda H_1.$$

In this case

$$\mathbf{r} = \frac{1}{2}H_1 \otimes H_0 + Y_0 \otimes X_1 + Y_1 \otimes X_0 + \lambda H_1 \wedge X_1.$$

Finally, an analysis shows that there is a contradiction if $C_i = \mathfrak{b}_i$. The whole argument is completed. To summarize,

Proposition 3.3.4. *Suppose* $\mathbf{r}$ *satisfies* (3.3.1) *and* (3.3.2). *Then either* $\mathbf{r}$ *or* $C - \mathbf{r}$ *can be uniquely transformed by a suitable automorphism into one of the following forms:*

(i) $\frac{1}{2}H_0 \otimes H_1 + X_0 \otimes Y_1 + Y_0 \otimes X_1 + \frac{1}{2}[H_0, c] \otimes H_0 + [X_0, c] \otimes Y_0 + [Y_0, c] \otimes X_0$ *for some* $c \in \mathfrak{sl}_2$ *such that* $\kappa(c, c) = 0$.

(ii) $\frac{1}{2}\lambda H_0 \otimes H_1 + \frac{1}{2}(1-\lambda)H_1 \otimes H_0 + Y_1 \otimes X_0 + Y_0 \otimes X_1$ *for some* $\lambda \neq 0, 1$.

(iii) $\frac{1}{2}H_1 \otimes H_0 + Y_0 \otimes X_1 + Y_1 \otimes X_0$.

(iv) $\frac{1}{2}H_1 \otimes H_0 + Y_0 \otimes X_1 + Y_1 \otimes X_0 + \lambda H_1 \wedge X_1$ *for some* $\lambda \neq 0$.

Chapter 4

A quantization of $\mathfrak{sl}_2[\mathsf{t}]/(\mathsf{t}^{m+1})$

4.1 Main result

Recall that the standard Lie bialgebra structure $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$ given in Section 3.2, which is induced by $\mathbf{r}_0 = \sum_{i+j=m} X_i \wedge Y_j$. In this chapter we will show that the Hopf algebra $U_h(\mathfrak{g}_m)$ in the following theorem is a quantization of $(\mathfrak{g}_m, \mathbf{r}_0)$ with respect to Definition 2.4.4, and $U_h(\mathfrak{g}_m)$ is a trivial algebra deformation (mod h^4).

Theorem 4.1.1. *Let $A = \mathbb{k}\{H_r, X_s, Y_t | 0 \le r, s, t \le m\}$ be the algebra of non-commutative polynomials in $3(m+1)$'s generators H_r, X_s, Y_t, and I be the two-sided ideal of $A[[h]]$ generated by*

$$[H_i, H_j], \quad [X_i, X_j], \quad [Y_i, Y_j], \tag{4.1.1}$$

$$[H_r, X_s] - 2X_{r+s}, \quad [H_r, Y_t] + 2Y_{r+t}, \quad [X_s, Y_t] - \frac{P^X_{m-s-t} - P^Y_{m-s-t}}{e^h - e^{-h}}, \tag{4.1.2}$$

where $H_k = X_k = Y_k = 0$ if $k > m$,

$$P^X_i = \sum_{j \ge 0} \frac{h^j}{j!} \sum_{a_1 + \cdots + a_j = jm-i} H_{a_1} \cdots H_{a_j}, \quad P^Y_i = \sum_{j \ge 0} \frac{(-h)^j}{j!} \sum_{a_1 + \cdots + a_j = jm-i} H_{a_1} \cdots H_{a_j}$$

for $0 \le i \le m$ and $P^X_i = P^Y_i = 0$ if $i < 0$. Let $\bar{I}$ be the closure of I in the h-adic topology. Then $U_h(\mathfrak{g}_m) = A[[h]]/\bar{I}$ is an algebra over $k[[h]]$.

There exists homomorphisms of $\mathbb{k}[[h]]$-algebras $\Delta_h : U_h(\mathfrak{g}_m) \to U_h(\mathfrak{g}_m) \otimes U_h(\mathfrak{g}_m)$, $\epsilon_h : U_h(\mathfrak{g}_m) \to \mathbb{k}[[h]]$ and $S_h : U_h(\mathfrak{g}_m) \to U_h(\mathfrak{g}_m)$ work on generators of $U_h(\mathfrak{g}_m)$ as follows:

$$\Delta_h(H_r) = H_r \otimes 1 + 1 \otimes H_r, \tag{4.1.3}$$

$$\Delta_h(X_s) = 1 \otimes X_s + \sum_{i=s}^{m} X_i \otimes P^X_{i-s}, \quad \Delta_h(Y_t) = Y_t \otimes 1 + \sum_{i=t}^{m} P^Y_{i-t} \otimes Y_i, \tag{4.1.4}$$

$$\epsilon_h(H_r) = \epsilon_h(X_s) = \epsilon_h(Y_t) = 0, \tag{4.1.5}$$

$$S_h(H_r) = -H_r, \quad S_h(X_s) = -\sum_{i=s}^{m} X_i P^Y_{i-s}, \quad S_h(Y_t) = -\sum_{i=t}^{m} P^X_{i-t} Y_i \tag{4.1.6}$$

for $0 \le r, s, t \le m$. Then Δ_h, ϵ_h and S_h as comultiplication, counit and antipode respectively define a topological Hopf structure of $U_h(\mathfrak{g}_m)$.

Moreover, $U_h(\mathfrak{g}_m)$ is a QUE algebra, whose classical limit is the Lie bialgebra structure on $\mathfrak{g}_m$ defined by $\mathbf{r}_0 = \sum_{i+j=m} X_i \wedge Y_j$.

This chapter is organized as follows. We prove that Δ_h in Theorem 4.1.1 is an algebra homomorphism in Section 4.2 and $U_h(\mathfrak{g}_m)$ is a topological Hopf algebra over $\Bbbk[[h]]$ in Section 4.3. In Section 4.4 we establish a topological PBW basis of $U_h(\mathfrak{g}_m)$, which completes the proof of Theorem 4.1.1 together with results in Section 4.2 and 4.3. Finally, an careful argument on cohomology in Section 4.5 shows that $U_h(\mathfrak{g}_m)$ is trivial as an algebra deformation (mod h^4) of $U(\mathfrak{g}_m)$.

4.2 Δ_h is an algebra homomorphism

Let us make some argument on $U_h(\mathfrak{g}_m)$ defined in Theorem 4.1.1. One should note that in both $P^X_{m-s-t} - P^Y_{m-s-t}$ and $e^h - e^{-h}$ each odd powerth of h appears but even powerth of h does not, hence $\dfrac{P^X_{m-s-t} - P^Y_{m-s-t}}{e^h - e^{-h}}$ is a well-defined element in $U_h(\mathfrak{g}_m)$. If $m = 0$ then $\mathfrak{g}_m$ is exactly $\mathfrak{sl}_2$ and $U_h(\mathfrak{g}_m)$ is the quantum group $U_h(\mathfrak{sl}_2)$ in Example 2.4.6. Hence we always set $m > 0$ in the rest of this chapter if there is no statement.

