Theorem 8.3

Let g be a complex simple finite dimensional Lie alge of type $X_N = D_{\ell+1}, A_{2\ell-1}, E_6, D_4$ or $A_{2\ell}$ and let $r = 2, 2, 2, 3$ or 2 respectively. Let μ be a diagram automorphism of g of order r. Then the Lie algebra $\hat{\mathcal{L}}(g,\mu)$ is a (twisted) affine Kac-Moody algebra $g(A)$ associated to the affine matrix A of type $X_N^{(r)}$ from Table Aff r ($r = 2$ or 3), $\mathcal{H}$ is its Cartan subalgebra, Δ the root system, Π and $\Pi^\vee$ the root basis and the coroot basis, and $e_0, \cdots, e_\ell, f_0, \cdots, f_\ell$ the Chevalley generators.

In other words, $(\hat{\mathcal{L}}(g,\mu), \mathcal{H}, \Pi, \Pi^\vee)$ is the quadruple associated to A

$$\hat{\mathcal{L}}(g,\mu) \cong g(A)$$

· Let $\hat{\mathcal{L}}(g,\mu) := \hat{\mathcal{L}}(g,\mu,r)$ for short, set $\mathcal{H} = \mathcal{H}_{\bar{0}} + \mathbb{C}K' + \mathbb{C}d'$ and define $\delta \in \mathcal{H}^*$ by $\delta|_{\mathcal{H}_{\bar{0}} + \mathbb{C}K'} = 0$, $\langle \delta, d' \rangle = 1$, set $e_\varepsilon = t \otimes E_\varepsilon$, $f_\varepsilon = t^{-1} \otimes F_\varepsilon$,
$e_i = 1 \otimes E_i$, $f_i = 1 \otimes F_i$ $(i \in I)$.

· we describe the root system and the root space decomposition of $\hat{\mathcal{L}}(g,\mu)$ with respect to $\mathcal{H}$:

$$\Delta = \{ j\delta + \gamma,\ \text{where } j \in \mathbb{Z},\ \gamma \in \Delta_{\bar{s}},\ j \equiv s \bmod r,\ s = 0, \cdots, r-1 \} \cup \{ j\delta,\ \text{where } j \in \mathbb{Z}, j \neq 0 \}$$

$$\hat{\mathcal{L}}(g,\mu) = \mathcal{H} \oplus \Big(\underline{\bigoplus_{\alpha \in \Delta} \mathcal{L}(g,\mu)_\alpha} \Big)$$

where $\mathcal{L}(g,\mu)_{s\delta+\gamma} = t^s \otimes g_{\bar{s},\gamma}$, $\mathcal{L}(g,\mu)_{s\delta} = t^s \otimes g_{\bar{s},0}$

(The restriction of $(\cdot|\cdot)$ to $H_{\bar{0}} = H' \cap g_{\bar{0}}$ is nondegenerate, and hence defines an isomorphism $\nu: \mathcal{H}_{\bar{0}} \to \mathcal{H}_{\bar{0}}^*$, Let $\Delta_{\bar{s}}$ $(s = 0, \cdots, r-1)$ be the set of nonzero weights of $\mathcal{H}_{\bar{0}}$ on $g_{\bar{s}}$ and let $g_{\bar{s}} = \bigoplus_{\alpha \in \Delta_{\bar{s}} \cup \{0\}} g_{\bar{s},\alpha}$ the the space decomposition).

prop 8.3 implies that $(\alpha|\alpha) \neq 0$, $\dim g_{\bar{s},\alpha} = 1$ and $[g_{\bar{s},\alpha}, g_{-\bar{s},-\alpha}] = \mathbb{C}\nu^{-1}(\alpha)$.

proof: · Since $[h^{(0)}, E_\alpha^{(j)}] = (h^{(0)}|\alpha)' E_\alpha^{(j)}$, $h^{(0)} \in \mathcal{H}_{\bar{0}}$, $\alpha \in \Delta$.

$\Delta_{\bar{0}}, \Delta_{\bar{1}}, \Delta_{\bar{2}} \cdots \subset \Delta$, mult $\alpha = 1$.

then $\dim g_{\bar{s},\alpha} = 1$.

