§ 8.7

Later we will need the following reformulation of thm 8.5 (which is a generalization of Thm 7.4 + 8.3)

Recall · $\sigma_{s,r}(E_j) := e^{\frac{2\pi i s_j}{m}} E_j$, $\sigma_{s,r} \in Aut(g)$ and $\sigma_{s,r}^m = 1$, where $s = (s_0, \dots s_l)$, $m = r\sum_{i=0}^{l} a_i s_i$

· $\sigma_{s,r}$ exhaust all m-th order automorphisms of g up to conjugate by an automorphism of g

$\left(\forall\ \sigma \in Aut(g),\ \sigma^m = 1 \Rightarrow \sigma \text{ is conjugate to } u\exp \operatorname{ad}(\varepsilon h) \right)$

reformulation of Thm 8.5

↓

Thm 8.7. $A \to X_N^{(r)}$

$g \longrightarrow$ a simple f.d. Lie algebra $\longrightarrow X_N$

$u \longrightarrow$ diagram automorphism of order r of g

E_i, F_i, H_i $(i = 0..l) \longrightarrow$ the elements of g introduced in § 8.2

$\sigma_{s,r} \longrightarrow \in Aut(g)$ and of type s

$g = \bigoplus_j g_j(s; r) \longrightarrow$ be the associated $\mathbb{Z}/m\mathbb{Z}$ -gradation

$m = r\sum_{i=0}^{l} a_i s_i$

Define the Lie alge. structure on $\sum_{j \in \mathbb{Z}} t^j \otimes g_{j \bmod m}(s,r) \oplus \mathbb{C}k$ by:

$[p_1(t) \otimes g_1 \oplus \lambda_1 k,\ p_2(t) \otimes g_2 \oplus \lambda_2 k]$

$$:= p_1(t)\, p_2(t) \otimes [g_1, g_2] \oplus \frac{1}{m} \left(\operatorname{Res} \frac{d\, (p_1(t))}{dt}\, p_2(t) \right) (g_1|g_2) K$$

「 $[a + \lambda K,\ b + \mu K] = [a, b]_0 + \psi(a, b) K$ §7.1 → p97

$\downarrow$ 2-cocycle

$$\psi(\underset{\substack{\| \\ p\otimes x}}{a}, \overset{\substack{Q\otimes y \\ \|}}{b}) = \operatorname{Res} \left(\frac{d\,(p\otimes x)}{dt}\, Q \otimes y \right)_t = \operatorname{Res}\left(\frac{dp}{dt} Q \right) (x|y)$$

这里与中心相差 $\frac{1}{m}$，2-cocycle ✓ 」

Then $\left(\sum_{j\in\mathbb{Z}} t^j \otimes g_j \bmod m\,(s;r) \right) \oplus \epsilon K \cong g'(A)$

proof same as Thm 8.3 (Thm 7.4)

with the chevalley generates $e_i = t^{s_i} \otimes E_i$ $(i = 0 \cdots l)$

$\downarrow$ $\deg = s_i$

$f_i = t^{-s_i} \otimes F_i$

the corrot basis : $\alpha_i^\vee = [e_i, f_i] = (1 \otimes H_i) \oplus K$

$\left(\frac{a_i s_i}{a_i^\vee m} K \right)$

and the canonial cortral element K

$\rightsquigarrow$ Extending this Lie algebra by $\epsilon d'$

where $d'(p(t) \otimes g) := \frac{a_0}{m} \left(t \frac{dp(t)}{dt} \right) \otimes g$ and

$[d', K] = 0$

「 作为导子，多一个系数 $\frac{a_0}{m}$ 」 ✓

then $\underbrace{\sum_{j\in\mathbb{Z}} t^j \otimes g_j \bmod m\,(s, v) \oplus \epsilon K + \epsilon d'}_{\mathcal{L}(g, \pi_{s,r}, m)} \cong g(A)$

with scaling elements $d = a_0 d' - a_0 H - \frac{1}{2} a_0 (H|H) K$

where $H \in \sum \mathbb{C} H_i$ is defined by $\langle \alpha_i, H \rangle = \frac{s_i}{m}$

