$\nu < \mu \; (\Lambda - \beta$

Prop 9.10. Let A be a symmetrizable matrix, possible infinite

(a) If $2\langle \Lambda+\rho, \nu^{-1}(\beta)\rangle \neq (\beta|\beta)$ for every $\beta \in Q_+ \setminus \{0\}$, then $\mathfrak{g}'(A)$-module $M(\lambda)$ is irreducible.

$(\Omega_0 - aI_V)^k(v)=0$

$\nu \longrightarrow \lambda$ $\qquad \Omega_0(v) = (2\Lambda+\rho \mid \Lambda)v \qquad v \in V_\lambda$

(b). Let V be a $\mathfrak{g}'(A)$-module s.t. the following three conditions are satisfied:

(i) for every $v \in V$, $e_i(v)=0$ for all but finite number of the e_i

restricted

(ii) for every $v \in V$, there exists $k>0$. Such that $e_{i_1}e_{i_2}\cdots e_{i_s}(v)=0$ whenever $s>k$

(iii) $V = \bigoplus_{\lambda \in (\mathfrak{h}')^*} V_\lambda \qquad V_\lambda = \{v \in V \mid h(v) = \langle \lambda, h\rangle v$ for all $h \in \mathfrak{h}'\}$

(iv) If λ and $\mu \in (\mathfrak{h}')^*$ are primitive weights such that $\lambda - \mu = \beta|_{\mathfrak{h}'}$ for some $\beta \in Q_+ \setminus \{0\}$, then $2\langle \lambda+\rho, \nu^{-1}(\beta)\rangle \neq (\beta|\beta)$

Then V is completely reducible. ie is isomorphic to a direct sum of $\mathfrak{g}'(A)$ module of the form $L(\Lambda)$, $\Lambda \in (\mathfrak{h}')^*$

$\Omega_0 = \sum e_{-\alpha} e_\alpha(v) \qquad v$

$v \quad n_+(v)=0 \qquad \Omega_0(v)=0 \qquad \Rightarrow a=0 \qquad \Rightarrow a=0$

Pf: Let $v \in V_\lambda$ be such that $(\Omega_0 - aI_V)^k(v)=0$ for some $k \in \mathbb{Z}_+$ and $a \in \mathbb{C}$, and let $\tilde{v} \in U'_{-\beta}(v)$ $(\beta \in Q_+)$

then we have $\qquad U'_{-\beta}(v)$

(9.10.2) $\left(\Omega_0 - (a + 2\langle \lambda+\rho, \nu^{-1}(\beta)\rangle - (\beta|\beta))I_V\right)^k(\tilde{v}) = 0 \qquad \Omega_0 L\{(v)\}$

Fact: If v is a weight vector with λ,

$\Omega_0(v) = 0 \Rightarrow a = 0$

$u = U'_{-\beta}(v)$, i.e $\exists\, \xi \in U'_{-\beta}$ let $u = \xi(v)$

$\Omega_0(u) = 0$

$\Rightarrow 2\langle \lambda + \rho, v'(\beta)\rangle = (\beta|\beta)$

$n_+(u) = 0$ $\quad u = \xi(v)$ $\quad \Omega_0(v) = 0 \Rightarrow a = 0$

$\Omega_0(u) = 0$

To prove (a): Suppose that J is a proper submodule of $M(\lambda)$ (nonzero)

take a nonzero element $v' = \sum_i v_i \in J$

where $v_i \in U_{-\beta_i}(v)$, v is a highest ~~weight ve~~ vector of $M(\lambda)$

$U(g'(A))_{-\beta_i}$ $\quad v' \to \lambda - \beta_i$

and $\beta_i \in Q_+ \setminus \{0\}$ s.t $\sum_i ht\,\beta_i$ is minimal. $\quad \beta_i = 0$

$\Rightarrow n_+(v_i) = 0$ and $n_+(v') = 0$ $\quad \Omega_0(v') = 0$ $\quad v \longleftrightarrow \lambda$

Let $V = M(\lambda)$, $v' = V_i$, then $M(\lambda)$ satisfy (b) (i) (ii)

We can use (9.10.2) $\quad v' = U_{-\beta_i}(v)$ $\quad \Omega_0(v') = 0$

Since $\Omega_0(v) = 0$ $\quad (\Omega_0 - aI_V)^k(v) = 0 \Rightarrow a = 0$ $\quad U_{-\beta}(v)$

