§8.6 An application to finite order automorphisms

· let μ be a diagram automorphism of g of order r, let E_i, F_i, H_i $(i=0,\cdots,l)$ be the elements of g (in 8.3) and let $\alpha_0,\cdots,\alpha_l$ be the roots attached the E_i.

Recall that the Element E_i $(i=0,\cdots,l)$ generate g and that there is a unique linear dependence $\sum_{i=0} a_i\alpha_i = 0$. (let $\alpha_\varepsilon = -\theta_0$, $a_\varepsilon = 1$). s.t. a_i are positive relatively prime integers.

Lemma 8.6: Every ideal of the lie algebra $\mathcal{L}(g,\mu)$ is of the form $p(t^r)\mathcal{L}(g,\mu)$, where $p(t)\in\mathcal{L}$. In particular, a maximal ideal is of the form: $(1-(at)^r)\mathcal{L}(g,\mu)$, where $a\in\mathbb{C}^\times$.

proof: let i be a nontrivial ideal of $\mathcal{L}(g,\mu)$.
and $x = \sum_{j,s} t^j \underline{P_{j,s}(t)} \otimes a_{\bar{j},s} \in i$. where $0\le j<r$ is such that

$\bar{j}\equiv j \bmod r$, $P_{\bar{j},s}(t)\in\mathcal{L}$, $P_{\bar{j},s}\ne 0$ and $a_{\bar{j},s}\in g_{\bar{j}}$ are linear indep.

· we show that $Q(t^r)P_{\bar{j},s}(t)\mathcal{L}(g,\mu)\subset i$ for all $Q(t)\in\mathcal{L}$.

$a_{\bar{j},s}$ x

let $H_{\bar{0}}$ be a Cartan subalgebra of $g_{\bar{0}}$, we can assume that x is an eigenvector for ad $H_{\bar{0}}$ with weight $\alpha\in H_{\bar{0}}^*$.

① if $\alpha=0$. $\underline{x\in H_{\bar{0}}.\ a_{\bar{j},s}\in H_{\bar{0}}}$

② if $\alpha\ne 0$.
taking $[x, t^{-j}\otimes a_{-\bar{j}}]$ with $a_{-\bar{j}}$ of weight $-\alpha$ instead of x, we may reduce the problem to the case $\alpha=0$ and $\bar{j}=0$
i.e. $a_{\bar{j},s}\in H_{\bar{0}}$

$\Rightarrow x = P_{\bar{j},s_j}(t)\otimes H_{\bar{0}} + tP_{\bar{j+1},s_{j+1}}\otimes g_{\bar{1}} + \cdots + t^{r-1}P_{\bar{j-1},s_{j-1}}\otimes g_{\bar{r-1}}$

· let $\gamma\in H_{\bar{0}}^*$ be root of $g_{\bar{0}}$ such that $\langle\gamma, a_{\bar{j},s}\rangle\ne 0$. then the element $y = [[x, Q(t^r)\otimes e_\gamma]e_\gamma]\in i$. where $e_{\pm\gamma}$ is a root vector with root $\pm\gamma$, has the following form:

$y = Q(t^r)\,\underline{(p\otimes h + tP_{\bar{1}}\otimes h_{\bar{1}} + \cdots + t^{r-1}P_{\bar{r-1}}\otimes h_{\bar{r-1}})}$.

where $\underline{P = P_{\bar{j},s}(t)}$, $P_{\bar{i}}\in\mathcal{L}$, $h\in H'$, $h\ne 0$, and $h_{\bar{i}}\in g_{\bar{i}}$ have zero weight with respect to H.

$H' = H_{\bar{0}}\oplus\cdots\oplus H_{\bar{r-1}}$

$[\ [x, Q(t^r)\otimes e_\gamma] = [P_{\bar{j},s_j}(t)\otimes H_{\bar{0}} + \cdots + t^{r-1}P_{\bar{j-1},s_{j-1}}\otimes g_{\bar{r-1}},\ Q(t\ )\otimes e_\gamma]$

$= Q(t^r)(P\otimes \underline{\langle\gamma, a_{\bar{j},s}\rangle e_\gamma} + tP_{\bar{1}}\otimes \underline{[g_{\bar{1}}, e_\gamma]} + \cdots + t^{r-1}P_{\bar{r-1}}\otimes[g_{\bar{r-1}}, e_\gamma]$

then $[[x, Q(t^r)\otimes e_\gamma], e_{-\gamma}]$

$= Q(t^r)\underbrace{(P\otimes\langle\gamma, -a_{\bar{j},s}\rangle \overbrace{[e_\gamma, e_{-\gamma}]}^{\gamma}}_{h\in H_{\bar{0}}\subset H'} + \cdots + t^{r-1}P_{\bar{r-1}}\otimes\underline{[[g_{\bar{r-1}}, e_\gamma]e_{-\gamma}]}$. $]$

· since $[y, e_j]\in i$ for all root vectors $e_j\in g_{\bar{0}}$.
we conclude that $Q(t^r)P\otimes \underline{H}\subset i$. therefore $\underline{Q(t^r)P\mathcal{L}(g,\mu)\subset i}$.