· Since $(\cdot|\cdot)_{\mathcal{H}_{\bar{0}}^*}$ is positive-definite

then $(\alpha|\alpha) \neq 0$, $\forall \alpha \in \Delta_{\bar{s}}$.

· $\forall x \in g_{\bar{s},\alpha}$, $y \in g_{-\bar{s},-\alpha}$, since $g = g_{\bar{0}} \oplus \cdots \oplus g_{\overline{r-1}}$ is a $\mathbb{Z}/r\mathbb{Z}$ gradation

then we have $[x,y] \in g_{\bar{0}}$, Actually $[x,y] \in \mathcal{H}_{\bar{0}}$.

by Thm 2.2. $[x,y] = (x|y)\nu^{-1}(\alpha)$, i.e. $[g_{\bar{s},\alpha}, g_{-\bar{s},-\alpha}] = \mathbb{C}\nu^{-1}(\alpha)$.

· we set: $\Pi = \{ \alpha_\varepsilon := \delta - \theta_0,\ \alpha_i\ (i \in I) \}$

$\Pi^\vee = \{ \alpha_\varepsilon^\vee := r a_0^{-1} K' + 1 \otimes H_\varepsilon,\ \alpha_i^\vee := 1 \otimes H_i\ (i \in I) \}$

· Using prop 8.3 we obtain that if g is of type X_N and r $(=2$ or $3)$ is that order of μ, then the matrix:

$$A = (\langle \alpha_j, \alpha_i^\vee \rangle)_{i,j=0}^{\ell} = (a_{ij})_{\ell+1 \times \ell+1}$$

is of type $X_N^{(r)}$ and the integers $a_0, \cdots, a_\ell$ are the labels at the diagram of this matrix in Tables Aff 2, 3.

· For $D_4^{(3)}$: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$ ① $\langle \alpha_0^\vee, \alpha_0 \rangle = \langle 3K' + 1 \otimes (-\theta^0 - \bar{\mu}(\theta^0) - \bar{\mu}^2(\theta^0)), \delta - \theta_0 \rangle$

$= \langle 1 \otimes \theta^0 + 1 \otimes \bar{\mu}(\theta^0) + \bar{\mu}^2(\theta^0), \theta_0 \rangle$

$= \langle \theta^0\ \theta^0 \rangle = 2.$

$\theta^0 = \alpha_1' + \alpha_2' + \alpha_3'$, $\bar{\mu}(\theta^0) = \alpha_3' + \alpha_2' + \alpha_4'$, $\bar{\mu}^2(\theta^0) = \alpha_4' + \alpha_2' + \alpha_1'$

$\theta_0 = \alpha_1 + 2\alpha_2 = \alpha_2' + \frac{2}{3}(\alpha_1' + \alpha_3' + \alpha_4')$

$\langle \theta^0, \theta_0 \rangle = \frac{2}{3} = \langle \bar{\mu}(\theta^0), \theta_0 \rangle = \langle \bar{\mu}^2(\theta^0), \theta_0 \rangle$.

② $\langle \alpha_0^\vee, \alpha_1 \rangle = \langle -(1 \otimes \theta^0 + 1 \otimes \bar{\mu}(\theta^0) + \bar{\mu}^2(\theta^0)), \frac{1}{3}(\alpha_1' + \alpha_3' + \alpha_4') \rangle = -1$

$\langle \alpha_1^\vee, \alpha_0 \rangle = -1$.

proof of Th 8.3: (based on prop 1.4 a)).

(1) H is a commutative subalgebra (i.e. $[H, H]=0$)

(2) $(H, \Pi, \Pi^\vee)$ is a realisation of X_N.

① both sets Π and $\Pi^\vee$ are linearly independent

② for case 1-4, if $i>0$ & $j>0$, $\alpha_i^\vee$ & α_i are both the simple coroots and roots of $g_{\bar{0}}$ which is of type (B_ℓ, C_ℓ, F_4, G_2) obtained from $X_N^{(r)}$ by removing the zero vertex.

