Chapter 10. Integrable Highest-weight Modules: the Character Formula

☆ Character formula for an integrable h.w.m $L(\Lambda)$ over a Kac-Moody algebra.

§10.1

GCM

$\mathfrak{g}(A) \to$ Kac-Moody algebra of rank n

$\mathfrak{h} \to$ be its Cartan subalgebra. $\to$ integral weight

Let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z} \ (i=1\cdots n)\}$

$\cup|$ $\to$ dominant integral ...

$P_+ = \{\lambda \in P \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \ (i=1\cdots n)\}$

$P_{++} = \{\lambda \in P \mid \lambda(\alpha_i^\vee) > 0 \ (i=1\cdots n)\}$

$\hookrightarrow$ regular dominant integral weight

The set P is called the weight lattice

Note that P contains the root lattice Q

☆ $\alpha_j(\alpha_i^\vee) = a_{ij} \in \mathbb{Z} \quad (i=1\cdots n)$

$Q \subseteq P$

recall: (defi): An $\mathfrak{h}$-diagonalizable module V over Kac-Moody alge $\mathfrak{g}(A)$ is called integrable if all e_i and f_i are locally nilpotent on V

Let V be a h.w.m over $\mathfrak{g}(A)$, and v a highest-weight vector. It follow from Lemmas 3.4(b), 3.5 $\Longrightarrow$

★ The module V is integrable $\Leftrightarrow$ if $f_i^{N_i}(v_\lambda)=0$. for some $N_i>0$ $(i=1\cdots n)$

Recall : Lemma 3.4 (b). $\{v_1, \cdots v_n\} \longrightarrow V$, let $x \in \mathfrak{g}$ be such that $\operatorname{ad} x$ is locally nilpotent on $\mathfrak{g}$. , $x^{N_i}(v_i)=0$ for all positive intergers N_i $(i=1\cdots)$, ★

Then x is locally nilpotent on ~~$\mathfrak{g}(A)$~~ V

Lemma 3.5 $\operatorname{ad} e_i$ $\operatorname{ad} f_i$ are locally nilpotent on $\mathfrak{g}(A)$,

$e^{N_i}(v_i)=0$ $f^{N_i}(v_i)=0$

"$\Leftarrow$" $\exists N_i$, s.t $f_i^{N_i}(v_\lambda)=0$ for $i=1\cdots n$

$\Rightarrow$ to prove e_i, f_i are locally nilpotent on V

For $v \in V_\mu = (U(n_-)(v_\lambda))_\mu$, let $\mu = \lambda - \sum_i k_i\alpha_i$

$e_i^m(v)=0$ where $m \geq k_i$, for all i

$V=U(n_-)(v_\lambda)$ is spaned by $\{f_{i_1}\cdots f_{i_k}(v_\lambda) \mid 1\leq i_1,\cdots i_k \leq n\}$

$\exists N_{i,i_1\cdots i_k}$ s.t $f_i^{N_{i,i_1\cdots i_k}}(f_{i_1}\cdots f_{i_k}(v_\lambda))=0$

By induction on k.

when $k=0$ $f_i^{N_i}(v_\lambda)=0$ for some $N_i>0$

Suppose $k>0$ ✓

In proof Lemma 3.4 , we get $\operatorname{ad} x = L_x - R_x$, $x \in \mathfrak{g}$

L_x, R_x are operator of left and right multi. by x

$L_{f_i} = \mathrm{ad} f_i + R_{f_i}$

$\Rightarrow f_i^{M} f_{i_1} = (L_{f_i})^m f_{i_1} = (\mathrm{ad} f_i + R_{f_i})^m f_{i_1}$

by (3.4.1) $\displaystyle\sum_{m=0}^{M} \binom{M}{m} (\mathrm{ad} f_i)^m f_{i_1} f_i^{M-m}$

$\Rightarrow f_i^M (f_{i_1} \cdots f_{i_{k+1}} (v_\lambda)) = \displaystyle\sum_{m=0}^{M} \binom{M}{m} \underbrace{(\mathrm{ad} f_i)^m f_{i_1}}_{0} f_i^{M-m} (f_{i_2} \cdots f_{i_{k+1}} (v_\lambda)$

By lemma 3.5 $\mathrm{ad} f_i$ $(i = 1 \dots n)$ are locally nilpotent on $\mathfrak{g}(A)$

