Recall: Lemma 10.6. Let V be a highest-weight module over $g(A)$.

Then: a) $Y(V)$ is a convex set

b) $Y(V) \supset Y \cap Y_0$

c) $Y(V) \supset \log n$

· Remark 10.6: Let $T \subset X_c$ be an open convex W-invariant set. Then $T \subset$ convex hull $\left(\bigcup_{w \in W} w(T \cap Y_0)\right)$

proof: $T_0 := \bigcup_{w \in W} w(\overline{Y_0} \setminus Y_0)$ is nowhere dense in $X_c = \bigcup_{w \in W} w(\overline{Y_0})$

($W \in GL(H^*)$)

Hence, every $h \in T$ lies in the convex hull of

$T \setminus T_0 = \bigcup_{w \in W} w(T \cap Y_0)$.

#

prop 10.6: Let $g(A)$ be a Kac-Moody algebra and let $L(\Lambda)$ be an irreducible $g(A)$-module with hight weight $\Lambda \in P_+$.

Then:

a) $Y(L(\Lambda))$ is a <u>solid</u> convex W-invariant set, which for every $x \in \mathrm{Int}\, X_c$ contains tx for all sufficiently large $t \in \mathbb{R}_+$.

(a subset S of a vector lattice is said to be solid).

b) $\mathrm{ch}\, L(\Lambda)$ is a holomorphic function on $\mathrm{Int}\, Y(L(\Lambda))$

c) $Y(L(\Lambda)) \supset \mathrm{Int}\, Y$.

d) The series $\sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)}$ converges absolutely on $\mathrm{Int}\, X_c$ to holomorphic function, and diverges absolutely on $H \setminus \mathrm{Int}\, X_c$

e) provided that the Cartan matrix A is symmetrizable, $\mathrm{ch}\, L(\Lambda)$ can be extended from $Y(L(\Lambda)) \cap X_c$ to a meromorphic function on Int X_c.

proof: set $T = \mathrm{Int}\, Y$. It is clear that T is open, convex, and W-invariant. [$w \in W$ induce a automorphism of H. w^* s.t. $\langle w^*(h), \lambda \rangle = \langle h, w(\lambda) \rangle$, $h \in H$, $\lambda \in H^*$,

$$Y = \{ h \in H \mid \sum_{\alpha \in \Delta_+} (\mathrm{mult}\,\alpha) |e^{-\langle \alpha, h \rangle}| < \infty \}.$$

$$\sum_{\alpha \in \Delta_+} \mathrm{mult}\,\alpha \, |e^{-\langle \alpha, w(h) \rangle}| = \sum_{\alpha \in \Delta_+} \mathrm{mult}\,\alpha \, |e^{-\langle w^*(\alpha), h \rangle}|$$]

by lemma 10.6 b), we have $Y(L(\Lambda)) \supset Y \cap Y_0$.

Furthermore, $Y(L(\Lambda))$ is a convex W-invariant set.

Now c) follows from Rem 10.6.

[$T = \mathrm{Int}\, Y \subset$ convex hull $(\bigcup_{w \in W} w(T \cap Y_0)) \subset$ convex hull $(\bigcup_{w \in W} (Y \cap Y_0))$

$\subset Y(L(\Lambda))$]

· To prove a), we have to show that:

$X' := \{ x \in \mathrm{Int}\, X_c \mid tx \in Y(L(\Lambda))$ for sufficiently large $t \in \mathbb{R}_+ \} = \mathrm{Int}\, X_c$

But again, X' is W-invariant and convex and contains Y_0.

$Y_0 \subset X'$ lemma: $Y(L(\Lambda)) \supset Y_{\log n}$.

$\forall s \in Y_{\log n}$, i.e. $t \cdot \frac{s}{t} \in Y(L(\Lambda))$, $\lim_{t \to \infty} \frac{s}{t} = 0$.

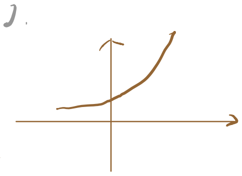
$\mathrm{Re} \langle \alpha_i, \frac{s}{t} \rangle > 0$. $\Rightarrow \frac{s}{t} \in Y_0$.

· Again, we can apply Remark 10.6.