Lemma 4.2.1. *For P^X_i and P^Y_i, $0 \le i \le m$, we have*

$$\Delta_h(P^X_i) = \sum_{k=0}^{i} P^X_k \otimes P^X_{i-k}, \quad \Delta_h(P^Y_i) = \sum_{k=0}^{i} P^Y_k \otimes P^Y_{i-k}. \tag{4.2.1}$$

Proof. We only prove (4.2.1) for P^X_i since the case for P^Y_i is similar. A direct calculation shows that

$$\begin{aligned}
\Delta_h(P^X_i) =& \sum_{j\ge 0} \frac{h^j}{j!} \sum_{a_1+\cdots+a_j=jm-i} \Delta_h(H_{a_1}\cdots H_{a_j}) \\
=& \sum_{j\ge 0} \frac{h^j}{j!} \sum_{a_1+\cdots+a_j=jm-i} \sum_{u=0}^{j} \binom{j}{u} H_{a_1}\cdots H_{a_u} \otimes H_{a_{u+1}}\cdots H_{a_j} \\
=& \sum_{k=0}^{i} \sum_{j\ge 0} \sum_{u=0}^{j} \left(\frac{h^u}{u!} \sum_{\substack{b_1+\cdots+b_u \\ =um-k}} H_{b_1}\cdots H_{b_u} \right) \otimes \left(\frac{h^{j-u}}{(j-u)!} \sum_{\substack{c_1+\cdots+c_{j-u} \\ =(j-u)m-(i-k)}} H_{c_1}\cdots H_{c_{j-u}} \right) \\
=& \sum_{k=0}^{i} \sum_{u\ge 0} \sum_{v\ge 0} \left(\frac{h^u}{u!} \sum_{b_1+\cdots+b_u=um-k} H_{b_1}\cdots H_{b_u} \right) \otimes \left(\frac{h^v}{v!} \sum_{c_1+\cdots+c_v=vm-(i-k)} H_{c_1}\cdots H_{c_v} \right) \\
=& \sum_{k=0}^{i} P^X_k \otimes P^X_{i-k},
\end{aligned}$$

which is exactly what we want. ☐

Lemma 4.2.2. *For $0 \le i, s, t \le m$, we have*

$$P^X_i X_s = \sum_{c\ge 0} X_{s+c} \sum_{j\ge 0} \frac{(2h)^j}{j!} N^c_j P^X_{i+c-jm}, \quad P^Y_i X_s = \sum_{c\ge 0} X_{s+c} \sum_{j\ge 0} \frac{(-2h)^j}{j!} N^c_j P^Y_{i+c-jm}, \tag{4.2.2}$$

$$P^X_i Y_t = \sum_{c\ge 0} Y_{t+c} \sum_{j\ge 0} \frac{(-2h)^j}{j!} N^c_j P^X_{i+c-jm}, \quad P^Y_i Y_t = \sum_{c\ge 0} Y_{t+c} \sum_{j\ge 0} \frac{(2h)^j}{j!} N^c_j P^Y_{i+c-jm}, \tag{4.2.3}$$

where $N^c_0 = \delta_{0,c}$ and $N^c_j = |\{0 \le a_1, \ldots, a_j \le m | a_1 + \cdots + a_j = c\}|$ for $j \ge 1$.

Proof. We only prove (4.2.2) for P_i^X, X_s, the rest are similar. Fix $a_i, \dots, a_j \in \mathbb{N}$, an easy induction shows that

$$H_{a_1} \cdots H_{a_j} X_s = \sum_{k=0}^{j} \sum_{\substack{\text{choose } b_1,\dots,b_k \\ \text{from } a_1,\dots,a_j, \\ c_{k+1},\dots,c_j \text{ left}}} 2^k X_{s+b_1+\cdots+b_k} H_{c_{k+1}} \cdots H_{c_j}.$$

Therefore we have

$$\begin{aligned} P_i^X X_s =& \sum_{j\geq 0} \frac{h^j}{j!} \sum_{a_1+\cdots+a_j=jm-i} \sum_{k=0}^{j} \sum_{\substack{\text{choose } b_1,\dots,b_k \\ \text{from } a_1,\dots,a_j, \\ c_{k+1},\dots,c_j \text{ left}}} 2^k X_{s+b_1+\cdots+b_k} H_{c_{k+1}} \cdots H_{c_j} \\ =& \sum_{j\geq 0} \frac{h^j}{j!} \sum_{c\geq 0} X_{s+c} \sum_{k=0}^{j} \sum_{c_{k+1}+\cdots+c_j=jm-i-c} 2^k \binom{j}{j-k} N_k^c H_{c_{k+1}} \cdots H_{c_j} \\ =& \sum_{c\geq 0} X_{s+c} \sum_{k\geq 0} \frac{(2h)^k}{k!} N_k^c \sum_{l\geq 0} \frac{h^l}{l!} \sum_{e_1+\cdots+e_l=lm-(i+c-km)} H_{e_1} \cdots H_{c_l} \\ =& \sum_{c\geq 0} X_{s+c} \sum_{k\geq 0} \frac{(2h)^k}{k!} N_k^c P_{i+c-km}^X, \end{aligned}$$

which is exactly what we want. ☐

Remark 4.2.3. *Since $P_k^X = 0$ if $k < 0$, $X_s = 0$ if $s > m$ and $i \leq m$, the formula in Lemma 4.2.2 for P_i^X and X_s can be reduced as follows:*

$$P_i^X X_s = \sum_{c=0}^{m} X_{s+c}(\delta_{0,c} P_i^X + 2h P_{i+c-m}^X + \delta_{i,m}\delta_{c,m} 2h^2(m+1)P_0^X). \tag{4.2.4}$$

In particular, if $i < s$ we have $P_i^X X_s = X_s P_i^X$. Similar reductions act for rest formulas.

Now we can prove Δ_h is a well-defined algebra homomorphism. An easy verification shows that

$$[\Delta_h(H_i), \Delta_h(H_j)] = 0, \quad [\Delta_h(H_r), \Delta_h(X_s)] = 2\Delta_h(X_{r+s}), \quad [\Delta_h(H_r), \Delta_h(Y_t)] = -2\Delta_h(Y_{r+t})$$

hold for $0 \leq i, j, r, s, t \leq m$.