$\quad b \quad \Omega_0(v) \quad \Omega_0(v')^k = 0$

Since $(\Omega_0 - (\langle 2(\lambda + \rho, v'(\beta_i)\rangle - (\beta_i|\beta_i)) I_V)v' = 0$

$\Omega_0(v') = 0$ $\quad J = M(\lambda)$ $\quad \beta_i \in Q_+ \setminus \{0\}$

v_i $\quad \Rightarrow 2\langle \lambda + \rho, v'(\beta_i)\rangle = (\beta_i|\beta_i)$ $\quad$ { our assumption $\quad v \to \lambda$

$\Rightarrow$ ~~$J = M(\lambda)$~~ $\quad v' \quad \lambda - \beta_i \quad \beta_i = 0$ $\quad e_i \in g_{\beta_i} \quad \beta_i \in Q$

$\forall v \in V$

(b) By (ii) $\exists k \in \mathbb{Z}^+$ s.t $e_{i_1} e_{i_2} \cdots e_{i_s}(v) = 0$

~~where~~ whenever $s > k$

$\Rightarrow x_1 \cdots x_p(v) = 0$ for $x_i \in g_{\beta_i}$ $\sum ht\,\beta_i > k$

$\beta_i \in Q_+$

$\Rightarrow$ Ω_0 is locally finite on V by (i) (ii).

$V = \bigoplus_{a \in \mathbb{C}} V^a$ $\quad V^a = \{v \in V \mid (\Omega_0 - a I_V)^k (v) = 0\}$

put $V^0 = \{v \in V \mid n_+(v) = 0\}$ $\quad v \in V^0$

Then by (iii) $V^0 = \bigoplus_{\lambda \in (\mathfrak{h}')^*} V_\lambda^0$, where $V_\lambda^0 = V^0 \cap V_\lambda$

for every $v \in V_\lambda^0$, then $M_v = U(\mathfrak{g}'(A))(v)$

M_v, $\exists k \longrightarrow \lambda + (iv)$

M_v, by the proof of (a) $\quad \{v \in V \mid n^+(v) = 0\}$

$\parallel$

Let V' be the $\mathfrak{g}'(A)$-module generated by V^0

then By the above argument, V' is completly reducible.

It suffice to prove $V = V'$

suppose to the contrary, then $v \in V \setminus V'$

i.e there exist $v \in V_\lambda^a \setminus V'$ for some $a \in \mathbb{C}$, $\lambda \in (\mathfrak{h}')^*$. s.t $n_+(v) \subset V'$ (i.e λ is primitive weight)

and $\Omega_0(v) \subset V' \Rightarrow v \in V^0 \Rightarrow a = 0$

$V' \leftarrow V^0$

there exist some $\beta \in Q_+ \setminus \{0\}$ and $u \in U'_\beta$ s.t

$u(v) \neq 0$ (by (ii)) $n_+(u(v)) = 0$

Since $\sqrt{2}_0(u(v)) = (2\langle\lambda+\rho, v^{-1}(\beta)\rangle - (\beta|\beta))\, u(v)$
$\qquad\qquad\; \overset{\shortparallel}{0}$

$\Rightarrow 2\langle\lambda+\rho, v^{-1}(\beta)\rangle \neq (\beta|\beta)$ { (iv)

$\Rightarrow V = V'$

$V' = \sum V_i$ $\qquad$ $U(g(A))(v) \to Q_+$ - graded

$b_{ij} = a_{ij}/\varepsilon_i$

Cor 9.10

Let A be a symmetric $\Rightarrow$ symmetrizable matrix with non positive real entries and Let V be a $g'(A)$-module Satisfying (i) (ii) (iii) of prop 9.10 (b)

Suppose that for every wight λ of V on

☆ $\lambda(\alpha_i^\vee) \geq 0$ for all i

Then V is a direct sum of irreducible $g'(A)$-modules which are free of rank 1 when viewed as $U(n_-)$-mod

Pf: For $\beta = \sum_i k_i\alpha_i \in Q_+\setminus\{0\}$ and a weight λ of V
$\lambda, \lambda-\beta$

We have: $\qquad \alpha_i = \alpha_i^\vee \qquad v^{-1}(\beta) = \sum_i k_i\alpha_i$

$2\langle\lambda+\rho, v^{-1}(\beta)\rangle - (\beta|\beta) \qquad \rho = \frac{1}{2}\sum_i \alpha_i$

