Twisted Affine algebra and Finite Order automorphism

$a(t) \in L(g)$ $\qquad x_i \in g_i$

§8.5 $\quad \gamma(t): \mathbb{C}^\times \to \mathrm{Aut}(g) \qquad z \mapsto (x_i \mapsto z^i x_i)$

$\gamma(t) \xrightarrow[\text{by pointwise action}]{\text{regard}} \in \mathrm{Aut}(L(g)) \qquad \gamma(t)(a(t)) = z^i t^i \otimes \gamma(t)(x_i)$

• We shall view $\sigma \in \mathrm{Aut}(g)$ as $\quad t^i \otimes z^i x_i$

an element of $\mathrm{Aut}(L(g))$ by lettin

$$\sigma(t^k \otimes a) = t^k \otimes \sigma(a) \qquad \mathrm{ad}(\ \ )\,\mathrm{ad}(\ \ )$$

• $(\sigma^m = 1)$ An equivalent definition of $\mathcal{L}(g,\sigma,m)$:

$$\mathcal{L}(g,\sigma,m) = \{ a(t) \in \mathcal{L}(g) \mid \sigma(a(\varepsilon^{-1}t)) = a(t) \}$$

pf "$\subseteq$" $\quad L(g,\sigma,m)_j \ni t^j \otimes x_j, \quad x_j \in g_j \qquad \varepsilon^j x_j = \varepsilon^{-j} t^j \otimes \sigma(x_j)$

$$\sigma(\varepsilon^{-j} t^j \otimes x_j) = t^j \otimes x_j$$

"$\supseteq$"

$a(t) \in \Delta \qquad a(t) = t^j \otimes x_j' \qquad \sigma(a(\varepsilon^{-1}t)) = \sigma(\varepsilon^{-j} t^j \otimes x_j') = t^j \otimes x_j'$

$\Rightarrow x_j' \in g_j$

Lemma 8.5. Let $\alpha, \beta \in \mathrm{Aut}(g)$ be such that $\alpha^m = \beta^m = 1$, $\gamma(t) \in \mathrm{Aut}(L(g))$. Then

$$\gamma(t)\, \mathcal{L}(g,\alpha,m) = L(g,\beta,m)$$

$$\Updownarrow$$

Kac's 1969 A Thm 1

(8.5.1) $\quad \beta(\gamma(\varepsilon^{-1}t)) = \gamma(t)\alpha \quad$ for all $t \in \mathbb{C}^\times$

pf $(\Downarrow)$

$$\begin{array}{ccc}
a(\varepsilon^{-1}t) & \overset{\alpha}{\longmapsto} & a(t) \\
L(g,\alpha,m) & \xrightarrow{\quad\quad} & L(g,\alpha,m) \\
\Big\downarrow \gamma(\varepsilon^{-1}t) & & \Big\downarrow \gamma(t) \\
L(g,\beta,m) & \xrightarrow{\ \beta\ } & L(g,\beta,m) \\
\gamma(\varepsilon^{-1}t)\, a(\varepsilon^{-1}t) & \xrightarrow{\ \beta\ } & \gamma(t)\, a(t) \in L(g,\beta,m) \\
\| & & \\
b(\varepsilon^{-1}t) & & b(t)
\end{array}$$

$a(t)$ $\quad \alpha \to \mathrm{Aut}(L(g))$

$\alpha(t^j \otimes a) = t^j \otimes \alpha(a)$

z^{-2i}

$L(g, \beta, m)$

P79. we regard $\gamma(t)$ as an element of $L(g)$

i.e $\gamma(t) \longleftrightarrow t^j \otimes x$, $x \in g_j$

(ア)

$\beta(\gamma(\varepsilon^{-1}t)a(\varepsilon^{-1}t)) = \gamma(t)a(t)$

$a(t) = t^i \otimes x_i$

$\gamma(z)a(t) = z^i t^i \otimes \gamma(z)(x_i) = z^i t^i \otimes z^i x_i$

$\gamma(\varepsilon^{-1}t)a(\varepsilon^{-1}t) = \varepsilon^{-i} z^i (\varepsilon^{-1}t)^i \otimes \gamma(\varepsilon^{-1}z) x_i$

$\| $

$\varepsilon^{-i} z^i (\varepsilon^{-1}t)^i \otimes \varepsilon^{-i} z^i x_i$

$\beta(\varepsilon^{-i} z^i (\varepsilon^{-1} z)^i$

If $\alpha, \beta \in Aut(g)$, and $\gamma : \mathbb{C}^\times \longrightarrow Aut(g)$ (a regular map) are such that (8.5.1) holds, we write $\alpha \xrightarrow{\gamma} \beta$, The following relation are immediate:

