§ 9.3.

Remark 9.3: BGG Category.

A module $\mathcal{V}$ from the category $\mathcal{O}$ is generated by its primitive vectors even as a n_--module.

Claim: a weight vector $v \in \mathcal{V}$ is not primitive iff $v \in U(n_-)U_0(n_+)v$ (= the submodule generated by $n_+(v)$)

where $U_0(g)$ denotes the augmentation ideal $gU(g)$ of $U(g)$.

· the augmentation map $\varepsilon_L : U(L) \longrightarrow F$ is the unique algebra homomorphism induced by $\varepsilon_L(x) = 0$ for every $x \in L$.

The kernel of ε_L is called the augmentation ideal of L.

· $\ker \varepsilon_L = LU(L) = U(L)L$

· $U(L) = \mathbb{C}1 \oplus \ker \varepsilon_L$.

(1) verify: a weight vector $v \in \mathcal{V}$ is not primitive iff $v \in \langle n_+(v) \rangle$

proof: "$\Rightarrow$" Assume v is not primitive, then submodule U in $\mathcal{V}$.

if $v \notin U$, then $n_+(v) \not\subset U$.

if $v \notin \langle n_+(v) \rangle$, but $n_+(v) \subset \langle n_+(v) \rangle$ Contradiction.

then $v \in \langle n_+(v) \rangle$.

"$\Leftarrow$" Assume $v \in \langle n_+(v) \rangle$.

if v is primitive, i.e. $\exists\, U < \mathcal{V}$. s.t. $v \notin U$. $n_+(v) \subset U$.

then $\langle n_+(v) \rangle \subset U$. contradiction.

(2) verify $U(n_-)U_0(n_+)v = \langle n_+(v) \rangle$

proof: we have known: $U_0(n_+) = n_+U(n_+) = U(n_+)n_+$.

then $U(n_-)U_0(n_+)v = U(n_-)U(n_+)n_+(v) = U(n_-)U(n_+)U(H)(n_+v)$.

$= U(g)\,n_+(v)$.

(3) $\mathcal{V}$ is generated by its primitive vector as $U(n_-)$-module.

proof: let U be the $U(n_-)$-submodule generated by the primitive vectors in $\mathcal{V}$.

since for each weight vector $v \in \mathcal{V}$, we have

$U(g(A))v = U(n_-)U(H)U(n_+)v = U(n_-)U(n_+)v$.

$= U(n_-)(\mathbb{C} + U_0(n_+))v = U(n_-)v + U(n_-)U_0(n_+)v$.

We have known $\mathcal{V}$ is generated its primitive vectors as $g(A)$-module then we can deduce $\mathcal{V}$ is generated as $U(g)$-module by U and the $U(n_-)$ submodule by $U_0(n_+)v$. for all primitive $v \in \mathcal{V}$.

Assume $\mathcal{V} \neq U$. there is a primitive v s.t. $U_0(n_+)v \not\subset U$.
let v have weight λ. then there is a weight vector $u_1 \in U_0(n_+)$ with $u_1 v \notin U$. $\Rightarrow u_1 v$ is not primitive. in $\mathcal{V}$. then we have $u_1 v \in U(n_-)U_0(n_+)u_1 v$. Hence $U_0(n_+)u_1 v \not\subset U$. so there is a weight vector $u_2 \in U_0(n_+)$ with $u_2 u_1 v \notin U$.

we obtain a sequence of weight vectors $u_1, u_2, \cdots$ in $U_0(n_+)$ s.t. $u_k \cdots u_1 v \notin \mathcal{V}$ for each k.

let the weight of u_i be μ_i, Then the weight of $u_k \cdots u_1 v$ is $\lambda + \mu_1 + \cdots + \mu_k$. we have $\lambda < \lambda + \mu_1 < \lambda + \mu_1 + \mu_2 < \cdots$.

$v \in \mathcal{O}$.