$= 2\langle\lambda+\rho, \sum_i k_i\alpha_i\rangle - (\sum_i k_i\alpha_i | \sum_i k_i\alpha_i)$

$= 2\sum_i k_i\lambda(\alpha_i^\vee) - \sum_{i\neq j} a_{ij}k_ik_j - \sum_i a_{ii}k_i^2 + \sum_i a_{ii}k_i$

$= 2\sum_i \underbrace{k_i}_{>0} \underbrace{\lambda(\alpha_i^\vee)}_{>0} - \sum_{i\neq j} \underbrace{a_{ij}}_{<0} k_i k_j - \sum_i \underbrace{a_{ii}}_{<0} \underbrace{(k_i^2 - k_i)}_{>0} > 0$

i.e $2(\lambda+\rho)(\bar{v}^{-1}\beta) \neq (\beta|\beta)$ for any $\beta = \sum_i k_i\alpha_i$

By prop 9.10 (b) ∇ is complety reducible.

and ∇ is a direct sum of $g'(A)$-mod $L(\lambda)$

by prop 9.10 (a) $M(\lambda)$ is irreducible.

$\underset{L(\Lambda)}{\overset{\cong}{\longrightarrow}}$

Λ

$L(i) \Rightarrow$ restited

(2.6.1) $\Rightarrow$ Fact

$u \in V_\lambda \quad (\Omega_0 - a I_V)^k = 0$ for $k \in \mathbb{Z}_+$ $a \in \mathbb{C}$

$v^i \in U_{-\beta}(u)$ the (see 3.4.1)

$(\Omega_0 - (a + \qquad))^k (v) = 0$

§9.11.

- $A \longrightarrow$ symmetrizable Cartan matrix
- $(\cdot|\cdot) \longrightarrow$ Them 2.2
- $\tilde{g}(A)$ and $g(A) = \tilde{g}(A)/\tau$ and $\tau = \tau_- \oplus \tau_+$
- Set $\tau_\alpha = \tilde{g}_\alpha \cap \tau$

aim: determin τ

Prop 9.11 (利用 Verma-模)

The ideal τ_+ (resp: τ_-) is generated as ideal in $\tilde{n}_+$ (resp $\tilde{n}_-$) by ~~the~~ theose τ_α (resp $\tau_{-\alpha}$)

For which $\alpha \in Q_+ \setminus \Pi$ and $2(\rho|\alpha) = (\alpha|\alpha)$

\Pf: For $\lambda \in \mathfrak{h}^*$, we define a Verma module $\widehat{M}(\lambda)$ over $\hat{\mathfrak{g}}(A)$ by $U(\hat{\mathfrak{g}}(A))/\hat{J}$, with the highest weight λ.

$$\hat{J} = \langle \hat{n}_+ , \underset{h \in \mathfrak{h}^*}{h - \lambda(h)} \rangle$$

Let $\widehat{M}'(\lambda)$ be the unique (proper) maximal submodule of $\widehat{M}(\lambda)$, Then we have an isomorphism of $\hat{\mathfrak{g}}(A)$-modules

$$(9.11.1) \qquad \widehat{M}'(0) \simeq \hat{\bigoplus_{i=1}^{n}} \widehat{M}(-\alpha_i)$$

the maximal submodule of $\widehat{M}(0)$, since $\hat{n}_-$ is free Lie algebra generated by $\{f_1 \cdots f_n\}$ (Thm 1.2 (b))

$U(\hat{n}_-)$ is freely asso. alge. generated by $\{f_1 \cdots f_n\}$

the $\widehat{M}'(0) = \bigoplus_{i=1}^{n} U(\hat{n}_-)(f_i + \hat{J}) \simeq \bigoplus_{i=1}^{n} \widehat{M}(-\alpha_i)$

$\supseteq \checkmark$ $\quad -\alpha_i$

The isomorphism of $\hat{\mathfrak{g}}(A)$-modues $\quad U(\hat{\mathfrak{g}}(A))$

~~#~~ Tensoring with $U(\mathfrak{g}) \otimes_{U(\hat{\mathfrak{g}})} \longrightarrow (9.11.1)$

$$(9.11.2) \longrightarrow U(\mathfrak{g}) \otimes_{U(\hat{\mathfrak{g}})} \widehat{M}'(0) \cong U(\mathfrak{g}) \otimes_{U(\hat{\mathfrak{g}})} \widehat{M}(-\alpha_i)$$

$$\cong \bigoplus_{i=1}^{n} M(-\alpha_i) \quad U(\mathfrak{g})$$

$\widehat{M}(-\alpha_i) / \cancel{\otimes}\ I\widehat{M}(-\alpha_i) \cong M(-\alpha_i)$