(8.5.2) $\alpha \xrightarrow{\gamma} \beta \Rightarrow \beta \xrightarrow{\gamma^{-1}} \alpha$

Pf: $\beta\gamma(\varepsilon^{-1}t) = \gamma(t)\alpha$ for all $t \in \mathbb{C}^\times$

$\Rightarrow \gamma^{-1}(t)\beta = \alpha\gamma^{-1}(\varepsilon^{-1}t)$ i.e $\beta \xrightarrow{\gamma^{-1}} \alpha$

(8.5.3) $\alpha \xrightarrow{\gamma} \beta \xrightarrow{\gamma_1} \beta_1 \Rightarrow \alpha \xrightarrow{\gamma_1\gamma} \beta_1$

$\beta\gamma(\varepsilon^{-1}t) = \gamma(t)\alpha$ $\quad$ $\beta_1\gamma_1(\varepsilon^{-1}t) = \gamma_1(t)\beta$

$\beta_1\gamma_1(\varepsilon^{-1}t) = \gamma_1(t)\beta$

$= \gamma_1(t)\gamma(t)\alpha\gamma^{-1}(\varepsilon^{-1}t)$

$\beta_1\gamma_1(\varepsilon^{-1}t)\gamma(\varepsilon^{-1}t) = \gamma_1(t)\gamma(t)\alpha$

$\Rightarrow \beta_1\gamma_1\gamma(\varepsilon^{-1}t) = \gamma_1\gamma(t)\alpha$

in particular

$(\alpha \xrightarrow{\gamma} \beta) \Rightarrow \alpha \xrightarrow{g\gamma} g\beta g^{-1}$ for $g \in Aut(g)$

Pf:

$$\begin{array}{ccccc} L(g) & \xrightarrow{\alpha} & L(g) & \leftarrow & L(g,\alpha,m) \\ g\gamma \downarrow & & \downarrow g\gamma & & \\ L(g) & \xrightarrow[g\beta g^{-1}]{} & L(g) & \leftarrow & L(g, g\beta g^{-1}, m) \end{array}$$

$g\beta g^{-1}\, g\gamma(\varepsilon^{-1}t) = g\beta\gamma(\varepsilon^{-1}t) = g\gamma(t)\alpha$

Prop 8.5. Let σ ~~be~~ $\in Aut(g)$, simple f-d Lie alge. of the form (8.12) $u\exp(\mathrm{ad}\frac{2\pi i}{m}h)$

$[\alpha_i(h) \in \mathbb{Z}$ $\quad u(h) = h]$ $\quad t \mapsto t^{\alpha(h)} \cdot g_\alpha$

$[\cdot, h$ $\quad$ $t^h : \mathbb{C}^\times \to Aut(g)$)

Denote by t^h the regular map $\mathbb{C}^\times \to \mathrm{Aut}(\mathfrak{g})$ $\quad \mathrm{Aut}(L(\mathfrak{g}))$

s.t t^h on $\mathfrak{g}_\alpha$ is an operator of

multiplication by $t^{\alpha(h)}$; Then $\quad t^{\alpha(h)} \otimes \mathfrak{g}_\alpha$

(8.5.5) $\quad t^r \otimes x_r \longrightarrow t^m \otimes x_m$

$$\longrightarrow t^h\,(L(\mathfrak{g}, u, m)) = L(\mathfrak{g}, \bar{\sigma}, m)$$

Pf: Lemma 8.5 + $t^h u = u t^h$ $\qquad \alpha(h) \sum a_i s_i$

$L(\mathfrak{g}, u, m) \xrightarrow{u} L(\mathfrak{g}, u, m)$ $\qquad \bar{\sigma} = u \exp(\mathrm{ad}(\varepsilon h))$

$(\varepsilon t)^h \downarrow \quad \circlearrowleft \quad \downarrow t^h \quad \Rightarrow \quad t^h u = \boxed{\bar{\sigma}\,(\varepsilon^{-1} t)^h = u t^h}$

$L(\mathfrak{g}, \bar{\sigma}, m) \xrightarrow{\bar{\sigma}} L(\mathfrak{g}, \sigma, m)$ $\qquad u \exp \mathrm{ad}(\varepsilon h)(\varepsilon^{-1} t)^h \stackrel{?}{=} u t^h$