$(i = 1 \cdots l)$

The normalized invariant form is defined by

$$(P_1(t) \otimes g_1 \mid P_2(t) \otimes g_2) = \frac{1}{\gamma} \operatorname{Res}(t^{-1} P_1(t) P_2(t)) (g_1 | g_2)$$

$$(\mathbb{C}K + \mathbb{C}d' \mid p(t) \otimes g) = 0 = (K|K) = (d'|d')$$

$$(K|d') = 1$$

Finally, setting $\deg(t) = 1$ $\deg g = 0$ for $g \in \mathfrak{g}$

$$\deg K = \deg d' = 0 \Rightarrow$$

the define the $\mathbb{Z}$-gradation of type S

#

Rmk: The realization of affine algebra of type $X_N^{(r)}$ provided by Thm 8.7 $\Rightarrow$ is called the realization of type $S = (s_0 \cdots s_l)$

Cor 8.7. Let $\hat{L}(\mathfrak{g}, \mu, r)$ be an affine algebra of type

$r = 1$, $\phi(x, y) = 2h^\vee (x, y)$

$X_N^{(r)}$, Then $\phi(x, y) = 2rh^\vee (x|y)$ for $x, y \in \mathfrak{g}$,

by ex 6.2 | $\phi = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$

where ϕ is the Kilig form on $\mathfrak{g}$.

$\phi(\ ,\ )$ $\phi(\ )$ ad: $\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ $\mathfrak{h}^*$

In particular:

$$\sum_{\alpha\in\Delta_{\mathfrak g}} (\lambda|\alpha)(\mu|\alpha) = 2rh^\vee(\lambda|\mu) \quad \lambda,\mu\in\mathfrak h^*$$

where $\mathfrak h$ is a Cartan subalgebra of $\mathfrak g$ and $\Delta_{\mathfrak g}\subset\mathfrak h^*$ the root system of $\mathfrak g$.

pf: (ex 6.2) Let $r=1$, show that $h=\sum_{i=0}^{l} a_i$ $= \mathring{\mathfrak g}$ is the coxeter number of the root system $\mathring\Delta$ of $\mathring{\mathfrak g}$

and $h^\vee = \sum_{i=0}^{l} a_i^\vee = \phi(\theta,\theta)^{-1} = 1+(\mathring\rho|\theta)$, where ϕ is the killing form of $\mathring{\mathfrak g}$, $\mathring\rho = \frac12\sum_{\alpha\in\mathring\Delta_+}\alpha$,

show that $\phi(x,y) = 2h^\vee(x|y)$ for $x,y\in\mathring{\mathfrak g}$

Cor 6.4. $\mathfrak g\to X_N^{(r)}$

note $\theta=\sum_{i=1}^{l} a_i\alpha_i$ $\quad\mathring\rho = \frac12\sum_{\alpha\in\Delta_+}\alpha_i$.

The the ratio of normalized invariant form of $\mathfrak g$ restric to $\mathring{\mathfrak g}$ the normalized invariant form of $\mathring{\mathfrak g}$ is equal to r

$\phi(x,y) = \mathrm{Tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y)$

$\{e_\alpha^{(i)}\}$ and $\{e_{-\alpha}^{(-i)}\}$ be basis of $\mathfrak g_\alpha$ and $\mathfrak g_{-\alpha}$

$\| \ x_i \qquad y_i$ dual basis $\phi(e_\alpha^{(i)}, e_{-\alpha}^{(j)}) = \delta_{ij}$

$\mathring{\mathfrak g}$, $\alpha\in\Delta_l \to (\alpha|\alpha)=2$

$$\Omega = \Omega_{ad} = \sum_i \mathrm{Tr}(\mathrm{ad}(x_i)\,\mathrm{ad}(y_i)) = \sum_i\phi(x_i,y_i)$$

Aff1 → long → 2

Aff2 → → 4

Aff3 → → 6

Ω commuted with ad: $\mathfrak g\to\mathfrak{gl}(\mathfrak g)$

$\Rightarrow$ Ω acts on $\mathfrak g$ by multi scaling element λ

$$\lambda = \frac{tr(\Omega)}{\dim(V)} = \frac{tr(\Omega)}{\dim \mathfrak{g}} = \frac{\sum_i tr(ad\, x_i\, ad\, y_i)}{\dim \mathfrak{g}}$$

$$= \frac{\sum_i \phi(x_i, y_i)}{\dim \mathfrak{g}} = \frac{\sum_i \delta_{ij}}{\dim \mathfrak{g}} = 1$$

i.e Ω action on $\mathfrak{g}$ by multi scaling $\lambda = 1$

note that $\phi(\theta + 2\rho, \theta)$ is the eigenvalues of Ω asso. to the killing form ϕ

Chapter 9. Highest-weight modules and Kac-Moody algebras.