Lemma 4.2.4. *For $0 \leq s < t \leq m$ we have*

$$[\Delta_h(X_s), \Delta_h(X_t)] = [\Delta_h(Y_s), \Delta_h(Y_t)] = 0$$

in $U_h(\mathfrak{g}_m)$.

Proof. We only prove Δ_h satisfies the relation between X_s and X_t. Since $P_i^X X_s = X_s P_i^X$ if $i < s$, one have

$$[\Delta_h(X_s), \Delta_h(X_t)] = \sum_{i=s}^{m} X_i \otimes [P_{i-s}^X, X_t] + \sum_{j=t}^{m} X_j \otimes [X_s, P_{j-t}^X] = \sum_{k=s+t}^{m} X_k \otimes ([P_{k-s}^X, X_t] - [P_{k-t}^X, X_s]).$$

Fix k, thanks to equation (4.2.4) we get

$$\begin{aligned}[P^X_{k-s}, X_t] - [P^X_{k-t}, X_s] =&\sum_{c=0}^{m} X_{t+c}(2hP^X_{k-s+c-m} + \delta_{k-s,m}\delta_{c,m}2h^2(m+1)P^X_0)\\ &- \sum_{d=0}^{m} X_{s+d}(2hP^X_{k-t+d-m} + \delta_{k-t,m}\delta_{d,m}2h^2(m+1)P^X_0)\\ =&\sum_{c=m-k+s}^{m-t} 2hX_{t+c}P^X_{k-s+c-m} - \sum_{d=m-k+t}^{m-s} 2hX_{s+d}P^X_{k-t+d-m}\\ =&0.\end{aligned}$$

The proof of $[\Delta_h(Y_s), \Delta_h(Y_t)] = 0$ is similar. □

Lemma 4.2.5. *For $0 \le s, t \le m$ we have*

$$[\Delta_h(X_s), \Delta_h(Y_t)] = \Delta_h\left(\frac{P^X_{m-s-t} - P^Y_{m-s-t}}{e^h - e^{-h}}\right)$$

in $U_h(\mathfrak{g}_m)$.

Proof. This proof is also a direct calculation. We compute

$$\begin{aligned}&[\Delta_h(X_s), \Delta_h(Y_t)]\\ =&\sum_{j=t}^{m} P^Y_{j-t} \otimes [X_s, Y_j] + \sum_{i=s}^{m}[X_i, Y_t] \otimes P^X_{i-s} + \sum_{i=s}^{m}\sum_{j=t}^{m}(X_iP^Y_{j-t} \otimes P^X_{i-s}Y_j - P^Y_{j-t}X_i \otimes Y_jP^X_{i-s})\\ =&\sum_{j=t}^{m} P^Y_{j-t} \otimes \frac{P^X_{m-s-j} - P^Y_{m-s-j}}{e^h - e^{-h}} + \sum_{i=s}^{m} \frac{P^X_{m-i-t} - P^Y_{m-i-t}}{e^h - e^{-h}} \otimes P^X_{i-s}\\ &+\sum_{i=s}^{m}\sum_{j=t}^{m}(X_iP^Y_{j-t} \otimes P^X_{i-s}Y_j - P^Y_{j-t}X_i \otimes Y_jP^X_{i-s})\\ =&\sum_{k=0}^{m-s-t} \frac{(P^X_k \otimes P^X_{m-s-t-k} - P^Y_k \otimes P^Y_{m-s-t-k})}{e^h - e^{-h}} + \sum_{i=s}^{m}\sum_{j=t}^{m}(X_iP^Y_{j-t} \otimes P^X_{i-s}Y_j - P^Y_{j-t}X_i \otimes Y_jP^X_{i-s}).\end{aligned}$$

Thanks to Lemma 4.2.1, we only need to check

$$\sum_{i=s}^{m}\sum_{j=t}^{m}(X_iP^Y_{j-t} \otimes P^X_{i-s}Y_j - P^Y_{j-t}X_i \otimes Y_jP^X_{i-s}) = 0. \tag{4.2.5}$$

Fix i and j, Lemma 4.2.2 and Remark 4.2.3 show us

$$\begin{aligned}X_iP^Y_{j-t} \otimes P^X_{i-s}Y_j =&\sum_{c\ge0} X_iP^Y_{j-t} \otimes Y_{j+c}(\delta_{0,c}P^X_{i-s+c} - 2hP^X_{i-s+c-m} + \delta_{i-s,m}\delta_{c,m}2h^2(m+1)P^X_0)\\ =&X_iP^Y_{j-t} \otimes Y_jP^X_{i-s} + \delta_{s,0}\delta_{t,0}\delta_{i,m}\delta_{j,0}2h^2(m+1)X_mP^Y_0 \otimes Y_mP^X_0\\ &- 2h\sum_{c=m-i+s}^{m-j} X_iP^Y_{j-t} \otimes Y_{j+c}P^X_{i-s+c-m},\end{aligned}$$

$$\begin{aligned}P^Y_{j-t}X_i \otimes Y_jP^X_{i-s} =&\sum_{d\ge0} X_{i+d}(\delta_{d,0}P^Y_{j-t+d} - 2hP^Y_{j-t+d-m} + \delta_{j-t,m}\delta_{d,m}2h^2(m+1)P^Y_0) \otimes Y_jP^X_{i-s}\\ =&X_iP^Y_{j-t} \otimes Y_jP^X_{i-s} + \delta_{s,0}\delta_{t,0}\delta_{i,0}\delta_{j,m}2h^2(m+1)X_mP^Y_0 \otimes Y_mP^X_0\\ &- 2h\sum_{d=m-j+t}^{m-i} X_{i+d}P^Y_{j-t+d-m} \otimes Y_jP^X_{i-s}.\end{aligned}$$

Therefore

$$\begin{aligned}\text{LHS of (4.2.5)} =&2h\sum_{i=s}^{m}\sum_{j=t}^{m}\left(\sum_{d=m-j+t}^{m-i} X_{i+d}P^Y_{j-t+d-m}\otimes Y_jP^X_{i-s} - \sum_{c=m-i+s}^{m-j} X_iP^Y_{j-t}\otimes Y_{j+c}P^X_{i-s+c-m}\right)\\ =&2h\sum_{k=0}^{m}\sum_{l=0}^{m}(X_k\otimes Y_l)\left(\sum_{i=s}^{m} P^Y_{l-t+k-i-m}\otimes P^X_{i-s} - \sum_{j=t}^{m} P^Y_{j-t}\otimes P^X_{k-s+l-j-m}\right).\end{aligned}$$