$H = 1\otimes H'$

零特征值对应的特征向量

两个特征向量特征值不同 ⇒ 两个特征

· For $x \in \mathfrak{g}_{\bar{j}}$, let.

r次 由于自同构造成的.

值部落理想.

Th. 8.6. Let $s = (s_0, \cdots, s_\ell)$ be a sequence of nonnegative relatively prime integers. Put $m = r \sum_{j=0}^{\ell} a_j s_j$. then.

保证了阶数是 m.

a) The relations:
$$\sigma_{s;r}(E_j) = e^{\frac{2\pi i s_j}{m}} E_j \quad (j = 0, \cdots, \ell)$$
define (uniquely) an m-th order automorphism $\sigma_{s;r}$ of $\mathfrak{g}$.

proof: Note that the root space decomposition:
$$\hat{\mathcal{L}}(\mathfrak{g}, \mu) = \mathcal{H} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathcal{L}(\mathfrak{g}, \mu)_\alpha\right). \quad (8.3.4).$$
where $\mathcal{L}(\mathfrak{g}, \mu)_{s\delta + \gamma} = t^s \otimes \mathfrak{g}_{\bar{s}, \gamma}$, $\mathcal{L}(\mathfrak{g}, \mu)_{s\delta} = t^s \otimes \mathfrak{g}_{\bar{s}, 0}$.

induces a gradation $\mathcal{L}(\mathfrak{g}, \mu) = \tilde{\mathcal{L}}(\mathfrak{g}, \mu) / \mathbb{C}K' = \bigoplus_\alpha \mathcal{L}_\alpha$

· Define the automorphism $\tilde{\sigma}_s$ of $\mathcal{L}(\mathfrak{g}, \mu)$ by.
$$\tilde{\sigma}_s : \mathcal{L}(\mathfrak{g}, \mu) \longrightarrow \mathcal{L}(\mathfrak{g}, \mu).$$
$$\text{s.t. } \tilde{\sigma}_s(e_\alpha) = \varepsilon^{\sum k_i s_i} e_\alpha, \text{ if } e_\alpha \in \mathcal{L}_\alpha, \text{ where } \underline{\alpha = \sum k_i \alpha_i}$$
$$(\varepsilon = e^{\frac{2\pi i}{m}}). \quad \underline{\sum k_i s_i}$$

· If $\mathcal{L}(\mathfrak{g}, \mu) = \bigoplus_{j \in \mathbb{Z}} \mathcal{L}(\mathfrak{g}, \mu)_j$ is the gradation of type s. then $\mathcal{L}(\mathfrak{g}, \mu)_j$ and $\mathcal{L}(\mathfrak{g}, \mu)_{j+m}$ lie in the eigenspace of $\tilde{\sigma}_s$ with eigenvalue $e^{\frac{2\pi i j}{m}}$.

· since $t^r \mathcal{L}(\mathfrak{g}, \mu)_j \subset \mathcal{L}(\mathfrak{g}, \mu)_{j+m}$, we deduce that the ideal $(1 - t^r)\mathcal{L}(\mathfrak{g}, \mu)$ is $\tilde{\sigma}_s$-invariant.

hence $\tilde{\sigma}_s$ induces (uniquely) an m-th order automorphism.
$$\sigma_{s;r} : \mathfrak{g} \longrightarrow \mathfrak{g}, \text{ s.t. } \sigma_{s;r}(E_j) = e^{\frac{2\pi i s_j}{m}} E_j. \quad (\deg E_j = s_j).$$

b). Up to conjugation by an automorphism of $\mathfrak{g}$, the automorphisms $\sigma_{s;r}$ exhaust all m-th order automorphisms of $\mathfrak{g}$.

i.e. $\forall f \in \mathrm{Aut}(\mathfrak{g})$, $f^m = 1$. then $\exists\, \theta \in \mathrm{Aut}(\mathfrak{g})$, s.t. $\theta^{-1} f \theta = \sigma_{s;r}$.

proof: Let now σ be an m-th order automorphism of $\mathfrak{g}$.