§9.1

- $A \in M_n(\mathbb{C}) \rightsquigarrow \mathfrak{g}(A)$
- $\mathfrak{g}(A)$ with a cartan subalgebra $\mathfrak{h}$ and a triangular decomposition :

$$\mathfrak{g}(A) = n_+ \oplus \mathfrak{h} \oplus n_-$$

$$U(\mathfrak{g}(A)) = U(n_+) \otimes U(\mathfrak{h}) \otimes U(n_-) \quad (9.11)$$

- $\mathfrak{g}(A)$-module V is called $\mathfrak{h}$-diagonalizable if it admits a wight space decomposition :

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

$$V_\lambda = \{ \underset{\neq 0}{v} \in V \mid h(v) = \lambda(h) \cdot v \quad \forall h \in \mathfrak{h} \}$$

is called a weight vector of weight λ

· Denote $P(V) = \{ \lambda \in \mathfrak{h}^* \mid V_\lambda \neq 0 \}$ the set of weights of V

$\lambda - u = \sum a_i \alpha_i$

$$D(\lambda) = \{ u \in \mathfrak{h}^* \mid u \leq \lambda \} \qquad \Delta_+$$

$g(A) \qquad n_+ \quad n_-$

$$\Delta = P(\underline{g(A)})$$

· Defi: the Category $\mathcal{O}$

obj : the $g(A)$-module V, which are $\mathfrak{h}$-diagonalizable

$(V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda) \qquad \dim V_\lambda < \infty$

$\exists$ finite numbers $\lambda_1 \ldots \lambda_s \in \mathfrak{h}^*$ s.t

$$P(V) \subset \bigcup_{i=1}^{s} D(\lambda_i)$$

morphsim: homomorphism of $g(A)$-modules

Rmk: ① $V \in \mathrm{obj}\, \mathcal{O}$, $U \subset V$ a submodule

By prop 1.5 $\quad U = \bigoplus_{\lambda \in \mathfrak{h}^*} (U \cap V_\lambda) = \underline{\bigoplus_{\lambda \in \mathfrak{h}^*} U_\lambda}$

$\underline{\dim(U_\lambda) \leq \dim(V_\lambda)} < \infty$

$P(U) \subset P(V) \subset \cdots$

② $\left(0 \to U \to V \to V/U \to 0\right)$

$U \in \mathcal{O}$, $V \in \mathcal{O}$

$$V/U = \left(\bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda\right) \Big/ \left(\bigoplus_{\lambda\in\mathfrak{h}^*} U_\lambda\right) = \left(\bigoplus_{\lambda\in\mathfrak{h}^*} U_\lambda + V_\lambda\right) \Big/ \bigoplus_{\lambda\in\mathfrak{h}^*} U_\lambda$$

$$\cong \left(\bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda \Big/ \bigoplus_{\lambda\in\mathfrak{h}^*} (\underbrace{U_\lambda \cap V_\lambda}_{U_\lambda}) \right)$$

$$= \bigoplus_{\lambda\in\mathfrak{h}^*} \left(V_\lambda / U_\lambda \right) \qquad V/U \in \mathcal{O}$$

③ $0 \to V_1 \to V \to V_2 \to 0$

$V_1 \in \mathcal{O}$, $V_2 \in \mathcal{O}$

$$(V_1 \oplus V_2)_\lambda = (V_1)_\lambda \oplus (V_2)_\lambda$$

$$(V_1 \oplus V_2) = \bigoplus_{\lambda\in\mathfrak{h}^*} (V_1 \oplus V_2)_\lambda$$

④ $V_1, V_2 \in \mathcal{O}$, $V_1 \otimes V_2 \in \mathcal{O}$

$$\left.\begin{aligned} V_1 &= \bigoplus_{\lambda_1\in\mathfrak{h}^*} (V_1)_{\lambda_1} \\ V_2 &= \bigoplus_{\lambda_2\in\mathfrak{h}^*} (V_2)_{\lambda_2} \end{aligned}\right\} \Rightarrow V_1 \otimes V_2 = \bigoplus_{\lambda_1,\lambda_2} \underbrace{\left\{(V_1)_{\lambda_1} \otimes (V_2)_{\lambda_2}\right\}}_{\cup \mathsf{I}}$$

now, $(V_1)_{\lambda_1} \otimes (V_2)_{\lambda_2} \subset (V_1 \otimes V_2)_{\lambda=\lambda_1+\lambda_2} \subset \bigoplus_{\lambda\in\mathfrak{h}^*} (V_1 \otimes V_2)_\lambda$

$$\Rightarrow (V_1 \otimes V_2) = \bigoplus_{\lambda\in\mathfrak{h}^*} (V_1 \otimes V_2)_\lambda$$