Since the two terms in the last parenthesis of the above equation are both

$$\sum_{a+b=k+l-s-t-m} P^Y_a\otimes P^X_b,$$

the proof is completed. $\square$

Combining Lemma 4.2.4 and Lemma 4.2.5 we have shown

Proposition 4.2.6. *$\Delta_h : U_h(\mathfrak{g}_m)\to U_h(\mathfrak{g}_m)\otimes U_h(\mathfrak{g}_m)$ is a well-defined $\Bbbk[[h]]$-algebra homomorphism.*

4.3 $U_h(\mathfrak{g}_m)$ is a topological Hopf algebra

Let $\mu_h : U_h(\mathfrak{g}_m)\otimes U_h(\mathfrak{g}_m)\to U_h(\mathfrak{g}_m)$ be the multiplication and $\eta_h : \Bbbk[[h]]\to U_h(\mathfrak{g}_m)$ be the unit of $U_h(\mathfrak{g}_m)$. We first show that $(U_h(\mathfrak{g}_m),\mu_h,\eta_h,\Delta_h,\epsilon_h)$ is a $\Bbbk[[h]]$-bialgebra. Thanks to Lemma 4.2.1, we have

$$\begin{aligned}(\Delta_h\otimes\mathrm{id})\Delta_h(X_s) =&1\otimes 1\otimes X_s + \sum_{i=s}^{m}(1\otimes X_i + \sum_{j=i}^{m} X_j\otimes P^X_{j-i})\otimes P^X_{i-s}\\ =&1\otimes(1\otimes X_s + \sum_{i=s}^{m} X_i\otimes P^X_{i-s}) + \sum_{j=s}^{m} X_j\otimes(\sum_{i=s}^{j} P^X_{j-i}\otimes P^X_{i-s})\\ =&(\mathrm{id}\otimes\Delta_h)\Delta_h(X_s).\end{aligned}$$

Similar calculations work for H_r and Y_t. Thus Δ_h satisfies coassociative law since it is an algebra homomorphism. It is clear that ϵ_h is a well-defined $\Bbbk[[h]]$-algebra homomorphism. One should note that $\epsilon_h(P^X_i)=\epsilon_h(P^Y_i)=\delta_{i,0}$, hence

$$\mu_h(\epsilon_h\otimes\mathrm{id})\Delta_h = \mu_h(\mathrm{id}\otimes\epsilon_h)\Delta_h = \mathrm{id}_{U_h(\mathfrak{g}_m)}.$$

Therefore $U_h(\mathfrak{g}_m)$ is a bialgebra.

Proposition 4.3.1. *The antipode $S_h : U_h(\mathfrak{g}_m)\to U_h(\mathfrak{g}_m)$ is a well-defined $\Bbbk[[h]]$-algebra antimorphism, satisfying*

$$\mu_h(S_h\otimes\mathrm{id})\Delta_h = \mu_h(\mathrm{id}\otimes S_h)\Delta_h = \epsilon_h.$$

Therefore $U_h(\mathfrak{g}_m)$ is a topological Hopf algebra over $\Bbbk[[h]]$.

Proof. It is easy to see that $S_h(P^X_i)=P^Y_i$, $S_h(P^Y_i)=P^X_i$ and $[S_h(H_i),S_h(H_j)]=0$. Since $\mu_h(S_h\otimes\mathrm{id})\Delta_h(H_r)=\mu_h(\mathrm{id}\otimes S_h)\Delta_h(H_r)=\epsilon_h(H_r)=0$, applying on equation (4.2.1) we have

$$\sum_{k=0}^{i} P^X_k P^Y_{i-k} = \epsilon_h(P^X_i) = \delta_{i,0} \tag{4.3.1}$$

for $0\le i\le m$.

We need to verify S_h holds the anti-relations in equations (4.1.1) and (4.1.2). There is no difficult to show

$$[S_h(H_r),S_h(X_s)]=-2S_h(X_{r+s}),\quad [S_h(H_r),S_h(Y_t)]=2S_h(Y_{r+t}).$$

Lemma 4.2.2 provides us

$$\begin{aligned}
[S_h(X_s),S_h(X_t)]=&\sum_{i=s}^{m}\sum_{j=t}^{m}(X_iP^Y_{i-s}X_jP^Y_{j-t}-X_jP^Y_{j-t}X_iP^Y_{i-s})\\
=&\sum_{i=s}^{m}\sum_{j=t}^{m}\Bigg(\sum_{d\geq 0}X_iX_{j+d}(\delta_{d,0}P^Y_{i-s}-2hP^Y_{i-s+d-m}+\delta_{i-s,m}\delta_{d,m}2h^2N_2^mP_0^Y)P^Y_{j-t}\\
&-\sum_{c\geq 0}X_jX_{i+c}(\delta_{c,0}P^Y_{j-t}-2hP^Y_{j-t+c-m}+\delta_{j-t,m}\delta_{c,m}2h^2N_2^mP_0^Y)P^Y_{i-s}\Bigg)\\
=&\sum_{k=0}^{m}\sum_{l=0}^{m}\Bigg[X_kX_l\left(P^Y_{k-s}P^Y_{l-t}-2h\sum_{j=t}^{m}P^Y_{k+l-s-j-m}P^Y_{j-t}\right)\\
&-X_lX_k\left(P^Y_{l-t}P^Y_{k-s}-2h\sum_{i=s}^{m}P^Y_{k+l-i-t-m}P^Y_{i-s}\right)\Bigg]\\
=&\sum_{k=0}^{m}\sum_{l=0}^{m}[X_k,X_l]\left(P^Y_{k-s}P^Y_{l-t}-2h\sum_{a+b=k+l-s-t-m}P^Y_aP^Y_b\right)=0.
\end{aligned}$$

Thanks to Remark 4.2.3, one can show that
(1) if $i<j$, then $[P_i^X,X_j]=[P_i^Y,X_j]=[P_i^X,Y_j]=[P_i^Y,Y_j]=0$;
(2) $[P_i^X,X_i]=2hX_mP_0^X$, $[P_i^Y,X_i]=-2hX_mP_0^Y$, $[P_i^X,Y_i]=-2hY_mP_0^X$, $[P_i^Y,Y_i]=2hY_mP_0^Y$.
Hence