$\Rightarrow \mathcal{V} = U$.

#

"Shur lemma"

Lemm. 9.6. $\mathrm{End}_{g(A)} L(\Lambda) = \mathbb{C} I_{L(\Lambda)}$.

proof: If $a \in \mathrm{End}_{g(A)} L(\Lambda)$ and v_Λ is a highest-weight vector of $L(\Lambda)$. Then by prop 9.3 b), we have

$$a(v_\Lambda) = \lambda v_\Lambda \quad \text{for some } \lambda \in \mathbb{C}$$

[since $\dim(v_\Lambda) = 1$, i.e. $V_\Lambda = \mathbb{C} v_\Lambda$, if $a(v_\Lambda) = v_\alpha$, $\alpha \neq \Lambda$.
$\Rightarrow n_+ a(v_\Lambda) = a(n_+(v_\Lambda)) = 0 = n_+(v_\alpha) \neq 0$. contradiction.]

· but then $a(u(v_\Lambda)) = \lambda u(v_\Lambda)$ for $u \in U(g)$.
Hence $a = \lambda I_{L(\Lambda)}$ $\quad U(g(A)) v_\Lambda = V$.

#

§9.4. A contravariant bilinear form.

· Let $L(\Lambda)^*$ be the $g(A)$-module contragredient to $L(\Lambda)$.

$g(A) \times L(\Lambda) \longrightarrow L(\Lambda)$; $\quad g(A) \times L(\Lambda)^* \longrightarrow L(\Lambda)^*$

$g.x \longmapsto gx$ $\quad\quad g.f \longmapsto gf : x \longmapsto -f(g(x))$

Then $L(\Lambda)^* = \prod_\lambda (L(\Lambda)_\lambda)^*$

· the subspace $L^*(\Lambda) := \bigoplus_\lambda (L(\Lambda)_\lambda)^*$ is submodule of the $g(A)$-module $L(\Lambda)^*$

· the module $L^*(\Lambda)$ is irreducible and for $v \in (L(\Lambda))^*_\Lambda$ one has
$n_-(v) = 0$, $h(v) = -\langle \Lambda, h \rangle v$ for $h \in H$.

· since $\dim L(\Lambda)_\lambda < \infty$, then $L(\Lambda) \xrightarrow{\cong} L^*(\Lambda)$.
$L(\Lambda)_\lambda \longmapsto (L(\Lambda)_\lambda)^*$

$\Rightarrow L^*(\Lambda)$ is irreducible. $\Rightarrow L^*(\Lambda) \in \mathcal{O}$.
and $n_-(v)(v_\Lambda) = -v(n_-(v_\Lambda)) = 0$ for $v_\Lambda \in L(\Lambda)_\Lambda$

· $h(v)(v_\Lambda) = -v(h(v_\Lambda)) = -\langle \Lambda, h \rangle v(v_\Lambda)$.
$\Rightarrow h(v) = -\langle \Lambda, h \rangle$.

· such a module is called an irreducible module with lowest weight $-\Lambda$.

· we have a bijective between H^* and irreducible lowest-weight modules: $\Lambda \longmapsto L^*(-\Lambda)$.

· Denote by π_Λ the action of $g(A)$ on $L(\Lambda)$, and the new action π^*_Λ on the space $L(\Lambda)$ by:

$$\pi^*_\Lambda(g) v = \pi_\Lambda(\omega(g)) v.$$

where ω is the chevalley involution of $g(A)$.

· It is clear that $(L(\Lambda), \pi^*_\Lambda)$ is an irreducible $g(A)$-module with lowest weight $-\Lambda$.

· the pairing between $L(\Lambda)$ and $L^*(\Lambda)$ gives us a nondegenerate bilinear form B on $L(\Lambda)$ s.t.

$$(9.4.2) \qquad B(g(x), y) = -B(x, \omega(g)(y)) \quad \text{for } g \in \mathfrak{g}(A),\ x, y \in L(\Lambda).$$

A bilinear form on $L(\Lambda)$ satisfies (9.4.2) is called a contravariant bilinear form.

