§7.12 The description of a completion and its central extension for an infinite rank affine algebra

· Let $A=(a_{ij})_{i,j\in\mathbb{Z}}$ be an infinite generalized Cartan matrix, s.t. every row (and hence column) contains only a finite number of nonzero entries.

Let $\mathfrak{g}'(A)=\bigoplus_\alpha \mathfrak{g}'_\alpha$ be the associated Kac-Moody algebra.

Denoted by $\bar{\mathfrak{g}}(A)=\{u=\sum_\alpha a_\alpha e_\alpha \mid a_\alpha\in\mathbb{C}, e_\alpha\in\mathfrak{g}'_\alpha \text{ s.t. } \#\{j\mid j=ht\,\alpha, a_\alpha\neq 0\}<\infty\}$

$\bar{\mathfrak{g}}(A)$ is the subspace of $\underline{\prod_\alpha \mathfrak{g}'_\alpha} \supsetneq \bigoplus_\alpha \mathfrak{g}'_\alpha$ P9 单根 直和的完备化是直积，
极限在里面.

不同高度有限多个，同一高度无限多个.

· we can extend the bracket from $\mathfrak{g}'(A)$ to $\bar{\mathfrak{g}}(A)$ by linearity.

· The Lie algebra $\bar{\mathfrak{g}}(A)$ contains $\bar{\mathfrak{H}}:=\prod_{i\in\mathbb{Z}}\mathbb{C}\alpha_i^\vee$, the completed Cartan subalgebra

· Denoted by $\overline{gl_\infty}$ the Lie algebra of all complex matrices $(a_{ij})_{i,j\in\mathbb{Z}}$, s.t. $a_{ij}=0$ for $\underline{|i-j|>>0}$, with the usual braket.
根的高度.

· If A is an infinite affine matrix of $X_\infty=A_\infty$ (resp B_∞, C_∞, or D_∞)
we denote $\bar{\mathfrak{g}}(A)$ by $\bar{x}_\infty$

· Clearly $\bar{x}_\infty\cong\overline{sl_\infty}$ (resp. the subalgebra of $\overline{sl_\infty}$, which consists of matrices perserving the the bilinear form B, C, D).

· Claim: A completed infinite rank affine algebra, denoted by x_∞ is the central extension of $\bar{x}_\infty$.

(1) The Lie algebra $\bar{a}_\infty$ has a 2-cocycle φ defined by:

① $\varphi(E_{ij},E_{ji})=1=-\varphi(E_{ji},E_{ij})$, if $i\le 0$ and $j\ge 1$. ✓

② $\varphi(E_{ij},E_{mn})=0$ otherwise. ○

Ex 7.17: Let $L_n=\sum_{i=1}^{n}\mathbb{Z}v_i$ be the standard lattice in the Eucildean space $\mathbb{R}^n$
Define on $L_n\times L_n$ a bimultiplicative function ε by letting

$$\varepsilon(v_j,v_k)\begin{cases}=1 & \text{if } j\le k\\ =-1 & \text{if } j>k.\end{cases}$$

Show that by restricting to root lattices $Q(A_\ell)\subset L_{\ell+1}$ and $Q(D_\ell)\subset L_\ell$ we obtain an asymmetry function.

· verify: (Co1): $\varphi(a,b)=-\varphi(b,a)$.

(Co2): $\varphi([a,b],c)+\varphi([b,c],a)+\varphi([c,a],b)=0$, $(a,b,c\in\bar{a}_\infty)$.

· if more than two of a,b,c in ② condition, then ✓.

· if $a=E_{ij}$, $b=E_{ji}$, $i\le 0$ and $j\ge1$, $c=E_{mn}$.

then $\underbrace{\varphi([E_{ij},E_{ji}],E_{m,n})}+\underbrace{\varphi([E_{ji},E_{m,n}],a)}_{0}+\underbrace{\varphi([E_{m,n}E_{ij}],b)}_{0}$.

$=\varphi(E_{ii}-E_{jj},E_{m,n})=0$.

(2) If $A=A_\infty$ (resp B_∞, C_∞ or D_∞) put $r=1$ (resp $\frac{1}{2}$, 1, or $\frac{1}{2}$). and
Let $x_\infty=\bar{x}_\infty\oplus\mathbb{C}K$ be the Lie algebra with the following bracket:

$[a\oplus\lambda K, b\oplus\mu K]=(ab-ba)\oplus r\varphi(a,b)K$, $(a,b\in\bar{x}_\infty; \lambda,\mu\in\mathbb{C})$.

$(K=\sum_{i\in\mathbb{Z}}a_i^\vee\alpha_i^\vee)$.