$\sigma \rightsquigarrow u \exp(\mathrm{ad} \frac{2\pi i}{2m} h)$ $\qquad$ action $L(\mathfrak{g}, u, m)$

$x_j \in \mathfrak{g}_{j, \gamma}$

$a(t) = t^j \otimes x_j$ $\quad t^h \to$ $\quad t^{\alpha(h)+j} \otimes \mathfrak{g}_\alpha$

$\gamma + j\delta \longmapsto \gamma + (j + \alpha(h))\delta$

$\Rightarrow t^{\frac{m}{r} h}:$ $\boxed{L(\mathfrak{g}, u, r) \xrightarrow{t^r} L(\mathfrak{g}, \bar{\sigma}, m)}$

$\wr\!\wr \qquad \varphi \qquad \Phi: L(\mathfrak{g}, \bar{\sigma}, m) \xrightarrow{s} \mathfrak{g}(A)$

$~~t^r \to x_m~~$ $\qquad \mathfrak{g}'(A)/\mathbb{C}K$

(Kac's 1969A) $\quad t^r \otimes x_r \longrightarrow t^m \otimes x_m$

Thm 1: We have an isomorphism:

$$L(\mathfrak{g}, u, r) \longrightarrow L(\mathfrak{g}, \bar{\sigma}, m) \qquad t^h$$

where $s_i = \deg(\cdots)$

$\downarrow \deg(e_i) = s_i \dots$

Pf: Let $s_0, \dots, s_l$ be a sequence of integer not all 0, define a new $\mathbb{Z}$-gradation

$$L(g,u) = \bigoplus_{j \in \mathbb{Z}} L(g,u)_j$$

as follows:

writing a root $\alpha = \sum_{i=0}^{l} k_i \alpha_i$,

and $L(g,u)_j = \sum_{\deg \alpha = j} L(g,u)_\alpha \longrightarrow$ is gradation of type $(s_0 \dots s_l)$

denote $(\alpha_0, s_0), \dots, (\alpha_l, s_l) = \hat{\alpha}_l$

to prove that $\exists$ an automorphism u of g induced by an automorphism of Dynkin diagram and a $\mathbb{Z}$-gradation of $L(g,u)$ of type $(s_0, \dots, s_l)$ in which $L(g,u)$ is isomorphism to the $\mathbb{Z}$-gradation of Lie algebra $L(g,\sigma)$ by an isomorphism under which two $\mathbb{Z}$ gradation correspond.

By (remark ①) We can choose u s.t
Thm 8.3

If $\beta_0 \dots \beta_l$ of the simple roots of $\underline{L(g,\sigma)}$ and $\deg e_i = s_i = -\deg f_i$

(in ~~a sui~~ a suitable order), then under the bijection: $\alpha_l \dots \alpha_1$

τ: $\widetilde{\beta_i} \longrightarrow \widetilde{\alpha_i}$ $(0 \le i \le l)$

$\widetilde{\beta_i} = (\beta_i, s_i')$, $\widetilde{\alpha_i} = (\alpha_i, s_i)$

$\Rightarrow$ then we have

$$L(g,\sigma) = \bigoplus_{j\in\mathbb{Z}} L(g,\sigma)_j \qquad (*)$$

$$L(g,\sigma)_j = \bigoplus_{\substack{\beta \\ \deg \beta = j}} L(g,\sigma)_\beta$$

$$L(g,u) = \bigoplus_{j\in\mathbb{Z}} L(g,u)_j \qquad (**)$$

$$L(g,u)_j = \bigoplus_{\substack{\alpha \\ \deg \alpha = j}} (L(g,u))_\alpha \qquad \alpha = j\delta + \gamma$$

The iso. Ψ: $L(g,\sigma,m) \longrightarrow L(g,u,r)$

$\underline{t^{j+m} \otimes g_j \text{ mod } u}$ $\Psi(g_\beta) = g_{\tau(\beta)} = g_\alpha$ $\quad (L(g,\sigma,m))_{\beta = j\delta+\gamma} \longrightarrow g^{L(g,u,r)}_{\beta'}$

where $\tau: \beta \longrightarrow \alpha$ give an $(*)$-gradation and $(**)$ gradation corresponding by τ $\quad\#$ $\underline{j}\delta+\gamma$ $\quad \underline{j}\delta + \gamma'$ $\quad \gamma' \in g_j$"

g_j

<u>Thm 8.5</u>. Let g be a simple f.d Lie alge. of type X_N.