$$\begin{aligned}
S_h(X_s)S_h(Y_t)=&\sum_{i=s}^{m}\sum_{j=t}^{m}X_iP^Y_{i-s}P^X_{j-t}Y_j\\
=&\sum_{i=s}^{m}\sum_{j=t}^{m}X_iP^Y_{i-s}(Y_jP^X_{j-t}-\delta_{t,0}2hY_mP_0^X)\\
=&\sum_{i=s}^{m}\sum_{j=t}^{m}\sum_{c\geq 0}X_iY_{j+c}(\delta_{c,0}P^Y_{i-s}+2hP^Y_{i-s+c-m}+\delta_{i-s,m}\delta_{c,m}2h^2(m+1)P_0^Y)P^X_{j-t}\\
&-\delta_{t,0}(m+1)\sum_{i=s}^{m}2hX_iP^Y_{i-s}Y_mP_0^X\\
=&\sum_{k=0}^{m}\sum_{l=0}^{m}X_kY_l\left(P^Y_{k-s}P^X_{l-t}+2h\sum_{a+b=k+l-s-t-m}P^Y_aP^X_b\right)\\
&-\delta_{s,0}\delta_{t,0}2h^2(m+1)X_mY_mP_0^XP_0^Y-\delta_{t,0}2h(m+1)\sum_{i=s}^{m}X_iY_mP^Y_{i-s}P_0^X.
\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}
S_h(Y_t)S_h(X_s)=&\sum_{k=0}^{m}\sum_{l=0}^{m}Y_lX_k\left(P^X_{l-t}P^Y_{k-s}+2h\sum_{a+b=k+l-s-t-m}P^X_aP^Y_b\right)\\
&-\delta_{s,0}\delta_{t,0}2h^2(m+1)Y_mX_mP_0^XP_0^Y-\delta_{t,0}2h(m+1)\sum_{i=s}^{m}Y_mX_iP_0^XP^Y_{i-s}.
\end{aligned}$$

(4.3.1) implies that

$$\begin{aligned}
[S_h(X_s), S_h(Y_t)] =& \sum_{k=0}^{m}\sum_{l=0}^{m}[X_k, Y_l](P^X_{l-t}P^Y_{k-s} + 2h\delta_{k+l,s+t+m}) - \delta_{t,0}2h(m+1)\sum_{i=s}^{m}[X_i, Y_m]P^X_0 P^Y_{i-s} \\
=& \sum_{u=0}^{m} \frac{P^X_{m-u} - P^Y_{m-u}}{e^h - e^{-h}}(\delta_{u,s+t} + 2h(u+1)\delta_{u,s+t+m}) - \delta_{s,0}\delta_{t,0}2h(m+1)\frac{P^X_0 - P^Y_0}{e^h - e^{-h}} \\
=& \frac{P^X_{m-s-t} - P^Y_{m-s-t}}{e^h - e^{-h}} = [X_s, Y_t].
\end{aligned}$$

Therefore S_h is a well-defined antimorphism. Finally we prove $\mu_h(S_h \otimes \mathrm{id})\Delta_h(X_s) = 0 = \mu_h(\mathrm{id} \otimes S_h)\Delta_h(X_s)$. The second equality is evident. For the first equality we have

$$\mu_h(S_h \otimes \mathrm{id})\Delta_h(X_s) = X_s - \sum_{i=s}^{m}\sum_{j=i}^{m} X_j P^Y_{j-i} P^X_{i-s} = X_s - \sum_{j=s}^{m}\delta_{j,s}X_j = 0,$$

where we use equation (4.3.1) again. Similar argument works for Y_t. We have shown that S_h is an antipode of bialgebra $U_h(\mathfrak{g}_m)$, hence $U_h(\mathfrak{g}_m)$ is a topological Hopf algebra. □

4.4 A topological PBW basis of $U_h(\mathfrak{g}_m)$

In this section we establish a topological PBW basis of $U_h(\mathfrak{g}_m)$ over $k[[h]]$, which provides an isomorphism $U_h(\mathfrak{g}_m) \cong U(\mathfrak{g}_m)[[h]]$ as $k[[h]]$-mods. Inspired by the classical theory of PBW basis for Lie algebra, we construct a topological $U_h(\mathfrak{g}_m)$-module structure on the trivial deformation of polynomial ring $V = \Bbbk[F_t, Z_r, E_s | 0 \le r, s, t \le m]$. All modules, algebras and bases are in the topological sense in the rest of this section.

Denote $F_k = Z_k = E_k = 0$ if $k > m$. Let $\mathbb{J}$ be the set consisting of all finite increasing integer sequences with each term between 0 and m (the empty sequence lies in $\mathbb{J}$). It is clear that $V[[h]]$ has a basis $\{F_{J_1}Z_{J_2}E_{J_3} | J_1, J_2, J_3 \in \mathbb{J}\}$ as a $k[[h]]$-mod, where $F_J = F_{a_1}\cdots F_{a_n}$ for any finite integer sequence $J = (a_1, \ldots, a_n)$ with $0 \le a_i \le m$. Z_J, E_J are defined in a similar way. Given an integer sequence $J = (a_1, \ldots, a_n)$, some integers $j, c_1, \ldots, c_k$ and a increasing subsequence $\omega = (i_1, \ldots, i_k)$ of $(1, \ldots, n)$ with length of k, we denote
(1) $(j, J) = (j, a_1, \ldots, a_n)$;
(2) $J_{\omega;c_1,\ldots,c_k}$ be the sequence whose i_l-th term is $a_{i_l} + c_l$ for $1 \le l \le k$ and others are corresponding terms in J.
If $\omega = (i)$ we briefly denote $J_{\omega;c}$ by $J_{i;c}$. Denote $\sigma(J)$ be the sum of all terms in J.

Let $U_h^{\le 0}$ be the topological subalgebra of $U_h(\mathfrak{g}_m)$ generated by $\{H_r, Y_t | 0 \le r, t \le m\}$. For $0 \le j \le m$ we define operators on $V[[h]]$ as follows:

$$\begin{aligned}
y_j(F_J Z_{J_1} E_{J_2}) &= F_{(j,J)} Z_{J_1} E_{J_2}, \\
h_j(F_J Z_{J_1} E_{J_2}) &= F_{I_1} Z_{(j,J_1)} E_{J_2} - 2\sum_{i=1}^{n} F_{J_{i;j}} Z_{J_1} E_{J_2},\ \forall J, J_1, J_2 \in \mathbb{J},
\end{aligned}$$

and denote $h_j = y_j = 0$ if $j > m$. One can easily show that $h_r h_t = h_t h_r$, $y_r y_t = y_t y_r$, $h_r y_t - y_t h_r = -2y_{r+t}$, hence $V[[h]]$ becomes a $U_h^{\le 0}$-mod via H_r acts as h_r and Y_t acts as y_t. Now define operators x_j on $V[[h]]$:

$$\begin{aligned}
x_j(F_J Z_{J_1} E_{J_2}) =& \sum_{i=1}^{n} F_{a_1}\cdots F_{a_{i-1}}\left(\frac{P^X_{m-j-a_i} - P^Y_{m-j-a_i}}{e^h - e^{-h}}.(F_{a_{i+1}}\cdots F_{a_n} Z_{J_1} E_{J_2})\right) \\
&+ F_J \sum_{J_1' \subseteq J_1} (-2)^{|J_1'|} Z_{J_1'} E_{(j+\sigma(J)-\sigma(J_1'),J_2)},\ \forall J = (a_1, \ldots, a_n), J_1, J_2 \in \mathbb{J}
\end{aligned}$$