· The elements $e_i, f_i \in g'(X_\infty) \subset X_\infty$ are called Chevalley generators of X_∞, and $\tilde{\mathcal{H}} = \bar{\mathcal{H}} + \mathbb{C}K$ is called the Cartan subalgebra, the element of the set:

$$\tilde{\Pi}^\vee = \{ \tilde{\alpha}_0^\vee = \alpha_0^\vee + K,\ \tilde{\alpha}_i^\vee = \alpha_i^\vee \text{ for } i \neq 0 \}$$ are called simple coroots.

we have the usual relations:

$(X_\infty, \tilde{\mathcal{H}}, \tilde{\Pi}, \tilde{\Pi}^\vee)$ is the quadruple associated to matrix A.

$-n < i, j < n$ 算.

$g'(A)$ 2 cocycle trivial. 0. boundary. $g'(A)$ 算 单李代数 还不存在非平凡的 2cocycle.

$\bar{g}(A)$ 2 cocycle.

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i} \qquad e_i, f_i \text{ 生成 } g'(A)$$

· we have the usual relations:

① $[\tilde{\alpha}_i^\vee, \tilde{\alpha}_j^\vee] = 0$, ② $[e_i, f_j] = \delta_{ij} \tilde{\alpha}_i^\vee$, ③ $[\tilde{\alpha}_i^\vee, e_j] = a_{ij} e_j$.

④ $[\tilde{\alpha}_i, f_j] = -a_{ij} f_j$. $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$, $(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$.

verify: ① $[\tilde{\alpha}_0^\vee, \tilde{\alpha}_j^\vee] = [\alpha_0^\vee + K, \alpha_j^\vee] = [\alpha_0^\vee, \alpha_j^\vee] + 2\psi(\alpha_0^\vee, \alpha_j^\vee)K$
$= [E_{00} - E_{11}, E_{jj} - E_{j+1,j+1}] + 0 = 0.$

② $[e_0, f_0] = [E_{01}, E_{10}] = E_{00} - E_{11} = \alpha_0^\vee$.

③ $[\alpha_0^\vee + K, e_j] = [\alpha_0^\vee, e_j] + 2\psi(\alpha_0^\vee, e_j)K$
$= (\varepsilon_j - \varepsilon_{j+1})(\alpha_0^\vee) e_j + 2\psi(\alpha_0^\vee, e_j)K$
$= (\varepsilon_j - \varepsilon_{j+1})(E_{00} - E_{11}) e_j + 2\psi(E_{00} - E_{11}, E_{j,j+1})K$
$= a_{0j} e_j$.

Remark 7.12:

1. The chevalley generators $\{e_i, f_i\}_{i \in \mathbb{Z}}$ generate a subalgebra $(\bigoplus_{i \in \mathbb{Z}} \mathbb{C}\tilde{\alpha}_i^\vee) \oplus (\bigoplus_{\alpha \in \Delta} g_\alpha)$ of X_∞, which is isomorphic to $g'(A)$.

$g'(A) \cong n_- \oplus H' \oplus n_+$, where $H' = \sum_{i \in \mathbb{Z}} \mathbb{C}\tilde{\alpha}_i^\vee$.

2. The pricipal gradation of $g'(A)$ extends in a natural way to a gradation of X_∞, called its pricipal gradation.

Recall $g_j(1) = \bigoplus_{\alpha:\, \mathrm{ht}\,\alpha = j} g_\alpha$, Note $g_0(1) = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}\tilde{\alpha}_i^\vee = H'$.

$g_{-1}(1) = \sum_{i \in \mathbb{Z}} \mathbb{C}f_i$, $g_1(1) = \sum_{i \in \mathbb{Z}} \mathbb{C}e_i$.

so that $n_\pm = \bigoplus_{j \geq 1} g_{\pm j}(1)$.

3. The canonical central element are as follows:

a_∞: $K = \sum_{i \in \mathbb{Z}} \tilde{\alpha}_i^\vee$

b_∞: $K = \tilde{\alpha}_0^\vee + 2\sum_{i \geq 1} \tilde{\alpha}_i^\vee$

c_∞: $K = \sum_{i \geq 0} \tilde{\alpha}_i^\vee$

d_∞: $K = \tilde{\alpha}_0^\vee + \tilde{\alpha}_1^\vee + 2\sum_{i \geq 2} \tilde{\alpha}_i^\vee$

$$\underline{A^T\delta^\vee = \delta^{\vee T}A = 0,}$$

- a_∞ : $A = \begin{bmatrix} \ddots & & & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & \ddots \\ & & & \ddots & -1 \\ & & & -1 & 2 & \ddots \end{bmatrix}$ $\qquad \delta^\vee = (\cdots, 1, \cdots, 1, \cdots)^T$