$\Downarrow$

$$(V_1 \otimes V_2)_\lambda = \sum_{\substack{\lambda_1,\lambda_2 \\ \lambda_1+\lambda_2=\lambda}} ((V_1)_{\lambda_1} \otimes (V_2)_{\lambda_2})$$

$\lambda \to (\lambda_1, \lambda_2)$

$P(V_1) \longrightarrow \xi$ $\quad \lambda = \lambda_1 + \lambda_2 < \xi + \eta$

$P(V_2) \longrightarrow \eta$

$\forall \lambda$, with $(V_1 \otimes V_2)_\lambda \neq 0$. $\Rightarrow$ $\exists$ only finite many pairs (λ_1, λ_2) with $\lambda_1 + \lambda_2 \in \lambda$ s.t. $(V_1)_{\lambda_1} \neq 0$ $\quad (V_2)_{\lambda_2} \neq 0$.

$\dim ((V_1 \otimes V_2)_\lambda) < \infty$

$\Rightarrow$ any submodule or quotient modules of a module $\in$ Obj $\mathcal{O}$
$\in$ Obj $\mathcal{O}$

§ 9.2.

(Example of category $\mathcal{O}$) $\longrightarrow$ highest wight moduls (h.w.m)

· Defi: A $g(A)$-module V is called a h.w.m with highest weight $\Lambda \in \mathfrak{h}^*$, If there exists a nonzero vector $v_\Lambda \in V$, s.t.

(1) (9.2.1) $\mathfrak{n}_+(v_\Lambda) = 0$, $h(v_\Lambda) = \Lambda(h)$ for $h \in \mathfrak{h}$

(2) (9.2.2) $U(g(A))(v_\Lambda) = V$

the vector v_Λ is called a highest-wight vector

(9.2.3) $U(g(A))(v_\Lambda) = U(\mathfrak{n}_-)(v_\Lambda)$

For $v_\Lambda \in V_\Lambda$, Λ is the highest wight

$\lambda + \alpha_i \in P(V) \Rightarrow n_+(v_\lambda) = 0$

$U(g(A))(v_\lambda) = U(n_-)(v_\lambda)$

It follows (9.2.1) and (9.2.3)

$$V = \bigoplus_{\lambda \preceq \Lambda} V_\lambda \ ; \quad V_\lambda = \mathbb{C} v_\lambda \ ; \ \dim V_\lambda < \infty$$

$\dim(V_\lambda) = 1$

In particular, h.w.m $\in \mathcal{O}$ $\quad P(V) \subset D(\Lambda)$

every two highest-wight vectors are proportional

$v_1, v_2 \in V_\lambda \qquad v_1 = k v_\lambda$

$v_2 = k_1 v_\lambda$

Def: (Verma module)

A $g(A)$-module $M(\Lambda)$ with hight wight Λ is called a Verma module, if every $g(A)$-module with highest weight Λ is a quotient of $M(\Lambda)$

V

Prop 9.2:

(a) For every $\Lambda \in \mathfrak{h}^*$, there exists a unique up to isomorphism Verma module $M(\Lambda)$

(唯一性)

Pf, $M_1(\Lambda)$, $M_2(\Lambda)$ are two Verma module

$\psi: M_1(\Lambda) \twoheadrightarrow M_2(\Lambda)$

$(M_1)_\lambda = \{ m_\lambda \in M(\lambda), \mid h(m) = \lambda(h) \cdot m \text{ for all } h \in \mathfrak{h} \}$

$(M_2)_\lambda \longrightarrow$

In particular, $\psi((M_1)_\lambda) = (M_2)_\lambda$

$\forall m_2 \in (M_2)_\lambda, \ \exists m_1 \in (M_1)_\lambda$ s.t $\psi(m_1) = m_2$