$\exists N$. s.t $(\mathrm{ad} f_i)^{N} f_{i_1} = 0$ $(i = 1 \dots n)$

$M \geq N + N_{i_1 i_2 \cdots i_{k+1}}$

$\Rightarrow f_i^M (f_{i_1} \cdots f_{i_{k+1}} (v_\lambda)) = 0$

$\Rightarrow f_i^{N_i} (v_\lambda) = 0$

Lemma 10.1

The $\mathfrak{g}(A)$-module $L(\Lambda)$ is integrable if and only if $\Lambda \in P_+$

Pf: Formular (3.2.4) implies ($\Rightarrow$) and the following formula:

$e_i f_i^{\Lambda(\alpha_i^\vee)+1} (v) = 0$ if $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+$ $\Lambda \in P_+$

(3.2.4) $e(v_j) = (\lambda - j + 1) v_{j-1}$, $v_j = \frac{1}{j!} f^j (v)$

$h \cdot v = \lambda \cdot v$

$v \to \lambda$ $\quad e_i(v_j) = (\lambda(h) - j + 1) \frac{1}{(j-1)!} f_i^{j-1}(v_\lambda)$ $\quad \to \lambda$ $\quad h_i \in g_{(i)}$

v_{j-1}

$e_i^2(v_j) = (\lambda(h) - j + 1)(\lambda(h) - j) \frac{1}{(j-2)!} f_i^{j-2}(v_\lambda)$

$\vdots$

$e_i^{m}(v_j) = f_i^{j-m+1}(v_\lambda) = 0$ $\quad N \quad e_i^n = 0 \quad v_{j-n}(v_\lambda)$

m

$\lambda(h_i) = (j - m) \in \mathbb{Z}_+$

$0 < m \leq n$

$\Rightarrow \lambda \in P_+$

$(\Leftarrow)$ $\quad e_i(v_j) = (\lambda(h_i) - j + 1) v_{j-1}$

$$e_i\left(\frac{f^j(v_\lambda)}{j!}\right) = (\lambda(h_i) - j + 1)\left(\frac{f^{j-1}(v_\lambda)}{(j-1)!}\right)$$

$$e_i(f^j(v_\lambda)) = j(\lambda(h_i) - j + 1) f^{j-1}(v_\lambda)$$

Let $j = \lambda(\alpha_i^\vee) + 1$

$\lambda(\alpha_i^\vee) - (\lambda(\alpha_i^\vee) + 1) + 1 = 0$

$e_i f_i^{\lambda(\alpha_i^\vee)+1}(v_\lambda) = 0$ $\quad$ if $\lambda(\alpha_i^\vee) \in \mathbb{Z}_+$ $(\lambda \in P_+)$

$\Rightarrow$ $\cancel{n_+ \to e_i}$ $\quad n_+(\quad) = 0$

$\to e_j\left(f_i^{\lambda(\alpha_i^\vee)+1}(v_\lambda)\right) = 0$

since $[e_j, f_i] = 0$ for $j \neq i$.

$(\forall j \neq i)$

$$e_j\left(f_i^{\lambda(\alpha_i^\vee)+1}(v_\lambda)\right) = \underbrace{[e_j, f_i] f_i^{\lambda(\alpha_i^\vee)+1}(v_\lambda)}_{0} + f_i e_j f_i^{\lambda(\alpha_i^\vee)}(v_\lambda)$$

$$[\quad] + f_i^2 + e_j f_i^{\lambda(\alpha_i^\vee)-1}(v_\lambda)$$

$$\Rightarrow \quad e_j\left(\underbrace{f_i^{\Lambda(\alpha_i^\vee)+1}(v_\lambda)}_{}\right) \overset{=0}{=} 0 \quad \forall j$$

$n_+\left(f_i^{\Lambda(\alpha_i^\vee)+1}(v_\lambda)\right)=0$ It is a primitive vector of $L(\lambda)$

wight $= \Lambda - (\Lambda(\alpha_i^\vee)+1)\alpha_i \neq \Lambda$

$\Lambda \in P_+ \quad \Lambda(\alpha_i^\vee) \geq 0$

By 9.3 (b)

$f_i^{\Lambda(\alpha_i^\vee)+1}(v_\lambda) = 0$ $\Rightarrow L(\Lambda)$ is integrable.. #