Recall : $K = rK' = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee$, $\delta = \sum_{i=0}^{\ell} a_i \alpha_i$

Now : ① $\langle \alpha_0, \alpha_i^\vee \rangle = \langle \delta - \sum_{j=1}^{\ell} a_j \alpha_j, \alpha_i^\vee \rangle = -\sum_{j=1}^{\ell} a_j \langle \alpha_j, \alpha_i^\vee \rangle = -\sum_{j=1}^{\ell} a_{ij} a_j$

$(A\delta = 0)$ $= a_0 a_{i0} = a_{i0}$ $(i = 1, \cdots, \ell)$

$(\delta^T A^T = 0)$ ② $\langle \alpha_j, \alpha_0^\vee \rangle = \langle \alpha_j, rK' - \sum_{i=1}^{\ell} a_i^\vee \alpha_i^\vee \rangle = \underline{-\sum_{i=1}^{\ell} a_i^\vee a_{ij} = a_0^\vee a_{0j}} = a_{0j}$

③ $\langle \alpha_0^\vee, \alpha_0 \rangle = \langle rK' + 1 \otimes H_0, \delta - \theta_0 \rangle$
$= \langle rK' - 1 \otimes \theta^0 - 1 \otimes \bar{\mu}(\theta^0) + \cdots + 1 \otimes \bar{\mu}^{r-1}(\theta^0), \delta - \theta_0 \rangle$
$= \langle rK', -\theta_0 \rangle - \langle 1 \otimes \theta^0 + 1 \otimes \bar{\mu}(\theta^0) + \cdots + 1 \otimes \bar{\mu}^{r-1}(\theta^0), -\theta_0 \rangle$

(3) e_i, f_j, H satisfy the following relation.

$[e_i, f_j] = \delta_{ij} \alpha_i^\vee$, $i, j = 0, 1, \cdots, \ell$.
$[h, e_i] = \langle h, \alpha_i \rangle e_i$, $h \in H$, $i = 0, \cdots, \ell$.
$[h, f_i] = -\langle h, \alpha_i \rangle f_i$, $h \in H$, $i = 0, \cdots, \ell$.

(4) e_i, f_i, $(i = 0, \cdots, \ell)$, H generate $\hat{\mathcal{L}}(g, \mu)$.

(5). $\hat{\mathcal{L}}(g, \mu)$ has nonzero ideals which intersect H trivially completing the proof #

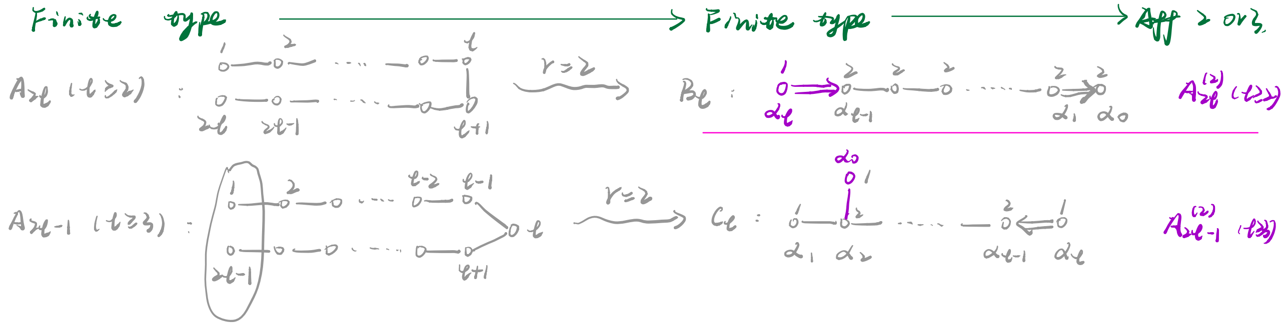
Summarize :

Let A be an affine matrix of type $X_N^{(r)}$, let g be a simple f.d. Lie algebra of type X_N and let μ be a diagram automorphism of g of order r ($=1, 2$ or 3) Then :