Prop 9.4. Every $\mathfrak{g}(A)$-module $L(\Lambda)$ carries a unique up to constant factor nondegenerate contravariant bilinear form B, This form is symmetric and $L(\Lambda)$ decomposes into an orthogonal direct sum of weight space w.r.t this form.

proof: • The existence of B.

since $L(\Lambda) \cong L^*(\Lambda)$ as $\mathfrak{g}(A)$-module.

then we have $f: L(\Lambda) \to L^*(\Lambda)$ as $\mathfrak{g}(A)$-module isomorphism.

Define: $B(x, y) = f(x)(y)$ where $x, y \in L(\Lambda)$

$$B(g(x), y) = -B(x, \omega(g)(y)) \Leftrightarrow f(g(x))(y) = -f(x)(\omega(g)(y)).$$

for $g \in \mathfrak{g}(A)$, $x, y \in L(\Lambda)$.

since $f(g(x)) = g.f(x)$. $\Rightarrow f(g(x))(y) = g.f(x)(y) = -f(x)g(y)$.

& we have $\pi_\Lambda^*(g)v = \pi_\Lambda(\omega(g))v$.

$\Rightarrow -f(x)g.y = -f(x)(\omega(g).y))$.

• The uniqueness follows from Lem. 9.3.

since $L(\Lambda) \cong L^*(\Lambda)$, then $f = c I_{L(\Lambda)}$.

• $B(x, y) = c\, x^*(y) = c\, y^*(x) = B(y, x)$

$f(x)(y) = a f(y)(x)$.

$f(v_\Lambda)(v_\Lambda) = a f(v_\Lambda)(v_\Lambda)$

$$\begin{array}{ccc} L(\Lambda) & \longrightarrow & L^*(\Lambda) \\ -\Lambda & \longmapsto & -\Lambda. \\ v_\Lambda & \longmapsto & k v_\Lambda^* \end{array}$$

• The fact that $B(L(\Lambda)_\lambda, L(\Lambda)_\mu) = 0$, if $\lambda \neq \mu$.

follows from (9.4.2) for $g \in \mathfrak{H}$, $x \in L(\Lambda)_\lambda$, $y = L(\Lambda)_\mu$.

$[\ B(g(x), y) = B(\lambda x, y) = \lambda B(x, y) = -B(x, -g(y))$

$= B(x, \mu(y)) = \mu B(x, y)$.

$\Rightarrow B(x, y) = 0 \quad \Rightarrow B(L(\Lambda)_\lambda, L(\Lambda)_\mu) = 0$

$\lrcorner$.

• A more explicit way to introduce the contravariant bilinear form is the following:

Let V be a highest-weight $\mathfrak{g}(A)$-module with a fixed highest-weight vector v_Λ.

Given $v \in V$, we define its expectation value $\langle v \rangle \in \mathbb{C}$.

by:
$$v = \langle v \rangle v_\Lambda + \sum_{\alpha \in Q_+ \setminus \{0\}} v_{\Lambda-\alpha} \quad \text{where } v_{\Lambda-\alpha} \in V_{\Lambda-\alpha}.$$

Extend the negative Chevalley involution $-\omega$ to an anti-involution $\hat{\omega}$ of $U(g(A))$ $\hat{\omega}$ $-\hat{\omega}$

$\hat{\omega}(xy) = \hat{\omega}(y)\hat{\omega}(x)$.