$\varphi: (M_1)_\lambda \longrightarrow (M_2)_\lambda$

$\dim((M_1)_\lambda) \geq \dim((M_2)_\lambda)$

Exchaging $M_1(\lambda)$ and $M_2(\lambda)$ $\Rightarrow$ φ is isomorphism

$\dim((M_1)_\lambda) = \dim((M_2)_\lambda)$

(存在性): $\{e_\alpha \mid \alpha \in \Delta\}$ $e_\alpha, \alpha \in \Delta^+$

Consider the left ideal $J(\Lambda)$ in $U(g(A))$

$$J(\Lambda) = \sum_{\alpha \in \Delta_+} U(g(A))(e_\alpha) + \sum_{i \in I} U(g(A))(h_i - \Lambda(h_i))$$

$\mathbb{C}$

$$M(\Lambda) := U(g(A)) / J(\Lambda)$$

$\Rightarrow$ then $M(\lambda)$ is a left $U(g(A))$-module with Λ

It is clear that $M(\Lambda)$ is a $g(A)$-module ①

with highest wight λ ②

①: $\varphi: g(A) \hookrightarrow U(g(A))$ $\quad g(A) \times M(\Lambda) \to M(\Lambda)$

$(x, m_\lambda) \mapsto \varphi(x) \cdot m_\Lambda$

②: $M(\Lambda) := U(g(A)) / \sum_{\alpha \in \Delta_+} U(g(A)) e_\alpha + \sum_i U(g(A))(h_i - \lambda(h_i))$

$1 \longmapsto 1$

$e_\alpha, \alpha \in \Delta^+ \longrightarrow 0$

$h_i \longmapsto \Lambda(h_i)$

the highest vecteev

$1 \in g(A)$

$n_+(\_)$

Λ is highest weight

$\Big((M(\lambda))_{\mu} \neq 0 \iff \underset{\text{root}}{\lambda - \mu}$ is a sum of positive $\Big)$

the highest-weight vector being the image of $1 \in U(g(A))$

If now V is a $g(A)$-module with highest weight $\underline{\Lambda}$,
annihilator of $\underbrace{V_\Lambda \subset V}$ is a left $\underline{\text{ideal } J_1}$
which contain $J(\Lambda)$

$\mathrm{Ann}_{g(A)}(V_\Lambda)$ is an ideal of $g(A)$

$J_1 \supset J(\Lambda)$

$\forall\, v_\lambda \in V_\Lambda$, $n_+(V_\Lambda) = 0$

$(h - \Lambda(h))(v_\Lambda) = \underline{\Lambda(h) v_\Lambda} - \underline{\Lambda(h) \cdot v_\Lambda} = 0$

$J(\Lambda) \subset \mathrm{Ann}_{g(A)}(V_\lambda)$

$U(g(A)) / \mathrm{ann}_{g(A)}(V_\Lambda) \cong V$

$V_\Lambda = \mathbb{C} v_\Lambda$

$U(g(A)) / J_1 \cong V$

$\cup|$

$M(\lambda) = U(g(A)) / J(\Lambda) \Rightarrow \exists\, \psi : M(\lambda) \twoheadrightarrow V$ is $\underline{\text{homo of } U(g)\text{-modules}}$
of $g(A)$-modules

$\Rightarrow \underline{V} = M(\lambda) / \ker \psi$

#

b). Viewed as a $U(n_-)$-module, $M(\lambda)$ is a free module of rank 1 generated by a highest-weight vector.

Pf: It follow from the explict construction of $M(\lambda)$ given above and PBW Thm #.

$$M(\lambda) = U(g(A)) \Big/ \sum_{\alpha \in \Delta^+} U(g(A)) e_\alpha + \sum_{i \in I} (h_i - \lambda(h_i)) /$$

Let m_λ be the image of $\underbrace{1 \in U(g(A))}$

$\downarrow$ highest weight vector

$n_+(m_\lambda) = 0$

By PBW Thm: $M(\lambda)$ is restricted to $U(n_-)$ is free mouddee of rank 1 on basis m_λ

c). $M(\lambda)$ contains a unique proper maximal submodule $M'(\lambda)$

Pf: It follow from the fact:

$\underbrace{\text{a sum of proper submodule}}$ in $M(\lambda)$ is a proper submodule, $\downarrow$ doesn't contain $M(\lambda)_\lambda$

$\downarrow$

By prop 1.5. every submodule in $M(\lambda)$ is graded with respect to the wight space decomposition and doesn't contain $n_-(v_\lambda)$

$\Rightarrow \dim(M'(\lambda)) \leq \dim(M(\lambda)) - 1$, Remark 9.2: ① One can also obtain via the construction of an induced module.