- $\sigma \in Aut(g)$, s.t $\sigma^m = 1$

- r be the least positive integer s.t. $\sigma^r \in \mathrm{Inn}(\mathfrak{g})$

$\Rightarrow$ the $r = 1, 2$ or 3

- $(\cdot|\cdot)$ be normalized invariant form on $\mathfrak{g}$
- μ be a diagram automorphism of $\mathfrak{g}$ of order r

- Choose a Cartan subalge $\mathfrak{h}_{\bar{0}}$ of the fixed point $\mathfrak{g}^{\sigma} = \mathfrak{g}_{\bar{0}}$ of σ
- $A \to$ affine matrix of type $X_N^{(r)}$

$\Rightarrow$ Then $\exists$ an isomorphism:

$$\hat{L}(\mathfrak{g}, \sigma, m) \longrightarrow \mathfrak{g}(A) \quad \text{st.}$$

(i) (ii) the $\mathbb{Z}$-gradation (8.22) of $\hat{L}(\mathfrak{g}, \sigma, m)$ induces a $\mathbb{Z}$-gradation of $\mathfrak{g}(A)$ of type $s = (s_0 \cdots s_l)$, where s_j are nonegative integers which satisfy the relation

$$(8.5.6) \qquad r \sum_{j=0}^{l} a_j s_j = m > 0$$

\pf: Due to prop 8.1 $\Rightarrow$ σ is conjugate to $\mu \exp(\mathrm{ad}(\frac{2\pi i}{m} h))$ $h \in \mathfrak{h}_{\bar{0}}$ $\mu(h) = h$

$\alpha_i(h) \in \mathbb{Z}$

By Lemma 8.15 $\Rightarrow$ t^h ($L(g,u,m) \xrightarrow{u} L(g,\bar{\sigma},m)$) σ

By (8.5.4) $\alpha \xrightarrow{\gamma} \beta$

$\alpha \xrightarrow{g\gamma} g\beta g^{-1}$

$\Rightarrow$ we can assume that $\sigma = u\exp(\mathrm{ad}\frac{2\pi i}{m}h)$

· σ^r ~~is $\in$ Inn~~ is an inner auto.

r: least positive integer

i.e. $r=1$ $\quad u = \mathrm{id}$ $\quad \cancel{\sigma} = \exp(\mathrm{ad}(\frac{2\pi i}{m}h)) \in \mathrm{Inn}(Lg)$

$r=2 \Rightarrow u = u^{-1} \Rightarrow$

$\underset{\in \mathrm{Aut}(g)}{\sigma^2} = u\exp(\mathrm{ad}(\frac{2\pi i}{m}h))\overset{u^{-1}}{\overset{\|}{u}}\exp(\text{---})$

$\theta\exp(\mathrm{ad}\,h)\theta^{-1} = \exp(\mathrm{ad}(\theta(h)))$

$= \exp(\mathrm{ad}(u(\frac{2\pi i}{m}h)))\exp(\quad) \in \mathrm{Inn}(Lg)$

$r=3 \Rightarrow u^{-1} = u^2$

$\sigma^3 = u \exp(\ \ ) \, u \cdot \exp(\ \ ) \, u \exp(\ \ )$

$u^2 \cdot u^2$

$\Rightarrow r \mid m$

[If not. r is prime number

$\Rightarrow am + br = 1$, $a, b \in \mathbb{Z}$, s.t

$\sigma = \sigma^{am+br} = (\sigma^r)^b \in \mathrm{Inn}(L(g))$

r

then By Lemma 8.5 we get

$$t^{\frac{m}{r}} h : L(g, u, r) \xrightarrow{\sim} L(g, \sigma, m)$$

thm 1

By ~~There~~ Thm 7.4 and 8.3, ~~whe~~ we have an isomorphism

$$\varphi : g'(A)/\mathbb{C}K \longrightarrow L(g, \sigma, m)$$

Thm 7.4 (8.3)