Due to $U(g(A)) = U(n_-)U(H)U(n_+)$. we can see that

$$(9.4.3) \qquad \langle \hat{\omega}(a) v_\lambda \rangle = \langle a v_\lambda \rangle \quad , a \in U(g(A))$$

we follows that $\langle \hat{\omega}(a) a' v_\lambda \rangle$ is symmetric in $a, a' \in U(g(A))$

i.e. $\langle \hat{\omega}(a) a' v_\lambda \rangle = \langle \hat{\omega}(a') a v_\lambda \rangle$.

$$\begin{matrix} \langle \hat{\omega}(a) a' v_\lambda \rangle & = & \langle \hat{\omega}(a') a v_\lambda \rangle \\ \| & & \| \\ \langle a a' v_\lambda \rangle & & \langle a' a v_\lambda \rangle \end{matrix}$$

hence the formula: $\underline{B(a v_\lambda, a' v_\lambda) = \langle \hat{\omega}(a) a' v_\lambda \rangle}$. gives

a well-defined symmetric bilinear form on V. which is contravariant and normalized by $B(v_\lambda, v_\lambda) = 1$.

· contravariant: i.e. $B(g(x), y) = B(x, \hat{\omega}(g)(y))$.

$$\begin{matrix} B(g(x), y) & & = & B(x, \hat{\omega}(g)(y)) \\ \| & & & \| \\ \langle \hat{\omega}(g(x)) y \rangle & & & \langle \hat{\omega}(x) \hat{\omega}(g)(y) \rangle \\ \| & & & \| \\ \langle g(x) y \rangle & & = & \langle \hat{\omega}(g(x)) y \rangle \end{matrix}$$

§9.5. complete reducibility lemma.

lemma 9.5: let V be a $g(A)$-module from the category $\mathcal{O}$. if for any two primitive weights λ and μ of V the inequality $\lambda \geq \mu$ implies $\lambda = \mu$. then the module V is completely reducible (i.e. V decomposes into a direct sum of irreducible modules)

proof: · set $V^0 = \{ v \in V \mid n_+(v) = 0 \}$.

this is H-invariant. hence we have the weight space decomposition $V^0 = \bigoplus_{\lambda \in L} V^0_\lambda$. where all elements from L are primitive weights.

let $\lambda \in L$ and $v \in V^0_\lambda$, $v \neq 0$, Then $g(A)$-module $U(g)(v)$ is irreducible by prop 9.3 b). $U(g)(v) \cong L(\lambda)$.

we have $U(n_-) v \cap V^0_\mu \neq 0$ for some $\mu < \lambda$. contradiction.

· the $g(A)$-submodule V' of V generated by V^0. is completely reducible.

· claim. $V' = V$.

if this is not the case, we consider the $g(A)$-module V/V', Then there is a weight vector $v \in V$ of weight

μ s.t. $v \notin V'$ but $e_i(v) \in V'$ and $e_i(v) \neq 0$ for some i.

But since $V \in \mathcal{O}$, there is $\lambda \in L$ s.t. $\lambda \geq \mu + \alpha_i$ and $\lambda > \mu$, which is contradicts the assumption of the lem.

#

§9.6. Substitute for composition series.

Lem 9.6: Let $V \in \mathcal{O}$ and $\lambda \in \mathfrak{H}^*$. Then there is a filtration by a sequence of submodules

$$V = V_t \supset V_{t-1} \supset \cdots \supset V_1 \supset V_0 = 0$$

and a subset $J \subset \{1, \cdots, t\}$, s.t.

(i). If $j \in J$, then $V_j / V_{j-1} \simeq L(\lambda_j)$ for some $\lambda_j \geq \lambda$.

(ii) If $j \notin J$, then $(V_j / V_{j-1})_\mu = 0$ for every $\mu \geq \lambda$.

proof. Let $a(V, \lambda) = \sum_{\mu \geq \lambda} \dim V_\mu$. We prove the lemma by induction on $a(V, \lambda)$.

· If $a(V, \lambda) = 0$, then $0 = V_0 \subset V_1 = V$ is the required filtration with $J = \phi$.