Let V be a left module a Lie algebra $\mathfrak{a}$, and suppose we are ~~given~~ given a Lie alge hom.

$$\psi : \mathfrak{a} \longrightarrow \mathfrak{b}$$

induced $\mathfrak{b}$-module is defined by

$$\boxed{U(\mathfrak{b}) \otimes_{U(\mathfrak{a})} V := \left(U(\mathfrak{b}) \otimes_{\mathbb{C}} V \right) \Big/ \sum_{a,b,v} \mathbb{C}\left(b\psi(a)\otimes v - b\otimes a(v)\right)}$$

where the summation is over $b \in U(\mathfrak{b})$ $a \in \mathfrak{a}$ $v \in V$. and the action of $\mathfrak{b}$ induced by left-multiplication in $U(\mathfrak{b})$

② Define the $(\mathfrak{n}_+ \oplus \mathfrak{h})$-module $\mathbb{C}_\lambda$ with underlying space $\mathbb{C}$, by $\mathfrak{n}_+(1)=0$

$h(1) = \lambda(h)\cdot 1$ for $h \in \mathfrak{h}$

λ

Then $M(\lambda) = U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{n}_+ \oplus \mathfrak{h})} \mathbb{C}_\lambda$

$$M(\lambda) = U(\mathfrak{g}(A)) \Big/ \left(\sum_{\alpha\in\Delta^+} U(\mathfrak{g}(A))\, e_\alpha + \sum_i U(\mathfrak{g}(A))\,(h - \lambda(h)) \right)$$

$$\psi : \mathfrak{a} \longrightarrow \mathfrak{b}$$

$$\big(\mathfrak{n}_+ + \mathfrak{h}\big) \qquad U\big(\mathfrak{g}(A)\big)$$

$$M(\lambda) = U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{n}_+ \oplus \mathfrak{h})} \mathbb{C}_\lambda$$

$$\mathbb{C}_\lambda \qquad \in \mathfrak{h}_+ \in \square$$

$$= U(\mathfrak{g}(A)) \otimes \mathbb{C}_\lambda \Big/ \underbrace{\sum U(\mathfrak{g}(A))}_{\square} \underbrace{e_\alpha}_{\square} + \sum \underbrace{U(\mathfrak{g}(A))}_{\square} \big(h_i - \lambda(h_i)\big)$$

$$V = \mathbb{C}_\lambda$$

$$\boxed{M(\lambda) = U(\mathfrak{g}(A)) \otimes_{(\mathfrak{n}_+ \oplus \mathfrak{h})} \mathbb{C}_\lambda} \qquad V_\lambda = \mathbb{C} v_\lambda$$

§. 9.3

It follow from prop 9.2 (c), that among the modules with highest wight λ, There is a unique irreducible one, namely the module

$$L(\lambda) = M(\lambda) / M'(\lambda)$$

clearly, $L(\lambda)$ is a quotient of any module with h.w. λ

$$L(\lambda) = M(\lambda)/M'(\lambda) \cong \big(M(\lambda)/\ker\psi\big) \Big/ \big(M'(\lambda)/\ker\psi\big)$$

$$V \longrightarrow \lambda \quad \Rightarrow \quad V \cong M(\lambda)/\ker\psi$$

$$\ker\psi \subset M'(\lambda)$$

/ $M(\lambda)$ / Rei4

$\tilde{V} \leftarrow M(\lambda)$

$\downarrow$

$L(V)$

Def: (primitive vector)

Let V be a $g(A)$-module, A vector $v \in V_\lambda$ is called primitive, if there exists a submodule U in V, s.t. $v \notin U$; $n_+(v) \subset U$

then λ is called a primitive weight

[similarly one defines primitive vectors and weight for a $g'(A)$-mod.]

Rmk: A weight vector v, s.t. $n_+(v) = 0$ is obviously primitive.