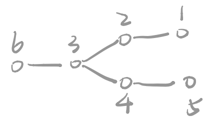
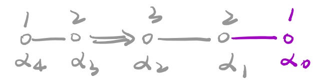
- $\hat{\mathcal{L}}(g, \mu) = \mathcal{L}(g, \mu) \oplus \mathbb{C}K' \oplus \mathbb{C}d' \cong g(A)$
- $\tilde{\mathcal{L}}(g, \mu) = \mathcal{L}(g, \mu) \oplus \mathbb{C}K' \cong g'(A)$
- $\mathcal{L}(g, \mu) \cong g'(A) / \mathbb{C}K'$

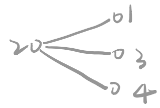
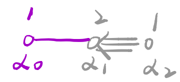
· Actually A_ℓ, D_ℓ, $E_{6,7,8}$ can produce other Finite and Aff type of 1,2,3 in some way.

Finite type ⟶ Finite type ⟶ Aff 2 or 3



D_{l+1} $(l \geq 2)$:  $\xrightarrow{r=2}$ B_l :  $D_{l+1}^{(2)}$ $(l \geq 2)$

E_6 :  $\xrightarrow{r=2}$ F_4 :  $E_6^{(2)}$

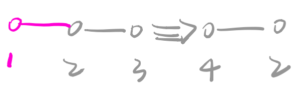
D_4 :  $\xrightarrow{r=3}$ G_2 :  $D_4^{(3)}$

Finite type $\longrightarrow$ Aff 1.

B_l : $(l \geq 3)$  $B_l^{(1)}$ Q: ${}^{T}B_l^{(1)} = A_{2l-1}^{(2)}$

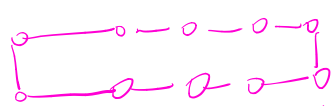
C_l : $(l \geq 2)$  $C_l^{(1)}$ $\longrightarrow$ ${}^{T}C_l^{(1)} = D_{l+1}^{(2)}$

G_2 :  $G_2^{(1)}$ $\longrightarrow$ ${}^{T}G_2^{(1)} = D_4^{(3)}$

F_4 :  $F_4^{(1)}$ $\longrightarrow$ ${}^{T}F_4^{(1)} = E_6^{(2)}$

A_l $(l \geq 2)$  $A_l^{(1)}$

仿射1型中图自同构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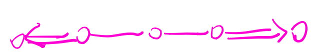
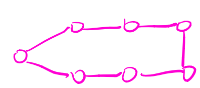


D_l $(l \geq 4)$:  $D_l^{(1)}$


长

Q：先作仿射再做图自同构，或先作图自同构再做仿射

Cor 8.3. Let $g(A)$ be an affine algebra of rank $\ell+1$ and let A be of type $X_N^{(r)}$, Then the multiplicity of the root $j r\delta$ is equal to ℓ, and the multiplicity of the root $s\delta$ for $s \not\equiv 0 \mod r$ is equal to $\frac{(N-\ell)}{(r-1)}$

proof: if $r=1$, then $\operatorname{mult} j\delta = \ell$ (for $j \neq 0$)
if $r=2$, or 3. by Th. 8.3. $\operatorname{mult} s\delta$ $(s \neq 0)$ is equal to the multiplicity of the eigenvalue $\exp(\frac{2\pi i s}{r})$ of μ operating on $\mathcal{H}'$, i.e. $\dim(\mathcal{H}_{\bar{s}})$.
$r=2$, $\dim(H_{\bar{1}}) = \dim(\mathcal{H}) - \dim(\mathcal{H}_{\bar{0}}) = (2N-N)-\ell = N-\ell$.
$r=3$. $X_N = D_4$, $\dim(\mathcal{H}_{\bar{s}}) = \frac{1}{2}(\dim(H) - \dim(\mathcal{H}_{\bar{0}}))$

$$= \frac{1}{2}(2\times 4 - 4 - 2) = 1 = \frac{4-2}{3-1}$$

$(N=4, \ell=2, r=3)$. #

· The normalized invariant form on $\hat{\mathcal{L}}(g,\mu)$ is given by.