$L(\Lambda)$

Denote $P(\Lambda)$ the set of weights $L(\Lambda)$

It is clear that $P(\Lambda) \subset P$ if $\Lambda \in P$

$\Lambda(\alpha_i^\vee) \in \mathbb{Z} \quad (i=1..n)$

$\lambda = \Lambda - \sum k_i \alpha_i \qquad \alpha_i(\alpha_i^\vee) \in \mathbb{Z}$

Prop 10.1 If $\lambda \in P_+$, then

$\mathrm{mult}_{L(\Lambda)}\lambda = \mathrm{mult}_{L(\Lambda)} w(\lambda)$ for $w \in W$

in particular, $P(\Lambda)$ is W-invariant

Pf: by Lemma 10.1 and prop 3.7(a)

prop 3.7(a) $\to$ V be an integrable module over $g(A)$

$\Rightarrow$ $\mathrm{mult}_V \lambda = \mathrm{mult}_V w(\lambda)$ for every $\lambda \in h^*$, $w \in W$

in particular, $P(V)$ is W-invariant

Lemma 10.1. $\Lambda \in P_+ \Rightarrow L(\Lambda)$ is integral..

Coro 10.1. If $\Lambda \in P_+$, then any $\lambda \in P(\Lambda)$ is W-equivalent to a unique $u \in P_+ \cap P(\Lambda)$

$\Downarrow L(\Lambda)$

Pf: (存在性) $\forall \lambda \in P(\Lambda)$

given $u \overset{\in P(\Lambda)}{=} W(\lambda)$, then $\Lambda - u \in Q_+$, s.t $\underline{ht(\Lambda - u) \text{ is minimal}}$

$\Downarrow \S$

$\Rightarrow$ $u + \alpha_i \notin P(\Lambda)$ for $i = 1 \cdots n$

since $r_i(u) = u - u(\alpha_i^\vee)\alpha_i$ $\Rightarrow$ $\underline{u(\alpha_i^\vee) \geq 0}$ for $i=1\cdots$

$u \in P_+$ $\Rightarrow$ $u \in \underline{P_+ \cap P(\Lambda)}$

(唯一性).

Lemma 3.12 (b)

$\underline{W(C) \cap X}$ $\overset{= W(C)}{\underset{u \in C \cup}{\Longrightarrow}}$ is exactly one point

$u(\alpha_i^\vee) \geq 0$ $\in$ C #

§ 10.2.

We let the Weyl group W act on the complex vector space $\tilde{\mathcal{E}}$ of all (possible infinite) linear combination of formal expontials by:

$$w\left(\sum_\lambda c_\lambda e(\lambda)\right) = \sum_\lambda c_\lambda e(w(\lambda)) \quad (w \in W$$

Then space $\tilde{\mathcal{E}}$ contains $\mathcal{E}$ as a subspace

then $\underline{w(P_1 P_2) = w(P_1) w(P_2)}$

$\Rightarrow$ $e(\lambda)e(u) = e(\lambda+u)$ $\Rightarrow$ $w(e(\lambda)e(u)) = w(e(\lambda+u))$

$= e(w(\lambda+\mu)) = e(w(\lambda) + w(\mu))$

$= e(w(\lambda))\, e(w(\mu)) = w(e(\lambda))\, w(e(\mu))$

Prop 10.1 says that:

(10.2.1) $\quad w(\underbrace{\mathrm{ch}\, L(\lambda)}_{\parallel}) = \underbrace{\mathrm{ch}\, L(\lambda)}$ for $w \in W$ and $\lambda \in P_+$

$\Big[\quad w\Big(\sum_{\lambda \in \mathfrak{h}^*} \mathrm{mult}_{L(\lambda)} \lambda\, e(\lambda)\Big) = \sum_{w(\lambda) \in \mathfrak{h}^*} \mathrm{mult}_{L(\lambda)}\, w(\lambda)\, e(w(\lambda))$

$\lambda \in P_+$

$P(\lambda) \in P_+$ $\qquad\qquad = \sum_{\lambda \in \mathfrak{h}^*} \mathrm{mult}_{L(\lambda)} \lambda\, e(\lambda) \quad \Big]$