for $0 \le j \le m$. Also denote $x_j = 0$ if $j > 0$. We claim that $V[[h]]$ is a $U_h(\mathfrak{g}_m)$-mod via H_r, X_s, Y_t act as h_r, x_s, y_t respectively. Since $[X_s, Y_t] \in U_h^{\le 0}$ we have

$$\begin{aligned}
&\sum_{i=1}^{n} F_{a_1} \cdots F_{a_{i-1}} \left(\frac{P^X_{m-j-a_i} - P^Y_{m-j-a_i}}{e^h - e^{-h}} (F_{a_{i+1}} \cdots F_{a_n} Z_{J_1} E_{J_2}) \right) \\
=&\left(\sum_{i=1}^{n} Y_{a_1} \cdots Y_{a_{i-1}} [X_j, Y_{a_i}] Y_{a_{i+1}} \cdots Y_{a_t} \right) .(Z_{J_1} E_{J_2}) \\
=&[X_j, Y_J].(Z_{J_1} E_{J_2}) \\
=&[X_j, Y_{J^\tau}].(Z_{J_1} E_{J_2}) \\
=&\sum_{i=1}^{n} F_{a_{\tau(1)}} \cdots F_{a_{\tau(i-1)}} \left(\frac{P^X_{m-j-a_{\tau(i)}} - P^Y_{m-j-a_{\tau(i)}}}{e^h - e^{-h}} (F_{a_{\tau(i+1)}} \cdots F_{a_{\tau(n)}} Z_{J_1} E_{J_2}) \right),
\end{aligned}$$

where $J = (a_1, \ldots, a_n) \in \mathbb{J}$, τ is an arbitrary n-element permutation and $J^\tau = (a_{\tau(1)}, \ldots, a_{\tau(n)})$. Therefore

$$x_j(F_{(t,J)} Z_{J_1} E_{J_2}) = \frac{P^X_{m-j-t} - P^Y_{m-j-t}}{e^h - e^{-h}} (F_J Z_{J_1} E_{J_2}) + F_a x_j (F_J Z_{J_1} E_{J_2}) \tag{4.4.1}$$

for arbitrary $0 \le t \le m$ and $J, J_1, J_2 \in \mathbb{J}$. In other words, $X_s Y_t - Y_t X_s = \frac{P^X_{m-j-t} - P^Y_{m-j-t}}{e^h - e^{-h}}$ as operators on $V[[h]]$. Moreover, using equation (4.4.1) one can verify that $h_r x_s - x_s h_r = 2x_{r+s}$ and $x_s x_t = x_t x_s$ by an induction on $|J|$. Hence the claim holds.

Finally, since $Y_J H_{J_1} X_{J_2}(1) = F_J Z_{J_1} E_{J_2}$ are linearly independent over $\Bbbk[[h]]$, $Y_J H_{J_1} X_{J_2}$ are also linearly independent. It is clear $U_h(\mathfrak{g}_m)$ is generated by $\{F_J Z_{J_1} E_{J_2} | J, J_1, J_2 \in \mathbb{J}\}$ as a $\Bbbk[[h]]$-mod, hence $\{F_J Z_{J_1} E_{J_2} | J, J_1, J_2 \in \mathbb{J}\}$ is a PBW basis of $U_h(\mathfrak{g}_m)$. Therefore

$$U_h(\mathfrak{g}_m) \cong V[[h]] \cong U(\mathfrak{g}_m)[[h]]$$

as $\Bbbk[[h]]$-mods since $U(\mathfrak{g}_m) \cong V$ as $\Bbbk$-vector spaces. One can even say $U_h(\mathfrak{g}_m) = U(\mathfrak{g}_m)[[h]]$. Therefore $U_h(\mathfrak{g}_m)$ is a topologically free module.

Since

$$P^X_i \equiv \delta_{0,i} + hH_{m-i} \pmod{h^2}, \quad P^Y_i \equiv \delta_{0,i} - hH_{m-i} \pmod{h^2}$$

for $0 \le i \le m$, we have

$$\frac{\Delta_h(X_s) - \Delta_h^{op}(X_s)}{h} \equiv \sum_{i+j=m+s} X_i \wedge H_j \pmod{h}, \quad \frac{\Delta_h(Y_t) - \Delta_h^{op}(Y_t)}{h} \equiv \sum_{i+j=m+t} Y_i \wedge H_j \pmod{h}.$$

Together with Sections 4.2 and 4.3, we have shown $U_h(\mathfrak{g}_m)$ is a quantization of $(\mathfrak{g}_m, \delta_{\mathbf{r}_0})$.

4.5 $U_h(\mathfrak{g}_m)$ is trivial as an algebra deformation $\pmod{h^4}$

Recall $A = \Bbbk\{H_r, X_s, Y_t \mid 0 \le r, s, t \le m\}$, let I' be the two sided ideal of $A[[h]]$ generated by

$$[H_i, H_j], \quad [X_i, X_j], \quad [Y_i, Y_j],$$

$$[H_r, X_s] - 2X_{r+s}, \quad [H_r, Y_t] + 2Y_{r+t}, \quad [X_s, Y_t] - \frac{P^X_{m-s-t} - P^Y_{m-s-t}}{2h}.$$

Define $B := A[[h]]/\overline{I'}$. Then $U_h(\mathfrak{g}_m) \cong B$ as $\Bbbk[[h]]$-algebras by the following isomorphism from $U_h(\mathfrak{g}_m)$ to B:

$$H_r \mapsto H_r, \quad X_s \mapsto \sqrt{\frac{2h}{e^h - e^{-h}}} X_s, \quad Y_t \mapsto \sqrt{\frac{2h}{e^h - e^{-h}}} Y_t.$$

Therefore B is also an algebra deformation of $U(\mathfrak{g}_m)$, and we can write the multiplication μ_B of B by

$$\mu_B = \sum_{n\geq 0} \mu_n h^n,$$

where each μ_n is a $\Bbbk$-bilinear map from $U(\mathfrak{g}_m)\times U(\mathfrak{g}_m)$ to $U(\mathfrak{g}_m)$. The key result proved later in this section is that for any $n\geq 1$, μ_n is a 2-coboundary on $U(\mathfrak{g}_m)$ with coefficients in $U(\mathfrak{g}_m)$, where $U(\mathfrak{g}_m)$ is a $U(\mathfrak{g}_m)$-bimodule in the natural way.