$I_- + \hat{J} \subset \hat{n}_-(\alpha)$

$\pi: \hat{\mathfrak{g}}(A) \longrightarrow \mathfrak{g}(A)$ $\qquad a \qquad 1 \otimes (a + \hat{J})$

We define a map: $\lambda_1 : I_- \longrightarrow U(\mathfrak{g}) \otimes_{U(\hat{\mathfrak{g}})} \widehat{M}'(0)$ by

$$\lambda_1(a) = 1 \otimes a(\hat{v}), \quad a \in \mathcal{I}_-$$

where $\hat{v}$ is a highest ~~weight~~ weight vector of $\hat{M}'(0)$

$$a \in \mathcal{I}_- \to a(\hat{v}) \in \hat{M}'(0)$$

This is a $\hat{g}$-module homomorphism:

Indeed, for $x \in \hat{g}$, $a \in \mathcal{I}_-$, we have

$$\lambda_1([x, a]) = 1 \otimes_{U(\hat{g})} \underbrace{(xa}_{\mathcal{I}} - ax)(\hat{v}) \qquad \pi : \hat{g} \xrightarrow{\mathcal{I}} g$$

$$= \pi(x) \otimes a(\hat{v}) + \underbrace{\pi(a)}_{=0} \otimes x(\hat{v})$$

$$= \pi(x) \otimes a(\hat{v})$$

$$= x(\lambda(\hat{v}))$$

$\lambda_1([\mathcal{I}_-, \mathcal{I}_-]) = 0$, So that we have a $g(A)$-mod homomorphism $\Leftarrow$ λ_1 induced.

$$\lambda : \mathcal{I}_- / [\mathcal{I}_-, \mathcal{I}_-] \longrightarrow \bigoplus_{i=1}^{n} M(-\alpha_i)$$

by (9.11.2)

Let v_i be a ~~high~~ highest-weight vector of $M(-\alpha_i)$

Any $a \in \mathcal{I}_-$ can be uniquely written ~~by~~ as

$$a = \sum_{i=1}^{n} u_i f_i, \quad u_i \in U_0(\hat{n}_-)$$

[$a \in \mathcal{I}^-$, $a \neq f_i$ $\quad U(\hat{n}_-) = \mathbb{C}1 + \underbrace{\hat{n}_- U(\hat{n}_-)}_{= U_0(\hat{n}_-)}$

$a \in \mathcal{I}^-$, $\quad a = \sum U_0(\hat{n}_-) \cdot f_i$]

Then $\lambda[a + [\mathcal{I}_-, \mathcal{I}_-]] = \sum_i \pi_i(u_i) v_i$

claim: λ is injective $\Leftarrow$

Indeed $\lambda(a + [\mathcal{I}_-, \mathcal{I}_-]) = 0$

$\Rightarrow \pi_i(u_i) = 0$ for all i, hence $u_i \in \mathcal{I}_- U_0(\hat{n}_-)$

and $a \in \mathfrak{t}_- U_0(\mathfrak{n}_-) \cap \mathfrak{t}_- = [\mathfrak{t}_-, \mathfrak{t}_-]$

1. $\mathfrak{g} \cap (U_0(\mathfrak{g})^2) = [\mathfrak{g}, \mathfrak{g}]$ ✓

2. $\mathfrak{t} \cap \mathfrak{t}\, U_0(\mathfrak{g}) = [\mathfrak{t}, \mathfrak{t}]$ ✓

$\supseteq$ ✓ $\quad \subseteq$?

Indeed. 1. "$\supseteq$" since $\mathfrak{g} \subset U_0(\mathfrak{g})$ and $[x\, y] = xy - yx$ in $U(\mathfrak{g})$

$\Rightarrow [\mathfrak{g}, \mathfrak{g}] \subset [U_0(\mathfrak{g})\ U_0(\mathfrak{g})] \subset (U_0(\mathfrak{g}))^2$

$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$

$\Rightarrow [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \cap (U_0(\mathfrak{g})^2)$

"$\subseteq$" Let $\bar{\mathfrak{g}} = \mathfrak{g} / [\mathfrak{g}, \mathfrak{g}]$