· Let $a(V, \lambda) > 0$. 某个weight [illegible] weight [illegible]

Choose a maximal element $\mu \in P(V)$ s.t. $\mu \geq \lambda$.
Choose a weight vector $v \in V_\mu$, and let $U = U(\mathfrak{g}) v$.
Clearly, U is a highest-weight module.

By prop 9.2 c) implies that U contains a maximal proper submodule $\bar{U}$.

[Since $\varphi : M(\mu) \twoheadrightarrow U$, i.e. $U \cong M(\mu) / \ker \varphi$.

since: $\underbrace{M(\mu) / M'(\mu)}$ $\cong$ $\underbrace{U / \bar{U}}$. $\Rightarrow M'(\mu) / \ker \varphi \cong \bar{U}$.]

We have $0 \subset \bar{U} \subset U \subset V$, $U / \bar{U} \simeq L(\mu)$, $\mu \geq \lambda$.

Since $a(\bar{U}, \lambda) < a(V, \lambda)$ and $a(V/U, \lambda) < a(V, \lambda)$. We induction to get a suitable filtration for $\bar{U}$ and V/U

$U_0 = 0 \subset U_1 \subset \cdots \subset U_s = \bar{U}$.

$0 = V/U \subset V_1/U \subset \cdots \subset V_t/U = V/U$. ; $\dfrac{V_j/U}{V_{j-1}/U} \cong V_j / V_{j-1}$.

$\Rightarrow 0 \subset U_1 \subset \cdots \subset U_{s-1} \subset \bar{U} \subset V_1 \subset \cdots \subset V_t = V$.

#

· Let $V \in \mathcal{O}$ and $\mu \in \mathfrak{H}^*$. Fix $\lambda \in \mathfrak{H}^*$ s.t. $\mu \geq \lambda$, and construct a filtration given by Lem. 9.6.

Def: Denote by $[V : L(\mu)]$ the number of times μ appears among $\{\lambda_j \mid j \in J\}$, this number is called the multiplicity of $L(\mu)$ in V.

Rem: (1) $[V : L(\mu)]$ is independent of the filtration furnished by Lem 9.6 and of the choice of λ.

(2). $L(\mu)$ has a nonzero multiplicity in V iff μ is a primitive weight of V.

proof: "$\Rightarrow$" If $[V : L(\mu)] \neq 0$, there is $j \in J$.

s.t. $V_j / V_{j-1} \cong L(\mu)$.

Let $v + V_{j-1}$ is the weight vector of μ in V_j/V_{j-1} then v is primitive and μ is a primitive weight:

$$\Big[\ \varphi : V_j / V_{j-1} \longrightarrow L(\mu), \quad \varphi(n_+(v + V_{j-1})) = n_+ \varphi(v + V_{j-1}) = \varphi(V_{j-1})\ \Big]$$

"$\Leftarrow$" If μ is a primitive weight, then there is $v \in V_\mu$ and a submodule U of V, s.t. $v \notin U$, $n_+(v) \subset U$.

hence $v + U \in (V/U)_\mu$ and $U(g(A))(v+U) = U(n_-)(v+U)$ is a highest-weight module with highest weight μ.

Consider a filtration for V/U & U satisfying Lem 9.6. Combining them we get the required filtration of V. and $L(\mu)$ is factor module.

Rem (1): $[V : L(\mu)]$ is independent of filtration.

#

. Rem: (1) $[V : L(\mu)]$ is independent of the filtration furnished by Lem 9.6 and of the choice of λ.

proof: We first observe that a filtration with respect to λ is also a filtration with respect to μ when $\mu > \lambda$. Also the multiplicity of $L(\mu)$ in such a filtration is the same whether it is regarded as a filtration with respect to λ or μ.

- Thus it will be sufficient to take two filtrations with respect to μ and show that $L(\mu)$ has the same multiplicity in each.

The following variant of the proof of the Jordan–Hölder theorem achieves this.

let
$$V = V_0 \supset V_1 \supset \cdots \supset V_{l_1} = 0 \qquad (1)$$
$$V = V'_0 \supset V'_1 \supset \cdots \supset V'_{l_2} = 0 \qquad (2)$$
be two such filtrations of lengths l_1, l_2.