- b_∞, $B = \begin{bmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & -1 & \\ & & -1 & 2 & \\ & & & -1 & \ddots \end{bmatrix}$ $\qquad \delta^\vee = (1, 2, \cdots\cdots)^T$

- c^∞ : $C = \begin{bmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ 0 & -1 & 2 & \ddots & \\ & & -1 & \ddots & \\ & & & & 2 & \ddots \end{bmatrix}$ $\qquad \delta^\vee = (1, 1, 1 \cdots\cdots)^T$

- d^∞ : $D = \begin{bmatrix} 2 & 0 & & & \\ 0 & 2 & -1 & & \\ -1 & -1 & 2 & -1 & \\ & & -1 & 2 & \\ & & & & \ddots \end{bmatrix}$ $\qquad \delta^\vee = (1, 1, 2, \cdots)^T$

$d_{\infty} = \langle \alpha_0^\vee, \alpha_0 \rangle$.

Chapter 8 : Twisted Affine Algebras and Finite order Automorphisms.

§8.1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebras and let σ be an automorphism of $\mathfrak{g}$ satisfying $\sigma^m = 1$ for a positive integer m. Then each eigenvalue of σ has the form ε^j, $j \in \mathbb{Z}/m\mathbb{Z}$, where $\varepsilon = \exp\frac{2\pi i}{m}$.

Since σ is diagonalizable, we have the decomposition:

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_j \qquad (8.1.1)$$

where $\mathfrak{g}_j$ is the eigenspace of σ for the eigenvalue ε^j.
i.e. $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \sigma(x) = \varepsilon^j x\}$.

- (8.1.1) is a $\mathbb{Z}/m\mathbb{Z}$-gradation of $\mathfrak{g}$.

$[x,y] \in [g_i, g_j]$, $\sigma[x,y] = [\sigma(x), \sigma(y)] = [\varepsilon^i x, \varepsilon^j y] = \varepsilon^{i+j}[x,y]$.
Then $[g_i, g_j] \subset g_{i+j}$.

· Conversely, if a $\mathbb{Z}/m\mathbb{Z}$ -gradation (8.11) is given, let $\sigma(x_j) = \varepsilon^j x_j$ where $x_j \in g_j$ & satisfies $\sigma^m = 1$, then $\sigma \in Aut(g)$.

· $\forall [x,y] \in [g_i, g_j]$, verify $\sigma[x,y] = [\sigma(x), \sigma(y)]$.
$\sigma[x,y] = \varepsilon^{i+j}[x,y] = [\varepsilon^i x, \varepsilon^j y]$.
· isomorphism :

· Let $\mathcal{H}_{\bar{0}}$ be a <u>maximal ad- diagonalizable</u> subalgebra of Lie algebra $g_{\bar{0}}$.

Lemm. 8.1.
a) Let $(\cdot|\cdot)$ be a nondegenerate invariant bilinear form on g, then $(g_i|g_j) = 0$ if $i+j \not\equiv 0 \pmod m$. and g_i and g_j are nondegenerately paired if $i+j \equiv 0 \pmod m$)

proof: · Given $x \in g_i$, $y \in g_j$, we have $(x|y) = (\sigma(x)|\sigma(y)) = \varepsilon^{i+j}(x|y)$.

then if $i+j \not\equiv 0 \pmod m$, $\varepsilon^{i+j} \neq 1$. $\Rightarrow (x|y) = 0 \Rightarrow (g_i|g_j) = 0$.

· if $i+j \equiv 0 \pmod m$, Assume $(\cdot|\cdot)|_{g_i+g_j}$ is degenerate.

Then $\exists\ x_i + y_j \neq 0$ where $x_i \in g_i$, $y_j \in g_j$. s.t. $\forall x_i' \in g_i$, $y_j' \in g_j$ we have $(x_i + y_j | x_i' + y_j') = 0$.

Then $\forall s = \sum s_j \in g$, $s_i \in g_i$, we have $(x_i + y_j | s) = 0$. Contradiction

b) The centralizer $\mathfrak{z}$ of $\mathcal{H}_{\bar{0}}$ in g is <u>a</u> Cartan subalgebra of g.
c) $g_{\bar{0}}$ is a reductive subalgebra of g.

proof: Note that $\mathfrak{z} = \mathcal{H} + \sum g_\alpha$. where $\mathcal{H}$ is a Cartan subalgebra of g. containing $\mathcal{H}_{\bar{0}}$. g_α are the root spaces with respect to $\mathcal{H}$, and $\alpha \in \Delta$ s.t. $\alpha|_{\mathcal{H}_{\bar{0}}} = 0$.