$$(P\otimes x \mid Q\otimes y) = r^{-1}\operatorname{Res}(t^{-1}PQ)(x|y) \quad (x,y \in g,\ P,Q \in \mathcal{L})$$

$$(\mathbb{C}K' + \mathbb{C}d' \mid \mathcal{L}(g,\mu)) = 0, \quad (K'|K') = (d'|d') = 0, \quad (K'|d') = 1.$$

where $(\cdot|\cdot)$ is the normalized invariant form on g.
(i.e. $(\alpha|\alpha) = 2r$, if $\alpha \in \Delta_\ell$).

· It is also easy to see that $K = rK'$ is the canonical central element and that $d = \frac{a_0}{r}d'$ is the scaling element.
proof: $K = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee = r a_0^{-1} K' + 1\otimes H_\varepsilon + a_1^\vee \alpha_1^\vee + \cdots + a_\ell^\vee \alpha_\ell^\vee$

Case 1: $= 2K' + 1\otimes \underbrace{(H_0 + a_1^\vee H_1 + \cdots + a_\ell^\vee H_\ell)}_{=0}$. $(a_0 = 1,\ r=2)$
$= 2K'$
$= 2K'$

Case 5: $= a_0^\vee K' + 1\otimes \underbrace{(a_\ell^\vee H_\ell + a_{\ell-1}^\vee H_{\ell-1} + \cdots + a_0^\vee H_0)}_{=0}$ $(a_0 = 2,\ r=2)$.
$= 2K' = rK'$

· Recall $(K|d) = a_0$. $(K'|d') = 1$.
$\Rightarrow d = \frac{a_0}{r}d'$.

Warning: The Lie algebra $g_{\bar{0}} \cong \mathring{g}$ introduced in §6.3. in all cases expect $A_{2\ell}^{(2)}$, in latter case $\mathring{g}$ is of type C_ℓ whereas $g_{\bar{0}}$ is of type B_ℓ.

§8.4. Another application of realization theorems.

prop 8.4. Let $g(A)$ be an affine algebra
(a): set $t = \mathbb{C}K + \sum_{s\in\mathbb{Z},\, s\neq 0} g_{s\delta}$, Then t is isomorphism to the infinite-d Heisenberg algebra (= Heisenberg Lie algebra of order ∞). with center $\mathbb{C}K$.

(b): The Hermitian form $(x|y)_0 = -(w_0(x)|y)$ is positive semifinite on $g'(A)$ with kernel $\mathbb{C}K$.

Def: The Heisenberg algebra $\mathcal{A}$ is a complex Lie algebra spanned by $\{a_n : n \in \mathbb{Z}\} \cup \{h\}$ where h is central and

$$[a_m, a_n] = m\,\delta_{m,-n}\, h$$
$$[h, a_n] = 0 \quad (n \in \mathbb{Z}).$$

非退化双线性型

proof:

a). By the realization theorem, $\mathfrak{g}'(A)/\mathbb{C}K \cong \mathcal{L}(\mathfrak{g}, \mu)$.
The gradation of $\mathfrak{g}$ which corresponds to μ induces the gradation of Cartan subalgebra $\mathcal{H}'$ of $\mathfrak{g}$. $\mathcal{H}' = \bigoplus_{j \in \mathbb{Z}/m\mathbb{Z}} \mathcal{H}_j$

we obtain the following isomorphism:

$$\mathfrak{t}/\mathbb{C}K = \Big(\mathbb{C}K + \sum_{\substack{s\in\mathbb{Z}\\ s\neq 0}} \mathfrak{g}_{s\delta}\Big)/\,\mathbb{C}K \cong \bigoplus_{\substack{s\in\mathbb{Z}\\ s\neq 0}} \mathfrak{g}_{s\delta} = \bigoplus_{\substack{s\in\mathbb{Z}\\ s\neq 0}} t^s \otimes H'_{\bar{s}} \quad (\text{by } 8.3.5).$$

It follows that $\mathfrak{t}/\mathbb{C}K$ is a commutative subalgebra.
and we have the restriction of the cocycle ψ to this subalgebra is nondegenerate.