- We shall use induction on $\min(l_1, l_2)$
- Suppose $\min(l_1, l_2) = 1$. Then either V is irreducible and the two filtrations are identical, or μ is not a weight of V and $L(\mu)$ does not appear in either filtration (by lem 9.6.)

Thus suppose $\min(l_1, l_2) > 1$ ① we suppose first that $V_1 = V'_1$. We then consider the two filtrations of V_1
$$V_1 \supset \cdots \supset V_{l_1} = 0 \quad ; \quad V'_1 \supset \cdots \supset V'_{l_2} = 0$$
By induction they give the same multiplicity for $L(\mu)$, and the filtrations for V are obtained by adding the additional factor V/V_1 which is the same for both. (by induction)

② We may therefore suppose that $V_1 \neq V'_1$, suppose first that $V_1 \subset V'_1$. Then V/V_1 is not irreducible and so μ is not a weight of V/V_1, Thus neither V/V_1 or V/V'_1 is isomorphic to $L(\mu)$.

let $V_1 \supset U_1 \supset \cdots \supset U_m = 0$ be a filtration of V_1 of the required type with respect to μ, we the consider the filtrations
$$V \supset V_1 \supset U_1 \supset \cdots \supset U_m = 0 \qquad (3)$$
$$V \supset V'_1 \supset V_1 \supset U_1 \supset \cdots \supset U_m = 0 \qquad (4)$$
These are filtrations of V of the required type with respect to μ.

$L(\mu)$ has the same multiplicity in filtrations (1) & (3) since they have the same leading term V_1. Similarly $L(\mu)$ has the same multiplicity in filtration (2) & (4) (by induction) so $L(\mu)$ has the same multiplicity in filtration (3) & (4) since none of V/V_1, V/V'_1, V'_1/V_1 is isomorphic to $L(\mu)$, Thus $L(\mu)$ has the same multiplicity in filtrations (1) & (2) as required.

③ We may therefor assume that neither of V_1, V'_1 is contained in the other. let $U = V_1 \cap V'_1$ and choose a filtration of U of the required kind with respect to μ. This has form $U \supset U_1 \supset \cdots \supset U_m = 0$

We then consider the filtrations
$$V \supset V_1 \supset U \supset U_1 \supset \cdots \supset U_m = 0 \qquad (5)$$
$$V \supset V'_1 \supset U \supset U_1 \supset \cdots \supset U_m = 0 \qquad (6)$$

(back side)

These are filtrations of ϑ of the required type w.r.t. μ. This is clear since: $\vartheta_1'/V \cong (\vartheta_1+\vartheta_1')/\vartheta_1$, $\vartheta_1/V \cong (\vartheta_1+\vartheta_1')/\vartheta_1'$

Now $L(\mu)$ has the same multiplicity in filtration (1) & (5) and the same multiplicity in filtrations (2) & (6) since the leading terms are the same. It is therefore sufficient to show that $L(\mu)$ has the same multiplicity in (5) & (6). These filtrations differ only in the first two factors.

If $\vartheta_1+\vartheta_1' = \vartheta$ then we have: $\vartheta/\vartheta_1 \cong \vartheta_1'/V$. $\vartheta/\vartheta_1' \cong \vartheta_1/V$ as required.

If $\vartheta_1+\vartheta_1' \neq \vartheta$, then ϑ/ϑ_1 and ϑ/ϑ_1' are not irreducible. In this case μ is not a weight of ϑ/ϑ_1 or ϑ/ϑ_1', so is not a weight of ϑ_1'/V. Thus none of ϑ/ϑ_1, ϑ_1/V, ϑ/ϑ_1', ϑ_1'/V is isomorphism to $L(\mu)$.

This completes the proof

$\blacksquare$