$\tilde{\varphi}$ $(t^{\frac{m}{r}} h)$

$\oplus L(g,\sigma,m)$

The gradation $(8,2,1)$ of $(L(g,\sigma,m))$ $\xrightarrow{\text{Thm 2}}$

the gradation of type $s' = (s_0' \cdots s_l')$

$\Gamma(8,2,1)$ $\quad L(g,\bar\sigma,m) = \bigoplus_j L(g,\sigma,m)_j$

$$L(g,\bar\sigma,m)_j = t^j \otimes g_{\bar j} \mod m$$

$$\Delta = \Delta_{\bar 0} \cup \Delta_{\bar 1} \cup \cdots \Delta_{\overline{m-1}}$$

then $\alpha \in \Delta_{\bar j}$ $(0 \le i \le m-1)$

then define $\deg \alpha = j$

i.e. $\alpha_0 \cdots \alpha_l \to$ have a degree

$(s_0 \cdots s_l) \rightsquigarrow g(A)$]

Where $\deg \bar{e_i} = s_i' = \deg \bar{f_i}$, for $x \in g(A)$, denote by $\bar x$ the coset $x + \epsilon K$.

note that $\deg g_\delta = \sum_{i=0}^{l} a_i s_i'$

On the other hand, multiplication by t^r increase the degree in $L(g,\sigma,m)$

by $m' > 0$

[i.e $t^r (L(\mathfrak{g}, \bar{\sigma}, m)_j) \subset L(\mathfrak{g}, \bar{\sigma}, m)_{j+\ldots}$

$L(\mathfrak{g}, \bar{\sigma}, m)_{j,\alpha} = \{ t^j \otimes x_j \mid x_j \in \mathfrak{g}_j \bmod m$

$[h, x_j] = \alpha(h) x_j$ for all $h \in \mathfrak{h} \}$

$t^r \otimes \mathfrak{h}_{\bar{0}} \longrightarrow \mathfrak{g}_{r\delta}$

$\deg(\mathfrak{g}_{r\delta}) = r \sum_{i=0}^{l} a_i s_i$

$\deg \mathfrak{h}_{\bar{0}} = 0$

$m' = \boxed{r \sum_{i=0} a_i s_i} > 0$]

$t^r \otimes \mathfrak{h}_{\bar{0}} \rightsquigarrow$ degree increase by m

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$\Rightarrow$ Show that s_i can be made non-nagative.

For pick $v \in \mathfrak{h} \to$ Catan subalge of $\mathfrak{g}(A)$

st $\alpha_i(v) = s_i \Rightarrow$ note $\delta(v) = \sum_{i=0}^{l} a_i \alpha_i(v)$

$= m'$

$\bar{r}\ (>0)$

By prop 5.8(b). If A is affine type

then $\underset{\text{Tits Cone}}{X} = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \delta(h) > 0$ or [illegible]$\}$

$\Rightarrow v \in X$ and By prop 3.12(b)

$\forall h \in X$, $W(h) \cap C$ is exactly one point.

$\exists\, w \in W$ s.t. $w(v) \in C$, then

$\alpha_i(w(v)) = s_i \geq 0$

i.e. there $w \in \hat{W}^{ad}$, s.t.

$\alpha_i(w(v)) = s_i \in \mathbb{Z}_+$

($\hat{W}^{ad}$ Remark 3.8)

$\pi : \mathfrak{g} \times V \to V$

$\mathfrak{g}_{(i)} = \langle e_i, f_i, h_i \rangle = sl_2(\mathbb{C})$

$\pi_i : sl_2(\mathbb{C}) \to GL(V)$

$G'' \subset Gl(V)$
$\parallel$

$\pi_i(Sl_2(\mathbb{C}))$ generate.

$W = \tilde{W}^{\pi} / D^{\pi} \to$ generate by

$r_j^{\pi} = \cancel{\exp(0}$

$\exp(f_i)\exp(-e_i)\exp(f_i)$

$S' \to S \to r\sum_{j=0} a_j s_j = m'$

(ii). the bilinear form on $\hat{L}(g, \sigma, m)$ defined $(8.38) \to g(A)$

pf: $\hat{L}(g, \sigma, m) \xrightarrow{\sim} \hat{L}(g, u, r) \xrightarrow[(8.3)]{(7.4)} g(A)$

(8.38)

$$(P \otimes x \mid Q \otimes y) = \frac{1}{r}\operatorname{Res}(t^{-1}PQ)(x|y)$$

$(x, y \in g,\ P, Q \in \mathcal{L})$

$(\mathbb{C}K' + \mathbb{C}d' \mid L(g, \sigma)) = 0$

$(\mathbb{C}k' | \mathbb{C}d' | L(g,\bar{\sigma},m)) = 0$

$(k'|k') = (d'|d') = 0 \qquad (k'|d') = 1$

By thm 1

$(p\otimes x \mid \alpha\otimes y) = \frac{1}{r}\,\mathrm{Reslt}\, p\otimes \mid (x|y)$