Recall: $\psi(a, b) = \mathrm{Res}\,\left(\frac{da}{dt} \middle| b\right)_t$, where $(P\otimes x \mid Q \otimes y)_t = \underline{r^{-1}} PQ\,(x|y)$.

if $t^s \otimes H'_{\bar{s}} \in \bigoplus_{\substack{s\in\mathbb{Z}\\ s\neq 0}} t^s \otimes H_{\bar{s}}$ and $\forall\, t^k \otimes H_{\bar{k}} \in \bigoplus_{\substack{s\in\mathbb{Z}\\ s\neq 0}} t^s \otimes H'_{\bar{s}}$

s.t. $\psi(t^s \otimes H'_{\bar{s}}, t^k \otimes H_{\bar{k}}) = 0$ $= \mathrm{Res}\,(s t^{-s} \otimes H'_{\bar{s}} \mid t^k \otimes H'_{\bar{k}})_t$

$= \mathrm{Res}\,(r^{-1} s\, t^{k-s})(H'_{\bar{s}} \mid H'_{\bar{k}})$.

since $(\cdot|\cdot)$ is nondegerate in $\mathcal{H}'$ $\Rightarrow$ $t^s \otimes H'_{\bar{s}} = 0$.

then we check: $[\mathbb{C}K, \mathfrak{g}_{s\delta}] = 0$ ✓

$[t^s \otimes H'_{\bar{s}}, t^k \otimes H'_{\bar{k}}] = \underbrace{m\,(H_{\bar{s}} \mid H_{\bar{k}})\,\delta_{s,-k}}\, K$

#.

- The subalgebra $\mathfrak{t}$ is called the Homogeneous Heisenberg subalgebra of the affine algebra $\mathfrak{g}(A)$.

§8.5

- Let $\mathfrak{g}$ be a simple f.d. Lie algebra, let m be a positive integer and let $\varepsilon = \exp\left(\frac{2\pi i}{m}\right)$, let $r(t) : \mathbb{C}^\times \longrightarrow \mathrm{Aut}\,\mathfrak{g}$ be regular map. 多项式 ⟶ 系是 $\mathbb{C}^\times$ 多项式函数.

保持连续

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多项式考虑定义拓扑结构

We may regard $r(t)$ as an element of $\mathrm{Aut}\,\mathcal{L}(\mathfrak{g})$ (by pointwise action of $r(t)$ on $a(t) \in \mathcal{L}(\mathfrak{g})$).

Recall: §7.2 P97.

The loop algebra $\mathcal{L}(\mathring{\mathfrak{g}}) := \mathcal{L} \otimes_{\mathbb{C}} \mathring{\mathfrak{g}}$ may be indentified with the Lie algebra of regular rational maps: $\mathbb{C}^\times \to \mathring{\mathfrak{g}}$.

$$\mathcal{L}(\mathring{\mathfrak{g}}) = \mathcal{L} \otimes_{\mathbb{C}} \mathring{\mathfrak{g}} \longleftrightarrow \mathbb{C}^\times \xrightarrow{f} \mathring{\mathfrak{g}}$$

$$t^j \otimes x_i \longleftrightarrow z \longmapsto z^j x_i$$

then $\gamma(t)$ can regarded as f. (where g can be considered as $\mathrm{Inn}(g)$ through $\theta_x : g \to g$, $y \longmapsto xyx^{-1}$).

then $\gamma(t) \in \mathrm{Inn}(\mathcal{L}(g)) \subset \mathrm{Aut}\,\mathcal{L}(g)$

where $t \in \mathbb{C}^\times$, i.e. $t^j \otimes x_i \longleftrightarrow t \xrightarrow{\gamma(t)} t^j x_i$.

$$\gamma(z)(t^j \otimes x_i) = (t^j \otimes y)(t^j \otimes x_i)(t^{-j} \otimes y^{-1})$$

$$= t^j \otimes y x_i y^{-1}$$

$$z^j t^j \otimes \nu(z)(x_i).$$