$(U = 0)$

§9.1

Rmk: every module from $\mathcal{O}$ is restricted (§2.5

§2.5

Recall that $g(A)$-module (resp $g'(A)$) V is called restricted if every $v \in V$, we have

$\mathfrak{g}_\alpha(v)=0$ for all but a finite number of positive roots α $\quad (\mathfrak{g}=\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$

submodule or quotient of restricted module is restricted.

pf: $g(A)$-$V\in\mathcal{O}$ $\quad \forall v\in V \to v\in V_\lambda$

$\alpha\in\Delta^+$ $(\mathfrak{g}_\alpha\cdot V_\lambda\subset V_{\lambda+\alpha})$

$\mathfrak{g}_\alpha(v)=0$ $\quad \lambda+\alpha,\ \lambda+2\alpha\in P(V)$

h.w.m $\in\mathcal{O}$

Prop 9.3:

Let V be a nonzero module from the category $\mathcal{O}$, Then:

(a) V contains a nonzero wight vector v, s.t $n_+(v)=0$

(b): TFAE:

(i) V is irreducible

(ii) V is a h.w.m. and any ~~primitve~~ primitiv vectors of V is a highest-weight vector.

(iii) $\tilde{V} \simeq L(\lambda)$ for some $\lambda \in \mathfrak{h}^\vee$

(c). $\tilde{V}$ is generated by primitive vectors as $\mathfrak{g}(A)$-mod

Pf: (a). Take a maximal $\lambda \in P(\tilde{V})$ (with respect to the ordering $\preceq$), Then on can take v to be a weight vector of wight λ. and $n_+(v)=0$

(b) (i $\Rightarrow$ ii)

Let $\tilde{V}$ be an irreducible module

By remark, ~~the~~ a wight vector $v \in \tilde{V}$ is primitive $\Longleftrightarrow$ $n_+(v)=0$

($\Leftarrow$ ✓

$\Rightarrow$ $\tilde{V} = \mathfrak{g}(A)(v)$

$U = \tilde{V}$, $U = 0$)

Take a primitive v of wight λ, Then $U(\mathfrak{g}(A))(v)$ is a submodule of $\tilde{V}$, $\Rightarrow$ $\tilde{V} = U(\mathfrak{g})(v)$

and $\tilde{V}$ is a module with highest wight λ. ✓

In particular, $P(V) \preceq \lambda$ and $\dim \tilde{V}_\lambda = \mathbb{C} v_\lambda = 1$

Hence, every primitive vector is proportial to v_λ

$\Rightarrow$ (i) $\Rightarrow$ (ii)

(ii) ⇒ (i)

If V is a h.w.m, and $U \subset V$ is a proper submodule then U ~~can~~ contain a highest-wight vector ↯

V is irreducible.

contain primitive vect $\in V$

by (a) $V \neq 0 \in \mathcal{O}$ $\quad U \subset V \quad U \in \mathcal{O}$

v, $n_+(v)=0$

权向量

~~(i) ⇒ (iii)~~ (iii) ⇒ (i)

(i) ⇒ (iii) V h.w.m

$$\varphi: M(\lambda) \twoheadrightarrow V$$

$$V \cong M(\lambda) / \ker\varphi$$

$\ker\varphi$ must be a maximal submodule of $M(\lambda)$

$\ker\varphi = M'(\lambda)$ i.e $V \cong M(\lambda)/\ker\varphi$

$$= M(\lambda)/M'(\lambda) = L(\lambda)$$

(c).

Pf: Let V' be the submodule in V generated by all primitive elements, If $V' \neq V$, then $g(A)$-module $V/V' \neq 0$ contains a vector $v+V'$ by (a) $v+V'$ is primitive vector.

But a weight vector in $\bar{V}$ which is peimage of v is primitive $\Big\{ \Rightarrow \bar{V} = V'$

$\#$

$\varphi \quad V \longrightarrow V/V'$

$v \longmapsto v + V' \to$ primitive

$\varphi^{-1}(U/V')$

$\begin{cases} U + V', \quad v + V' \notin U/V' \\ \quad\quad\quad\quad \in V' \\ n_+(v + V') \in U/V' \end{cases}$

$n^+(v) \in \boxed{\varphi^{-1}(U/V')}$ $\quad n_+(v) + V' \in U + V'$

$v \notin \varphi^{-1}(U/V') \quad n_+(v) \in$

$v \in \varphi^{-1}(U/V')$

$\varphi(v) \in U/V' \quad \Big\{$

$n^+(v) \in \varphi^{-1}(U/V')$

Rmk: Thus, we have abijection ~~betwe~~ between $\mathfrak{h}^*$ and irreducible from the category $\mathcal{O}$

given by $\lambda \longmapsto \underbrace{L(\lambda)}_{\text{不可约}} = M(\lambda) / M'(\lambda)$

#