[$\mathrm{Int}\, X_c \subset X'$, since let $T = \mathrm{Int}\, X_c$, then:

$\mathrm{Int}\, X_c \subset$ convex hull $(\bigcup_{w \in W} w(\mathrm{Int}\, X_c \cap Y_0))$. $\subset X'$]

· The convexity of $|e^{\lambda}|$ implies that the absolute converges in uniform on compact sets, which proves b).



[· if S is compact interval (or in general a compact topological space), and (f_n) is a monotone increasing sequence (meaning $f_n(x) < f_{n+1}(x)$ for all n and x) of continuous functions with a pointwise limit f which is also continuous, the the convergence is necessarily uniform].

$(\sum \underline{f_n})' = \sum f_n'$. 复变

- In order to prove d) Remark that: $w(\Lambda+\rho)-(\Lambda+\rho)$ are distinct for distinct $w \in W$. by prop 3.12 b).

It is clear from the proof of prop. 3.12 d) that:

$\underline{(\Lambda+\rho)-w(\Lambda+\rho) \in Q_+}$.

[prop 3.12 d) $C = \{h \in H_R \mid$ for every $w \in W$, $h - w(h) = \sum_i c_i \alpha_i^\vee$, where $c_i \geq 0\}$.

The definition of $C = \{h \in H_R \mid \langle \alpha_i, h\rangle \geq 0$, for $i = 1, \cdots, n\}$.

Since $\Lambda \in P_+$, $\langle \rho, \alpha_i^\vee\rangle = \frac{1}{2} a_{ii}$ $\Rightarrow$ $\Lambda + \rho \in C^\vee$]

Hence we have for $h \in Y_0$:

$$\left|\sum_{w \in W} \varepsilon(w) e^{\langle w(\Lambda+\rho)-(\Lambda+\rho), h\rangle}\right| \leq \underline{\sum_{\alpha \in Q_+} \left|e^{-\langle\alpha, h\rangle}\right| < \infty}.$$

$Re\langle \alpha_i, h\rangle > 0$. $\left|e^{-\langle\alpha, h\rangle}\right| < 1$

$1 + e^{-\alpha_1} + e^{-2\alpha_1} + \cdots + e^{\cdots}$

$\prod_{i=1}^{n} (1 - e^{-\alpha_i})^{-1}$

Thus the region of absolute convergence of the series in question contains Y_0. and is clearly convex and W-invariant, so it contains $\operatorname{Int} X_{\mathbb{C}}$.

- Let $\underline{h \in H \setminus \operatorname{Int} X_{\mathbb{C}}}$, Then the set:

$|\Delta_0| := |\{\alpha \in \Delta_+^{re} \mid Re\langle\alpha, h\rangle \leq 0\}| = \infty$. by prop 3.12 c) and f).

[prop 3.12 c) $X = \{h \in H_R \mid \langle\alpha, h\rangle < 0$ only for a finite number of $\alpha \in \Delta_+\}$

$r_i(h) = h$ $\langle \alpha_i, h\rangle = 0$.

$r_\alpha r_\beta(h) = h -$

f) if $h \in X$, then $|W_h| < \infty$ iff h lies in the interior of X, where $W_h = \{ w \in W \mid w(h) = h \}$. $\Rightarrow |W_h| = \infty$

and for every $\alpha \in \Delta_0$, we have

$$|e^{\langle r_\alpha(\Lambda+\rho), h\rangle}| \geq |e^{\langle \Lambda+\rho, h\rangle}|$$

[$r_\alpha(\Lambda+\rho) = \Lambda+\rho - \langle \Lambda+\rho, \alpha^\vee\rangle\alpha$, & $\langle \Lambda+\rho, \alpha^\vee\rangle \geq 0$.

$\Rightarrow \langle r_\alpha(\Lambda+\rho), h\rangle > \langle \Lambda+\rho, h\rangle$.].

$C = \{ h \in \mathfrak{h}_\mathbb{R} \mid \langle \alpha_i, h\rangle \geq 0$ for $i = 1, \cdots, n \}$ fundamental chamber.

prop 3.12 b) any orbit $W \cdot h$ of $h \in X$ intersects C is exactly one point.