$\pi : \mathfrak{g} \to \bar{\mathfrak{g}}$

we have a natural homo: $U(\mathfrak{g}) \longrightarrow U(\bar{\mathfrak{g}})$

which $\mathfrak{g} \cap U_0(\mathfrak{g})^2$ maps to $\bar{\mathfrak{g}} \cap (U_0(\bar{\mathfrak{g}})^2)$

since $\bar{\mathfrak{g}}$ is abelian $\Rightarrow U(\bar{\mathfrak{g}})$ is a polynomial alge

in such alge $\bar{\mathfrak{g}} \cap (U_0(\bar{\mathfrak{g}})^2) = 0$

$\Rightarrow \mathfrak{g} \cap (U_0(\mathfrak{g})^2) \subset \ker \pi$

$\Rightarrow \mathfrak{g} \cap U_0(\mathfrak{g})^2 \subset [\mathfrak{g}, \mathfrak{g}]$

Prove that $[\mathfrak{t}, \mathfrak{t}] \supseteq \mathfrak{t} \cap \mathfrak{t}\, U_0(\mathfrak{g})$

Let $\{v_i\}$ be a basis of $\mathfrak{t}$ and extend it to a basis $\{v_i, u_j\}$ of $\mathfrak{g}$. using PBW Thm

$\prod_{i,j} v_i^{m_i} u_j^{n_j}$ with $(\sum m_i + \sum n_j > 0) \xrightarrow{\text{basis of}} U_0(\mathfrak{g})$

$\cancel{\mathfrak{t}}$

$\prod_{i,j}$ ~~ with $(\sum m_i + \sum n_j) \geq 2$ $\qquad \mathfrak{t}\, U_0(\mathfrak{g})$

$\sum m_i \geq 1$

the each elements of $\mathcal{I} \cap (\mathcal{I}\, U_0(g))$ is a linear combination of monomial: $\prod_{i,j} r_i^{m_i} u_j^{n_j}$

with $\sum m_i \geq 2$, $\sum n_j = 0$

hence $\mathcal{I} \cap \mathcal{I}\, U_0(g) \subset \mathcal{I} \cap (U_0(\mathcal{I}))^2 = [\mathcal{I}, \mathcal{I}]$

$$\lambda : \mathcal{I}_- / [\mathcal{I}_-, \mathcal{I}_-] \longrightarrow \bigoplus_i M(-\alpha_i)$$

is an embeding in the category $\mathcal{O}$ of $g(A)$-mod.

Now, Let $-\alpha$ $(\alpha \in Q)$ be a primitive weight of the $g(A)$-module of $\mathcal{I}_- / [\mathcal{I}_-, \mathcal{I}_-]$

Note $\alpha \notin \Pi$, since no f_i lies in $\mathcal{I}$.

using the embeding, We conclude by Lemma 9.8

$$\left(\begin{array}{l} -\alpha \longrightarrow \text{primitive weight} \\ -\alpha_i \longrightarrow \text{highest weight} \end{array}\right)$$

$\Rightarrow |-\alpha_i + \rho|^2 = |-\alpha + \rho|^2$ (1) for some i

and since $2(\rho|\alpha_i) = (\alpha_i|\alpha_i)$ (2)

(1)+(2)

$\Rightarrow (\alpha_i|\alpha_i) - 2(\alpha_i|\rho) + (\rho|\rho) = (\alpha|\alpha) - 2(\rho|\alpha) + (\rho|\rho)$

Appling remark 9.3 A-module V $\in \mathcal{O}$ is generated by it's primitive vector

$\Rightarrow (\alpha|\alpha) = 2(\rho|\alpha)$

$\Rightarrow \mathcal{I}_- / [\mathcal{I}_-, \mathcal{I}_-] \longrightarrow \mathcal{I}_\alpha$, $\alpha \in Q_+ \setminus \Pi$

$\downarrow \mathcal{I}_-$

$2(\rho|\alpha) = (\alpha|\alpha)$

enter n as $\mathfrak{n}$-module

$\Rightarrow$ Let P be the $\tilde{n}_-$-submodule generated by such $\mathfrak{t}_{-\alpha}$

then $\mathfrak{t}_- / [\mathfrak{t}_- \ \mathfrak{t}_-] = P$

$\Rightarrow \mathfrak{t}_- = [\mathfrak{t}_- \ \mathfrak{t}_-] + P$

Suppose $P \neq \mathfrak{t}_-$, then $\mathfrak{t}_-/P$ is an $\tilde{n}_-$ module