$P(\lambda) \to W$-invariant

Consider now the ~~men~~ element (cf §9.7)

$$\mathrm{ch}(M(\lambda)) = e(\lambda) \underbrace{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{-\mathrm{mult}\,\alpha}} = e(\lambda) R^{-1}$$

$$R = \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\mathrm{mult}\,\alpha} \in \mathcal{E}$$

fixed an element $\rho \in \mathfrak{h}^*$ s.t (cf §2.5)

$\rho(\alpha_i^\vee) = 1 \quad (i = 1 \ldots n)$

$\Big[\quad (\rho | \alpha_i) = \dfrac{(\alpha_i | \alpha_i)}{2}$ for $i = 1 \ldots n$

$$\rho(\alpha_i^\vee) = \frac{2}{(\alpha_i | \alpha_i)}\, \rho(\nu^{-1}(\alpha_i)) = \frac{2}{(\alpha_i | \alpha_i)} \underbrace{(\rho | \alpha_i)}_{= \frac{(\alpha_i|\alpha_i)}{2}} = 1 \quad \Big]$$

For $w \in W$ set $\varepsilon(w) = (-1)^{l(w)}$. By (3.11) we have $\varepsilon(w) = \det_{\mathfrak{h}} w$

Furthermore, one has:

(10.2.2) $w(e(\rho)R) = \cancel{\varepsilon(w)}\ (-1)^{l(w)} e(\rho) R$ for $w \in W$

$r_i(\rho) = \rho - \rho(\alpha_i^\vee)\alpha_i = \rho - \alpha_i$

$r_i(e(\rho)) = e(\rho) e(-\alpha_i)$

~~It is~~ Indeed, It is sufficient to check (10.2.2) for each fundamental reflection r_i:

By Lemma 3.7 $\Delta_+ \setminus \{\alpha_i\}$ is r_i-invariant.

Prop 3.7: $\mathrm{mult}\, r_i(\alpha) = \mathrm{mult}\, \alpha$ for $\alpha \in \Delta_+$

$$r_i(R) = r_i\Big(\prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(-\alpha))^{\mathrm{mult}\,\alpha} \cdot (1 - e(-\alpha_i))\Big)$$

$$= \Big(\prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(-r_i(\alpha))^{\mathrm{mult}\, r_i(\alpha) = \mathrm{mult}\,\alpha}\Big)(1 - e(\alpha_i))$$

$$= \prod_{r_i(\alpha) \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(-r_i(\alpha))^{\mathrm{mult}(r_i(\alpha))} (1 - e(\alpha_i))$$

$$= \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(-\alpha))^{\mathrm{mult}(\alpha)} (1 - e(\alpha_i))$$

$$r_i(e(\rho)R) = r_i(e(\rho))\, r_i(R)$$

$$= e(\rho) e(-\alpha_i) \frac{(1 - e(\alpha_i))}{1 - e(-\alpha_i)} R$$

$$= e(\rho) \frac{e(-\alpha_i) - 1}{1 - e(-\alpha_i)} R = (-1) e(\rho) R$$

$l(r_i)=1$ ✓

By induction on $l(w)$

$\Rightarrow w(e(e), R) = \underbrace{(-1)^{l(w)}}_{\varepsilon(w)} e(e) R$, $w \in W$

§ 10.3

Recall $(\alpha_i | \alpha_i) > 0$ $(i = 1 \cdots n)$

$\mathrm{supp}(\alpha) \to$ For $\alpha = \sum_i k_i \alpha_i$

$k_i \neq 0$ $S(A)$ i

The following is key fact in the ...

Lemma 10.3

Let $\Lambda, \lambda \in P$ be such that $\lambda \leq \Lambda$ and $\underline{\Lambda + \lambda \in P_+}$

$\Rightarrow$ Then either $(\Lambda + \lambda)(\alpha_i^\vee) = 0$ for $i \in \mathrm{Supp}(\Lambda - \lambda)$
or $(\Lambda|\Lambda) - (\lambda|\lambda) > 0$. In particular, if $\Lambda \in P_+$, $\lambda \in P_+$ and $\lambda < \Lambda$ $\Rightarrow$ $\cancel{(\Lambda|\Lambda) > (\lambda|\lambda)}$ $(\Lambda|\Lambda) - (\lambda|\lambda) > 0$