$\Rightarrow$ divergence at h.

e). follows from d) and (10.4.5)

$$(10.4.5) \quad ch\, L(\Lambda) = \sum_{w \in W} \varepsilon(w) e(w(\Lambda+\rho)) \Big/ \sum_{w\in W} \varepsilon(w) e(w(\rho))$$

#

- Denote by $\mathcal{O}_c$ the full subcategory the category $\mathcal{O}$ of $g(A)$-modules V such that $ch V$ converges absolutely on Y_N for some $N > 0$.
- Also denote by $\mathcal{E}_c$ the subalgebra of the series from $\mathcal{E}$ which converge absolutely on Y_N for some N.
- every highest-weight module lies in $\mathcal{O}_c$.
- we have a homomorphism φ of $\mathcal{E}_c$ into the algebra of functions, which are holomorphic on Y_N for some N, defined by $\varphi : e(\lambda) \longmapsto e^\lambda$.
- Applying φ to both sides of formulas (10.4.5) and (10.4.4) we obtain on Y:

(10.6.2) $\quad ch_{L(\Lambda)} = \sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\rho)} \Big/ \sum_{w\in W} \varepsilon(w) e^{w(\rho)}$

(10.6.3) $\quad \prod_{\alpha\in\Delta_+} (1-e^{-\alpha})^{\mathrm{mult}\,\alpha} = \sum_{w\in W} \varepsilon(w) e^{w(\rho)-\rho}$.

§10.7. complete reducibility theorem.

Theorem 10.7. Let A be a symmetrizable generalized Cartan matrix

a) Suppose that a $g'(A)$-module V satisfies the following two conditions:

(10.7.1) for every $v\in V$, there exists N such that:

$e_{i_1}\cdots e_{i_k}(v) = 0$ whenever $k > N$.

(10.7.2) for every $v\in V$ and every i there exists N such that: $f_i^N(v) = 0$ $\Rightarrow$ V is integrable

Then V is isomorphic to a direct sum of $g'(A)$-module $L(\Lambda)$ s.t. $\langle\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_+$ for all i.

b) Every integrable $g(A)$-module V from the category $\mathcal{O}$ 最高权模之和. is isomorphic to a direct sum of modules $L(\Lambda)$, $\Lambda\in P_+$.

proof: Thanks to prop 9.10 b) (P155) and Remark 3.6 (P35) in the statement a) (resp. prop 9.9 b) P154) in case of b).

It suffices to check that if λ and μ are primitive weights and $\beta\in Q_+\setminus\{0\}$ is such that $\lambda-\mu-\beta$, then

$2\langle\lambda+\rho, \nu^{-1}(\beta)\rangle \neq (\beta|\beta)$. primitive $\to$ highest weight

. since the module V is integrable, we have by (3.2.4) P31

$\langle\lambda, \alpha_i^\vee\rangle \geq 0$ $(i=1,\dots,n)$ for every primitive weight λ.

[(3.2.4). $e(v_j) = (\lambda-j+1)v_{j-1}$, where $v_j = (j!)^{-1} f^j(v)$, $h(v) = \lambda v$.

Cor: 3.6: a) If λ is a weight of an integral $g(A)$-module V.

and $\lambda + \alpha_i$ is not a weight, then $\langle \lambda, \alpha_i^\vee \rangle \geq 0$.

prop 3.6 b) ii) if both λ and $\lambda + \alpha_i$ are weights, then

$e_i(v_\lambda) \neq 0$.

$v \notin V \quad n_+(v) \in V$.

上去下不来

$j = 0$, then $e(v) = 0$.

But then we can write:

$$\geq \langle \lambda + \rho, \nu^{-1}(\beta) \rangle - (\beta | \beta) = \langle \lambda + (\lambda - \beta) + 2\beta, \nu^{-1}(\beta) \rangle$$

$$= \langle \lambda + \mu + 2\rho, \nu^{-1}(\beta) \rangle > 0.$$

#

Consider verma module dual.

$M(\Lambda) / M'(\Lambda)$ 达不到最高权回不来.

As a $g_{(i)}$-module, $V = V_1 \oplus \cdots \oplus V_n \oplus \cdots$

$V_1 \uparrow sl_2 \qquad V_n \uparrow sl_2$

$M'(\Lambda) \quad v_1 \quad v_2 \quad \cdots \quad v_n$

$x_1 \quad x_2$ 次高

$v_1 \wedge v_2$ 最高权.

$sl_2 \quad$ 5维 $\quad$ 7维.

$\uparrow$ 2维