Th. 8.5 gives us an isomorphism:
$$\Phi : \mathcal{L}(\mathfrak{g}, \sigma, m) \longrightarrow \mathcal{L}(\mathfrak{g}, \mu).$$
such that the $\mathbb{Z}$-gradation of $\mathcal{L}(\mathfrak{g}, \sigma, m)$ induces a $\mathbb{Z}$-gradation type s of $\mathcal{L}(\mathfrak{g}, \mu)$ with $s_i \in \mathbb{Z}_+$ satisfying $r = \sum_{j=0}^{\ell} a_j s_j = m$.

· Denote by τ_a the automorphism of $\mathcal{L}(\mathfrak{g}, \mu)$ s.t. $t \longmapsto at$, $a \in \mathbb{C}^\times$

Then since by Lem. 8.6 any maximal ideal of $\mathcal{L}(\mathfrak{g}, \mu)$ is of the form $(1 - (at)^r)\mathcal{L}(\mathfrak{g}, \mu)$, we have the following com. diagram for a suitable automorphism ψ of $\mathfrak{g}$ and $a \in \mathbb{C}^\times$.

$$\begin{tikzcd}
\mathcal{L}(\mathfrak{g}, \sigma, m) \arrow[r, "\Phi"] \arrow[d, "\varphi_\sigma"'] & \mathcal{L}(\mathfrak{g}, \mu) \arrow[r, "\tau_a"] & \mathcal{L}(\mathfrak{g}, \mu) \arrow[d, "\varphi_\mu"] \\
\mathfrak{g} \arrow[rr, "\psi"] & & \mathfrak{g}
\end{tikzcd}$$

where the covering homomorphism $\varphi_\sigma : L(g,\sigma) \longrightarrow g$. $t \longmapsto 1$.
i.e. $\varphi_\sigma(\sum p_j \otimes g_j) = \sum p_j(1) g_j$.

$\ker \varphi_\sigma = (1-t^m) L(g,\sigma)$.

· Noting that the automorphism $\tau_\sigma : L(g,\mu) \longrightarrow L(g,\mu)$ preserves each L_α. we deduce that:

$$\psi(\varphi_\sigma(L(g,m,\sigma)_j)) = \varphi_\mu(\bigoplus_{\deg \alpha = j} L_\alpha).$$

such ψ s.t. the above diagram commuting.

Thus ψ maps the ε^j-eigenspace of σ onto the ε^j-eigenspace of $\sigma_{s;r}$.

Hence $\psi \sigma \psi^{-1} = \sigma_{s;r}$. $(\psi\sigma = \sigma_{s;r}\psi)$.

c). The element $\sigma_{s;r}$ and $\sigma_{s';r'}$ are conjugate by an automorphism of g iff and only if $r = r'$ and the sequence s can be transformed into the sequence s' by an automorphism of the diagram X_N^r.

pf: "$\Rightarrow$" suppose $\sigma = \sigma_{s;r}$ and $\sigma' = \sigma_{s';r'}$ are conjugate, i.e. $\tau\sigma\tau^{-1} = \sigma'$ for some $\tau \in \mathrm{Aut}\, g$.

Note that by prop 8.6 b). $\mathcal{H}_{\bar{\sigma}}$ (the cartan subalgebra of g^μ) is the Cartan subalgebra of g^σ and $g^{\sigma'}$ (回顾上文).

[Thm 8.5. we have $L(g,\sigma,m) \cong L(g,\mu,r)$ & $L(g,\sigma',m) \cong L(g,\mu',r')$

Hence $L(g,\mu,r)$ & $L(g,\mu',r')$ have the same diagram, so, $\mu = \mu'$, and $r = r'$].

· since $\tau\sigma\tau^{-1}(g) = \tau(g^\sigma) = \sigma'(g) = g^{\sigma'}$. $\tau(\mathcal{H}_{\bar{\sigma}})$ is another subalgebra of $g^{\sigma'}$, let τ_1 be an inner automorphism of $g^{\sigma'}$ such. $\tau_1(\tau(\mathcal{H}_{\bar{\sigma}})) = \mathcal{H}_{\bar{\sigma}}$.

· Replacing τ by $\tau_1\tau$, we may assume that $\tau(\mathcal{H}_{\bar{\sigma}}) = \mathcal{H}_{\bar{\sigma}}$ and the sets of positive roots of g^σ and $g^{\sigma'}$ correspond to each other under τ.

· The extension $\tilde{\tau}$ of τ given $\tilde{\tau}(t^j \otimes a) = t^j \otimes \tau(a)$.

s.t. $\tilde{\tau} : L(g,\sigma,m) \xrightarrow{\sim} L(g,\sigma',m)$

$L(g,\sigma,m)_j \longmapsto L(g,\sigma',m)_j$

Thus the simple root $(\alpha_0, s_0), \cdots, (\alpha_n, s_n)$ of $L(g,\sigma)$ and the simple root $(\alpha_0', s_0'), \cdots, (\alpha_n', s_n')$ of $L(g,\sigma',m)$. correspond under τ.