Pf: We have $\lambda = \Lambda - \beta$, where $\beta = \sum_i k_i \alpha_i$, $k_i \geq 0$

Hence $(\Lambda|\Lambda) - (\lambda|\lambda)$

$(\Lambda|\Lambda) - (\lambda|\lambda) = (\Lambda + \lambda | \Lambda - \lambda)$

$= (\Lambda + \lambda | \beta) = (\Lambda + \lambda | \sum_{i=1}^n k_i \alpha_i)$

$= \sum_i^n k_i \langle \nu^{-1}(\alpha_i), \Lambda + \lambda \rangle$ $\to \in P_+$

$$= \sum_{i=1}^{n} \underbrace{k_i}_{\geq 0} \underbrace{\frac{(\alpha_i|\alpha_i)}{2}}_{>0} \underbrace{\langle \alpha_i^\vee , \Lambda + \lambda \rangle}_{\geq 0} \geq 0$$

Since $(\alpha_i|\alpha_i) > 0$ for all i

If $(\Lambda|\Lambda) - (\lambda|\lambda) = 0 \Rightarrow \langle \alpha_i^\vee , \Lambda + \lambda \rangle = 0$

$k_i = 0$

$k_i \neq 0$. for all $i \in \operatorname{supp}(\Lambda - \lambda)$, $\langle \alpha_i^\vee , \Lambda + \lambda \rangle = 0$

#

§10.4. (Weyl – Kac Character Formula)

Thm 10.4 , let $g(A)$ be a symmetrizable Kac- Moody algebra let $L(\Lambda)$ be an irreducible $g(A)$-module with highest weight $\Lambda \in P_+$, Then

$$(10.4.1) \quad ch(L(\Lambda)) = \frac{\sum_{w \in W} \varepsilon(w) e(w(\Lambda+\rho) - \rho)}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\operatorname{mult} \alpha}}$$

$V \longrightarrow \Lambda$

pf: $ch V = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c_\lambda \, ch\, M(\lambda)$, where $c_\lambda \in \mathbb{Z}$, $c_\Lambda = 1$

(9.8.1)

multiplying both sides of (9.8.1) by $e(\rho) R$ using (9.7.2)

(9.7.2) $ch\, M(\Lambda) = e(\Lambda) R^{-1}$

ε

$$e(\rho) R \, ch V = \sum_{\substack{\lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} e(\rho) R \, c_\lambda \, ch\, M(\lambda)$$

$$= \sum c_\lambda\, e(\lambda+\rho)$$

We obtain (10.4.2) $e(\rho) R\, ch\, L(\Lambda) = \sum_{\substack{\lambda \le \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} c_\lambda\, e(\lambda+\rho)$

$$w(e(\rho) R\, ch(L(\lambda)))$$
$$= \varepsilon(w)\, e(\rho) R\, ch L(\lambda) \qquad \varepsilon(w) = (-1)^{l(w)}$$

By (10.2.1) (10.2.2) $\Longrightarrow$ the left-hand side of (10.4.2)
W-skew-invariant

(an element $L \in \tilde{\varepsilon}$ is called W-skew-invariant if
$w(L) = \varepsilon(w) L$)

(10.2.1) $w(ch(L(\Lambda))) = ch L(\Lambda)$

(10.2.1) $w(e(\rho) R) = \varepsilon(w) e(\rho) R$

Hence the cofficents in the right-hand side of (10.4.2) have the following property:

(10.4.3) If $w(e(\lambda+\rho)) = e(w(\lambda+\rho))$

Let $u+\rho = w(\lambda+\rho)$

$u \le \Lambda \qquad |u+\rho|^2 = |\lambda+\rho|^2$

(10.4.3) $c_\lambda = \varepsilon(w)\, c_u$ if $u+\rho = w(\lambda+\rho)$ for some $w \in W$ $c_u \neq 0$

$\lambda \longrightarrow$ 存在 $w \longrightarrow u$

Let λ s.t $c_\lambda \neq 0$, then by (10.4.3),
$c_{w(\lambda+\rho)-\rho} \neq 0$ for every $w \in W$

Hence, it follow from (10.4.2) that
$w(\lambda+\rho) \le \lambda+\rho \qquad (u \le \rho)$

Let $u \in \{ w(\lambda+\rho)-\rho, w \in W \}$ be such that $ht(\Lambda - u)$ minimal $\{ C_\lambda = \varepsilon(w)$