$\downarrow \quad \Psi^{-1}(g_\beta) = g_{\tau(\beta)} = g_\alpha$

$p'\otimes x' \in g_\alpha \qquad \alpha'\otimes y' \in g_{\alpha'}$

$(p'\otimes x' \mid \alpha'\otimes y') := (p\otimes x \mid \alpha\otimes y)$

$\varphi : g(A)/\mathbb{C}K \simeq L(g,\mu,r)$

$\varphi : g'(A)/\mathbb{C}K \hookrightarrow L(g,\bar{\sigma},m) \quad \oplus\mathbb{C}d \quad \oplus\mathbb{C}K$

$g'(A) = L(g,\mu,r) \oplus \mathbb{C}K$

The iso. $\varphi^{-1} : L(g,\bar{\sigma},m) \longrightarrow g'(A)/\mathbb{C}K$

$\mathbb{C}k' \oplus \mathbb{C}d'$

can be lifted to a (unique) linear isomorphism

$\varphi' : \underset{\oplus\, \mathbb{C}k' \,\oplus\, \mathbb{C}d'}{L(g,\bar{\sigma},m)} \longrightarrow \underset{\oplus\, \mathbb{C}k \,\oplus\, \mathbb{C}d}{g'(A)/\mathbb{C}K.}$

$\bar{\sigma} \mapsto \bar{\phi}$

$\mathbb{C}k' \longmapsto$

$\mathbb{C}d' \longrightarrow$

$\Phi' : \hat{L}(g,\bar{\sigma},m) \longrightarrow g(A)$

by (iv) :

v

$\sigma = u \exp(\operatorname{ad} 2\pi i h)$

$\Phi'(mK') = K \longrightarrow$ the canonical central element of $g(A)$

by (v)

$$\Phi'\left(\frac{d'}{m}\right) = \frac{d}{a_0} + v + \frac{1}{2}(v|v)K$$

where d is the scaling element of $g(A)$

~~ad~~ and $v \in h$ is defined by

$$\alpha_i(u) = \frac{rs_i}{m} \quad (i = 1 \cdots l)$$

$$\left(r\sum_{j=0}^{l} a_j s_j = m \Rightarrow \sum_{j=0}^{l} a_i \alpha_i(u) = 1 \right)$$

and the following condition

$$\underbrace{\left(\Phi^{-1}(\alpha_i^\vee) = \varphi(\alpha_i^\vee) + v\left(\frac{a_i s_i}{a_i^\vee}\right)K \quad i = 0 \cdots l \right)}$$

$\Rightarrow$ (iii)

$\Rightarrow$ this is a Lie algebra iso.

$\Phi([k', d']) = \Phi([mk', \frac{d'}{m}])$ //?

(1969 A)

ω

$\mathrm{Aut}(g)/\mathrm{Inn}(g) = \mathrm{Aut}$ of Dynkin diagram

$[\Phi(mk)]$ // $t^{\frac{m}{r}h}$: $L(g, \mu, r) \to L(g, \sigma, m)$

$[\Phi(\frac{d'}{m})] \in \mathrm{Aut}(g)$

d

$[K, a_0 + v - \frac{1}{2}(u|u)K] = 0$

$\sigma^m = 1$

$L(g, u, r) \rightsquigarrow g(A)$

$g \xrightarrow{\mathrm{Inn}(L)} g'$ $\mathrm{ad}(\ )$ $x_i^{(v)}$

$g'(A) \to g(A)$ affine Φ alg

$\in \mathrm{Aut}(g)$

$u \exp(\mathrm{ad}\, \varepsilon h)$

$m = 1$

$\sigma \in \mathrm{Aut}(g) \rightsquigarrow \sigma$ is conjugate to

§8.6

- Theorem 8.5 $\xrightarrow{\text{deduce}}$ a classification of finite order automorphism of g of type X_N

$\Phi: \hat{\mathcal{L}}(g, \sigma, m) \xrightarrow{\sim} g(A)$

$u \to X_N \longrightarrow X_N^{(r)}$

- Let u be a diagram automorphism of g of order r

$\sigma \longmapsto \Phi$

- E_i, H_i, F_i $(i = 0 \cdots l)$ be the element of g in §8.3

$\sigma' \longrightarrow u' \longrightarrow g(A')$

· $\alpha_0 \dots \alpha_r$ → be the roots attached to the E_i $\Phi,$

(prop 8.31a) L