Hence sequences s and s' correspond under an automorphism of diagram $X_N^{(r)}$.

· Def: Given a nonzero sequence $s = (s_0, \cdots, s_\ell)$ of nonnegative integers and a number $r = 1, 2$ or 3. we call automorphism $\sigma_{s;r}$ of g defined by (8.6.1) i.e. $\sigma_{s;r}(E_j) = e^{\frac{2\pi i s_j}{m}} E_j$ $(j = 0, \cdots, \ell)$ the automorphism of type $(s;r)$.

· let $g = \bigoplus_j g_j(s;r)$ be the $\mathbb{Z}/m\mathbb{Z}$-gradation associated to the Aut. of type $(s;r)$.

prop. 8.6.

a) r is the least positive integer for which $\sigma_{s;r}^{r}$ is an inner automorphism.

proof: a) follows from the fact a finite order automorphism σ of g is inner iff there is a Cartan subalgebra which is pointwise fixed under σ.

$\sigma_{s;r}^{r}$ $\quad r = 1, 2, 3.$

$\sigma_{s;r} = \mu \exp(- \quad).$ $\quad r = 1, 2, \text{or } 3.$

$\forall\, h \in \mathfrak{H}_{\bar 0}$

b).

proof of b): is immediate from $g^{\sigma}_{\bar 0} \simeq \mathcal{L}(g,\sigma)_0$ and Cor 5.12 b).

· Consider the $\mathbb{Z}/m\mathbb{Z}$-gradation $g = \bigoplus_j g_j(s;r)$ associated to $\sigma_{s;r}$. we have for each $r \in \mathbb{Z}$,

$$\varphi(\mathcal{L}_{j+km}) = \varphi(x^{rk}\mathcal{L}_j) = \varphi(\mathcal{L}_j).$$

where φ is the covering map in Th. 8.6. so $\varphi(\mathcal{L}_j) = g_{\bar j}$.
so φ gives an isomorphism of $\mathcal{L}_j$ onto $g_{j \bmod m}$.
In particular:

$$g_{\bar 0} \simeq \bigoplus_{\deg\alpha = 0} \mathcal{L}(g,\sigma)_\alpha .$$

But $\deg\alpha = 0$ iff $\alpha = \sum_{t=1}^{p} k_{i_t}\alpha_{i_t}$

Hence the algebra $\bigoplus_{\deg\alpha=0} \mathcal{L}(g,\sigma)_\alpha$ is generated by H_i, e_{i_t}, f_{i_t} $(1 \le i \le l,\ 1 \le t \le p)$.

· putting $Y_j = \varphi(f_j)$, $X_j = \varphi(e_j)$.

the X_{i_t}, Y_{i_t} $(1 \le t \le p)$ generate semisimple part of $g_{\bar 0}$.
It's Cartan matrix is the submatrix $(a_{i_s i_t})_{1 \le s,t \le p}$ of the Cartan matrix (a_{ij}) of $X_N^{(r)}$

The center of $g_{\bar 0}$ is spanned by the vectors: H_j $(j \ne i_1, \dots, i_p)$.

c) proof: Note that the $g_{\bar 0}(s;r)$-module $g_{\bar 1}(s;r)$ is isomorphism to $\mathcal{L}(g,\sigma)_0$-module $\mathcal{L}(g,\sigma)_1$

Furthermore, $\mathcal{L}(g,\sigma)_1$ is spanned by elements of the form:

$[\cdots[[e_{i_1}, e_{i_2}]e_{i_3}]\cdots e_{i_r}]$, s.t. $s_{i_1} + \cdots + s_{i_r} = 1$. $\quad (P_j)$.

$\Rightarrow$ $s_{i_1} = 1$ and $s_{i_t} = 0$ for $t > 1$

then $\mathcal{L}(g,\sigma)_0$-module $\mathcal{L}(g,\sigma)_1$ is generated by e_{j_t}, f_{j_t} $(1 \le t \le n)$.

$\Rightarrow$ $g_{\bar 1}(s;r)$ is generated by $X_{j_t} = \varphi(e_{j_t})$, $Y_{j_t} = \varphi(f_{j_t})$ $(1 \le t \le n)$

since $g_{\bar{0}}(s;r)$ is semisimple. the Weyl complete reducibility theorem.

$\Rightarrow g_{\bar{1}}(s;r) \cong g_{-\alpha_{j_1}} \oplus \cdots \oplus g_{-\alpha_{j_n}}$

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