$u = \Lambda$

Then clearly $u+\rho \in P_+$ $(u+\rho)(\alpha_i^\vee) \geq 0$

$\Lambda \in P_+$ $\lambda(\alpha_i^\vee) \geq 0$ $\Lambda + \rho \in P_{++}$

Apply Lemma 10.3 to the elements $\lambda+\rho \in P_{++}$ and $u+\rho \in P_+$, if $u+\rho < \Lambda+\rho$

then $(\lambda+\rho \mid \lambda+\rho) > (u+\rho \mid u+\rho)$ $\{$

$(\cancel{\lambda+\lambda}$

$|\Lambda+\rho|^2 = |u+\rho|^2$

$u = \Lambda$

$(10.4.3) \Longrightarrow C_\lambda = \varepsilon(w) = (-1)^{l(w)}$

$C_\lambda = \varepsilon(w) C_u = \varepsilon(w) C_\Lambda = \varepsilon(w)$

$\shortparallel$

But $\Lambda+\rho \in P_{++}$, hence prop 3.12 (b) $\cancel{u+\rho}$

$\cancel{with}$ $W \curvearrowright C$

$\forall \lambda \in P$. $\to u = \lambda$ s.t $w(\lambda+\rho) = \Lambda+\rho \Rightarrow C_\lambda = \varepsilon(w)$

$$(10.4.2) \Rightarrow e(\rho) R \, ch L(\Lambda) = \sum_{\substack{\cancel{\lambda \leq \Lambda} \\ \cancel{|\lambda+\rho|^2 = |\Lambda+\rho|^2} \\ w \in W}} \varepsilon(w) e(w(\lambda+\rho))$$

$\lambda = \Lambda$

$w = 1$ $C_\Lambda = 1$ $\varepsilon(1) = 1$

$C_\lambda = \varepsilon(w)$ for $\lambda \leq \Lambda$ and $|\lambda+\rho|^2 = |\Lambda+\rho|^2$

$w(\lambda+\rho) = \lambda+\rho \Rightarrow w=1$

$c_\lambda = 1 \qquad \varepsilon(1)=1$

$$e(\rho) R \,\mathrm{ch}\, L(\lambda) = \sum_{w\in W} \varepsilon(w)\, e(w(\lambda+\rho))$$

$$\mathrm{ch} L(\lambda) = \frac{\sum_{w\in W} \varepsilon(w)\, e(w(\lambda+\rho))}{R\, e(\rho)}$$

$$= \frac{\sum_{w\in W} \varepsilon(w)\, e(w(\lambda+\rho)-\rho)}{R}$$

(10.4.1)

Weyl – Kac . . . #

Rmk: ① $\Lambda = 0$ in (10.4.1) since $L(0)$ is the trivial 1-dimensional module over $g(A)$

$L(\Lambda)$ $\cong$ $\mathbb{C}$

$\dim(L(\Lambda)) = 1 \Leftarrow \Lambda|_{\mathfrak{h}} = 0$

$h \mapsto 0 \quad \forall\, h \in \mathfrak{h}$

$L(\lambda) \cong \mathbb{C}$

$$\mathrm{ch}(L(0)) = e^0 = 1$$

$$\mathrm{ch} L(\lambda) = \frac{\sum_{w\in W} \varepsilon(w)\, e(w(\lambda+\rho)-\rho)}{\prod_{\alpha\in\Delta_+} (1-e(-\alpha))^{\mathrm{mult}\,\alpha}}$$

$\forall \Lambda$

$\Lambda = 0$

$\mathrm{ch}\, L(0) \Rightarrow \prod_{\alpha\in\Delta_+} (1-e(-\alpha))^{\mathrm{mult}\,\alpha} = \sum_{w\in W} \varepsilon(w)\, e(w(\rho)-\rho)$

$$\mathrm{ch}\, L(\Lambda) = \frac{\sum_{w\in W} \varepsilon(w)\, e(w(\Lambda+\rho)-\rho)}{\sum_{w\in W} \varepsilon(w)\, e(w(\rho)-\rho)} \quad (10.4.5)$$

②. ~~We~~ $\mathfrak{g}(A)$ is a finite-dimensional Lie alg.