· $\forall\ h' + \sum f_\alpha \in \mathcal{H} + \sum g_\alpha$. $[h' + \sum f_\alpha, h] = [h', h] + \sum [f_\alpha, h]$
$= 0 + \sum \alpha(h) f_\alpha = 0 \Rightarrow \mathcal{H} + \sum g_\alpha \subset \mathfrak{z}$

· conversly. with respect to $\mathcal{H}$, g has root space decomposition.
$g = \mathcal{H} \oplus (\bigoplus_\beta g_\beta)$
then $\forall z \in \mathfrak{z}$, $z = h' + \sum g_\beta$.
since $[z, h] = 0 \Rightarrow z \in \mathcal{H} + \sum g_\alpha$.

$\mathfrak{z} = \sum_\alpha g_\alpha + [g_\alpha\ g_{\alpha}]$

· it follows that $\mathfrak{z} = \mathcal{H} + s$ where s is a σ-invariant semisimple subalgebra (i.e. $s =$ the derived subalgebra of $\mathfrak{z}$). ?
· ~~s is simple~~ $[g_\alpha\ g_\beta] \subset g_{\alpha+\beta}$ $\alpha+\beta|$

$\mathfrak{z} = \{x \in g \mid [x, h] = 0, \forall h \in \mathcal{H}_{\bar{0}}\}$.

$K(x,y) = tr(ad x\, ad y)$. $K(\alpha_i^\vee,$
$K(x, h) = 0 \Rightarrow x = 0.$

$K(\alpha_i^\vee, \alpha_i^\vee) = tr(ad\,\alpha_i^\vee\, ad\,\alpha_i^\vee) \neq 0.$

$\underline{g = [g\, g]}$

- $\mathfrak{s} \cap g_{\bar{0}} = 0.$

$\forall\, x_i \in \mathfrak{s} \cap g_{\bar{0}}.$ if $x_i \neq 0.$

$[x_i, h] = \langle \alpha_i\, h \rangle x_i = 0. \quad \forall\, h \in H_{\bar{0}}.$

$\Rightarrow \underline{x_i \in H_{\bar{0}}}.$ by $\underline{H_{\bar{0}} = C_{H_{\bar{0}}}(g_{\bar{0}})}$ Contradiction

$\sigma(x_i) = x_i = [a\ b] \quad a \in \mathfrak{z},\ b \in \mathfrak{z}.$

$\sigma[a\ b] = [\sigma(a)\ \sigma(b)] = [a\ b].$

- Thus (8.1.1) induces a $\mathbb{Z}/m\mathbb{Z}$ gradation $\mathfrak{s} = \bigoplus_j \mathfrak{s}_j$, s.t. $\mathfrak{s}_{\bar{0}} = \{0\}$.
  Numbering the elements of $\mathbb{Z}/m\mathbb{Z}$ by corresponding integers in the set $N_m = \{0, \cdots, m-1\}$. and defining $\mathfrak{s}_a = \mathfrak{s}_b$ if $b \in N_m$ and $a \equiv b \mod m$.
  - claim: $\mathfrak{s}_n = \mathfrak{s}_{-n} = 0.$

  we know $\mathfrak{s}_0 = 0$, let $n > 0$. & $x \in \mathfrak{s}_n$. Then $(ad\,x)^r \mathfrak{s}_i \subset \mathfrak{s}_{nr+i}$, $i, n \in N_m$.
  select a positive integer r, s.t. $\underline{n(r-1) < m-i \leq nr}$. $\Rightarrow nr+i < m+n$.
  Then $nr+i = m+t$ with $0 \leq t < n$.
  so by the inductive assumption $\mathfrak{s}_{nr+i} = \mathfrak{s}_t = 0.$
  Thus $ad\,x|_{\mathfrak{s}}$ is nilpotent
  Similarly $ad\,y|_{\mathfrak{s}}$ is nilpotent if $y \in \mathfrak{s}_{-n}$.

  But $[\mathfrak{s}_n, \mathfrak{s}_{-n}] \subset \mathfrak{s}_0 = 0.$ so $ad\,x$ and $ad\,y$ commute on $\mathfrak{s}$.
  by nilpotency, $tr_{\mathfrak{s}}(ad\,x\, ad\,y) = 0.$

  Since $\underline{n-n = 0}$, Then $tr_{\mathfrak{s}}(ad\,x\, ad\,y) = K(\underline{x}, \underline{y}) = 0.$
  by a) $\Rightarrow x = y = 0.$
  Thus $\mathfrak{s}_n = 0 \Rightarrow \mathfrak{s} = 0$
  $\mathfrak{z} = H$ is a Cartan subalgebra of g.

#