Cosider $[\mathfrak{t}_-/P, \ \mathfrak{t}_-/P]$ of $\mathfrak{t}_-/P$

this is an $\tilde{n}_-$-module weight are of the form $\beta+\gamma$ and β, γ are weigh of $\mathfrak{t}_-/P$

Thus. If α is a weight of $\mathfrak{t}_-/P$ for which $|ht\,\alpha|$ minimal, then α can't be a weight of $[\mathfrak{t}/P \ \ \mathfrak{t}_-/P] \Rightarrow [\mathfrak{t}_-/P, \mathfrak{t}_-/P] \neq \mathfrak{t}_-/P$ ~~FG~~

~~$[\mathfrak{t}_- \ \mathfrak{t}_-]$~~

$$[\mathfrak{t}_-/P, \ \mathfrak{t}_-/P]$$
$$= [\mathfrak{t}_-, \mathfrak{t}_-] + P \neq \mathfrak{t}_- + P$$

$$\Rightarrow [\mathfrak{t}_-, \mathfrak{t}_-] + P \neq \mathfrak{t}_-$$

$$\Rightarrow P = \mathfrak{t}_-$$

$\mathfrak{t}_{-\alpha}$. $\alpha \in Q_+ \setminus \pi$ $\quad 2(\rho|\alpha) = (\alpha|\alpha)$

$T_+ \to \tilde{W}$

Thm 9.11. (3.3.1)

Let $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau$ be a Kac-Moody alge with symmetrizable Cartan matrix A. Then the elements

(9.11.5) $(\operatorname{ad} e_i)^{1-a_{ij}} e_j \quad (i \neq j) \quad (i,j = 1 \cdots n)$

(9.11.6) $(\operatorname{ad} f_i)^{1-a_{ij}} f_j \quad (i \neq j) \quad (i,j = 1 \cdots n)$

generated the ideals τ_+ and τ_- respectively.

$= 0$

(3.3.1)

Pf: Denote $\mathfrak{g}_1(A) = \tilde{\mathfrak{g}}(A) / \langle 9.11.5 + 9.11.6 \rangle$

these relation hold by (3.3.1), we have $\tilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$, $\tilde{\mathfrak{g}}(A) \to \mathfrak{g}_1(A) \to \mathfrak{g}(A)$

We have induced Q-gradation $\mathfrak{g}_1(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}'_\alpha$

Let τ'_+ denote the image of τ_+ in $\mathfrak{g}_1(A)$

Suppose $\tau'_+ \neq 0$, choose a root α of minimal height among the $\alpha \in Q_+ \setminus \{0\}$ s.t $\tau'_+ \cap \mathfrak{g}'_\alpha \neq 0$

and Let $\alpha = \sum_i k_i \alpha_i$

$\tau'_+ \cap \mathfrak{g}'_\alpha = (\tau'_+)_\alpha$

It is clear that $(\tau'_+)_\alpha$ must occur in any system of homogenous generators of τ'_+

$(\tau_+)_\alpha$ occur in ----- generators of τ_+ as an ideal ~~of n_+~~

$$(\alpha|\alpha) = 2(\rho|\alpha) \quad , \alpha \in Q_+ \setminus \tau_i$$

Kac-Moody.

We know that the weyl group W act on the weight of $\mathfrak{g} = \tilde{\mathfrak{g}}/\tau$ and in the same

w – orbit have the same multiplicity

The same argument can be applied to $g_1(A)$ to give a similar result.

(the proofs in §3.8, used only the relation 3.3.1)

Since $\dim g_1(A)_\alpha = \dim g_\alpha + \dim T_\alpha'$

$\tilde{r}_i(T_\alpha') = T'_{r_i(\alpha)}$

$(T'_+)_\alpha \neq 0$

$(\hat{r}_i(T_\alpha')) = T'_{r_i(\alpha)} \neq 0$

for any i

Note $|ht(r_i(\alpha))| \geq |ht(\alpha)|$

$(\alpha_i|\alpha) \leq 0 \Rightarrow (\alpha|\alpha) \leq 0$

But $2(\rho|\alpha) > 0$ $\}$

$(\rho|\alpha_i) = \frac{1}{2}(\alpha_i|\alpha_i)$

$= 2(\rho|\alpha) =$

$= k_i \sum_i (\alpha_i|\alpha_i)$

$\rho = \frac{1}{2}\Sigma$ P93 $\frac{7}{2}$