Lemma 4.5.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{g}_m = \mathfrak{g}[\mathbf{t}]/(\mathbf{t}^{m+1})$. Let $\{e_k\}$ be a basis of $\mathfrak{g}$, take the dual basis $\{e^k\}$ of $\{e_k\}$ respect to the Killing form of $\mathfrak{g}$. Then $\sum_k e_k e^k$ is the Casimir element of $\mathfrak{g}$. Let M be a finite dimensional $\mathfrak{g}_m$-mod and f be an n-cocycle ($n>0$) on $\mathfrak{g}_m$ with coefficients in M. Suppose N is a $\mathfrak{g}$-submodule of M such that all irreducible direct summands of N are non-trivial and isomorphic to each other, f satisfies* $\operatorname{Im} f\subseteq N$ *and*

$$e_k.f(e^k\mathbf{t}^r, a_1,\ldots,a_{n-1}) = e_k\mathbf{t}^r.f(e^k,a_1,\ldots,a_{n-1}),\quad \forall a_1,\ldots,a_{n-1}\in\mathfrak{g}_m, 0\leq r\leq m.$$

Then f is an n-coboundary.

Proof. For any $x\in\mathfrak{g}$, denote

$$[e_k,x]=\sum_k \alpha_{k,l}(x)e_l,\quad [e^k,x]=\sum_k \beta_{k,l}(x)e^l$$

in $\mathfrak{g}$. Then for any $0\leq r,s\leq m$,

$$[e_k\mathbf{t}^r, x\mathbf{t}^s]=\sum_k \alpha_{k,l}(x)e_l\mathbf{t}^{r+s},\quad [e^k\mathbf{t}^r, x\mathbf{t}^s]=\sum_k \beta_{k,l}(x)e^l\mathbf{t}^{r+s}$$

in $\mathfrak{g}_m$. Since Killing form is invariant, we have $\alpha_{k,l}=-\beta_{l,k}$ for any k,l. For any p-cochain ($p>0$) g on $\mathfrak{g}_m$ with coefficients in M, We define an $(p-1)$-cochain πg as follows:

$$(\pi g)(a_1,\ldots,a_{p-1}) := \sum_k e_k.f(e^k,a_1,\ldots,a_{p-1}),\quad \forall a_1,\ldots,a_{p-1}\in\mathfrak{g}_m.$$

Then

$$\begin{aligned}
&(d_{\mathfrak{g}_m}\pi g)(a_1,\ldots,a_p)\\
=&\sum_{i=1}^{p}(-1)^{i+1}a_i.(\pi g)(a_1,\ldots,\widehat{a_i},\ldots,a_p)+\sum_{1\leq i<j\leq p}(-1)^{i+j}(\pi g)([a_i,a_j],a_1,\ldots,\widehat{a_i},\ldots,\widehat{a_j},\ldots,a_p)\\
=&\sum_k\Bigg(\sum_{i=1}^{p}(-1)^{i+1}a_i.(e_k.g(e^k,a_1,\ldots,\widehat{a_i},\ldots,a_p))\\
&\qquad+\sum_{1\leq i<j\leq p}(-1)^{i+j}e_k.g(e^k,[a_i,a_j],a_1,\ldots,\widehat{a_i},\ldots,\widehat{a_j},\ldots,a_p)\Bigg),
\end{aligned}$$

and

$$\begin{aligned}
&(\pi d_{\mathfrak{g}_m} g)(a_1,\ldots,a_p)\\
=&\sum_k e_k.(d_{\mathfrak{g}_m}g)(e^k,a_1,\ldots,a_p)\\
=&\sum_k e_k.\Bigg(e^k.g(a_1,\ldots,a_p)+\sum_{i=1}^{p}(-1)^i g([e^k,a_i],a_1,\ldots,\widehat{a_i},\ldots,a_p)\\
&+\sum_{i=1}^{p}(-1)^i a_i.g(e^k,a_1,\ldots,\widehat{a_i},\ldots,a_p)+\sum_{1\leq i<j\leq p}(-1)^{i+j}g([a_i,a_j],e^k,a_1,\ldots,\widehat{a_i},\ldots,\widehat{a_j},\ldots,a_p)\Bigg).
\end{aligned}$$

Therefore

$$\begin{aligned}&(d_{\mathfrak{g}_m}\pi g+\pi d_{\mathfrak{g}_m}g)(a_1,\dots,a_p)\\ =&\left(\sum_k e_ke^k\right).g(a_1,\dots,a_p)\\ &+\sum_{i=1}^p(-1)^i\sum_k\Big([e_k,a_i].g(e^k,a_1,\dots,\widehat{a_i},\dots,a_p)+e_k.g([e^k,a_i],a_1,\dots,\widehat{a_i},\dots,a_p)\Big).\end{aligned}$$

For any $x\in\mathfrak{g},x_1,\dots,x_{n-1}\in\mathfrak{g}_m$ and $0\le r\le m$, by conditions we have

$$\begin{aligned}&\sum_k\Big([e_k,x\mathbf{t}^r].f(e^k,x_1,\dots,x_{n-1})+e_k.f([e^k,x\mathbf{t}^r],x_1,\dots,x_{n-1})\Big)\\ =&\sum_{k,l}\Big(\alpha_{k,l}(x)e_l\mathbf{t}^r.f(e^k,x_1,\dots,x_{n-1})+\beta_{k,l}(x)e_k.f(e^l\mathbf{t}^r,x_1,\dots,x_{n-1})\Big)\\ =&\sum_{k,l}(\alpha_{k,l}(x)+\beta_{l,k}(x))e_l\mathbf{t}^r.f(e^k,x_1,\dots,x_{n-1})\\ =&0.\end{aligned}$$

Due to the decomposition of N as a $\mathfrak{g}$-mod, the Casimir element $\sum_k e_ke^k$ acts on N by a non-zero scalar c. Then for any $a_1,\dots,a_n\in\mathfrak{g}_m$,

$$(d_{\mathfrak{g}_m}\pi f+\pi d_{\mathfrak{g}_m}f)(a_1,\dots,a_n)=cf(a_1,\dots,a_n)$$

since $\operatorname{Im}f\subseteq N$. Therefore $f=c^{-1}(d_{\mathfrak{g}_m}\pi f)$ is an n-coboundary. □

Let n be a positive integer. Define an antisymmetric bilinear map $F_n:\mathfrak{g}_m\times\mathfrak{g}_m\to S^n\mathfrak{g}_m$ as follows:

$$F_n(H_s,H_t)=F_n(X_s,X_t)=F_n(Y_s,Y_t)=F_n(H_r,X_s)=F_n(H_r,Y_t)=0,$$
$$F_n(X_s,Y_t)=\sum_{\substack{a_1+\dots+a_n\\=(n-1)m+s+t}}H_{a_1}\otimes\dots\otimes H_{a_n}$$

for any $0\le r,s,t\le m$.