→ Weyl character formula

→ Weyl demominator identity. #

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§9.12 → 9.11

Recall Prop 9.11 $\mathfrak{g}(A) \cong \tilde{\mathfrak{g}}(A)/\tau$

$\tau = \tau_- \oplus \tau_+$ $\quad \tau_\alpha = \tilde{\mathfrak{g}}_\alpha \cap \tau$

The ideal τ_+ (resp τ_-) is generated as an ideal in $\tilde{n}_+$ by those τ_α (resp $\tau_{-\alpha}$) for which $\alpha \in Q_+ \setminus \Pi$ and $2(\rho|\alpha) = (\alpha|\alpha)$

τ

$\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau \quad \tau = 0$

prop 9.12 $A = (a_{ij})$ be symm ...

$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

(a) If (a_{ij}) ~~are positive~~ are nonzero real numbers of same sign for all $\alpha, \beta \in L$

$\Rightarrow \mathfrak{n}_+$ $(\mathfrak{n}_-)$ is a free Lie algebra on generators $e_1 \cdots e_n$ (resp $f_1 \cdots f_n$)

$\alpha \notin \Pi$ $\alpha \neq 0$

Pf: For $\alpha = \sum_i k_i \alpha_i \in Q_+$, we have

$$2(\rho|\alpha) - (\alpha|\alpha) = \sum_i \underbrace{a_{ii}}_{} \underbrace{(k_i - k_i^2)}_{<0} - \underbrace{\sum_{i \neq j} a_{ij} k_i k_j}_{<0}$$

<0

(b). $L \subset \Delta_+$ be such that

(i) $(\alpha|\beta) \neq 0$ 符号一样 for $\alpha, \beta \in L$

(ii) $\alpha, \beta \in L$, $\alpha - \beta \in \Delta_+ \Rightarrow \alpha - \beta \in L$

$\mathfrak{n}_\pm^L$ to be a subalgebra of $\mathfrak{n}_\pm$ generated by $\bigoplus_{\alpha \in L} \mathfrak{g}_{\pm\alpha}$

$\Rightarrow \mathfrak{n}_\pm^L$ is free generated by $\bigoplus_{\alpha \in L} \mathfrak{g}_{\pm\alpha}^0$

(1) $\mathfrak{g}_\alpha \times \mathfrak{g}_\beta = 0$, $\alpha + \beta \neq 0$

$\mathfrak{g}_\alpha \in \{x \in \mathfrak{g}_\alpha \mid (x \mid [\mathfrak{n}^L, \mathfrak{n}_-^L]) = 0\}$

a

pf (b)

$|L|=1$ $\alpha\in\Delta_+$ $(\alpha|\alpha)$ $d\alpha$ $\mathfrak{g}_{-2\alpha}$

$n_+^L \subseteq n_+$ is generated by $\mathfrak{g}_\alpha$, $\mathfrak{h}^L = C\nu^{-1}(\alpha)$

For $\alpha\in L$ $\mathfrak{g}_\alpha^0 = \mathfrak{g}_\alpha \cap [n_-^L, n_-^L]^\perp$

the matrix $B = ((\alpha|\beta))_{\alpha,\beta\in I}$

mult α = dim $\mathfrak{g}_\alpha$

$\mathfrak{g}^L = n_+^L \oplus \mathfrak{h}^L \oplus n_+^L$

$\mathfrak{h}^L = C\nu^{-1}(\alpha)$

$\dim \mathfrak{g}_\alpha = d \longrightarrow \{x_\alpha^{(1)} \dots x_\alpha^{(d)}\}$

$\dim \mathfrak{g}_{-\alpha} = d$ $\{y_\alpha^{(1)} \dots y_\alpha^{(d)}\}$,

$(x_\alpha^{(i)} | x_\alpha^{(j)}) = \delta_{ij}$

$B = (b_{ij})_{i,j\in I}$

$b_{ij} = (\alpha|\beta)$ if $(i,j = 1 \dots d)$

$$= \begin{pmatrix} (\alpha|\alpha) & (\alpha|\alpha \\ (\alpha|\alpha) & \dots \end{pmatrix}_{d\times d}$$

$\mathfrak{g}(B)$ / (center ideal) $\Leftarrow$ $\mathfrak{g}^L$

Cartan subalgebra of $\mathfrak{g}(B)$