Lemma 4.5.2. *Let $n\ge1$ be an odd integer. Then the antisymmetric map F_n defined above is a 2-coboundary on $\mathfrak{g}_m$ with coefficients in $S^n\mathfrak{g}_m$.*

Proof. We claim that for any $a\in\mathfrak{g}_m$ and $0\le r\le m$,

$$H_0.F_n(H_r,a)=H_r.F_n(H_0,a),\quad X_0.F_n(Y_r,a)=X_r.F_n(Y_0,a),\quad Y_0.F_n(X_r,a)=Y_r.F_n(X_0,a).$$

The first equality is easy since both sides are always zero by definition of F_n. The proof of the last two equalities are similar, we only prove the third one. For convenience, denote $v_{0,r}=-X_r$, $v_{1,r}=H_r$ and $v_{2,r}=Y_r$. Note that if $a_1+\dots+a_n=(n-1)m+k$ and $0\le a_i\le m$, then $a_i\ge k$. Hence a direct computation shows that for any $0\le s\le m$,

$$\begin{aligned}Y_0.F_n(X_r,Y_s)=&Y_0.\sum_{\substack{a_1+\dots+a_n\\=(n-1)m+r+s}}v_{1,a_1}\otimes\dots\otimes v_{1,a_n}\\ =&2\sum_{i=1}^n\sum_{\substack{a_1+\dots+a_n\\=(n-1)m+r+s}}v_{1,a_1}\otimes\dots\otimes v_{2,a_i}\otimes\dots\otimes v_{1,a_n}\\ =&2\sum_{i=1}^n\sum_{\substack{a_1+\dots+a_n\\=(n-1)m+s}}v_{1,a_1}\otimes\dots\otimes v_{2,a_i+r}\otimes\dots\otimes v_{1,a_n}\\ =&Y_r.F_n(X_0,Y_s).\end{aligned}$$

Therefore the claim holds and a routine check illustrates that F_n is a 2-cocycle.

Note that the $\mathfrak{sl}_2$-submodule of $S^n\mathfrak{g}_m$ generated by $\operatorname{Im} F$ is exactly M defined in Proposition 3.1.4. Recall $M = \oplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} M^i$ and as a $\mathfrak{sl}_2$-mod, each direct summand of M^i is isomorphic to the $(2n-4i+1)$-dimensional irreducible $\mathfrak{sl}_2$-mod. Thus the projection $p_i : S^n\mathfrak{g}_m \to M^i$ is a $\mathfrak{g}_m$-mod map. Denote $G_i = p_i \circ F_i$, then each G_i is a 2-cocycle with coefficients in M^i, such that

$$H_0.G_i(H_r, a) = H_r.G_i(H_0, a), \quad X_0.G_i(Y_r, a) = X_r.G_i(Y_0, a), \quad Y_0.G_i(X_r, a) = Y_r.G_i(X_0, a)$$

for any $a \in \mathfrak{g}_m$ and $0 \le r \le m$. Since

$$\frac{1}{4}X_0Y_0 + \frac{1}{8}H_0H_0 + \frac{1}{4}Y_0X_0$$

is the Casimir element of $\mathfrak{sl}_2$ and n is odd, Lemma 4.5.1 implies that each G_i is a 2-coboundary. Finally,

$$F_n = \sum_{i=0}^{\frac{n-1}{2}} p_i \circ G_i$$

is a 2-coboundary on $\mathfrak{g}_m$ with coefficients in $S^n\mathfrak{g}_m$. ☐

Since $\mathrm{HH}^i(U(\mathfrak{g}_m), U(\mathfrak{g}_m)) \cong \mathrm{H}^i(\mathfrak{g}_m, U(\mathfrak{g}_m)^{\mathrm{ad}})$, we only need to show that each μ_n^σ is a 2-coboundary on $\mathfrak{g}_m$ with coefficients in $U(\mathfrak{g}_m)^{\mathrm{ad}}$. Let $S(\mathfrak{g}_m)$ be the symmetric algebra of $\mathfrak{g}_m$ and we equip $S(\mathfrak{g}_m)$ with the $\mathfrak{g}$-mod structure given by:

$$x.(x_1 \cdots x_n) := \sum_{i=1}^{n} x_1 \cdots x_{i-1}[x, x_i]x_{i+1} \cdots x_n.$$

By [10, Chapter XVIII, Section 3], there is a $\mathfrak{g}_m$-mod isomorphism

$$\eta : S(\mathfrak{g}_m) \to U(\mathfrak{g}_m)^{\mathrm{ad}}, \quad x_1 \cdots x_n \mapsto \frac{1}{n!}\sum_{\tau \in S_n} x_{\tau(1)} \cdots x_{\tau(n)}, \quad \forall x_i \in \mathfrak{g}_m.$$

Moreover, the quotient map $\pi : T(\mathfrak{g}_m) \to S(\mathfrak{g}_m)$ restricted to $\oplus_{n\ge0}S^n\mathfrak{g}_m$ gives a $\mathfrak{g}_m$-mod isomorphism

$$\pi : \oplus_{n\ge0}S^n\mathfrak{g}_m \to S(\mathfrak{g}_m), \quad \frac{1}{n!}\sum_{\tau \in S_n} x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)} \mapsto x_1 \cdots x_n, \forall x_i \in \mathfrak{g}_m.$$

Let $\iota_j : S^j\mathfrak{g}_m \to \oplus_{n\ge0}S^n\mathfrak{g}_m$ be the inclusion. Since $(\eta\circ\pi)^{-1}\circ\mu_n^\sigma = \iota_n\circ F_n$, each μ_n^σ is a 2-coboundary by Lemma 4.5.2, hence also μ_n. Finally,

Proposition 4.5.3. *$U_h(\mathfrak{g}_m)$ is trivial as an algebra deformation* (mod h^4).

Proof. Obviously $\mu_n = 0$ if n is odd by definition of P_i^X and P_i^Y. Since μ_2 is a 2-coboundary, there exists an 1-cochain $\alpha_2 : U(\mathfrak{g}_m) \to U(\mathfrak{g}_m)$ such that $\mu_2 = d^1_{U(\mathfrak{g}_m)}(\alpha_2)$. Since

$$\left(\sum_{n\ge0}(-1)^n\alpha_2^n h^{2n}\right) \circ (\mu_0 + \mu_2h^2) \circ ((1+\alpha h^2) \otimes (1+\alpha_2h^2)) \equiv \mu_0 \ (\mathrm{mod}\ h^4),$$

B is trivial as an algebra deformation (mod h^4). The algebra isomorphism $U(\mathfrak{g}_m) \cong B$ completes the proof. ☐

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